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Session 6

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Definition. Let α be an infinite ordinal. We say that $\langle a_\beta \mid \beta < \alpha \rangle$, where $a_\beta \subseteq \beta$ for each $\beta < \alpha$, is a $\diamondsuit_\alpha^-$**-sequence** iff for every $A \subseteq \alpha$ there is some $\beta \geq \omega$ such that $a_\beta = A \cap \beta$. We say that $\diamondsuit_\alpha^-$ holds if there is a $\diamondsuit_\alpha^-$-sequence.

Definition. Let α be an ordinal. We say that $\langle a_\beta \mid \beta < \alpha \rangle$, where $a_\beta \subseteq \beta$ for each $\beta < \alpha$, is a $\diamondsuit_\alpha^{--}$**-sequence** iff for every $A \subseteq \alpha$ there is some $\beta > 0$ such that $a_\beta = A \cap \beta$. We say that $\diamondsuit_\alpha^{--}$ holds if there is a $\diamondsuit_\alpha^{--}$-sequence.

Definition. For $f, g : \omega_1 \to \omega_1$ we set

$$f :=^* g \Leftrightarrow \{\alpha < \omega_1 \mid f(\alpha) = g(\alpha)\} \text{ contains a club,}$$
$$f :\leq^* g \Leftrightarrow \{\alpha < \omega_1 \mid f(\alpha) \leq g(\alpha)\} \text{ contains a club and}$$
$$f :<^* g \Leftrightarrow \{\alpha < \omega_1 \mid f(\alpha) < g(\alpha)\} \text{ contains a club.}$$

Definition. We say that $\langle f_\alpha \in {}^{\omega_1}\omega_1 \mid \alpha < \omega_2 \rangle$ is a sequence of **canonical functions** iff

- $f_\alpha <^* f_\beta$ for all $\alpha < \beta < \omega_2$ and
- if $g \in {}^{\omega_1}\omega_1$ is such that $g <^* f_\beta$ for some $\beta < \omega_2$, then there is an $\alpha < \beta$ and a stationary set $S \subseteq \omega_1$ such that $g \restriction S = f_\alpha \restriction S$.

1) (a) Let κ be a regular uncountable cardinal. Show that $\diamondsuit_\kappa$ implies that for every infinite cardinal $\lambda < \kappa$, $2^\lambda \leq \kappa$ holds.
   (b) Prove that $\diamondsuit_\omega^{--}$ is false,
   (c) but $\diamondsuit_{\omega+1}^{--}$ is true.
   (d) Show that $\diamondsuit_\alpha^-$ is false for $\omega < \alpha < \omega_1$.

2) (*) Show that $\diamondsuit_{\omega_1}^-$ is equivalent to $\diamondsuit_{\omega_1}$.
   Hint: First show that $\diamondsuit_{\omega_1}^-$ implies that there is a $\diamondsuit_{\omega_1}^-$-sequence which guesses every subset of ω_1 correctly on a limit ordinal, say $\langle a_\beta \mid \beta < \omega_1 \rangle$ is such a sequence. Let $\gamma \in a_\beta^{\text{even}}$ iff $2 \cdot \gamma \in a_\beta$ and show that $\langle a_\beta^{\text{even}} \mid \beta < \omega_1 \rangle$ is a $\diamondsuit_{\omega_1}$-sequence.

3) (a) Show that there is no sequence $\langle f_n \in {}^{\omega_1}\omega_1 \mid n < \omega\rangle$ that is strictly $<^*$-decreasing.

(b) Show that $({}^{\omega_1}\omega_1, <^*)$ is wellfounded.

(c) Let $\tau_0, \tau_1 : \omega_1 \to \alpha$ be onto. Define

$$f_i : \omega_1 \to \omega_1, \gamma \mapsto otp(\tau_i[\gamma])$$

for $i = 0, 1$. Show that $f_0 =^* f_1$.

(d) For every $\alpha < \omega_2$, fix $\pi_\alpha : \omega_1 \to \alpha$ onto. Define $f_\alpha : \omega_1 \to \omega_1$ as above. Show that $\langle f_\alpha \mid \alpha < \omega_2\rangle$ is a sequence of canonical functions.

4) (a) Let κ be a regular uncountable cardinal and $S \subseteq \kappa$ stationary. If $\mathcal{M} = (M, \in, \dots)$ is a structure of a countable language such that $\kappa \subseteq M$, then there is a substructure $(X, \in, \dots) \prec \mathcal{M}$ such that $X \cap \kappa \in S$.

(b) Assume $V = L$ and let

$$\begin{aligned} g : \omega_1 &\to \omega_1 \\ \alpha &\mapsto \text{ least } \beta < \omega_1 \text{ such that } \alpha \text{ is countable in } L_\beta. \end{aligned}$$

Show that if $\langle f_\alpha \mid \alpha < \omega_2\rangle$ is the sequence of canonical functions from 3)d, then $f_\alpha <^* g$ for all $\alpha < \omega_2$.