Cost-Optimal Adaptive FEM for Semilinear Elliptic PDEs

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Abstract We present an adaptive finite element method for vector-valued semilinear elliptic PDEs that is rate-optimal with respect to computational cost, i.e., computation time. To ensure linear complexity of the individual building blocks of the adaptive algorithm, we adaptively linearize the underlying semilinear PDE and solve the arising symmetric positive definite system by means of a norm-contractive algebraic solver, e.g., an optimally preconditioned conjugate gradient method or an optimal geometric multigrid method. To deal with the local Lipschitz continuity of the problem, we prove that the norm of all computed iterates of the proposed adaptive iteratively linearized finite element method (AILFEM) are uniformly bounded. Owing to an equibalance of discretization, linearization, and algebraic solver errors, the algorithm guarantees optimal convergences rates with respect to the number of degrees of freedom, computational cost, and computation time.

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1 Introducing the Main Results

By employing conforming finite elements, we aim at the rate-optimal approximation of the solution $u^\star \in H^1_0(\Omega)$ to the *semilinear* elliptic model problem

$$-\operatorname{div}(\boldsymbol{A}\nabla u^\star) + b(u^\star) = F \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega, \tag{1}$$

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with a polygonal Lipschitz domain $\Omega \subset \mathbb{R}^d$ for $d \in \{1, 2, 3\}$, a uniformly elliptic diffusion coefficient $\boldsymbol{A} \colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, a monotone nonlinearity $b \colon \Omega \to \mathbb{R}$, and sufficiently regular data $F \in H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the topological dual space of $H^1_0(\Omega)$. Under the assumption of a smooth b with polynomial (and compact) growth (see [1, Section 3] for the precise statement), the Browder–Minty theorem [2, Theorem 25.B] proves existence and uniqueness of the weak solution $u^\star$ to (1).

On each mesh level (with mesh index ℓ), the arising discrete nonlinear system cannot be solved exactly as supposed in standard adaptive FEM. To deal with this issue, we develop an algorithm coined as *adaptive iteratively linearized FEM* (AILFEM) (detailed in Algorithm 1 below) that steers the decision to refine the mesh adaptively, to compute an additional step of linearization, or a further algebraic solver step instead. More precisely, we follow [3, 4] and consider the so-called *Zarantonello iteration* from [3, 5] as a linearization method (with linearization index k). To ensure linear complexity of the proposed AILFEM algorithm, we solve the arising symmetric positive definite Zarantonello system with a norm-contractive algebraic solver [6] (with algebraic solver index i). The schematic algorithm reads

$$\boxed{\texttt{ITERATIVELY SOLVE \& ESTIMATE } (\ell, k, i)} \longrightarrow \boxed{\texttt{MARK } (\ell)} \longrightarrow \boxed{\texttt{REFINE } (\ell)}$$

Inherent to semilinear problems is that the Lipschitz constant L of b depends on the considered functions v and w in the sense that for $\vartheta > 0$, it holds only that

$$\|b(v)-b(w)\|_{H^{-1}(\Omega)} \le L[\vartheta]\,\|v-w\|_{H^1_0(\Omega)} \;\forall v, w \in H^1_0(\Omega) \text{ with } \max\left\{|||v|||, |||w|||\right\} \le \vartheta.$$

This dependence is passed on to the stability constant of the residual-based *a posteriori* error estimator [7, 8]. By modifying the stopping criteria compared to existing literature [5, 9, 10, 11], we prove the first main result: All iterates $u_\ell^{k,i} \approx u^\star$ that appear in Algorithm 1 are uniformly bounded (see Theorem 1 below).

Once uniform boundedness is established, we prove full R-linear convergence (Theorem 2 below) of an appropriate quasi-error quantity $\mathrm{H}_\ell^{k,i}$ that consists of the discretization error, the linearization error, and the algebraic error. Full R-linear convergence essentially states uniform contraction in each step of the algorithm regardless of the algorithmic decision. As a consequence of uniform boundedness and full R-linear convergence, we prove optimal convergence rates understood with respect to the number of degrees of freedom and with respect to the overall computational cost (Corollary 1 and Theorem 3).

Contrary to our work [12] that employs energy arguments for scalar semilinear PDEs, the arguments in this work are solely based on norm contraction. This allows for an extension to vector-valued semilinear model problems at the expense of introducing new technicalities in the form of a mesh-refinement index $\ell_0 \in \mathbb{N}_0$ such that R-linear convergence holds only for $\ell \ge \ell_0$ (Theorem 2 below), which also affects the subsequent optimality results.

2 Problem Setting

Strong monotonicity and local Lipschitz continuity. Associated to a mesh $\mathcal{T}_H$ of Ω (understood throughout as conforming simplicial triangulation), let $\mathcal{X}_H \subset H^1_0(\Omega)$ denote the finite element space of piecewise polynomials of degree $\le p$. The discrete formulation of the model problem (1) reads: Find $u^\star_H \in \mathcal{X}_H$ such that

$$\langle \mathcal{A}u^\star_H\,, v_H\rangle := \langle \boldsymbol{A}\nabla u^\star_H\,, v_H\rangle_\Omega + \langle b(u^\star_H)\,, v_H\rangle_\Omega = \langle F\,, v_H\rangle \quad \text{for all } v_H \in \mathcal{X}_H, \tag{2}$$

where $\langle \cdot\,,\,\cdot\rangle_\Omega$ is the $L^2(\Omega)$-scalar product and $\langle \cdot\,,\,\cdot\rangle$ the dual pairing on $H^{-1}(\Omega) \times H^1_0(\Omega)$. We use $|||\cdot|||^2 := \langle \boldsymbol{A}\nabla\cdot\,,\,\nabla\cdot\rangle$ for the equivalent energy norm on $H^1_0(\Omega)$.

We suppose that the model problem (1) (cf. [1, Section 3]) is **strongly monotone**

$$\alpha\,|||v-w|||^2 \le \langle \mathcal{A}v - \mathcal{A}w\,,\, v-w\rangle \quad \text{for all } v, w \in H^1_0(\Omega). \tag{SM}$$

and **locally Lipschitz continuous**, i.e., for all $\vartheta > 0$, there exists $L[\vartheta] > 0$ such that

$$\langle \mathcal{A}v - \mathcal{A}w, \varphi\rangle \le L[\vartheta]\,|||v-w|||\,|||\varphi||| \;\; \forall v, w, \varphi \in H^1_0(\Omega):\; \max\big\{|||v|||, |||w|||\big\} \le \vartheta. \tag{LIP}$$

The Browder–Minty theorem [2, Theorem 25.B] proves existence and uniqueness of the exact solution $u^\star \in H^1_0(\Omega)$ and its Galerkin approximation $u^\star_H \in \mathcal{X}_H$ to (2). As in the linear case, there holds a Céa-type best-approximation result of the form

$$|||u^\star - u^\star_H||| \le L[2M]/\alpha \min_{v_H \in \mathcal{X}_H} |||u^\star - v_H||| \quad \text{with} \quad M := \|F - \mathcal{A}0\|_{H^{-1}(\Omega)}. \tag{3}$$

Zarantonello iteration. To iteratively linearize the nonlinear problem (2), we employ the Zarantonello iteration, cf. [1, Sections 2.2–2.4]: For a damping parameter $\delta > 0$, the Zarantonello mapping $\Phi_H(\delta;\cdot) : \mathcal{X}_H \to \mathcal{X}_H$ is defined via

$$\langle \nabla\Phi_H(\delta; u_H), \nabla v_H\rangle_\Omega = \langle \nabla u_H, \nabla v_H\rangle_\Omega + \delta\big[\langle F - \mathcal{A}u_H\,,\, v_H\rangle\big] \;\; \forall v_H \in \mathcal{X}_H. \tag{4}$$

For sufficiently small $0 < \delta < \delta^\star := \alpha/L[8M]^2$ with M from (3), the Zarantonello iteration is norm-contractive [1, Proposition 4] for all $|||u_H||| \le 4M$:

$$\exists\, 0 < q^\star_{\mathrm{Zar}} < 1: \quad |||u^\star_H - \Phi_H(\delta; u_H)||| \le q^\star_{\mathrm{Zar}}\, |||u^\star_H - u_H|||. \tag{5}$$

Algebraic solver. To obtain linear complexity of the AILFEM algorithm, we solve (4) by a uniformly contractive geometric multigrid method [6]. In abstract terms, the solver $\Psi_H(u^{k,\star}_H;\cdot) : \mathcal{X}_H \to \mathcal{X}_H$ approximates $u^{k,\star}_H := \Phi_H(\delta; u_H)$, i.e.,

$$\exists\, 0 < q_{\mathrm{alg}} < 1: \;\; |||u^{k,\star}_H - \Psi_H(u^{k,\star}_H; w_H)||| \le q_{\mathrm{alg}}\, |||u^{k,\star}_H - w_H||| \quad \forall w_H \in \mathcal{X}_H. \tag{6}$$

A posteriori error estimation. For $F = f - \operatorname{div} \boldsymbol{f}$ in (1) with $f \in L^2(\Omega)$ and $\boldsymbol{f} \in [L^2(\Omega)]^d$, the residual-based error estimator $\eta_H(\cdot)$ reads

$$\eta_H(T, v_H)^2 := |T|^{2/d} \, \|f + \operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) - b(v_H)\|^2_{L^2(T)} + |T|^{1/d} \, \|[\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) \cdot n]\!]\|^2_{L^2(\partial T \cap \Omega)} \quad \text{for } T \in \mathcal{T}_H \text{ and } v_H \in \mathcal{X}_H, \tag{7a}$$

where $[\![\cdot]\!]$ denotes the jump across $(d-1)$-dimensional faces of $\mathcal{T}_H$. To abbreviate notation, we define, for all $\mathcal{U}_H \subseteq \mathcal{T}_H$ and all $v_H \in \mathcal{X}_H$,

$$\eta_H(v_H) := \eta_H(\mathcal{T}_H, v_H) \quad \text{with} \quad \eta_H(\mathcal{U}_H, v_H) := \Big(\sum_{T \in \mathcal{U}_H} \eta_H(T, v_H)^2 \Big)^{1/2}. \tag{7b}$$

As proven in [8, Proposition 15], the estimator satisfies the so-called *axioms of adaptivity* (A1)–(A4) from [13] with a modified stability property (A1). To this end, for a mesh $\mathcal{T}_H$, let $\mathcal{T}_h$ denote a mesh that is generated by finitely many steps of newest vertex bisection (NVB) from the mesh $\mathcal{T}_H$; see, e.g., [14] for NVB.

(A1) stability: For all $\vartheta > 0$ and all $\mathcal{U}_H \subseteq \mathcal{T}_h \cap \mathcal{T}_H$, there exists $C_{\rm stab}[\vartheta] > 0$ such that for all $v_h \in \mathcal{X}_h$ and $v_H \in \mathcal{X}_H$ with $\max\{|||v_h|||, |||v_H|||\} \le \vartheta$, it holds that

$$\big|\eta_h(\mathcal{U}_H, v_h) - \eta_H(\mathcal{U}_H, v_H)\big| \le C_{\rm stab}[\vartheta] \, |||v_h - v_H|||. \quad \square \tag{A1}$$

3 Adaptive Algorithm

Algorithm 1 (**AILFEM**). ***Input:*** *Conforming initial mesh* $\mathcal{T}_0$*, marking parameters* $0 < \theta \le 1$*,* $C_{\rm mark} \ge 1$*, solver parameters* $\lambda_{\rm lin}, \lambda_{\rm alg} > 0$*, minimal number of algebraic solver steps* $i_{\min} \in \mathbb{N}$*, initial guess* $u_0^{0,0} := u_0^{0,\star} := u_0^{0,\underline{i}} \in \mathcal{X}_0$ *with* $|||u_0^{0,0}||| \le 2M$ *with* M *from* (3)*, and Zarantonello damping parameter* $\delta > 0$*.*
Adaptive loop: *For all* $\ell = 0, 1, 2, \ldots,$ *repeat the following steps* (I)–(III)*:*

(I) `SOLVE & ESTIMATE.` *For all* $k = 1, 2, 3, \ldots,$ *repeat steps* (a)–(c)*:*

- (a) *Define* $u_\ell^{k,0} := u_\ell^{k-1,\underline{i}}$ *and, only for theoretical reasons,* $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k,0})$.
- (b) *For all* $i = 1, 2, 3, \ldots$ *repeat steps* (i)–(ii)*:*
  - (i) *Compute* $u_\ell^{k,i} := \Psi_\ell(u_\ell^{k,\star}; u_\ell^{k,i-1})$ *and error estimator* $\eta_\ell(u_\ell^{k,i})$.
  - (ii) *Terminate the* i*-loop and define* $\underline{i}[\ell, k] := i$ *if*

$$|||u_\ell^{k,i-1} - u_\ell^{k,i}||| \le \lambda_{\rm alg} \left[\lambda_{\rm lin} \, \eta_\ell(u_\ell^{k,i}) + |||u_\ell^{k,i} - u_\ell^{k,0}|||\right] \quad \texttt{AND} \quad i_{\min} \le i. \tag{8}$$

- (c) *Terminate the* k*-loop and define* $\underline{k}[\ell] := k$ *if*

$$\|u_\ell^{k,\underline{i}} - u_\ell^{k,0}\| \le \lambda_{\rm lin} \, \eta_\ell(u_\ell^{k,\underline{i}}) \quad \texttt{AND} \quad |||u_\ell^{k,\underline{i}}||| \le 2M. \tag{9}$$

(II) `MARK.` *Find* $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{\underline{k},\underline{i}}] := \{\mathcal{U}_\ell \subseteq \mathcal{T}_\ell \mid \theta \, \eta_\ell(u_\ell^{\underline{k},\underline{i}})^2 \le \eta_\ell(\mathcal{U}_\ell, u_\ell^{\underline{k},\underline{i}})^2\}$ *with*

$$\#\mathcal{M}_\ell \le C_{\rm mark} \min_{\mathcal{U}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{\underline{k},\underline{i}}]} \#\mathcal{U}_\ell. \tag{10}$$

(III) `REFINE.` *Bisect all* $T \in \mathcal{M}_\ell$ *by NVB to generate the new mesh* $\mathcal{T}_{\ell+1}$ *from* $\mathcal{T}_\ell$ *and define* $u_{\ell+1}^{0,0} := u_{\ell+1}^{0,\underline{i}} := u_{\ell+1}^{0,\star} := u_\ell^{\underline{k},\underline{i}}$ *(nested iteration).*

Output: *Refined meshes* $\mathcal{T}_\ell$*, discrete approximations* $u_\ell^{k,i}$*, and estimators* $\eta_\ell(u_\ell^{k,i})$*.*

We define the countably infinite index set $\mathcal{Q}$ and the total step counter $|\cdot,\cdot,\cdot|$ by

$$\mathcal{Q} := \{(\ell,k,i) \in \mathbb{N}_0^3 \colon u_\ell^{k,i} \text{ is used in Algorithm 1}\},$$
$$|\ell,k,i| := \{(\ell',k',i') \in \mathcal{Q} \mid (\ell',k',i') < (\ell,k,i)\} \quad \forall(\ell,k,i) \in \mathbb{Q}.$$

We denote the stopping indices of the mesh level by $\underline{\ell} \in \mathbb{N}_0 \cup \{\infty\}$, the linearization by $\underline{k} = \underline{k}[\ell] \in \mathbb{N} \cup \{\infty\}$, and the algebraic solver by $\underline{i} = \underline{i}[\ell,k] \in \mathbb{N}$; cf. [12, Lemma 6].

By enforcing at least $i_{\min} \in \mathbb{N}$ algebraic solver steps, we guarantee that the inexact Zarantonello iteration is contractive (despite finitely many algebraic solver steps).

Lemma 1 ([15, Lemma 5.1, Corrigendum, Section 2)]) *There exist* $\lambda_{\mathrm{alg}}^\star > 0$ *and* $i_{\min} \in \mathbb{N}$ *such that* $q_{\mathrm{alg}}^{i_{\min}} < (1-q_{\mathrm{Zar}}^\star[\delta,4M])/(1+q_{\mathrm{Zar}}^\star[\delta,4M])$ *and for all* $0 < \lambda_{\mathrm{alg}} \le \lambda_{\mathrm{alg}}^\star$ *and* $0 < \lambda_{\mathrm{lin}}$, *there exists* $0 < q_{\mathrm{per}} < 1$ *with*

$$|||u_\ell^\star - u_\ell^{k,\underline{i}}||| \le q_{\mathrm{per}}\, |||u_\ell^\star - u_\ell^{k-1,\underline{i}}||| \quad \forall(\ell,k,0) \in \mathcal{Q} \text{ with } 1 \le k \le \underline{k}[\ell]. \quad \square \tag{11}$$

We stress that every step of Algorithm 1 is realized in linear complexity, since all building blocks can be computed in linear complexity:

- ▷ `SOLVE & ESTIMATE.` The algebraic solver is an hp-robust multigrid [6]. On a mesh $\mathcal{T}_\ell$, each algebraic solver step requires only $\mathcal{O}(\#\mathcal{T}_\ell)$ operations. Moreover, the simultaneous computation of the standard error indicators $\eta_\ell(T, u_\ell^{k,i})$ for all $T \in \mathcal{T}_\ell$ can be done at the cost of $\mathcal{O}(\#\mathcal{T}_\ell)$.
- ▷ `MARK.` The determination of $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ by Dörfler marking is indeed a linear complexity problem; see, e.g., [16] for $C_{\mathrm{mark}} = 1$.
- ▷ `REFINE.` The refinement of $\mathcal{T}_\ell$ is based on NVB and, owing to the the finite number of children [17] and the mesh-closure estimate [18, 14], requires only linear cost $\mathcal{O}(\#\mathcal{T}_\ell)$.

Therefore, the total work until and including the computation of $u_\ell^{k,i}$ is given by

$$\mathtt{cost}(\ell,k,i) := \sum_{\substack{(\ell',k',i') \in \mathcal{Q} \\ |\ell',k',i'| \le |\ell,k,i|}} \#\mathcal{T}_{\ell'}, \tag{12}$$

reflecting the adaptive nature of Algorithm 1, since the computation of $u_\ell^{k,i}$ depends on the entire computational history of the algorithm. Uniform boundedness of all iterates $u_\ell^{k,i}$ follows along the lines of [12, Theorem 6] by use of norm-contraction.

Theorem 1 *Suppose that* $\mathcal{A}$ *satisfies* (SM) *and* (LIP) *with* M *from* (3). *Let* $\lambda_{\mathrm{lin}}, \lambda_{\mathrm{alg}} > 0$ *and* $0 < \theta \le 1$ *be arbitrary. Suppose that* $|||u_\ell^{0,0}||| \le 2M$ *with* M *from* (3) *and that* $i_{\min} \in \mathbb{N}$ *as in Lemma 1. Then, for any* $\delta > 0$ *satisfying* $0 < \delta < 2\delta^\star = 2\alpha/L[8M]^2$, *all iterates satisfy the uniform bound*

$$|||u_\ell^{k,i}||| \le 4M \quad \text{for all } (\ell,k,i) \in \mathcal{Q}. \quad \square \tag{UB}$$

4 Main Results

Full R-linear convergence is a key improvement to plain convergence and constitutes the second main result. It states that Algorithm 1 contracts an appropriate quasi-error $\mathrm{H}_\ell^{k,i}$ (up to a constant) regardless of the algorithmic decisions.

Theorem 2 (full R-linear convergence of Algorithm 1) *With the assumptions of Theorem 1 and the estimator axioms stability* (A1)*, reduction* (A2)*, and reliability* (A3)*, Algorithm 1 guarantees full R-linear convergence of the quasi-error*

$$\mathrm{H}_\ell^{k,i} := |||u_\ell^\star - u_\ell^{k,i}||| + |||u_\ell^{k,\star} - u_\ell^{k,i}||| + \eta_\ell(u_\ell^{k,i}), \tag{13}$$

i.e., there exist constants $0 < q_{\mathrm{lin}} < 1$, $C_{\mathrm{lin}} > 0$*, and* $\ell_0 \in \mathbb{N}_0$ *such that*

$$\mathrm{H}_\ell^{k,i} \le C_{\mathrm{lin}} q_{\mathrm{lin}}^{|\ell,k,i|-|\ell',k',i'|}\, \mathrm{H}_{\ell'}^{k',i'} \tag{14}$$

for all $(\ell',k',i'), (\ell,k,i) \in \mathcal{Q}$ *with* $|\ell',k',i'| < |\ell,k,i|$ *and* $\ell, \ell' \ge \ell_0$.

Full R-linear convergence from Theorem 2 yields that convergence rates with respect to the number of degrees of freedom $\dim \mathcal{X}_\ell \simeq \#\mathcal{T}_\ell$ coincide with rates with respect to the overall computational cost; cf. [11, Corollary 15].

Corollary 1 (rates = complexity) *Suppose full R-linear convergence* (14)*. Recall* $\mathtt{cost}(\ell,k,i)$ *from* (12)*. Then, for any rate* $s > 0$*, it holds that*

$$\sup_{(\ell,k,i)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \mathrm{H}_\ell^{k,i} < \infty \quad \Longleftrightarrow \quad \sup_{(\ell,k,i)\in\mathcal{Q}} \mathtt{cost}(\ell,k,i)^s\, \mathrm{H}_\ell^{k,i} < \infty. \quad \square \tag{15}$$

Algorithm 1 equilibrates various error sources captured in the quasi-error $\mathrm{H}_\ell^{k,i}$ and each step generates computational cost (12) for the iterates $u_\ell^{k,i}$.

The subsequent Theorem 3 proves that Algorithm 1 reproduces the best possible rate $s > 0$ over the computational cost (right-hand side in (16) is finite), if $u^\star$ can theoretically be approximated at the rate s in the sense that the error estimators of the exact Galerkin solutions along optimal meshes decay at rate s with respect to the number of degrees of freedom (left-hand side in (16) is finite). More precisely, to tame the cost $\mathtt{cost}(\ell,k,i)^s$ that increases with rate s in the right-hand side of (16) below, the quasi-error $\mathrm{H}_\ell^{k,i}$ must decay asymptotically at least with rate s. Due to the linear complexity of all its parts, these rates can also be understood (and measured) with respect to the overall computation time for an optimal implementation.

Theorem 3 (optimal complexity) *Suppose the assumptions of Theorem 1 and discrete reliability* (A4)*. Then, there exist* $\lambda_{\mathrm{lin}}^\star$ *and* $\theta^\star$ *such that, for all* $0 < \lambda_{\mathrm{lin}} < \lambda_{\mathrm{lin}}^\star$ *and* $0 < \theta < \theta^\star$*, Algorithm 1 guarantees that*

$$\sup_{N\in\mathbb{N}_0} \Big[(N+1)^s \min_{\mathcal{T}_{\mathrm{opt}}\in\mathbb{T}_N} \eta_{\mathrm{opt}}(u_{\mathrm{opt}}^\star)\Big] < \infty \Longrightarrow \sup_{(\ell,k,i)\in\mathcal{Q}} \mathtt{cost}(\ell,k,i)^s\, \mathrm{H}_\ell^{k,i} < \infty. \quad \square \tag{16}$$

5 Proof of Full R-linear Convergence

To extend [12] to vector-valued semilinear PDEs, we replace energy arguments that use Pythagorean-type energy identities with quasi-Pythagorean estimates in the energy norm. The following result is essentially included in [8, Lemma 29] and [15, Lemma 5.4] and follows from the fact that, by assumption, the nonlinear reaction term is only a compact perturbation when compared to the linear principal part.

Lemma 2 (quasi-orthogonality) *Suppose* (SM) *and* (LIP). *Then, for all* $0 < \varepsilon < 1$, *there exists* $\ell_0 \in \mathbb{N}$ *such that for all* $\ell_0 = \ell_0(\varepsilon) \le \ell \le \underline{\ell}$ *and the discrete limit space* $\mathcal{X}_\infty := \mathrm{closure}\big(\bigcup_{\ell=0}^{\ell} \mathcal{X}_\ell\big)$ *with corresponding Galerkin solution* $u_\infty^\star$, *it holds that*

$$\frac{1}{1+\varepsilon}\,|||u_\infty^\star - v_\ell|||^2 \le |||u_\infty^\star - u_\ell^\star|||^2 + |||u_\ell^\star - v_\ell|||^2 \le \frac{1}{1-\varepsilon}\,|||u_\infty^\star - v_\ell|||^2 \;\forall v_\ell \in \mathcal{X}_\ell. \tag{QO}$$

If $\ell < \underline{\ell}$, *the statement holds accordingly with* $u_\infty^\star$ *being replaced with* $u_{\ell+1}^\star$. □

Proof of Theorem 2. The proof consists of three steps, where we prove summability of the quasi-error $\mathrm{H}_\ell^{k,i}$ from (13) via summability of the simplified quantities

$$\mathrm{H}_\ell^k := |||u_\ell^\star - u_\ell^{k,\underline{i}}||| + \eta_\ell(u_\ell^{k,\underline{i}}) \quad \text{for all } (\ell,k,\underline{i}) \in \mathcal{Q} \quad \text{and} \tag{17}$$

$$\mathrm{H}_\ell := \big[\alpha_\ell^2 + \gamma\,\eta_\ell(u_\ell^{\underline{k},\underline{i}})^2\big]^{1/2} \quad \text{for all } (\ell,\underline{k},\underline{i}) \in \mathcal{Q}, \tag{18}$$

where $\alpha_\ell := |||u_\ell^\star - u_\ell^{\underline{k},\underline{i}}|||$ and the constant $\gamma > 0$ will be chosen below.

Step 1 (summability of H_ℓ). This step is subdivided into two substeps.

Step 1.1. First, we prove the perturbed contraction $\mathrm{H}_{\ell+1} \le q\mathrm{H}_\ell + R_\ell$ with a suitable remainder R_ℓ. With $\mathrm{A}_{\ell+1} := |||u_{\ell+1}^\star - u_\ell^\star|||$, the inexact Zarantonello contraction (11) with $0 < q_{\mathrm{per}} < 1$ and nested iteration $u_\ell^{\underline{k},\underline{i}} = u_{\ell+1}^{0,0}$ yield

$$|||u_{\ell+1}^{\underline{k},\underline{i}} - u_\ell^{\underline{k},\underline{i}}||| \le |||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k},\underline{i}}||| + |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{i}}||| \overset{(11)}{\lesssim} |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{i}}||| \le \mathrm{A}_{\ell+1} + \alpha_\ell. \tag{19}$$

With estimator reduction [5, Equation (52)] with $0 < q_\theta < 1$ depending only on θ and reduction (A2), the Young inequality with $0 < \mu < 1$ verifies

$$\begin{aligned}\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{i}})^2 &\le (1+\mu)\,q_\theta\,\eta_\ell(u_\ell^{\underline{k},\underline{i}})^2 + (1+\mu^{-1})C_{\mathrm{stab}}[8M]^2\,|||u_{\ell+1}^{\underline{k},\underline{i}} - u_\ell^{\underline{k},\underline{i}}|||^2 \\ &\overset{(19)}{\le} (1+\mu)\,q_\theta\,\eta_\ell(u_\ell^{\underline{k},\underline{i}})^2 + C_1\alpha_\ell^2 + C_1\,\mathrm{A}_{\ell+1}^2,\end{aligned} \tag{20}$$

where C_1 depends only on $C_{\mathrm{stab}}[8M]^2$, (19), μ^{-1}, and q_{per}. The Céa lemma (3), reliability (A3), stability (A1), and the Young inequality lead us to

$$\mathrm{A}_{\ell+1}^2 \overset{(3)}{\lesssim} |||u^\star - u_\ell^\star|||^2 \overset{(\mathrm{A3})}{\lesssim} \eta_\ell(u_\ell^\star)^2 \overset{(\mathrm{A1})}{\lesssim} \eta_\ell(u_\ell^{\underline{k},\underline{i}})^2 + \alpha_\ell^2 \simeq \mathrm{H}_\ell^2. \tag{21}$$

Inequality (20), contraction (11), and the Young inequality for $|||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{i}}|||^2$ prove

$$
\begin{aligned}
\mathrm{H}_{\ell+1}^2 &\overset{(11)}{\leq} q_{\mathrm{per}}^2 \, |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{i}}|||^2 + \gamma \, \eta_\ell(u_{\ell+1}^{\underline{k},\underline{i}})^2 \\
&\overset{(20)}{\leq} [(1+\mu)\, q_{\mathrm{per}}^2 + C_1\gamma] \, \alpha_\ell^2 + (1+\mu)\gamma q_\theta \, \eta_\ell(u_\ell^{\underline{k},\underline{i}})^2 + [(1+\mu^{-1}) + C_1\gamma] \, \mathrm{A}_{\ell+1}^2 .
\end{aligned}
\tag{22}
$$

Let $\sigma > 0$ that is fixed below in (24). Define $C_2 := (1+\mu^{-1}) + C_1\gamma$. By adding $\pm C_2\sigma\eta_\ell(u_\ell^\star)^2$ in (22) and by using $\eta_\ell(u_\ell^\star)^2 \leq 2\eta_\ell(u_\ell^{\underline{k},\underline{i}})^2 + 2C_{\mathrm{stab}}[8M]^2\alpha_\ell^2$, we get

$$
\begin{aligned}
\mathrm{H}_{\ell+1}^2 &\overset{(22)}{\leq} [(1+\mu)q_{\mathrm{per}}^2 + C_1\gamma + 2\sigma C_2 C_{\mathrm{stab}}[8M]^2] \, \alpha_\ell^2 \\
&\qquad + [(1+\mu)q_\theta + 2\sigma\gamma^{-1}C_2] \, \gamma \, \eta_\ell(u_\ell^{\underline{k},\underline{i}})^2 + C_2[\mathrm{A}_{\ell+1}^2 - \sigma\eta_\ell(u_\ell^\star)^2] \\
&\overset{(21)}{\leq} \max\{[(1+\mu)\, q_{\mathrm{per}}^2 + C_1\gamma + 2\sigma C_2 C_{\mathrm{stab}}[8M]^2], (1+\mu)q_\theta + 2\sigma\gamma^{-1}C_2\} \, \mathrm{H}_\ell^2 \\
&\qquad + C_2[\mathrm{A}_{\ell+1}^2 - \sigma\eta_\ell(u_\ell^\star)^2] =: q\, \mathrm{H}_\ell^2 + R_\ell^2 .
\end{aligned}
\tag{23}
$$

To obtain quasi-contraction with remainder of the form (23) with $0 < q < 1$, choose

- $\mu > 0$ such that $(1+\mu)q_{\mathrm{per}}^2 < 1$ and $(1+\mu)q_\theta < 1$;
- $\gamma > 0$ such that $(1+\mu)q_{\mathrm{per}}^2 + \gamma C_1[\mu^{-1}] < 1$;
- $\sigma > 0$ such that q from (23) is contractive, i.e., $0 < q < 1$.

Step 1.2. In this step, we prove summability of R_ℓ^2 from (23) and conclude summability of H_ℓ. To this end, we first choose $0 < \varepsilon < 1$ such that

$$
q_\varepsilon := \frac{1+\varepsilon}{1-\varepsilon}\, q_{\mathrm{per}}^2 < 1 \quad \text{ and } \quad \frac{\varepsilon}{1-\varepsilon}\,(1+C_{\text{Céa}})^2 C_{\mathrm{rel}}^2 \leq \sigma, \tag{24}
$$

which determines the index $\ell_0 = \ell_0(\varepsilon) \in \mathbb{N}_0$ from Lemma 2. Reliability (A3) verifies

$$
\frac{\varepsilon}{1-\varepsilon}\, |||u_\infty^\star - u_\ell^\star|||^2 \overset{(3)}{\leq} \frac{\varepsilon}{1-\varepsilon}\,(1+C_{\text{Céa}})^2\, |||u^\star - u_\ell^\star|||^2 \overset{(A3),\,(24)}{\leq} \sigma\, \eta_\ell(u_\ell^\star)^2. \tag{25}
$$

For $\ell_0 \leq \ell'$, quasi-orthogonality (QO) and the Céa lemma (3) prove

$$
\begin{aligned}
\sum_{\ell=\ell'}^{\ell-1} \big[\mathrm{A}_{\ell+1}^2 - \sigma\,\eta_\ell(u_\ell^\star)^2\big] &\overset{(QO)}{\leq} \sum_{\ell=\ell'}^{\ell-1} \Big[\frac{1}{1-\varepsilon}\, |||u_\infty^\star - u_\ell^\star|||^2 - |||u_\infty^\star - u_{\ell+1}^\star|||^2 - \sigma\,\eta_\ell(u_\ell^\star)^2\Big] \\
&\overset{(25)}{\leq} \sum_{\ell=\ell'}^{\ell-1} \Big[\frac{1}{1-\varepsilon}\, |||u_\infty^\star - u_\ell^\star|||^2 - |||u_\infty^\star - u_{\ell+1}^\star|||^2 - \frac{\varepsilon}{1-\varepsilon}\, |||u_\infty^\star - u_\ell^\star|||^2\Big] \\
&\leq |||u_\infty^\star - u_{\ell'}^\star|||^2 \overset{(3)}{\lesssim} |||u^\star - u_{\ell'}^\star|||^2 \lesssim \mathrm{H}_{\ell'}^2 .
\end{aligned}
$$

Hence, $\sum_{\ell=\ell'}^{\ell-1} R_\ell^2 \lesssim \mathrm{H}_\ell^2$, i.e., the remainder R_ℓ is square-summable. The summability criterion from [13, Lemma 4.7] concludes (square-)summability of H_ℓ.

Step 2 (summability of $\mathbf{H}_\ell^k$ over indices (ℓ, k)). By using the perturbed Zarantonello contraction (11) (instead of energy contraction), we conclude tail summability of H_ℓ^k owing to [12, Step 4–5].

Step 3 (summability of $\mathbf{H}_\ell^{k,i}$ over index (ℓ, k, i)). The tail summability for $\mathrm{H}_\ell^{k,i}$ is a consequence of [12, Step 6–7], and with [13, Lemma 4.9], we conclude R-linear convergence. This completes the proof. □

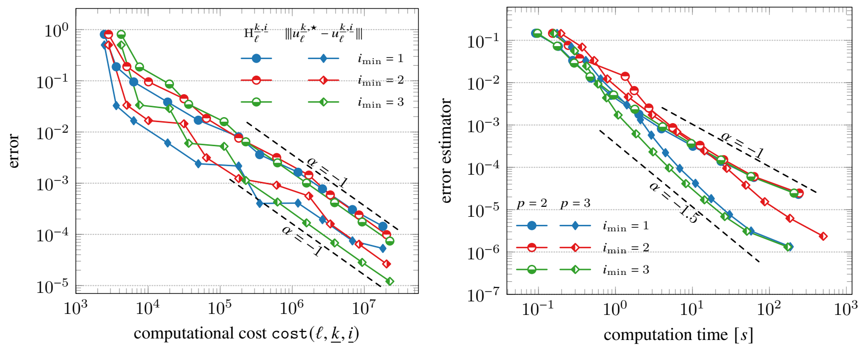

Fig. 1 Semilinear experiment (26) with $i_{\min} \in \{1, 2, 3\}$: Left: Convergence plots of the error $\mathrm{H}_\ell^{\underline{k},\underline{i}}$ (circle) and the algebraic error $|||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},\underline{i}}|||$ (diamond) over $\mathtt{cost}(\ell, \underline{k}, \underline{i})$ for $p = 2$. Right: We plot $\eta_\ell(u_\ell^{\underline{k},\underline{i}})$ over computation time in seconds for $p = 2$ (circle) and $p = 3$ (diamond).

6 Numerical Experiments

We consider a semilinear problem, where our implementation relies on MooAFEM [19]. On the L-shape $\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0] \subset \mathbb{R}^2$, we approximate

$$-\Delta u^\star + (u^\star)^3 = f - \operatorname{div} \boldsymbol{f} \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \text{ on } \partial\Omega \tag{26}$$

with the monotone semilinearity $b(v) = v^3$, which is locally Lipschitz continuous (by the same reasoning as in [1, Experiment 26]). We choose

$$f = 0 \quad \text{and} \quad \boldsymbol{f}(x_1, x_2) = (-1, 0)^\intercal \begin{cases} 1, & \text{if } \|(x_1, x_2)\|_1 > 1, \\ 0, & \text{else} . \end{cases}$$

For the computations, we consider a damping parameter $\delta = 0.3$, the number of minimal algebraic steps $i_{\min} \in \{1, 2, 3\}$, and $\lambda_{\mathrm{lin}} = \lambda_{\mathrm{alg}} = \theta = 0.5$. In Figure 1 (left), we plot the quasi-error $\mathrm{H}_\ell^{\underline{k},\underline{i}}$ and the algebraic error $|||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},\underline{i}}|||$ over the computational cost $\mathtt{cost}(\ell, k, i)$ from (12) with $p = 2$. We observe the optimal decay rate $s = 1$ for the quasi-error $\mathrm{H}_\ell^{\underline{k},\underline{i}}$ for all choices of $i_{\min}$. However, we observe that the algebraic error is more erratic (and slightly suboptimal) for lower $i_{\min}$. In Figure 1 (right), we plot the estimator $\eta_\ell(u_\ell^{\underline{k},\underline{i}})$ for $p \in \{2, 3\}$ over the computation time in seconds. This is justified since $\mathrm{H}_\ell^{\underline{k},\underline{i}} \simeq \eta_\ell(u_\ell^{\underline{k},\underline{i}})$ by virtue of the stopping criteria in Algorithm 1. As before, we also observe optimal convergence rates $s = p/2$.

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