UE Mengenlehre SoSe2024

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Session 7

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Definition. A **Suslin line** is a counterexample to Suslin's Hypothesis, i.e. $(S, \leq)$ is a **Suslin line** iff $(S, \leq)$ is a dense linear order without endpoint which is complete and satisfies the c.c.c., but is not separable.

Definition. A tree T is an **Aronszajn tree** iff $ht(T) = \omega_1$, $|T_\alpha| \leq \aleph_0$ for each $\alpha < \omega_1$ and $[T] = \emptyset$.

Definition. Let $\mathcal{F}$ be a filter on κ. For every $A \subseteq \kappa$, let

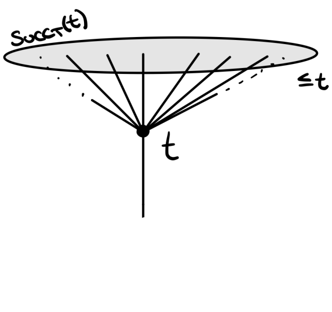
$$A \in \mathcal{F}^+ \Leftrightarrow \forall B \in \mathcal{F}(A \cap B \neq \emptyset).$$

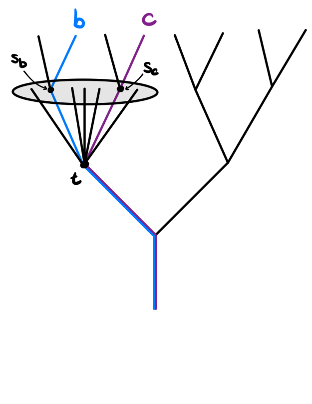
We call the sets in $\mathcal{F}^+$ the positive sets w.r.t. $\mathcal{F}$.

1) Let $(T, \leq_T)$ be a normal extensional Suslin tree. Show that there is a Suslin line by proving the following:

   (a) For every $t \in T$, let $\leq_t$ be a linear order on $Succ_T(t) := \{s \in T \mid t = pred_T(s)\}$ such that $(Succ_T(t), \leq_t) \cong (\mathbb{Q}, \leq_{\mathbb{Q}})$, where $\leq_{\mathbb{Q}}$ is the usual order on $\mathbb{Q}$.
   Now define a linear order $\leq$ on ∂T. For $b, c \in \partial T$, let $t := max(b \cap c)$, i.e. b and c agree up to t but not further. Let s_b be the unique element in $b \cap Succ_T(t)$ and s_c be the unique element in $c \cap Succ_T(t)$. Set

$$b \leq c \Leftrightarrow s_b \leq_t s_c.$$





Show that $(\partial T, \leq)$ is a dense linear order without endpoints which satisfies the c.c.c. and $(\partial T, \leq)$ is not separable.

(b) Let $(L, \leq_L)$ be the completion of $(\partial T, \leq)$, e.g. $L = \{D \subseteq \partial T \mid D \neq \emptyset,\ D$ is open, D is bounded from above and D is downwards closed w.r.t. $\leq\}$ and $\leq_L = \subseteq$. The sets in L are called Dedekind cuts of $(\partial T, \leq)$.

Show that $(L, \leq_L)$ satisfies the c.c.c. and is not separable.

2) Show that there is an Aronszajn tree $(T, \leq_T)$.

Hint: Construct T such that for every $\alpha < \omega_1$, $t \in T_\alpha$ is an embedding $t : (\alpha, \leq) \to (\mathbb{Q}, \leq_{\mathbb{Q}})$. Let $\leq_T$ be end-extension. Arrange that the following strengthening of normality holds for T:

$$\forall\, \alpha < \omega_1\, \forall\, t \in T_\alpha\, \forall\, \alpha \leq \beta < \omega_1\, \forall\, \varepsilon > 0\, \exists\, s \in T_\beta (t \leq_T s, sup(ran(s)) \leq sup(ran(t)) + \varepsilon).$$

3) Let κ be an uncountable cardinal.

(a) Let $\mathcal{F}$ be a filter on κ. Show that the following are equivalent

i. $\mathcal{F}$ is normal.

ii. For every regressive $f : A \to \kappa$ with $A \in \mathcal{F}^+$, there is some $B \in \mathcal{F}^+$ such that $f \upharpoonright B$ is constant.

(b) Let U be a measure on κ. Show that the following are equivalent

i. U is normal.

ii. $(\{[f]_U \mid f : \kappa \to \kappa \text{ regressive}\}, E_U)$ is a wellorder of ordertype κ.

4) Let κ be an uncountable cardinal.

(a) Show that the following are equivalent

i. κ is inaccessible.

ii. For every $S \subseteq \mathcal{P}(\kappa)$ with $|S| < \kappa$, there is a filter $\mathcal{F}$ on κ such that

- $\mathcal{F}$ measures all sets in S and
- $\bigcap \mathcal{F} \subseteq \kappa$ is unbounded.

(b) Conclude that any measurable cardinal is inaccessible.