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Session 5

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Definition. For a set $A \subseteq \omega_1$ we define a game in the following way. Player I chooses some $\alpha_0 \in \omega_1$. Then Player II chooses $\beta_0 \in \omega_1$. It's Player I's turn again and she chooses $\alpha_1 \in \omega_1$ and so on. We end up with

Player I	α_0		α_1		...	
Player II		β_0		β_1		...

We say that **Player I wins** iff $sup\{\alpha_n, \beta_n \mid n < \omega\} \in A$ and otherwise **Player II wins**.

Definition. A **strategy for Player I** is a function

$$\sigma : \bigcup_{k \in \omega} (\omega_1)^{2k} \to \omega_1.$$

The strategy tells I what she should do in the next step based on the previous choices of I and II. If we have

$$(\alpha_0, \beta_0, \dots, \alpha_{k-1}, \beta_{k-1})$$

after k-many rounds, then σ says that I should play

$$\sigma(\alpha_0, \beta_0, \dots, \alpha_{k-1}, \beta_{k-1}) \in \omega_1$$

next. We say that σ is a **winning strategy** iff I always wins, as long as she does what σ suggests. (Winning) strategies for II are defined analogously.

1) Show that

$$\text{I has a winning strategy} \Leftrightarrow \text{there is } C \subseteq A \text{ which is club in } \omega_1 \text{ and}$$
$$\text{II has a winning strategy} \Leftrightarrow \text{there is } C \subseteq \omega_1 \setminus A \text{ which is club in } \omega_1.$$

2) Let κ be a regular uncountable cardinal and let $\alpha < \kappa^+$ be a limit ordinal. Show that there is a sequence $\langle S_\beta \mid \beta < \alpha \rangle$ of sets which are stationary subsets of κ such that $S_\beta \supseteq S_\gamma$ for $\beta < \gamma < \alpha$ and $\bigcap_{\beta < \alpha} S_\beta = \emptyset$.

3) Let κ be a singular cardinal of uncountable cofinality. Show that if $2^\lambda \leq \lambda^{++}$ for each infinite cardinal $\lambda < \kappa$, then $2^\kappa \leq \kappa^{++}$.

4) (a) Show that for every ordinal α, $V_\alpha \cap Ord = \alpha$.

   (b) Prove Lemma 6.5: If $\alpha \in Lim$ and α is uncountable, then $V_\alpha \models ZFC - (Replacement)$.

   (c) Show that if there is an ordinal α such that $V_\alpha \models ZFC$, then the smallest such α has countable cofinality.