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A-posteriori-steered and hp-robust multigrid solvers

Ani Miraçi

joint work with

Jan Papež, Dirk Praetorius, Julian Streitberger, Martin Vohralík

TU Wien
Institute for Analysis and Scientific Computing

Outline

Setting

A-posteriori-steered multigrid: p-robustness

A-posteriori-steered multigrid: hp-robustness

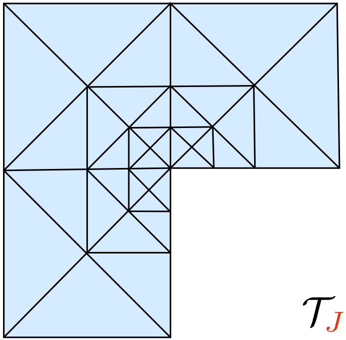
Conclusion

Setting

Setting: domain $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, source term $f \in L^2(\Omega)$, s.p.d. diffusion coefficient $\mathbf{K} \in [L^\infty(\Omega)]^{d\times d}$.
Model problem: find $u \in H_0^1(\Omega)$ such that $(\mathbf{K}\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$.

Fix $p \geq 1$, let $\mathbb{P}_p(\mathcal{T}_J) := \{v_J \in L^2(\Omega), v_J|_K \in \mathbb{P}_p(K)\ \forall K \in \mathcal{T}_J\}$, and define

$$V_J^p := \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega).$$



$\mathcal{T}_J$

Discrete problem: Find $u_J \in V_J^p$ such that

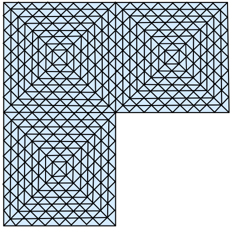
$$(\mathbf{K}\nabla u_J, \nabla v_J) = (f, v_J) \quad \forall v_J \in V_J^p. \qquad \text{(FE)}$$

By introducing a basis of V_J^p: $\mathbb{A}_J \mathrm{U}_J = \mathrm{F}_J$. We work with the *basis-independent* functional formulation (FE).

Algebraic residual functional: $v_J \mapsto (f, v_J) - (\mathbf{K}\nabla u_J^i, \nabla v_J) \in \mathbb{R}, \quad v_J \in V_J^p.$

A hierarchy of meshes and spaces

Example: Two different hierarchies with $J = 3$ refinements.



Assumptions: The meshes $\{\mathcal{T}_j\}_{1\leq j\leq J}$ can be generated through *uniform* or *adaptive* refinement, satisfying:

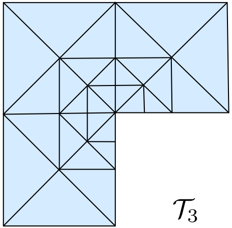
- (C_{qu}-)quasi-uniform $\mathcal{T}_0$,
- ($\kappa_{\mathcal{T}}$-)shape-regularity,
- (C_{ref}-)maximum strength of refinement.

For given p and J, choose *increasing* polynomial degrees p_j, $j \in \{1, \ldots, J\}$,

$$1 = p_0 \leq p_1 \leq p_2 \leq \ldots \leq p_J = p,$$

and define the spaces

$$V_j^{p_j} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\Omega).$$



$\mathcal{T}_3$

Economical choice: $p_0 = p_1 = \ldots = p_{J-1} = 1,\ p_J = p$.

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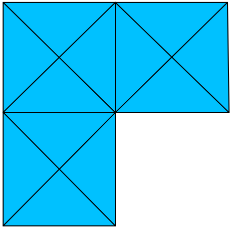

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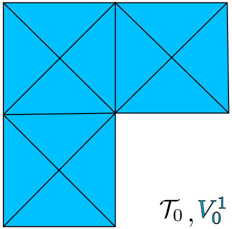

$\mathcal{T}_0$, V_0^1

Economical choice: $p_0 = p_1 = \ldots = p_{J-1} = 1$, $p_J = p$.

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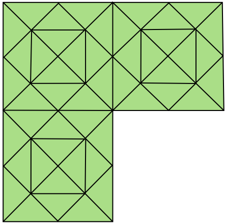

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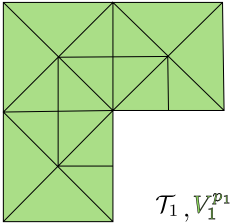

$\mathcal{T}_1$, $V_1^{p_1}$

Economical choice: $p_0 = p_1 = \ldots = p_{J-1} = 1, \; p_J = p.$

A hierarchy of meshes and spaces

Example: Two different hierarchies with $J = 3$ refinements.

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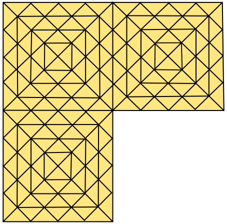

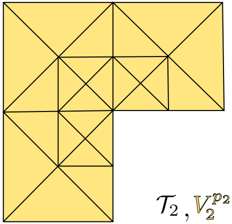

$\mathcal{T}_2$, $V_2^{p_2}$

A hierarchy of meshes and spaces

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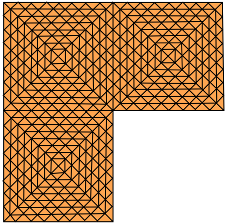

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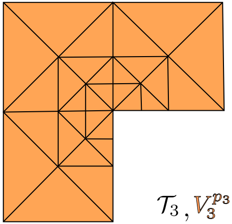

$\mathcal{T}_3$, $V_3^{p_3}$

Economical choice: $p_0 = p_1 = \dots = p_{J-1} = 1,\ p_J = p.$

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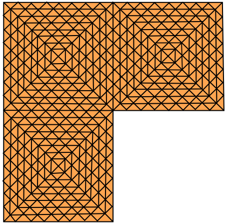

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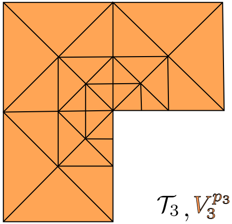


$\mathcal{T}_3, V_3^{p_3}$

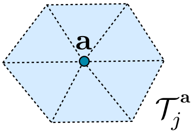
Economical choice: $p_0 = p_1 = \dots = p_{J-1} = 1,\ p_J = p.$

Patches

Let $\mathcal{V}_j$ be the set of vertices of the mesh $\mathcal{T}_j$, $j \in \{1, \ldots, J\}$.

Given a vertex $\mathbf{a} \in \mathcal{V}_j$, we denote:

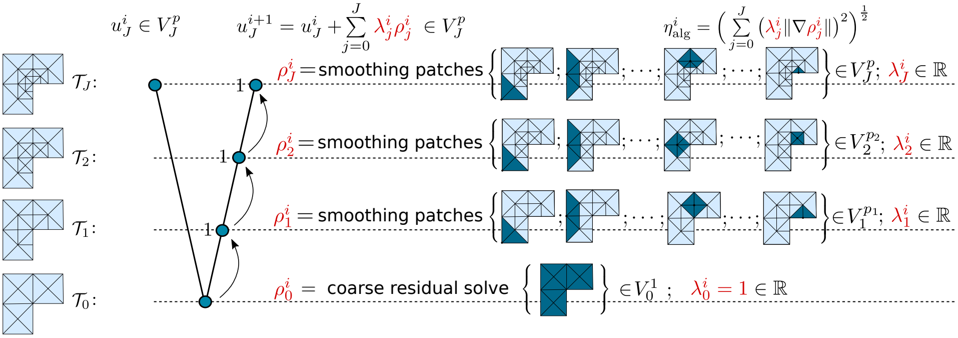
- $\mathcal{T}_j^{\mathbf{a}}$ the patch of elements sharing vertex $\mathbf{a}$,
- $\omega_j^{\mathbf{a}}$ the corresponding patch subdomain,
- $V_j^{\mathbf{a}} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\omega_j^{\mathbf{a}})$ the associated local space.



patch subdomain $\omega_j^{\mathbf{a}}$ for a vertex $\mathbf{a} \in \mathcal{V}_j$

A-posteriori-steered multigrid: p-robustness

A-posteriori-steered multigrid



- V-cycle of geometric multigrid: coarse grid solve and level-wise smoothing,
- zero pre- and a single post-smoothing step,
- *additive Schwarz* / *block Jacobi* smoothing: fully *parallel* on each level,
- level-wise step-sizes in multigrid error correction stage: optimally chosen by *line search*[1].

[1] Heinrichs. "Line relaxation for spectral multigrid methods". *J. Comput. Phys.* 1988.

Multilevel lifting of the algebraic residual

Let $u_J^i \in V_J^p$ be *arbitrary*. We construct its associated *level-wise algebraic residual liftings* $\{\rho_j^i\}_{j=0}^J$ and *level-wise step-sizes* $\{\lambda_j^i\}_{j=0}^J$ as follows:

Coarse solve: Define $\rho_0^i \in V_0$ by: $(\mathbf{K}\nabla\rho_0^i, \nabla v_0) = (f, v_0) - (\mathbf{K}\nabla u_J^i, \nabla v_0), \quad \forall v_0 \in V_0$ and set $\lambda_0^i := 1$.

Level-wise local solves: For $j = 1:J$, for all $\mathbf{a} \in \mathcal{V}_j$, define $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$ by :

$$(\mathbf{K}\nabla\rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - (\mathbf{K}\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \sum_{k=0}^{j-1} \lambda_k^i (\mathbf{K}\nabla\rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}}, \quad \forall v_{j,\mathbf{a}} \in V_j^{\mathbf{a}}.$$

Level-wise algebraic residual liftings: Define $\rho_j^i \in V_j^{p_j}$ by: $\rho_j^i := \sum_{\mathbf{a}\in\mathcal{V}_j} \rho_{j,\mathbf{a}}^i$.

Level-wise step-sizes: If $\rho_j^i \neq 0$, set $\lambda_j^i := \dfrac{(f, \rho_j^i) - (\mathbf{K}\nabla(u_J^i + \sum_{k=0}^{j-1} \lambda_k^i \rho_k^i), \nabla\rho_j^i)}{\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|^2}$, otherwise set $\lambda_j^i := 1$.

Proposition (Pythagorean error representation of one solver step)

For $u_J^i \in V_J^p$, let $u_J^{i+1} \in V_J^p$ be the next iterate constructed from u_J^i by our solver. Then

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2 - \underbrace{\sum_{j=0}^{J}\left(\lambda_j^i\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|\right)^2}_{=\,(\eta_{\mathrm{alg}}^i)^2}.$$

Proof: Going from the finest level to the coarsest and by construction of the optimal step-sizes λ_j^i:

$$\begin{aligned}
\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 &= \left\|\mathbf{K}^{\frac{1}{2}}\nabla\Big(u_J - \big(u_J^i + \sum_{j=0}^{J}\lambda_j^i\rho_j^i\big)\Big)\right\|^2 \\
&= \left\|\mathbf{K}^{\frac{1}{2}}\nabla\big(u_J - u_J^i - \sum_{j=0}^{J-1}\lambda_j^i\rho_j^i\big)\right\|^2 - 2\lambda_J^i\left[(f,\rho_J^i) - \Big(\mathbf{K}^{\frac{1}{2}}\nabla\big(u_J^i + \sum_{j=0}^{J-1}\lambda_j^i\rho_j^i\big),\nabla\rho_J^i\Big)\right] + \left(\lambda_J^i\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_J^i\right\|\right)^2 \\
&= \left\|\mathbf{K}^{\frac{1}{2}}\nabla\big(u_J - u_J^i - \sum_{j=0}^{J-1}\lambda_j^i\rho_j^i\big)\right\|^2 - \left(\lambda_J^i\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_J^i\right\|\right)^2 = \ldots = \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 - \sum_{j=0}^{J}\left(\lambda_j^i\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|\right)^2 \\
&= \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 - (\eta_{\mathrm{alg}}^i)^2.
\end{aligned}$$

Main results

Theorem (p-robust reliable and efficient bound on the algebraic error)

Let $u_J^i \in V_J^p$ be arbitrary. Let η_{alg}^i be the associated a posteriori estimator on the algebraic error. Then, there holds:

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\| \geq \eta_{\mathrm{alg}}^i \quad \textit{and} \quad \eta_{\mathrm{alg}}^i \geq \beta\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \quad 0 < \beta(\kappa_{\mathcal{T}}, J, d, \mathbf{K}) < 1.$$

Theorem (p-robust error contraction of the multilevel solver)

For $u_J^i \in V_J^p$, let $u_J^{i+1} \in V_J^p$ be constructed from u_J^i using one step of the solver. There holds:

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\| \leq \alpha\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \qquad \alpha = \sqrt{1-\beta^2}.$$

Remark:

- The dependence on J is at most *linear* under minimal H^1-regularity.
- Complete *independence* from J is obtained in H^2-regularity setting.
- $\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 - \left(\eta_{\mathrm{alg}}^i\right)^2.$

Schöberl, Melenk, Pechstein, and Zaglmayr. *IMA J. Numer. Anal.* 2008.

p-robustness: iteration numbers for uniform meshes

Stopping criterion: $$\frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{i_s}\|}{\|\mathrm{F}_J\|} \le 10^{-5} \frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{0}\|}{\|\mathrm{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

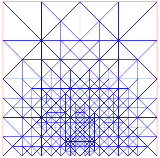
			H^2-regular			H^1-regular			
			Sine	Peak	L-shape	Checkerboard		Skyscraper	
			$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathcal{J}(\mathbf{K})=O(10^6)$	$\mathcal{J}(\mathbf{K})=O(1)$	$\mathcal{J}(\mathbf{K})=O(10^7)$
			$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	21	18	18	19	19
	3	$1e^5$	29	28	29	27	28	31	31
	6	$6e^5$	30	30	26	24	25	28	28
	9	$1e^6$	31	30	23	23	23	26	26
4	1	$6e^4$	21	20	21	19	19	19	19
	3	$6e^5$	29	29	28	26	27	30	30
	6	$2e^6$	31	30	25	24	24	27	27
	9	$5e^6$	32	31	23	22	23	25	25

Numerical $\mathbf{K}$- and J-robustness is observed even in low-regularity cases.

p-robustness: iteration numbers for graded meshes

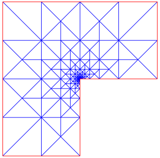
Peak, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
4	1	14	8	1	16	16	1	16
	3	11		3	9		3	9
	6	9		6	8		6	8
	9	8		9	8		9	9



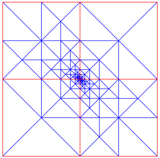
L-shape, $\mathbf{K} = I$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4



Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15



These low-regularity test cases indicate the possibility of J-*dependence*, in accordance with the theoretical results.

Comparison with other multilevel solvers

We compare our methods with [2,3,4] in terms of the number of iterations (and CPU times[5]).

		∼MG(0,1) -bJ $1, p \to p$		∼MG(0,adapt) -bJ (wRAS) $1 \nearrow p$		PCG(MG (3,3)-bJ) $p \to p$		MG(1,1)-PCG(iChol) $1 \nearrow p$		MG(0,1)-bGS $1 \to 1, p$		MG(3,3)-GS $1 \nearrow p$	
J	p	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time
4	1	19	0.12 s	9	0.11 s	11	0.20 s	16	0.74 s	11	0.06 s	4	0.05 s
	3	11	2.07 s	7	1.62 s	3	2.34 s	44	27.48 s	10	9.64 s	5	1.37 s
	6	9	20.19 s	4	12.54 s	3	38.40 s	>80	>6.87m	9	34.78 s	6	14.44 s
	9	9	2.13m	3	49.84 s	2	2.24m	>80	>23.08m	8	1.72m	9	1.21m

MG(1,1)-PCG(iChol): not p-robust

MG(3,3)-GS: not p-robust

[2] Antonietti et al. *J. Sci. Comput.* 2017.

[3] Botti et al. *J. Comput. Phys.* 2017.

[4] Schöberl. "C++11 Implementation of Finite Elements in NGSolve". *Tech. report.* 2014.

[5] The experiments were run on one Dell C6220 dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée "NEF" computation cluster, however, in a sequential Matlab script.

A-posteriori-steered multigrid: hp-robustness

Fitting into AFEM setting

Optimal convergence rates wrt to overall computational cost[6] for contractive solvers[7,8].

Algorithm

Input $\mathcal{T}_0$, u_0^0, $0 < \theta \leq 1$ *sufficiently small*, $\mu > 0$ *sufficiently small*

For each $j = 0, 1, 2, \ldots$ *do*

- **SOLVE & ESTIMATE** *For* $i = 1, 2, \ldots, i_{\mathrm{s}}$, *repeat*
  - *compute* u_j^i, $\eta_{\mathrm{alg}}^i =: \eta_{\mathrm{alg}}(u_j^i)$
  - *compute* $\eta_{\mathrm{disc}}(T, u_j^i)$ *for all* $T \in \mathcal{T}_j$

  until $\eta_{\mathrm{alg}}(u_j^{i_{\mathrm{s}}}) \leq \mu \eta_{\mathrm{disc}}(u_j^{i_{\mathrm{s}}})$ $\longrightarrow$ *idea: equilibrate algebraic and discretization errors*
- **MARK** *choose* $\mathcal{M}_j \subset \mathcal{T}_j$ *such that* $\theta \sum_{T \in \mathcal{T}_j} \eta_{\mathrm{disc}}(T, u_j^{i_{\mathrm{s}}})^2 \leq \sum_{T \in \mathcal{M}_j} \eta_{\mathrm{disc}}(T, u_j^{i_{\mathrm{s}}})^2$
- **REFINE** $\mathcal{T}_{j+1} := \mathtt{refine}(\mathcal{T}_j, \mathcal{M}_j)$, $u_{j+1}^0 := u_j^{i_{\mathrm{s}}}$ $\longrightarrow$ *nested iterations: a posteriori error control on all* u_j^i *except* u_0^0

Output *Discrete solutions* $u_j^{i_{\mathrm{s}}}$ *and corresponding estimators* $\eta_{\mathrm{alg}}(u_j^{i_{\mathrm{s}}}), \eta_{\mathrm{disc}}(u_j^{i_{\mathrm{s}}})$

[6] Gantner, Haberl, Praetorius, Schimanko. *Math. Comp.* 2021.

[7] Chen, Nochetto, Xu. *Numer. Math.* 2012.

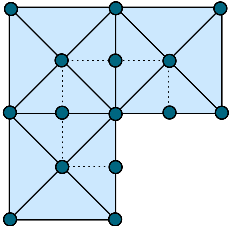
[8] Wu, Zheng. *Appl. Numer. Math.* 2017.

Key to obtaining J-robustness

Remark: From now on, consider $p_0 = \ldots = p_{J-1} = 1$ and $p_J = p$.

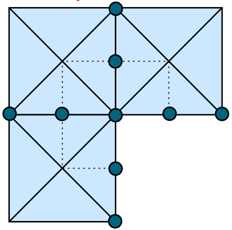
For intermediate levels $j \in \{1, \ldots, J-1\}$:

previously



smoothing on *all* patches

improvement



smoothing *locally*

For the finest level J: smoothing on all patches *when* $p > 1$.

Takeaway message:

- J-robustness by local smoothing on lowest-order levels.
- p-robustness by smoothing on all patches of the high-order level.

Theorem (hp-robust reliable and efficient bound on the algebraic error)

Let $u_J^i \in V_J^p$ be arbitrary. Let η_{alg}^i be the associated a posteriori estimator on the algebraic error. Then, there holds:

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\| \geq \eta_{\mathrm{alg}}^i \quad \text{and} \quad \eta_{\mathrm{alg}}^i \geq \beta \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \quad 0 < \beta(\kappa_{\mathcal{T}}, d, \mathbf{K}) < 1.$$

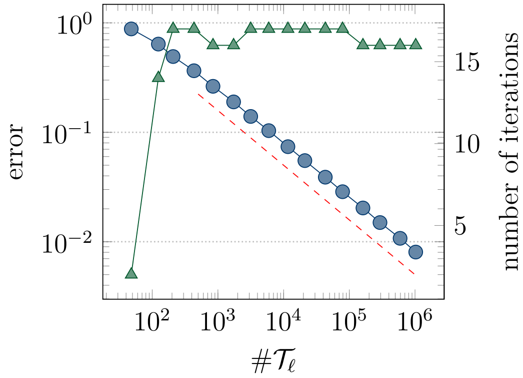
Theorem (hp-robust error contraction of the multilevel solver)

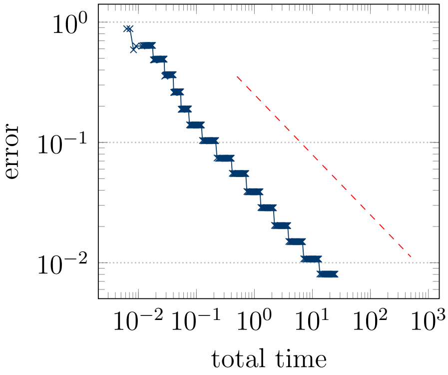
For $u_J^i \in V_J^p$, let $u_J^{i+1} \in V_J^p$ be constructed from u_J^i using one step of the solver. There holds:

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\| \leq \alpha \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \qquad \alpha = \sqrt{1-\beta^2}.$$

Remark: Complete *independence* from J is obtained even under minimal H^1-regularity setting.

L-shape problem





Conclusion

Contributions and perspectives

NumPDEs ASC TUWIEN

We presented:

- A p-**robustly efficient** a posteriori algebraic error estimator.
- A minimalistic and non-symmetric p-**robust contractive** multigrid solver which is steered by the a posteriori estimator.
- Optimal level-wise **step-sizes** used in the multigrid error correction stage.
- An hp-**robust contractive** extension satisfying the requirements of the SOLVE module in AFEM.

Future work would explore:

- Extension of the approach to BEM.
- Study of the robustness with respect to the jumps of the diffusion coefficient $\mathbb{K}$.

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Thank you for your attention!

- Miraçi, Papež, and Vohralík. A multilevel algebraic error estimator and the corresponding iterative solver with p-robust behavior. *SIAM J. Numer. Anal.* (2020)
- Miraçi, Papež, and Vohralík. A-posteriori-steered p-robust multigrid with optimal step-sizes and adaptive number of smoothing steps. *SIAM J. Sci. Comput.* (2021)
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Dr. Ani Miraçi

TU Wien

Institute of Analysis
and Scientific Computing

NumPDEs group

ani.miraci@asc.tuwien.ac.at

https://www.asc.tuwien.ac.at/∼amiraci

NumPDEs
Workgroup on Numerics of PDEs