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Session 8

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The goal of these exercises is so show the non-trivial direction of the following theorem.

Theorem. *The following are equiconsistent.*

(i). ZFC + there is a total probability measure which vanishes on points.

(ii). ZFC + there is a measurable cardinal.

Definition. Let X be a set. $\pi : \mathcal{P}(X) \to [0,1] \subseteq \mathbb{R}$ is a **total probability measure on** X iff

- $A \subseteq B \subseteq X \Rightarrow \pi(A) \leq \pi(B)$,
- $\pi(X) = 1$ and $\pi(\emptyset) = 0$ and
- $\pi(\bigcup_{n<\omega} A_n) = \Sigma_{n<\omega}\pi(A_n)$ for $\{A_n \subseteq X \mid n < \omega\}$ pairwise disjoint.

A total probability measure π on X **vanishes on points** iff $\pi(\{x\}) = 0$ for each $x \in X$.

Definition. Let $\mathcal{F}$ be a filter on some set X and λ a cardinal. $\mathcal{A} \subseteq \mathcal{F}^+$ is an **antichain** in $\mathcal{F}$ iff $A \cap B \notin \mathcal{F}^+$ for $A, B \in \mathcal{A}$ with $A \neq B$. We say that $\mathcal{F}$ is **λ-saturated** iff $\mathcal{F}$ doesn't have antichains of size λ.

Definition. Let π be a total probability measure on some set X. We set $\mathcal{F}_\pi := \{A \subseteq X \mid \pi(A) = 1\}$.

Definition. Let $\mathcal{F}$ be a filter on some set X and $A \in \mathcal{F}^+$. We say that A is an **atom of** $\mathcal{F}$ iff for every $B \subseteq A$, either $B \notin \mathcal{F}^+$ or $A \setminus B \notin \mathcal{F}^+$.

1) Suppose that there is a total probability measure π on some cardinal μ which vanishes on points. Show that if κ is the least such cardinal, then κ is uncountable and $\mathcal{F}_\pi$ is a $<\kappa$-closed ω_1-saturated non-principal filter.

2) Let $\lambda < \kappa$ be infinite cardinals. Let $\mathcal{F}$ be a $<\kappa$-closed λ^+-saturated non-principal filter on κ. Show the following:

(a) If $\mathcal{F}$ has an atom then κ is measurable.

(b) If $\mathcal{F}$ doesn't have an atom then $\kappa \leq 2^\lambda$.
HINT: Build an extensional tree T for as long as possible and a function $f\colon T \to \mathcal{F}^+$ so that f maps the root of T to κ, each $s \in T$ has two immediate successors t_0, t_1 so that $\{f(t_i) \mid i < 2\}$ is a partition of $f(s)$ and if $t \in T$ is of limit height then $f(t) = \bigcap_{s <_T t} f(s)$. Show that $|\partial T| \leq 2^\lambda$ and that $\kappa = \bigcup_{b \in \partial T} \bigcap_{t \in b} f(t)$ is a partition of κ with $\bigcap_{t \in b} f(t) \notin \mathcal{F}^+$ for each $b \in \partial T$.

3) (*) Let $\lambda \leq \kappa$ be uncountable cardinals. Let $\mathcal{F}$ be a $<\kappa$-closed λ-saturated non-principal filter on κ. Show that there is a normal $<\kappa$-closed λ-saturated non-principal filter on κ by proving the following:

(a) There is some $B \in \mathcal{F}^+$ and a function $f : B \to \kappa$ such that

I. $f \restriction A$ is not constant for each $A \in \mathcal{F}^+$ with $A \subseteq B$ and

II. if $g : B \to \kappa$ satisfies I., then $\{\alpha \in B \mid g(\alpha) < f(\alpha)\} \notin \mathcal{F}^+$.

HINT: Assume this fails and construct a sequence $(f_n)_{n<\omega}$ of functions $f_n\colon \kappa \to \kappa$ satisfying I. so that $\{\alpha < \kappa \mid f_{n+1}(\alpha) < f_n(\alpha)\} \in \mathcal{F}$ for all $n < \omega$.

(b) Let $f\colon B \to \kappa$ be as in (a). Show that $\mathcal{G} := \{A \subseteq \kappa \mid (\kappa \setminus B) \cup f^{-1}[A] \in \mathcal{F}\}$ is a normal $<\kappa$-closed λ-saturated non-principal filter on κ.

(c) If $\mathcal{H}$ is any normal $<\kappa$-closed λ-saturated filter then for $A \in \mathcal{H}$ and $f\colon A \to \kappa$ regressive, there is $B \subseteq A$, $B \in \mathcal{H}$ with $|f[B]| < \lambda$.

4) Let $\lambda < \kappa$ be infinite cardinals. Let $\mathcal{F}$ be a normal $<\kappa$-closed λ^+-saturated non-principal filter on κ. Show that

(a) Let M be a transitive model of ZFC^- with $\mathcal{F} \in M$, $\mathcal{P}(\kappa) \subseteq M$. If $X \preceq M$ with $\lambda+1 \cup \{\mathcal{F}, \kappa\} \subseteq X$ and $|X|, \delta < \kappa$. Then there is $Y \preceq M$ such that $X \subseteq Y$, $Y \cap \kappa \in \mathcal{F}$ and $Y \cap \delta = X \cap \delta$.

(b) Set $\mathcal{G} := L[\mathcal{F}] \cap \mathcal{F}$. Then $L[\mathcal{F}] \models$ "$\mathcal{G}$ is a normal $<\kappa$-closed γ-saturated non-principal filter on κ" where γ is the true λ^+.

(c) $L[\mathcal{F}] \models 2^\lambda = \lambda^+ < \kappa$. Using Exercise 2, conclude that $L[\mathcal{F}] \models$ "κ is measurable".

HINT: Follow section 9.2 in the lecture notes. For (a), use Exercise 3 (c). To see that $\lambda^+ < \kappa$, show that if M is as in (a) and $Z \preceq M$ with $\lambda + 1 \subseteq Z$ then $Z \cap \lambda^+ \in \mathrm{Ord}$.