VSM Retreat 2024

# Should the axiom of choice be our axiom of choice?

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# HISTORY OF SET THEORY

- Modern set theory started in 1870s with Cantor and Dedekind
  ↳ there was no precice definition of sets
  ↳ called "naive set theory"

- around 1900 they discovered flaws
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- (1908) Zermelo introduced first axiomatic system

- after a few years of refinement together with Fraenkel, nowaday's system was born, called "Zermelo-Fraenkel set theory"

A special axiom: **THE AXIOM OF CHOICE**

**Definition**

The Axiom of Choice (AC) says that for every family of non-empty sets $\{X_i\}_{i\in J}$ indexed by a set $J$, there is a choice set $\{a_i\}_{i\in J}$ st $a_i \in X_i$ for each $i \in J$.

I.e. we can <u>simultaniously</u> choose an element from each set of the family.

ZFC is short for Zermelo-Fraenkel set theory with (AC) and

ZF — " — without (AC).

# A special axiom: THE AXIOM OF CHOICE

- Zermelo came up with the Axiom of Choice and it is part of his axiomatization of set theory
- In the beginning, the axiom was heavily discussed
- Meanwhile most mathematicians are not worried about using (AC) and ZFC is the most used axiom system
- But (AC) also has some unpleasant consequences.

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**Theorem**

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# WHAT IS NICE ABOUT (AC)?

**Theorem**

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If a partially ordered set $(X, \leq)$ satisfies "every totally ordered subset has an upper bound in $X$", then $X$ has a $\leq$-maximal element.

Every model of ZF satisfies (AC) iff it satisfies Zorn's Lemma

In ZF: $\boxed{\text{(AC)} \implies \text{Every vector space has a basis}}$

Proof:

- Fix $V$ vector space
- basis of $V$ = maximal linear independent subset of $V$

Use Zorn's Lemma:

- $X := \{L \subseteq V \mid L \text{ is lin. indep.}\}$ partially ordered by $\subseteq$
- For $C \subseteq X$ totally ordered: $\bigcup_{L \in C} L$ is an upper bound of $C$, $\in X$

$\Rightarrow X$ has a $\subseteq$-max. el $B$

$\Rightarrow B$ is a basis of $V$ $\square$

In ZF: $\boxed{\text{(AC)} \implies \text{Every vector space has a basis}}$

But also

In ZF: $\boxed{\text{Every vector space has a basis} \implies \text{(AC)}}$

So (AC) is equivalent to "every space has a basis" !

# WHAT IS BAD ABOUT (AC)?

$\boxed{\text{Definition}}$

- define relation on $\mathbb{R}\times\mathbb{R}$ by $r \sim s \Leftrightarrow r-s \in \mathbb{Q}$
- $\sim$ is equivalence relation
- denote equ. classes by $[r]$
- Note: $[r] \cap [0,1] \neq \emptyset$ for each $r \in \mathbb{R}$
- By (AC), there is a choice set $M \subseteq [0,1]$

  i.e. $|M \cap [r]| = 1$ for each $r \in \mathbb{R}$

The set $M$ is called a Vitali Monster.

## Theorem

Let $M$ be a Vitali Monster. Then $M$ is not Lebesgue-measurable.

Proof:

Sps $M$ is L-measurable.

- enumerate $[-1,1] \cap \mathbb{Q} = \{q_k \mid k \in \mathbb{N}\}$
- set $M_k := M + q_k$ for each $k \in \mathbb{N}$
- Note: $M_k$'s are pw. disjoint

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$\boxed{\text{Theorem}}$

Let $M$ be a Vitali Monster. Then $M$ is not Lebesgue-measurable.

Proof:

$$\lambda(M) = 0 \implies 1 \le \sum_{k=0}^{\infty} \lambda(M) \le 3$$

$$\sum_{k=0}^{\infty} \lambda(M) = 0$$

$$\lightning$$

## $\boxed{\text{Theorem}}$

Let $M$ be a Vitali Monster. Then $M$ is not Lebesgue-measurable.

<u>Proof</u>:

$$1 \le \sum_{k=0}^{\infty} \lambda(M) \le 3$$

$$\lambda(M) = 0 \implies 0 = \sum_{k=0}^{\infty} \lambda(M) \quad ↯$$

$$\lambda(M) > 0 \implies \sum_{k=0}^{\infty} \lambda(M) = \infty \quad ↯$$

$\square$

# WHAT CAN WE DO?

- Study alternatives to (AC)
  e.g. Axiom of Determinacy
- Study weakenings of (AC)
  e.g. Axiom of Countable Choice
  Axiom of Dependent Choice