Optimal Cost of (Goal-oriented) Adaptive FEM for General Second-Order Linear Elliptic PDEs

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Abstract The ultimate goal of any numerical scheme for partial differential equations (PDEs) is to compute an approximation of user-prescribed accuracy at quasi-minimal computational time. To this end, algorithmically, the standard adaptive finite element method (AFEM) and goal-oriented AFEM (GOAFEM) must integrate an inexact solver and nested iterations with discerning stopping criteria balancing the different error components. The algorithms require several fine-tuned parameters in order to make the underlying analysis work. We review recent developments in the field, recall the up-to-date optimal algorithm and investigate the choice of adaptivity parameters for a prototypical GOAFEM example.

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1 State of the Art

Adaptive finite element algorithms strive to approximate the unknown and possibly singular exact PDE solution $u^\star$ on the basis of *a posteriori* error estimation and adaptive mesh-refinement strategies. With $\ell \in \mathbb{N}_0$ denoting the mesh level and $k \in \mathbb{N}$ denoting the inexact solver counter, the adaptive feedback loop with *inexact solver* generates a sequence $\mathcal{T}_\ell$ of successively refined meshes and finite element approximations $u_\ell^k \approx u^\star$ together with error estimators $\eta_\ell(u_\ell^k)$ by iterating the scheme in Figure 1. Over the past three decades, the mathematical understanding of adaptive

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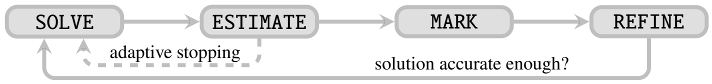

Fig. 1: Modules of the standard AFEM algorithm with inexact solver.

finite element methods (AFEMs) has matured; see, e.g., [10, 2, 20, 8, 11] and [7] for an axiomatic framework summarizing the earlier references. In most of the cited works, the focus is on (*plain*) *convergence* [10] and optimal convergence rates with respect to the number of degrees of freedom [2, 8, 11]. Nevertheless, owing to the incremental nature of adaptivity (i.e., $\mathcal{T}_\ell$ and u_ℓ^k depend on all prior computed $\mathcal{T}_{\ell'}$ and $u_{\ell'}^{k'}$), the mathematical question regarding optimal convergence rates should instead pertain to the overall computation cost (or the cumulative computation time). This, termed as *optimal complexity* in the context of adaptive wavelet methods [9], was later adopted in AFEM [20, 6]. In these works, optimal complexity is ensured for AFEM with an inexact solver, contingent on the condition that the computed iterates u_ℓ^k closely approximate the (unavailable) exact discrete solutions $u_\ell^\star$.

Motivated by the interest in AFEMs for nonlinear problems [15], recent works [13, 5, 4] aimed to integrate a combination of linearization and algebraic solver within a nested adaptive algorithm. Following this approach, the algorithmic decision for mesh-refinement, linearization, or algebraic solver steps is guided by *a-posteriori*-based stopping criteria with suitable stopping parameters. This allows for balancing the error components and computing inexact approximations $u_\ell^k \approx u_\ell^\star$. Since an algebraic solver for the nonsymmetric second-order linear elliptic PDE

$$-\mathrm{div}(\boldsymbol{A}\,\nabla u^\star) + \boldsymbol{b}\cdot\nabla u^\star + c\,u^\star = f - \mathrm{div}\boldsymbol{f} \text{ in } \Omega \subset \mathbb{R}^d \quad \text{subject to } u^\star = 0 \text{ in } \partial\Omega \quad (1)$$

that is contractive in the PDE-related energy norm is not available at the moment, we adapt this strategy for linear but nonsymmetric problems. In the work [5], we have shown that this leads to contraction in each step of the proposed AFEM algorithm and, with a geometric series argument, to optimal computational complexity for sufficiently small adaptivity parameters. Moreover, the recent preprint [4] presents a new proof strategy for full linear convergence relaxing the parameter bounds.

In many applications, the key focus is not the approximation of the exact PDE solution $u^\star$ but rather a function value $G(u^\star)$ for a continuous functional $G(u^\star)$. A naive approach allows to control the goal error by the energy norm of the approximation. However, a duality approach in the spirit of [14] allows to essentially double the convergence rates by solving so-called primal and dual problems simultaneously. For the extension of the adaptive algorithm with nested iterative solver above, it is natural to apply the inexact solver from [5, 4] to the decoupled problems in parallel together with a combined marking strategy [12].

The remainder of the work reads as follows. Section 2 introduces the basic notation and the problem setting, before Section 3 presents the AFEM algorithm of Figure 1 to illustrate the strategy. The main results of this work in Section 4, full linear convergence and optimal complexity, are followed by a discussion of the difficulties for GOAFEM in Section 5. Finally, Section 6 illustrates the theoretical findings with a numerical study for a variant of the prototypical example from [18].

2 Preliminaries

In this section, we present the model problem, the components of the nested solver, and the residual-based *a posteriori* error estimator.

Model Problem. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and given data $\boldsymbol{A} \in \left[L^\infty(\Omega)\right]^{d\times d}_{\mathrm{sym}}$, $\boldsymbol{b} \in \left[L^\infty(\Omega)\right]^d$, $c \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$, $\boldsymbol{f} \in \left[L^2(\Omega)\right]^d$, we seek the solution $u^\star \in \mathcal{X} := H^1_0(\Omega)$ to (1) in its weak formulation

$$b(u^\star, v) := \langle \boldsymbol{A}\, \nabla u^\star, \nabla v\rangle + \langle \boldsymbol{b}\cdot\nabla u^\star + c\, u^\star, v\rangle = \langle f, v\rangle + \langle \boldsymbol{f}, \nabla v\rangle =: F(v) \;\; \forall v \in \mathcal{X}. \quad (2)$$

The principal part $a(u, v) := \langle \boldsymbol{A}\, \nabla u, \nabla v\rangle$ induces an equivalent norm $|\!|\!| v |\!|\!| := a(v, v)^{1/2}$ on $\mathcal{X}$. We suppose that $b(\cdot,\cdot)$ fits into the Lax–Milgram framework which guarantees existence and uniqueness of $u^\star \in \mathcal{X}$. For a conforming triangulation $\mathcal{T}_H$ into compact simplices and fixed $p \in \mathbb{N}$, the Lax–Milgram lemma also applies to the FEM space

$$\mathcal{X}_H := \{v_H \in \mathcal{X} : \forall T \in \mathcal{T}_H, v_H|_T \in \mathbb{P}_p(T)\}$$

and assures existence and uniqueness of $u^\star_H \in \mathcal{X}_H$ to

$$b(u^\star_H, v_H) = F(v_H) \quad \text{for all } v_H \in \mathcal{X}_H. \quad (3)$$

While (3) corresponds to a linear system, the exact computation of $u^\star_H$ would prevent optimal complexity of the later AFEM algorithm. Since (3) is nonsymmetric, we cannot apply a PCG or multigrid solver to the associated linear system and use a nested iterative solver consisting of symmetrization and algebraic solver instead.

Zarantonello Iteration. The Zarantonello iteration stems from the proof of the Lax–Milgram lemma. For $\delta > 0$, the iteration function $\Phi_H(\delta;\cdot) : \mathcal{X}_H \to \mathcal{X}_H$ solves

$$\langle \nabla\Phi_H(\delta; u_H), \nabla v_H\rangle = \langle \nabla u_H, \nabla v_H\rangle + \delta\big[F(v_H) - b(u_H, v_H)\big] \;\; \forall v_H \in \mathcal{X}_H. \quad (4)$$

We stress that (4) corresponds to a linear system with SPD matrix. Moreover, for sufficiently small $0 < \delta < \delta^\star$, the Zarantonello iteration is contractive [15]. In particular, there exists $0 < q_{\mathrm{sym}} < 1$ depending only on $b(\cdot,\cdot)$ and δ such that

$$|\!|\!| u^\star_H - \Phi_H(\delta; u_H) |\!|\!| \le q_{\mathrm{sym}}\, |\!|\!| u^\star_H - u_H |\!|\!| \quad \forall u_H \in \mathcal{X}_H. \quad (5)$$

Indeed, the Banach fixed-point theorem thus proves that (4) admits a unique solution (being the unique fixed-point of $\Phi_H(\delta,\cdot)$) and so proves the Lax–Milgram lemma.

Algebraic Solver. Since large SPD linear systems like (4) are still computationally expensive, we apply an iterative algebraic solver. More precisely, we employ a uniformly contractive geometric multigrid [17] with iteration function $\Psi_H(u^\sharp_H;\cdot) \colon \mathcal{X}_H \to \mathcal{X}_H$ to approximate the solution $u^\sharp_H := \Phi_H(\delta; u_H)$ to the SPD system (4), i.e., there exists $0 < q_{\mathrm{alg}} < 1$ independent of $\mathcal{X}_H$ such that

$$|||u_H^\sharp - \Psi_H(u_H^\sharp; w_H)||| \le q_{\mathrm{alg}}\, |||u_H^\sharp - w_H||| \quad \text{for all } w_H \in \mathcal{X}_H \text{ and all } \mathcal{T}_H \in \mathbb{T}. \tag{6}$$

Here, $\mathbb{T}$ is the set of all conforming triangulations intro compact simplices that can be obtained from a fixed initial triangulation $\mathcal{T}_0$ by newest vertex bisection (NVB) [21].

A Posteriori Error Estimation. We assume additional data regularity $\boldsymbol{A}|_T$, $\boldsymbol{f}|_T \in W^{1,\infty}(T)$ for all $T \in \mathcal{T}_0$ and use the standard residual error estimator $\eta_H(\cdot)$ defined, for $\mathcal{T}_H \in \mathbb{T}$, $T \in \mathcal{T}_H$, and $v_H \in \mathcal{X}_H$, by

$$\begin{aligned} \eta_H(T, v_H)^2 := |T|^{2/d}\, \|\mathrm{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) - \boldsymbol{b}\cdot\nabla v_H - c\, v_H + f\|_{L^2(T)}^2 \\ + |T|^{1/d}\, \|[[(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\cdot n]]\|_{L^2(\partial T \cap \Omega)}^2, \end{aligned} \tag{7a}$$

where $[[\cdot]]$ denotes the jump over the $(d-1)$-dimensional faces of $\mathcal{T}_H$. To abbreviate notation, we define, for all $\mathcal{U}_H \subseteq \mathcal{T}_H$ and all $v_H \in \mathcal{X}_H$,

$$\eta_H(v_H) := \eta_H(\mathcal{T}_H, v_H) \quad \text{with} \quad \eta_H(\mathcal{U}_H, v_H) := \Big(\sum_{T \in \mathcal{U}_H} \eta_H(T, v_H)^2\Big)^{1/2}. \tag{7b}$$

Then, it is well-known that the estimator satisfies standard properties, like (discrete) reliability, nowadays called the *axioms of adaptivity* [7].

3 Adaptive Algorithm

The adaptive algorithm embedding a nested iterative solver takes the following form.

Algorithm 1 (AFEM with nested contractive solvers)**.** *Given an initial conforming mesh $\mathcal{T}_0$ into compact simplices, the Zarantonello parameter $\delta > 0$, adaptivity parameters $0 < \theta \le 1$ and $C_{\mathrm{mark}} \ge 1$, solver-stopping parameters $\lambda_{\mathrm{sym}}, \lambda_{\mathrm{alg}} > 0$, and an initial guess $u_0^{0,0} := u_0^{0,J} \in \mathcal{X}_0$, iterate the following steps* (i)–(iv) *for all $\ell = 0, 1, 2, 3, \dots$:*

(i) ***Solve & estimate:*** *For all $k = 1, 2, 3, \dots,$ repeat the following steps* (a)–(b) *until*

$$|||u_\ell^{k,J} - u_\ell^{k-1,J}||| \le \lambda_{\mathrm{sym}}\, \eta_\ell(u_\ell^{k,J}). \tag{8}$$

(a) ***Inner solver loop:*** *For all $j = 1, 2, 3, \dots,$ repeat the steps* (I)–(II) *until*

$$|||u_\ell^{k,j} - u_\ell^{k,j-1}||| \le \lambda_{\mathrm{alg}}\, \big[\lambda_{\mathrm{sym}}\eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,J}|||\big]. \tag{9}$$

(I) *Compute one step of the contractive SPD solver $u_\ell^{k,j} := \Psi_\ell(u_\ell^{k,\star}; u_\ell^{k,j-1})$, where $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,J}) \in \mathcal{X}_\ell$ is only a theoretical quantity.*

(II) *Compute the refinement indicators $\eta_\ell(T, u_\ell^{k,j})$ for all $T \in \mathcal{T}_\ell$.*

(b) *Upon termination of the j-loop, define $J[\ell, k] := j \in \mathbb{N}$ and $u_\ell^{k,J} := u_\ell^{k,j}$.*

(ii) *Upon termination of the k-loop, define $K[\ell] := k \in \mathbb{N}$ and $u_\ell^{K,J} := u_\ell^{k,J}$.*

(iii)***Mark:*** *Determine a set* $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{K,J}] := \{\mathcal{M}_\ell \subseteq \mathcal{T}_\ell : \theta\,\eta_\ell(u_\ell^{K,J})^2 \le \eta_\ell(\mathcal{U}_\ell, u_\ell^{K,J})^2\}$ *satisfying* $\#\mathcal{M}_\ell \le C_{\mathrm{mark}} \min_{\mathcal{U}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{K,J}]} \#\mathcal{U}_\ell$.

(iv)***Refine:*** *Employ NVB to refine at least all marked element* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *to generate the refined triangulation* $\mathcal{T}_{\ell+1}$ *and use nested iteration* $u_{\ell+1}^{0,0} := u_{\ell+1}^{0,J} := u_\ell^{K,J}$.

To formulate convergence, we introduce the index set $\mathcal{Q} := \{(\ell, k, j) \in \mathbb{N}_0^3 : u_\ell^{k,j}$ is utilized in Algorithm 1$\}$ along with the lexicographic ordering

$$(\ell', k', j') \le (\ell, k, j) \quad :\Longleftrightarrow \quad u_{\ell'}^{k',j'} \text{ is defined not later than } u_\ell^{k,j} \text{ in Algorithm 1,}$$

and the total step counter

$$|\ell, k, j| := \#\{(\ell', k', j') \in \mathcal{Q} : (\ell', k', j') \le (\ell, k, j)\} \in \mathbb{N}_0 \ \ \forall (\ell, k, j) \in \mathcal{Q}. \quad (10)$$

4 Main Results

The key ingredient in the proof of optimal complexity is full R-linear convergence, which essentially states contraction in each step of the adaptive algorithm. We note that Theorem 1 has first been proved in [5, Theorem 4.1], before an alternative proof from [3] led to sharper constants and weaker assumptions.

Theorem 1 (full R-linear convergence of Algorithm 1, [3, Theorem 13]) *Define the quasi-error*

$$\mathrm{H}_\ell^{k,j} := |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j}). \quad (11)$$

Let $0 < \theta \le 1$*,* $C_{\mathrm{mark}} \ge 1$*,* $\lambda_{\mathrm{sym}}, \lambda_{\mathrm{alg}} > 0$*, and* $u_0^{0,0} \in \mathcal{X}_0$*. Then, there exist* $0 < \lambda_{\mathrm{alg}}^\star$ *and* $0 < \lambda^\star$ *(and we refer to [3] for details) such that, for all* $0 < \lambda_{alg} \le \lambda_{\mathrm{alg}}^\star$ *and* $0 < \lambda_{\mathrm{sym}}\,\lambda_{\mathrm{alg}} < \lambda^\star$*, Algorithm 1 guarantees R-linear convergence of the quasi-error, i.e., there exist* $0 < q_{\mathrm{lin}} < 1$ *and* $C_{\mathrm{lin}} > 0$ *such that, for all* $(\ell', k', j'), (\ell, k, j) \in \mathcal{Q}$*,*

$$\mathrm{H}_\ell^{k,j} \le C_{\mathrm{lin}} q_{\mathrm{lin}}^{|\ell,k,j|-|\ell',k',j'|}\, \mathrm{H}_{\ell'}^{k',j'} \text{ whenever } |\ell', k', j'| \le |\ell, k, j|. \quad \square \quad (12)$$

Since each module of Algorithm 1 can be realized in linear complexity $\mathcal{O}(\#\mathcal{T}_\ell)$, see [17] for the optimal geometric multigrid, [19] for marking, and [21] for NVB, the overall cost of the adaptive algorithm up to the computation of $u_\ell^{k,j}$ is given by

$$\mathtt{cost}(u_\ell^{k,j}) \simeq \sum_{\substack{(\ell',k',j') \in \mathcal{Q} \\ |\ell',k',j'| \le |\ell,k,j|}} \#\mathcal{T}_{\ell'} \quad \text{for all } (\ell, k, j) \in \mathcal{Q}.$$

An important consequence of full R-linear convergence (following with elementary calculations from the geometric series) is the equivalence of the convergence rates with respect to the number of degrees of freedom and the overall computational cost. More precisely, note that $M(s) < \infty$ (with $M(s)$ from (13) below) holds if and only if $\mathrm{H}_\ell^{k,j} = \mathcal{O}\big((\#\mathcal{T}_\ell)^{-1/s}\big)$ as $|\ell,k,j| \to \infty$, i.e., the quasi-error $\mathrm{H}_\ell^{k,j}$ decays with rate $1/s$ over the number $\#\mathcal{T}_\ell$ of elements, which is proportional to $\dim \mathcal{X}_\ell$.

Corollary 1 (rates = complexity) *For $s > 0$, full R-linear convergence* (12) *yields*

$$M(s) := \sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \mathrm{H}_\ell^{k,j} \le \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \mathrm{H}_\ell^{k,j} \le C_{\mathrm{cost}}(s)\, M(s), \quad (13)$$

where the constant $C_{\mathrm{cost}}(s) > 0$ depends only on C_{lin}, q_{lin}, and s. Moreover, there exists $s_0 > 0$ such that $M(s) < \infty$ for all $0 < s \le s_0$. □

Finally, for sufficiently small adaptivity parameters, Algorithm 1 even guarantees optimal complexity, i.e., optimal convergence rates with respect to the overall computation cost (resp. the total computation time).

Theorem 2 (optimal complexity [5, Theorem 4.3]) *Recall $\lambda_{\mathrm{alg}}^\star$ and $\lambda^\star$ from Theorem 1. Then, there exist $0 < \lambda_{\mathrm{sym}}^\star \le 1$ and $0 < \theta^\star < 1$ (and we refer to [5] for details) such that, if θ, λ_{sym}, and λ_{alg} are sufficiently small in the sense of*

$$\begin{aligned} &0 < \lambda_{\mathrm{alg}} \le \lambda_{\mathrm{alg}}^\star, \quad 0 < \lambda_{\mathrm{sym}} < \lambda_{\mathrm{sym}}^\star, \quad \textit{and} \quad \lambda_{\mathrm{alg}}\,\lambda_{\mathrm{sym}} < \lambda^\star, \\ &0 < \theta_{\mathrm{opt}} := \frac{(\theta^{1/2} + \lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star)^2}{(1-\lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star)^2} < \theta^\star < 1, \end{aligned} \quad (14)$$

then Algorithm 1 has optimal complexity: Suppose that $u^\star$ can be approximated at rate $s > 0$ (formally stated via approximation classes $\|u^\star\|_{\mathbb{A}_s} < \infty$; see [5]), i.e., there exist (unavailable) optimal meshes $\mathcal{T}_\ell^{\mathrm{opt}}$ with corresponding exact solutions u_ℓ^{opt} and error estimators $\eta_\ell(u_\ell^{\mathrm{opt}})$ such that $\eta_\ell(u_\ell^{\mathrm{opt}}) \to 0$ as $\ell \to \infty$ with $\sup_{\ell\in\mathbb{N}_0} \big(\#\mathcal{T}_\ell^{\mathrm{opt}}\big)^s\, \eta_\ell(u_\ell^{\mathrm{opt}}) < \infty$. Then, Algorithm 1 guarantees that

$$\sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \mathrm{H}_\ell^{k,j} < \infty, \quad (15)$$

which can be phrased explicitly as follows: If $u^\star$ can be approximated at rate $s > 0$ with respect to the number of degrees of freedom, then Algorithm 1 approximates $u^\star$ with rate $s > 0$ with respect to the overall computational cost.

5 Extension to GOAFEM

In this section, we discuss the extension of Algorithm 1 to GOAFEM and highlight the changes and difficulties. Let $G : H_0^1(\Omega) \to \mathbb{R}$ be a linear and continuous goal functional. Additionally to the so-called discrete primal solution $u_H^\star$ from (3), GOAFEM invokes the discrete dual solution $z_H^\star \in \mathcal{X}_H$ solving

$$b(v_H, z_H^\star) = G(v_H) \quad \text{for all } v_H \in \mathcal{X}_H. \tag{16}$$

Hence, the **Solve & estimate** loop in Algorithm 1(i) consists of the parallel treatment of the primal problem in (3) and the dual problem in (16). Therefore, the index set $\mathcal{Q}$ is the union of the indices coming from both loops and the quasi-error quantities have to be extended to the full index set $\mathcal{Q}$. Then, for approximations $u_H, z_H \in \mathcal{X}_H$, the error estimator ζ_H for the dual problem (16) similar to (7), and the computable discrete goal $G_H(u_H, z_H) := G(u_H) + \big[F(z_H) - b(u_H, z_H)\big]$, we arrive at the goal error estimate

$$|G(u^\star) - G_H(u_H, z_H)| \lesssim \big[\eta_H(u_H) + |||u_H^\star - u_H|||\big]\,\big[\zeta_H(z_H) + |||z_H^\star - z_H|||\big]. \tag{17}$$

Thus, an adaptive algorithm has to reduce the quasi-error product $\mathrm{H}_\ell^{k,j}\,\mathrm{Z}_\ell^{k,j}$ rather than the primal quasi-error $\mathrm{H}_\ell^{k,j}$ from (11), where the dual quasi-error $\mathrm{Z}_\ell^{k,j}$ is defined analogously, i.e., $\mathrm{Z}_\ell^{k,j} := |||z_\ell^\star - z_\ell^{k,j}||| + |||z_\ell^{k,\star} - z_\ell^{k,j}||| + \zeta_\ell(z_\ell^{k,j})$ for some indices (ℓ, k, j).

A possible marking criterion for the estimator product originates from the work [18] and was enhanced in [12]. First, the marking determines set $\overline{\mathcal{M}}_\ell^u \in \mathbb{M}_\ell^u[\theta, u_\ell^{K,J}]$ and $\overline{\mathcal{M}}_\ell^z \in \mathbb{M}_\ell^z[\theta, z_\ell^{K,J}]$ as in Algorithm 1(iii) and then merges them to $\mathcal{M}_\ell := \mathcal{M}_\ell^u \cup \mathcal{M}_\ell^z$, where $\mathcal{M}_\ell^u \subseteq \overline{\mathcal{M}}_\ell^u$ and $\mathcal{M}_\ell^z \subseteq \overline{\mathcal{M}}_\ell^z$ satisfy $\#\mathcal{M}_\ell^u = \#\mathcal{M}_\ell^z = \min\{\#\overline{\mathcal{M}}_\ell^u, \#\overline{\mathcal{M}}_\ell^z\}$.

Analyzing the proposed GOAFEM algorithm faces two main challenges. First, there is a complex nonlinear structure due to a combined quasi-error product. This makes the proofs more involved compared to dealing with the primal problem alone. The second challenge comes from the marking strategy resulting in a mixed marked set. Here, only either the primal or the dual estimator ensures the estimator reduction property. This leads to subtle technicalities as the estimator is part of the quasi-error, causing a failure of contraction for one of the quasi-errors. While [1] solves this in the symmetric case, adding a symmetrization loop to handle nonsymmetric PDEs like (1) leads to further problems due to the lack of a Pythagorean identity (as opposed to the symmetric case). To overcome these challenges, [4] adapts the innovative *tail-summability* criterion from [3], proving full linear convergence (12) and optimal complexity (13) for the nonlinear quasi-error product, i.e., Theorem 1 and 2 hold verbatim with $\mathrm{H}_\ell^{k,j}\,\mathrm{Z}_\ell^{k,j}$ replacing $\mathrm{H}_\ell^{k,j}$.

6 Numerical Experiments

The experiments employ the MATLAB object-oriented AFEM software package from [16] embedding the hp-robust local multigrid solver from [17]. We consider a nonsymmetric variant of [18, Example 7.3]. On the Z-shaped domain $\Omega = (-1,1)^2 \setminus \mathrm{conv}\{(-1,-1,),(-1,0),(0,0)\} \subset \mathbb{R}^2$ and $T_1 := \{x \in \Omega : x_1 + x_2 \geq 1/2\}$, we seek a solution $u^\star \in \mathcal{X}$ to

$$\langle \nabla u^\star, \nabla v \rangle + \langle x \cdot \nabla u^\star + u^\star, v \rangle = \int_{T_1} (1,0)^\top \cdot \nabla v \quad \text{for all } v \in \mathcal{X} = H^1_0(\Omega).$$

The quantity of interest $G(u^\star)$ reads

$$G(u^\star) = \int_{T_2} (1,0)^\top \cdot \nabla u^\star \, \mathrm{d}x \quad \text{where } T_2 = \{x \in (0,1)^2 : x_1 + x_2 \leq -1/2\}.$$

In Figure 2, we display a mesh generated by the GOAFEM algorithm and the supports T_1 of the primal right-hand side in blue and T_2 of the dual right-hand side in green. In particular, the adaptive algorithm captures the singularities induced by the jumps

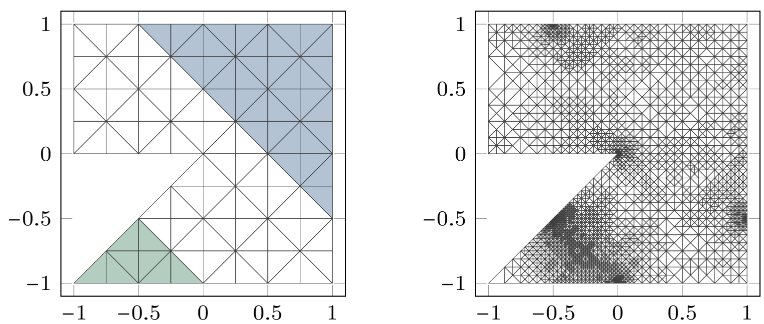

Fig. 2: Left: Initial mesh $\mathcal{T}_0$ with the support of the right-hand functions. Right: Mesh $\mathcal{T}_{12}$ generated by the adaptive algorithm with $\#\mathcal{T}_{12} = 4967$.

in the right-hand side as well as the geometric singularity at the reentrant corner. Since the exact solution $u^\star$ is unknown, we approximate the unavailable goal value by a computation on a uniform mesh with polynomial degree $p = 3$ resulting in $G(u^\star) \approx 0.0018701367282$. Table 1 displays the weighted costs of the GOAFEM algorithm and suggests that moderate $\theta \in \{0.3, 0.4, 0.5\}$ together with larger solver-stopping parameters is beneficial. Figure 3 illustrates that the adaptive algorithm with $\theta = 0.3$, $\lambda_{\mathrm{sym}} = 0.5$, $\lambda_{\mathrm{alg=0.9}}$, and $\delta = 0.5$ leads to optimal convergence rates $-p$ both with respect to the number of degrees of freedom and the overall computation time for the estimator product $\eta_\ell(u_\ell^{K,J})\,\zeta_\ell(z_\ell^{K,J})$ (which is, due to the stopping criteria (8)–(9), equivalent to the quasi-error product $\mathrm{H}_\ell^{K,J}\,\mathrm{Z}_\ell^{K,J}$) and the goal error $|G(u^\star) - G_\ell(u_\ell^{K,J}, z_\ell^{K,J})|$.

References

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$\cdot 10^{-6}$	$\theta = 0.1$					$\theta = 0.2$					$\theta = 0.3$				
λ_{alg} \ λ_{sym}	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
0.1	82.8	83.85	64.26	64.23	63.78	33.31	25.06	19.56	21.25	20.73	21.85	17.9	14.48	14.68	14.38
0.3	83.45	64.41	63.44	63.48	63.72	25.74	19.19	18.7	18.9	19.54	13.67	13.06	12.14	12.76	12.27
0.5	70.82	63.38	64.31	66.05	63.45	26.37	18.72	19.09	19.61	19.21	13.2	12.74	11.98	12.2	12.38
0.7	64.68	64.22	63.38	63.49	63.35	26.06	18.58	19.09	19.32	19.06	13.18	12.48	11.29	11.96	11.61
0.9	63.83	63.3	63.25	63.59	66.02	20.11	19.31	19.2	18.81	19.21	12.43	11.3	10.66	11	11.11
	$\theta = 0.4$					$\theta = 0.5$					$\theta = 0.6$				
0.1	20.75	15.12	12.86	11.89	12.12	25.72	17.36	13.99	13.18	13.17	32.35	19.24	15.16	14.28	13.85
0.3	13.78	14.37	11.9	11.68	11.65	18.6	14.86	11.97	11.97	12.03	21.3	17.03	13.03	12.93	12.8
0.5	13.74	13.21	11.41	11.78	11.78	14.99	15.77	12.32	11.87	12.08	16.19	16.18	14.82	13.76	14.89
0.7	13.31	13.28	11.32	11.63	11.61	14.21	14.5	12.65	11.6	12.01	16.37	17.68	14.89	14.51	13.55
0.9	13.58	14.77	10.99	11.76	11.78	14.34	16.56	11.95	12.48	12.32	15.5	19.69	12.74	12.69	12.78
	$\theta = 0.7$					$\theta = 0.8$					$\theta = 0.9$				
0.1	38.98	22.49	17.52	16.89	16.86	48.56	28.32	20.98	19.7	19.46	71.49	43.94	38.28	31.11	31.28
0.3	24.6	20.65	13.81	14.3	14.8	31.68	27.52	17.12	19.35	19.28	46.29	42.33	29.62	30.62	30.37
0.5	20.15	20.63	14.77	16.05	16.13	29.16	25.46	19.75	17.99	18.22	46.41	45.45	28.09	29.98	30.44
0.7	17.77	19.39	15.95	15.31	16.1	22.47	26.46	20.62	20.25	20	34	45.56	37.88	42.65	43.2
0.9	17.74	22.96	15.32	16.02	15.98	21.49	32.29	19.96	21.66	21.34	33.91	55.87	40.76	51.52	51.12

Table 1: Optimal selection of parameters with respect to the computational costs for the GOAFEM experiment with $p = 2$ and $\delta = 0.5$. For the comparison, we consider the weighted cumulative time $\big[\eta_\ell(u_\ell^{K,J})\,\zeta_\ell(z_\ell^{K,J})\sum_{|\ell',k',j'|\le|\ell,K,J|}\mathrm{time}(\ell')\big]$ (values in 10^{-6}) with stopping criterion $\eta_\ell(u_\ell^{K,J})\,\zeta_\ell(z_\ell^{K,J}) < 5\cdot 10^{-10}$ for various choices of λ_{sym}, λ_{alg}, and θ. In each θ-block, we mark in yellow the best choice per column, in blue the best choice per row, and in green when both choices coincide. The best choices for λ_{alg} and λ_{sym} are observed for $\theta \in \{0.3, 0.4, 0.5\}$.

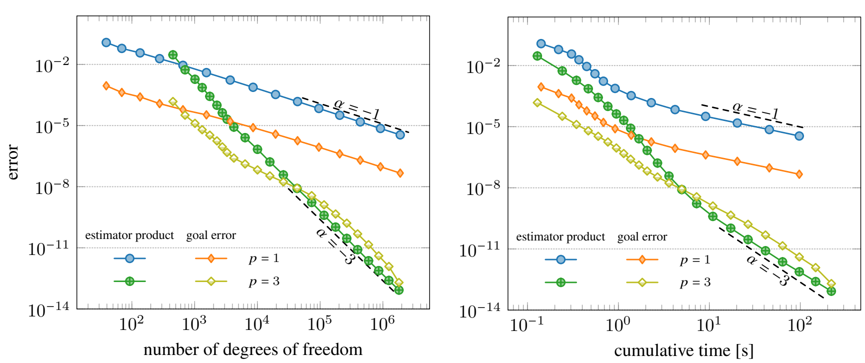

Fig. 3: Convergence history plot of estimator product $\eta_\ell(u_\ell^{K,J})\,\zeta_\ell(z^{K,J})$ indicated by bullets and goal error from (17) indicated by diamonds with respect to the cumulative computational work (left) and with respect to the cumulative computational time (right) for the benchmark problem

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