# UE Mengenlehre SoSe2024

lena.wallner@tuwien.ac.at

## Session 10

updated on May 30 at 17:40

1) Let $\mathcal{U}$ be an ultrafilter. Show that the following are equivalent:

   i. $\mathrm{Ult}(V,\mathcal{U})$ is wellfounded.

   ii. $\mathcal{U}$ is $\sigma$-closed (or in other words $<\omega_1$-closed).

2) Suppose that $(I,\preceq)$ is a directed partial order and $\langle \mathcal{M}_i, \pi_{i,j} \mid i \preceq j\rangle$ is a directed system of elementary embeddings and $\mathcal{M}_i$ is an $\in$-structure for all $i \in I$. Let $\langle \mathcal{M}_\infty, \pi_{i,\infty} \mid i \in I\rangle$ be the direct limit along this system. Show that the following are equivalent:

   i. $\mathcal{M}_\infty$ is illfounded (as an $\in$-structure).

   ii. There is a $\preceq$-increasing sequence $(i_n)_{n<\omega}$ through $I$ and a sequence $(a_n)_{n<\omega}$ with $a_n \in \mathcal{M}_{i_n}$ so that $\mathcal{M}_{i_{n+1}} \models a_{n+1} \in \pi_{i_n,i_{n+1}}(a_n)$.

   Conclude that if $I$ is countably directed (i.e. whenever $\{j_n \mid n < \omega\} \subseteq I$ there is some $j_* \in I$ with $j_n \preceq j_*$ for all $n$) and all $\mathcal{M}_i$ are wellfounded for $i \in I$ then $\mathcal{M}_\infty$ is wellfounded.

3) Suppose that $\mathcal{U}$ is a measure. Let $\vec{\mathcal{M}} := \langle \mathcal{M}_n, j_{n,m} \mid n \leq m < \omega\rangle$ be the directed system so that $M_0 = V$, $M_{n+1} = \mathrm{Ult}(M_n,\mathcal{U})$ and $j_{n,n+1}$ is the resulting ultrapower map. Show that $\mathcal{M}_n$ is wellfounded for all $n < \omega$, yet the direct limit along $\vec{\mathcal{M}}$ is illfounded.

4) Suppose $\mathcal{U}$ is a normal measure on $\kappa$ and write $\kappa_\alpha$ for $j^{\mathcal{U}}_{0,\alpha}(\kappa)$. Show that:

   (a) $[f]_{\mathcal{U}} = j_{\mathcal{U}}(f)(\kappa)$ for[1] any $f\colon \kappa \to V$.

   (b) $\mathrm{Ult}^{(\alpha)}(V,\mathcal{U}) = \bigcup_{n<\omega}\{j^{\mathcal{U}}_{0,\alpha}(f)(\kappa_{i_0},\ldots,\kappa_{i_{n-1}}) \mid f\colon \kappa^n \to V \wedge i_0 < \cdots < i_{n-1} < \alpha\}$.

   (c) $\kappa_\theta = \theta$ for all cardinals $\theta > 2^\kappa$.

   Hint: For (a), first use Session 7 Exercise 3 (b) to prove $\mathcal{U} = \{A \subseteq \kappa \mid \kappa \in j_{\mathcal{U}}(A)\}$.

[1] Recall that we identify $[f]_{\mathcal{U}}$ with its image under the collapsing map.