AFEM for nonsymmetric linear elliptic PDEs: optimal complexity

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joint work with

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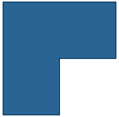
Outline

Introduction

Motivation

- $\Omega \subset \mathbb{R}^d$ with $d \in \{1,2,3\}$ a polyhedral Lipschitz domain
- $\boldsymbol{A} \in [L^\infty(\Omega)]^{d\times d}_{\mathrm{sym}}$ a symmetric diffusion matrix,
  $\boldsymbol{b} \in [L^\infty(\Omega)]^d$ a convection coefficient,
  $c \in L^\infty(\Omega)$ a reaction coefficient
- $f \in L^2(\Omega)$ and $\boldsymbol{f} \in [L^2(\Omega)]^d$ the given data



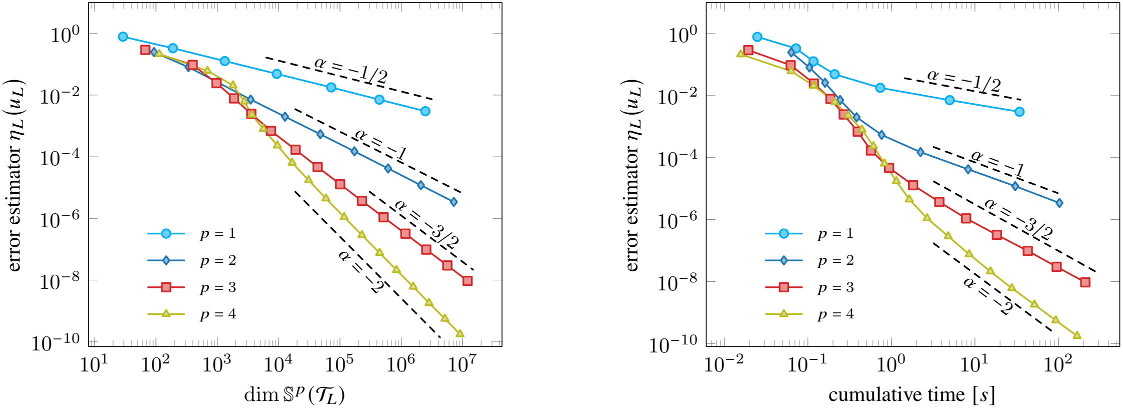
Model problem

Consider the *nonsymmetric* second order linear elliptic PDE

$$
\begin{aligned}
-\mathrm{div}(\boldsymbol{A}\nabla u^\star) + \boldsymbol{b}\cdot\nabla u^\star + cu^\star &= f - \mathrm{div}\,\boldsymbol{f} && \text{in } \Omega \subset \mathbb{R}^d \\
u^\star &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Aims and approach

- accurate approximation of $u^\star$ with minimal computational cost
- adaptivity and contractive inexact solvers



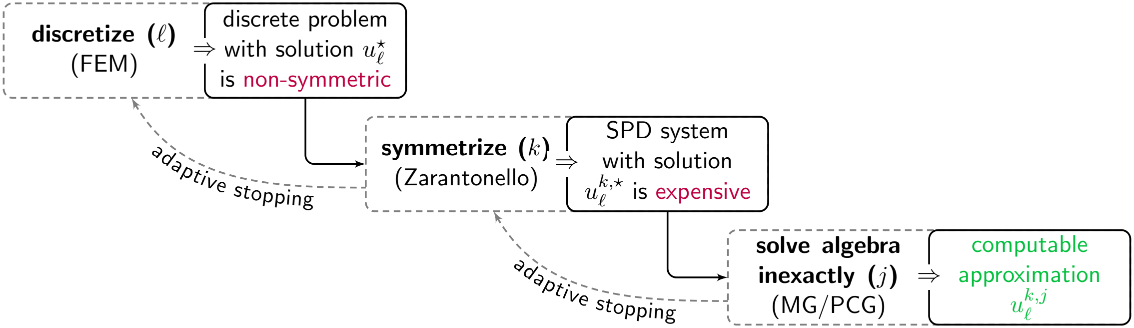
- Optimal decay wrt. dof and time
- Both rates coincide

Some(!) literature

- Optimal rates with exact solver: Binev, Dahmen, DeVore (2004) / Stevenson (2007) / Cascon, Kreuzer, Nochetto, Siebert (2008) / Belenki, Diening, Kreuzer (2012) / Garau, Morin, Zuppa (2012) / Cascon, Nochetto (2012) / Feischl, Führer, Praetorius (2014) / Carstensen, Feischl, Page, Praetorius (2014) / ...
- Optimal rates with inexact solver: Stevenson (2007) / Becker, Mao (2010) / Carstensen, Feischl, Page, Praetorius (2014) / Gantner, Haberl, Praetorius, Stiftner (2018) / ...
- Optimal complexity with inexact solver for symmetric problems: Stevenson (2007) / Gantner, Haberl, Praetorius, Schimanko (2021) / Haberl, Praetorius, Schimanko, Vohralik (2021) / ...
- Optimal complexity with inexact solver for nonsymmetric problems: now

Stucture of AISFEM

Adaptive iteratively symmetrized finite element method:



Setting

Abstract setting: posing the problem

- **Continuous problem:** Find $u^\star \in \mathbb{V}$ solving

$$b(u^\star, v) = a(u^\star, v) + \langle \mathcal{K}u^\star \,,\, v\rangle = F(v) \text{ for all } v \in \mathbb{V}.$$

Assumption: $b(\cdot,\cdot)$ is elliptic $(\alpha > 0)$ and continuous $(\alpha \le L < \infty)$ on $\mathbb{V}$ (Lax–Milgram).

  - $\mathbb{V} := H^1_0(\Omega)$
  - $a(u, v) := \langle \boldsymbol{A}\nabla u \,,\, \nabla v\rangle_\Omega$ principal part
  - energy norm $|||v|||^2 := a(v, v)$
  - $\langle \mathcal{K}u^\star \,,\, v\rangle := \langle \boldsymbol{b}\cdot\nabla u + cu \,,\, v\rangle_\Omega$
  - $F(v) := \langle f \,,\, v\rangle_\Omega + \langle \boldsymbol{f} \,,\, \nabla v\rangle_\Omega$

- **Discrete problem:** Find $u^\star_\ell \in \mathbb{V}_\ell$ solving

$$b(u^\star_\ell, v_\ell) = F(v_\ell) \quad \text{for all } v_\ell \in \mathbb{V}_\ell.$$

  - $\mathcal{T}_\ell$ conforming simplicial triangulation of Ω
  - $p \in \mathbb{N}$ a fixed polynomial degree
  - $\mathbb{V}_\ell := \{v_\ell \in H^1_0(\Omega) \mid v_\ell|_T \text{ is a polynomial of degree} \le p, \text{ for all } T \in \mathcal{T}_\ell\}$

Abstract setting: iterative solution

- **Symmetrization:** approximate the discrete solution $u_\ell^\star$ by $u_\ell^k \in \mathbb{V}_\ell$ and iterate thanks to Zarantonello mapping $\Phi_\ell(\delta;\cdot)\colon \mathbb{V}_\ell \to \mathbb{V}_\ell$, $\delta > 0$

$$a(\Phi_\ell(\delta; u_\ell^k), v_\ell) = a(u_\ell^k, v_\ell) + \delta\big[F(v_\ell) - b(u_\ell, v_\ell)\big] \quad \text{for all } v_\ell \in \mathbb{V}_\ell. \tag{1}$$

  Note: the unique fix-point is the discrete solution $u_\ell^\star = \Phi_\ell(\delta; u_\ell^\star)$.
  Assumption: damping parameter $\delta > 0$ is small to guarantee contraction, in particular:

$$|||u_\ell^\star - \Phi_\ell(\delta; u_\ell^k)||| \le q_{\mathsf{sym}}[\delta]\, |||u_\ell^\star - u_\ell^k||| \quad \text{with} \quad q_{\mathsf{sym}}[\delta] := 1 - \delta(2\alpha - \delta L^2) < 1.$$

- **Iteration:** approximate the solution $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^k)$ of the SPD linear system (1) by $u_\ell^{k,j}$ and iterate thanks to a contractive algebraic solver
  - CG with optimal preconditioner
  - optimal multigrid method

  One solver step mapping $\Psi_\ell(\cdot)\colon \mathbb{V}_\ell \to \mathbb{V}_\ell$ ensures

$$|||u_\ell^{k,\star} - \Psi_\ell(u_\ell^{k,j})||| \le q_{\mathsf{alg}}\, |||u_\ell^{k,\star} - u_\ell^{k,j}||| \quad \text{with} \quad q_{\mathsf{alg}} < 1.$$

Algorithm and main results

Algorithm

Input $\mathcal{T}_0,\ u_0^{0,0},\ 0 < \theta \le 1,\ \delta > 0,\ \lambda_{\mathsf{sym}}, \lambda_{\mathsf{alg}} > 0$

For $\ell = 0, 1, 2, \ldots$ *repeat steps 1–3:*

1 SOLVE & ESTIMATE

For $k = 1, 2, \ldots,$ *repeat*

For $j = 1, 2, \ldots,$ *repeat*

compute $u_\ell^{k,j}$

compute $\eta_\ell(T, u_\ell^{k,j})$ *for all* $T \in \mathcal{T}_\ell$

until $|||u_\ell^{k,j} - u_\ell^{k,j-1}||| \le \lambda_{\mathsf{alg}} \left[\lambda_{\mathsf{sym}}\, \eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,J}|||\right] \longrightarrow$ *idea: equilibrate algebraic error*

$J := j$

until $|||u_\ell^{k,J} - u_\ell^{k-1,J}||| \le \lambda_{\mathsf{sym}}\, \eta_\ell(u_\ell^{k,J}) \longrightarrow$ *idea: equilibrate symmetrization error*

$K := k$

2 MARK *choose* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *such that* $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^{K,J})^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^{K,J})^2$

3 REFINE $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell), \quad u_{\ell+1}^{0,0} := u_\ell^{K,J} \longrightarrow$ *nested iterations with error control on all* $u_\ell^{k,j}$ *except* $u_0^{0,0}$

Output *Discrete solutions* $u_\ell^{k,j}$ *and corresponding estimators* $\eta_\ell(u_\ell^{k,j})$

Main result 1: Full linear convergence

- $\mathcal{Q} := \{(\ell, k, j) \in \mathbb{N}_0^3 \mid u_\ell^{k,j}$ computed by algorithm$\}$
- $|\ell, k, j| \ := \#\{(\ell', k', j') \in \mathcal{Q} \mid u_{\ell'}^{k',j'}$ computed earlier than $u_\ell^{k,j}\}$
- quasi-error $\Delta_\ell^{k,j} := |||u^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j})$ for all $(\ell, k, j) \in \mathcal{Q}$

Theorem

Let $0 < \theta \le 1$, $\lambda_{\mathsf{sym}} > 0$, and $0 < \lambda_{\mathsf{alg}} \le \lambda_{\mathsf{alg}}^\star$, such that $\lambda_{\mathsf{alg}}^\star > 0$ and $0 < \overline{q}_{\mathsf{sym}} := \dfrac{q_{\mathsf{sym}} + 2\,\frac{q_{\mathsf{alg}}}{1-q_{\mathsf{alg}}}\,\lambda_{\mathsf{alg}}^\star}{1 - 2\,\frac{q_{\mathsf{alg}}}{1-q_{\mathsf{alg}}}\,\lambda_{\mathsf{alg}}^\star} < 1$.
Then, there exist $C_{\mathsf{lin}} > 0$, $0 < q_{\mathsf{lin}} < 1$, and $\ell_0 \in \mathbb{N}_0$ with $\ell_0 \le L$ such that

$$\Delta_\ell^{k,j} \le C_{\mathsf{lin}} q_{\mathsf{lin}}^{|\ell,k,j| - |\ell',k',j'|}\, \Delta_{\ell'}^{k',j'}. \tag{2}$$

- note: linear convergence (2) $\iff$ $\displaystyle\sum_{\substack{(\ell,k,j)\in\mathcal{Q} \\ |\ell,k,j| > |\ell',k',j'|}} \Delta_\ell^{k,j} \lesssim \Delta_{\ell'}^{k',j'}$
- the threshold level $\ell_0 \in \mathbb{N}_0$ arises from the lack of Galerkin orthogonality wrt $a(\cdot,\cdot)$: *analysis more involved*
- the condition on the parameter $\lambda_{\mathsf{alg}}^\star$ arises from ensuring that the *perturbed* Zarantonello iteration is also contractive

Brunner, Heid, Innerberger, Miraçi, Praetorius, Streitberger. arXiv:2212.00353 (2022)

Main result 2: Rate = complexity

Corollary

Suppose full linear convergence. Let $s > 0$, then there holds

$$\sup_{\substack{(\ell,k,j)\in\mathcal{Q} \\ \ell' \geq \ell_0}} (\#\mathcal{T}_\ell)^s \Delta_\ell^{k,j} \simeq \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\leq|\ell,k,j| \\ \ell' \geq \ell_0}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j}$$

- $M := \sup_{\substack{(\ell,k,j)\in\mathcal{Q} \\ \ell' \geq \ell_0}} (\#\mathcal{T}_\ell)^s \, \Delta_\ell^{k,j} \leq \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\leq|\ell,k,j| \\ \ell' \geq \ell_0}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} \overset{?}{\lesssim} M$
- definition of M $\Longrightarrow$ $\#\mathcal{T}_{\ell'} \leq M^{1/s} \, (\Delta_{\ell'}^{k',j'})^{-1/s}$
- full linear convergence $\Longrightarrow$ $\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\leq|\ell,k,j| \\ \ell' \geq \ell_0}} \#\mathcal{T}_{\ell'} \leq M^{1/s} \sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\leq|\ell,k,j| \\ \ell' \geq \ell_0}} \big(\Delta_{\ell'}^{k',j'}\big)^{-1/s} \lesssim M^{1/s} \, (\Delta_\ell^{k,j})^{-1/s}$

Main result 3: Optimal complexity

NumPDEs
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Theorem

Let $0 < \theta \leq 1$ and $\lambda_{\mathsf{sym}} > 0, \lambda_{\mathsf{alg}} > 0$ be sufficiently small. For all $s > 0$, let

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T})} := \sup_{N \geq \#\mathcal{T}} \min_{\#\mathcal{T}_{\mathrm{opt}} \leq N} N^s \Delta^\star_{\mathrm{opt}}.$$

There holds

$$c_{\mathsf{opt}} \, \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} \leq \sup_{(\ell,k,j) \in \mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j') \in \mathcal{Q} \\ |\ell',k',j'| \leq |\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j},$$

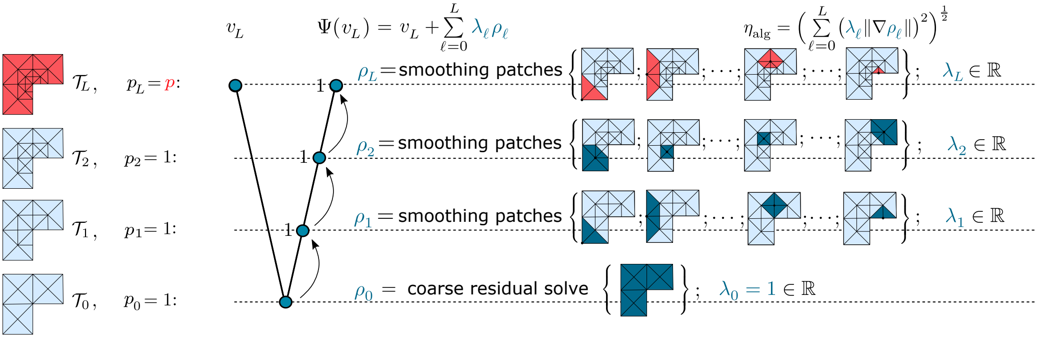
$$\sup_{\substack{(\ell,k,j) \in \mathcal{Q} \\ \ell \geq \ell_0}} \Big(\sum_{\substack{(\ell',k',j') \in \mathcal{Q} \\ |\ell',k',j'| \leq |\ell,k,j| \\ \ell' \geq \ell_0}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} \leq C_{\mathsf{opt}} \max\{\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}, \Delta_{\ell_0}^{0,0}\}.$$

Brunner, Heid, Innerberger, Miraçi, Praetorius, Streitberger. arXiv:2212.00353 (2022)

Choice of algebraic solver: geometric multigrid

Wishlist: one solver step is robustly contractive and of linear cost

Construction:



Takeaway message: construction of a-posteriori-steered multigrid ensures

- L-robustness by local smoothing on lowest-order levels
- p-robustness by smoothing on all patches of the high-order level

Choice of algebraic solver: robustness

- Denote the true solution of the algebraic system to be solved by $u_L \in \mathbb{V}_\ell$.

Theorem (h- and p-robust error contraction of the a-posteriori-steered multigrid)

Let $v_L \in \mathbb{V}_\ell$ be arbitrary. Construct $\Psi(v_L) \in \mathbb{V}_\ell$ from v_L using one step of the solver.

$$\Longrightarrow \qquad |||u_L - \Psi(v_L)||| \leq \alpha |||u_L - v_L||| \qquad 0 < \alpha < 1.$$

Theorem (h- and p-robust reliable and efficient bound on the algebraic error)

Let $v_L \in \mathbb{V}_\ell$ be arbitrary. Let η_{alg} be the associated estimator on the algebraic error.

$$\Longrightarrow \qquad |||u_L - v_L||| \geq \eta_{\mathrm{alg}} \quad \textit{and} \quad \eta_{\mathrm{alg}} \geq \beta |||u_L - v_L||| \quad \textit{with} \quad \beta = \sqrt{1-\alpha^2}$$

Remark: a-posteriori-steering means a layer of adaptivity can be added in the SOLVE module

Innerberger, Miraçi, Praetorius, Streitberger. Optimal computational costs of AFEM with optimal local hp-robust multigrid solver, arXiv:2210.10415 (2022)

Miraçi, Papež, Vohralík. A-posteriori-steered p-robust multigrid with optimal step-sizes and adaptive number of smoothing steps. SIAM J. Sci. Comput. 43, 5 (2021), S117–S145

Numerical experiments

Test case

Diffusion-convection-reaction on L-shaped domain

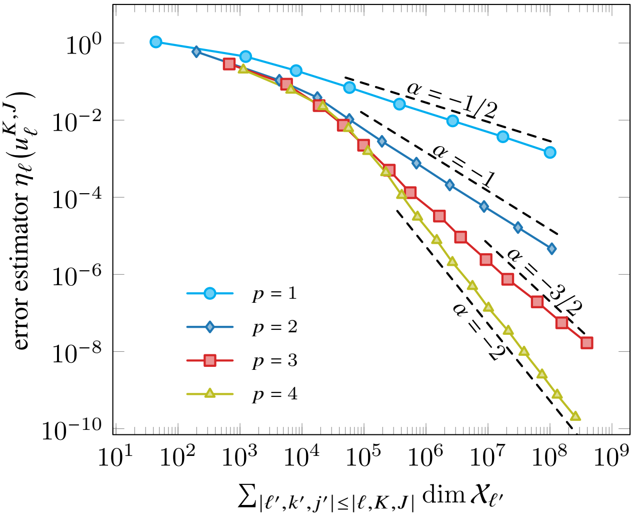
- $\Omega = (-1,1)^2 \setminus \big([0,1] \times [-1,0]\big) \subset \mathbb{R}^2$
- Consider

$$\begin{aligned} -\Delta u^\star(x) + x \cdot \nabla u^\star(x) + u^\star(x) &= 1 \quad \text{for } x \in \Omega \\ u^\star(x) &= 0 \quad \text{for } x \in \partial\Omega, \end{aligned}$$

i.e., $\boldsymbol{A} = \mathrm{Id}, \quad \boldsymbol{b}(x) = x, \quad c(x) = 1, \quad f(x) = 1$

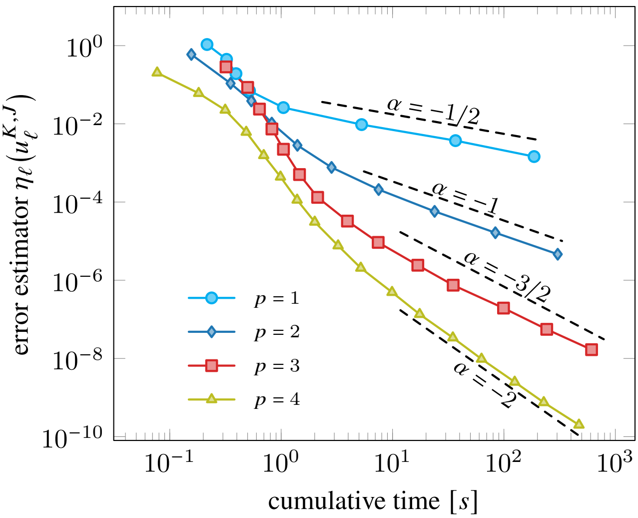
Optimal convergence wrt. computational cost

- $\theta = 0.5$,
- $\delta = 0.5$,
- $\lambda_{\mathsf{sym}} = \lambda_{\mathsf{alg}} = 0.1$



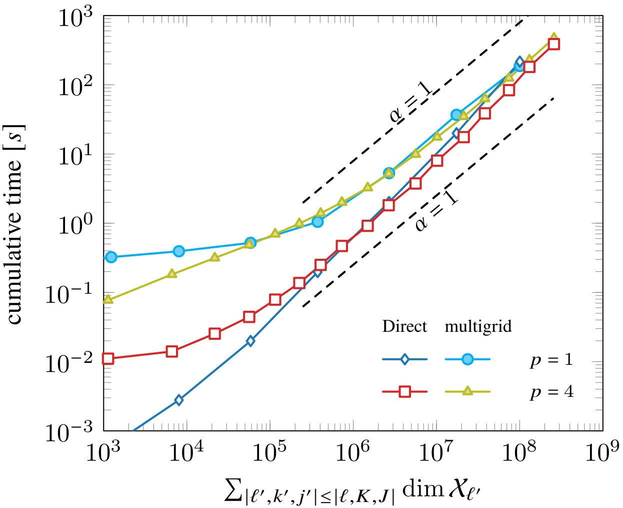
Optimal convergence wrt. time

- $\theta = 0.5$,
- $\delta = 0.5$,
- $\lambda_{\mathsf{sym}} = \lambda_{\mathsf{alg}} = 0.1$



Optimal geometric multigrid vs. direct solve

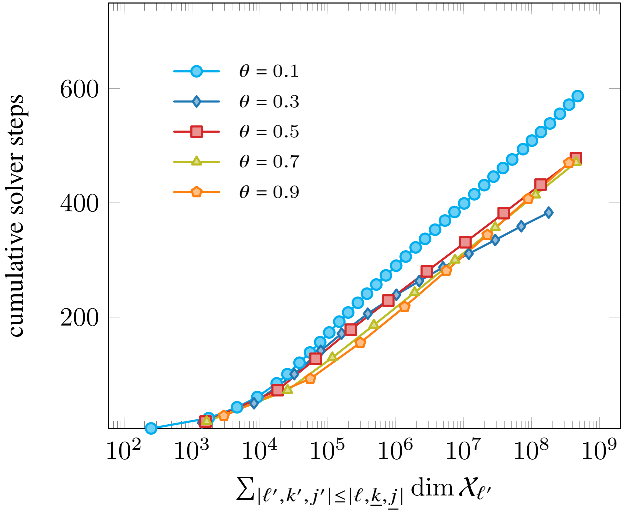
- $\theta = 0.5$,
- $\delta = 0.5$,
- $\lambda_{\mathsf{sym}} = \lambda_{\mathsf{alg}} = 0.1$



Innerberger, Miraçi, Praetorius, Streitberger: 2210.10415 (2022)

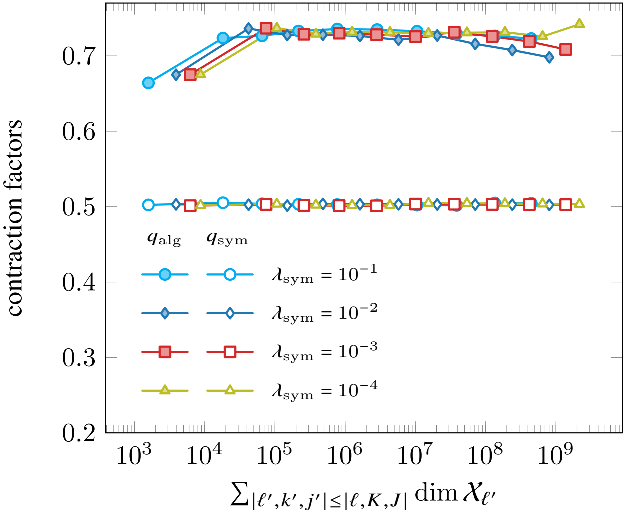
Choice of θ?

- $\delta = 0.5$,
- $\lambda_{\mathsf{sym}} = 0.1$
- $\lambda_{\mathsf{alg}} = 0.01$
- $p = 2$



Contraction factors wrt. λ_{sym}

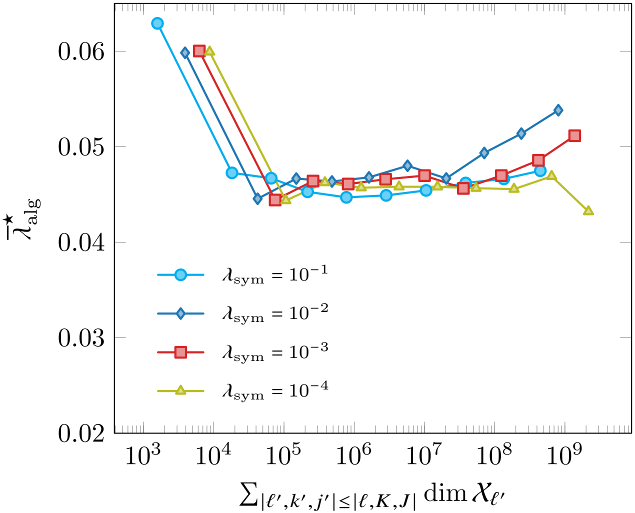
- $\theta = 0.5$,
- $\delta = 0.5$,
- $\lambda_{\mathsf{alg}} = 0.01$
- $p = 2$



Behavior of $\overline{\lambda}^{\star}_{\text{alg}}$?

- $\theta = 0.5$,
- $\delta = 0.5$,
- $\lambda_{\text{alg}} = 0.01$
- $p = 2$

- Recall
$\overline{\lambda}^{\star}_{\text{alg}} = \frac{(1-q_{\text{alg}})(1-q_{\text{sym}})}{(4q_{\text{alg}})}$



Conclusion

Conclusion

We presented an adaptive FEM algorithm:

- steered via natural equidistribution of estimated error components
- using nested iterations
- using combined contractive solvers for nonsymmetric linear problems:
  - Zarantonello iteration to symmetrize
  - optimal geometric multigrid for the remaining SPD linear system

Analysis of optimal complexity (quasi-optimal computational cost):

- main concept: full linear convergence

Thank you for your attention!

- Michael Innerberger, Ani Miraçi, Dirk Praetorius, Julian Streitberger
  Optimal computational costs of AFEM with optimal local hp-robust multigrid solver
  arXiv:2210.10415 (2022)
- Maximilian Brunner, Pascal Heid, Michael Innerberger, Ani Miraçi, Dirk Praetorius, Julian Streitberger
  Adaptive FEM with quasi-optimal overall cost for nonsymmetric linear elliptic PDEs
  arXiv:2212.00353 (2022)

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