UE Mengenlehre SoSe2024

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Session 3

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1) Show that for every $\alpha < \omega_1$ there is a tree $T_\alpha \subseteq 2^{<\omega}$ such that $\forall \beta < \alpha\,(T^{(\alpha)} \neq T^{(\beta)})$ but $T^{(\alpha)} = T^{(\alpha+1)}$.

2) (a) Let α be an ordinal. Show that ω^α is closed under addition, i.e. if $\beta, \gamma < \omega^\alpha$ then $\beta + \gamma < \omega^\alpha$.

   (b) Fix $n < \omega, l \leq \omega$. We define a function $F_{n,l} : \omega \to V$. Write $k < \omega$ in hereditary base n (i.e. write k in base n and then write every exponent in base n and so on). Then exchange every n by l. For example 17 in hereditary base 3 is $3^{3^0 \cdot 2} + 3^{3^0} \cdot 2 + 3^0 \cdot 2$. So $F_3(17) = l^{l^0 \cdot 2} + l^{l^0} \cdot 2 + l^0 \cdot 2$. Show that $F_{n,l}$ is strictly increasing.

   (c) Show that the Goodstein sequence of every $m < \omega$ is zero after finitely many steps.

   Definition. Fix $m < \omega$. We construct the **Goodstein sequence** $G_m = \langle G_m(n) \mid n < \omega \rangle$ of n by recursion as follows

   - $G_m(0) := m$ and
   - $G_m(n+1) = F_{n+1,n+2}(G_m(n)) - 1$.

   Example. The Goodstein sequence of 3 is computed as follows

   - $G_3(0) = 3$
   - $G_3(0) = 2^{2^0} + 2^0$. So $G_3(1) = (3^{3^0} + 3^0) - 1 = (3^1 + 1) - 1 = 3$
   - $G_3(1) = 3 = 3^{3^0}$. So $G_3(2) = 4^{4^0} - 1 = 3$
   - ...

3) (a) Show that the Axiom of Choice is equivalent to "every surjective function splits".

   Definition. A surjective function $f : a \to b$ **splits** iff there is a function $g : b \to a$ such that $f \circ g = id_y$.

   (b) Show that the Axiom of Choice implies the Ultrafilter Lemma.

   Definition. Let x be a set. $G \subseteq \mathcal{P}(x)$ is a **filter** on x iff $\emptyset \notin G$, G is closed under intersections (i.e. $a, b \in G \Rightarrow a \cap b \in G$) and G is upwards closed (i.e. $a \in G \wedge b \in \mathcal{P}(x) \wedge a \subseteq b \Rightarrow b \in G$).

A filter G is an **ultrafilter** G is maximal w.r.t. inclusion (i.e. if F is a filter on x and $G \subseteq F$, then $G = F$). The **Ultrafilter Lemma** is the statement, that for every set x and every filter G on x, there is an ultrafilter on x that contains G.

4) Do the following sets have cardinality $2^{\aleph_0}$ or $2^{2^{\aleph_0}}$?

   (a) $\mathbb{R}$,

   (b) $\{f : \mathbb{R} \to \mathbb{R}\}$ and

   (c) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$.