Theory of Pontryagin spaces

Geometry and Operators

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Contents

1 Some linear algebra **1**

1.1 Scalar Product Spaces 1
1.2 Orthogonality 8
1.3 Orthocomplemented Subspaces 16
1.4 Definiteness Properties 21
1.5 Angular Operators 25
1.6 Index of Positivity and Negativity 28
1.7 Neutral Subspaces 36

2 Scalar Product Spaces with Topology **45**

2.1 Basic Consequences of Continuity 45
2.2 Gram Spaces 49
2.3 Krein Spaces 56
2.4 Pontryagin Spaces 63
2.5 Almost Pontryagin Spaces 70
2.6 Completions 80
2.7 *Almost Pontryagin Space Completions 86
2.8 To become part of the Appendix 93

3 Linear Relations **95**

3.1 Operators as linear relations 95
3.2 Transformations of linear Relations 98
3.3 Möbius-Calculus for Linear Relations 101
3.4 Linear Relations on normed Spaces 105
3.5 Spectrum, resolvent set and points of regular type 106
3.6 Functional Calculus for rational functions 114
3.7 Finite dimensional perturbations of linear relations 117
3.8 Adjoint linear Relations 122
3.9 Special types of linear relations and their connection 127
3.10 The Potapov-Ginzburg transform 131
3.11 More on contractive relations 133
3.12 Moving linear relations 139
3.13 Definitizable linear relations 143
3.14 Functional Calculus for seladjoint definitizable relations 147

4 Reproducing kernel spaces **153**
4.1 Kernel functions 153
4.2 Constructions with hermitian kernels 165
4.3 Analytic kernels 169
4.4 Some classes of kernels associated with analytic functions 172
4.4.1 Analytic mappings between general disks 172
4.4.2 Hilbert space valued Nevanlinna functions 176
4.4.3 De Branges' spaces of entire functions 176
4.4.4 Entire J-inner matrix functions 185
4.5 *Reproducing kernel almost Pontryagin spaces 185

Bibliography **187**

Index **187**

NOTATION...

Before we start the exposition, let us fix some standard notation.

– We denote by $\mathbb{Z}$ the set of *integer numbers*, and by $\mathbb{N}$ the set of *positive integers*,

$$\mathbb{Z} := \{\dots, -1, 0, 1, 2, \dots\}, \quad \mathbb{N} := \{1, 2, 3, \dots\}.$$

– We denote by $\mathbb{R}$ and by $\mathbb{C}$ the fields of *real*- and *complex*, respectively, numbers.

– We denote by $\mathbb{C}^n$, $n \in \mathbb{N}$, the set of all *n-vectors* with complex entries. We write an n-vector as a column of n entries.

– We denote by $\mathbb{C}^{n\times m}$, $n, m \in \mathbb{N}_0$, the set of all *n×m-matrices* with complex entries (n rows and m columns).

– For $A = (\alpha_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,m}} \in \mathbb{C}^{n\times m}$, we denote by $A^t \in \mathbb{C}^{m\times n}$ and $A^* \in \mathbb{C}^{m\times n}$ the *transpose* and *conjugate transpose*, respectively, of A. That is,

$$A^t = (\beta_{ij})_{\substack{i=1,\dots,m,\\ j=1,\dots,n}} \quad \beta_{ij} := \alpha_{ji},\ i = 1, \dots, m, j = 1, \dots, n.$$

$$A^* = (\gamma_{ij})_{\substack{i=1,\dots,m,\\ j=1,\dots,n}} \quad \beta_{ij} := \overline{\gamma_{ji}},\ i = 1, \dots, m, j = 1, \dots, n.$$

– For a subspace M of a linear space $\mathcal{V}$, we denote by $\operatorname{span} M$ the *linear span of M*, i.e., the smallest linear subspace of $\mathcal{V}$ which contains M. If $\mathcal{L}$ and $\mathcal{M}$ are linear subspaces of $\mathcal{V}$ such that $\mathcal{L} \cap \mathcal{M} = \{0\}$ then we denote the *direct sum of $\mathcal{L}$ and $\mathcal{M}$* as

$$\mathcal{L} \dot{+} \mathcal{M}.$$

Chapter 1

Some linear algebra

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We lay out the algebraic basics of indefinite scalar product spaces. Topics include: orthocomplemented subspaces, angular operators, semidefinite subspaces, index of positivity and negativity, skewly linked neutral subspaces. Much of the material is standard linear algebra; the experienced reader may skip this chapter and return when necessary.

§1. Scalar product spaces	1
§2. Orthogonality	8
§2. Orthocomplemented subspaces	16
§4. Definiteness properties	21
§5. Angular Operators	25
§6. Index of positivity and negativity	28
§7. Neutral subspaces	36

1.1 Scalar Product Spaces

calarProductSpaces

Unless explicitly stated, all linear spaces are understood over the scalar field $\mathbb{C}$ of complex numbers.

scalarprod.

1.1.1 Definition. Let $\mathcal{V}$ be a linear space, and let $[.,.]:\mathcal{V}\times\mathcal{V}\to\mathbb{C}$. We call $[.,.]$ a *scalar product on* $\mathcal{V}$, if

(1) $[.,.]$ is *linear in the first argument*, i.e.,

$$[\alpha x+\beta y,z]=\alpha[x,z]+\beta[y,z],\quad x,y,z\in\mathcal{V},\ \alpha,\beta\in\mathbb{C}\,.$$

(2) $[.,.]$ is *hermitian*, i.e.,

$$[x,y]=\overline{[y,x]},\quad x,y\in\mathcal{V}\,.$$

We call a tuple $(\mathcal{V}, [.,.])$ a *scalar product space*, if $\mathcal{V}$ is a linear space and $[.,.]$ is a scalar product on $\mathcal{V}$. $\Diamond$

When no confusion may occur, we will often drop explicit notation of the scalar product $[.,.]$ in $(\mathcal{V}, [.,.])$ and shortly speak of a scalar product space $\mathcal{V}$.

Let us point out that in contrast to most of Functional Analysis' literature we do not assume that $[x, x] > 0$, $x \in \mathcal{V} \setminus \{0\}$.

Simple manipulations with the axioms show that each scalar product satisfies

$$[x, 0] = [0, x] = 0, \quad x \in \mathcal{V}.$$

and is *conjugate linear in the second argument*, i.e.,

$$[z, \alpha x + \beta y] = \overline{\alpha}[z, x] + \overline{\beta}[z, y], \quad x, y, z \in \mathcal{V},\ \alpha, \beta \in \mathbb{C}.$$

Moreover, the *polar identity*

$$[x, y] = \underbrace{[x + y, x + y] - [x - y, x - y] + i[x + iy, x + iy] - i[x - iy, x - iy]}_{=\sum_{k=0}^{3} i^k [x+i^k y, x+i^k y]}, \quad x, y \in \mathcal{V}, \tag{1.1.1}$$ polarid

holds true.

polidrem. *1.1.2 Remark.* The polar identity, (1.1.1), also holds true for all sesquilinear forms, i.e., mappings $\langle .,. \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ which are linear in the first and conjugate linear in the second argument.

In fact, $[.,.]$ being a scalar products means exactly that $[.,.]$ is a hermitian sesquilinear form. Hermitian sesquilinear forms $[.,.]$ always satisfy $[x, x] \in \mathbb{R}$, $x \in \mathcal{V}$. Conversely, if $[.,.]$ is a sesquilinear form satisfying $[x, x] \in \mathbb{R}$, $x \in \mathcal{V}$, then it follows from the polar identity, (1.1.1), that $[.,.]$ is hermitian. $\Diamond$

As a first example we consider finite dimensional spaces.

I.2. *1.1.3 Example.* Let $m \in \mathbb{N}$, and consider the space $\mathbb{C}^m$. Moreover, let a matrix $G = (\gamma_{ij})_{i,j=1}^m \in \mathbb{C}^{m \times m}$ with $G = G^*$ be given. Here G^* denotes the conjugate transpose of the matrix G. Then

$$\left[\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \right] := \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}^* G \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \sum_{i,j=1}^{m} \overline{\beta_i} \cdot \gamma_{ij} \cdot \alpha_j, \quad \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \in \mathbb{C}^m. \tag{1.1.2}$$ I.19

constitutes a scalar product on $\mathbb{C}^m$, because linearity in the first argument is obvious and the assumption that $G = G^*$ yields that $[x, y] = \overline{[y, x]}$, $x, y \in \mathbb{C}^m$.

Conversely, any scalar product $[.,.]$ on $\mathbb{C}^m$ can be obtained in this way. To see this, denote by $e_1, \dots, e_m$ the *canonical basis vectors*, i.e., $e_k := (\delta_{kj})_{j=1}^m$. Here δ_{kj} stands for the *Kronecker-Delta*, i.e., $\delta_{kj} = 1$ if $k = j$ and 0 otherwise. Set

$$\gamma_{ij} := [e_j, e_i], \quad i, j = 1, \dots, m.$$

The matrix $G := (\gamma_{ij})_{i,j=1}^m$ then satisfies $G = G^*$, and

$$\left[\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \right] = \Big[\sum_{j=1}^{m} \alpha_j e_j, \sum_{i=1}^{m} \beta_i e_i \Big] = \sum_{i,j=1}^{m} \overline{\beta_i} \cdot [e_j, e_i] \cdot \alpha_j = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}^* G \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}.$$

The matrix G is also called the *Gram matrix* of $[.,.]$.

Denote by $(.,.)$ the euclidean scalar product on $\mathbb{C}^m$, i.e.,

$$\left(\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix},\begin{pmatrix}\beta_1\\ \vdots\\ \beta_m\end{pmatrix}\right) := \sum_{i=1}^m \alpha_i\overline{\beta_i}\,.$$

Then the relation (1.1.2) can be written in the form

$$\left[\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix},\begin{pmatrix}\beta_1\\ \vdots\\ \beta_m\end{pmatrix}\right] = \left(G\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix},\begin{pmatrix}\beta_1\\ \vdots\\ \beta_m\end{pmatrix}\right).$$

Thus we may say that the Gram matrix of $[.,.]$ realizes the switch from $(.,.)$ to $[.,.]$.
◊

Our second example is similarly straightforward, but not anymore finite dimensional.

I.3. *1.1.4 Example.* For a nonempty set M we denote by $\mathcal{F}(M)$ the linear space of all *finitely supported complex valued functions on* M, i.e.,

$$\mathcal{F}(M) := \Big\{f\in\mathbb{C}^M\ :\ \{\zeta\in M : f(\zeta)\neq 0\}\text{ is finite}\Big\}.$$

Let $K : M\times M\to\mathbb{C}$ be a function which satisfies

$$K(\zeta,\eta) = \overline{K(\eta,\zeta)},\quad \zeta,\eta\in M\,. \tag{1.1.3}$$ I.4

We set

$$[f,g] := \sum_{\zeta,\eta\in M}\overline{g(\zeta)}\cdot K(\zeta,\eta)\cdot f(\eta),\quad f,g\in\mathcal{F}(M)\,. \tag{1.1.4}$$ I.20

This expression is well-defined since the sum on the right side contains only finitely many nonzero summands. It is straightforward to check that $[.,.]$ is a scalar product on $\mathcal{F}(M)$. Here linearity in the first argument is obvious and (1.1.3) ensures that $[.,.]$ is hermitian.

We refer to a function K with (1.1.3) as a *hermitian kernel on* M, and to $(\mathcal{F}(M),[.,.])$ where $[.,.]$ is as in (1.1.4) as the *scalar product space generated by the kernel* K. Let us notice explicitly that the functions δ_ξ defined for each $\xi\in M$ as

$$\delta_\xi(\zeta) := \begin{cases}1, & \zeta=\xi\\ 0, & \zeta\neq\xi\end{cases},\quad \zeta\in M\,, \tag{1.1.5}$$ I.50

form a basis of $\mathcal{F}(M)$.

This example of scalar product spaces includes the above Example 1.1.3. To see this, assume a matrix $G=(\gamma_{ij})_{i,j=1}^m$ with $G=G^*$ is given. For

$$M := \{1,\dots,m\},\quad K(\zeta,\eta) := \gamma_{\zeta\eta},\ \zeta,\eta\in M\,,$$

the space $\mathcal{F}(M)$ is nothing else but $\mathbb{C}^m$ and the scalar product (1.1.4) coincides with the scalar product defined by (1.1.2). The basis $\{\delta_1,\dots,\delta_m\}$ of $\mathcal{F}(M)$ is nothing but the canonical basis of $\mathbb{C}^m$.
◊

I.3post. *1.1.5 Example.* Let $D\subseteq\mathbb{R}^p$ be an open set, and denote by $C^\infty(D)$ the linear space of all complex valued, infinitely often differentiable functions. Let $\mathcal{E}$ be a linear subspace of $C^\infty(D)$, which is closed under complex conjugation and multiplication.

If $\phi : \mathcal{E} \to \mathbb{C}$ is a linear functional such that $\phi(f) \in \mathbb{R}$ for all real valued f from $\mathcal{E}$, then

$$[f, g] := \phi(\, f \cdot \bar{g}\,)\,,$$

constitutes a scalar product on $\mathcal{E}$; see Remark 1.1.2.

For example if $D = \mathbb{R}$, $\mathcal{E} = C^\infty_{00}(\mathbb{R})$ is the space of functions in $C^\infty(\mathbb{R})$ with compact support, and $\phi(f) = \int f(x)\, dx$, then $(\mathcal{E}, [.,.])$ is a dense subspace of $L^2(\mathbb{R})$. $\Diamond$

I.5. **1.1.6 Definition.** Let $(\mathcal{V}, [.,.])$ and $(\mathcal{W}, [\![.,.]\!])$ be two scalar product spaces, and let $\varphi : \mathcal{V} \to \mathcal{W}$. Then we call φ *isometric*, if

$$[\![\varphi x, \varphi y]\!] = [x, y], \quad x, y \in \mathcal{V}\,.$$

If $\varphi : \mathcal{V} \to \mathcal{W}$ is isometric, we also call φ an *isometry of $\mathcal{V}$ into $\mathcal{W}$*. $\Diamond$

Note that we do not include the requirement that φ is linear into the definition of an isometry.

Let us exhibit an interesting example of an isometric map.

I.6. *1.1.7 Example.* Let r be a rational function with real coefficients, and denote by $M \subseteq \mathbb{C}$ the set of all points where r is analytic. Then M is symmetric with respect to the real axis, i.e., $\zeta \in M$ if and only if $\overline{\zeta} \in M$, and

$$r(\overline{\zeta}) = \overline{r(\zeta)}, \quad \zeta \in M\,.$$

Moreover, $\mathbb{C} \setminus M$ is finite. Consider the function $K : M \times M \to \mathbb{C}$ defined as

$$K(\zeta, \eta) := \begin{cases} \frac{r(\eta) - \overline{r(\zeta)}}{\eta - \overline{\zeta}}\,, & \zeta \neq \overline{\eta} \\ r'(\eta) \quad\;\, , & \eta = \overline{\zeta} \end{cases}$$

Clearly, K is a hermitian kernel, i.e., satisfies $K(\zeta, \eta) = \overline{K(\eta, \zeta)}$, $\zeta, \eta \in M$. Hence, we may consider the scalar product space $(\mathcal{F}(M), [.,.])$ generated by K. We refer to this kernel K as the *Nevanlinna kernel* of the function r.

Choose relatively prime polynomials p and q with real coefficients, such that $r = \frac{p}{q}$. Then $M = \{\zeta \in \mathbb{C} : q(\zeta) \neq 0\}$. The polynomial $p(\eta)q(\zeta) - p(\zeta)q(\eta)$ in the two variables ζ and η vanishes whenever $\zeta = \eta$. Hence, it is divisible by $(\eta - \zeta)$ in the ring of all polynomials in two variables with real coefficients and we can write
$(m := \max\{\deg p, \deg q\})$

$$L(\zeta, \eta) := \frac{p(\eta)q(\zeta) - p(\zeta)q(\eta)}{\eta - \zeta} = \sum_{i,j=0}^{m-1} \zeta^i \gamma_{ij} \eta^j \tag{1.1.6}$$ gammadef

with some coefficients $\gamma_{ij} \in \mathbb{R}$. Since $L(\zeta, \eta) = L(\eta, \zeta)$, we have $\gamma_{ij} = \gamma_{ji}$, $i, j = 0, \ldots, m-1$. Therefore, we may consider the scalar product space $(\mathbb{C}^m, [\![.,.]\!])$ defined as in Example 1.1.3 using the matrix $G := (\gamma_{ij})_{i,j=0}^{m-1}$.

We define the linear mapping $\theta : \mathcal{F}(M) \to \mathbb{C}^m$ as

$$\theta f := \Big(\sum_{\zeta \in M} \frac{f(\zeta)}{q(\zeta)} \zeta^i \Big)_{i=0}^{m-1}, \quad f \in \mathcal{F}(M)\,. \tag{1.1.7}$$ I.theta

This expression is well-defined since the sums on the right side contain only finitely many nonzero summands, and since $q(\zeta) \neq 0$, $\zeta \in M$. Clearly, θ is linear. We are going to check that θ is isometric. To this end, let us show that

$$K(\zeta,\eta) = \frac{1}{\overline{q(\overline{\zeta})}q(\eta)} L(\overline{\zeta},\eta), \quad \zeta,\eta \in M\,. \tag{1.1.8}$$ I.23

For $\eta \neq \overline{\zeta}$ this is obvious. Both functions K and L are continuous in ζ for each fixed η. Hence the relation (1.1.8) extends by continuity to all $\zeta,\eta \in M$. Using (1.1.8), we compute (here $[\theta f]_i$ denotes the i-th component of the vector θf)

$$\begin{aligned}
[f,g] &= \sum_{\zeta,\eta\in M} \overline{g(\zeta)} \cdot K(\zeta,\eta) \cdot f(\eta) = \sum_{\zeta,\eta\in M} \overline{\Big(\frac{g(\zeta)}{q(\zeta)}\Big)} \cdot L(\overline{\zeta},\eta) \cdot \Big(\frac{f(\eta)}{q(\eta)}\Big) = \\
&= \sum_{\zeta,\eta\in M} \sum_{i,j=0}^{m-1} \overline{\Big(\frac{g(\zeta)}{q(\zeta)}\Big)} \cdot \overline{\zeta}^i \cdot \gamma_{ij} \cdot \eta^j \cdot \Big(\frac{f(\eta)}{q(\eta)}\Big) = \sum_{i,j=0}^{m-1} \gamma_{ij} \sum_{\zeta,\eta\in M} \overline{\Big(\frac{g(\zeta)}{q(\zeta)}\zeta^i\Big)} \Big(\frac{f(\eta)}{q(\eta)}\eta^j\Big) = \\
&= \sum_{i,j=0}^{m-1} \overline{[\theta g]_i} \cdot \gamma_{ij} \cdot [\theta f]_j = [\![\theta f, \theta g]\!], \quad f,g \in \mathcal{F}(M)\,,
\end{aligned}$$

i.e., θ is isometric.

As a conclusion, for a rational function r, the scalar product space generated by the Nevanlinna kernel of r can be taken isometrically into a finite dimensional space. ◊

Next, we provide some standard constructions which can be carried out with scalar product spaces. Verification of these facts is straightforward; we leave the details to the reader.

I.8. **1.1.8 Proposition.** *The following constructions can be carried out within the class of scalar product spaces.*

(1) *Let* $(\mathcal{V},[.,.])$ *be a scalar product space, and let* $\mathcal{L}$ *be a linear subspace of* $\mathcal{V}$. *Then* $(\mathcal{L},[.,.]|_{\mathcal{L}\times\mathcal{L}})$ *is a scalar product space, where* $\mathcal{L}$ *is endowed with the natural linear operations inherited from* $\mathcal{V}$ *and where* $[.,.]|_{\mathcal{L}\times\mathcal{L}}$ *denotes the restricton of* $[.,.]$ *to vectors from* $\mathcal{L}$.

The set-theoretic inclusion map of $\mathcal{L}$ *into* $\mathcal{V}$ *is linear and isometric.*

(2) *For each* $j \in \{1,\dots,n\}$ *let* $(\mathcal{V}_j,[.,.]_j)$ *be a scalar product space, and denote by* $[.,.]$ *the* sum scalar product on $\prod_{j=1}^n \mathcal{V}_j$, *i.e.,*

$$[(x_1;\dots;x_n),(y_1;\dots;y_n)] := \sum_{j=1}^n [x_j,y_j]_j,$$

$$(x_1;\dots;x_n),(y_1;\dots;y_n) \in \prod_{j=1}^n \mathcal{V}_j\,. \tag{1.1.9}$$ I.7

Then $(\prod_{j=1}^n \mathcal{V}_j,[.,.])$ *is a scalar product space, where* $\prod_{j=1}^n \mathcal{V}_j$ *is endowed with the natural linear operations defined in a componentwise manner.*

For each $j \in \{1,\dots,n\}$ *the embedding* $\iota_j : \mathcal{V}_j \to \prod_{j=1}^n \mathcal{V}_j$ *defined as*

$$\iota_j x := (0,\dots,\underset{\substack{\uparrow\\ j\text{-th place}}}{x},\dots,0), \quad x \in \mathcal{V}_j\,,$$

is linear and isometric.

(3) *Let* $(\mathcal{V}, [.,.])$ *be a scalar product space, and let* $\mathcal{N}$ *be a linear subspace of* $\mathcal{V}$ *with*

$$[x, y] = 0, \quad x \in \mathcal{V}, y \in \mathcal{N}.$$

Then a scalar product $[\![.,.]\!]$ *on* $\mathcal{V}/_{\mathcal{N}}$ *is well-defined by*

$$[\![x + \mathcal{N}, y + \mathcal{N}]\!] := [x, y], \quad x, y \in \mathcal{V}. \tag{1.1.10}$$ I.9

The canonical projection $\pi : \mathcal{V} \to \mathcal{V}/_{\mathcal{N}}$, $x \mapsto x + \mathcal{N}$, *is linear and isometric. Here* $\mathcal{V}/_{\mathcal{N}}$ *is endowed with the natural linear operations defined via representants. We refer to* $[\![.,.]\!]$ *as the* factor scalar product on $\mathcal{V}/_{\mathcal{N}}$.

(4) *Let* $\mathcal{V}$ *be a linear space, and let* $(\mathcal{W}, [\![.,.]\!])$ *be a scalar product space. Moreover, let* $\varphi : \mathcal{V} \to \mathcal{W}$ *be a linear map, and set*

$$[x, y] := [\![\varphi x, \varphi y]\!], \quad x, y \in \mathcal{V}.$$

Then $[.,.]$ *is a scalar product on* $\mathcal{V}$, *and* φ *is a linear and isometric map of* $(\mathcal{V}, [.,.])$ *into* $(\mathcal{W}, [\![.,.]\!])$.

The scalar product $[.,.]$ *is the unique scalar product on* $\mathcal{V}$ *such that* φ *becomes isometric. We speak of* $[.,.]$ *as the scalar product* defined by requiring isometry of φ.

❑

I.8post. *1.1.9 Example.* A rather elementary way to obtain isometries is to start with a scalar product space $(\mathcal{V}, [.,.])$. Take any indexed subset $\{z_i : i \in I\}$ of vectors from $\mathcal{V}$ and define $\psi : \mathcal{F}(I) \to \mathcal{V}$ by

$$\psi f = \sum_{i \in I} f(i)\, z_i,$$

which is possible since $f(i) \neq 0$ only for finitely many $i \in I$. By the last assertion in Proposition 1.1.8 this obviously linear mapping induces a scalar product $[\![.,.]\!]$ on $\mathcal{F}(I)$. From

$$[\![f, g]\!] = [\psi f, \psi g] = \sum_{i,j \in I} \overline{g(i)} \cdot [z_j, z_i] \cdot f(j)$$

we see that by defining $K(i, j) := [z_j, z_i] = \overline{[z_i, z_j]}$ the scalar product $[.,.]$ coincides with the one obtained from K as in Example 1.1.4.

Note here, that for $I = \{1, \ldots, m\}$ and $\{z_1, \ldots, z_m\} \subseteq \mathcal{V}$ we already saw, that for the initial space of our mapping ψ we have $\mathcal{F}(I) = \mathbb{C}^m$ and $[\![(\alpha_j)_{j=1}^m, (\beta_i)_{i=1}^m]\!] = (G(\alpha_j)_{j=1}^m, (\beta_i)_{i=1}^m)$, where $G = (K(i, j))_{i,j=1}^m = ([z_j, z_i])_{i,j=1}^m$. ◊

Taking an indexed basis $\{z_i : i \in I\}$ of $\mathcal{V}$ from Example 1.1.9 we immediately obtain

I.8postps. **1.1.10 Corollary.** *Every scalar product space* $(\mathcal{V}, [.,.])$ *is isometric, isomorphic to a space* $(\mathcal{F}(I), [\![.,.]\!])$ *where* $[\![f, g]\!] = \sum_{i,j \in I} \overline{g(i)} \cdot K(i, j) \cdot f(j)$ *for a certain index set* I *and a certain hermitian kernel* K *on* I.

Let us consider again a scalar product space generated by a Nevanlinna kernel as in Example 1.1.7. But this time we do not use a rational function.

I.49. *1.1.11 Example.* Let μ be a finite real Borel measure on $\mathbb{R}$, and consider the function

$$q(\zeta) := \int_{\mathbb{R}} \frac{1+t\zeta}{t-\zeta}\, d\mu, \quad \zeta \in \mathbb{C} \setminus \operatorname{supp}|\mu| .$$

Here we denote by $|\mu|$ the total variation of μ, and by $\operatorname{supp}|\mu|$ its support which is the smallest closed subset of $\mathbb{R}$ whose complement is a $|\mu|$-zero set. This function is analytic on $\mathbb{C} \setminus \operatorname{supp}|\mu|$. To see this, use the bounded convergence theorem to show that q is continuous and Fubini's theorem to show that the integral along each triangular path vanishes. Clearly, the function q satisfies $q(\overline{\zeta}) = \overline{q(\zeta)}$, $\zeta \in \mathbb{C} \setminus \operatorname{supp}|\mu|$. Let $K(\zeta, \eta)$, $\zeta, \eta \in \mathbb{C} \setminus \operatorname{supp}|\mu|$, be the Nevanlinna kernel of q:

$$K(\zeta,\eta) := \begin{cases} \frac{q(\eta)-\overline{q(\zeta)}}{\eta-\overline{\zeta}} \quad, & \zeta \neq \overline{\eta} \\ q'(\eta) \quad, & \eta = \overline{\zeta} \end{cases}$$

Under an additional hypothesis, we can obtain some knowledge on the scalar product space $\mathcal{F}(\mathbb{C} \setminus \operatorname{supp}|\mu|)$ generated by K. Namely, let μ_+ and μ_- be the finite positive measures in the Jordan decomposition $\mu = \mu_+ - \mu_-$ of μ, and assume that

$$\operatorname{supp}\mu_+ \cap \operatorname{supp}\mu_- = \emptyset .$$

Denote by $L^2(\mu_+)$ the usual L^2-space associated with the measure μ_+, that is the space of all (equivalence classes of) square integrable functions endowed with the positive definite scalar product $[f,g]_+ := \big(\int_{\mathbb{R}} f\overline{g}\, d\mu_+\big)^{\frac{1}{2}}$. Denote by $L^2(-\mu_-)$ the linear space $L^2(\mu_-)$ endowed with the negative definite scalar product $[f,g]_- := -\big(\int_{\mathbb{R}} f\overline{g}\, d\mu_-\big)^{\frac{1}{2}}$. Now choose continuous functions $\chi_+, \chi_- : \mathbb{R} \to [0,1]$ with $\chi_+ + \chi_- = 1$, such that

$$\chi_+|_{\operatorname{supp}\mu_+} = 1,\ \chi_+|_{\operatorname{supp}\mu_-} = 0, \qquad \chi_-|_{\operatorname{supp}\mu_+} = 0,\ \chi_-|_{\operatorname{supp}\mu_-} = 1 ,$$

and define a map ϕ from $\mathcal{F}(M)$ into $L^2(\mu_+) \times L^2(\mu_-)$ by linearity and

$$\phi(\delta_\xi) = \Big(\tfrac{\sqrt{1+t^2}}{t-\xi}\chi_+(t); \tfrac{\sqrt{1+t^2}}{t-\xi}\chi_-(t)\Big), \quad \xi \in \mathbb{C} \setminus \operatorname{supp}|\mu| , \qquad (1.1.11)$$ I.49eq

where the basis elements δ_ξ, $\xi \in M$, are as in (1.1.5). In order to show that ϕ is isometric, by linearity it suffices to check the isometry property for these basis vectors δ_ξ. Given $\xi, \eta \in \mathbb{C} \setminus \operatorname{supp}|\mu|$ with $\xi \neq \overline{\eta}$ we compute

$$\begin{aligned}
[\delta_\xi, \delta_\eta] =& K(\eta,\xi) = \frac{q(\xi) - \overline{q(\eta)}}{\xi - \overline{\eta}} = \frac{1}{\xi-\overline{\eta}} \int_{\mathbb{R}} \Big(\frac{1+t\xi}{t-\xi} - \frac{1+t\overline{\eta}}{t-\overline{\eta}}\Big) d\mu = \\
=& \int_{\mathbb{R}} \frac{1+t^2}{(t-\xi)(t-\overline{\eta})}\, d\mu = \int_{\mathbb{R}} \frac{1+t^2}{(t-\xi)(t-\overline{\eta})}\, d\mu_+ + \int_{\mathbb{R}} \frac{1+t^2}{(t-\xi)(t-\overline{\eta})}\, d\mu_- = \\
=& \int_{\mathbb{R}} \tfrac{\sqrt{1+t^2}}{t-\xi}\chi_+(t) \cdot \tfrac{\sqrt{1+t^2}}{t-\overline{\eta}}\chi_+(t)\, d\mu_+ + \int_{\mathbb{R}} \tfrac{\sqrt{1+t^2}}{t-\xi}\chi_-(t) \cdot \tfrac{\sqrt{1+t^2}}{t-\overline{\eta}}\chi_-(t)\, d\mu_- = \\
=& \Big[\tfrac{\sqrt{1+t^2}}{t-\xi}\chi_+(t), \tfrac{\sqrt{1+t^2}}{t-\eta}\chi_+(t)\Big]_+ + \Big[\tfrac{\sqrt{1+t^2}}{t-\xi}\chi_-(t), \tfrac{\sqrt{1+t^2}}{t-\eta}\chi_-(t)\Big]_- .
\end{aligned}$$

To settle also the case $\xi = \overline{\eta}$, we pass to the limit $\xi \to \overline{\eta}$ in this relation. The leftmost term tends to $[\delta_{\overline{\eta}}, \delta_\eta]$ by analyticity of q. The rightmost sum tends to

$$\Big[\tfrac{\sqrt{1+t^2}}{t-\overline{\eta}}\chi_+(t), \tfrac{\sqrt{1+t^2}}{t-\eta}\chi_+(t)\Big]_+ + \Big[\tfrac{\sqrt{1+t^2}}{t-\overline{\eta}}\chi_-(t), \tfrac{\sqrt{1+t^2}}{t-\eta}\chi_-(t)\Big]_-$$

by bounded convergence. Alltogether, we conclude that indeed ϕ is isometric. $\Diamond$

1.2 Orthogonality

section--Orthogonality

orthogo. **1.2.1 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space.

(1) We call two elements $x, y \in \mathcal{V}$ *orthogonal w.r.t.* $[.,.]$, for short $[.,.]$-*orthogonal*, if

$$[x, y] = 0\,,$$

and write $x[\perp]y$ to express this fact.

(2) We call two subsets $M, N \subseteq \mathcal{V}$ *orthogonal w.r.t.* $[.,.]$ or shortly $[.,.]$-*orthogonal*, if

$$x[\perp]y, \quad x \in M, y \in N\,,$$

and write $M[\perp]N$ to express this fact.

(3) For a subset $M \subseteq \mathcal{V}$ we call

$$M^{[\perp]} := \{x \in \mathcal{V} : x[\perp]y \text{ for all } y \in M\}$$

the *orthogonal complement of M* in $(\mathcal{V}, [.,.])$.

(4) We call

$$\mathcal{V}^{[\circ]} := \mathcal{V}^{[\perp]} = \{x \in \mathcal{V} : x[\perp]y \text{ for all } y \in \mathcal{V}\}$$

the *isotropic part of* $\mathcal{V}$.

(5) The scalar product space $(\mathcal{V}, [.,.])$ is called *nondegenerated* if $\mathcal{V}^{[\circ]} = \{0\}$, and *degenerated* otherwise. We set

$$\operatorname{ind}_0(\mathcal{V}, [.,.]) := \dim \mathcal{V}^{[\circ]} \in \mathbb{N}_0 \cup \{\infty\}\,,$$

and call this number the *index of nullity* of $\mathcal{V}$. Here, and always when talk about dimensions, unless the contrary is explicitly mentioned, we do not distinguish different cardinalities of ∞.

As usual, when no confusion is possible we sometimes drop explit notation of the scalar product $[.,.]$ under consideration. ◊

Concerning orthogonal complements, one word of caution is in order: As we see from the definition, the orthogonal complement $M^{[\perp]}$ must always be understood w.r.t. a given scalar product space, and not only w.r.t. the scalar product $[.,.]$. In fact, as M could be a subset of different scalar product spaces, one should be specific about the base space. However, to avoid cumbersome notation like "$M^{\perp(\mathcal{V},[.,.])}$", except from the following remark we do not indicate the base space explicitly.

subspaceortho. *1.2.2 Remark.* Let $\mathcal{L}$ be a linear subspace of a scalar product space $(\mathcal{V}, [.,.])$. For a subset M of $\mathcal{L}$ the orthogonal complement of M with respect to the scalar product space $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}})$ is obviously $M^{[\perp]} \cap \mathcal{L}$, hence,

$$M^{\perp(\mathcal{L},[.,.]|_{\mathcal{L}\times\mathcal{L}})} = M^{[\perp]} \cap \mathcal{L}\,.$$

In particular, the *isotropic part* $\mathcal{L}^{[\circ]}$ of $\mathcal{L}$, which is understood as the isotropic part of the scalar product space $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}})$, is given by

$$\mathcal{L}^{[\circ]} = \mathcal{L} \cap \mathcal{L}^{[\perp]}\,, \qquad (1.2.1)$$ I.isotrop

where the orthogonal complement on the right is understood w.r.t. the base space $\mathcal{V}$. Obviously,

$$\mathcal{L} \cap \mathcal{V}^{[\circ]} \subseteq \mathcal{L}^{[\circ]} .$$

The notation *(non-)degenerated* and $\operatorname{ind}_0 \mathcal{L}$ is defined correspondingly viewing $\mathcal{L}$ as a scalar product space. ◊

I.neutral. *1.2.3 Example.* Let $(\mathcal{V}, [.,.])$ be a scalar product space, and consider any linear subspace $\mathcal{L}$ with

$$\mathcal{L} \subseteq \{x \in \mathcal{V} : [x,x] = 0\} .$$

According to the polar identity (1.1.1) we obtain

$$[x,y] = \sum_{k=0}^{3} i^k [x + i^k y, x + i^k y] = 0, \ \ x, y \in \mathcal{L},$$

i.e., $\mathcal{L}^{[\circ]} = \mathcal{L}$. Subspaces with this property are called *neutral* subspaces. Finally, note that in general the subset $\{x \in \mathcal{V} : [x,x] = 0\}$ is not a linear subspace. ◊

The verification of the following statement is immediate; we leave the details to the reader.

orthissubsp. **1.2.4 Lemma.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space.*

(1) *Let $M \subseteq \mathcal{V}$. Then $M^{[\perp]}$ is a linear subspace of $\mathcal{V}$ with* $(\operatorname{span} M)^{[\perp]} = M^{[\perp]}$ *and*

$$\mathcal{V}^{[\circ]} \subseteq M^{[\perp]} = (M + \mathcal{V}^{[\circ]})^{[\perp]} .$$

Here "span" *stands for "linear span", i.e.,* span M *is the smallest linear subspace which contains M. Moreover,*

$$M + \mathcal{V}^{[\circ]} \subseteq (M^{[\perp]})^{[\perp]} .$$

(2) *Let $M, N \subseteq \mathcal{V}$. If $M \subseteq N$, then $N^{[\perp]} \subseteq M^{[\perp]}$.*

(3) *Let $M_i \subseteq \mathcal{V}$, $i \in I$. Then*

$$\Big(\bigcup_{i\in I} M_i\Big)^{[\perp]} = \Big(\operatorname{span} \bigcup_{i \in I} M_i\Big)^{[\perp]} = \bigcap_{i\in I} M_i^{[\perp]} .$$

(4) *For a subspace $\mathcal{M}$ of $\mathcal{V}$ and a vector $x \in \mathcal{V} \setminus \mathcal{M}$ the* $(\operatorname{span}\{x\} + \mathcal{M})^{[\perp]} = \{x\}^{[\perp]} \cap \mathcal{M}^{[\perp]}$ *of $\mathcal{M}^{[\perp]}$ either coincides with $\mathcal{M}^{[\perp]}$ or constitutes a hyperplane in $\mathcal{M}^{[\perp]}$.*

❑

Let $\mathcal{L}_1, \dots \mathcal{L}_n$ be linear subspaces of a scalar product space $\mathcal{V}$. If $\mathcal{L}_i [\perp] \mathcal{L}_j$ for $i \neq j$, then we will write

$$\mathcal{L}_1 [+] \dots [+] \mathcal{L}_n$$

for the sum $\mathcal{L}_1 + \dots + \mathcal{L}_n$ and speak of an *orthogonal sum*. If in addition $\mathcal{L}_i \cap \mathcal{L}_j = \{0\}$ for $i \neq j$, then we will write

$$\mathcal{L}_1 [\dot{+}] \dots [\dot{+}] \mathcal{L}_n$$

and speak of an *orthogonal and direct sum*.

I.56. *1.2.5 Remark.* We often tacitly identify direct and orthogonal sums and direct products. Let us make explicit that this is justified. If $\mathcal{L}_1, \dots, \mathcal{L}_n$ are linear subspaces of a scalar product space $\mathcal{V}$ with

$$\mathcal{L}_i[\perp]\mathcal{L}_j,\ \mathcal{L}_i \cap \mathcal{L}_j = \{0\}, \quad i \neq j,$$

Then the map

$$\varphi : \begin{cases} \prod_{i=1}^n \mathcal{L}_i & \to \quad \mathcal{L}_1[\dot{+}] \dots [\dot{+}]\mathcal{L}_n \\ (x_1; \dots; x_n) & \mapsto \quad x_1 + \dots + x_n \end{cases}$$

is linear, bijective, and isometric when $\prod_{i=1}^n \mathcal{L}_i$ is endowed with the sum scalar product.

Conversely, if $\mathcal{V}_1, \dots, \mathcal{V}_n$ are scalar product spaces, then $\iota_1(\mathcal{V}_1), \dots, \iota_n(\mathcal{V}_n)$ are pairwise orthogonal subspace of $\prod_{i=1}^n \mathcal{V}_i$ with pairwise trivial intersection, and (see Proposition 1.1.8)

$$\iota_1(\mathcal{V}_1)[\dot{+}] \dots [\dot{+}]\iota_n(\mathcal{V}_n) = \prod_{i=1}^n \mathcal{V}_i\,.$$

◊

I.18a. **1.2.6 Lemma.** *For each $j \in \{1, \dots, n\}$ let $(\mathcal{V}_j, [.,.]_j)$ be a scalar product space, and consider the space $\prod_{j=1}^n \mathcal{V}_j$ endowed with the sum scalar product. For subsets $M_j \subseteq \mathcal{V}_j$, $j = 1, \dots, n$, we have*

$$\Big(\prod_{j=1}^n M_j\Big)^{[\perp]} = \prod_{j=1}^n M_j^{[\perp]}\,.$$

In particular,

$$\Big(\prod_{j=1}^n \mathcal{V}_j\Big)^{[\circ]} = \prod_{j=1}^n \mathcal{V}_j^{[\circ]}\,.$$

Proof. Since the orthogonal complement of a subset coincides with the orthogonal complement of the linear span of this subset, we see from $\operatorname{span}(\prod_{j=1}^n M_j) = \prod_{j=1}^n \operatorname{span}(M_j)$, that we may assume the subsets M_j to be linear subspaces. In particular, $0 \in M_j$ for $j = 1, \dots, n$. For $(x_1; \dots; x_n) \in \prod_{i=1}^n \mathcal{V}_i$ we then have

$$(x_1; \dots; x_n) \in \Big(\prod_{j=1}^n M_j\Big)^{[\perp]} \Leftrightarrow \forall (y_1; \dots; y_n) \in \prod_{i=1}^n M_i : \sum_{i=1}^n [x_i, y_i]_i = 0 \Leftrightarrow$$
$$\forall i \in \{1, \dots, n\} \forall y_i \in M_i : [x_i, y_i]_i = 0 \Leftrightarrow \forall i \in \{1, \dots, n\} : x_i \in M_i^{[\perp]}$$

❑

direktsum. *1.2.7 Remark.* Either direcktly or as an immediate consequence of Lemma 1.2.6 in combination with Remark 1.2.5 we see that for any linear subspace $\mathcal{L}$ of a scalar product space $\mathcal{V}$ with $\mathcal{V} = \mathcal{L}[\dot{+}]\mathcal{V}^{[\circ]}$ the scalar product space $\mathcal{L}$ is nondegenerated.

More general, for a nondegnerated subspace $\mathcal{M}$ and a subspace $\mathcal{L}$ of $\mathcal{V}$ with $\mathcal{L} \supseteq \mathcal{M}$ and $\mathcal{V} = \mathcal{L}[\dot{+}]\mathcal{V}^{[\circ]}$ we obtain from Lemma 1.2.6 applied to the subspace $\mathcal{M} \times \{0\}$ of $\mathcal{L} \times \mathcal{V}^{[\circ]}$ in combination with Remark 1.2.5

$$\mathcal{M}^{[\perp]} = (\mathcal{M}^{[\perp]} \cap \mathcal{L})[\dot{+}]\mathcal{V}^{[\circ]} \tag{1.2.2}$$ orthoform

◊

orthissubspendl. **1.2.8 Lemma.** *Let* $(\mathcal{V}, [.,.])$ *be a nondegenerated scalar product space with* $\dim \mathcal{V} < \infty$ *and let* $\mathcal{M}$ *be a linear subspace of* $\mathcal{V}$. *Then*

$$\dim \mathcal{M} + \dim \mathcal{M}^{[\perp]} = \dim \mathcal{V}$$

and $(M^{[\perp]})^{[\perp]} = \mathcal{M}$. *If in addition* $\mathcal{M}$ *is nondegenerated, then*

$$\mathcal{V} = \mathcal{M}[\dot{+}]\mathcal{M}^{[\perp]} . \tag{1.2.3}$$ firstorthocompl

Proof. For $\mathcal{M} = \{0\}$ the assertion is clear, and for $\mathcal{M} = \mathcal{V}$ it follows from our assumption that $\mathcal{V}^{[\circ]} = \{0\}$.

For a nontrivial subspace $\mathcal{M}$ of dimension $0 < k < \dim \mathcal{V} =: m$ let $\{a_1, \dots, a_m\}$ be a basis of $\mathcal{V}$ such that $\{a_1, \dots, a_k\}$ is a basis of $\mathcal{M}$. By Lemma 1.2.4 we have

$$\mathcal{V} \supseteq \{a_1\}^{[\perp]} \supseteq \cdots \supseteq \{a_1, \dots, a_k\}^{[\perp]} \supseteq \cdots \supseteq \{a_1, \dots, a_m\}^{[\perp]} = \{0\} ,$$

where $\{a_i : i < j+1\}^{[\perp]} \subseteq \{a_i : i < j\}^{[\perp]}$ with codimension at most one. If this codimension were zero at least once in this chain, than $\{0\}$ would have codimension strictly less than m in $\mathcal{V}$, which contradicts $\dim \mathcal{V} =: m$. Thus, $\mathcal{M}^{[\perp]} = \{a_1, \dots, a_k\}^{[\perp]}$ has codimension k in $\mathcal{V}$, i.e., $\dim \mathcal{V} - \dim \mathcal{M}^{[\perp]} = k = \dim \mathcal{M}$. $(M^{[\perp]})^{[\perp]} = \mathcal{M}$ now follows from the general fact that $(M^{[\perp]})^{[\perp]} \supseteq \mathcal{M}$ and from a comparison of dimensions. ❑

direktsumpost. *1.2.9 Remark.* If $(\mathcal{V}, [.,.])$ is finite dimensional and not necessarily nondegenerated and if $\mathcal{M}$ is a nondegnerated subspace, then we can choose a subspace $\mathcal{L}$ of $\mathcal{V}$ with $\mathcal{L} \supseteq \mathcal{M}$ and $\mathcal{V} = \mathcal{L}[\dot{+}]\mathcal{V}^{[\circ]}$. Since $\mathcal{L}$ is nondegenerated, (1.2.3) yields $\mathcal{L} = \mathcal{M}[\dot{+}](\mathcal{M}^{[\perp]} \cap \mathcal{L})$. Together with (1.2.2) we see that also here

$$\mathcal{V} = \mathcal{M}[\dot{+}]\mathcal{M}^{[\perp]} .$$

◊

I.15def. **1.2.10 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space with $m := \dim \mathcal{V} < \infty$. Then a basis $\{b_1, \dots, b_m\}$ of $\mathcal{V}$ satisfying

$$[b_i, b_j] = 0,\ i, j \in \{1, \dots, m\}, i \neq j, \quad [b_i, b_i] \in \{0, +1, -1\},\ i = 1, \dots, m , \tag{1.2.4}$$ I.32

is called an *orthonormal basis* of $\mathcal{V}$. ◊

In the situation of Definition 1.2.10, we can identity $\mathcal{V}$ with $\operatorname{span}\{b_1\} \times \cdots \times \operatorname{span}\{b_m\}$; cf. Remark 1.2.5. Thus, we get from Lemma 1.2.6

$$\mathcal{V} = \operatorname{span}\{b_i : i \in \{1, \dots, m\}, [b_i, b_i] \neq 0\}[\dot{+}]\underbrace{\operatorname{span}\{b_i : i \in \{1, \dots, m\}, [b_i, b_i] = 0\}}_{=\mathcal{V}^{[\circ]}} . \tag{1.2.5}$$ I.32post

In finite dimensional scalar product spaces, one can always choose an orthonormal basis.

I.15. **1.2.11 Lemma.** *Let* $(\mathcal{V}, [.,.])$ *be a scalar product space with* $m := \dim \mathcal{V} < \infty$. *Then there exists an orthonormal basis* $\{b_1, \dots, b_m\}$ *of* $\mathcal{V}$, *i.e., a basis satisfying* (1.2.4).

Proof. If $\mathcal{V}^{[\circ]} = \mathcal{V}$, then any linear basis of $\mathcal{V}$ constitutes an orthonormal basis. The case $\mathcal{V}^{[\circ]} \neq \mathcal{V}$ will be settled by induction on m. For $m = 1$ take a nonzero vector $x \in \mathcal{V}$ and define $b_1 := \frac{1}{\sqrt{|[x,x]|}} x$. From $[b_1, b_1] = \operatorname{sgn}[x, x]$ we see that $\{b_1\}$ is an orthonormal basis.

Assume the assertion is true for all scalar product spaces with dimension less or equal m. Let $\mathcal{V} \neq \mathcal{V}^{[\circ]}$ be a scalar product space of dimension $m + 1$. If $\mathcal{V}^{[\circ]} \neq \{0\}$, we can write $\mathcal{V} = \mathcal{M}[\dot{+}]\mathcal{V}^{[\circ]}$ for a proper linear subspace $\mathcal{M}$ of $\mathcal{V}$ with $0 < \dim \mathcal{M} \leq m$. By induction hypothesis we find an orthonormal basis of $\mathcal{M}$. Joining this basis with any algebraic basis of $\mathcal{V}^{[\circ]}$ one easily checks, that the resulting basis is orthonormal.

It remains to deal with a nondegenerated scalar product space $\mathcal{V}$ of dimension $m + 1$. By the polar identity (1.1.1) there must be an $x \in \mathcal{V}$ satisfying $[x, x] \neq 0$; cf. Example 1.2.3. The vector $b_1 := \frac{1}{\sqrt{|[x,x]|}} x$ then satisfies $[b_1, b_1] = \operatorname{sgn}[x, x]$. In particular, $\operatorname{span}\{b_1\}$ is nondegenerated. Due to (1.2.3) we have $\mathcal{V} = \operatorname{span}\{b_1\}[\dot{+}]b_1^{[\perp]}$. By induction hypothesis we find an orthonormal basis of $b_1^{[\perp]}$ which in union with b_1 gives an orthonormal basis of $\mathcal{V}$. ❑

From a computational point of view, it is useful to know how to construct an orthonormal basis. The following algorithm is a variant of the *Gram-Schmidt orthogonalisation process*.

I.21. *1.2.12 Remark.* Let $(\mathcal{V}, [.,.])$ be a scalar product space with $m := \dim \mathcal{V} < \infty$ and let $\{a_1, \dots, a_k\}$ be linearly independent vectors in $\mathcal{V}$ such that such that $\operatorname{span}\{a_1, \dots, a_j\}^{[\circ]} = \{0\}$, $j = 1, \dots, k$. By Remark 1.2.9 we have

$$\mathcal{V} = \operatorname{span}\{a_1, \dots, a_k\}[\dot{+}] \operatorname{span}\{a_1, \dots, a_k\}^{[\perp]},$$

where $\operatorname{span}\{a_1, \dots, a_k\}^{[\perp]}$ contains $\mathcal{V}^{[\circ]}$. If $\operatorname{span}\{a_1, \dots, a_k\}^{[\perp]} \neq \mathcal{V}^{[\circ]}$, then, by the polar identity (1.1.1), we find an $a_{k+1} \in \operatorname{span}\{a_1, \dots, a_k\}^{[\perp]}$ satisfying $[a_{k+1}, a_{k+1}] \neq 0$; cf. Example 1.2.3. Consequently, $\operatorname{span}\{a_1, \dots, a_k, a_{k+1}\}$ is not degenerated. Again employing Remark 1.2.9 and the polar identity either $\operatorname{span}\{a_1, \dots, a_k, a_{k+1}\}^{[\perp]} = \mathcal{V}^{[\circ]}$ or we find an $a_{k+2} \in \operatorname{span}\{a_1, \dots, a_{k+1}\}^{[\perp]}$ with $[a_{k+2}, a_{k+2}] \neq 0$.

We continue this procedure until $\operatorname{span}\{a_1, \dots, a_k, \dots, a_l\}^{[\perp]} = \mathcal{V}^{[\circ]}$. Then $\{a_1, \dots, a_l\}$ are linearly independent vectors in $\mathcal{V}$ such that such that $\operatorname{span}\{a_1, \dots, a_j\}^{[\circ]} = \{0\}$, $j = 1, \dots, l$ and

$$\mathcal{V} = \operatorname{span}\{a_1, \dots, a_l\}[\dot{+}]\mathcal{V}^{[\circ]}.$$

Now we are going to provide an algorithm for the construction of an orthonormal basis $\{b_1, \dots, b_m\}$ such that

$$\operatorname{span}\{a_1, \dots, a_j\} = \operatorname{span}\{b_1, \dots, b_j\}, \ j = 1, \dots, l. \tag{1.2.6}$$ orthoerzeug

For $b_{l+1}, \dots, b_m$ we simply take any algebraic basis of $\mathcal{V}^{[\circ]}$.

Define $b_1 := \frac{1}{\sqrt{|[a_1,a_1]|}} a_1$ which is possible by $\operatorname{span}\{a_1\}^{[\circ]} = \{0\}$. According to (1.2.3) for the orthogonal complement $b_1^{[\perp]} \cap \operatorname{span}\{a_1, a_2\}$ of b_1 within the nondegenerated $\operatorname{span}\{a_1, a_2\}$ we have

$$\operatorname{span}\{a_1, a_2\} = (b_1^{[\perp]} \cap \operatorname{span}\{a_1, a_2\})[\dot{+}] \operatorname{span}\{b_1\}.$$

The first space on the right hand side is nondegenerated and spanned by $x_2 := a_2 - [a_2, b_1] \cdot [b_1, b_1] b_1$. Therefore, we can define $b_2 := \frac{1}{\sqrt{|[x_2,x_2]|}} x_2$.

Continuing in this way we obtain a basis $\{b_1, \dots, b_j\}$ of $\operatorname{span}\{a_1, \dots, a_j\}$ satisfying (1.2.4). If $j = l$, we are finished. Otherwise we have again

$$\operatorname{span}\{a_1, \dots, a_{j+1}\} = (\{b_1, \dots, b_j\}^{[\perp]} \cap \operatorname{span}\{a_1, \dots, a_{j+1}\})[\dot{+}] \operatorname{span}\{b_1, \dots, b_j\}.$$

The first space on the right hand side is nondegenerated and spanned by

$$x_{j+1} := a_{j+1} - \sum_{r=1}^{j} [a_{j+1}, b_r] \cdot [b_r, b_r] b_r ,$$

since $[x_{j+1}, b_r] = [a_{j+1}, b_r] - [a_{j+1}, b_r] \cdot [b_r, b_r][b_r, b_r] = 0$. Set $b_{j+1} := \frac{1}{\sqrt{|[x_{j+1}, x_{j+1}]|}} x_{j+1}$. Repeating this process until $j = l$, we end up with an orthonormal basis of $\mathcal{V}$ satisfying (1.2.6). ◊

There is an alternativ approach to construct an orthonormal basis using the Spectral Theorem for selfadjoint matrices. Missing Reference

I.15ex. *1.2.13 Example.* Choose a linear bijection ψ of $\mathbb{C}^m$ onto $\mathcal{V}$, and define a scalar product $[\![.,.]\!]$ on $\mathbb{C}^m$ by requiring isometry of ψ; see Example 1.1.9. Let $G \in \mathbb{C}^{m \times m}$ be the Gram matrix of $[\![.,.]\!]$, so that (we denote by $(.,.)$ the euclidean scalar product on $\mathbb{C}^m$)

$$[\![(\alpha_j)_{j=1}^m, (\beta_j)_{j=1}^m]\!] = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}^* G \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = (G(\alpha_j)_{j=1}^m, (\beta_j)_{j=1}^m), \quad (\alpha_j)_{j=1}^m, (\beta_j)_{j=1}^m \in \mathbb{C}^m .$$

Clearly, an element $a \in \mathbb{C}^m$ belongs to the isotropic part of $(\mathbb{C}^m, [\![.,.]\!])$ if and only if $Ga = 0$. Since ψ is bijective and isometric,

$$\dim \ker G = \dim (\mathbb{C}^m)^{[\![\circ]\!]} = \dim \mathcal{V}^{[\circ]} = m - n .$$

Since $G = G^*$, there exists a basis $a_1, \dots, a_m$ consisting of eigenvectors of G with corresponding eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, i.e.,

$$\mathbb{C}^m = \operatorname{span}\{a_1, \dots, a_m\}, \quad Ga_k = \lambda_k a_k, \ k = 1, \dots, m ,$$

which are pairwise $(.,.)$-orthogonal and normalized by $(a_k, a_k) = 1$. We choose the enumeration such that $\lambda_1, \dots, \lambda_n \neq 0$ and $\lambda_{n+1}, \dots, \lambda_m = 0$. Now we set

$$b_k := \begin{cases} \frac{1}{\sqrt{|\lambda_k|}} \psi a_k , & k = 1, \dots, n \\ \psi a_k \quad , & k = n+1, \dots, m \end{cases}$$

Then $\operatorname{span}\{b_{n+1}, \dots, b_m\} = \mathcal{V}^{[\circ]}$, and

$$[b_k, b_j] = \frac{1}{\sqrt{|\lambda_k|}} \frac{1}{\sqrt{|\lambda_j|}} [\psi a_k, \psi a_j] = \frac{1}{\sqrt{|\lambda_k|}} \frac{1}{\sqrt{|\lambda_j|}} [\![a_k, a_j]\!] = \frac{1}{\sqrt{|\lambda_k|}} \frac{1}{\sqrt{|\lambda_j|}} (Ga_k, a_j) =$$
$$= \frac{1}{\sqrt{|\lambda_k|}} \frac{1}{\sqrt{|\lambda_j|}} \lambda_k \underbrace{(a_k, a_j)}_{=\delta_{kj}} = \frac{\lambda_k}{|\lambda_k|} \delta_{kj} = \pm \delta_{kj}, \quad k, j = 1, \dots, n .$$

◊

Next, we investigate how orthogonal complements and isotropic parts behave when performing constructions as in Proposition 1.1.8, (3) and (4).

I.18. **1.2.14 Proposition.** *The following statements hold.*

(1) *Let* $(\mathcal{V}, [.,.])$ *be a scalar product space, let* $\mathcal{N}$ *be a linear subspace of* $\mathcal{V}^{[\circ]}$*, and consider the the space* $\mathcal{V}/_{\mathcal{N}}$ *endowed with the scalar product* (1.1.10)*. For any* $M \subseteq \mathcal{V}$ *we then have*

$$\big(\mathcal{V}/_{\mathcal{N}}\big)^{[\circ]} = \mathcal{V}^{[\circ]}/_{\mathcal{N}}\,.$$

(2) *Let* $(\mathcal{V}, [.,.])$ *and* $(\mathcal{W}, [\![.,.]\!])$ *be scalar product spaces, and let* $\varphi : \mathcal{V} \to \mathcal{W}$ *be a linear and isometric map. For any* $M \subseteq \mathcal{V}$ *we have*

$$M^{[\perp]} = \varphi^{-1}\big(\varphi(M)^{[\![\perp]\!]}\big)\,.$$

In particular,

$$\mathcal{V}^{[\circ]} = \varphi^{-1}\big([\operatorname{ran}\varphi]^{[\![\circ]\!]}\big) \supseteq \ker\varphi\,. \tag{1.2.7}$$ I.57

Here $\operatorname{ran}\varphi := \{y \in \mathcal{W} :\ \exists x \in \mathcal{V}$ *with* $\varphi x = y\}$ *denotes the* range of φ *and* $\ker\varphi := \{x \in \mathcal{V} : \varphi x = 0\}$ *denotes the* kernel of φ*.*

Proof. First we prove item (2). For $x \in \mathcal{V}$ we have

$$x \in M^{[\perp]} \ \Leftrightarrow\ \forall y \in M : [x,y] = 0 \ \Leftrightarrow\ \forall y \in M : [\![\varphi x, \varphi y]\!] = 0 \ \Leftrightarrow$$
$$\forall z \in \varphi(M) : [\![\varphi x, z]\!] = 0 \ \Leftrightarrow\ \varphi x \in \varphi(M)^{[\![\perp]\!]}$$

Item (1) now follows from (2) applied to the canoncial projection:

$$\big(\mathcal{V}/_{\mathcal{N}}\big)^{[\circ]} = \pi\Big(\pi^{-1}\big(\mathcal{V}/_{\mathcal{N}}\big)^{[\circ]}\Big) = \pi(\mathcal{V}^{[\circ]})\,.$$

❑

These facts are supplemented by the following homomorphy theorem; again we skip the details.

I.22. **1.2.15 Lemma.** *Let* $(\mathcal{V}, [.,.])$ *and* $(\mathcal{W}, [\![.,.]\!])$ *be scalar product spaces, and* $\varphi : \mathcal{V} \to \mathcal{W}$ *a linear and isometric map. Then there exists a unique linear, bijective, and isometric map* $\tilde{\varphi} : \mathcal{V}/_{\ker\varphi} \to \operatorname{ran}\varphi/_{[\operatorname{ran}\varphi]^{[\![\circ]\!]}}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\varphi"] \arrow[d, "\pi_{\mathcal{V}}"'] & \operatorname{ran}\varphi \arrow[d, "\pi_{\operatorname{ran}\varphi}"] \\
\mathcal{V}/_{\ker\varphi} \arrow[r, dotted, "\tilde{\varphi}"'] & \operatorname{ran}\varphi/_{[\operatorname{ran}\varphi]^{[\![\circ]\!]}}
\end{tikzcd}$$

where the downwards arrows are the respective canonical projections.

Here we say that a diagram commutes*, if the action of any map between two vertices does not depend on the choice of the path. For the above diagram this just means that* $\tilde{\varphi} \circ \pi_{\mathcal{V}} = \pi_{\operatorname{ran}\varphi} \circ \varphi$*.* ❑

Let us revisit the scalar product space generated by the Nevanlinna kernel of a rational function.

I.34. *1.2.16 Example.* We use without further notice the notation from in Example 1.1.7. Since $\dim\mathcal{F}(M)=\infty$ but $\operatorname{ran}\varphi\subseteq\mathbb{C}^m$, certainly $\ker\varphi\neq\{0\}$. Hence, also $\mathcal{F}(M)^{[\circ]}\neq\{0\}$. In the first part of this example we show that the function r can be reconstructed using isotropic elements of $\mathcal{F}(M)$.

Choose $g\in\mathcal{F}(M)^{[\circ]}\setminus\{0\}$, and write $\operatorname{supp} g=\{\zeta_1,\dots,\zeta_n\}$. For δ_ξ as in (1.1.5) we then have

$$0=[\delta_\xi,g]=\sum_{\zeta,\eta\in M}\overline{g(\zeta)}\cdot K(\zeta,\eta)\delta_\xi(\eta)=\sum_{i=1}^{n}\overline{g(\zeta_i)}\frac{r(\xi)-\overline{r(\zeta_i)}}{\xi-\overline{\zeta_i}},\quad \xi\in M\setminus\{\overline{\zeta_1},\dots,\overline{\zeta_n}\}.$$

Solving for $r(\xi)$ gives

$$r(\xi)=\Big[\sum_{i=1}^{n}\frac{\overline{r(\zeta_i)}\,\overline{g(\zeta_i)}}{\xi-\overline{\zeta_i}}\Big]\cdot\Big[\sum_{i=1}^{n}\frac{\overline{g(\zeta_i)}}{\xi-\overline{\zeta_i}}\Big]^{-1}=$$

$$=\Big[\sum_{i=1}^{n}\overline{r(\zeta_i)}\,\overline{g(\zeta_i)}\prod_{\substack{j=1\\ j\neq i}}^{n}(\xi-\overline{z\zeta_j})\Big]\cdot\Big[\sum_{i=1}^{n}\overline{g(\zeta_i)}\prod_{\substack{j=1\\ j\neq i}}^{n}(\xi-\overline{\zeta_j})\Big]^{-1},\quad(1.2.8)$$ I.35

provided that $\xi\in M\setminus\{\overline{\zeta_1},\dots,\overline{\zeta_n}\}$ and the second sum is nonzero at ξ. The set of zeros of this sum, however, is finite. This formula shows that r can be fully recovered knowing a nontrivial isotropic element of $\mathcal{F}(M)$ and the function values of r at the points of $\operatorname{supp} g$.

In the second part of the example, we analyse further the space $(\mathbb{C}^m,[\![.,.]\!])$, where $[\![(\alpha_j)_{j=1}^m,(\beta_i)_{i=1}^m]\!]=\sum_{i,j=1}^m\overline{\beta_i}\cdot\gamma_{ij}\cdot\alpha_j$ with the number γ_{ij} defined by (1.1.6). Our aim is to show that it is nondegenerated. To this end we first prove the existence of a right inverse for the mapping θ as defined in (1.1.7).

Choose pairwise different points $\xi_1,\dots,\xi_m\in M$. Then the matrix

$$A:=\begin{pmatrix}1&\cdots&1\\ \xi_1&\cdots&\xi_m\\ \vdots&&\vdots\\ \xi_1^{m-1}&\cdots&\xi_m^{m-1}\end{pmatrix}$$

is invertible. Moreover, the mapping $\psi:\mathbb{C}^m\to\mathcal{F}(M),\ (\alpha_j)_{j=1}^m\mapsto\sum_{j=1}^m\alpha_j\cdot\delta_{\xi_j}$ is injective. Thus $\vartheta:\mathbb{C}^m\to\mathcal{F}(M)$ defined by

$$\vartheta(\alpha_j)_{j=1}^m=\psi\Big(\operatorname{diag}(q(\xi_1),\dots,q(\xi_1))\,A^{-1}(\alpha_j)_{j=1}^m\Big)$$

is also injective. Denoting by $[(\beta_j)_{j=1}^m]_i$ the i-th entry of $(\beta_j)_{j=1}^m\in\mathbb{C}^m$ we have

$$[\theta\circ\vartheta(\alpha_j)_{j=1}^m]_i=\sum_{k=0}^{m-1}\frac{\vartheta((\alpha_j)_{j=1}^m)(\xi_k)}{q(\xi_k)}\xi_k^i=$$

$$\sum_{k=0}^{m-1}\xi_k^i\frac{q(\xi_k)\cdot[A^{-1}(\alpha_j)_{j=1}^m]_k}{q(\xi_k)}=\sum_{k=0}^{m-1}\xi_k^i[A^{-1}(\alpha_j)_{j=1}^m]_k=[AA^{-1}(\alpha_j)_{j=1}^m]_i=\alpha_i\,.$$

Therefore, $\theta\circ\vartheta$ is the identity map on $\mathbb{C}^m$, i.e., ϑ is a right inverse von θ. In particular, θ is onto.

Now assume that $(\mathbb{C}^m, [\![.,.]\!])$ were degenerated. Choose a nonzero isotropic vector $a \in \mathbb{C}^m$, and set $g = \vartheta a \in \mathcal{F}(M)$ so that $\varphi g = a$. By (1.2.7) we have $g \in \mathcal{V}^{[\circ]}$. Moreover, $\operatorname{supp} g$ $(\subseteq \{\xi_1, \dots, \xi_m\})$ contains at most the m points. The polynomials in the numerator and denominator of the representation (1.2.8) of r both have degree at most $m - 1$. Since the representation of a rational function as a quotient of two relatively prime polynomials is unique up to scalar multiples, we get

$$\deg p, \deg q \leq m - 1 .$$

This contradicts the definition of m, and we see that $(\mathbb{C}^m, [\![.,.]\!])$ is nondegenerated.

Finally, let us point out that by the definition of ϑ, by $\theta \circ \vartheta = \mathrm{id}_{\mathbb{C}^m}$ and by $(\mathbb{C}^m)^{[\![\circ]\!]} = \{0\}$ we have

$$\operatorname{ran} \vartheta = \operatorname{ran} \psi, \quad \ker \theta = \mathcal{F}(M)^{[\circ]}, \quad \mathcal{F}(M) = \operatorname{ran} \psi [\dot{+}] \mathcal{F}(M)^{[\circ]} .$$

$\lozenge$

1.3 Orthocomplemented Subspaces

a--OrthocomplementedSubspaces

I.10. **1.3.1 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}$ be a linear subspace of $\mathcal{V}$. Then we call $\mathcal{L}$ *orthocomplemented* in $(\mathcal{V}, [.,.])$, if $\mathcal{L} + \mathcal{L}^{[\perp]} = \mathcal{V}$.

$\lozenge$

Notice that we do not require that $\mathcal{L} \cap \mathcal{L}^{[\perp]} = \{0\}$ in this definition.

For the notion of orthocomplemented subspaces, the same word of caution applies as for the notion of orthogonal complements: The property of being orthocomplemented not only depends on $[.,.]$ but also on the base space $\mathcal{V}$.

We call a linear map $P : \mathcal{V} \to \mathcal{V}$ a *projection* if $P^2 = P$, and speak of an *orthogonal projection* if in addition $\ker P [\perp] \operatorname{ran} P$, i.e., $\ker P \subseteq (\operatorname{ran} P)^{[\perp]}$.

Recall that for any projection one has $\ker P = \operatorname{ran}(I - P), \ker(I - P) = \operatorname{ran} P$ and $\mathcal{V} = \ker P \dot{+} \operatorname{ran} P$. Moreover, P is uniquely determined by its range and its kernel.

I.11. **1.3.2 Proposition.** *For a scalar product space $(\mathcal{V}, [.,.])$ the following statements hold true.*

(1) *Let $\mathcal{L}$ be a linear subspace of $\mathcal{V}$. Then $\mathcal{L}$ is orthocomplemented, if and only if there exists an orthogonal projection $P : \mathcal{V} \to \mathcal{V}$ whose range equals $\mathcal{L}$.*

If $\mathcal{L}$ is nondegenerated and orthocomplemented, then the orthogonal projection with range $\mathcal{L}$ is unique.

(2) *Let $\mathcal{L}$ be an orthocomplemented linear subspace of $\mathcal{V}$. Then*

  (*i*) $\mathcal{L}^\circ \subseteq \mathcal{V}^\circ$.

  (*ii*) $\mathcal{L}^{[\perp]}$ *is orthocomplemented.*

  (*iii*) $(\mathcal{L}^{[\perp]})^{[\circ]} = \mathcal{V}^{[\circ]}$.

  (*iv*) $\mathcal{L} + \mathcal{V}^\circ = (\mathcal{L}^{[\perp]})^{[\perp]}$.

In particular, for nondegenerated scalar product spaces $\mathcal{V}$ all orthocomplemented subspaces are nondegenerated.

(3) *If $\mathcal{L}$ is orthocomplemented, then any subspace $\mathcal{M}$ of $\mathcal{L}$ with $\mathcal{L} = \mathcal{M}[\dot{+}]\mathcal{L}^{[\circ]}$ is also orthocomplemented with $\mathcal{V} = \mathcal{M}[\dot{+}]\mathcal{M}^{[\perp]}$, where $\mathcal{M}^{[\perp]} = \mathcal{L}^{[\perp]}$.*

(4) *Let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be linear subspaces of $\mathcal{V}$ with $\mathcal{L}_i[\perp]\mathcal{L}_j$, $i \neq j$, and set $\mathcal{M} := \mathcal{L}_1 + \cdots + \mathcal{L}_n$. Then $\mathcal{M}$ is orthocomplemented if and only if each of the spaces $\mathcal{L}_i$ is orthocomplemented.*

Proof. For the proof of (1) assume first that P is an orthogonal projection with $\operatorname{ran} P = \mathcal{L}$. Then
$$\mathcal{L} + \mathcal{L}^{[\perp]} \supseteq \operatorname{ran} P + \ker P = \mathcal{V},$$
and hence $\mathcal{L}$ is orthocomplemented. Conversely, if $\mathcal{L} + \mathcal{L}^{[\perp]} = \mathcal{V}$, choose a linear subspace $\mathcal{M}$ of $\mathcal{L}^{[\perp]}$ with
$$\mathcal{L}^{[\perp]} = \mathcal{M} \dot{+} \underbrace{(\mathcal{L} \cap \mathcal{L}^{[\perp]})}_{=\mathcal{L}^{[\circ]}}.$$
Then $\mathcal{L} \cap \mathcal{M} = \{0\}$ and $\mathcal{L} \dot{+} \mathcal{M} = \mathcal{V}$. Hence, there exists a projection P with $\operatorname{ran} P = \mathcal{L}$ and $\ker P = \mathcal{M}$. Clearly, P is an orthogonal projection.

Assume that $\mathcal{L}$ is nondegenerated and orthocomplemented, and let P be an orthogonal projection with range $\mathcal{L}$. We have $\mathcal{V} = \mathcal{L}[+]\mathcal{L}^{[\perp]}$, and this sum is direct. On the other hand, we have the direct sum decomposition $\mathcal{V} = \mathcal{L}[\dot{+}]\ker P$. Both sums being direct, $\ker P \subseteq \mathcal{L}^{[\perp]}$ already implies $\ker P = \mathcal{L}^{[\perp]}$.

We come to the proof of (2). The inclusion (i) is a consequence of
$$\mathcal{L}^{\circ} = \mathcal{L} \cap \mathcal{L}^{[\perp]} \subseteq (\mathcal{L}^{[\perp]})^{[\perp]} \cap \mathcal{L}^{[\perp]} = (\underbrace{\mathcal{L}^{[\perp]} + \mathcal{L}}_{=\mathcal{V}})^{[\perp]} = \mathcal{V}^{[\circ]}.$$

The assertion in (ii) follows from
$$\mathcal{L}^{[\perp]} + (\mathcal{L}^{[\perp]})^{[\perp]} \supseteq \mathcal{L}^{[\perp]} + \mathcal{L} = \mathcal{V}.$$

To see (iii), note first that
$$\mathcal{V}^{[\circ]} \subseteq \mathcal{L}^{[\perp]} \cap (\mathcal{L}^{[\perp]})^{[\perp]} = (\mathcal{L}^{[\perp]})^{[\circ]}.$$

Since by (ii) the subspace $\mathcal{L}^{[\perp]}$ is orthocomplemented, the reverse inclusion $(\mathcal{L}^{[\perp]})^{[\circ]} \subseteq \mathcal{V}^{[\circ]}$ holds by (i).

The inclusion "$\subseteq$" in (iv) is obvious. In fact, it holds without any assumptions on $\mathcal{L}$; cf. Lemma 1.2.4. For the proof of the reverse inclusion, let $x \in (\mathcal{L}^{[\perp]})^{[\perp]}$ be given. Since $\mathcal{L}$ is orthocomplemented, we can write $x = y + z$ with some $y \in \mathcal{L}$ and $z \in \mathcal{L}^{[\perp]}$. This gives
$$z = x - y \in \mathcal{L}^{[\perp]} \cap (\mathcal{L}^{[\perp]})^{[\perp]} = (\mathcal{L} + \mathcal{L}^{[\perp]})^{[\perp]} = \mathcal{V}^{[\circ]},$$
and in turn $x \in \mathcal{L} + \mathcal{V}^{[\circ]}$.

For (3) let $\mathcal{L} = \mathcal{M}[\dot{+}]\mathcal{L}^{[\circ]}$ be orthocomplemented. Then $\mathcal{L} = \mathcal{L}[+]\mathcal{L}^{[\perp]} = \mathcal{M}[\dot{+}]\mathcal{L}^{[\perp]}$. Moreover, $\mathcal{M}^{[\perp]} \supseteq \mathcal{L}^{[\perp]}$. Since $\mathcal{M}$ is nondegenerated, we have $\mathcal{M}^{[\perp]} \cap \mathcal{M} = \{0\}$, and in turn $\mathcal{M}^{[\perp]} = \mathcal{L}^{[\perp]}$.

Finally, for the proof of (4), first assume that $\mathcal{M}$ is orthocomplemented. Any given $x \in \mathcal{V}$ can be written as $x = y + z$ with $y \in \mathcal{M}$ and $z \in \mathcal{M}^{[\perp]}$. In turn, $y = y_1 + \cdots + y_n$ with $y_i \in \mathcal{L}_i$. Consider the decompositions
$$x = y_i + \Big(\sum_{\substack{j=1\\ j\neq i}}^{n} y_j + z\Big), \quad i = 1, \dots, n.$$

Since the spaces $\mathcal{L}_i$ are pairwise orthogonal and since $\mathcal{M}^{[\perp]} = \bigcap_{i=1}^n \mathcal{L}_i^{[\perp]}$, the second summand belongs to $\mathcal{L}_i^{[\perp]}$. Thus $x \in \mathcal{L}_i + \mathcal{L}_i^{[\perp]}$, and we conclude that $\mathcal{L}_i$ is orthocomplemented.

Conversely, assume that each space $\mathcal{L}_i$ is orthocomplemented, and let again $x \in \mathcal{V}$ be given. Then, for each $i \in \{1, \dots, n\}$, we can write $x = y_i + z_i$ with $y_i \in \mathcal{L}_i$ and $z_i \in \mathcal{L}_i^{[\perp]}$. Then $y := y_1 + \cdots + y_n \in \mathcal{M}$ and we have

$$x = y + \Big(z_i - \sum_{\substack{j=1\\ j\neq i}}^n y_j\Big), \quad i = 1, \dots, n,$$

implying

$$x - y = z_i - \sum_{\substack{j=1\\ j\neq i}}^n y_j \in \mathcal{L}_i^{[\perp]}, \quad i = 1, \dots, n.$$

Again appealing to $\mathcal{M}^{[\perp]} = \bigcap_{i=1}^n \mathcal{L}_i^{[\perp]}$, it follows that $x - y \in \mathcal{M}^{[\perp]}$. Therefore, $\mathcal{V} = \mathcal{M}[+]\mathcal{M}^{[\perp]}$. ❑

For scalar product space $\mathcal{V}$ with $\dim \mathcal{V} < \infty$ we saw in Remark 1.2.9 that nondegenerated subspaces are orthocomplemented. In general it is hard to decide whether a given subspace is orthocomplemented. As a rule of thumb, orthocomplemented subspaces are rare.

For finite dimensional subspaces of arbitrary scalar product spaces, however, it can easily be decided whether the subspace is orthocomplemented. Indeed, the necessary condition Proposition 1.3.2, (2),(*i*), turns out to be also sufficient. Remark 1.2.9

I.12. **1.3.3 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}$ be a finite dimensional linear subspace of $\mathcal{V}$. Then $\mathcal{L}$ is orthocomplemented if and only if $\mathcal{L}^\circ \subseteq \mathcal{V}^\circ$.*

Proof. As already noted, we only need to establish sufficiency of the stated condition. Assuming $\mathcal{L}^{[\circ]} \subseteq \mathcal{V}^{[\circ]}$ set $m := \dim \mathcal{L}$ and $n := \dim \mathcal{L} - \dim \mathcal{L}^{[\circ]}$. Choose an orthonormal basis $\{b_1, \dots, b_m\}$ of $\mathcal{L}$ such that $\operatorname{span}\{b_{n+1}, \dots, b_m\} = \mathcal{L}^{[\circ]}$; cf. Lemma 1.2.11 and (1.2.5). Now we define $P : \mathcal{V} \to \mathcal{V}$ by

$$Px := \sum_{i=1}^n \frac{[x, b_i]}{[b_i, b_i]} b_i, \quad x \in \mathcal{V}.$$

Then, using (1.2.4), we obtain $Pb_j = b_j$, $j = 1, \dots, n$. It follows that

$$\operatorname{ran} P = \operatorname{span}\{b_1, \dots, b_n\}, \quad P^2 = P.$$

Moreover, since $\{b_1, \dots, b_n\}$ is linearly independent,

$$\ker P = \{b_1, \dots, b_n\}^{[\perp]} = \operatorname{span}\{b_1, \dots, b_n\}^{[\perp]}.$$

Employing Proposition 1.3.2, (1), we see that $\operatorname{span}\{b_1, \dots, b_n\}$ is orthocomplemented. Finally, since $\mathcal{L}^{[\circ]} \subseteq \mathcal{V}^{[\circ]}$, we have $(\operatorname{span}\{b_1, \dots, b_n\})^{[\perp]} = (\operatorname{span}\{b_1, \dots, b_n\} + \mathcal{L}^{[\circ]})^{[\perp]} = \mathcal{L}^{[\perp]}$; cf. Lemma 1.2.4. Hence,

$$\mathcal{L} + \mathcal{L}^{[\perp]} \supseteq \operatorname{span}\{b_1, \dots, b_n\}\dot{+}(\operatorname{span}\{b_1, \dots, b_n\})^{[\perp]} = \mathcal{V}.$$

❑

I.12post. *1.3.4 Remark.* From the previous proof let us point out the fact that given a finite dimensional subspace $\mathcal{L}$ of a scalar product space $\mathcal{V}$ – the assumption $\mathcal{L}^{[\circ]} \subseteq \mathcal{V}^{[\circ]}$ is not necessary – and an orthonormal basis $\{b_1, \dots, b_m\}$ of $\mathcal{L}$ such that $\operatorname{span}\{b_{n+1}, \dots, b_m\} = \mathcal{L}^{[\circ]}$, the linear mapping $P : \mathcal{V} \to \mathcal{V}$ defined by

$$Px := \sum_{i=1}^{n} \frac{[x, b_i]}{[b_i, b_i]} b_i, \quad x \in \mathcal{V}.$$

constitutes an orthogonal projection with $\operatorname{ran} P = \operatorname{span}\{b_1, \dots, b_n\}$ and $\ker P = (\operatorname{span}\{b_1, \dots, b_n\})^{[\perp]}$. If $\mathcal{L}$ is nondegenerated, then due to $\operatorname{ran} P = \operatorname{span}\{b_1, \dots, b_n\} = \mathcal{L}$ this is a representation of the unique orthogonal projection onto $\mathcal{L}$. ◊

If a finite dimensional subspace $\mathcal{M}$ of $(\mathcal{V}, [.,.])$ does not satisfy $\mathcal{M}^\circ \subseteq \mathcal{V}^\circ$, then we at least find a larger subspaces $\mathcal{L}$ satisfying $\mathcal{L}^\circ \subseteq \mathcal{V}^\circ$.

I.58. **1.3.5 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{M}$ be a finite dimensional linear subspace of $\mathcal{V}$ with $\mathcal{M}^\circ \nsubseteq \mathcal{V}^\circ$. Then there exists a subspace $\mathcal{L}$ satisfying*

$$\mathcal{L} \supseteq \mathcal{M}, \ \mathcal{L}^\circ \subseteq \mathcal{M}^\circ \cap \mathcal{V}^\circ, \ \dim \mathcal{L} = \dim \mathcal{M} + (\dim \mathcal{M}^{[\circ]} - \dim \mathcal{M}^\circ \cap \mathcal{V}^\circ).$$

Proof. First we additionally assume that $\mathcal{M} \cap \mathcal{V}^\circ = \{0\}$ (or equivalently $\mathcal{M}^\circ \cap \mathcal{V}^\circ = \{0\}$), and prove the assertion by induction on $m = \dim \mathcal{M}^{[\circ]}$. For $m = 0$ we are done. Assume that the assertion is true for m, and let $\mathcal{M}$ be a finite dimensional subspace of $\mathcal{V}$ satisfying $\mathcal{M} \cap \mathcal{V}^{[\circ]} = \{0\}$ with $\dim \mathcal{M}^{[\circ]} = m + 1$.

For any nonzero $a \in \mathcal{M}^{[\circ]}$ we have $a \notin \mathcal{V}^{[\circ]}$. Thus, there exists a vector $b \in \mathcal{V}$ such that $[a, b] = 1$. Obviously, $b \notin \mathcal{M}$. Moreover, $\mathcal{M} = \{x \in \mathcal{M} : [x, b] = 0\}[\dot{+}] \operatorname{span}\{a\}$. Hence,

$$\mathcal{M}_1 := \{x \in \mathcal{M} : [x, b] = 0\}[\dot{+}] \operatorname{span}\{a, b\}$$

is subspace with dimension $\dim \mathcal{M} + 1$ containing $\mathcal{M}$.

For $\alpha a + \beta b \in \operatorname{span}\{a, b\}^{[\circ]}$ we have $0 = [\alpha a + \beta b, a] = \bar{\beta}$ and $0 = [\alpha a, b] = \alpha$. Hence, $\operatorname{span}\{a, b\}^{[\circ]} = \{0\}$, and in turn (see Lemma 1.2.6 in combination with Remark 1.2.5)

$$\mathcal{M}_1^\circ = \{x \in \mathcal{M} : [x, b] = 0\}^{[\circ]} = \{x \in \mathcal{M}^{[\circ]} : [x, b] = 0\},$$

This space is contained in $\mathcal{M}^{[\circ]}$ with codimension one, i.e., $\dim \mathcal{M}_1^{[\circ]} = m$. By induction hypothesis there exists a nondegenerated subspace $\mathcal{L} \supseteq \mathcal{M}_1 \supseteq \mathcal{M}$ with $\dim \mathcal{L} = \dim \mathcal{M}_1 + \dim \mathcal{M}_1^{[\circ]} = \dim \mathcal{M} + 1 + \dim \mathcal{M}^{[\circ]} - 1$.

Finally, if $\mathcal{M}^\circ \cap \mathcal{V}^\circ \neq \{0\}$, then we decompose $\mathcal{V}$ as $\mathcal{V} = \mathcal{V}'[\dot{+}](\mathcal{M}^\circ \cap \mathcal{V}^\circ)$ and correspondingly $\mathcal{M} = \mathcal{M}'[\dot{+}](\mathcal{M}^\circ \cap \mathcal{V}^\circ)$. Then $(\mathcal{M}')^\circ \cap (\mathcal{V}')^\circ = \{0\}$. From the special case we infer the existence of a nondegenerated subspace $\mathcal{L}'$ of $\mathcal{V}'$ with $\mathcal{L}' \supseteq \mathcal{M}'$ and

$$\dim \mathcal{L}' = \dim \mathcal{M}' + \dim(\mathcal{M}')^{[\circ]} = \dim \mathcal{M}' + (\dim \mathcal{M}^{[\circ]} - \dim \mathcal{M}^\circ \cap \mathcal{V}^\circ).$$

With $\mathcal{L} := \mathcal{L}'[\dot{+}](\mathcal{M}^\circ \cap \mathcal{V}^\circ)$ we obtain the a subspace with the desired properties. ❑

Let us provide two examples of subspaces which are not orthocomplemented. First, we illustrate the finite dimensional situation.

I.13. *1.3.6 Example.* Consider the space $\mathbb{C}^2$ endowed with the scalar product $[.,.]$ generated by the Gram matrix

$$G := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Explicitly, this is

$$\left[\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right), \left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)\right] := \alpha_1\overline{\beta_2} + \alpha_2\overline{\beta_1}, \quad \left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right), \left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right) \in \mathbb{C}^2 .$$

As $\left[\left(\begin{smallmatrix}1\\ i\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\ i\end{smallmatrix}\right)\right] = 0$ the subspace

$$\mathcal{L} := \operatorname{span}\left\{\left(\begin{smallmatrix}1\\ i\end{smallmatrix}\right)\right\},$$

is neutral, i.e., $\mathcal{L}^{[\circ]} = \mathcal{L}$; see Example 1.2.3. However, $\mathcal{V}^{[\circ]} = \{0\}$. According to Proposition 1.3.2, (2), (i), $\mathcal{L}$ cannot be orthocomplemented.

Of course, in this example, we could just compute the orthogonal complement of $\mathcal{L}$ and observe that $\mathcal{L} + \mathcal{L}^{[\perp]} \neq \mathcal{V}$. In fact, $\mathcal{L}^{[\perp]} = \mathcal{L}$. ◊

Second, we bring an example in order to show that for infinite dimensional subspaces the condition Proposition 1.3.2, (2),(i), is indeed not anymore sufficient.

I.14. *1.3.7 Example.* Consider the linear space of all komplex *left-finite two-sided sequences*

Ich habs mal two-sided sequences genannt, weil bouble seuqence schon vergeben!

$$\mathcal{V} := \{(\alpha_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : \ \exists N \in \mathbb{N} : \alpha_j = 0, j < -N\},$$

and define a scalar product on $\mathcal{V}$ by

$$[(\alpha_j)_{j\in\mathbb{Z}}, (\beta_j)_{j\in\mathbb{Z}}] := \sum_{j\in\mathbb{Z}} \alpha_j \cdot \overline{\beta_{-j-1}}, \quad (\alpha_j)_{j\in\mathbb{Z}}, (\beta_j)_{j\in\mathbb{Z}} \in \mathcal{V}.$$

This expression is well-defined since the sum on the right side contains only finitely many nonzero summands.

Setting $e_k := (\delta_{kj})_{j\in\mathbb{Z}}$ for any $k \in \mathbb{Z}$ we consider the subspace

$$\mathcal{L} := \operatorname{span}\{e_k + e_{-k-1} : k \in \mathbb{N}_0\},$$

of $\mathcal{V}$. From $[e_k, e_l] = \delta_{k(-l-1)}$ we conclude

$$[e_k + e_{-k-1}, e_l + e_{-l-1}] = 2\delta_{kl}, \quad k, l \in \mathbb{N}_0 . \tag{1.3.1}$$ Lbasprod

Clearly, the set $\{e_k + e_{-k-1} : k \in \mathbb{N}_0\}$ consists of linearly independent elements. Moreover, (1.3.1) implies $[x, x] > 0$ for all $x \in \mathcal{L} \setminus \{0\}$. In particular, $\mathcal{L}$ is nondegenerated.

We are going to determin the orthogonal complement of $\mathcal{L}$. For $x = (\alpha_j)_{j\in\mathbb{Z}} \in \mathcal{L}^{[\perp]}$ we have

$$0 = [x, e_k + e_{-k-1}] = \alpha_{-k-1} + \alpha_k, \quad k \in \mathbb{N}_0 .$$

As x is a left finite two-sided sequence this implies that only finitely many terms α_j are nonzero, and in turn that $x \in \operatorname{span}\{e_k - e_{-k-1} : k \in \mathbb{N}_0\}$. By $[e_k, e_l] = \delta_{k(-l-1)}$ each element $e_k - e_{-k-1}$, $k \in \mathbb{N}_0$, belongs to $\mathcal{L}^{[\perp]}$. We end up with

$$\mathcal{L}^{[\perp]} = \operatorname{span}\{e_k - e_{-k-1} : k \in \mathbb{N}_0\}.$$

We see that each element from $\mathcal{L} + \mathcal{L}^{[\perp]}$ is a two-sided sequence with only finitely many entries being nonzero. Thus,

$$\mathcal{L} + \mathcal{L}^{[\perp]} \neq \mathcal{V}.$$

◊

1.4 Definiteness Properties

nitenessProperties

Scalar products for which the quadratic form $[x, x]$ retains sign play a particular role.

I.1. **1.4.1 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space. We call $(\mathcal{V}, [.,.])$

- *positive definite*, if $[x, x] > 0$, $x \in \mathcal{V} \setminus \{0\}$.
- *positive semidefinite*, if $[x, x] \geq 0$, $x \in \mathcal{V}$.
- *negative definite*, if $[x, x] < 0$, $x \in \mathcal{V} \setminus \{0\}$.
- *negative semidefinite*, if $[x, x] \leq 0$, $x \in \mathcal{V}$.
- *neutral*, if $[x, x] = 0$, $x \in \mathcal{V}$.
- *definite*, if it is positive- or negative definite.
- *semidefinite*, if is is positive- or negative semidefinite.

$\Diamond$

Definiteness properties of a subspace $\mathcal{L}$ of a scalar product space $(\mathcal{V}, [.,.])$ are defined in the obvious manner as follows. We say that $\mathcal{L}$ is a *positive definite*, *positive semidefinite*, *negative definite*, *negative semidefinite*, *neutral*, *definite*, *semidefinite*, subspace of $\mathcal{V}$, if the scalar product space $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}})$ has the respective property according to Definition 1.4.1.

I.neutral2 *1.4.2 Remark.* Obviously, $\mathcal{L}$ being a neutral subspace is equivalent to the fact, that $\mathcal{L}$ is positive semidefinite and negative semidefinite. Using the polar identity we already saw in (1.2.3) that $\mathcal{L}$ being a neutral subspace is also equivalent to $[x, y] = 0$ for all $x, y \in \mathcal{L}$. $\Diamond$

I.poskern. *1.4.3 Example.* Let M be a set, let K be a hermitian kernel on M, and consider as in Example 1.1.4 the scalar product space $(\mathcal{F}(M), [.,.])$ generated by K, where

$$[f, g] := \sum_{\zeta,\eta\in M} \overline{g(\zeta)} \cdot K(\zeta, \eta) \cdot f(\eta), \quad f, g \in \mathcal{F}(M).$$

If the scalar product space $(\mathcal{F}(M), [.,.])$ is positive semidefinite (negative semidefinite), then we call the hermitian kernel K on M *positive* (*negative*). Explicitely, this means

$$\sum_{i,j=1}^{m} \bar{\alpha}_i \cdot K(\zeta_i, \zeta_j) \cdot \alpha_j \ \geq \ (\leq)\, 0\,,$$

for any $m \in \mathbb{N}$ and any finite samples $\zeta_1, \dots, \zeta_m \in M$, $\alpha_1, \dots, \alpha_m \in \mathbb{C}$. This is the same as saying that the selfadjoint matrix

$$(\ K(\zeta_i, \zeta_j)\ )_{i,j=1}^{m}$$

is positive semidefinite (negative semidefinite) for any $m \in \mathbb{N}$ and any finite sample $\zeta_1, \dots, \zeta_m \in M$. $\Diamond$

Let us notice explicitly that taking orthogonal sum preserves definiteness properties.

I.31. **1.4.4 Lemma.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be positive definite linear subspaces of $\mathcal{V}$. If $\mathcal{L}_i[\perp]\mathcal{L}_j,\ i \neq j$, then their sum $\mathcal{L}_1 + \dots + \mathcal{L}_n$ is positive definite.*

The same statement holds when "positive definite" is everywhere replaced with one of "positive semidefinite, negative definite, negative semidefinite, neutral".

Proof. Let $x \in \mathcal{L}_1 + \dots + \mathcal{L}_n$, and write $x = x_1 + \dots + x_n$ with $x_j \in \mathcal{L}_j$. Due to our hypothesis $\mathcal{L}_i[\perp]\mathcal{L}_j,\ i \neq j$, we have

$$[x, x] = [x_1, x_1] + \dots + [x_n, x_n] .$$

All summands on the right side are nonnegative. Moreover, if $x \neq 0$, at least one of the x_j must be nonzero. Hence, at least one of these summands must be positive.

The corresponding assertions for the other listed definiteness properties follow in the same way. ❑

Let us observe that if in the previous assertion all spaces $\mathcal{L}_j$ are positive definite or all spaces are negative definite, then $\mathcal{L}_i[\perp]\mathcal{L}_j$ yields $\mathcal{L}_i \cap \mathcal{L}_j = \emptyset$ for $i \neq j$.

It is a basic fact that in semidefinite scalar product spaces the *Schwarz inequality* holds.

I.16. **1.4.5 Lemma.** *For a semidefinite scalar product space $(\mathcal{V}, [.,.])$ we have*

$$|[x, y]|^2 \leq [x, x] \cdot [y, y], \quad x, y \in \mathcal{V} .$$

Proof. It is enough to provide explicit proof for the case that $\mathcal{V}$ is positive semidefinite. The case that $\mathcal{V}$ is negative semidefinite is reduced to this one by considering $(\mathcal{V}, -[.,.])$.

Let $x, y \in \mathcal{V}$ be given. Set $\alpha := [x, x], \beta := |[x, y]|$, and $\gamma := [y, y]$, and let $\lambda \in \mathbb{C}$, $|\lambda| = 1$ be such that $\lambda[y, x] = |[y, x]|$. Then we get

$$0 \leq [x - \xi\lambda y, x - \xi\lambda y] = [x, x] - \xi\lambda[y, x] - \xi\overline{\lambda}[x, y] + \xi^2[y, y], \quad \xi \in \mathbb{R} ,$$

i.e. $\alpha - 2\xi\beta + \xi^2\gamma \geq 0$ for all $\xi \in \mathbb{R}$. This inequality yields $\beta = 0$, in the case that $\gamma = 0$, and with the choice $\xi := \frac{\beta}{\gamma}$ in the case $\gamma > 0$, it yields $\alpha\gamma - \beta^2 \geq 0$. ❑

I.17. **1.4.6 Corollary.** *In a semidefinite scalar product space $(\mathcal{V}, [.,.])$ every neutral element belongs to the isotropic part of $\mathcal{V}$, i.e.,*

$$\mathcal{V}^{[\circ]} = \{x \in \mathcal{V} : [x, x] = 0\} .$$

Proof. By the Schwarz inequality $[x, x] = 0$ for an $x \in \mathcal{V}$ implies

$$|[x, y]|^2 \leq [x, x] \cdot [y, y] = 0, \quad y \in \mathcal{V} .$$

❑

The set of all linear subspaces of a linear space $\mathcal{V}$ is ordered with respect to set-theoretic inclusion. Hence, also the set of all positive definite (positive semidefinite, negative definite, etc.) linear subspaces is.

I.25. **1.4.7 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}$ be a linear subspace of $\mathcal{V}$. Then we call $\mathcal{L}$ *maximal positive definite*, if it maximal in the set of all positive definite linear subspaces of $\mathcal{V}$ w.r.t. set-theoretic inclusion, i.e., there is no strictly larger positive definite linear subspaces of $\mathcal{V}$ than $\mathcal{L}$.

Terminology *maximal positive semidefinite*, *maximal negative definite*, *maximal negative semidefinite*, and *maximal neutral* is defined in the same way. ◊

maxenth. *1.4.8 Remark.* Since $\mathcal{L} + \mathcal{V}^{[\circ]}$ is positive semidefinite (negative semidefinite, neutral), if $\mathcal{L}$ has this property, every maximal positive semidefinite (negative semidefinite, neutral) subspace contains $\mathcal{V}^{[\circ]}$.

Decomposing $\mathcal{V}$ as $\mathcal{V} = \mathcal{V}'[\dot{+}]\mathcal{V}^{[\circ]}$ with a nondegenerated $\mathcal{V}'$ and correspondingly $\mathcal{L} = \mathcal{L}'[\dot{+}]\mathcal{V}^{[\circ]}$ with $\mathcal{L}' \subseteq \mathcal{V}'$ the space $\mathcal{L}'$ is maximal positive semidefinite (negative semidefinite, neutral) in $\mathcal{V}'$. In fact, any prper positive semidefinite (negative semidefinite, neutral) extension $\mathcal{M}'$ would deliver the proper extension $\mathcal{M}'[\dot{+}]\mathcal{V}^{[\circ]}$ of $\mathcal{L}$. ◊

Zorn's lemma ensures that maximal elements always exist.

I.24. **1.4.9 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}$ be a positive definite linear subspace of $\mathcal{V}$. Then there exists a maximal positive definite linear subspace which contains $\mathcal{L}$.*

The same statement holds true when "positive definite" is everywhere replaced with one of "positive semidefinite", "negative definite", "negative semidefinite", or "neutral".

Proof. The set of all positive definite linear subspaces of $\mathcal{V}$ which contain $\mathcal{L}$ is nonempty since it contains $\mathcal{L}$ itself. Moreover, the union of an increasing chain of positive definite linear subspaces is again a positive definite linear subspace. Hence, each increasing chain has an upper bound. Zorn's lemma provides us with a maximal element $\mathcal{M}$. Clearly, $\mathcal{M}$ is even maximal among all positive definite linear subspaces of $\mathcal{V}$.

The word-by-word same argument applies if we substitute "positive definite" by any other of the stated definiteness properties. ❑

The set of all positive definite linear subspaces of $\mathcal{V}$ is contained in the set of all positive semidefinite linear subspaces. Hence, each positive definite subspace which is maximal positive semidefinite is also maximal positive definite. However, not each maximal positive definite subspace is necessarily maximal positive semidefinite.

I.27. *1.4.10 Example.* Consider $\mathbb{C}^2$ endowed with the scalar product defined by $[x, y] := (Gx, y),\ x, y \in \mathbb{C}^2$, with the Gram matrix

$$G := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The linear subspace

$$\mathcal{L} := \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}.$$

is maximal positive definite but not maximal positive semidefinite, because $(\mathbb{C}^2, [.,.])$ is positive semidefinite. ◊

I.33a. **1.4.11 Proposition.** *If $\mathcal{L}$ is a maximal positive definite or maximal positive semidefinite subspace of a scalar product space $(\mathcal{V}, [.,.])$, then $\mathcal{L}^{[\perp]}$ is negative semidefinite.*

If $\mathcal{L}$ is positive definite and maximal positive semidefinite, then $\mathcal{L}^{[\perp]}$ is even negative definite.

The same statements hold true when we replace everywhere "positive" by "negative" and vica versa.

Proof. For a maximal positive definite or maximal positive semidefinite subspace $\mathcal{L}$ assume on the contrary that $x \in \mathcal{L}^{[\perp]}$ and $[x, x] > 0$. If we had $x \notin \mathcal{L}$, then $\mathcal{L} + \operatorname{span}\{x\}$ ($\supsetneq \mathcal{L}$) would have the same definiteness property (positive semidefinite or positive definite) as $\mathcal{L}$, in contradiction to its maximality. Thus $x \in \mathcal{L}^{[\circ]}$. In particular, $[x, x] = 0$ contradicting $[x, x] > 0$.

For a positive definite and maximal positive semidefinite $\mathcal{L}$ we repeat these argumentation for an $x \in \mathcal{L}^{[\perp]} \setminus \{0\}$ with $[x, x] \geq 0$. In fact, $x \notin \mathcal{L}$ would give the strictly larger positive semidefinite subspace $\mathcal{L} + \operatorname{span}\{x\}$ contradicting the maximality of $\mathcal{L}$, and $x \in \mathcal{L}^{[\circ]}$ would imply $[x, x] = 0$ in contradiction to the definiteness of $\mathcal{L}$. ❑

For orthocomplemented subspaces $\mathcal{L}$ a much better characterization of maximality can be given.

Since $\mathcal{L}^{[\circ]} = \mathcal{L} \cap \mathcal{L}^{[\perp]}$, the assumption $\mathcal{V} = \mathcal{L}_1[+]\mathcal{L}_2$ with a nondegenerated $\mathcal{L}_1$ in the following proposition is equivalent to the fact that $\mathcal{L}_1$ is orthocomplemented with $\mathcal{L}_1^{[\circ]} = \{0\}$ and $\mathcal{L}_1^{[\perp]} = \mathcal{L}_2$. It is also equivalent to the fact that $\mathcal{L}_2$ is orthocomplemented with $\mathcal{L}_2^{[\perp]} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2^{[\circ]}$.

I.33b. **1.4.12 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}_1, \mathcal{L}_2$ be subspaces such that $\mathcal{V} = \mathcal{L}_1[+]\mathcal{L}_2$ with a nondegenerated $\mathcal{L}_1$. Then the following statments are equivalent.*

(*a*) *$\mathcal{L}_1$ is positive definte and $\mathcal{L}_2$ is negative semidefinite.*

(*b*) *$\mathcal{L}_1$ is maximal positive definte.*

(*c*) *$\mathcal{L}_2$ is maximal negative semidefinte.*

The same is true when we replace everywhere "positive" by "negative" and vica versa.

Proof. We show $(a) \Rightarrow (b)$. Assume that $\mathcal{M}$ is positive definite and $\mathcal{M} \supseteq \mathcal{L}_1$. Then $\mathcal{M} = \mathcal{L}_1[\dot{+}](\mathcal{M} \cap \mathcal{L}_2)$. The second summand, however, must be equal to $\{0\}$, since $\mathcal{L}_2$ is negative semidefinite. Hence, $\mathcal{L}_1 = \mathcal{M}$, and $\mathcal{L}_1$ is maximal positive definite. Almost the same reasoning shows $(a) \Rightarrow (c)$.

$(b) \Rightarrow (c)$ follows from Proposition 1.4.11 and $\mathcal{L}_1^{[\perp]} = \mathcal{L}_2$. By Proposition 1.4.11, assuming (c) we get that $\mathcal{L}_2^{[\perp]} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2^{[\circ]}$ is positive semidefinite, and hence $\mathcal{L}_1$ is positive definite; cf. Corollary 1.4.6. Thus, also $(c) \Rightarrow (a)$ holds true. ❑

1.5 Angular Operators

--AngularOperators

We start with a result not necessarily dealing with scalar products.

I.38. **1.5.1 Lemma.** *Assume that a vector space $\mathcal{V}$ is decomposed as $\mathcal{V} = \mathcal{L}_1 \dot{+} \mathcal{L}_2$ with two subspaces $\mathcal{L}_1, \mathcal{L}_2$ of $\mathcal{V}$, and let $P : \mathcal{V} \to \mathcal{L}_1$ be the corresponding projection, i.e., $P^2 = P$,* $\operatorname{ran} P = \mathcal{L}_1$ *and* $\ker P = \mathcal{L}_2$.

(1) *For a linear subspace $\mathcal{M}$ of $\mathcal{V}$ with $\mathcal{M} \cap \mathcal{L}_2 = \{0\}$ the mapping $P|_{\mathcal{M}} : \mathcal{L} \to P(\mathcal{M})$ $(\subseteq \mathcal{L}_1)$ is a linear bijection. In particular,* $\dim \mathcal{M} = \dim P(\mathcal{M})$.

Moreover, defining the so called angular operator *W with respect to the decomposition $\mathcal{V} = \mathcal{L}_1 \dot{+} \mathcal{L}_2$ associated with $\mathcal{M}$ on* $\operatorname{dom} W := P(\mathcal{M})$ $(\subseteq \mathcal{L}_1)$ *by*

$$W := (I - P)(P|_{\mathcal{M}})^{-1} : \operatorname{dom} W \to \mathcal{L}_2 \,,$$

the space $\mathcal{M}$ can be recovered from W by

$$\mathcal{M} = \{x + Wx : x \in \operatorname{dom} W\} \,.$$

(2) *Let W be a linear operator defined on some linear subspace* $\operatorname{dom} W$ *of $\mathcal{L}_1$ and mapping into $\mathcal{L}_2$. Then the linear subspace $\mathcal{M}$ defined as*

$$\mathcal{M} := \{x + Wx : x \in \operatorname{dom} W\}$$

satisfies $\mathcal{M} \cap \mathcal{L}_2 = \{0\}$ and has W as its angular operator.

Proof. Due to $\mathcal{M} \cap \mathcal{L}_2 = \mathcal{M} \cap \ker P = \{0\}$ the mapping $P|_{\mathcal{M}} : \mathcal{L} \to \mathcal{L}_1$ is injective. Hence, $P|_{\mathcal{M}} : \mathcal{L} \to P(\mathcal{M})$ $(\subseteq \mathcal{L}_1)$ is a linear bijection, and W is well defined. Clearly, $\dim \mathcal{M} = \dim P(\mathcal{M})$.

For $y \in \mathcal{M}$ the vector $x := Py$ belongs to $\operatorname{dom} W$ and

$$y = Py + (I - P)y = x + Wx \,. \tag{1.5.1}$$ I.75

Conversely, for $x \in \operatorname{dom} W$ we have $y := (P|_{\mathcal{M}})^{-1} \in \mathcal{M}$, and reading (1.5.1) from right to left yields $x + Wx = y$.

For (2), let $x \in \operatorname{dom} W$ and set $y := x + Wx$. From $x = Px = Py$ we conclude that $y \in \mathcal{L}_2 = \ker P$ implies $x = 0$ and, in turn, $y = 0$. Thus, $\mathcal{M} \cap \mathcal{L}_2 = \{0\}$. Moreover, $P\{x + Wx : x \in \operatorname{dom} W\} = \operatorname{dom} W$. For $x \in \operatorname{dom} W$ and $y := x + Wx$ we have $Py = x$. Hence, $(P|_{\mathcal{M}})^{-1} x = y$ and

$$(I - P)(P|_{\mathcal{M}})^{-1} x = (I - P)y = Wx \,.$$

❑

Angular operators are particularly interesting, if $(\mathcal{V}, [.,.])$ is a scalar product space and our decomposition $\mathcal{V} = \mathcal{L}_1 \dot{+} \mathcal{L}_2$ is orthogonal, because then definiteness properties of $\mathcal{M}$ can characterized by means of W.

I.39. **1.5.2 Lemma.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space such that $\mathcal{V} = \mathcal{L}_1 [\dot{+}] \mathcal{L}_2$. Moreover, let $\mathcal{M}$ be a linear subspace of $\mathcal{V}$ which satisfies*

$$\mathcal{M} \cap \mathcal{L}_2 = \{0\} \,,$$

and let $W : \operatorname{dom} W \to \mathcal{L}_2$ *be the angular operator of* $\mathcal{M}$ *w.r.t.* $\mathcal{V} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2$ *as in Lemma* 1.5.1. *Then the definiteness properties of* $\mathcal{M}$ *can be characterized by W as follows:*

$$\begin{array}{lcl} \mathcal{M} \text{ is positive definite} & \Longleftrightarrow & [Wx, Wx] > -[x,x],\ x \in \operatorname{dom} W \setminus \{0\} \\ \mathcal{M} \text{ is positive semidefinite} & \Longleftrightarrow & [Wx, Wx] \geq -[x,x],\ x \in \operatorname{dom} W \\ \mathcal{M} \text{ is negative definite} & \Longleftrightarrow & [Wx, Wx] < -[x,x],\ x \in \operatorname{dom} W \setminus \{0\} \\ \mathcal{M} \text{ is negative semidefinite} & \Longleftrightarrow & [Wx, Wx] \leq -[x,x],\ x \in \operatorname{dom} W \\ \mathcal{M} \text{ is neutral} & \Longleftrightarrow & [Wx, Wx] = -[x,x],\ x \in \operatorname{dom} W \end{array}$$

Proof. According to Lemma 1.5.1, $\mathcal{M} = \{x + Wx : x \in \operatorname{dom} W\}$. As $\operatorname{ran} W \subseteq mcL_2$ and $\operatorname{dom} W = P(\mathcal{M}) \subseteq \mathcal{L}_1$ we can compute

$$[x + Wx, x + Wx] = [x,x] + [Wx, Wx], \quad x \in \operatorname{dom} W.$$

The asserted equivalences easily follow from this equation. ❑

I.37. *1.5.3 Example.* In order to visualize the action of an angular operator, we consider – as an exception – a linear space over the scalar field $\mathbb{R}$. Let $\mathcal{V} := \mathbb{R}^2$, and let $\mathcal{V}$ be endowed with the euclidean scalar product $[\binom{\alpha_1}{\alpha_2}, \binom{\beta_1}{\beta_2}] = \alpha_1\beta_1 - \alpha_2\beta_2$. Obviously, $\mathcal{V} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2$ for the positve definite subspace $\mathcal{L}_1 := \operatorname{span}\{\binom{1}{0}\}$ and negative definite subspace $\mathcal{L}_2 := \operatorname{span}\{\binom{0}{1}\}$. The orthogonal projection of $\mathcal{V}$ onto $\mathcal{L}_1$ is given as

$$P : \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\ 0\end{pmatrix}.$$

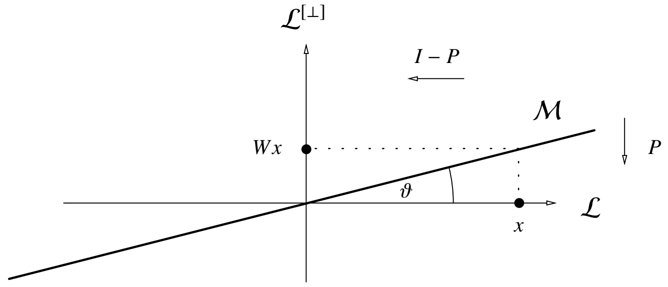
For $\vartheta \in (0, \frac{\pi}{2})$ consider the linear subspace

$$\mathcal{M} := \left\{\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{R}^2 : \beta = \tan\vartheta \cdot \alpha\right\}.$$

Then $\mathcal{M} \cap \mathcal{L}_2 = \{0\}$ and $P(\mathcal{M}) = \mathcal{L}_1$. The angular operator W associated with $\mathcal{M}$ acts as

$$W\begin{pmatrix}\alpha\\ 0\end{pmatrix} := \begin{pmatrix}0\\ \tan\vartheta\cdot\alpha\end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

For $\vartheta \in [0, \frac{\pi}{4})$ we have $[Wx, Wx] \geq 0 > -[x,x],\ x \in \mathcal{L}_1 \setminus \{0\}$. Hence, the corresponding space $\mathcal{M}$ is positive definite. For $\vartheta = \frac{\pi}{4}$ the space $\mathcal{M}$ is neutral, because $[Wx, Wx] = -[x,x],\ x \in \mathcal{L}_1$. For $(\frac{\pi}{4}, \frac{\pi}{2})$ the space $\mathcal{M}$ is negative definite.



Starting with a decomposition $\mathcal{V} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2$ with a definite (semidefinite) $\mathcal{L}_2$ we can be even more specific. Indeed, if, for example, $\mathcal{L}_2$ is positive definte, then obviously $\mathcal{M} \cap \mathcal{L}_2 = \{0\}$ for all negative semidefinite subspaces $\mathcal{M}$.

I.66. **1.5.4 Corollary.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space such that $\mathcal{V} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2$.*

(1) *If $\mathcal{L}_2$ is positive definite, then $\mathcal{M} \cap \mathcal{L}_2 = \{0\}$, and hence $\dim \mathcal{M} \leq \dim \mathcal{L}_1$ for any negative semidefinite subspace.*

Moreover, $\mathcal{M} = \{x + Wx : x \in \operatorname{dom} W\}$ constitutes a bijection between all negative semidefinite (neutral) subspaces and all linear operators $W : \operatorname{dom} W \to \mathcal{L}_2$ with $\operatorname{dom} W \subseteq \mathcal{L}_1$ and $[Wx, Wx] \leq -[x, x]$ $([Wx, Wx] = -[x, x])$ for all $x \in \operatorname{dom} W$. Hereby, $\dim \mathcal{M} = \dim(\operatorname{dom} W)$.

(2) *If $\mathcal{L}_2$ is positive semidefinite, then $\mathcal{M} \cap \mathcal{L}_2 = \{0\}$, and hence $\dim \mathcal{M} \leq \dim \mathcal{L}_1$ for any negative definite subspace.*

Moreover, $\mathcal{M} = \{x + Wx : x \in \operatorname{dom} W\}$ constitutes a bijection between all negative definite subspaces and all linear operators $W : \operatorname{dom} W \to \mathcal{L}_2$ with $\operatorname{dom} W \subseteq \mathcal{L}_1$ and $[Wx, Wx] < -[x, x]$, $x \in \operatorname{dom} W \setminus \{0\}$. Hereby, $\dim \mathcal{M} = \dim(\operatorname{dom} W)$.

The same statements hold true with "$\leq$", "$<$" replaced by "$\geq$","$>$" and "positive" replaced by "negative" and vica versa.

By the previous result we can also relate maximal subspaces with maximal angular operators.

In fact, in the situation of (1), $\mathcal{M}$ is maximal negative semidefinite (maximal neutral) if and only if the corresponding angular operator $W : \operatorname{dom} W\ (\subseteq \mathcal{L}_1) \to \mathcal{L}_2$ satisfies $[Wx, Wx] \leq -[x, x]$ $([Wx, Wx] = -[x, x])$ for all $x \in \operatorname{dom} W$ and has no proper extension W' with the $[W'x, W'x] \leq -[x, x]$ $([W'x, W'x] = -[x, x])$ for all $x \in \operatorname{dom} W'$.

In the situation of (2), $\mathcal{M}$ is maximal negative definite if and only if the corresponding angular operator $W : \operatorname{dom} W\ (\subseteq \mathcal{L}_1) \to \mathcal{L}_2$ satisfies $[Wx, Wx] < -[x, x]$, $x \in \operatorname{dom} W \setminus \{0\}$ and has no proper extension W' with the $[W'x, W'x] < -[x, x]$, $x \in \operatorname{dom} W' \setminus \{0\}$.

The same is true with '$\leq$", "$<$" replaced by "$\geq$","$>$" and "positive" replaced by "negative" and vica versa.

In particular, $\operatorname{dom} W = \mathcal{L}_1$ always implies maximality. If $\dim \mathcal{L}_1 < \infty$, we can say even more.

I.66post. **1.5.5 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space such that $\mathcal{V} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2$.*

(1) *Let $\mathcal{L}_1$ be finite dimensional and negative semidefinite, and let $\mathcal{L}_2$ be positive definite. Then $\mathcal{M}$ is maximal negative semidefinite if and only if the corresponding angular operator $W : \operatorname{dom} W\ (\subseteq \mathcal{L}_1) \to \mathcal{L}_2$ satisfies $[Wx, Wx] \leq -[x, x]$, $x \in \operatorname{dom} W$, and $\operatorname{dom} W = \mathcal{L}_1$.*

(2) *Let $\mathcal{L}_1$ be finite dimensional and negative definite, and let $\mathcal{L}_2$ be positive semidefinite. Then $\mathcal{M}$ is maximal negative definite if and only if the corresponding angular operator $W : \operatorname{dom} W\ (\subseteq \mathcal{L}_1) \to \mathcal{L}_2$ satisfies $[Wx, Wx] < -[x, x]$, $x \in \operatorname{dom} W \setminus \{0\}$, and $\operatorname{dom} W = \mathcal{L}_1$.*

The same statements hold true with "$\leq$", "$<$" replaced by "$\geq$","$>$" and "positive" replaced by "negative" and vica versa.

Proof. We already mentioned that in any case $\operatorname{dom} W = \mathcal{L}_1$ implies maximality. For the converse we consider the cases separately.

(1) Let $\mathcal{M}$ be negative semidefinite such that the corresponding W satisfies $\operatorname{dom} W \neq \mathcal{L}_1$. Due to Corollary 1.4.6 we have $(\operatorname{dom} W)^{[\circ]} \subseteq \mathcal{L}_1^{[\circ]}$. Therefore, $\operatorname{dom} W$ is orthocomplemented, and for any subspace $\mathcal{N}$ with $\operatorname{dom} W = \mathcal{N}[\dot+](\operatorname{dom} W)^{[\circ]}$ we have

$$\mathcal{L}_1 = \mathcal{N}\,[\dot+]\,\underbrace{\mathcal{N}^{[\perp]}}_{=(\operatorname{dom} W)^{[\perp]}}$$

Obviously, $(\operatorname{dom} W)^{[\circ]} \subsetneq \mathcal{N}^{[\perp]}$. Take a $z \in \mathcal{N}^{[\perp]} \setminus \operatorname{dom} W = \mathcal{N}^{[\perp]} \setminus (\operatorname{dom} W)^{[\circ]}$ and define the extension $W' : \operatorname{span}\{z\}[\dot+]\operatorname{dom} W \to \mathcal{L}_2$ by $W'(x) = 0$. For $\lambda z + x \in \operatorname{dom} W' = \operatorname{span}\{z\}[\dot+]\operatorname{dom} W$ we then have

$$[W'(\lambda z + x), W'(\lambda z + x)] = [Wx, Wx] \leq -[x, x] = -[\lambda z + x, \lambda z + x]\,.$$

Hence, W' is the angular operator of a negative semidefinite subspace $\mathcal{M}'$ containing $\mathcal{M}$ properly. This menas that $\mathcal{M}$ is not maximal.

(2) Let $\mathcal{M}$ be negative definite such that the corresponding W satisfies $\operatorname{dom} W \neq \mathcal{L}_1$. $\mathcal{L}_1$ being positive definite, $\operatorname{dom} W$ is orthocomplemented with

$$\mathcal{L}_1 = \operatorname{dom} W[\dot+](\operatorname{dom} W)^{[\perp]}\,.$$

Take a $z \in (\operatorname{dom} W)^{[\perp]}$ and define the extension $W' : \operatorname{span}\{z\}[\dot+]\operatorname{dom} W \to \mathcal{L}_2$ by $W'(x) = 0$. For $\lambda z + x \in \operatorname{dom} W' = \operatorname{span}\{z\}[\dot+]\operatorname{dom} W$ we then have

$$[W'(\lambda z + x), W'(\lambda z + x)] = [Wx, Wx] \quad\text{and}\quad -[\lambda z + x, \lambda z + x] = -|\lambda|^2[z, z] - [x, x]\,.$$

From $[Wx, Wx] < -[x, x]$ for $x \neq 0$ and $[z, z] > 0$ we get $[W'(\lambda z + x), W'(\lambda z + x)] <$ for $\lambda z + x \neq 0$. Hence, W' is the angular operator of a negative definite subspace $\mathcal{M}'$ containing $\mathcal{M}$ properly. This menas that $\mathcal{M}$ is not maximal.

❑

1.6 Index of Positivity and Negativity

dexOfPositivityAndNegativity

negposdef.

1.6.1 Definition. Let $(\mathcal{V}, [.,.])$ be a scalar product space. We set

$$\operatorname{ind}_+(\mathcal{V}, [.,.]) := \sup\{\dim \mathcal{L} : \mathcal{L} \text{ positive definite subspace of } \mathcal{V}\} \in \mathbb{N}_0 \cup \{\infty\}\,,$$
$$\operatorname{ind}_-(\mathcal{V}, [.,.]) := \sup\{\dim \mathcal{L} : \mathcal{L} \text{ negative definite subspace of } \mathcal{V}\} \in \mathbb{N}_0 \cup \{\infty\}\,,$$

and call $\operatorname{ind}_+(\mathcal{V}, [.,.])$ the *index of positivity* of $\mathcal{V}$, and $\operatorname{ind}_-(\mathcal{V}, [.,.])$ the *index of negativity* of $\mathcal{V}$.

Again we do not distinguish different cardinalities of ∞. Moreover, when no confusion may occur, we drop explicit notation of $[.,.]$ and write $\operatorname{ind}_+ \mathcal{V}$ and $\operatorname{ind}_- \mathcal{V}$. ◊

The sets over which the suprema in this definition are taken can be narrowed in two different ways. First, since an infinite dimensional positive (negative) definite subspace contains positive (negative) subspaces with arbitrary finite dimension, we have

$$\begin{aligned}
\operatorname{ind}_+(\mathcal{V}, [.,.]) &= \sup\{\dim\mathcal{L} : \mathcal{L} \text{ positive definite subspace of } \mathcal{V},\ \dim\mathcal{L} < \infty\}\,,\\
\operatorname{ind}_-(\mathcal{V}, [.,.]) &= \sup\{\dim\mathcal{L} : \mathcal{L} \text{ negative definite subspace of } \mathcal{V},\ \dim\mathcal{L} < \infty\}\,.
\end{aligned}$$

Second, since each positive (negative) definite subspace is contained in a maximal positive (negative) definite subspace, we have

$$\begin{aligned}
\operatorname{ind}_+(\mathcal{V}, [.,.]) &= \sup\{\dim\mathcal{L} : \mathcal{L} \text{ maximal positive definite subspace of } \mathcal{V}\}\,,\\
\operatorname{ind}_-(\mathcal{V}, [.,.]) &= \sup\{\dim\mathcal{L} : \mathcal{L} \text{ maximal negative definite subspace of } \mathcal{V}\}\,.
\end{aligned}$$

The next result tells us that when using maximal definite subspaces, taking a supremum is not necessary at all.

I.40. **1.6.2 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space. Then for each maximal positive definite subspace $\mathcal{L}_+$*

$$\dim\mathcal{L}_+ = \operatorname{ind}_+\mathcal{V}\,.$$

In the case that $\operatorname{ind}_+\mathcal{V} < \infty$*, a subspace $\mathcal{L}_+$ is maximal positive definite, if and only if it is positive definite and* $\dim\mathcal{L}_+ = \operatorname{ind}_+\mathcal{V}$*. Moreover, $\mathcal{L}_+$ being maximal positive definite and taking any subspace $\mathcal{L}_-$ with $\mathcal{L}_+^{[\perp]} = (\mathcal{L}_+^{[\perp]})^{[\circ]}[\dot+]\mathcal{L}_-$, we have $(\mathcal{L}_+^{[\perp]})^{[\circ]} = \mathcal{V}^{[\circ]}$ and*

$$\mathcal{V} = \mathcal{L}_+[\dot+]\underbrace{\mathcal{V}^{[\circ]}[\dot+]\mathcal{L}_-}_{=\mathcal{L}_+^{[\perp]}}\,, \tag{1.6.1}$$ I.52eq

with a maximal negative semidefinite $\mathcal{L}_+^{[\perp]}$ and a maximal negative definite $\mathcal{L}_-$.

As usual, the same holds when everywhere "positive", "ind_+*" and "negative", "*ind_-*" are exchanged.*

Proof. If all maximal positive definite subspaces of $\mathcal{V}$ have infinite dimension, there is nothing to prove.

Assume that $\mathcal{L}_+$ is a maximal positive definite subspace with $\dim\mathcal{L}_+ < \infty$. Then $\mathcal{L}_+$ is orthocomplemented and there exists a unique orthogonal projection P with $\operatorname{ran}P = \mathcal{L}_+$; see Proposition 1.3.2. This projection satisfies $\ker P = \mathcal{L}_+^{[\perp]}$. By Proposition 1.3.2 we have $(\mathcal{L}_+^{[\perp]})^{[\circ]} = \mathcal{V}^{[\circ]}$. Moreover, choosing a subspace $\mathcal{L}_-$ with $\mathcal{L}_+^{[\perp]} = \mathcal{L}_-\dot+\mathcal{V}^{[\circ]}$ we obtain the decomposition in (1.6.1). From Proposition 1.4.12 we conclude that $\mathcal{L}_+^{[\perp]}$ is maximal negative semidefinite and that $\mathcal{L}_-$ is maximal negative definite.

By Corollary 1.5.4, $\dim\mathcal{L} \leq \dim\mathcal{L}_+$ for any positive definite $\mathcal{L}$, and in turn

$$\operatorname{ind}_+\mathcal{V} = \dim\mathcal{L} < \infty\,.$$

Knowing that $\operatorname{ind}_+\mathcal{V} < \infty$, each maximal positive definite subspace has finite dimension and, hence, the above argument applied to any maximal positive definite subspace shows that their dimensions must all equal to $\operatorname{ind}_+\mathcal{V}$. Finally, if $\mathcal{L}$ is positive definite and $\dim\mathcal{L} = \operatorname{ind}_+\mathcal{V}$ $(<\infty)$, then any positive definite space $\mathcal{M} \supseteq \mathcal{L}$ must satisfy $\dim\mathcal{M} \leq \operatorname{ind}_+\mathcal{V}$. Hence, $\mathcal{M} = \mathcal{L}$ and $\mathcal{L}$ must be maximal.

The assertion concerning the index of negativity is verified analogously, or follows by considering $(\mathcal{V}, -[.,.])$ instead of $(\mathcal{V}, [.,.])$. ❑

I.40post. *1.6.3 Remark.* By (1.6.1) $\operatorname{ind}_+ \mathcal{V} = 0$ ($\operatorname{ind}_- \mathcal{V} = 0$) means nothing else than the negative semidefiniteness (positive semidefiniteness) of $[.,.]$ on $\mathcal{V}$. ◊

I.52. **1.6.4 Corollary.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space. Then*

$$\dim \mathcal{V} = \operatorname{ind}_+ \mathcal{V} + \operatorname{ind}_- \mathcal{V} + \operatorname{ind}_0 \mathcal{V}\,.$$

Proof. If one of the quantities $\operatorname{ind}_+ \mathcal{V}$, $\operatorname{ind}_- \mathcal{V}$, or $\operatorname{ind}_0 \mathcal{V}$, is infinite, there is nothing to prove. Hence, assume that all three of these numbers are finite. Choosing a maximal positive definite subspace $\mathcal{L}_+$ of $\mathcal{V}$ we have a decomposition like in (1.6.1) with a maximal negative definite $\mathcal{L}_-$. Hence,

$$\dim \mathcal{V} = \dim \mathcal{L}_+ + \dim \mathcal{M} + \dim \mathcal{V}^{[\circ]} = \operatorname{ind}_+ \mathcal{V} + \operatorname{ind}_- \mathcal{V} + \operatorname{ind}_0 \mathcal{V}\,.$$

❑

I.45. *1.6.5 Example.* Let $[.,.]$ be a scalar product on $\mathbb{C}^m$, and let G be the Gram matrix of $[.,.]$. Choose an orthonormal basis $\{a_1, \dots, a_m\}$ consisting of eigenvectors of G, where orthonormality is understood w.r.t. the euclidean scalar product $(.,.)$. Denote the eigenvalues corresponding to $a_1, \dots, a_m$ by $\lambda_1, \dots, \lambda_m$, and set

$$\mathcal{L}_+ := \operatorname{span}\{a_i : \lambda_i > 0\}, \quad \mathcal{L}_- := \operatorname{span}\{a_i : \lambda_i < 0\}\,.$$

From

$$\Big[\sum_{i=1}^m \alpha_i a_i, \sum_{i=1}^m \beta_i a_i\Big] = \Big(\sum_{i=1}^m \lambda_i \alpha_i a_i, \sum_{i=1}^m \beta_i a_i\Big) = \sum_{i=1}^n \lambda_i \alpha_i \overline{\beta_i}\,.$$

we see that $\mathcal{L}_+$ is positive definite, $\mathcal{L}_-$ is negative definite. Hence,

$$\mathbb{C}^m = \mathcal{L}_+ [\dot{+}] \mathcal{L}_- [\dot{+}] (\mathbb{C}^m)^{[\circ]}\,.$$

It follows that $\mathcal{L}_+$ is maximal positive definite and $\mathcal{L}_-$ is maximal negative definite, cf. Proposition 1.4.12. Thus

$$\begin{aligned}
\operatorname{ind}_+(\mathbb{C}^m, [.,.]) &= \#\{i \in \{1, \dots, m\} : \lambda_i > 0\}\,,\\
\operatorname{ind}_-(\mathbb{C}^m, [.,.]) &= \#\{i \in \{1, \dots, m\} : \lambda_i < 0\}\,,\\
\operatorname{ind}_0(\mathbb{C}^m, [.,.]) &= \#\{i \in \{1, \dots, m\} : \lambda_i = 0\}\,.
\end{aligned}$$

Defining for a given selfadjoint matrix G, by $\operatorname{ind}_+ G$ ($\operatorname{ind}_- G$) the number of positive (negative) eigenvalues of G counted according to their multiplicities, and by $\operatorname{ind}_0 G$ the multiplicity of the eigenvalue zero, we have $\operatorname{ind}_\pm(\mathbb{C}^m, [.,.]) = \operatorname{ind}_\pm G$ and $\operatorname{ind}_0(\mathbb{C}^m, [.,.]) = \operatorname{ind}_0 G$. ◊

I.43. **1.6.6 Corollary.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space. Then for each maximal positive semidefinite subspace $\mathcal{L}$ of $\mathcal{V}$ we have*

$$\dim \mathcal{L} = \operatorname{ind}_+ \mathcal{V} + \operatorname{ind}_0 \mathcal{V}\,. \tag{1.6.2}$$ I.73

If this equality holds true for a positive semidefinite subspace $\mathcal{L}$ with $\dim \mathcal{L} < \infty$, then $\mathcal{L}$ is maximal positive semidefinite. Moreover,

$$\sup\{\dim \mathcal{L} : \mathcal{L} \textit{ positive semidefinite, } \mathcal{L} \cap \mathcal{V}^{[\circ]} = \{0\}\} = \operatorname{ind}_+ \mathcal{V}\,. \tag{1.6.3}$$ I.74

The same statements hold when everywhere "positive" is replaced with "negative" and "ind_+" by "ind_-"

Proof. First we establish the inequalities "≤" in (1.6.2) and (1.6.3). If $\operatorname{ind}_+ \mathcal{V} = \infty$ there is nothing to prove. So we assume $\operatorname{ind}_+ \mathcal{V} < \infty$. Choose a maximal positive definite subspace $\mathcal{L}_+$ of $\mathcal{V}$. Then we have have (1.6.1) with a maximal negative definite $\mathcal{L}_-$. Applying Corollary 1.5.4, with $\mathcal{L}_2 = \mathcal{L}_-$ and $\mathcal{L}_1 = \mathcal{V}^{[\circ]}[\dot{+}]\mathcal{L}_+$ shows that

$$\dim \mathcal{L} \leq \dim(\mathcal{V}^{[\circ]}[\dot{+}]\mathcal{L}_+) = \operatorname{ind}_+ \mathcal{V} + \operatorname{ind}_0 \mathcal{V},$$

for any positive semidefinite subspace $\mathcal{L}$. If in addition $\mathcal{L} \cap \mathcal{V}^{[\circ]} = \{0\}$, then $\mathcal{L}$ is even positive definite, and hence $\dim \mathcal{L} \leq \operatorname{ind}_+ \mathcal{V}$.

Moreover, if equality holds in in (1.6.2) for a finite dimensional and positive semidefinite $\mathcal{L}$, then we conclude from Proposition 1.5.5, (1), that $\mathcal{L}$ must be maximal.

The inequality "≥" in (1.6.3) is a consequence of the fact that any maximal positive definite subspace $\mathcal{L}$ satisfies $\mathcal{L} \cap \mathcal{V}^{[\circ]} = \{0\}$.

Finally, we show "≥" in (1.6.2). As mentioned in Remark 1.4.8, we have $\mathcal{V}^{[\circ]} \subseteq \mathcal{L}$ and we can write $\mathcal{V} = \mathcal{V}'[\dot{+}]\mathcal{V}^{[\circ]}$ and $\mathcal{L} = \mathcal{L}'[\dot{+}]\mathcal{V}^{[\circ]}$ with a maximal positive semidefinite subspace $\mathcal{L}'$ of the nondegenerated $\mathcal{V}'$. Hence, for the verification of (1.6.2) we can assume that $\mathcal{V}^{[\circ]} = \{0\}$. Clearly, we can also assume that $\dim \mathcal{L} < \infty$.

Take any finite dimensional positive definite subspace $\mathcal{N}$ of $\mathcal{V}$ with $\dim \mathcal{N} \leq \operatorname{ind}_+ \mathcal{V}$. Due to Proposition 1.3.5 we find a finite dimensional subspace $\mathcal{M} \supseteq \mathcal{L} + \mathcal{N}$ with $\mathcal{M}^{[\circ]} \subseteq \mathcal{V}^{[\circ]} = \{0\}$, i.e., $\mathcal{M}$ is nondegenerated. According to (1.6.1),

$$\mathcal{M} = \mathcal{M}_+[\dot{+}]\mathcal{M}_-,$$

with a maximal positive (negative) definite subspace $\mathcal{M}_+$ ($\mathcal{M}_-$) in $\mathcal{M}$. Clearly, $\mathcal{L}$ is also maximal positive semidefinite as a subspace of $\mathcal{M}$. By Proposition 1.5.5 we have

$$\dim \mathcal{L} = \dim \mathcal{M}_+ = \operatorname{ind}_+ \mathcal{M} \geq \operatorname{ind}_+ \mathcal{N}.$$

Since $\mathcal{N}$ was arbitrarily chosen, we get $\dim \mathcal{L} \geq \operatorname{ind}_+ \mathcal{V}$.

❑

Next, let us investigate how index of positivity and negativity behave when performing constructions with scalar product spaces.

I.44. **1.6.7 Proposition.** *The following statements hold true.*

(1) *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}$ be a linear subspace of $\mathcal{V}$ endowed with the scalar product inherited from $\mathcal{V}$. Then*

$$\operatorname{ind}_+ \mathcal{L} \leq \operatorname{ind}_+ \mathcal{V}, \quad \operatorname{ind}_- \mathcal{L} \leq \operatorname{ind}_- \mathcal{V}.$$

(2) *For each $j \in \{1, \dots, n\}$ let $(\mathcal{V}_j, [.,.]_j)$ be a scalar product space, and consider the space $\prod_{j=1}^n \mathcal{V}_j$ endowed with the sum scalar product. Then*

$$\operatorname{ind}_+ \prod_{j=1}^n \mathcal{V}_j = \sum_{j=1}^n \operatorname{ind}_+ \mathcal{V}_j, \quad \operatorname{ind}_- \prod_{j=1}^n \mathcal{V}_j = \sum_{j=1}^n \operatorname{ind}_- \mathcal{V}_j.$$

(3) *Let $(\mathcal{V}, [.,.])$ be a scalar product space, let $\mathcal{N}$ be a linear subspace of $\mathcal{V}^{[\circ]}$, and consider the the space $\mathcal{V}/_{\mathcal{N}}$ endowed with the factor scalar product. Then*

$$\operatorname{ind}_+ \mathcal{V}/_{\mathcal{N}} = \operatorname{ind}_+ \mathcal{V}, \quad \operatorname{ind}_- \mathcal{V}/_{\mathcal{N}} = \operatorname{ind}_- \mathcal{V}.$$

(4) *Let* $(\mathcal{V},[.,.])$ *and* $(\mathcal{W},[\![.,.]\!])$ *be scalar product spaces, and let* $\varphi:\mathcal{V}\to\mathcal{W}$ *be a linear and isometric map. Then*

$$\operatorname{ind}_+\mathcal{V}=\operatorname{ind}_+\operatorname{ran}\varphi,\quad \operatorname{ind}_-\mathcal{V}=\operatorname{ind}_-\operatorname{ran}\varphi\,.$$

Proof. We establish the assertions concerned with index of positivity. The index of negativity is treated in the same way.

Item (1) holds since each positive definite subspace of $\mathcal{L}$ is also one of $\mathcal{V}$. For (2), let $\mathcal{L}_j$, $j=1,\dots,n$, be positive definite subspaces of $\mathcal{V}_j$. Then

$$\mathcal{L}:=\prod_{j=1}^n\mathcal{L}_j$$

is a positive definite subspace of $\prod_{j=1}^n\mathcal{V}_j$; cf. Lemma 1.4.4. Hence, the inequality "$\geq$" follow. If one of $\operatorname{ind}_+\mathcal{V}_j$, $j=1,\dots,n$, is infinite, we are done. Otherwise, choose for each $j\in\{1,\dots,n\}$ a maximal positive definite subspace $\mathcal{L}_j$ of $\mathcal{V}_j$. Then $\mathcal{L}_j^{[\perp]}$ is negative semidefinite, and $\mathcal{V}_j=\mathcal{L}_j[\dot{+}]\mathcal{L}_j^{[\perp]}$. It follows that

$$\prod_{j=1}^n\mathcal{V}_j=\prod_{j=1}^n\mathcal{L}_j[\dot{+}]\prod_{j=1}^n\mathcal{L}_j^{[\perp]}\,.$$

The first summand is positive definite, and second is negative semidefinite. By Proposition 1.4.12 the first summand is maximal positive definite. Consequently,

$$\operatorname{ind}_+\prod_{j=1}^n\mathcal{V}_j=\dim\prod_{j=1}^n\mathcal{L}_j=\sum_{j=1}^n\dim\mathcal{L}_j=\sum_{j=1}^n\operatorname{ind}_+\mathcal{L}_j\,.$$

In the next step, we prove (4). If $\mathcal{L}$ is a positive definite subspace of $\mathcal{V}$, then $\varphi|_{\mathcal{L}}$ is injective, and hence $\dim\varphi(\mathcal{L})=\dim\mathcal{L}$. Thus $\operatorname{ind}_+\operatorname{ran}\varphi\geq\operatorname{ind}_+\mathcal{V}$. For the converse, choose a right inverse ψ of φ, i.e., a linear map ψ with $\varphi\circ\psi=\mathrm{id}_{\operatorname{ran}\varphi}$. Clearly, ψ is again isometric. Using what we already showed,

$$\operatorname{ind}_+\operatorname{ran}\varphi\leq\operatorname{ind}_+\operatorname{ran}\psi\leq\operatorname{ind}_+\mathcal{V}\,.$$

Item (3) is now immediate, since the canonical projection $\pi:\mathcal{V}\to\mathcal{V}/\mathcal{N}$ is linear, isometric, and surjective. ❑

Sometimes it is useful to describe the index of positivity (negativity) with the help of the mapping ψ as introduced in Example 1.1.9. Recall from Example 1.6.5 that, for a given selfadjoint matrix G, ind_+G (ind_-G) is the number of positive (negative) eigenvalues of G counted according to their multiplicities.

I.8postinddesc. **1.6.8 Proposition.** *For any scalar product space* $(\mathcal{V},[.,.])$ *we then have*

$$\operatorname{ind}_\pm(\mathcal{V},[.,.])=\sup\{\operatorname{ind}_\pm([z_j,z_i])_{i,j=1}^m:m\in\mathbb{N},\ z_1,\dots,z_m\in\mathcal{V}\}\,.\qquad(1.6.4)$$ I.8postinddesceq1

More generally,

$$\operatorname{ind}_\pm(\mathcal{V},[.,.])=\sup\{\operatorname{ind}_\pm([z_j,z_i])_{i,j=1}^m:m\in\mathbb{N},\ z_1,\dots,z_m\in M\}\,,\qquad(1.6.5)$$ I.8postinddesceq2

for any subset M *of* $\mathcal{V}$ *such that* $\operatorname{span}M=\mathcal{V}$.

Hereby, $\operatorname{ind}_\pm([z_j,z_i])_{i,j=1}^m=m$ *if and only if* $z_1,\dots,z_m$ *is the basis of a positive (negative) definite subspace.*

Proof. For $m \in \mathbb{N}$ and $z_1, \dots, z_m \in \mathcal{V}$ providing $\mathbb{C}^m$ with the scalar product

$$[f, g] = [\psi f, \psi g] = \sum_{i,j=1}^{m} \overline{g(i)} \cdot [z_j, z_i] \cdot f(j), \ f, g \in \mathcal{F}(\{1, \dots, m\}),$$

let $\psi : \mathcal{F}(\{1, \dots, m\}) (= \mathbb{C}^m) \to \mathcal{V}$ be the isometric mapping $\psi f = \sum_{j=1}^{m} f(j) \cdot z_j$ with $\operatorname{ran} \psi = \operatorname{span}\{z_1, \dots, z_m\}$. Considering $f, g \in \mathcal{F}(\{1, \dots, m\})$ as vectors we have $[f, g] = (([z_j, z_i])_{i,j=1}^{m} f, g)$; see Example 1.1.9.

As discussed in Example 1.6.5, $\operatorname{ind}_\pm(\mathbb{C}, [.,.]) = \operatorname{ind}_\pm([z_j, z_i])_{i,j=1}^{m}$, and by Proposition 1.6.7 we have $\operatorname{ind}_\pm(\mathbb{C}, [.,.]) = \operatorname{ind}_\pm \operatorname{ran} \psi \le \operatorname{ind}_\pm(\mathcal{V}, [.,.])$. Hence, the inequality $\ge$ holds in (1.6.4) and (1.6.5).

If $z_1, \dots, z_m$ is a basis of an m-dimensional positive (negative) definite subspace $\mathcal{L}$, then $\operatorname{ind}_\pm([z_j, z_i])_{i,j=1}^{m} = \operatorname{ind}_\pm \operatorname{ran} \psi = \dim \mathcal{L} = m$. Conversely, if $\operatorname{ind}_\pm([z_j, z_i])_{i,j=1}^{m} = \operatorname{ind}_\pm \operatorname{ran} \psi = \operatorname{ind}_\pm \operatorname{span}\{z_1, \dots, z_m\}$ equals to $\operatorname{ind}_\pm(\mathcal{V}, [.,.]) = m$, then by Corollary 1.6.4 necessarily $\dim \operatorname{span}\{z_1, \dots, z_m\} = m$ and, hence, $z_1, \dots, z_m$ is a basis of the positive (negative) definite subspace $\operatorname{span}\{z_1, \dots, z_m\}$; see Corollary 1.6.4.

In order to show equality in (1.6.4) and (1.6.5), it plainly suffices to show equality in (1.6.5). Choose a maximal positive (negative) definite subspace $\mathcal{L}$ of $\mathcal{V}$, and let $\mathcal{M}$ be a finite dimensional subspace of $\mathcal{L}$. Then there exist $z_1, \dots, z_m \in M$ such that $\mathcal{M} \subseteq \operatorname{span}\{z_1, \dots, z_m\}$. By Proposition 1.6.7 applied to the corresponding mapping ψ we get $\operatorname{ind}_\pm([z_j, z_i])_{i,j=1}^{m} = \operatorname{ind}_\pm(\mathbb{C}, [.,.]) = \operatorname{ind}_\pm \operatorname{ran} \psi \ge \operatorname{ind}_\pm \mathcal{M}$, and in turn

$$\sup\{ \operatorname{ind}_\pm([z_j, z_i])_{i,j=1}^{m} : m \in \mathbb{N}, \ z_1, \dots, z_m \in M \} \ge \operatorname{ind}_\pm \mathcal{M}.$$

Since $\mathcal{M}$ was arbitrarily chosen, we get equality in (1.6.5). ❑

We apply Proposition 1.6.8 to the scalar product space generated by a hermitian kernel.

I.46. *1.6.9 Example.* Let M be a set, let K be a hermitian kernel on M, and consider the scalar product space $(\mathcal{F}(M), [.,.])$ generated by K, cf. Example 1.1.4. Since $[\delta_\xi, \delta_\eta] = K(\eta, \xi)$ for the basis $\{\delta_\xi : \xi \in M\}$ of $\mathcal{F}(M)$ defined in (1.1.5), we get from (1.6.5)

$$\operatorname{ind}_\pm \mathcal{F}(M) = \sup_{m \in \mathbb{N}, \ \xi_1, \dots, \xi_m \in M} \operatorname{ind}_\pm(K(\xi_i, \xi_j))_{i,j=1}^{m}.$$

◊

We continue our study of the scalar product space generated by the Nevanlinna kernel of a rational function $r = \frac{p}{q}$ from Example 1.1.7 and Example 1.2.16.

I.51a. *1.6.10 Example.* Notation is, without further notice, as in Example 1.1.7. In particular, m denotes the degree of the rational function r, i.e., the maximum degree of the polynomials in numerator and denominator of a relatively prime quotient representation of r.

We have already shown that the map θ defined as in (1.1.7) is a linear, surjective, and isometric mapping of $(\mathcal{F}(M), [.,.])$ onto $(\mathbb{C}^m, [\![.,.]\!])$, and that $(\mathbb{C}^m, [\![.,.]\!])$ is nondegenerated. It follows that (see Proposition 1.6.7 and Corollary 1.6.4)

$$\operatorname{ind}_+ \mathcal{F}(M) = \operatorname{ind}_+(\mathbb{C}^m, [\![.,.]\!]), \ \operatorname{ind}_- \mathcal{F}(M) = \operatorname{ind}_-(\mathbb{C}^m, [\![.,.]\!]), \tag{1.6.6}$$ I.53

$$\operatorname{ind}_+ \mathcal{F}(M) + \operatorname{ind}_- \mathcal{F}(M) = m.$$

We will show in this example that given $m \in \mathbb{N}_0$ and $N_+, N_- \in \mathbb{N}_0$ with $N_+ + N_- = m$ there always exists a rational function r of degree m such that such that $\operatorname{ind}_+ \mathcal{F}(M) = N_+$ and $\operatorname{ind}_- \mathcal{F}(M) = N_-$. For this purpose we employ the construction of a rational function as in Example 1.1.11. Choose pairwise different real points $\lambda_1, \dots, \lambda_m$, and consider the measure

$$\mu := \sum_{i=1}^{N_+} \delta_{\lambda_i} - \sum_{i=N_+ +1}^{m} \delta_{\lambda_i} ,$$

where δ_{λ_i} denotes the *unit point mass*, also called the *Dirac measure*, concentrated at the point λ_i. The function built from this measure is

$$q(\zeta) := \int_{\mathbb{R}} \frac{1+t\zeta}{t-\zeta}\, d\mu = \sum_{i=1}^{N_+} \frac{1+\lambda_i\zeta}{t-\lambda_i} - \sum_{i=N_+ +1}^{m} \frac{1+\lambda_i\zeta}{t-\lambda_i} .$$

Obviously, q is rational with degree at most m. It has a simple pole at each point λ_i. Hence, its degree in fact must be equal to m. We are going to show that the scalar product space generated by the Nevanlinna kernel of q has index of positivity equal to N_+ and index of negativity equal to N_-.

The measures in the Jordan decomposition of μ are

$$\mu_+ := \sum_{i=1}^{N_+} \delta_{\lambda_i}, \quad \mu_- := \sum_{i=N_+ +1}^{m} \delta_{\lambda_i} .$$

Thus $\dim L^2(\mu_+) = N_+$ and $\dim L^2(-\mu_-) = m - N_+ = N_-$. (1.1.11) provides us with an isometric map ϕ from $\mathcal{F}(M)$ into $L^2(\mu_+) \times L^2(-\mu_-)$. Since $\theta : \mathcal{F}(M) \to \mathbb{C}^m$ is linear and surjective, we can choose a linear map $\vartheta : \mathbb{C}^m \to \mathcal{F}(M)$ with $\theta \circ \vartheta = \mathrm{id}_{\mathbb{C}^m}$. In fact, in Example 1.2.16 we explicitely constructed such a mapping $\vartheta : \mathbb{C}^m \to \mathcal{F}(M)$.

Clearly, as a right inverse of an isometry, ϑ is also isometric. The composition $\phi \circ \vartheta$ thus maps $\mathbb{C}^m$ linearly and isometrically into $L^2(\mu_+) \times L^2(-\mu_-)$. Since $\mathbb{C}^m$ is nondegenerated, $\phi \circ \vartheta$ must be injective; cf. (1.2.7). By equality of dimensions, it is bijective. Hence,

$$\begin{aligned} &\operatorname{ind}_+(\mathbb{C}^m, [\![.,.]\!]) = \operatorname{ind}_+ (L^2(\mu_+) \times L^2(-\mu_-)), \\ &\qquad \operatorname{ind}_-(\mathbb{C}^m, [\![.,.]\!]) = \operatorname{ind}_- (L^2(\mu_+) \times L^2(-\mu_-)) . \end{aligned} \tag{1.6.7}$$ I.54

However, $L^2(\mu_+)$ is positive definite and $L^2(-\mu_-)$ is negative definite, and $L^2(\mu_+) \times L^2(-\mu_-)$ is the direct and orthogonal sum of these two spaces. Thus $L^2(\mu_+)$ is maximal positive definite, $L^2(-\mu_-)$ is maximal negative definite; cf. Proposition 1.4.12. Therefore,

$$\begin{aligned} \operatorname{ind}_+ (L^2(\mu_+) \times L^2(-\mu_-)) &= \dim L^2(\mu_+) = N_+ , \\ \operatorname{ind}_- (L^2(\mu_+) \times L^2(-\mu_-)) &= \dim L^2(\mu_-) = N_- . \end{aligned} \tag{1.6.8}$$ I.55

Putting together (1.6.6), (1.6.7), and (1.6.8), we see that for $r := q$ we indeed have $\operatorname{ind}_+ \mathcal{F}(M) = N_+$ and $\operatorname{ind}_- \mathcal{F}(M) = N_-$. ◊

I.51b. *1.6.11 Example.* We will show that, contrasting the general case, a neat formula for $\operatorname{ind}_+ \mathcal{F}(M)$ and $\operatorname{ind}_- \mathcal{F}(M)$ can be given when $q = 1$, i.e., r is a polynomial

$$r(\zeta) = \lambda_m \zeta^m + \lambda_{m-1}\zeta^{m-1} \cdots + \lambda_0 ,$$

with $\lambda_i \in \mathbb{R}$, $\lambda_m \neq 0$. In fact, (here $\lfloor x \rfloor$ denotes the largest integer not exceeding x)

$$\operatorname{ind}_+ \mathcal{F}(M) = \Big\lfloor \frac{m}{2} \Big\rfloor + \begin{cases} 1\,, & m \text{ odd}, \ \lambda_m > 0 \\ 0\,, & \text{otherwise} \end{cases}$$

$$\operatorname{ind}_- \mathcal{F}(M) = \Big\lfloor \frac{m}{2} \Big\rfloor + \begin{cases} 1\,, & m \text{ odd}, \ \lambda_m < 0 \\ 0\,, & \text{otherwise} \end{cases}$$

To show these formulas, we investigate the Gram matrix G which was used to define $[\![.,.]\!]$ on $\mathbb{C}^m$ in Example 1.1.7 such that

$$[f,g] = [\![\theta f, \theta g]\!] = (G\,\theta f, \theta g), \ f,g \in \mathcal{F}(M)\,, \tag{1.6.9}$$

erst34

where $\theta : \mathcal{F}(M) \to \mathbb{C}^m$, $\theta f = \big(\sum_{\zeta\in M} f(\zeta)\zeta^i\big)_{i=0}^{m-1}$ is isometric and onto; see Example 1.2.16. For $\zeta \in M$ we have $\theta\delta_\zeta = (\zeta^i)_{i=0}^{m-1}$. Since the Nevanlinna kernel of r computes as

$$[\delta_\eta, \delta_\zeta] = K(\zeta,\eta) = \frac{r(\eta) - \overline{r(\zeta)}}{\eta - \overline{\zeta}} = \sum_{i=1}^{m} \frac{\lambda_i(\eta^i - \overline{\zeta}^i)}{\eta - \overline{\zeta}} = \sum_{i=0}^{m-1} \lambda_{i+1} \sum_{k=0}^{i} \eta^k (\overline{\zeta})^{i-k}\,,$$

we get

$$[\delta_\eta, \delta_\zeta] = (B\,\theta\delta_\eta, \theta\delta_\zeta)\,,$$

and hence by linearity $[f,g] = (B\,\theta f, \theta g)$, $f,g \in \mathcal{F}(M)$, where

$$B = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m-1} & \lambda_m \\ \lambda_2 & & & \lambda_m & 0 \\ \vdots & & \iddots & & \vdots \\ \lambda_{m-1} & \lambda_m & & & \\ \lambda_m & 0 & \cdots\cdots & & 0 \end{pmatrix}$$

Since θ is onto, we obtain $B = G$ from (1.6.9). Therefore, G is of *Hankel form*, i.e., has constant entries along each skew-diagonal.

As a subspace of $(\mathbb{C}^m, [\![.,.]\!])$, where $[\![.,.]\!] = (G.,.)$,

$$\mathcal{L} := \operatorname{span}\Big\{e_k : k > \frac{m+1}{2}\Big\}$$

is neutral. It is thus in the same time positive semidefinite and negative semidefinite. Due to (1.6.2), its dimension does not exceed either of $\operatorname{ind}_+ \mathbb{C}^m$ and $\operatorname{ind}_- \mathbb{C}^m$. If m is even, we have $\dim \mathcal{L} = \frac{m}{2}$ and hence $\operatorname{ind}_+ \mathbb{C}^m, \operatorname{ind}_- \mathbb{C}^m \geq \frac{m}{2}$. However, the sum of these two number equals m, and it follows that

$$\operatorname{ind}_+ \mathbb{C}^m = \operatorname{ind}_- \mathbb{C}^m = \frac{m}{2}\,.$$

If m is odd, we have $\dim \mathcal{L} = \lfloor \frac{m}{2} \rfloor$. Moreover, the space $\mathcal{L}$ is orthogonal to $\mathcal{M} := \operatorname{span}\{e_{\frac{m+1}{2}}\}$. The space $\mathcal{M}$ is positive definite or negative definite, depending on the sign of $\lambda_m = [e_{\frac{m+1}{2}}, e_{\frac{m+1}{2}}]$. Hence, with $(\mathbb{C}^m, [\![.,.]\!])$ also $\mathcal{M}^{[\![\perp]\!]}$ is nondegenerated. Since $\mathcal{L}$ is a neutral and, hence, a positive and negative semidefinite subspace of $\mathcal{M}^{[\![\perp]\!]}$, (1.6.3) gives $\lfloor \frac{m}{2} \rfloor = \dim \mathcal{L} \leq \min(\operatorname{ind}_+ \mathcal{M}^{[\![\perp]\!]}, \operatorname{ind}_- \mathcal{M}^{[\![\perp]\!]})$. As the dimension of

$\mathcal{M}^{[\![\perp]\!]}$ is $m-1 = 2\cdot\lfloor\frac{m}{2}\rfloor$ and equals to $\operatorname{ind}_+ \mathcal{M}^{[\![\perp]\!]} + \operatorname{ind}_- \mathcal{M}^{[\![\perp]\!]}$, it follows that $\operatorname{ind}_+ \mathcal{M}^{[\![\perp]\!]} = \operatorname{ind}_- \mathcal{M}^{[\![\perp]\!]} = \lfloor\frac{m}{2}\rfloor$. Hence,

$$\operatorname{ind}_+ \mathcal{V} = \operatorname{ind}_+(\mathbb{C}^m, [\![.,.]\!]) = \operatorname{ind}_+ \mathcal{M}^{[\![\perp]\!]} + \operatorname{ind}_+ \mathcal{M} = \left\lfloor\frac{m}{2}\right\rfloor + \begin{cases} 1, & m \text{ odd}, \ \lambda_m > 0 \\ 0, & \text{otherwise} \end{cases}$$

and correspondingly for the index of negativity. ◊

1.7 Neutral Subspaces

section--NeutralSubspaces

We start the present section with an example which provides us with a procedure to construct neutral subspaces.

I.61pre. *1.7.1 Example.* Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}_+, \mathcal{L}_-$ be two subspaces with $m := \dim \mathcal{L}_+ = \dim \mathcal{L}_- < \infty$ and $\mathcal{L}_+[\perp]\mathcal{L}_-$ such that $\mathcal{L}_+$ is positive and $\mathcal{L}_-$ is negative definite. Moreover, let $a_1, \dots, a_m$ be an orthonormal basis of $\mathcal{L}_+$ and $b_1, \dots, b_m$ be an orthonormal basis of $\mathcal{L}_-$; cf. Definition 1.2.10.

For $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ with $|\lambda_j| = 1$, $j = 1, \dots, m$, i.e., $\lambda_1, \dots, \lambda_m \in \mathbb{T}$. Set

$$\mathcal{M} := \operatorname{span}\{a_1 + \lambda_1 b_1, \dots, a_m + \lambda_m b_m\}. \tag{1.7.1}$$ I.65pre

Clearly, $\dim \mathcal{M} = m$, and from

$$[a_i + \lambda_i b_i, a_j + \lambda_j b_j] = \delta_{ij} - \lambda_i \bar{\lambda}_j \delta_{ij} = 0, \quad i, j = 1, \dots, m,$$

we see that $\mathcal{M}$ is neutral.

The construction of this space $\mathcal{M}$ can also be viewed with the help of angular operators. In fact, defining $W : \mathcal{L}_+ \to \mathcal{L}_-$ by $W(a_j) = \lambda_j b_j$, $j = 1, \dots, m$, $\mathcal{M}$ is exactly the subspace of $\mathcal{L}_+[\dot+]\mathcal{L}_-$ whose angular operator is W; see Lemma 1.5.1. From $|\lambda_j| = 1$ we conclude $-[Wx, Wx] = [x, x]$, $x \in \mathcal{L}_+$. With the help of Corollary 1.5.4 we again see that $\mathcal{M}$ is neutral. ◊

The existence of neutral subspaces $\mathcal{M}$ of $\mathcal{L}_+[\dot+]\mathcal{L}_-$ with $\dim \mathcal{M} = \dim \mathcal{L}_+ = \dim \mathcal{L}_-$ is used in the following assertion.

I.67. **1.7.2 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space. Then for each maximal neutral subspace $\mathcal{M}$ of $\mathcal{V}$ we have*

$$\dim \mathcal{M} = \min\{\operatorname{ind}_+ \mathcal{V}, \operatorname{ind}_- \mathcal{V}\} + \operatorname{ind}_0 \mathcal{V}.$$

If this equality holds true for a neutral subspace $\mathcal{M}$ with $\dim \mathcal{M} < \infty$, then $\mathcal{M}$ is maximal neutral. Moreover,

$$\sup\{\dim \mathcal{M} : \mathcal{M} \textit{ neutral subspace of } \mathcal{V} \textit{ with } \mathcal{M} \cap \mathcal{V}^{[\circ]} = \{0\}\} = \\ = \min\{\operatorname{ind}_+ \mathcal{V}, \operatorname{ind}_- \mathcal{V}\}.$$

Proof. Since any neutral subspace is negative and positive semidefinite, the inequalities "≤" here are consequences of (1.6.2) and (1.6.3).

Moreover, for a finite dimensional neutral subspace $\mathcal{M}$ with $\dim \mathcal{M} = \min\{\operatorname{ind}_+ \mathcal{V}, \operatorname{ind}_- \mathcal{V}\} + \operatorname{ind}_0 \mathcal{V}$ we obtain from Corollary 1.6.6 in the case

$\mathrm{ind}_+ \mathcal{V} \leq \mathrm{ind}_- \mathcal{V}$ that $\mathcal{M}$ is maximal positive semidefinite, and hence maximal neutral. In case $\mathrm{ind}_+ \mathcal{V} \geq \mathrm{ind}_- \mathcal{V}$, we argue analogously.

For the converse inequalities "$\geq$", pick $m \in \mathbb{N}_0$ with $m \leq \min\{\mathrm{ind}_+ \mathcal{V}, \mathrm{ind}_- \mathcal{V}\}$. Then there exists a positive definite subspace $\mathcal{L}_+$ of $\mathcal{V}$ with $\dim \mathcal{L}_+ = m$. We have $\mathcal{V} = \mathcal{L}_+[\dot{+}]\mathcal{L}_+^{[\perp]}$, and hence $\mathrm{ind}_- \mathcal{L}_+^{[\perp]} = \mathrm{ind}_- \mathcal{V} \geq m$; cf. Proposition 1.6.7. Choose a negative definite subspace $\mathcal{L}_-$ of $\mathcal{L}_+^{[\perp]}$ with $\dim \mathcal{L}_- = m$.

We saw in Example 1.7.8 that $\mathcal{L}_+[\dot{+}]\mathcal{L}_-$ contains a neutral subspace of dimension m. By Corollary 1.6.4 we have $\dim(\mathcal{L}_+[\dot{+}]\mathcal{L}_-)^{[\circ]} = 0$. In particular, $\mathcal{M} \cap \mathcal{V}^{[\circ]} = \{0\}$.

Passing to the supremum over m, this readily establishes the inequality "$\geq$" in the second asserted relation. To obtain the corresponding inequality in the first asserted relation, notice that the subspace $\mathcal{M} + \mathcal{V}^{[\circ]}$ is neutral and $\dim(\mathcal{M} + \mathcal{V}^{[\circ]}) = m + \mathrm{ind}_0 \mathcal{V}$. ❑

I.61post. *1.7.3 Remark.* With $\mathcal{L}_+$ and $\mathcal{L}_-$ as in Example 1.7.1 let $\mathcal{M}$ be any neutral subspace of $\mathcal{L}_+[\dot{+}]\mathcal{L}_-$ with $\dim \mathcal{M} = \dim \mathcal{L}_+ = \dim \mathcal{L}_- = m$. By Corollary 1.5.4 the corresponding angular operator $W : \mathrm{dom}\, W\ (\subseteq \mathcal{L}_+) \to \mathcal{L}_-$, in fact, satisfies $\mathrm{dom}\, W = \mathcal{L}_+$ and $-[Wx, Wx] = [x, x],\ x \in \mathcal{L}_+$.

Choosing any orthonormal basis $a_1, \ldots, a_m$ of $\mathcal{L}_+$, we define $b_j := Wa_j$, $j = 1, \ldots, m$. Since by polar identity, (1.1.1), we have, $-[Wx, Wy] = [x, y],\ y \in \mathcal{L}_+$, $b_1, \ldots, b_m$ is an orthonormal bases of $\mathcal{L}_-$.

Because of $\mathcal{M} = \{x + Wx : x \in \mathcal{L}_+\} = \mathrm{span}\{a_j + b_j : j = 1, \ldots, m\}$, we are in the situation of (1.7.1) with $\lambda_1, \ldots, \lambda_m = 1$. ◊

Now recall Proposition 1.3.5. In the special case that $\mathcal{M}$ is a finite dimensional, neutral subspace of a scalar product space $(\mathcal{V}, [.,.])$ such that $\mathcal{M} \cap \mathcal{V}^{[\circ]} = \{0\}$ this result provides us with a nondegenerated subspace $\mathcal{L} \supseteq \mathcal{M}$ of dimension $2 \dim \mathcal{M}$.

Take any nondegenerated subspace $\mathcal{L} \supseteq \mathcal{M}$ of dimension at most $2 \dim \mathcal{M}$. According to Corollary 1.6.4 we have then $2 \dim \mathcal{M} \geq \dim \mathcal{L} = \mathrm{ind}_+ \mathcal{L} + \mathrm{ind}_- \mathcal{L}$. But from Proposition 1.7.2 we see that $\dim \mathcal{M} \leq \min\{\mathrm{ind}_+ \mathcal{L}, \mathrm{ind}_- \mathcal{L}\}$. Both inequalities are only possible, if $\mathrm{ind}_+ \mathcal{L} = \mathrm{ind}_- \mathcal{L} = \dim \mathcal{M}$.

Take any decomposition $\mathcal{L} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-$ with a positive (negative) definite $\mathcal{L}_+$ ($\mathcal{L}_-$). We just saw in Remark 1.7.3 that $\mathcal{M} = \mathrm{span}\{a_j + b_j : j = 1, \ldots, m\}$ for certain orthonormal basises $a_1, \ldots, a_m$ of $\mathcal{L}_+$ and $b_1, \ldots, b_m$ of $\mathcal{L}_-$. Now set

$$\mathcal{N} := \mathrm{span}\{a_j - b_j : j = 1, \ldots, m\}.$$

By Example 1.7.8 this is also an m-dimensional, neutral subspace of $\mathcal{L}$. It is easy to check that $\mathcal{M} + \mathcal{N}$ coincides with the nondegenerated subspace $\mathcal{L}$. Thus, we verified the following assertion.

I.60pre. **1.7.4 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{M}$ be a finite dimensional, neutral subspace of $\mathcal{V}$. Assume that $\mathcal{M} \cap \mathcal{V}^{[\circ]} = \{0\}$. Then there exists a subspace $\mathcal{N}$ of $\mathcal{V}$, such that*

$$\mathcal{N} \textit{ neutral}, \quad \dim \mathcal{N} = \dim \mathcal{M}, \quad (\mathcal{M} + \mathcal{N})^{[\circ]} = \{0\}.$$

More exactly, if $\mathcal{L}$ is a nondegenerated subspace of $\mathcal{V}$ containing the neutral subspace $\mathcal{M}$ with $\dim \mathcal{L} \leq 2 \dim \mathcal{M}$, then $\mathcal{N}$ can be choosen such that $\mathcal{L} = \mathcal{M} + \mathcal{N}$. In this situation, in fact, we always have $\dim \mathcal{L} = 2 \dim \mathcal{M}$, $\mathrm{ind}_+ \mathcal{L} = \mathrm{ind}_- \mathcal{L} = \dim \mathcal{M}$.

Pairs of neutral subspaces having the property that their span is nondegenerated play an important role.

I.59. **1.7.5 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{M}$ and $\mathcal{N}$ be two neutral subspaces of $\mathcal{V}$. Then we call $\mathcal{M}$ and $\mathcal{N}$ *skewly linked* if

$$\dim \mathcal{M} < \infty,\ \dim \mathcal{N} < \infty, \qquad (\mathcal{M} + \mathcal{N})^{[\circ]} = \{0\}\,.$$

$\Diamond$

First, we collect some simple properties of skewly linked subspaces.

I.60. **1.7.6 Lemma.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, let $\mathcal{M}$ and $\mathcal{N}$ be two skewly linked neutral subspaces of $\mathcal{V}$. Then we have*

$$\mathcal{M} \cap \mathcal{N} = \{0\}, \quad \dim \mathcal{M} = \dim \mathcal{N} = \operatorname{ind}_\pm(\mathcal{M} + \mathcal{N}) = \frac{1}{2}\dim(\mathcal{M} + \mathcal{N})\,, \tag{1.7.2}$$ I.71

$$\mathcal{M}^{[\perp]} \cap (\mathcal{M} + \mathcal{N}) = \mathcal{M}, \quad \mathcal{N}^{[\perp]} \cap (\mathcal{M} + \mathcal{N}) = \mathcal{N}\,. \tag{1.7.3}$$ I.72

Proof. By symmetry, we can assume that $\dim \mathcal{M} \leq \dim \mathcal{N}$. For $\mathcal{L} := \mathcal{M} + \mathcal{N}$ the assumptions on $\mathcal{M}$ and $\mathcal{L}$ in the second part of Proposition 1.7.4 are satisfied. Therefore, $\dim \mathcal{L} = 2 \dim \mathcal{M}$, $\operatorname{ind}_+ \mathcal{L} = \operatorname{ind}_- \mathcal{L} = \dim \mathcal{M}$. From $\dim \mathcal{M} \leq \dim \mathcal{N}$ and $\mathcal{L} = \mathcal{M} + \mathcal{N}$ we then get $\dim \mathcal{M} = \dim \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$. This proves (1.7.2).

We come to the proof of (1.7.3). The inclusions "$\supseteq$" are obvious. Assume that $x \in \mathcal{M}^{[\perp]} \cap (\mathcal{M} + \mathcal{N})$. Write $x = y + z$ with $y \in \mathcal{M}$ and $z \in \mathcal{N}$. Since $\mathcal{M} \subseteq \mathcal{M}^{[\perp]}$, it follows that also $z \in \mathcal{M}^{[\perp]}$. Hence,

$$z \in \mathcal{M}^{[\perp]} \cap \mathcal{N} \subseteq \mathcal{M}^{[\perp]} \cap \mathcal{N}^{[\perp]} = (\mathcal{M} + \mathcal{N})^{[\perp]}\,,$$

and nondegeneracy of $\mathcal{M} + \mathcal{N}$ implies that $z = 0$. Thus $x = y \in \mathcal{M}$. The inclusion "$\subseteq$" in the second relation in (1.7.3) follows in the same way. ❑

jonsp1. *1.7.7 Example.* For $\beta \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ und an open subset U of $\mathbb{C}$ with $U = U^*$ $(= \{\bar{z} : z \in U\})$ and $\beta \in U$ consider the space $H(U)$ of all holomorphic, $\mathbb{C}$-valued functions defined on U. For $f \in H(U)$ the function

$$f^*(z) := \overline{f(\bar{z})},\ z \in U\,,$$

is again holomorphic on U, i.e., $f^* \in H(U)$. Clearly, $(f^*)^* = f$ and $(fg)^* = f^* g^*$ for $f, g \in H(U)$. Moreover, $(f^*)^{(j)} = (f^{(j)})^*$ for all degrees of derivation $j \in \mathbb{N}_0$.

For given $m \in \mathbb{N}$ and $d_0, \dots, d_{m-1} \in \mathbb{C}$ with $d_{m-1} \neq 0$

$$[f, g] := \sum_{j=0}^{m-1} \left(\frac{d_j}{j!} (f \cdot g^*)^{(j)}(\beta) + \frac{\bar{d}_j}{j!} (f \cdot g^*)^{(j)}(\bar{\beta}) \right),\ f, g \in H(U)\,,$$

satisfies

$$\overline{[f, g]} = \sum_{j=0}^{m-1} \left(\frac{\bar{d}_j}{j!} \overline{(f \cdot g^*)^{(j)}(\beta)} + \frac{d_j}{j!} \overline{(f \cdot g^*)^{(j)}(\bar{\beta})} \right) =$$

$$\sum_{j=0}^{m-1} \left(\frac{\bar{d}_j}{j!} ((f \cdot g^*)^{(j)})^*(\bar{\beta}) + \frac{d_j}{j!} ((f \cdot g^*)^{(j)})^*(\beta) \right) =$$

$$\sum_{j=0}^{m-1} \left(\frac{\bar{d}_j}{j!} ((f \cdot g^*)^*)^{(j)}(\bar{\beta}) + \frac{d_j}{j!} ((f \cdot g^*)^*)^{(j)}(\beta) \right) = [g, f]\,.$$

Hence, $(H(U), [.,.])$ is a scalar product space. The mapping $\theta : H(U) \to \mathbb{C}^{2m}$ defined by

$$\theta f := \Big(\frac{1}{0!} f^{(0)}(\beta), \dots, \frac{1}{(m-1)!} f^{(m-1)}(\beta), \frac{1}{0!} f^{(0)}(\bar\beta), \dots, \frac{1}{(m-1)!} f^{(m-1)}(\bar\beta)\Big)$$

is clearly linear. By induction on m it is not hard to verify, that θ maps the space $\mathbb{C}[z]_{<2m}$ of all polynomials of degree less than $2m$ bijectively onto $\mathbb{C}^{2m}$. Therefore, θ is onto. Since

$$\begin{aligned}
[f,g] &= \sum_{j=0}^{m-1} \left(\frac{d_j}{j!} (f \cdot g^*)^{(j)}(\beta) + \frac{\bar d_j}{j!} (f \cdot g^*)^{(j)}(\bar\beta) \right) = \\
&= \sum_{j=0}^{m-1} \left(\frac{d_j}{j!} \sum_{\substack{k,l\in\mathbb{N}_0 \\ k+l=j}} \frac{j!}{k!l!} f^{(k)}(\beta) \cdot (g^*)^{(l)}(\beta) + \frac{\bar d_j}{j!} \sum_{\substack{k,l\in\mathbb{N}_0 \\ k+l=j}} \frac{j!}{k!l!} f^{(k)}(\beta) \cdot (g^*)^{(l)}(\bar\beta) \right) \\
&= \sum_{\substack{k,l\in\mathbb{N}_0 \\ k+l\le m-1}} \frac{1}{l!} \overline{g^{(l)}(\bar\beta)} \cdot d_{k+l} \cdot \frac{1}{k!} f^{(k)}(\beta) + \sum_{\substack{k,l\in\mathbb{N}_0 \\ k+l\le m-1}} \frac{1}{l!} \overline{g^{(l)}(\beta)} \cdot \bar d_{k+l} \cdot \frac{1}{k!} f^{(k)}(\bar\beta) \\
&= (G\,\theta f, \theta g)\,,
\end{aligned}$$

we see that θ is isometric providing $\mathbb{C}^{2m}$ with $(G.,.)$, where G is the selfadjoint matrix of the block form

$$G = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$$

with

$$B = \begin{pmatrix} d_0 & d_1 & \cdots & d_{m-2} & d_{m-1} \\ d_1 & & & d_{m-1} & 0 \\ \vdots & & \iddots & & \vdots \\ d_{m-2} & d_{m-1} & & & \\ d_{m-1} & 0 & \cdots\cdots & & 0 \end{pmatrix}$$

If $f_0, \dots, f_{m-1}, g_0, \dots, g_{m-1} \in H(U)$ such that $f_k^{(j)}(\beta) = 0 = g_k^{(j)}(\bar\beta)$ for $j, k = 0, \dots, m-1$, then we see that $\operatorname{span}\{f_0, \dots, f_{m-1}\}$ and $\operatorname{span}\{g_0, \dots, g_{m-1}\}$ are two neutral subspaces of $H(U)$ which are skewly linked. ◊

Next, we come to a description of pairs of skewly linked neutral subspaces in the spirit of Example 1.7.1.

I.61. *1.7.8 Example.* Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{L}_+, \mathcal{L}_-$ be two subspaces with $m := \dim \mathcal{L}_+ = \dim \mathcal{L}_- < \infty$ and $\mathcal{L}_+[\perp]\mathcal{L}_-$ such that $\mathcal{L}_+$ is positive and $\mathcal{L}_-$ is negative definite. Moreover, let $a_1, \dots, a_m$ be an orthonormal basis of $\mathcal{L}_+$ and $b_1, \dots, b_m$ be an orthonormal basis of $\mathcal{L}_-$.

Again let $\lambda_1, \dots, \lambda_m \in \mathbb{T}$, and let $\alpha_1, \dots, \alpha_m \in \mathbb{T}$ be a second sample of such numbers. Set

$$\mathcal{M} := \operatorname{span}\{a_1+\lambda_1 b_1, \dots, a_m+\lambda_m b_m\}, \quad \mathcal{N} := \operatorname{span}\{a_1+\alpha_1 b_1, \dots, a_m+\alpha_m b_m\}\,. \quad (1.7.4)$$ I.65

If $\lambda_i \ne \alpha_i$ for all $i = 1, \dots, m$, then $\mathcal{M} + \mathcal{N} = \mathcal{L}_+[\dot+]\mathcal{L}_-$. Hence, $\mathcal{M} + \mathcal{N}$ is nondegenerated. Since $(i, j = 1, \dots, m)$

$$\begin{aligned}
[a_i + \lambda_i b_i, a_j + \lambda_j b_j] = [a_i + \alpha_i b_i, a_j + \alpha_j b_j] = 0, \\
[a_i + \lambda_i b_i, a_j + \alpha_j b_j] = (1 - \lambda_i \bar\alpha_j)\delta_{ij}\,, \quad (1.7.5)
\end{aligned}$$ I.61eq

these spaces are always neutral, and the condition $\lambda_i \neq \alpha_i$ for all $i = 1, \dots, m$, is not only sufficient but also necessary for $\mathcal{M}$ and $\mathcal{N}$ to be skewly linked. ◊

All skewly linked subspaces can be obtained in this way.

I.62. **1.7.9 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, let $\mathcal{M}$ and $\mathcal{N}$ be two skewly linked neutral subspaces of $\mathcal{V}$, and set $m := \dim \mathcal{M}$.*

Let $\mathcal{L}_+$ be a maximal positive definite subspace of $\mathcal{M} + \mathcal{N}$. Then there exist orthonormal bases $\{a_1, \dots, a_m\}$ of $\mathcal{L}_+$ and $\{b_1, \dots, b_m\}$ of $\mathcal{L}_+^{[\perp]}$, and numbers $\lambda_1, \dots, \lambda_m \in \mathbb{T} \setminus \{1\}$, such that the subspaces defined in (1.7.4) with $\alpha_1 = \cdots = \alpha_m = 1$ coincide with the given spaces $\mathcal{M}$ and $\mathcal{N}$.

Proof. We have $\dim(\mathcal{M} + \mathcal{N}) = 2m$, $(\mathcal{M} + \mathcal{N})^{[\circ]} = \{0\}$, and $\operatorname{ind}_+(\mathcal{M} + \mathcal{N}) = \operatorname{ind}_-(\mathcal{M} + \mathcal{N}) = m$; cf. Lemma 1.7.6. Let $\mathcal{L}_+$ be a maximal positive definite subspace of $\mathcal{M} + \mathcal{N}$. Then $\mathcal{M} + \mathcal{N} = \mathcal{L}_+[\dotplus]\mathcal{L}_-$, where $\mathcal{L}_- = \mathcal{L}^{[\perp]}$ is negative definite and $\dim \mathcal{L}_+ = \dim \mathcal{L}_- = m$.

In Remark 1.7.3 we saw that the angular operator $W : \operatorname{dom} W \ (\subseteq \mathcal{L}_+) \to \mathcal{L}_-$ corresponding to $\mathcal{M}$ is defined on $\mathcal{L}_+$, i.e., $\operatorname{dom} W = \mathcal{L}_+$. The same is true for angular operator $V : \operatorname{dom} V \ (\subseteq \mathcal{L}_+) \to \mathcal{L}_-$ corresponding to $\mathcal{N}$. The composition

$$U = V^{-1} \circ W \ : \mathcal{L}_+ \to \mathcal{L}_+$$

is thus a bijective linear map. For all $x \in \mathcal{L}_+$ we have

$$[Ux, Ux] = -[VUx, VUx] = -[Wx, Wx] = [x, x] \,,$$

i.e., it is unitary on the finite dimensional Hilbert space $(\mathcal{L}_+, [.,.])$. Due to the the Spectral Theorem for unitary matrices (see) we find an orthonormal basis $\{a_1, \dots, a_m\}$ of $\mathcal{L}_+$ which consists of eigenvalues of U. Set $b_i := Wa_i$, then $\{b_1, \dots, b_m\}$ is an orthonormal basis of $\mathcal{L}_-$.

Missing Reference

Let us denote the eigenvalues corresponding to $a_1, \dots, a_m$ by $\bar{\lambda}_1, \dots, \bar{\lambda}_m \in \mathbb{T}$. None of these eigenvalues can be equal to 1. Assume on the contrary that, say, $\bar{\lambda}_1 = 1$. Then we have $Wa_1 = VUa_1 = Va_1$ and therefore,

$$a_1 + Wa_1 = a_1 + Va_1 \in \mathcal{M} \cap \mathcal{N} = \{0\} \,.$$

Since $a_1 \in \mathcal{L}_+$ and $Wa_1 \in \mathcal{L}_-$ are linearly independent, we have reached a contradiction. Finally,

$$\mathcal{M} = \{x + Wx : x \in \mathcal{L}_+\} = \operatorname{span}\{a_i + Wa_i : i = 1, \dots, m\} = \operatorname{span}\{a_i + b_i : i = 1, \dots, m\} \,,$$

and

$$\begin{aligned} \mathcal{N} = \{x + Vx : x \in \mathcal{L}_+\} &= \operatorname{span}\{a_i + V(\lambda_i U a_i) : i = 1, \dots, m\} = \\ &= \operatorname{span}\{a_i + \lambda_i W a_i : i = 1, \dots, m\} = \operatorname{span}\{a_i + \lambda_i b_i : i = 1, \dots, m\} \,. \end{aligned}$$

❑

There is also another way to describe skewly linked subspaces with the help basises.

I.61post2. *1.7.10 Example.* Let $(\mathcal{V},[.,.])$ be a scalar product space. Let $m\in\mathbb{N}$ and $a_1,\dots,a_m,b_1,\dots,b_m\in\mathcal{V}$ be such that

$$[a_i,a_j]=[b_i,b_j]=0,\ [a_i,b_j]=\delta_{ij},\quad i,j=1,\dots,m\,. \tag{1.7.6}$$ I.63

Then it is easy to check that

$$\mathcal{M}:=\operatorname{span}\{a_1,\dots,a_m\},\quad \mathcal{N}:=\operatorname{span}\{b_1,\dots,b_m\}\,,$$

$\mathcal{M}$ and $\mathcal{N}$ are skewly linked neutral subspaces of $\mathcal{V}$. ◊

As an immediate corollary of Proposition 1.7.9 and (1.7.5) we get

I.62post. **1.7.11 Corollary.** *Let $(\mathcal{V},[.,.])$ be a scalar product space, let $\mathcal{M}$ and $\mathcal{N}$ be two skewly linked neutral subspaces of $\mathcal{V}$, and set $m:=\dim\mathcal{M}$. Then there exist basises $\{a_1,\dots,a_m\}$ and $\{b_1,\dots,b_m\}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, such that* (1.7.6) *holds.*

Given a pair of skewly linked neutral subspace $\mathcal{M}$ and $\mathcal{N}$, we refer to each pair of bases $\{a_1,\dots,a_m\}$ and $\{b_1,\dots,b_m\}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, which satisfy (1.7.6), as *skewly linked bases*.

I.64. *1.7.12 Remark.* Any finite dimensional neutral scalar product space $(\mathcal{M},[.,.])$ can be viewed as a subspace of a nondegenerated scalar product space $(\mathcal{V},[.,.])$ with $\dim\mathcal{V}=2\dim\mathcal{M}$. In fact, take a copy $\mathcal{N}$ of $\mathcal{M}$ and set $\mathcal{V}:=\mathcal{M}\times\mathcal{N}$. Now take bases $\{a_1,\dots,a_m\}$ and $\{b_1,\dots,b_m\}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, and define $[.,.]$ by linearity on $\mathcal{V}$ such that these basises are skewly linked. ◊

Each neutral subspace is at the same time positive semidefinite and negative semidefinite. Hence, each neutral subspace which is maximal positive semidefinite or maximal negative semidefinite is automatically maximal neutral. Interestingly, each maximal neutral subspace must already be maximal in at least one of these larger sets.

I.69. **1.7.13 Proposition.** *Let $(\mathcal{V},[.,.])$ be a scalar product space, and let $\mathcal{M}$ be a neutral subspace of $\mathcal{V}$. Then the following assertions are equivalent.*

(1) *$\mathcal{M}$ is maximal neutral.*

(2) *$\mathcal{M}^{[\perp]}$ is semidefinite and $\mathcal{M}=(\mathcal{M}^{[\perp]})^{[\perp]}$.*

(3) *$\mathcal{M}$ is maximal positive semidefinite or maximal negative semidefinite.*

Proof. For the proof of "(1) ⇒ (2)" assume that $\mathcal{M}$ is maximal neutral. Being neutral means $\mathcal{M}\subseteq\mathcal{M}^{[\perp]}$. Hence $\mathcal{M}^{[\perp][\perp]}\subseteq(\mathcal{M}^{[\perp][\perp]})^{[\perp]}$, and we see that $\mathcal{M}^{[\perp][\perp]}$ is again neutral. By maximality, thus $\mathcal{M}^{[\perp][\perp]}=\mathcal{M}$. If $\mathcal{M}^{[\perp]}$ were not semidefinite, the second assertion in Proposition 1.7.2 applied to $\mathcal{M}^{[\perp]}$ would provide us with a nonzero neutral subspace $\mathcal{N}$ satisfying $\mathcal{N}\cap(\mathcal{M}^{[\perp]})^{[\circ]}=\{0\}$. However, $\mathcal{M}\subseteq\mathcal{M}^{[\perp]}\cap\mathcal{M}^{[\perp][\perp]}=(\mathcal{M}^{[\perp]})^{[\circ]}$. Consequently, $\mathcal{M}[\dot{+}]\mathcal{N}$ would again be neutral and would contain $\mathcal{M}$ properly.

Next, we show "(2) ⇒ (3)". For definiteness, assume that $\mathcal{M}^{[\perp]}$ is positive semidefinite. Let $\mathcal{N}$ be a negative semidefinite subspace with $\mathcal{M}\subseteq\mathcal{N}$. Due to Corollary 1.4.6, we get $\mathcal{M}\subseteq(\mathcal{N})^{[\circ]}\subseteq\mathcal{N}^{[\perp]}$, and in turn

$$\mathcal{N}\subseteq\mathcal{N}^{[\perp][\perp]}\subseteq\mathcal{M}^{[\perp]}\,.$$

$\mathcal{M}^{[\perp]}$ being positive semidefinitem and $\mathcal{N}$ being negative semidefinite yields that $\mathcal{N}$ is neutral. Consequently,

$$\mathcal{N} \subseteq (\mathcal{M}^{[\perp]})^{[\circ]} \subseteq \mathcal{M}^{[\perp][\perp]} = \mathcal{M}.$$

We conclude that $\mathcal{M}$ is maximal negative semidefinite.

The implication "(3) ⇒ (1)" follows from the fact, that any neutral space containing $\mathcal{M}$ is both, negative and positive semidefinite. ❑

I.76. *1.7.14 Remark.* Notice that in the proof of "(2) ⇒ (3)" we have shown that $\mathcal{M}^{[\perp]}$ being positive semidefinite (and $\mathcal{M} = (\mathcal{M}^{[\perp]})^{[\perp]}$) implies that $\mathcal{M}$ is maximal negative semidefinite. The same holds, of course, with "positive" and "negative" exchanged. ◊

In view of this result, the following stronger maximality property of a neutral subspace appears to be natural.

I.26. **1.7.15 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{M}$ be a linear subspace of $\mathcal{V}$. Then we call $\mathcal{M}$ *hypermaximal neutral*, if it is maximal positive semidefinite and maximal negative semidefinite. ◊

Clearly, each hypermaximal neutral subspace is maximal neutral. The converse, however, need not hold. For instance consider the subspace span $\{\binom{0}{1}\}$ in Example 1.4.10.

The following characterization of hypermaximality becomes especially appealing when one remembers that a subspace $\mathcal{M}$ is neutral if and only if $\mathcal{M} \subseteq \mathcal{M}^{[\perp]}$; see Corollary 1.4.6.

I.30. **1.7.16 Lemma.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $\mathcal{M}$ be a linear subspace of $\mathcal{V}$. Then $\mathcal{M}$ is hypermaximal neutral if and only if $\mathcal{M} = \mathcal{M}^{[\perp]}$.*

Proof. Clearly, $\mathcal{M}^{[\perp]} = \mathcal{M}$ yields $\mathcal{M}^{[\perp][\perp]} = \mathcal{M}$. Moreover, $\mathcal{M}^{[\perp]}$ is neutral. By what we said in Remark 1.7.14, $\mathcal{M}$ is maximal positive semidefinite and maximal negative semidefinite.

Conversely, assume that $\mathcal{M}$ is maximal positive semidefinite and maximal negative semidefinite. Then $\mathcal{M}$ is neutral, i.e. $\mathcal{M} \subseteq \mathcal{M}^{[\perp]}$. Due to Proposition 1.7.13, $\mathcal{M}^{[\perp]}$ is semidefinite. By the maximality assumptions on $\mathcal{M}$, we finally get $\mathcal{M} = \mathcal{M}^{[\perp]}$. ❑

I.30post. *1.7.17 Remark.* Due to Corollary 1.6.6 in combination with Remark 1.4.8, $\operatorname{ind}_+ \mathcal{V} = \operatorname{ind}_- \mathcal{V}$ is a necessary condition for the existence of hypermaximal neutral subspaces.

In a space $\mathcal{V}$ satisfying $\operatorname{ind}_+ \mathcal{V} = \operatorname{ind}_- \mathcal{V}$ every maximal neutral subspace is hypermaximal neutral, as can be seen from Proposition 1.7.2 and Corollary 1.6.6. ◊

I.28. *1.7.18 Example.* Let $(\mathcal{H}, [.,.])$ be a Hilbert space, and consider the linear space $\mathcal{H} \times \mathcal{H}$. We define

$$[\![(x;y),(a;b)]\!] := i([x,b] - [y,a]),\ (x;y),(a;b) \in \mathcal{H} \times \mathcal{H}.$$

It is easy to check that $[\![.,.]\!]$ is a scalar product.

Let T be a bounded linear operator on $\mathcal{H}$ and denote by T^* its adjoint, i.e., the unique bounded linear operator on $\mathcal{H}$ which satisfies

$$[Tx, y] = [x, T^*y], \quad x, y \in \mathcal{H}. \tag{1.7.7}$$ I.29

Consider the graphs

$$\begin{aligned}\operatorname{graph} T &:= \{(x;y) \in \mathcal{H} \times \mathcal{H} : y = Tx\} \\ \operatorname{graph} T^* &:= \{(x;y) \in \mathcal{H} \times \mathcal{H} : y = T^*x\}\end{aligned}$$

of these operators. (1.7.7) gives

$$[\![(x;y),(a;b)]\!] = i([x,b]-[y,a]) = i([x,T^*a]-[Tx,a]) = 0, \\ (x;y) \in \operatorname{graph} T,\ (a;b) \in \operatorname{graph} T^*,$$

i.e., $\operatorname{graph} T^* \subseteq (\operatorname{graph} T)^{[\![\perp]\!]}$. Conversely, if $(a;b) \in (\operatorname{graph} T)^{[\![\perp]\!]}$, then

$$[x,b] = [Tx,a] = [x,T^*a], \quad x \in \mathcal{H},$$

and hence $b = T^*a$, i.e., $(a;b) \in \operatorname{graph} T^*$. Alltogether, we have

$$\operatorname{graph} T^* = (\operatorname{graph} T)^{[\![\perp]\!]}.$$

In particular, the operator T is symmetric if and only if $\operatorname{graph} T$ is a neutral subspace of $(\mathcal{H} \times \mathcal{H}, [\![.,.]\!])$, and T is selfadjoint if and only if $\operatorname{graph} T$ is hypermaximal neutral.

◊

Chapter 2

Scalar Product Spaces with Topology

chapter_SPSWT

Preliminary version Tue 7 Jan 2014 10:34

We consider scalar product spaces which in addition carry a topology such that the scalar product is continuous. Our focus lies on three particular kinds of such spaces: Pontryagin-, almost Pontryagin-, and Krein spaces. Concerning the geometric setup, these are the main players throughout the book, and we investigate them in some detail. A uniformising concept is the notion of Gram spaces.

§1. Basic consequences of continuity 45
§2. Gram spaces 49
§3. Krein spaces 56
§4. Pontryagin spaces 63
§5. Almost Pontryagin spaces 70
§6. Completions 80
§7. *Almost Pontryagin space completions 86

2.1 Basic Consequences of Continuity

section2.1

Let $(\mathcal{A}, [.,.])$ be a scalar product space, and let O be a topology on $\mathcal{A}$. We are interested in situations when the scalar product is continuous as a map of $\mathcal{A}\times\mathcal{A}$ into $\mathbb{C}$, where $\mathcal{A}\times\mathcal{A}$ is endowed with the product topology $O\times O$. To simplify language, we say that $[.,.]$ *is continuous w.r.t.* $\mathcal{O}$ to express this property. If $\mathcal{O}$ is induced by a positive definite scalar product $(.,.)$ or , more generally, by a norm $\|.\|$ on $\mathcal{A}$, we say that $[.,.]$ *is continuous w.r.t.* $(.,.)$ or $[.,.]$ *is continuous w.r.t.* $\|.\|$, respectively. Note that, if $\mathcal{O}$ is induced by a norm $\|.\|$, the product topology is induced by $\|(x;y)\| := (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}$, the maximum norm $\|(x;y)\| := \max\{\|x\|, \|y\|\}$ or any other equivalent norm.

To start off, a characterization of continuity in concrete terms.

II.1. **2.1.1 Proposition.** *Let* $(\mathcal{A}, [.,.])$ *be a scalar product space, and let* $\|.\|$ *be a norm on* $\mathcal{A}$*. Then the following assertions are equivalent.*

(1) *The scalar product* $[.,.]$ *is continuous w.r.t.* $\|.\|$.

(2) *There exists a constant* $C \geq 0$*, such that*

$$\big|[x,y]\big| \leq C\|x\| \cdot \|y\|, \quad x, y \in \mathcal{A}. \tag{2.1.1}$$ II.2

(3) *There exists a constant* $C \geq 0$*, such that*

$$\big|[x,x]\big| \leq C\|x\|^2, \quad x \in \mathcal{A}. \tag{2.1.2}$$ II.3

Proof. For "(1) $\Leftrightarrow$ (2)" assume that the scalar product is continuous w.r.t. $\|.\|$. Then it is, in particular, continuous at the point $(0;0)$. Hence, there exists a $\delta > 0$ such that

$$\big|[x,y]\big| \leq 1, \quad \|x\|, \|y\| \leq \delta.$$

Set $C := \frac{1}{\delta^2}$. If $x, y \in \mathcal{A}$ with $x = 0$ or $y = 0$, then (2.1.1) holds trivially. If $x, y \neq 0$, we have

$$\Big|\Big[\frac{\delta}{\|x\|}x, \frac{\delta}{\|y\|}y\Big]\Big| \leq 1,$$

and again (2.1.1) follows.

For the implication "(2) $\Leftrightarrow$ (1)", assume that (2.1.1) for some $C \geq 0$. The continuity of $[.,.]$ at any point $(x_0;y_0) \in \mathcal{A}\times\mathcal{A}$ is a consequence of

$$\begin{aligned}\big|[x,y] - [x_0,y_0]\big| &\leq \big|[x - x_0, y]\big| + \big|[x_0, y - y_0]\big| \leq \\ &\leq C\|x - x_0\| \cdot \|y\| + C\|x_0\| \cdot \|y - y_0\| \leq \\ &\leq C\|y_0\| \cdot \|x - x_0\| + C\|x - x_0\| \cdot \|y - y_0\| + C\|x_0\| \cdot \|y - y_0\|.\end{aligned}$$

The implication "(2) $\Rightarrow$ (3)" is trivial. In fact, in (2.1.2) we may use the same constant $C \geq 0$ as provided by (2.1.1). For "(3) $\Rightarrow$ (2)" let $C \geq 0$ be such that (2.1.2) holds true. We are going to show (2.1.1) with the constant $4C$. By sesquilinearity it is enough to show that

$$\big|[x,y]\big| \leq 4C, \quad \|x\|, \|y\| \leq 1, \quad x, y \in \mathcal{A}.$$

Let $x, y \in \mathcal{A}$ with $\|x\|, \|y\| \leq 1$. Using the polar identity, (1.1.1), we have

$$\begin{aligned}4\big|[x,y]\big| &= \big|[x+y, x+y] - [x-y, x-y] + i[x+iy, x+iy] - i[x-iy, x-iy]\big| \leq \\ &\leq C(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2) \leq 4C(\|x\| + \|y\|)^2 \leq 16C.\end{aligned}$$

❑

We will see a large class of examples for continuous scalar products later on, cf. Example 2.2.4. Right now let us only observe that not for every scalar product $[.,.]$ there exists a norm $\|.\|$ such that $[.,.]$ is continuous w.r.t. $\|.\|$.

II.6. *2.1.2 Example.* As in Example 1.3.7 consider the linear space of all left-finite two-sided sequences

Ich habs mal two-sided sequences genannt, weil bouble seuqence schon vergeben!

$$\mathcal{V} := \{(\alpha_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : \ \exists N \in \mathbb{N} : \alpha_j = 0, j < -N\},$$

with the scalar product

$$[(\alpha_j)_{j\in\mathbb{Z}}, (\beta_j)_{j\in\mathbb{Z}}] := \sum_{j\in\mathbb{Z}} \alpha_j \cdot \overline{\beta_{-j-1}}, \quad (\alpha_j)_{j\in\mathbb{Z}}, (\beta_j)_{j\in\mathbb{Z}} \in \mathcal{V}.$$

Assume that $\|.\|$ is a norm on $\mathcal{V}$ such that $[.,.]$ is continuous w.r.t. $\|.\|$, and let $C \geq 0$ be as in (2.1.1). For $e_k := (\delta_{kj})_{j\in\mathbb{Z}}$ consider the sequence $a := (\alpha_j)_{j\in\mathbb{Z}}$ defined as

$$\alpha_j := \begin{cases} (j+1)\|e_{-j-1}\| & , \quad j \geq 0 \\ 0 & , \quad j < 0 \end{cases}.$$

For each $k \in \mathbb{N}_0$,

$$(k+1)\|e_{-k-1}\| = \alpha_k = [a, e_{-k-1}] \leq C\,\|a\| \cdot \|e_{-k-1}\|.$$

Hence, we get the obvious contradiction $k+1 \leq C\|a\|$ for all $k \in \mathbb{N}$. ◊

A slight modification of the above argument shows that there exists no locally convex topology $\mathcal{O}$ on $\mathcal{V}$, such that $[.,.]$ is continuous w.r.t. $\mathcal{O}$.

Next, some immediate properties of continuous scalar products.

II.4. **2.1.3 Lemma.** *Let $(\mathcal{A}, [.,.])$ be a scalar product space, let $\|.\|$ be a norm on $\mathcal{A}$, and assume that $[.,.]$ is continuous w.r.t. $\|.\|$. Then the following assertions hold.*

(1) *For each $y \in \mathcal{A}$ the functional*

$$[.,y] : \begin{cases} \mathcal{A} & \to & \mathbb{C} \\ x & \mapsto & [x,y] \end{cases}$$

is continuous.

(2) *For each subset $M \subseteq \mathcal{A}$, the orthogonal complement $M^{[\perp]}$ is a closed subspace. Moreover, $M^{[\perp]} = c\ell(M)^{[\perp]}$, where $c\ell(M)$ denotes the closure of M w.r.t. $\|.\|$.*

Fuer den Abschluss nun der Befehl $c\ell$ statt dem Strich drueber!

(3) *The isotropic part $\mathcal{L}^{[\circ]}$ of any closed linear subspace $\mathcal{L}$ of $\mathcal{A}$ is closed.*

(4) *Let $\mathcal{L}$ be a finite dimensional and nondegenerated linear subspace of $\mathcal{A}$. Then the orthogonal projection P of $\mathcal{A}$ onto $\mathcal{L}$ is continuous.*

Proof. The first assertion follows from (2.1.1). In fact, the norm of $[.,y]$ does not exceed $C\|y\|$. Item (2) follows from

$$M^{[\perp]} = \bigcap_{y\in M} \ker[.,y],$$

and the fact that, due to continuity $[x,y] = 0,\ y \in M$ implies $[x,y] = 0,\ y \in c\ell(M)$ for any fixed $x \in \mathcal{A}$. Item (3) follows from $\mathcal{L}^{[\circ]} = \mathcal{L} \cap \mathcal{L}^{[\perp]}$, and (4) from the representation of P in Remark 1.3.4. ❑

The following continuity property of the index of positivity (negativity) is less obvious.

topvecsp. **2.1.4 Proposition.** *Let* $(\mathcal{A}, [.,.])$ *be a scalar product space, let* $\|.\|$ *be a norm on* $\mathcal{A}$*, and assume that* $[.,.]$ *is continuous w.r.t.* $\|.\|$*. Let* $\mathcal{L}$ *be a linear subspace of* $\mathcal{A}$*, and denote by* $c\ell(\mathcal{L})$ *the closure of* $\mathcal{L}$ *w.r.t.* $\|.\|$*. Then* $\operatorname{ind}_+ c\ell(\mathcal{L}) = \operatorname{ind}_+ \mathcal{L}$ *and* $\operatorname{ind}_- c\ell(\mathcal{L}) = \operatorname{ind}_- \mathcal{L}$*.*

We give the proof along the ideas presented in Proposition 1.6.8, and use the determinantal criterion of Sylvester for positive definiteness of a quadratic form; cf. .

Missing Reference

Proof (of Proposition 2.1.4*).* The inequalities "≥" is an immediate consequence of $\mathcal{L} \subseteq c\ell(\mathcal{L})$; cf. Proposition 1.6.7. We show the reverse inequality for the index of positivity. The reverse inequality for the index of negativity is treated in the same way. Let $n \in \mathbb{N}$ with $n \leq \operatorname{ind}_+ c\ell(\mathcal{L})$, and choose a positive definite subspace $\mathcal{M}$ of $\mathcal{A}$ with $\dim \mathcal{M} = n$. For any basis $\{x_1, \dots, x_n\}$ of $\mathcal{M}$ by Proposition 1.6.8 we have $\operatorname{ind}_+([x_j, x_i])_{i,j=1}^n = n$, i.e., the quadratic form

$$Q(\alpha_1, \dots, \alpha_n) := \sum_{i,j=1}^{n} \overline{\alpha_i} \cdot [x_j, x_i] \cdot \alpha_j$$

is positive definite. Sylvester's criterion gives

$$D_m := \det\big([x_j, x_i]\big)_{i,j=1}^m > 0, \quad m = 1, \dots, n\,.$$

Since $\mathcal{L}$ is dense in $c\ell(\mathcal{L})$ and D_m depends continuously on $x_1, \dots, x_n$, we find elements $x'_1, \dots, x'_n \in \mathcal{L}$ with

$$D'_m := \det\big([x'_j, x'_k]\big)_{j,k=1}^m > 0, \quad m = 1, \dots, n\,.$$

Again appealing to Sylvester's criterion, $\operatorname{ind}_+([x'_j, x'_i])_{i,j=1}^n = n$ and hence, the space $\operatorname{span}\{x'_1, \dots, x'_n\}$ $(\subseteq \mathcal{L})$ is positive definite. We conclude that $n \leq \operatorname{ind}_+ \mathcal{L}$. Passing to the supremum over n, yields $\operatorname{ind}_+ c\ell(\mathcal{L}) \leq \operatorname{ind}_+ \mathcal{L}$. ❑

closvondef. *2.1.5 Remark.* Since positive (negative) semidefiniteness of a subspace $\mathcal{L}$ can be characterized by $\operatorname{ind}_- \mathcal{L} = 0$ $(\operatorname{ind}_+ \mathcal{L} = 0)$, we get from Proposition 2.1.4 that the closure of positive (negative) semidefinite subspaces is again positive (negative) semidefinite. ◊

We close this section with a somewhat more specific lemma which is often useful. Thereby, we call a family $\mathcal{F}$ of linear functionals on a linear space $\mathcal{A}$ *point separating*, if for each $x \in \mathcal{A} \setminus \{0\}$ there exists $\varphi \in \mathcal{F}$ with $\varphi(x) \neq 0$. By linearity, this is equivalent to

$$\forall\, x, y \in \mathcal{A}, x \neq y\ \exists \varphi \in \mathcal{F} : \quad \varphi(x) \neq \varphi(y) \tag{2.1.3}$$

II.5pre

II.5. **2.1.6 Lemma.** *Let* $\mathcal{A}$ *be a linear space, and let* $\|.\|_1$ *and* $\|.\|_2$ *be norms on* $\mathcal{A}$ *which both turn* $\mathcal{A}$ *into a Banach space. If there exists a point separating family of functionals on* $\mathcal{A}$ *which are all continuous with respect to both norms, then* $\|.\|_1$ *and* $\|.\|_2$ *are equivalent.*

Proof. We check that the identity map $\mathrm{id}_{\mathcal{A}} : (\mathcal{A}, \|.\|_1) \to (\mathcal{A}, \|.\|_2)$ has closed graph. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $x_n \to x$ w.r.t. $\|.\|_1$ and $x_n \to y$ w.r.t. $\|.\|_2$. Let φ be a functional on $\mathcal{A}$ which is continuous w.r.t. to both norms. Then we have $\varphi(x) = \lim_{n\to\infty} \varphi(x_n) = \varphi(y)$. By the present assumption, the totality of all such functionals is sufficiently rich to ensure that $x = y$.

The closed graph yields the continuity of $\mathrm{id}_{\mathcal{A}} : (\mathcal{A}, \|.\|_1) \to (\mathcal{A}, \|.\|_2)$. By symmetry also $\mathrm{id}_{\mathcal{A}} : (\mathcal{A}, \|.\|_2) \to (\mathcal{A}, \|.\|_1)$ is continuous. ❑

The significance of this fact at the present stage is the following application. Another interesting application will be seen later, cf. . Missing Lokal Reference

nureinetopo. **2.1.7 Corollary.** *Let $(\mathcal{A}, [.,.])$ be a scalar product space, let $\|.\|_1$ and $\|.\|_2$ be norms on $\mathcal{A}$ which both turn $\mathcal{A}$ into a Banach space, and assume that $[.,.]$ is continuous w.r.t. both of these norms. If $\mathcal{A}^{[\circ]} = \{0\}$, then $\|.\|_1$ and $\|.\|_2$ are equivalent.*

Proof. Since $\mathcal{A}$ is nondegenerated, the family $\{[.,y] : y \in \mathcal{A}\}$ is point separating. ❑

2.2 Gram Spaces

section2.2

Concerning the geometrical setup, the main players in this book are scalar product spaces whose scalar product is continuous with respect to a well-behaved topology.

Definition and Examples

II.8. **2.2.1 Definition.** Let $\mathcal{A}$ be a linear space, and let $\mathcal{O}$ be a topology on $\mathcal{A}$. We call $\mathcal{O}$ a *Hilbert space topology* on $\mathcal{A}$, if there exists a scalar product $(.,.)$ on $\mathcal{A}$ such that $(\mathcal{A}, (.,.))$ is a Hilbert space and the topology induced by $(.,.)$ equals $\mathcal{O}$.

Given a Hilbert space topology $\mathcal{O}$ on $\mathcal{A}$, we call a scalar product $(.,.)$ *compatible with* $\mathcal{O}$, if it turns $\mathcal{A}$ into a Hilbert space and induces $\mathcal{O}$. ◊

Of course, given Hilbert space topology $\mathcal{O}$ on $\mathcal{A}$, the Hilbert space scalar product $(.,.)$ which is compatible with $\mathcal{O}$ is not uniquely determined – only the corresponding norms are equivalent.

g-space. **2.2.2 Definition.** We call a triple $(\mathcal{A}, [.,.], \mathcal{O})$ a *Gram space*, if

(1) $(\mathcal{A}, [.,.])$ is a scalar product space.

(2) $\mathcal{O}$ is a Hilbert space topology on $\mathcal{A}$.

(3) $[.,.]$ is continuous w.r.t. $\mathcal{O}$.

◊

If it is clear from the context to which topology $\mathcal{O}$ and scalar product $[.,.]$ we refer it, then we will drop the explicit notation of $\mathcal{O}$ and $[.,.]$, and speak of a Gram space $\mathcal{A}$.

The terminology “Gram space” originates from the existence of Gram operators. Indeed, given a Gram space $(\mathcal{A}, [.,.], \mathcal{O})$ and a Hilbert space scalar product $(.,.)$ be compatible with $\mathcal{O}$, by the Lax-Milgram theorem (see), there exists a unique bounded and selfadjoint operator G on $(\mathcal{A}, (.,.))$, such that Missing Reference

$$[x, y] = (Gx, y), \quad x, y \in \mathcal{A}. \tag{2.2.1}$$ gramdef

This operator is called the *Gram operator of* $[.,.]$ *w.r.t.* $(.,.)$.

II.9. *2.2.3 Remark.* Our focus lies on the scalar product $[.,.]$, its continuity, and the fact that the topology $\mathcal{O}$ is well-behaved. An actual concrete compatible scalar product $(.,.)$, and with it, a concrete Gram operator G, is not viewed as an intrinsic object. Making a choice of a compatible scalar product is often of great help and will be used heavily, but essential, intrinsic properties of a Gram space should be independent of this choice. ◊

The following example is basic. It exhibits a large class of Gram spaces. The spaces we deal with later on, namely Krein-, almost Pontryagin-, and Pontryagin spaces, are of this kind.

II.7. *2.2.4 Example.* Let $(\mathcal{A}_+, (.,.)_+)$, $(\mathcal{A}_-, (.,.)_-)$, and $(\mathcal{A}_0, (.,.)_0)$, be Hilbert spaces. Set

$$\begin{aligned} &\mathcal{A} := \mathcal{A}_+ \times \mathcal{A}_- \times \mathcal{A}_0 \\ &\big[(x;y;z),(u;v;w)\big] := (x,u)_+ - (y,v)_-, \quad (x;y;z),(u;v;w) \in \mathcal{A}, \\ &\big((x;y;z),(u;v;w)\big) := (x,u)_+ + (y,v)_- + (z,w)_0, \quad (x;y;z),(u;v;w) \in \mathcal{A}. \end{aligned} \tag{2.2.2}$$ II.28

Then $[.,.]$ and $(.,.)$ are scalar products on $\mathcal{A}$, and $(\mathcal{A}, (.,.))$ is a Hilbert space such that

$$\Big|\big[(x;y;z),(x;y;z)\big]\Big| \leq \big((x;y;z),(x;y;z)\big), \quad (x;y;z) \in \mathcal{A}.$$

Hence, $[.,.]$ is continuous w.r.t. to $(.,.)$. Denote by $\mathcal{O}$ the topology induced by $(.,.)$. Then the triple $(\mathcal{A}, [.,.], \mathcal{O})$ forms a Gram space.

The spaces $\mathcal{A}_+, \mathcal{A}_-, \mathcal{A}_0$ are naturally embedded in $\mathcal{A}$, and we always think of them as subspaces of $\mathcal{A}$. In other words, we tacitly identify the direct product $\mathcal{A}_+ \times \mathcal{A}_- \times \mathcal{A}_0$ with the direct sum $\mathcal{A}_+ \dot{+} \mathcal{A}_- \dot{+} \mathcal{A}_0$. As subspaces of $\mathcal{A}$, each of $\mathcal{A}_+$, $\mathcal{A}_-$, and $\mathcal{A}_0$, is closed. Moreover, these subspaces are pairwise orthogonal w.r.t. both scalar products $[.,.]$ and $(.,.)$. We have $\mathcal{A}^{[\circ]} = \mathcal{A}_0$, and $(\mathcal{A}_+, [.,.])$ and $(\mathcal{A}_-, -[.,.])$ are both Hilbert spaces.

Denote by P_+ and P_- the projections of $\mathcal{A}$ onto $\mathcal{A}_+$ and $\mathcal{A}_-$ with kernel $\mathcal{A}_- + \mathcal{A}_0$ and $\mathcal{A}_+ + \mathcal{A}_0$, respectively. Then the Gram operator G of $[.,.]$ w.r.t. $(.,.)$ is given as $G = P_+ - P_-$. ◊

II.29. *2.2.5 Remark.* If $(\mathcal{A}, [.,.])$ is a nondegenerated scalar product space, then there exists at most one topology which turns $\mathcal{A}$ into a Gram space, cf. Corollary 2.1.7. Hence, for nondegenerated spaces, being a Gram space can be seen as a property of the scalar product alone. ◊

It is interesting to observe that generically, for a degenerated space the topology is not unique. This also justifies notating a Gram space as triple $(\mathcal{A}, [.,.], \mathcal{O})$, rather than as tuple $(\mathcal{A}, [.,.])$ and requiring mere existence of $\mathcal{O}$.

II.10. *2.2.6 Example.* Let $(\mathcal{A}_+, (.,.)_+)$, $(\mathcal{A}_-, (.,.)_-)$, and $(\mathcal{A}_0, (.,.)_0)$, be Hilbert spaces, and assume that

$$\dim \mathcal{A}_+ = \infty, \quad \mathcal{A}_0 \neq \{0\}.$$

Let $(\mathcal{A}, [.,.], O)$ be the Gram space defined in (2.2.2).

Choose $c \in \mathcal{A}_0$ with $(c, c)_0 = 1$, and a linear functional $\varphi : \mathcal{A}_+ \to \mathbb{C}$ which is not continuous with respect to $(.,.)_+$. For the existence of such a functional, simply take a linearly independent, countable system $\{b_n : n \in \mathbb{N}\}$ of vectors with $(b_n, b_n)_+ = 1,\ n \in \mathbb{N}$, set $\varphi(b_n) = n$, and continue φ to a linear functional on $\mathcal{A}_+$.

The linear mappings $\Phi, \Psi : \mathcal{A} \to \mathcal{A}$ defined by

$$\Phi(x; y; z) := (x; y; z + \varphi(x)c),\ \Psi(x; y; z) := (x; y; z - \varphi(x)c), \quad (x; y; z) \in \mathcal{A},$$

satisfy $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{id}_{\mathcal{A}}$. In particular, Φ is bijective with Ψ as its inverse. Next, we define a scalar product $\llparenthesis .,. \rrparenthesis$ on $\mathcal{A}$ by

$$\llparenthesis (x; y; z), (u; v; w) \rrparenthesis := (\Phi(x; y; z), \Phi(u; v; w)), \quad (x; y; z), (u; v; w) \in \mathcal{A}.$$

Then $(\mathcal{A}, \llparenthesis .,. \rrparenthesis)$ is a Hilbert space, and Φ is a unitary operator from $(\mathcal{A}, \llparenthesis .,. \rrparenthesis)$ onto $(\mathcal{A}, (.,.))$. Denote by $\mathcal{T}$ the topology induced by $\llparenthesis .,. \rrparenthesis$. Since,

$$\big|[(x; y; z), (x; y; z)]\big| \leq (x, x)_+ + (y, y)_- \leq \llparenthesis (x; y; z), (x; y; z) \rrparenthesis, \quad (x; y; z) \in \mathcal{A},$$

the scalar product $[.,.]$ is continuous w.r.t. $\mathcal{T}$. Thus $(\mathcal{A}, [.,.], \mathcal{T})$ is a Gram space. However,

$$\llparenthesis (x; 0; 0), (0; 0; c) \rrparenthesis = \varphi(x), \quad x \in \mathcal{A}_+,$$

yields the continuity of φ w.r.t. $\mathcal{T}|_{\mathcal{A}_+}$. This shows $\mathcal{T} \neq O$. ◊

finitedimgram. *2.2.7 Remark.* Since all norms on a finite dimensional vector space are equivalent, for finite dimensional Gram spaces $(\mathcal{A}, [.,.], O)$ the topology O is uniquely determined. In particular, if $[.,.]$ is a nondegenerated scalar product on $\mathcal{A}$ and $b_1, \ldots, b_n$ is a basis of $\mathcal{A}$, then

$$\langle x, y \rangle := \sum_{j=1}^{n} [x, b_n][b_n, y], \quad x, y \in \mathcal{A},$$

constitutes a Hilbert space scalar product on $\mathcal{A}$ inducing O. ◊

Properties and Constructions

laxmilgr. **2.2.8 Proposition.** *The following constructions can be carried out within the class of Gram spaces.*

(1) *Let $(\mathcal{A}, [.,.], O)$ be a Gram space, and let $\mathcal{L}$ be a closed linear subspace of $\mathcal{A}$. Then $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}}, O|_{\mathcal{L}})$ is a Gram space.*

Let $(.,.)$ be a compatible scalar product, G the Gram operator of $[.,.]$ w.r.t. $(.,.)$, and denote by P the $(.,.)$-orthogonal projection of $\mathcal{A}$ onto $\mathcal{L}$. Then $(.,.)|_{\mathcal{L}\times\mathcal{L}}$ is compatible with the subspace topology $O|_{\mathcal{L}}$, and the Gram operator of $[.,.]|_{\mathcal{L}\times\mathcal{L}}$ w.r.t. $(.,.)|_{\mathcal{L}\times\mathcal{L}}$ is $PG|_{\mathcal{L}}$.

(2) *For each $j \in \{1,\dots,n\}$ let $(\mathcal{A}_j,[.,.]_j,\mathcal{O}_j)$ be a Gram space, and denote by $[.,.]$ the sum scalar product on $\prod_{j=1}^n \mathcal{A}_j$, i.e.,*

$$[(x_1;\dots;x_n),(y_1;\dots;y_n)] := \sum_{j=1}^n [x_j,y_j]_j, \quad x_j,y_j \in \mathcal{A}_j . \tag{2.2.3}$$ II.31

Then $\left(\prod_{j=1}^n \mathcal{A}_j,[.,.],\prod_{j=1}^n \mathcal{O}_j\right)$ is a Gram space.

For $j \in \{1,\dots,n\}$ let $(.,.)_j$ be a compatible scalar product on $\mathcal{A}_j$, and let G_j be the Gram operator of $[.,.]_j$ w.r.t. $(.,.)_j$. Then the sum scalar product

$$((x_1;\dots;x_n),(y_1;\dots;y_n)) := \sum_{j=1}^n (x_j,y_j)_j, \quad x_j,y_j \in \mathcal{A}_j ,$$

is a Hilbert space scalar product on $\prod_{j=1}^n \mathcal{A}_j$, and it is compatible with the product topology $\prod_{j=1}^n \mathcal{O}_j$. The Gram operator of $[.,.]$ w.r.t. $(.,.)$ is, written as a block operator matrix, $\operatorname{diag}(G_1,\dots,G_n)$.

vielleicht eine art von homomorphiesatz anfuegen

(3) *Let $(\mathcal{A},[.,.],\mathcal{O})$ be a Gram space, and let $\mathcal{B}$ be a closed subspace of $\mathcal{A}^{[\circ]}$. Then a scalar product $[\![.,.]\!]$ on $\mathcal{A}/_{\mathcal{B}}$ is well-defined by*

$$[\![x+\mathcal{B},y+\mathcal{B}]\!] := [x,y], \quad x,y \in \mathcal{A} . \tag{2.2.4}$$ II.32

Moreover, there is a unique Hilbert space topology $\mathcal{O}/_{\mathcal{B}}$ on $\mathcal{A}/_{\mathcal{B}}$ such that the canonical projection $\pi : \mathcal{A} \to \mathcal{A}/_{\mathcal{B}}$ is continous and maps open subsets onto open subsets, and $(\mathcal{A}/_{\mathcal{B}},[\![.,.]\!],\mathcal{O}/_{\mathcal{B}})$ is a Gram space.

Let $(.,.)$ be a compatible scalar product, G the Gram operator of $[.,.]$ w.r.t. $(.,.)$, and denote by $P_{\mathcal{B}^{(\perp)}}$ the $(.,.)$-orthogonal projection of $\mathcal{A}$ onto $\mathcal{B}^{(\perp)}$. Then

$$(x+\mathcal{B},y+\mathcal{B}) := (P_{\mathcal{B}^{(\perp)}}x,P_{\mathcal{B}^{(\perp)}}y), \ \ x,y \in \mathcal{A} , \tag{2.2.5}$$ hilbprodafac

constitutes a Hilbert space scalar product on $\mathcal{A}/_{\mathcal{B}}$ compatible with $\mathcal{O}/_{\mathcal{B}}$, and the Gram operator of $[\![.,.]\!]$ w.r.t. $(.,.)$ on $\mathcal{A}/_{\mathcal{B}}$ is given by $x+\mathcal{B} \mapsto (Gx)+\mathcal{B}$.

Proof.
Subspaces: Since $\mathcal{L}$ is closed, $(\mathcal{L},(.,.)|_{\mathcal{L}\times\mathcal{L}})$ is a Hilbert space. Moreover,

$$[x,y] = (Gx,y) = (PG|_{\mathcal{L}}x,y)|_{\mathcal{L}\times\mathcal{L}}, \quad x,y \in \mathcal{L} .$$

Products: It is enough to compute

$$[(x_1;\dots;x_n),(y_1;\dots;y_n)] = \sum_{j=1}^n [x_j,y_j]_j = \\ = \sum_{j=1}^n (G_jx,y)_j = ((G_1x_1;\dots;G_nx_n),(y_1;\dots;y_n)) .$$

Factors: For a compatible scalar product $(.,.)$ on $\mathcal{A}$, denote by $P_{\mathcal{B}^{(\perp)}}$ the $(.,.)$-orthogonal projection of $\mathcal{A}$ onto $\mathcal{B}^{(\perp)}$. Then the factor space $\mathcal{A}/_{\mathcal{B}}$ can be identified with the closed subspace $\mathcal{B}^{(\perp)}$ via the map

$$\psi : x+\mathcal{B} \mapsto P_{\mathcal{B}^{(\perp)}}x, \quad x \in \mathcal{A} .$$

Consequently, (2.2.5) defines a Hilbert space scalar product inducing the Hilbert space topology $\mathcal{O}/_{\mathcal{B}}$ on $\mathcal{A}/_{\mathcal{B}}$. Since $\mathcal{A}$ as a Hilbert space can be identified with $\mathcal{B}\times\mathcal{B}^{(\perp)}$, $P_{\mathcal{B}^{(\perp)}} = \psi\circ\pi$ is continuous and open. Hence π also has these properties. The uniqueness of $\mathcal{O}/_{\mathcal{B}}$ with these properties immediately follows from the Open Mapping Theorem; see . Finally, $x+\mathcal{B}\mapsto(Gx)+\mathcal{B}$ is well defined because of Missing Reference $\mathcal{B}\subseteq\mathcal{A}^{[\circ]} = \ker G$, and due to

$$[\![x+\mathcal{B}, y+\mathcal{B}]\!] = [x,y] = (Gx,y) = (Gx, P_{\mathcal{B}^{(\perp)}}y) = (P_{\mathcal{B}^{(\perp)}}Gx, P_{\mathcal{B}^{(\perp)}}y) = ((Gx)+\mathcal{B}, y+\mathcal{B}),$$

it is the Gram operator of $[\![.,.]\!]$ w.r.t. $(.,.)$ on $\mathcal{A}/_{\mathcal{B}}$. ❑

II.56gram. *2.2.9 Remark.* As mentioned in Remark 1.2.5 for linear subspaces $\mathcal{L}_1,\dots,\mathcal{L}_n$ of a scalar product space $\mathcal{V}$ satisfying $\mathcal{L}_i[\perp]\mathcal{L}_j$, $\mathcal{L}_i\cap\mathcal{L}_j=\{0\}$, $i\neq j$, the scalar product subspace $\mathcal{L}_1[\dot+]\dots[\dot+]\mathcal{L}_n$ of $\mathcal{V}$ can be identified in a natural way with the scalar product space $\prod_{i=1}^n\mathcal{L}_i$.

For Gram spaces a similar identification can be made. Indeed, let $(\mathcal{A},[.,.],\mathcal{O})$ be a Gram space, and let $\mathcal{L}_1,\dots,\mathcal{L}_n$ be closed subspaces satisfying $\mathcal{L}_i[\perp]\mathcal{L}_j$, $\mathcal{L}_i\cap\mathcal{L}_j=\{0\}$, $\quad i\neq j$ such that $\mathcal{L}_1[\dot+]\dots[\dot+]\mathcal{L}_n$ is a closed subspace of $\mathcal{A}$. By Remark 1.2.5 the map

$$\varphi:\begin{cases}\prod_{i=1}^n\mathcal{L}_i & \to \quad \mathcal{L}_1[\dot+]\dots[\dot+]\mathcal{L}_n\\ (x_1;\dots;x_n) & \mapsto \quad x_1+\dots+x_n\end{cases}$$

is linear, bijective, and isometric when $\prod_{i=1}^n\mathcal{L}_i$ is endowed with the sum scalar product. Providing $\mathcal{L}_i$ with the subspace topologies $\mathcal{O}|_{\mathcal{L}_i}$, φ is clearly continuous, when $\mathcal{L}_1[\dot+]\dots[\dot+]\mathcal{L}_n$ carries the subspace topology $\mathcal{O}|_{\mathcal{L}_1[\dot+]\dots[\dot+]\mathcal{L}_n}$. Since all subspaces under consideration are closed, and hence carry a Hilbert space topology, by the open mapping theorem, , the mapping φ is also bi-continuous. Thus, φ not only retains the Missing Reference algebraic structure, but also the topological structure. ◊

II.11. **2.2.10 Lemma.** *Let $(\mathcal{A},[.,.],\mathcal{O})$ be a Gram space. Let $(.,.)$ be a compatible scalar product, and G the Gram operator of $[.,.]$ w.r.t. $(.,.)$. For each subset $M\subseteq\mathcal{A}$ we have*

$$M^{[\perp]} = \big(G(M)\big)^{(\perp)} = G^{-1}(M^{(\perp)}),$$

where $(M^{(\perp)})$ denotes the inverse image $\{x\in\mathcal{A}: Gx\in M^{(\perp)}\}$ of $M^{(\perp)}$ under G. In particular, we have $\ker G=\mathcal{A}^{[\circ]}$.

Proof. For (1) compute

$$x\in\mathcal{A}^{[\circ]} \Leftrightarrow \forall y\in\mathcal{A}:[x,y]=0 \Leftrightarrow \forall y\in\mathcal{A}:(Gx,y)=0 \Leftrightarrow Gx=0$$

An element x belongs to $M^{[\perp]}$, if and only if $[x,y]=0$ for all $y\in M$. Now we have

$$\forall y\in M:[x,y]=0 \Leftrightarrow \forall y\in M:(x,Gy)=0 \Leftrightarrow x\in(G(M))^{(\perp)}$$
$$\forall y\in M:[x,y]=0 \Leftrightarrow \forall y\in M:(Gx,y)=0 \Leftrightarrow Gx\in M^{(\perp)} \Leftrightarrow x\in G^{-1}(M^{(\perp)})$$

For $M=\mathcal{A}$ we get $\mathcal{A}^{[\circ]}=\mathcal{A}^{[\perp]}=G^{-1}(\mathcal{A}^{(\perp)})=G^{-1}(\{0\})=\ker G$. ❑

One of the great advantages of Gram spaces $(\mathcal{A},[.,.],\mathcal{O})$ is to split up the space $\mathcal{A}$ into subspaces with the help of the spectral measure E of the Gram operator G of $[.,.]$

w.r.t. a given O-compatible Hilbert space scalar product $(.,.)$ on $\mathcal{A}$. Let us recall in the following some of the most important details on spectral measures .
A spectral measure on $\mathbb{R}$ is a mapping E, which assignes to any Borel subset Δ of $\mathbb{R}$ a bounded, linear operator $E(\Delta)$ on $\mathcal{A}$ with the following properties:

Was vom folgen-den in den Appendix und was noch ueber Spectral measure

(*i*) $E(\Delta)$ is a $(.,.)$-orthogonal projection.

(*ii*) $E(\emptyset) = 0$ and $E(\mathbb{R})$ is the identity mapping on $\mathcal{A}$.

(*iii*) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for two Borel subset Δ_1, Δ_2 of $\mathbb{R}$.

(*iv*) For any sequence of pairwise disjoint Borel subsets $\Delta_n,\ n \in \mathbb{N}$, of $\mathbb{R}$ one has

$$E\Big(\dot{\bigcup_{n\in\mathbb{N}}}\Delta_n\Big) = \sum_{n=1}^{\infty} E(\Delta_n)$$

in the strong sense, i.e., $E\Big(\bigcup_{n=1}^{\infty}\Delta_n\Big)x = \sum_{n=1}^{\infty} E(\Delta_n)x$ for all $x \in \mathcal{A}$.

Given spectral measure E on $\mathbb{R}$ for any measureable bounded function $\phi : \mathbb{R} \to \mathbb{C}$ there exists a unique bounded operator $E(\phi)$ on $\mathcal{A}$ such that

$$(E(\phi)x, x) = \int_{\mathbb{R}} \phi(t)\, dE_{x,x}(t), \quad x \in \mathcal{A},$$

where $E_{x,x}$ is the nonnegative measure on $\mathbb{R}$ defined by

$$E_{x,x}(\Delta) := (E(\Delta)x, x), \qquad \Delta \text{ Borel subset of } \mathbb{R}.$$

Denoting by $\mathbb{1}_\Delta$ the indicator function of the set Δ, in particular, $E(\mathbb{1}_\Delta) = E(\Delta)$ for any Borel subset Δ of $\mathbb{R}$. The functional calculus $\phi \mapsto E(\phi)$ is linear and multiplicative.
It is a well known result in Functional Analysis (see) that for any bounded selfadjoint operator G there exists a spectral measure E on $\mathbb{R}$ such that E is supported on the spectrum $\sigma(G)$ of G, i.e., $E(\mathbb{R} \setminus \sigma(G)) = 0$, and such that $E(\mathrm{id}_{\mathbb{R}}) = G$.

Missing Reference

Note here that since $E(\mathbb{R} \setminus \sigma(G)) = 0$, we have

$$\int_{\mathbb{R}} t\, dE_{x,x}(t) = \int_{\mathbb{R}} \mathbb{1}_{\sigma(G)} \cdot t\, dE_{x,x}(t).$$

Hence, $\mathrm{id}_{\mathbb{R}}$ can be identified with the bounded function $t \mapsto \mathbb{1}_{\sigma(G)} \cdot t$, which makes it possible for us to define $E(\mathrm{id}_{\mathbb{R}})$.

Finally, recall that if $E(U) = 0$ for an open subset of $\mathbb{R}$ and $\lambda \in U$, then $\phi : t \mapsto \mathbb{1}_{\mathbb{R}\setminus U}(t) \cdot \frac{1}{t-\lambda}$ satisfies $E(\phi)E(\mathrm{id}_{\mathbb{R}} - \lambda) = E(\mathbb{1}_{\mathbb{R}\setminus U}) = E(\mathbb{R}) = I$. Hence, $\lambda \in \mathbb{R} \setminus \sigma(G)$ and in turn $U \subseteq \mathbb{R} \setminus \sigma(G)$. Thus, $\mathbb{R} \setminus \sigma(G)$ is in fact the maximum of open subsets U of $\mathbb{R}$ with $E(U) = 0$.

II.30pre.

2.2.11 Lemma. *Let $(\mathcal{A}, [.,.], O)$ be a Gram space. Let $(.,.)$ be a compatible scalar product, G the Gram operator of $[.,.]$ w.r.t. $(.,.)$, and denote by E the spectral measure of G. For any Borel subset M of $\mathbb{R}$ and $x \in \operatorname{ran} E(M)$ we have*

$$\inf(M \cap \sigma(G))\,(x,x) \le [x,x] \le \sup(M \cap \sigma(G))\,(x,x).$$

If on the left or on the right hand side we have equality, then $x \in \operatorname{ran} E(\{\inf(M \cap \sigma(G))\})$ or $x \in \operatorname{ran} E(\{\sup(M \cap \sigma(G))\})$, respectively. Under the additional assumption that $\inf(M \cap \sigma(G)) \notin M \cap \sigma(G)$ or $\sup(M \cap \sigma(G)) \notin M \cap \sigma(G)$, respectively, we even have $x = 0$.

Proof. If $M \subseteq \mathbb{R}$ is a Borel set and $x \in \operatorname{ran} E(M)$, then

$$(x,x) = (E(M)x,x) = \int_{\sigma(G)} \mathbb{1}_M \, dE_{x,x} = \int_{M\cap\sigma(G)} \mathbb{1} dE_{x,x} = (E(M\cap\sigma(G))x,x)\,,$$
$$[x,x] = (Gx,x) = (E(M)Gx,x) = \int_{\sigma(G)} t\cdot \mathbb{1}_M(t)\, dE_{x,x}(t) = \int_{M\cap\sigma(G)} t\cdot \mathbb{1}\, dE_{x,x}(t)\,.$$

Since $\inf(M\cap\sigma(G)) \le t \le \sup(M\cap\sigma(G))$ for $t\in M\cap\sigma(G)$, we obtain the mentioned inequalities.

If, say, the right inequality is an equality, i.e., $[x,x] = \tau\,(x,x)$ with $\tau := \sup(M\cap\sigma(G))$, then

$$\int_{M\cap\sigma(G)\setminus\{\tau\}} (\tau - t)\, dE_{x,x}(t) = \int_{M\cap\sigma(G)} (\tau - t)\, dE_{x,x}(t) = 0\,.$$

Since $(\tau - t) > 0$ on $M\cap\sigma(G)\setminus\{\tau\}$, the measure of integration must be the zero measure on $M\cap\sigma(G)\setminus\{\tau\}$. Hence,

$$(x,x) = (E(M\cap\sigma(G))x,x) = (E\{\tau\}x,x) = (E\{\tau\}x, E\{\tau\}x)\,,$$

showing that $x\in \operatorname{ran} E(\{\tau\})$. In case $\tau\notin M\cap\sigma(G)$, we get

$$x \in \operatorname{ran}\Big(E(M\cap\sigma(G))\Big) \cap \operatorname{ran} E(\{\tau\}) = \{0\}\,.$$

Equality on the left is treated in the same way. ❑

II.30. **2.2.12 Proposition.** *Let $(\mathcal{A}, [.,.], O)$ be a Gram space. Let $(.,.)$ be a compatible scalar product, G the Gram operator of $[.,.]$ w.r.t. $(.,.)$, and denote by E the spectral measure of G. Then $\mathcal{A}$ admits the decomposition*

$$\mathcal{A} = \operatorname{ran} E(0,\infty)\ [\dot{+}]\ \operatorname{ran} E(-\infty,0)\ [\dot{+}]\ \underbrace{\operatorname{ran} E\{0\}}_{=\ker G = \mathcal{A}^{[\circ]}}\,. \tag{2.2.6}$$ II.27

(1) *The spaces in the decomposition* (2.2.6) *are closed, and the topology O coincides with the product topology of the subspace topologies $O|_{\operatorname{ran} E(0,\infty)}$, $O|_{\operatorname{ran} E(-\infty,0)}$, and $O|_{\mathcal{A}^{[\circ]}}$, where we identify the above direct sum with the corresponding direct product. Moreover, the sum* (2.2.6) *is also orthogonal w.r.t.* $(.,.)$.

(2) *The space* $(\operatorname{ran} E(0,\infty), [.,.])$ *is positive definite, and* $(\operatorname{ran} E(-\infty,0), [.,.])$ *is negative definite. Consequently,*

$$\operatorname{ind}_+(\mathcal{A},[.,.]) = \dim\operatorname{ran} E(0,\infty),$$
$$\operatorname{ind}_-(\mathcal{A},[.,.]) = \dim\operatorname{ran} E(-\infty,0), \quad \operatorname{ind}_0(\mathcal{A},[.,.]) = \dim\operatorname{ran} E\{0\}\,.$$

(3) *The space* $(\operatorname{ran} E(0,\infty), [.,.])$ *is a Hilbert space if and only if* $\inf(\sigma(G)\cap(0,\infty)) > 0$. *This is certainly true, if* $\operatorname{ind}_+(\mathcal{A},[.,.]) < \infty$.

The space $(\operatorname{ran} E(-\infty,0), -[.,.])$ *is a Hilbert space, if and only if* $\sup(\sigma(G)\cap(-\infty,0)) < 0$. *This is certainly true, if* $\operatorname{ind}_-(\mathcal{A},[.,.]) < \infty$.

Proof. Since $(0,\infty)$, $(-\infty,0)$ and $\{0\}$ constitutes a partition of $\mathbb{R}$, it follows from the properties of spectral measures that we have

$$\mathcal{A} = \operatorname{ran} E(0,\infty) \,(\dot{+})\, \operatorname{ran} E(-\infty,0) \,(\dot{+})\, \operatorname{ran} E\{0\},$$

and item (1) follows; see also Remark 2.2.9.

By Lemma 2.2.11 applied to $(0,+\infty)$ we see that $(\operatorname{ran} E(0,\infty), [.,.])$ is positive definite. Similarly, $(\operatorname{ran} E(-\infty,0)), [.,.])$ is negative definite and $(\operatorname{ran} E\{0\}, [.,.])$ is neutral. (2) now follows after a glance at Proposition 1.6.7.

As $G|_{\operatorname{ran} E((0,\infty))}$ is the Gram operator of $[.,.]$ on $\operatorname{ran} E(0,\infty)$ by Lemma 2.2.10 $\ker G|_{\operatorname{ran} E(0,\infty)} = \{0\}$. Similarly, $\ker G|_{\operatorname{ran} E(-\infty,0)} = \{0\}$. Therefore, $\mathcal{A}^{[\circ]} = \ker G = E\{0\}$.

We come to the proof of the assertions about $\operatorname{ran} E(0,\infty)$ in (3). The assertions about $\operatorname{ran} E(-\infty,0)$ are verified very similar. If $\inf((0,\infty)\cap\sigma(G)) > 0$ by Lemma 2.2.11 the norms $[.,.]^{\frac{1}{2}}$ and $(.,.)^{\frac{1}{2}}$ are equivalent. Since $(.,.)^{\frac{1}{2}}$ turns $\operatorname{ran} E(0,\infty)$ into a Hilbert space, so does $[.,.]^{\frac{1}{2}}$.

Assume conversely that $[.,.]^{\frac{1}{2}}$ turns $\operatorname{ran} E(0,\infty)$ into a Hilbert space. The functionals $[.,y]$, $y \in \operatorname{ran} E(0,\infty)$ are continuous w.r.t. both norms, $[.,.]^{\frac{1}{2}}$ and $(.,.)^{\frac{1}{2}}$, and the family of all these functionals is point separating. Hence, the norms $[.,.]^{\frac{1}{2}}$ and $(.,.)^{\frac{1}{2}}$ are equivalent; cf. Corollary 2.1.7. Let $\varepsilon > 0$ be such that

$$[x,x] \geq \varepsilon(x,x), \quad x \in \operatorname{ran} E(0,\infty).$$

By Lemma 2.2.11 applied to $M = (0,\varepsilon)$ we infer $\operatorname{ran} E(0,\varepsilon) = 0$. From $\sigma(G|_{\operatorname{ran} E(M)}) \subseteq c\ell(M)$ applied for $M = (-\infty,0] \cup [\varepsilon,+\infty)$ we get $(0,\varepsilon)\cap\sigma(G) = 0$.

Finally, assume that $\operatorname{ind}_-(\mathcal{A},[.,.]) = \dim \operatorname{ran} E((0,\infty)) < \infty$. It follows from $E(0,\infty)x = E[1,\sigma(G)]x + \sum_{n=1}^{\infty}[\frac{1}{n+1},\frac{1}{n})x$, that

$$\operatorname{ran} E(0,\infty) = c\ell\,\Big(\bigcup_{n\in\mathbb{N}} \operatorname{ran} E[\frac{1}{n},\sigma(G)]\,\Big).$$

Since the left hand side is finite dimensional, the closure is not necessary, and this union of increasing subspaces, in fact, must concide with one space of the form $\operatorname{ran} E[\frac{1}{n},\sigma(G)]$. This, in turn, yields $E(0,\frac{1}{n}) = 0$. ❑

The basic difference between the general situation and the situation exhibited in Example 2.2.4 is now apparent. In fact, every Gram space can be constructed as a direct and orthogonal sum of a positive definite, a negative definite, and a neutral space, but the two definite summands need not be complete with respect to the norm induced by $[.,.]$ or $-[.,.]$, respectively.

2.3 Krein Spaces

section2.3

We specify a subclass of the class of all Gram spaces.

Definition and Examples

kreinspace. **2.3.1 Definition.** We call a triple $(\mathcal{A}, [.,.], \mathcal{O})$ a *Krein space*, if

(1) $(\mathcal{A}, [.,.], \mathcal{O})$ is a Gram space.

(2) There exists a compatible Hilbert space scalar product, such that the corresponding Gram operator G is bijective, i.e., the point 0 belongs to its resolvent set $\rho(G)$.

$\Diamond$

kreinspacepost. *2.3.2 Example.* Let $(\mathcal{A}, (.,.))$ be a Hilbert space and denote by $\mathcal{O}$ the topology induced by $(.,.)$. For any bounded and selfadjoint operator $K : \mathcal{A} \to \mathcal{A}$ we can consider $G := I + K$ and $[.,.] := (G.,.)$. Clearly, $(\mathcal{A}, [.,.], \mathcal{O})$ is a Gram space.

Assume in addition that K is finite dimensional, i.e., $\dim \operatorname{ran} K < \infty$ or equivalently $\ker K = (\operatorname{ran} K)^{(\perp)}$ is of finite codimension in $\mathcal{A}$. In this case we have $\operatorname{ind}_-(\mathcal{A}, [.,.]) \leq \dim \operatorname{ran} K < \infty$. In fact, if $\mathcal{L}$ is a negative definite subspace, and $x \in \mathcal{L} \cap \ker K$, then

$$[x, x] = ((I + K)x, x) = (x, x)\,,$$

and in turn $x = 0$. Since $\ker K = (\operatorname{ran} K)^{(\perp)}$ has codimension $\dim \operatorname{ran} K$ in $\mathcal{A}$, $\mathcal{L}$ be of dimension at most $\dim \operatorname{ran} K$.

Moreover, for finite dimensional K the Gram operator G is bijective, i.e., $(\mathcal{A}, [.,.], \mathcal{O})$ is a Krein space, if and only if $\ker(I + K) = \{0\}$, i.e. $[.,.]$ is nondegenerated; cf. Lemma 2.2.10.

In fact, since $\ker(I + K) = (\operatorname{ran}(I + K))^{(\perp)}$, the condition $\ker(I + K) = \{0\}$ is equivalent to the density of $\operatorname{ran}(I + K)$ in $\mathcal{A}$. But $\operatorname{ran}(I + K)$ always contains the closed subspace $(I + K)(\ker K) = \ker K$. Hence, as a finite dimensional extension of a closed subspace also $\operatorname{ran}(I + K)$ is closed; see . Therefore, $\ker(I + K) = \{0\}$ is necessary and sufficient for $\operatorname{ran}(I + K) = \mathcal{A}$. $\Diamond$ Missing Reference

The first thing to do, is to show that the property of being a Krein space is an intrinsic property of $(\mathcal{A}, [.,.], \mathcal{O})$. Thereby, we denote by $(\mathcal{A}, \mathcal{O})'$ the *topological dual space* of $\mathcal{A}$ with respect to the topology $\mathcal{O}$, i.e.,

$$(\mathcal{A}, \mathcal{O})' := \{\varphi : \mathcal{A} \to \mathbb{C} \ : \ \varphi \text{ linear and } \mathcal{O}\text{-continuous}\}\,.$$

II.14. **2.3.3 Theorem.** *Let $(\mathcal{A}, [.,.], \mathcal{O})$ be a Gram space. Then the following assertions are equivalent.*

(1) *$(\mathcal{A}, [.,.], \mathcal{O})$ is a Krein space.*

(2) *The topological dual space of $\mathcal{A}$ is given as*

$$(\mathcal{A}, \mathcal{O})' = \{[., y] : \ y \in \mathcal{A}\}\,. \tag{2.3.1}$$ subspgl

(3) *The space $\mathcal{A}$ admits a decomposition $\mathcal{A} = \mathcal{A}_+ [\dot{+}] \mathcal{A}_-$ where $\mathcal{A}_+$ and $\mathcal{A}_-$ are closed, and $(\mathcal{A}_+, [.,.])$ and $(\mathcal{A}_-, -[.,.])$ are Hilbert spaces.*

(4) *There exists a certain O-compatible Hilbert space scalar product* $(.,.)$*, such that for the corresponding Gram operator G of* $[.,.]$ *w.r.t.* $(.,.)$ *and its spectral measure E we have $E(-\epsilon,\epsilon)=0$ for a certain $\epsilon>0$.*

If $(\mathcal{A},[.,.],O)$ is a Krein space, then the condition formulated in Definition 2.3.1*,* (2)*, and assertion* (4) *here hold for every compatible Hilbert space scalar product.*

Proof. The implication "(1) ⇒ (3)" follows from Proposition 2.2.12. Next, assume that (3) holds. We are going to show (2.3.1). Thereby, the inclusion "⊇" is clear since $[.,.]$ is continuous w.r.t. O. In order to show the reverse inclusion, choose a decomposition $\mathcal{A}=\mathcal{A}_+[\dot{+}]\mathcal{A}_-$ with the properties stated in (3). Then

$$(x_+ + x_-, y_+ + y_-) := [x_+, y_+] - [x_-, y_-], \quad x_+, y_+ \in \mathcal{A}_+,\ x_-, y_- \in \mathcal{A}_-$$

defines a Hilbert space scalar product on $\mathcal{A}$. Because of

$$\big|[x_+ + x_-, y_+ + y_-]\big| \le \big|[x_+, y_+]\big| + \big|[x_-, y_-]\big| = (x_+ + x_-, y_+ + y_-)\,, \qquad x_+, y_+ \in \mathcal{A}_+,\ x_-, y_- \in \mathcal{A}_-\,,$$

$[.,.]$ is continuous w.r.t. $(.,.)$. Since $\mathcal{A}$ is nondegenerated (see Lemma 1.2.6), the topology induced by $(.,.)$ coincides with O; see Corollary 2.1.7. Due to the
Missing Reference Riesz-Fischer Theorem, , for any $\varphi\in(\mathcal{A},O)'$ there exists a $y\in\mathcal{A}$ such that $\varphi=(.,y)$. Write $y=y_+ + y_-$ with $y_+\in\mathcal{A}_+$ and $y_-\in\mathcal{A}_-$. Then

$$\varphi(x_+ + x_-) = (x_+ + x_-, y) = [x_+, y_+] - [x_-, y_-] = [x_+ + x_-, y_+ - y_-], \quad x_+\in\mathcal{A}_+, x_-\in\mathcal{A}_-\,,$$

i.e., $\varphi=[., y_+ - y_-]$.

Finally, for the proof of "(2) ⇒ (1)", assume that (2.3.1) holds. Let $(.,.)$ be a compatible scalar product, and G the Gram operator of $[.,.]$ w.r.t. $(.,.)$. For $y\in\mathcal{A}$ we have $(.,y)\in(\mathcal{A},O)'$. Hence, we find $z\in\mathcal{A}$ with $(.,y)=[.,z]$. This implies

$$(x,y) = [x,z] = (x,Gz), \quad x\in\mathcal{A}\,,$$

and hence $Gz=y$. Thus, G is surjective, and from $\ker G = (\operatorname{ran} G^*)^{(\perp)} = \{0\}$ we even see that G is bijective.

Finally, "(1) ⇔ (4)" follows from the fact that $\mathbb{R}\setminus\sigma(G)$ is the maximum of all open subsetes U of $\mathbb{R}$ with $E(U)=0$. ❑

II.33. *2.3.4 Remark.* Let $(\mathcal{A},[.,.])$ be a scalar product space. An obviously necessary condition for a topology O to exist such that $(\mathcal{A},[.,.],O)$ is a Krein space, is that $\mathcal{A}$ is nondegenerated. Hence, if there exists such a topology, it is unique. Consequently, if $(\mathcal{A},[.,.],O)$ is a Krein space, we may drop explicit notation of O, and speak of a Krein space $(\mathcal{A},[.,.])$. The unique topology O which turns $\mathcal{A}$ into a Krein space is called the *Krein space topology* of $(\mathcal{A},[.,.])$. Unless explicitly stated, all topological notions in a Krein space refer to this topology. ◊

Of course, also here our usual abuse of language applies, and we speak of a Krein space $\mathcal{A}$ if it is clear from the context which inner product we refer to.

Let $(\mathcal{A},[.,.])$ be a scalar product space. From the characterization Theorem 2.3.3, (3), we see that $(\mathcal{A},[.,.])$ is a positive definite Krein space if and only if $(\mathcal{A},[.,.])$ is a

Hilbert space. Similarly, $(\mathcal{A}, [.,.])$ is a negative definite Krein space if and only if $(\mathcal{A}, [.,.])$ is a anti-Hilbert space. In terms of the Gram operator G w.r.t. a compatible Hilbert space scalar product this means $\sigma(G) \subseteq [\varepsilon, +\infty)$ $(\sigma(G) \subseteq (-\infty, -\varepsilon])$ for some $\varepsilon > 0$; see Proposition 2.2.12.

gramverkehrt. *2.3.5 Remark.* Let $(\mathcal{A}, [.,.])$ be a Krein space, $(.,.)$ a Hilbert space scalar product which is compatible with the Krein space topology, and G the Gram operator of $[.,.]$ w.r.t. $(.,.)$. Since G is invertible, we have

$$[x, G^{-1}y] = (x, GG^{-1}y) = (x, y) = (GG^{-1}x, y) = [G^{-1}x, y], \ x, y \in \mathcal{A}.$$

Therefore, G^{-1} can be viewed as the *Gram operator of* $(.,.)$ *w.r.t. the Krein space scalar product* $[.,.]$. It easily follows from the nondegeneracy of $[.,.]$, that G^{-1} is the unique operator H on $\mathcal{A}$ satisfying $(x, y) = [Hx, y]$, $x, y \in \mathcal{A}$. ◊

Each decomposition of a Krein space $(\mathcal{A}, [.,.], O)$ as in Theorem 2.3.3, (3), is called a *fundamental decomposition* of $\mathcal{A}$. As seen in Proposition 1.4.12 for every fundamental decomposition $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$ the space $\mathcal{A}_+$ $(\mathcal{A}_-)$ is maximal positive (negative) definite. From Remark 2.2.9 and Proposition 2.2.8 we immediately get the following assertion.

funddecomptop. **2.3.6 Lemma.** *If $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$ is any fundamental decomposition of the Krein space $(\mathcal{A}, [.,.])$, then*

$$(x_+ + x_-, y_+ + y_-) := [x_+, y_+] - [x_-, y_-], \ \ x_+, y_+ \in \mathcal{A}_+, \ x_-, y_- \in \mathcal{A}_-,$$

constitutes a Hilbert space scalar product on $\mathcal{A}$ inducing the unique Hilbert space topology on $\mathcal{A}$. The Gram operator of $[.,.]$ w.r.t. $(.,.)$ then is $G = P_+ - P_-$, where $P_\pm$ is the projection of $\mathcal{A}$ onto $\mathcal{A}_\pm$ with kernel $\mathcal{A}_\mp$. These projections are orthogonal w.r.t. $[.,.]$ and $(.,.)$, are continuous, and satisfy $P_+ = I - P_-$.

We should note that a Krein space has many different fundamental decompositions unless it is positive or negative definite. Still, given a Krein space $(\mathcal{A}, [.,.])$, it may be a hard task to explicitly find a fundamental decomposition.

Let $(\mathcal{A}_+, (.,.))$ and $(\mathcal{A}_-, (.,.)_-)$ be two Hilbert spaces. Set $\mathcal{A}_0 := \{0\}$, and consider the space (2.2.2). Then $(\mathcal{A}, [.,.])$ is a Krein space; a fundamental decomposition being $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$.

II.35. *2.3.7 Example.* Let Ω be a set, Σ a σ-algebra on Ω, and let μ_+ and μ_- be two mutually singular positive measures on Ω. Set $\mathcal{A}_+ := L^2(\mu_+)$, $\mathcal{A}_- := L^2(\mu_-)$, and $\mathcal{A}_0 := \{0\}$, and consider the Krein space (2.2.2). Explicitly $(\mu := \mu_+ + \mu_-)$

$$\mathcal{A} := L^2(\mu_+)\dot{+}L^2(\mu_-) = L^2(\mu),$$
$$[f, g] := \int_\Omega f\bar{g}\, d\mu_+ - \int_\Omega f\bar{g}\, d\mu_- = \int_\Omega f\bar{g}\, d(\mu_+ - \mu_-), \quad f, g \in \mathcal{A}.$$

The Krein space topology of $\mathcal{A}$ is induced by the usual $L^2(\mu)$-scalar product.

A particular case of this example are weighted L^2-spaces. Denote by λ the Lebesgue measure on the real line, and let $w \in L^1(\lambda)$ be real-valued. In the above construction we use

$$d\mu_+ := \max\{w, 0\}\, d\lambda, \quad d\mu_- := -\min\{w, 0\}\, d\lambda.$$

Then

$$\mathcal{A} = L^2(|w|d\lambda)\,,$$
$$[f,g] = \int_{\mathbb{R}} f\bar{g}\,wd\lambda, \quad f,g \in \mathcal{A}\,.$$

◊

Properties and Constructions

In a Hilbert space the double orthogonal complement of a linear subspace equals the closure of this subspace. It is often useful to know that this property also holds in Krein spaces.

II.17. **2.3.8 Lemma.** *Let* $(\mathcal{A},[.,.])$ *be a Krein space, and let* $\mathcal{L}$ *be a linear subspace of* $\mathcal{A}$. *Then*

$$(\mathcal{L}^{[\perp]})^{[\perp]} = c\ell(\mathcal{L})\,.$$

Proof. We use the formulas provided in Lemma 2.2.10 to compute orthogonal complements. Let $(.,.)$ be a compatible scalar product on $\mathcal{A}$, and let G be the Gram operator of $[.,.]$ w.r.t. $(.,.)$. Then

$$(\mathcal{L}^{[\perp]})^{[\perp]} = \Big(G^{-1}(\mathcal{L}^{(\perp)})\Big)^{[\perp]} = \Big[G\Big(G^{-1}(\mathcal{L}^{(\perp)})\Big)\Big]^{(\perp)} = (\mathcal{L}^{(\perp)})^{(\perp)} = c\ell(\mathcal{L})\,.$$

❑

Let us now investigate the constructions of Proposition 2.2.8 within the class of Krein spaces. First, taking factors is meaningless, since a Krein space is always nondegenerated. Second, direct products are straightforward.

II.34. **2.3.9 Lemma.** *For each* $j \in \{1,\dots,n\}$ *let* $(\mathcal{A}_j,[.,.]_j)$ *be a Krein space, and denote by* $[.,.]$ *the sum inner product* (2.2.3) *on* $\prod_{j=1}^n \mathcal{A}_j$. *Then* $(\prod_{j=1}^n \mathcal{A}_j,[.,.])$ *is a Krein space.*

Proof. Choose compatible scalar products $(.,.)_j$ on $\mathcal{A}_j$, then the corresponding Gram operators G_j satisfy $0 \in \rho(G_j)$. Thus also $0 \in \rho(\operatorname{diag}(G_1,\dots,G_n))$. ❑

Taking subspaces is delicate matter. It is a basic observation that a closed subspace of a Krein space, though certainly being a Gram space, is not necessarily a Krein space. The obvious obstacle is that a subspace may be degenerated. However, this is not the only obstacle, and this fact is responsible for many unpleasant geometric pecularities of Krein spaces.

II.15. *2.3.10 Example.* Let $\Omega := \mathbb{N}$ and $\Sigma := \mathcal{P}(\mathbb{N})$, and let $\mu_\pm$ be the measures ($\delta_{\{j\}}$ denotes the Dirac measure concentrated at the point j)

$$\mu_+ := \sum_{j\text{ even}} \delta_{\{j\}}, \quad \mu_- := \sum_{j\text{ odd}} \delta_{\{j\}}\,.$$

Consider the Krein space constructed as in Example 2.3.7 from these data. Explicitly,

$$\mathcal{A} := \ell^2(\mathbb{N})\,,$$
$$[(\alpha_j)_{j\in\mathbb{N}},(\beta_j)_{j\in\mathbb{N}}] := \sum_{j\in\mathbb{N}} (-1)^n \alpha_j\overline{\beta_j}, \quad (\alpha_j)_{j\in\mathbb{N}},(\beta_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N})\,.$$

Consider the linear subspace of $\mathcal{A}$ defined as

$$\mathcal{L} := \Big\{(\alpha_j)_{j\in\mathbb{N}} \in \mathcal{A} : \ \alpha_{2k} = \frac{2k}{2k-1}\alpha_{2k-1}, k \in \mathbb{N}\Big\}.$$

For each $k \in \mathbb{N}$ the projection of a sequence $(\alpha_j)_{j\in\mathbb{N}}$ onto its k-th element α_k is continuous. Hence, $\mathcal{L}$ is closed. Because of

$$[(\alpha_j)_{j\in\mathbb{N}}, (\alpha_j)_{j\in\mathbb{N}}] = \sum_{j\in\mathbb{N}}(-1)^j|\alpha_j|^2 = \sum_{j\in\mathbb{N}}\Big[\Big(\frac{2j}{2j-1}\Big)^2 - 1\Big] \cdot |\alpha_{2j-1}|^2 > 0\,,$$

$$(\alpha_j)_{j\in\mathbb{N}} \in \mathcal{L} \setminus \{0\}\,,$$

$\mathcal{L}$ is positive definite.

We are going to show that $(\mathcal{L}, [.,.])$ is not a Krein space. Denote by $(.,.)$ the usual $\ell^2(\mathbb{N})$-scalar product, and let G be the Gram operator of $[.,.]|_{\mathcal{L}\times\mathcal{L}}$ with respect to $(.,.)|_{\mathcal{L}\times\mathcal{L}}$. Then $G \geq 0$. Setting $\delta := \min\sigma(G)$, we have

$$[x, x] = (Gx, x) \geq \delta(x, x), \quad x \in \mathcal{L}\,.$$

For $n \in \mathbb{N}$ let the sequence $a_n = (\alpha_j^{(n)})_{j\in\mathbb{N}} \in \mathcal{L}$ be defined as

$$\alpha_j^{(n)} := \delta_{2n-1,j} + \frac{2n}{2n-1}\delta_{2n,j}, \quad j \in \mathbb{N}\,.$$

Then $a_n \in \mathcal{L}$, and

$$(a_n, a_n) = 1 + \Big(\frac{2n}{2n-1}\Big)^2, \quad [a_n, a_n] = -1 + \Big(\frac{2n}{2n-1}\Big)^2 = \frac{4n-1}{(2n-1)^2}\,.$$

Hence,

$$\lim_{n\to\infty}(a_n, a_n) = 2, \quad \lim_{n\to\infty}[a_n, a_n] = 0\,.$$

We conclude that $\delta = 0$, i.e., $0 \in \sigma(G)$. ◊

Although primarily interested in Krein spaces, we state the the following definition in the context of Gram spaces for practical reasons.

II.18. **2.3.11 Definition.** Let $(\mathcal{A}, [.,.], \mathcal{O})$ be a Gram space, and let $\mathcal{L}$ be a linear subspace of $\mathcal{A}$. Then we call $\mathcal{L}$ a *Krein subspace* of $(\mathcal{A}, [.,.], \mathcal{O})$, if $\mathcal{L}$ is closed w.r.t. $\mathcal{O}$ and $(\mathcal{L}, [.,.])$ is a Krein space. ◊

A Krein subspace of a Gram space naturally carries its Krein space topology. It is a simple but important fact that this topology coicides with the subspace topology inherited from the Gram space.

II.49. **2.3.12 Lemma.** *Let $(\mathcal{A}, [.,.], \mathcal{O})$ be a Gram space, and let $\mathcal{L}$ be a Krein subspace of $\mathcal{A}$. Then the Krein space topology of $\mathcal{L}$ is equal to $\mathcal{O}|_{\mathcal{L}}$.*

Proof. With both topologies the space $\mathcal{L}$ is a nondegenerated Gram space. Hence, Corollary 2.1.7 yields the assertion. ❑

It is a natural task to geometrically characterize the Krein subspaces of a Krein space (or of a Gram space). This task can be completed in several ways; we give one of them.

existprojection. **2.3.13 Proposition.** *Any Krein subspace $\mathcal{L}$ of a Gram space $(\mathcal{A}, [.,.], \mathcal{O})$ is orthocomplemented. In fact, we have $\mathcal{A} = \mathcal{L}[\dot{+}]\mathcal{L}^{[\perp]}$.*

For a Krein space $(\mathcal{A}, [.,.])$ a subspace $\mathcal{L}$ of $\mathcal{A}$ is a Krein subspace, if and only if $\mathcal{L}$ is orthocomplemented.

Proof. Let $(.,.)$ be a compatible scalar product $(.,.)$ on $\mathcal{A}$, let G be the Gram operator of $[.,.]$ w.r.t. $(.,.)$, and denote by Q the $(.,.)$-orthogonal projection of $\mathcal{A}$ onto $\mathcal{L}$. Moreover, denote the Gram operator of the subspace $\mathcal{L}$ by $G_{\mathcal{L}}$, i.e., $G_{\mathcal{L}} := Q\,G|_{\mathcal{L}}$; see Proposition 2.2.8.

Assume that $\mathcal{L}$ is a Krein subspace. Then $\mathcal{L}$ is closed and $0 \in \rho(G_{\mathcal{L}})$. Define $P : \mathcal{A} \to \mathcal{A}$ as

$$Px := G_{\mathcal{L}}^{-1} Q\,Gx, \quad x \in \mathcal{A},$$

where $G_{\mathcal{L}}^{-1}$ in this formula has to be considered as a mapping from $\mathcal{L}$ into $\mathcal{A}$. Then (see Lemma 2.2.10)

$$P^2 = G_{\mathcal{L}}^{-1}(\underbrace{Q\,G\,G_{\mathcal{L}}^{-1}}_{=\mathrm{id}_{\mathcal{L}}})Q\,G = P$$

$$\ker P = \ker(G_{\mathcal{L}}^{-1} Q\,G) = \ker(Q\,G) = G^{-1}(\ker Q) = G^{-1}(\mathcal{L}^{(\perp)}) = \mathcal{L}^{[\perp]},$$

and

$$\mathcal{L} = \operatorname{ran}(G_{\mathcal{L}}^{-1}(\underbrace{Q\,G|_{\mathcal{L}}}_{=G_{\mathcal{L}}})) \subseteq \operatorname{ran}(G_{\mathcal{L}}^{-1} Q\,G) \subseteq G_{\mathcal{L}}^{-1}(\mathcal{L}) = \mathcal{L}.$$

Thus, P is a projection with range $\mathcal{L}$ and kernel $\mathcal{L}^{[\perp]}$, and it follows that $\mathcal{A} = \mathcal{L}[+]\mathcal{L}^{[\perp]}$; see Proposition 1.3.2. Due to $\{0\} = \mathcal{L}^{[\circ]}$ this sum is direct.

For the second part, assume that $(\mathcal{A}, [.,.])$ is a Krein space, and that $\mathcal{L}$ is an orthocomplemented subspace. Then $\mathcal{L}^{[\circ]} \subseteq \mathcal{A}^{[\circ]} = \{0\}$ and $\mathcal{L}^{[\perp][\perp]} = \mathcal{L} + \mathcal{A}^{[\circ]} = \mathcal{L}$; cf. Proposition 1.3.2. Consequently, $\mathcal{L}$ is closed, and thus, $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}}, \mathcal{O}|_{\mathcal{L}})$ is a Gram space. We check condition (2) of Theorem 2.3.3:

For $\varphi \in (\mathcal{L}, \mathcal{O}|_{\mathcal{L}})'$, $\varphi \circ Q : \mathcal{A} \to \mathbb{C}$ is a linear extension of φ, and hence belongs to $(\mathcal{A}, \mathcal{O})'$. $(\mathcal{A}, [.,.])$ being a Krein space yields the existence of an element $y \in \mathcal{A}$ with $\varphi \circ Q = [.,y]$. Write $y = u + v$ with $u \in \mathcal{L}$ and $v \in \mathcal{L}^{[\perp]}$. Then

$$\varphi(x) = \varphi \circ Q(x) = [x,y] = [x,u], \quad x \in \mathcal{L}.$$

Hence, $\varphi = [.,u]$ with $u \in \mathcal{L}$. ❑

orthgramproj. *2.3.14 Remark.* We'd like to point out from the proof of Proposition 2.3.13, that

$$Px := G_{\mathcal{L}}^{-1} Q\,G\,x, \; x \in \mathcal{A}, \tag{2.3.2}$$ projcont

is the uniquely determined (see Proposition 1.3.2) $[.,.]$-orthogonal projection from a Gram space $(\mathcal{A}, [.,.], \mathcal{O})$ onto the Krein subspace $\mathcal{L}$, where G is the Gram operator of $[.,.]$ on $\mathcal{A}$ w.r.t. to a given, $\mathcal{O}$-compatible Hilbert space scalar product, where Q is the $(.,.)$-orthogonal projection onto $\mathcal{L}$, and where $G_{\mathcal{L}} := Q\,G|_{\mathcal{L}}$ is the Gram operator of $[.,.]|_{\mathcal{L}\times\mathcal{L}}$ on $\mathcal{L}$ w.r.t. to $(.,.)|_{\mathcal{L}\times\mathcal{L}}$. ◊

We know that the direct and orthogonal sum of two orthocomplemented subspaces of a scalar product space is again orthocomplemented; cf. Proposition 1.3.2. Hence, by Proposition 2.3.13 the direct and orthogonal sum of two Krein subspaces of a Krein space is again a Krein subspace. Let us show that this is true also for Krein subspaces of a Gram space.

sumofregsub. **2.3.15 Proposition.** *Let $(\mathcal{A}, [.,.], O)$ be a Gram space. If $\mathcal{L}$ and $\mathcal{M}$ are Krein subspaces of $\mathcal{A}$ with $\mathcal{L}[\perp]\mathcal{M}$, then $\mathcal{L}[\dot{+}]\mathcal{M}$ is a Krein subspace of $\mathcal{A}$.*

Proof. Let $P_{\mathcal{L}}$ and $P_{\mathcal{M}}$ be the $[.,.]$-orthogonal projections of $\mathcal{A}$ onto $\mathcal{L}$ and $\mathcal{M}$, respectively. Then

$$\operatorname{ran} P_{\mathcal{L}} = \mathcal{L} \subseteq \mathcal{M}^{[\perp]} = \ker P_{\mathcal{M}}, \quad \operatorname{ran} P_{\mathcal{M}} = \mathcal{M} \subseteq \mathcal{L}^{[\perp]} = \ker P_{\mathcal{L}},$$

and hence $P_{\mathcal{L}}P_{\mathcal{M}} = P_{\mathcal{M}}P_{\mathcal{L}} = 0$. Consequently,

$$P := P_{\mathcal{L}} + P_{\mathcal{M}}$$

is a projection with $\ker(I - P) = \operatorname{ran} P = \mathcal{L}[\dot{+}]\mathcal{M}$. According to (2.3.2) $P_{\mathcal{L}}$, $P_{\mathcal{M}}$, and hence P are continuous. Hence, $\ker(I - P) = \mathcal{L}[\dot{+}]\mathcal{M}$ is closed.

We saw in Remark 2.2.9 that for the closed subspaces $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{L}[\dot{+}]\mathcal{M}$ of the Gram space $\mathcal{A}$ the product space $\mathcal{L} \times \mathcal{M}$ provided with the sum scalar product and the product topology of the respective subspace topologies is isomorphic (as a Gram space) to the subspace $\mathcal{L}[\dot{+}]\mathcal{M}$ provided with the scalar product inherited from $\mathcal{A}$ and the subspace topology. Therefore, with $\mathcal{L} \times \mathcal{M}$ (see Lemma 2.3.9) also $\mathcal{L}[\dot{+}]\mathcal{M}$ is a Krein space. ❑

2.4 Pontryagin Spaces

section2.4

Definition

II.21. **2.4.1 Definition.** We call a triple $(\mathcal{A}, [.,.], O)$ a *Pontryagin space*, if

(1) $(\mathcal{A}, [.,.], O)$ is a Krein space.

(2) $\operatorname{ind}_-(\mathcal{A}, [.,.]) < \infty$.

◊

The following equivalent characterizations of Pontryagin spaces can be given.

II.22. **2.4.2 Theorem.** *Let $(\mathcal{A}, [.,.], O)$ be a Gram space. Then the following are equivalent.*

(1) *$(\mathcal{A}, [.,.], O)$ is a Pontryagin space.*

(2) *The space $\mathcal{A}$ admits a decomposition $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$ where $\mathcal{A}_+$ is closed, $(\mathcal{A}_+, [.,.])$ is a Hilbert space, and $\mathcal{A}_-$ is finite dimensional and negative definite.*

(3) *There exists a certain O-compatible Hilbert space scalar product $(.,.)$, such that for the corresponding Gram operator G of $[.,.]$ w.r.t. $(.,.)$ and its spectral measure E we have $\dim \operatorname{ran} E(-\infty, 0) < \infty$ and $E[0, \epsilon) = 0$ for a certain $\epsilon > 0$.*

If $(\mathcal{A}, [.,.], O)$ *is a Pontryagin space, then* (4) *is true for every compatible Hilbert space scalar product.*

Proof. The equivalences of (1) to (3) immediately follow from Theorem 2.3.3 and Proposition 2.2.12. ❑

As for a Krein space, the topology of a Pontryagin space $(\mathcal{A}, [.,.], O)$ is uniquely determined by its inner product. Hence, we may again drop explicit notation of O. If $(\mathcal{A}, [.,.])$ is a Pontryagin space, we refer to its Krein space topology also as its *Pontryagin space topology*. Unless explicitly stated, all topological notions refer to this topology.

If the scalar product $[.,.]$ is clear from the context, we again just speak of a Pontryagin space $\mathcal{A}$.

Clearly, a Pontryagin space $(\mathcal{A}, [.,.])$ is a Hilbert space if and only if $\operatorname{ind}_-(\mathcal{A}, [.,.]) = 0$.

founddecpont. *2.4.3 Remark.* Let $(\mathcal{A}, [.,.])$ be a Pontryagin space. Given any fundamental decomposition $\mathcal{A} = \mathcal{A}_+[+]\mathcal{A}_-$, we clearly have $\dim \mathcal{A}_- = \operatorname{ind}_-(\mathcal{A}, [.,.])$, i.e., $\mathcal{A}_-$ is maximal negative definite.

Moreover, as seen in Lemma 2.3.6 every such fundamental decomposition gives rise compatible Hilbert space scalar product $(.,.)$ on $\mathcal{A}$ such that the Gram operator of $[.,.]$ w.r.t. $(.,.)$ is of the form $G = I - 2P_-$ where P_- is the projection of $\mathcal{A}$ onto $\mathcal{A}_-$ along $\mathcal{A}_+$. In particular, G is a finite dimensional perturbation of the identity. ◊

These considerations give raise to the following characterization of Pontryagin spaces.

II.22post. **2.4.4 Theorem.** *A Gram space* $(\mathcal{A}, [.,.], O)$ *is a Pontryagin space if and only if there exists a certain* O*-compatible Hilbert space scalar product* $(.,.)$*, such that the corresponding Gram operator* G *of* $[.,.]$ *w.r.t.* $(.,.)$ *is of the form* $G = I + K$ *for a bounded,* $(.,.)$*-selfadjoint and finite dimensional* $K : \mathcal{A} \to \mathcal{A}$ *with* $\ker(I + K) = \{0\}$.

Proof. As mentioned in Remark 2.4.3 this condition is necessary for $\mathcal{A}$ being a Pontryagin space. Sufficency follows after a glance at Example 2.3.2. ❑

Properties and Constructions

Let us investigate the basic constructions of Proposition 2.2.8. As for Krein spaces, taking factors with respect to subspaces of $\mathcal{A}^{[\circ]}$ is meaningless. Direct products are straightforward to treat.

II.48. **2.4.5 Lemma.** *For each* $j \in \{1, \ldots, n\}$ *let* $(\mathcal{A}_j, [.,.]_j)$ *be a Pontryagin space, and denote by* $[.,.]$ *the sum inner product* (2.2.3) *on* $\prod_{j=1}^n \mathcal{A}_j$. *Then* $(\prod_{j=1}^n \mathcal{A}_j, [.,.])$ *is a Pontryagin space.*

Proof. We know that $(\prod_{j=1}^n \mathcal{A}_j, [.,.])$ is Krein space (see Lemma 2.3.9), and that (see Proposition 1.6.7)

$$\operatorname{ind}_- \prod_{j=1}^n \mathcal{A}_j = \sum_{j=1}^n \operatorname{ind}_- \mathcal{A}_j < \infty .$$

❏

As the class of Krein spaces, also the class of Pontryagin spaces is not closed w.r.t. taking subspaces. Unlike for Krein spaces, the obvious obstacle that $\mathcal{L}$ may be degenerated is the only one. This obstacle however is in some sense not as bad as in Krein spaces since due to Proposition 1.7.2 the index of nullity of $(\mathcal{L}, [.,.])$ is bounded from above by $\operatorname{ind}_-(\mathcal{A}, [.,.])$.

II.45. **2.4.6 Proposition.** *Let $(\mathcal{A}, [.,.])$ be a Pontryagin space, and let $\mathcal{L}$ be a closed linear subspace of $\mathcal{A}$. Then $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}})$ is a Pontryagin space if and only if $\mathcal{L}^{[\circ]} = \{0\}$. In this case, the Pontryagin space topology of $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}})$ coincides with the restriction to $\mathcal{L}$ of the Pontryagin space topology of $(\mathcal{A}, [.,.])$.*

Proof. Clearly, in order that $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}})$ is a Pontryagin space, it is necessary that $\mathcal{L}$ is nondegenerated.

Assume that $\mathcal{L}^{[\circ]} = \{0\}$. We saw in Theorem 2.4.2 that the Gram operator of $[.,.]$ on $\mathcal{A}$ w.r.t. a certain $\mathcal{O}$-compatible Hilbert space scalar product $(.,.)$ is of the form $G = I + K$ with a selfadjoint and finite dimensional operator K. Hence, the Gram operator $G_{\mathcal{L}} = PG|_{\mathcal{L}}$ of $[.,.]|_{\mathcal{L}\times\mathcal{L}}$ with respect to $(.,.)|_{\mathcal{L}\times\mathcal{L}}$ coincides with

$$G_{\mathcal{L}} = I + PK|_{\mathcal{L}}\,;$$

see Proposition 2.2.8. Here P is the $(.,.)$-orthogonal projection onto $\mathcal{L}$. $PK|_{\mathcal{L}}$ is a finite dimensional operator in $\mathcal{L}$. Due to $\ker G_{\mathcal{L}} = \mathcal{L}^{[\circ]} = \{0\}$ we see that $(\mathcal{L}, [.,.]_{\mathcal{L}\times\mathcal{L}}, \mathcal{O}|_{\mathcal{L}})$ is a Pontryagin space; cf. Theorem 2.4.2.

The fact that the Pontryagin space topology of $\mathcal{L}$ equals $\mathcal{O}|_{\mathcal{L}}$ follows from Lemma 2.3.12. ❏

The next assertion is often useful.

II.52. **2.4.7 Lemma.** *Let $(\mathcal{A}, [.,.])$ be a Pontryagin space. Then the following assertions hold true.*

(1) *A closed subspace $\mathcal{L}$ of $\mathcal{A}$ is nondegenerated if and only if $\mathcal{L}$ is orthocomplemented, i.e.,*
$$\mathcal{A} = \mathcal{L}[\dot{+}]\mathcal{L}^{[\perp]}\,. \tag{2.4.1}$$ II.46

(2) *If $\mathcal{L}$ and $\mathcal{M}$ are closed and nondegenerated subspaces of $\mathcal{A}$ with $\mathcal{L}[\perp]\mathcal{M}$, then $\mathcal{L}[+]\mathcal{M}$ is closed and nondegenerated.*

Proof. For (1) apply Proposition 2.3.13, and for (2) Proposition 2.3.15. ❏

The following result continues the discussion of fundamental decompositions of Pontryagin spaces; see Remark 2.4.3.

II.52post. **2.4.8 Corollary.** *Let $(\mathcal{A}, [.,.])$ be a Pontryagin space. Then any maximal negative definite subspace $\mathcal{A}_-$ is of dimension $\operatorname{ind}_-(\mathcal{A}, [.,.])$, and gives raise to a fundamental decomposition $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$, where $\mathcal{A}_+ := \mathcal{A}_-^{[\perp]}$.*

If $\mathcal{D}$ is a dense linear subspace of $\mathcal{A}$, then every maximal negative definite subspace $\mathcal{A}_-$ of $\mathcal{D}$ is maximal negative definite in $\mathcal{A}$. Moreover, for the corresponding fundamental decomposition $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$ we have

$$\mathcal{D} = (\mathcal{A}_+ \cap \mathcal{D})[\dot{+}]\mathcal{A}_- ,$$

where $\mathcal{A}_+ \cap \mathcal{D}$ is densely contained in the Hilbert space $(\mathcal{A}_+, [.,.])$.

Proof. Given a maximal negative definite subspace $\mathcal{A}_-$ we know from Proposition 1.6.2 that $\mathrm{ind}_-(\mathcal{A}, [.,.])$ and that $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$ with a positive definite and closed $\mathcal{A}_+ = \mathcal{A}_-^{[\perp]}$. As a positive definite Pontryagin space $(\mathcal{A}_+, [.,.])$ is a Hilbert space. Cleary, $(\mathcal{A}_-, -[.,.])$ is also a Hilbert space; see Remark 2.2.7. Therefore, $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$ is a fundamental decomposition.

For a dense linear subspace $\mathcal{D}$ of $\mathcal{A}$ we know from Proposition 2.1.4 that $\mathrm{ind}_-(\mathcal{D}, [.,.]) = \mathrm{ind}_-(\mathcal{A}, [.,.]) < \infty$. Hence, any maximal negative definite subspace $\mathcal{A}_-$ of $\mathcal{D}$ is maximal negative definite in $\mathcal{A}$. $\mathcal{D} = \mathcal{A}_-[\dot{+}](\mathcal{A}_+ \cap \mathcal{D})$ follows from a simple algebraic argument, and for the density of $\mathcal{A}_+ \cap \mathcal{D}$ in $\mathcal{A}_+$ see Lemma 2.8.2. ❑

With the help of Corollary 2.4.8 convergence in a Pontryagin space $(\mathcal{A}, [.,.])$ can be characterized only by means of $[.,.]$. The analogous statement in Hilbert spaces is well-known.

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2.4.9 Corollary. *For a Pontryagin space $(\mathcal{A}, [.,.])$ the following assertions hold true.*

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(1) *A subset B of $\mathcal{A}$ is bounded w.r.t. any norm inducing the unique Hilbert space topology $\mathcal{O}$ on $\mathcal{A}$ if and only if the sets $\{[x,x] : x \in B\}$ and $\{[x,y] : x \in B\}$ are bounded in $\mathbb{C}$ for all $y \in M$, where $M \subseteq \mathcal{A}$ with $\mathrm{ind}_-(\mathrm{span}\, M, [.,.]) = \mathrm{ind}_-(\mathcal{A}, [.,.])$.*

(2) *A sequence $(x_n)_{n\in\mathbb{N}}$ converges in $\mathcal{A}$ to an $x \in \mathcal{A}$ w.r.t. the unique Hilbert space topology $\mathcal{O}$ if and only if $\lim_{n\to\infty}[x_n, x_n] = [x,x]$ and $\lim_{n\to\infty}[x_n, y] = [x,y]$ for all $y \in M$, where $M \subseteq \mathcal{A}$ with $c\ell(\mathrm{span}\, M) = \mathcal{A}$.*

Proof. Since $[.,.]$ is $\mathcal{O}$-continuous, the necessity of the condition for boundedness follows from Proposition 2.1.1. The $\mathcal{O}$-continuity of $[.,.]$ also yields the necessity of the condition for the convergence of $(x_n)_{n\in\mathbb{N}}$.

To prove sufficency in (1) and (2), the respective assumptions on M yield the existence of a negative definite subspace $\mathcal{A}_- \subseteq \mathrm{span}\, M$ of dimension $\mathrm{ind}_-(\mathcal{A}, [.,.])$. For (1) this follows from Proposition 1.6.2 and for (2) this follows from Corollary 2.4.8. In any case we have the fundamental decomposition

$$\mathcal{A} = \mathcal{A}_-[\dot{+}]\mathcal{A}_+ ,$$

where $\mathcal{A}_+ = \mathcal{A}_-^{[\perp]}$. Recall that the projections $P_\pm$ onto the respective components are continous. Let $(.,.)$ be the Hilbert space scalar product introduced in Lemma 2.3.6.

In the situation of (1) by linearity

$$\{[x,y] : x \in B\} = \{[P_-x, y] : x \in B\}$$

is bounded for all $y \in \mathcal{A}_-$. Since the norm induced by $(.,.)$ on $\mathcal{A}_-$ is equivalent to the norm induced by the Hilbert space scalar product in Remark 2.2.7 we have $-[P_-x, P_-x] = (P_-x, P_-x) \le C$, $x \in B$, for a certain $C \ge 0$. Boundedness of B now follows from

$$(x,x) = [x,x] - 2[P_-x, P_-x] \le \sup_{x\in B}[x,x] + 2C < +\infty, \quad x \in B.$$

Finally, we turn to the sufficency in (1). By linearity $\lim_{n\to\infty}[x_n, y] = [x, y]$ for all $y \in \mathcal{A}_- \subseteq \operatorname{span} M$. Again employing Remark 2.2.7 yields $\lim_{n\to\infty} P_-x_n = P_-x$. Consequently,

$$(P_+x_n, P_+x_n) = [x_n, x_n] - 2(P_-x_n, P_-x_n) \to [x,x] - 2(P_-x, P_-x) = (P_+x, P_+x),$$

and

$$(P_+x_n, y) = [x_n, y] \to [x, y] = (P_+x, y), \quad \text{for all } \ y \in \mathcal{A}_+ \cap \mathcal{D}.$$

Due to Corollary 2.4.8 the space $\mathcal{A}_+ \cap \mathcal{D}$ is dense in $\mathcal{A}_+$. By the classical Hilbert space version of (2) (see) we get $\lim_{n\to\infty} P_+x_n = P_+x$, and hence

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$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} P_+x_n + P_-x_n = P_+x + P_-x = x.$$

❑

If $\mathcal{L}$ is a degenerated subspace of a Pontryagin space, of course, $\mathcal{L}$ cannot be orthocomplemented. In the next statement we give an analogue of the decomposition (2.4.1) which is valid for degenerated subspaces.

II.47. **2.4.10 Theorem.** *Let $(\mathcal{A}, [.,.])$ be a Pontryagin space, and let $\mathcal{L}$ be a closed linear subspace of $\mathcal{A}$. Then the following hold true.*

(1) *Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces of $\mathcal{A}$ with*

$$\mathcal{L} = \mathcal{M}[\dot+]\mathcal{L}^{[\circ]}, \quad \mathcal{L}^{[\perp]} = \mathcal{N}[\dot+]\mathcal{L}^{[\circ]}. \tag{2.4.2}$$ II.50

Then there exists a subspace $\mathcal{W}$ of $\mathcal{A}$ which is skewly linked with $\mathcal{L}^{[\circ]}$, such that

$$\mathcal{A} = \mathcal{M}[\dot+](\mathcal{L}^{[\circ]}\dot+\mathcal{W})[\dot+]\mathcal{N}. \tag{2.4.3}$$ II.51

(2) *Let $\mathcal{W}$ be a subspace of $\mathcal{A}$ which is skewly linked with $\mathcal{L}^{[\circ]}$. Then there exist unique closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{A}$ such that* (2.4.2) *and* (2.4.3) *hold.*

Note that, since $\mathcal{L}^{[\circ]}$ is a closed subspace of the closed subspace $\mathcal{L}$, closed subspaces $\mathcal{M}$ and $\mathcal{N}$ as in Theorem 2.4.10, (1) always can be found; simply take, for example, the $(.,.)$-orthogonal complement of $\mathcal{L}^{[\circ]}$ within $\mathcal{L}$ for any Hilbert space scalar product $(.,.)$. Since $\mathcal{L}^{[\circ]}$ is finite dimensional and neutral, due to Proposition 1.7.4 also a subspace $\mathcal{W}$ as in Theorem 2.4.10, (2), can be found. Indeed, given a closed subspace, there always exist many corresponding decompositions of $\mathcal{A}$.

Proof (of Theorem 2.4.10*).* For the proof of (1), let $\mathcal{M}$ and $\mathcal{N}$ be given with the properties stated in (1). Then $\mathcal{M}$ and $\mathcal{N}$ are nondegenerated and $\mathcal{M}[\perp]\mathcal{N}$. Hence, the space

$$C := \mathcal{M}[\dot+]\mathcal{N}$$

is closed and orthocomplemented. The space $\mathcal{C}^{[\perp]}$ is nondegenerated and contains $\mathcal{L}^{[\circ]}$. Hence, there exists a subspace $\mathcal{W} \subseteq \mathcal{C}^{[\perp]}$ which is skewly linked with $\mathcal{L}^{[\circ]}$; see Proposition 1.7.4.

The space $\mathcal{L}^{[\circ]} \dot{+} \mathcal{W}$ is finite dimensional, and hence the sum

$$\mathcal{D} := \mathcal{C}[\dot{+}](\mathcal{L}^{[\circ]} \dot{+} \mathcal{W})$$

is closed; see Lemma 2.4.7. In order to establish the equality (2.4.3), it is thus enough to show that $\mathcal{D}^{[\perp]} = \{0\}$; cf. Lemma 2.3.8. To this end, let $x \in \mathcal{D}^{[\perp]}$ be given. Then $x[\perp](\mathcal{M} + \mathcal{L}^{[\circ]}) = \mathcal{L}$. Hence $x \in \mathcal{L}^{[\perp]}$, and we may write $x = x_1 + x_0$ with $x_1 \in \mathcal{N}$ and $x_0 \in \mathcal{L}^{[\circ]}$. We have $x, x_0[\perp]\mathcal{N}$ and thus $x_1 = 0$. We conclude that $x \in \mathcal{L}^{[\circ]}$, and now $x[\perp]\mathcal{W}$ implies $x = 0$.

We come to the proof of (2). Let a subspace $\mathcal{W}$, which is skewly linked with $\mathcal{L}^{[\circ]}$, be given. Set

$$\mathcal{M} := \mathcal{L} \cap (\mathcal{L}^{[\circ]} \dot{+} \mathcal{W})^{[\perp]}, \quad \mathcal{N} := \mathcal{L}^{[\perp]} \cap (\mathcal{L}^{[\circ]} \dot{+} \mathcal{W})^{[\perp]} .$$

Then $\mathcal{M}$ and $\mathcal{N}$ are closed. Since $\mathcal{M}, \mathcal{N}[\perp]\mathcal{W}$, we also have $\mathcal{M} \cap \mathcal{L}^{[\circ]} = \mathcal{N} \cap \mathcal{L}^{[\circ]} = \{0\}$.

Next we show the equalities (2.4.2). To this end, notice that the space $\mathcal{L}^{[\circ]} \dot{+} \mathcal{W}$ is finite dimensional and nondegenerated, and therefore orthocomplemented. Hence, every element $x \in \mathcal{A}$ can be written as $x = x_1 + x_2$ with

$$x_1 \in \mathcal{L}^{[\circ]} \dot{+} \mathcal{W}, \quad x_2 \in (\mathcal{L}^{[\circ]} \dot{+} \mathcal{W})^{[\perp]} .$$

Assume $x \in \mathcal{L}$. Then we have $x, x_2[\perp]\mathcal{L}^{[\circ]}$, hence also $x_1[\perp]\mathcal{L}^{[\circ]}$. Since $\mathcal{W}$ is skewly linked with $\mathcal{L}^{[\circ]}$, we get $x_1 \in \mathcal{L}^{[\circ]}$, and in turn $x_2 = x - x_1 \in \mathcal{L}$. Thus $x_2 \in \mathcal{M}$. We see that $x \in \mathcal{L}^{[\circ]} + \mathcal{M}$, and this is the first equality in (2.4.2). To show the second, we argue similarly. If $x \in \mathcal{L}^{[\perp]}$, then $x, x_2[\perp]\mathcal{L}^{[\circ]}$, hence $x_1[\perp]\mathcal{L}^{[\circ]}$ and in turn $x_1 \in \mathcal{L}^{[\circ]}$. Now we have $x_2 = x - x_1 \in \mathcal{L}^{[\perp]}$. Hence, $x_2 \in \mathcal{N}$.

Applying the already proved item (1) with the spaces $\mathcal{M}$ and $\mathcal{N}$ just constructed, provides us with a space $\mathcal{W}'$ which is skewly linked with $\mathcal{L}^{[\circ]}$ such that $\mathcal{A}$ decomposes as in (2.4.3). Hence,

$$\dim (\mathcal{M}[\dot{+}]\mathcal{N})^{[\perp]} = \dim \mathcal{L}^{[\circ]} + \dim \mathcal{W}' = 2 \dim \mathcal{L}^{[\circ]} .$$

We know that $\mathcal{L}^{[\circ]} \dot{+} \mathcal{W} \subseteq (\mathcal{M}[\dot{+}]\mathcal{N})^{[\perp]}$, and due to the equality of dimensions, in this inclusion equality must hold.

It remains to show the uniqueness part of (2). Assume that $\mathcal{M}'$ and $\mathcal{N}'$ are closed subspaces with (2.4.2) and (2.4.3), and let $\mathcal{M}$ and $\mathcal{N}$ be the subspaces constructed above. We have

$$\mathcal{M}' \subseteq \mathcal{L},\ \mathcal{N}' \subseteq \mathcal{L}^{[\perp]}, \quad \mathcal{M}[\dot{+}]\mathcal{N} = (\mathcal{L}^{[\circ]} \dot{+} \mathcal{W})^{[\perp]} .$$

It follows that $\mathcal{M}' \subseteq \mathcal{M}$ and $\mathcal{N}' \subseteq \mathcal{N}$. Due to (2.4.2), in these inclusions equality must hold. ❑

II.54. *2.4.11 Example.* For $\nu \in \mathbb{R}$ let $\mathcal{D}_\nu$ be the set of all sequences $(a_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} (\nu + n)|a_n|^2 < \infty ,$$

and let $\mathcal{D}_\nu$ be endowed with the scalar product

$$[(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}]_\nu := \sum_{n=1}^{\infty}(\nu+n)a_n\overline{b_n}, \quad (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathcal{D}_\nu .$$

We call the space $\mathcal{D}_\nu$ obtained in this way the *generalized Dirichlet space*.

Set $e_j := (\delta_{jn})_{n\in\mathbb{N}}$, and

$$\begin{aligned}\mathcal{A}_+ &:= \{(a_n)_{n\in\mathbb{N}} \in \mathcal{D}_\nu : n > -\nu\}\\ \mathcal{A}_- &:= \{(a_n)_{n\in\mathbb{N}} \in \mathcal{D}_\nu : n < -\nu\}\\ \mathcal{A}_0 &:= \{(a_n)_{n\in\mathbb{N}} \in \mathcal{D}_\nu : n = -\nu\}\end{aligned}$$

The space $\mathcal{A}_+$ is naturally isomorphic to the space $L^2(\mu)$ with ($\delta_{\{n\}}$ denotes the *Dirac measure* concentrated at the point n)

$$\mu := \sum_{\substack{n\in\mathbb{N}\\ n>-\nu}}(\nu+n)\delta_{\{n\}} .$$

Hence, $\mathcal{A}_+$ is a Hilbert space. The space $\mathcal{A}_-$ is finite dimensional and negative definite, and the space $\mathcal{A}_0$ is (at most) one dimensional and neutral.

The space D_ν decomposes into the direct and orthogonal sum

$$\mathcal{D}_\nu = \mathcal{A}_+[\dot{+}]\mathcal{A}_-[\dot{+}]\mathcal{A}_0 .$$

In the case $\nu > -1$ we have $\mathcal{A}_- = \mathcal{A}_0 = \{0\}$, and hence $(\mathcal{D}_\nu, [.,.])$ is a Hilbert space.

For $\nu \leq -1, \nu \notin \mathbb{Z}$ still $\mathcal{A}_0 = \{0\}$, but $\dim \mathcal{A}_- = \lfloor -\nu \rfloor$. Providing $\mathcal{A}_\pm$ with the Hilbert space scalar product $\pm[.,.]$ and $\mathcal{D}_\nu = \mathcal{A}_+[\dot{+}]\mathcal{A}_- \cong \mathcal{A}_+ \times \mathcal{A}_-$ with the corresponding product Hilbert space topology, according to Theorem 2.4.2, (2), $(\mathcal{D}_\nu, [.,.])$ is a Pontryagin space.

It remains to deal with $\mathcal{D}_\nu$ in the case $\nu \in \{-1,-2,-3,\dots\}$, where $\dim \mathcal{A}_- = \lfloor -\nu \rfloor - 1$ and $\dim \mathcal{A}_0 = 1$. Like in the previous case $(\mathcal{A}_+[\dot{+}]\mathcal{A}_-, [.,.])$ is a Pontryagin space, but since $\mathcal{D}_\nu^{[\circ]} = \mathcal{A}_0 \neq \{0\}$, $\mathcal{D}_\nu$ cannot be provided with a Hilbert space topology such that it becomes a Pontryagin space.

Despite the fact that the construction of these Dirichlet spaces may seem artificial, these spaces appear naturally in complex analysis; we will see some more details later in . Therefore, it is disirable to provide $\mathcal{D}_\nu$ with a Gram space structure also in the case $\nu \in \{-1,-2,-3,\dots\}$.

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One way to do this, is to view $\mathcal{A}_0$ as a neutral subspace of a two dimensional nondegenerated scalar product space $\mathcal{V}$ as mentioned in Remark 1.7.12. As a finite dimensional space $\mathcal{V}$ carries a unique Hilbert space topology. Considering $\mathcal{A} \cong \mathcal{A}_+ \times \mathcal{A}_- \times \mathcal{A}_0$ as a closed subspace of $\mathcal{A}_+ \times \mathcal{A}_- \times \mathcal{V}$, where the latter space is equipped with the sum scalar product, we see that $\mathcal{D}_\nu$ can be identified as a closed and degenerated subspace of a Pontryagin space. It is the aim of the next section, to study such subspaces of Pontryagin spaces and to characterize them independently from the Pontryagin superspace. ◊

2.5 Almost Pontryagin Spaces

section2.5

Definition and Examples

We alraedy saw that subspaces of Krein spaces are not necessarily Krein spaces. The same is true for Pontryagin spaces, which are a special kind of Krein spaces. But in contrast to the Krein spaces case, whether a subspace $\mathcal{L}$ of a Pontryagin space is a Pontryagin space or not, can be decided just by the algebraic condition $\mathcal{L}^{[\circ]} = \{0\}$; see Proposition 2.4.6.

II.36. **2.5.1 Definition.** We call a triple $(\mathcal{A}, [.,.], O)$ an *almost Pontryagin space*, if

(1) $(\mathcal{A}, [.,.], O)$ is a Gram space.

(2) There exists a Pontryagin space $(\mathcal{P}, [.,.])$, such that $(\mathcal{A}, [.,.], O)$ is a subspace in the sense Gram spaces, i.e., $[.,.]$ on $\mathcal{A}$ is the restriction of $[.,.]$ on $\mathcal{P}$ to $\mathcal{A}$ and O is the subspace topology on $\mathcal{A}$ induced by the unique Pontryagin space topology on $\mathcal{P}$; see Proposition 2.2.8.

◊

If the topology O and scalar product $[.,.]$ is clear from the context, we shall again just speak of an almost Pontryagin space $\mathcal{A}$. Nevertheless, as the spaces in Example 2.2.6 can be identified with the help of Theorem 2.5.2 as almost Pontryagin spaces, we see that the topology O of an almost Pontryagin space $(\mathcal{A}, [.,.], O)$ is generically not uniquely determined by the scalar product.

By Proposition 2.4.6 an almost Pontryagin space is a Pontryagin space if and only if it is nondegenerated. Therefore, the class of Pontryagin spaces is the intersection of the classes of Krein- and almost Pontryagin spaces. Moreover, an almost Pontryagin space $(\mathcal{A}, [.,.], O)$ is a Hilbert space if and only if $\operatorname{ind}_0(\mathcal{A}, [.,.]) = 0$ and $\operatorname{ind}_-(\mathcal{A}, [.,.]) = 0$.

Almost Pontryagin spaces can be characterized without reference to the Pontryagin space $\mathcal{P}$.

II.38. **2.5.2 Theorem.** *Let $(\mathcal{A}, [.,.], O)$ be a Gram space. Then the following are equivalent.*

(1) *$(\mathcal{A}, [.,.], O)$ is an almost Pontryagin space.*

(2) *There exists a certain O-compatible Hilbert space scalar product $(.,.)$, such that for the corresponding Gram operator G of $[.,.]$ w.r.t. $(.,.)$ is of the form $G = I + K$ for a bounded, $(.,.)$-selfadjoint and finite dimensional $K : \mathcal{A} \to \mathcal{A}$.*

(3) *There exists a closed subspace $\mathcal{L}$ of $\mathcal{A}$ with finite codimension in $\mathcal{A}$, such that $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}})$ is a Hilbert space.*

(4) *We have $\operatorname{ind}_0(\mathcal{A}, [.,.]) < \infty$, and the space $\mathcal{A}$ admits a decomposition $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-[\dot{+}]\mathcal{A}^{[\circ]}$ where $\mathcal{A}_+$ is closed, $(\mathcal{A}_+, [.,.])$ is a Hilbert space, and $\mathcal{A}_-$ is finite dimensional and negative definite.*

If $(\mathcal{A}, [.,.], O)$ is an almost Pontryagin space, then there always exists a Pontryagin space $(\mathcal{P}, [.,.])$, which contains $(\mathcal{A}, [.,.], O)$ as a Gram subspace of codimension $\operatorname{ind}_0(\mathcal{A}, [.,.])$.

Proof. Assume (1), and we will show (2) very much along the arguments given in the proof of Proposition 2.4.6. Indeed, by Theorem 2.4.2 applied to $(\mathcal{P}, [.,.])$ we have $[.,.] = (G.,.)$ with a certain Hilbert space scalar product $(.,.)$ and the corresponding Gram operator of the form $G = I + K$ with a finite dimensional K. Denoting by P the $(.,.)$-orthogonal projection from $\mathcal{P}$ onto the subspace $\mathcal{A}$ we infer from Proposition 2.2.8 that the Gram operator of $[.,.]$ on $\mathcal{A}$ w.r.t. $(.,.)$ is

$$G_{\mathcal{A}} = PG|_{\mathcal{A}} = I + PK|_{\mathcal{A}},$$

where P is the $(.,.)$-orthogonal projection onto $\mathcal{A}$. This gives (2) because $PK|_{\mathcal{A}}$ is a finite dimensional operator in $\mathcal{A}$.

Assuming (2) we consider the closed subspace $\mathcal{L} := \ker K$, which has finite codimension in $\mathcal{A}$. As $(.,.)$ and $[.,.]$ coincide on $\mathcal{L}$, we get (3).

Assuming (3) we know from Proposition 2.3.13 that $\mathcal{A} = \mathcal{L}[\dot{+}]\mathcal{L}^{[\perp]}$, where $\dim \mathcal{L}^{[\perp]} < \infty$. Hence, $\mathcal{L}^{[\perp]} = \mathcal{L}_+[\dot{+}]\mathcal{A}_-[\dot{+}]\mathcal{A}_\circ$ with positve definite, negative definite and neutral subspaces $\mathcal{L}_+, \mathcal{A}_-, \mathcal{A}_\circ$; see Proposition 1.6.2. Clearly, $\mathcal{A}_\circ = \mathcal{A}^{[\circ]}$. $\mathcal{L}_+$ being finite dimensional shows that $\mathcal{A}_+ := \mathcal{L}[\dot{+}]\mathcal{L}_+$ is closed and, provided with $[.,.]$, constitutes a Hilbert space. Therefore, (4) hold true.

If we can decompose $\mathcal{L}$ as in (4), then $(\mathcal{A}^{[\circ]}, [.,.])$ can be viewed as a neutral subspace of a nondegenerated space $(C, [.,.])$ with two times the dimension of $\mathcal{A}^{[\circ]}$; see Remark 1.7.12. Providing C with the unique Hilbert space topology (see Remark 2.2.7), $\mathcal{A}^{[\circ]}$ becomes a Gram subspace of C. Hence, the product Gram space $\mathcal{A}_+ \times \mathcal{A}_- \times \mathcal{A}^{[\circ]}$ is a closed subspace of the product Gram space $\mathcal{A}_+ \times \mathcal{A}_- \times C$; see Proposition 2.2.8. As seen in Remark 2.2.9, the former as a Gram space can be identified with $\mathcal{A}$.

Writting C as $C = C_+[\dot{+}]C_-$ with positive (negative) definite $C_\pm$ (see Proposition 1.6.2) we infer from Theorem 2.4.2, (3), applied with the Hilbert space $\mathcal{A}_+ \times \{0\} \times C_+$ and the finite dimensional anti Hilbert space $\{0\} \times \mathcal{A}_- \times C_-$, that $\mathcal{A}_+ \times \mathcal{A}_- \times C$ is a Pontryagin space. Thus, (1) holds true. ❑

A decomposition of an almost Pontryagin space $(\mathcal{A}, [.,.], O)$ as in Theorem 2.5.2, (3), is called a *fundamental decomposition* of $\mathcal{A}$.

There is also a characterization involving spectral properties of the corresponding Gram operator.

II.38post. **2.5.3 Proposition.** *A Gram space $(\mathcal{A}, [.,.], O)$ is an almost Pontryagin space if and only if there exists a certain O-compatible Hilbert space scalar product $(.,.)$, such that for the corresponding Gram operator G of $[.,.]$ w.r.t. $(.,.)$ and its spectral measure E we have* $\dim \operatorname{ran} E(-\infty, 0] < \infty$ *and* $E(0, \epsilon) = 0$ *for a certain* $\epsilon > 0$.

In this case this condition holds for every O-compatible scalar product.

Proof. For any O-compatible Hilbert space scalar product $(.,.)$ and the corresponding Gram operator G let E be the spectral measure of G. $\dim \operatorname{ran} E(-\infty, 0] < \infty$ is the same as saying $\dim\operatorname{ran} E\{0\} < \infty$ together with $\dim\operatorname{ran} E(-\infty, 0) < \infty$. But then due to Proposition 2.2.12, the assumption $\dim\operatorname{ran} E(-\infty, 0] < \infty$ and $E(0, \epsilon) = 0$ imply the validity of Proposition 2.5.3, (3).

Conversely, if $(\mathcal{A}, [.,.], \mathcal{O})$ is an almost Pontryagin space, then due to Proposition 2.2.12, $\operatorname{ind}_-(\mathcal{A}, [.,.]) < \infty$ and $\operatorname{ind}_0(\mathcal{A}, [.,.]) < \infty$ imply $\dim\operatorname{ran} E(-\infty, 0] < \infty$. Moreover, $\operatorname{ran} E(0, \infty)$ being a positve definite and closed subspace of the almost Pontryagin space $\mathcal{A}$, due to Proposition 2.4.6, $(\operatorname{ran} E(0, \infty), [.,.])$ is a positve definite Pontryagin space, i.e., a Hilbert space. By Proposition 2.2.12 we then have $E(0, \varepsilon) = 0$ for a certain $\varepsilon > 0$. ❑

II.55. *2.5.4 Example.* A first, admittedly very simple example of almost Pontryagin spaces are finite dimensional spaces as treated in Example 1.1.3, where for a hermitian $n \times n$-matrix G we defined a scalar product on $\mathbb{C}^n$ as

$$\left[\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right] := \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}^* G \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n .$$

Moreover, let $\mathcal{O}$ be the euclidean topology on $\mathbb{C}^n$. Then $(\mathbb{C}^n, [.,.], \mathcal{O})$ is an almost Pontryagin space. The positive index (negative index) of this space equals the number of positive (negative) eigenvalues of G counted according to their multiplicities. The degree of degeneracy equals the multiplicity of the point 0 as an eigenvalue of G; see Example 1.6.5. ◊

II.53. *2.5.5 Example.* Let $(\mathcal{A}, (.,.))$ be a Hilbert space, and let $x_1, \dots, x_n \in \mathcal{A}$. We define a new scalar product $[.,.]$ on $\mathcal{A}$ as

$$[x, y] := (x, y) - \sum_{i=1}^{n} (x, x_i)(x_i, y), \quad x, y \in \mathcal{A} .$$

Denote by $\mathcal{O}$ the Hilbert space topology of $\mathcal{A}$. Since

$$\big|[x, y]\big| \le \Big(1 + \sum_{i=1}^{n} \|x_i\|^2\Big) \cdot \|x\| \cdot \|y\|, \quad x, y \in \mathcal{A} ,$$

$(\mathcal{A}, [.,.], \mathcal{O})$ is a Gram space. The Gram operator G of $[.,.]$ w.r.t. $(.,.)$ is given as

$$G = I - \sum_{i=1}^{n} (., x_i) x_i .$$

By Theorem 2.5.2 $(\mathcal{A}, [.,.], \mathcal{O})$ is an almost Pontryagin space. ◊

II.57. *2.5.6 Example.* Let us instanciate Example 2.5.6 in a concrete situation. Denote by $\hat{.} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the *Fouriertransform*, i.e., the unitary operator obtained as the extension of

$$f(\tau) \quad \mapsto \quad \hat{f}(\eta) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\tau)\, e^{-i\tau\eta}\, d\tau, \ \eta \in \mathbb{R}$$

from the dense subspace $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to all of $L^2(\mathbb{R})$. For each $a \ge 0$, we define the *Paley-Wiener space* $\mathcal{PW}_a$ as the Fourier image of the set of all functions in $L^2(\mathbb{R})$ which are supported on the interval $[-a, a]$. Since

$$\{f \in L^2(\mathbb{R}) : \operatorname{supp} f \subseteq [-a, a]\}$$

is a closed subspace of $L^2(\mathbb{R})$, and $\hat{.}$ is unitary, $\mathcal{P}W_a$ is a closed subspace of $L^2(\mathbb{R})$. Hence, when endowed with the $L^2(\mathbb{R})$-scalar product, $\mathcal{P}W_a$ becomes a Hilbert space.

If $f \in L^2(\mathbb{R})$ is compactly supported, then $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Its Fouriertransform $\hat{f}$ is thus given by the formula

$$\hat{f}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\tau)\, e^{-i\tau\eta}\, d\tau = \frac{1}{\sqrt{2\pi}} \int_{\operatorname{supp} f} f(\tau)\, e^{-i\tau\eta}\, d\tau, \quad \eta \in \mathbb{R}. \tag{2.5.1}$$ II.72

Obviously, the integral on the right side makes sense also if η is nonreal. In fact, it represents a function defined and analytic on all of $\mathbb{C}$. Consequently the space $\mathcal{P}W_a$, originally being a subspace of $L^2(\mathbb{R})$ can be considered as a space of entire functions on $\mathbb{C}$:

$$\mathcal{P}W_a = \Big\{ f : \mathbb{C} \to \mathbb{C} :\ \exists g \in L^2(\mathbb{R}) \text{ with } \operatorname{supp} g \subseteq [-a,a],\ f(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\tau)\, e^{-i\tau\eta}\, d\tau, \eta \in \mathbb{C} \Big\}.$$

In particular, we naturally have the *point evaluation functionals*

$$\chi_\eta : \begin{cases} \mathcal{P}W_a & \to \quad \mathbb{C} \\ \quad g & \mapsto \quad g(\eta) \end{cases}$$

Let us show that χ_η is continuous. To this end notice that, since the Fouriertransform is isometric, we can compute for each $g \in L^2(\mathbb{R})$ with $\operatorname{supp} g \subseteq [-a,a]$ ($(.,.)$ denotes the $L^2(\mathbb{R})$-scalar product)

$$\hat{g}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\tau)\, e^{-i\tau\eta}\, d\tau = \Big(f(\tau), \frac{1}{\sqrt{2\pi}} \mathbb{1}_{[-a,a]}(\tau)\, e^{i\tau\overline{\eta}} \Big) = \\ = \Big(\hat{f}, \hat{.}\Big[\frac{1}{\sqrt{2\pi}} \mathbb{1}_{[-a,a]}(\tau)\, e^{i\tau\overline{\eta}} \Big] \Big), \quad \eta \in \mathbb{C}.$$

Clearly, the element

$$k_a(\eta, .) := \hat{.}\Big[\frac{1}{\sqrt{2\pi}} \mathbb{1}_{[-a,a]}(\tau)\, e^{i\tau\overline{\eta}} \Big]$$

belongs to $\mathcal{P}W_a$. The above formula says that point evaluation is represented as inner product with $k_a(\eta, .)$, hence, χ_η is continuous. In fact, we have

$$\Big\| \chi_\eta|_{\mathcal{P}W_a} \Big\| = \| k_a(\eta, .) \|_{\mathcal{P}W_a}.$$

Explicitly,

$$k_a(\eta, \zeta) = \frac{\sin a(\zeta - \overline{\eta})}{\pi(\zeta - \overline{\eta})}.$$

$$\| k_a(\eta, .) \|^2 = k_a(\eta, \eta) = \frac{\sinh(2a \operatorname{Im} \eta)}{2\pi \operatorname{Im} \eta}.$$

Thereby, for $\zeta = \overline{\eta}$ or $\operatorname{Im} \eta = 0$, respectively, the expression on the right are interpreted in the naturally way (both have a continuous extension to such points).

Let $\gamma \in \mathbb{C}$, and apply Example 2.5.5 with the Hilbert space $\mathcal{P}W_a$ and "$n = 1, x_1 = \gamma k_a(0, .)$". The scalar product $[.,.]$ defined there computes explicitly as

$$[f, g] = \int_{\mathbb{R}} f(\eta)\overline{g(\eta)}\, d\eta - |\psi|^2 f(0)\overline{g(0)}, \quad f, g \in \mathcal{P}W_a.$$

The space $(\mathcal{P}W_a, [.,.], \mathcal{O})$, where $\mathcal{O}$ is the Hilbert space topology induced by the $L^2(\mathbb{R})$-scalar product $(.,.)$, is an almost Pontryagin space. The spectrum of the Gram operator of $[.,.]$ w.r.t. $(.,.)$ consists of the point 1 and one simple eigenvalue λ. Hence, depending on the location of λ, one of the following alternatives takes place:

(1) $\operatorname{ind}_-(\mathcal{P}W_a, [.,.]) = \operatorname{ind}_0(\mathcal{P}W_a, [.,.]) = 0$ and hence $\mathcal{P}W_a$ is a Hilbert space.

(2) $\operatorname{ind}_-(\mathcal{P}W_a, [.,.]) = 0$, $\operatorname{ind}_0(\mathcal{P}W_a, [.,.]) = 1$.

(3) $\operatorname{ind}_-(\mathcal{P}W_a, [.,.]) = 1$, $\operatorname{ind}_0(\mathcal{P}W_a, [.,.]) = 0$ and hence $\mathcal{P}W_a$ is a Pontryagin space.

We have $[f, f] = (f, f) - |\gamma|^2 |f(0)|^2$, $f \in \mathcal{P}W_a$, and hence

$$\min_{\substack{f\in\mathcal{P}W_a\\ \|f\|=1}} [f, f] = 1 - |\gamma|^2 \|k_a(0, .)\|^2 = 1 - |\gamma|^2 \frac{a}{\pi} .$$

Thus,

$$\text{alternative} \left\{ \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \right\} \text{ takes place} \quad \Longleftrightarrow \quad |\gamma| \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \sqrt{\frac{a}{\pi}}$$

◊

Properties and Constructions

It is an important structural property that in contrast to Pontryagin spaces the class of almost Pontryagin spaces is closed with respect to taking subspaces.

II.37. **2.5.7 Proposition.** *The following constructions can be carried out within the class of almost Pontryagin spaces.*

(1) *Let* $(\mathcal{A}, [.,.], \mathcal{O})$ *be an almost Pontryagin space, and let* $\mathcal{L}$ *be a closed linear subspace of* $\mathcal{A}$*. Then* $(\mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}}, \mathcal{O}|_{\mathcal{L}})$ *is an almost Pontryagin space. In the case of a nondegenerated* $\mathcal{L}$ *it is even a Pontryagin space.*

(2) *For each* $j \in \{1, \dots, n\}$ *let* $(\mathcal{A}_j, [.,.]_j, \mathcal{O}_j)$ *be an almost Pontryagin space, and denote by* $[.,.]$ *the sum scalar product* (2.2.3) *on* $\prod_{j=1}^n \mathcal{A}_j$*. Then* $(\prod_{j=1}^n \mathcal{A}_j, [.,.], \prod_{j=1}^n \mathcal{O}_j)$ *is an almost Pontryagin space.*

Proof. The assertions about subspaces immediately follows from Definition 2.5.1. In combination with the fact that the class of Pontryagin spaces is closed under the formation of finite product spaces as shwon in Lemma 2.4.5, Definition 2.5.1 also yields the assertion about the finite product of almost Pontryagin spaces. ❑

The following assertion yields that almost Pontryagin spaces are closed under the formation of factor spaces.

II.37post. **2.5.8 Lemma.** *Let* $(\mathcal{A}, [.,.], \mathcal{O})$ *and* $(\mathcal{B}, [.,.], \mathcal{T})$ *be a Gram spaces, and assume that* $\varphi : \mathcal{A} \to \mathcal{B}$ *is a bounded linear and surjective mapping which is isometric with respect to* $[.,.]$.

If $(\mathcal{A}, [.,.], \mathcal{O})$ *is an almost Pontryagin space, then* $(\mathcal{B}, [.,.], \mathcal{T})$ *is also an almost Pontryagin space. The converse is true, if we additionally assume that* $\dim\ker\varphi < \infty$.

Proof. For an $\mathcal{O}$-compatible Hilbert space scalar produt $(.,.)$ on $\mathcal{A}$ and for the closed subspace $C := (\ker\varphi)^{(\perp)}$ the restriction $\varphi|_C : C \to \mathcal{B}$ is linear, isometric, bijective and, by the open mapping Theorem , continuous in both directions, i.e., as Gram spaces C and $\mathcal{B}$ are isomorphic. Missing Reference

If $(\mathcal{A}, [.,.], \mathcal{O})$ is an almost Pontryagin space, so is C. For example from Theorem 2.5.2, (3), it then easily follows that the isomorphic copy $\mathcal{B}$ of C is an almost Pontryagin space, too.

Conversely, if $\mathcal{B}$ is an almost Pontryagin space, so is C. Moreover, since by (1.2.7), $\ker\varphi$ is contained in $\mathcal{A}^{[\circ]} \subseteq C^{[\perp]}$, $\mathcal{A}$ is isomorphic to the product Gram space $C \times \ker\varphi$; cf. Remark 2.2.9. In case $\dim\ker\varphi < \infty$, $(\ker\varphi, [.,.])$ clearly constitutes an almost Pontryagin space; see Example 2.5.4. By Proposition 2.5.7 also $C \times \ker\varphi$, and hence $\mathcal{A}$, is an almost Pontryagin space. ❑

Applying Lemma 2.5.8 to the factor mapping we immediately get

II.37post2. **2.5.9 Corollary.** *Let $(\mathcal{A}, [.,.], \mathcal{O})$ be an almost Pontryagin space, and let $\mathcal{M}$ be a subspace of $\mathcal{A}^{[\circ]}$. Denote by $[\![.,.]\!]$ the scalar product* (2.2.4) *on $\mathcal{A}/_{\mathcal{M}}$ and by $\mathcal{O}/_{\mathcal{M}}$ the Hilbert space topology defined as in Proposition* 2.2.8. *Then $(\mathcal{A}/_{\mathcal{M}}, [\![.,.]\!], \mathcal{O}/_{\mathcal{M}})$ is an almost Pontryagin space. For $\mathcal{M} = \mathcal{A}^{[\circ]}$ it is in fact a Pontryagin space.*

If $(\mathcal{A}, [.,.], \mathcal{O})$ is an almost Pontryagin space with $\mathcal{A}^{[\circ]} = \{0\}$, then it is a Krein space, and hence its topological dual is equal to $\{[.,y] : y \in \mathcal{A}\}$. If $\mathcal{A}^{[\circ]} \neq \{0\}$, this cannot be true anymore as we certainly have $\mathcal{A}^{[\circ]} \subseteq \ker[.,y]$, $y \in \mathcal{A}$, and hence $\{[.,y] : y \in \mathcal{A}\}$ is not point separating whereas the dual space $\mathcal{A}'$ is.

II.73. **2.5.10 Lemma.** *Let $(\mathcal{A}, [.,.], \mathcal{O})$ be an almost Pontryagin space. Then*

$$\dim\, (\mathcal{A}, \mathcal{O})'/_{\{[.,y]\,:\,y \in \mathcal{A}\}} = \operatorname{ind}_0(\mathcal{A}, [.,.])\,.$$

Proof. As we already noticed above, the asserted equality holds if $\mathcal{A}$ is nondegenerated. Hence, assume that $\mathcal{A}^{[\circ]} \neq \{0\}$.

Let $(.,.)$ be a compatible scalar product on $\mathcal{A}$, and denote by G the Gram operator of $[.,.]$ w.r.t. $(.,.)$. If $\varphi \in (\mathcal{A}, \mathcal{O})'$, we find $z \in \mathcal{A}$ with $\varphi = (.,z)$. Write $z = z_0 + z_1$ with $z_0 \in \mathcal{A}^{[\circ]}$ and $z_1 \in (\mathcal{A}^{[\circ]})^{(\perp)}$.

Since $\mathcal{A}^{[\circ]} = \ker G = \operatorname{ran} G^{(\perp)}$, the Gram operator of $[.,.]$ on $(\mathcal{A}^{[\circ]})^{(\perp)}$ is $G|_{(\mathcal{A}^{[\circ]})^{(\perp)}}$, and as a nondegenerated subspace $(\mathcal{A}^{[\circ]})^{(\perp)}$ is a Pontryagin space. Hence, $G|_{(\mathcal{A}^{[\circ]})^{(\perp)}} : (\mathcal{A}^{[\circ]})^{(\perp)} \to (\mathcal{A}^{[\circ]})^{(\perp)}$ is bijevtive, and we may choose $y \in \mathcal{A}$ with $Gy = z_1$. Then

$$\varphi(x) = (x,z) = (x,z_1) + (x,z_0) = (x,Gy) + (x,z_0) = [x,y] + (x,z_0), \quad x \in \mathcal{A}\,.$$

Clearly, functionals of the forms $[.,y]$, $y \in \mathcal{A}$, and $(.,z_0)$, $z_0 \in \mathcal{A}^{[\circ]}$, belong to $(\mathcal{A}, \mathcal{O})'$. It follows that

$$(\mathcal{A}, \mathcal{O})' = \{[.,y] : y \in \mathcal{A}\} + \{(.,z_0) : z_0 \in \mathcal{A}^{[\circ]}\}\,.$$

In order to show, that this sum is direct, let $y \in \mathcal{A}$, $z_0 \in \mathcal{A}^{[\circ]}$, and assume that $[.,y] = (.,z_0)$. Then

$$(x,z_0) = [x,y] = 0, \quad x \in \mathcal{A}^{[\circ]}\,,$$

and hence $z_0 = 0$. It follows that

$$\dim(\mathcal{A}, O)'/\{[., y] : y \in \mathcal{A}\} = \dim\{(., z_0) : z_0 \in \mathcal{A}^{[\circ]}\} = \dim \mathcal{A}^{[\circ]}.$$

❑

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2.5.11 Lemma. *Let $(\mathcal{A}, [.,.], O)$ be an almost Pontryagin space, and let $[\![.,.]\!]$ be another O-continuous scalar product on $\mathcal{A}$ such that $\mathcal{A}^{[\![\circ]\!]}$ has finite codimension in $\mathcal{A}$. Setting $\langle .,. \rangle := [.,.] + [\![.,.]\!]$ also $(\mathcal{A}, \langle .,. \rangle, O)$ constitutes an almost Pontryagin space.*

Proof. By Lemma 2.1.3, $\mathcal{A}^{[\![\circ]\!]}$ is O-closed. If $\mathcal{L}$ is an O-closed subspace of $\mathcal{A}$ of finite codimension, such that $(\mathcal{L}, [.,.])$ is a Hilbert space as in Theorem 2.5.2, (3), then also $\mathcal{L} \cap \mathcal{A}^{[\![\circ]\!]}$ is O-closed and of finite codimension. Since $[.,.]$ and $\langle .,. \rangle$ coincide on $\mathcal{L} \cap \mathcal{A}^{[\![\circ]\!]}$, due to Theorem 2.5.2, $(\mathcal{A}, \langle .,. \rangle, O)$ is an almost Pontryagin space. ❑

The following assertion in particular shows that any almost Pontryagin space scalar product can be obtained as in Example 2.5.5. For this we are going to employ Lemma 2.5.11 in the case when $\langle .,. \rangle$ is positive definite. Recall that then $(\mathcal{A}, \langle .,. \rangle, O)$ becomes a Hilbert space.

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2.5.12 Proposition. *Let $(\mathcal{A}, [.,.], O)$ be an almost Pontryagin space, and let $\mathcal{F}$ be a point separating family of continuous linear functionals on $\mathcal{A}$; see* (2.1.3)*. Then there exist $N \in \mathbb{N}_0$, linearly independent functionals $\phi_1, \ldots, \phi_N \in \mathcal{F}$, and a number $\gamma_0 \in \mathbb{R}$, such that for each $\gamma > \gamma_0$ the scalar product defined as*

$$(x, y)_+ := [x, y] + \gamma \sum_{j=1}^{N} \phi_j(x)\, \overline{\phi_j(y)}, \quad x, y \in \mathcal{A}, \tag{2.5.2}$$ III.23

is a compatible Hilbert space scalar product on $\mathcal{A}$.

Proof. According to Theorem 2.5.2 there exists an O-compatible Hilbert space scalar product $(.,.)$ on $\mathcal{A}$, such that the Gram operator G of $[.,.]$ w.r.t. $(.,.)$ is of the form $G = I + K$ for some bounded, $(.,.)$-selfadjoint K with $d := \dim \operatorname{ran} K < \infty$. By the Spectral Theorem for selfadjoint matrices we have

$$K(x) = \sum_{k=1}^{d} \lambda_k (., a_k) \cdot a_k, \tag{2.5.3}$$ disgfor

for $x \in \operatorname{ran} K$ with an $(.,.)$-orthogonal basis $a_1, \ldots, a_d$ of $\operatorname{ran} K$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$. From $\ker K = (\operatorname{ran} K)^{(\perp)}$ the validity of (2.5.3) follows for all $x \in \mathcal{A}$.

For $\phi \in \mathcal{F}$ let $b(\phi) \in \mathcal{A}$ be the element which represents ϕ as $\phi = (., b(\phi))$. Since $\mathcal{F}$ is point separating, we have

$$\Big[\operatorname{span}\{b(\phi) : \phi \in \mathcal{F}\}\Big]^{(\perp)} = \bigcap_{\phi \in \mathcal{F}} \{b(\phi)\}^{(\perp)} = \bigcap_{\phi \in \mathcal{F}} \ker \phi = \{0\}.$$

This means that $\operatorname{span}\{b(\phi) : \phi \in \mathcal{F}\}$ is dense in $\mathcal{A}$. For $c_1, \ldots, c_d \in \operatorname{span}\{b(\phi) : \phi \in \mathcal{F}\}$ let $K_1 : \mathcal{A} \to \mathcal{A}$ be defined by $K_1 := \sum_{k=1}^{d} (., c_k) c_k$. Clearly, K_1 is $(.,.)$-selfadjoint.

Comparing K and K_1 yields ($\|.\|$ denotes the norm induced by $(.,.)$)

$$\|(K-K_1)x\| \leq \sum_{k=1}^{d} \Big\|(x,a_k)(a_k-c_k)+(x,a_k-c_k)c_k\Big\| \leq$$
$$d(1+\max_{k=1,\dots,d}\|c_k\|)\max_{k=1,\dots,d}\|a_k-c_k\|\cdot\|x\|\,.$$

Due to $\operatorname{span}\{b(\phi):\phi\in\mathcal{F}\}$'s density we can choose $c_1,\dots,c_d$ such that $\|K-K_1\|<1$, and hence

$$[x,x]-(K_1x,x)=(x,x)+((K-K_1)x,x)>0,\ x\in\mathcal{A}\setminus\{0\}\,. \tag{2.5.4}$$

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Choose functionals $\phi_1,\dots,\phi_N\in\mathcal{F}$, such that $c_k\in\operatorname{span}\{b(\phi_1),\dots,b(\phi_N)\}$, $k=1,\dots,d$, and let α_{ij} be coefficients with

$$c_k=\sum_{j=1}^{N}\alpha_{kj}b(\phi_j),\quad k=1,\dots,d\,.$$

We then have

$$(K_1x,y)=\sum_{k=1}^{d}(x,c_k)(c_k,y)=$$
$$\sum_{k=1}^{d}\sum_{i,j=1}^{N}\overline{\alpha_{kj}}\,(x,b(\phi_j)(b(\phi_i),y)\,\alpha_{ki}=\sum_{i,j=1}^{N}\overline{\phi_i(y)}\,\underbrace{\sum_{k=1}^{d}\overline{\alpha_{kj}}\,\alpha_{ki}}_{=:\beta_{ij}}\,\phi_j(x)\,.$$

Clearly, the $N\times N$-matrix $(\beta_{ij})_{i,j}$ is selfadjoint. If $-\gamma_0$ denotes the smallest eigenvalue of $(\beta_{ij})_{i,j}$, then we know from Example 1.6.5 that $(\beta_{ij})_{i,j}+\gamma I$ is the Gram matrix of a positive definite scalar product on $\mathbb{C}^N$ w.r.t. the euclidean scalar product whenever $\gamma>\gamma_0$. Thus, together with (2.5.4) we have

$$(x,x)_+=[x,x]+\gamma\sum_{j=1}^{N}\phi_j(x)\,\overline{\phi_j(x)}=[x,x]-(K_1x,x)+(K_1x,x)+\gamma\sum_{j=1}^{N}\phi_j(x)\,\overline{\phi_j(x)}=$$
$$[x,x]-(K_1x,x)+\sum_{i,j=1}^{N}\overline{\phi_i(y)}\,(\beta_{ij}+\delta_{ij})\,\phi_j(x)>0,\ x\in\mathcal{A}\setminus\{0\}\,.$$

As already mentioned, Lemma 2.5.11 shows that $(\mathcal{A},(.,.)_+)$ is a Hilbert space. ❑

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2.5.13 Remark. The functionals ϕ_j, $j=1,\dots,N$, in Proposition 2.5.12 can be represented as $\phi_j=[.,b_j]$, $j=1,\dots,N$, with some $b_j\in\mathcal{A}$ if and only if $(\mathcal{A},[.,.])$ is a Pontryagin space. In fact, if $(\mathcal{A},[.,.])$ is nondegenerated, then this follows from Lemma 2.5.10.

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On the other hand, $x\in\operatorname{span}\{b_j:j=1,\dots,N\}^{[\circ]}$ yields $(x,x)_+=[x,x]+\gamma\sum_{j=1}^{N}[x,b_j]\,[b_j,x]=0$, and hence $x=0$. Beacuse of $\operatorname{span}\{b_j:j=1,\dots,N\}^{[\circ]}\supseteq\mathcal{A}^{[\circ]}$ our $\mathcal{A}$ must be a Pontryagin space, where in addition $\mathcal{L}:=\operatorname{span}\{b_j:j=1,\dots,N\}$ is nondegenerated.

Moreover, assuming the Pontryagin space case, we have

$$(x, y)_+ = [x, y] + \gamma \sum_{j=1}^{N} [x, b_j]\ [b_j, y] = [x + \gamma \sum_{j=1}^{N} [x, b_j]\ b_j, y],\ \ x, y \in \mathcal{A}\,.$$

This means that $I + \gamma \sum_{j=1}^{N} [., b_j] b_j$ is the unique Gram operator of $(.,.)_+$ w.r.t. $[.,.]$ as discussed in Remark 2.3.5. Hence, it coincides with G^{-1} if G denotes the Gram operator of $[.,.]$ w.r.t. $(.,.)_+$. Since $\mathcal{L}$ is nondegenerated, and hence $\mathcal{A} = \mathcal{L}[\dot{+}]\mathcal{L}^{[\perp]}$, $G^{-1} = I + \gamma \sum_{j=1}^{N} [., b_j] b_j$ has diagonal form w.r.t. this decomposition. Thus, the same ist true for G. In particular, considering a basis of $\mathcal{L}$, which is a subset of $\{b_j : j = 1, \ldots, N\}$, G can be written in the form

$$G = I + \sum_{j,k=1}^{N} \alpha_{jk} [., b_j] b_k$$

with $\alpha_{j,k} = \overline{\alpha_{k,j}} \in \mathbb{C}$. ◊

If $\mathcal{M}$ and $\mathcal{N}$ are linear subspaces of Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\varphi : \mathcal{M} \to \mathcal{N}$ is a linear and isometric map, then φ is continuous and has an extension to a bi-continuous of $c\ell(\mathcal{M})$ onto $c\ell(\mathcal{N})$. In the indefinite setting this statement does not remain true in general. For instance in Example 2.2.6 the identity map $\mathrm{id} : (\mathcal{A}, [.,.], \mathcal{O}) \to (\mathcal{A}, [.,.], \mathcal{T})$ is an isometric bijection, but is not continuous. We will meet another instance later in Example 2.6.11. There the map $\iota' \circ \iota^{-1}$ is an isometric and linear bijection between dense subspaces of two Krein spaces. However, again, it cannot be extended in a bi-continuous way.

In the following theorem and its corollaries we provide a condition when an analogue of the mentioned fact from Hilbert space theory does hold true. For the sake of ease in application, we choose a rather general formulation readily anticipating the language of linear relations which we will introduce in Chapter .

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For the following recall the notion of the *graph* of a map. Let M and N be two sets, and let $\varphi : M \to N$. Then we denote

$$\operatorname{graph} \varphi := \{(x; \varphi x) : x \in M\} \subseteq M \times N\,.$$

For subsets M and N of normed spaces $\mathcal{A}$ and $\mathcal{B}$, respectively, and for a bounded, linear $\varphi : M \to N$ it is straight forward to check that $\operatorname{graph} \varphi$ is a closed linear subspace of $M \times N$; see .

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If $\mathcal{L}$ is a linear subspace of a $\mathcal{A} \times \mathcal{B}$ we introduce the notation

$$\begin{aligned} \operatorname{dom}(\mathcal{L}) &:= \{x \in \mathcal{A} :\ \exists y \in \mathcal{B} : (x; y) \in \mathcal{L}\}, \\ \operatorname{ran}(\mathcal{L}) &:= \{y \in \mathcal{B} :\ \exists x \in \mathcal{A} : (x; y) \in \mathcal{L}\}\,, \end{aligned}$$

It is easy to check that $\operatorname{dom} \mathcal{L}$ and $\operatorname{ran} \mathcal{L}$ are indeed linear subspaces.

II.41. **2.5.14 Theorem.** *Let $(\mathcal{A}, [.,.], \mathcal{O})$ and $(\mathcal{B}, [\![.,.]\!], \mathcal{T})$ be two almost Pontryagin spaces, and let $\mathcal{L}$ be a linear subspace of $\mathcal{A} \times \mathcal{B}$ with*

$$[x, x'] = [\![y, y']\!], \quad (x; y), (x'; y') \in \mathcal{L}\,. \tag{2.5.5}$$ II.42

Then (2.5.5) *is true even for all* $(x; y), (x'; y') \in c\ell(\mathcal{L})$.

Moreover, there exist closed subspaces $\operatorname{dom} R, \mathcal{E} \subseteq \mathcal{A}$, $\operatorname{ran} R, \mathcal{F} \subseteq \mathcal{B}$ *and a linear, isometric and bi-continuous bijection* $R : \operatorname{dom} R \to \operatorname{ran} R$ *such that*

$$\mathcal{E} \subseteq c\ell(\operatorname{dom} L)^{[\circ]} \ \ \textit{and} \ \ \mathcal{F} \subseteq c\ell(\operatorname{ran} L)^{[\![\circ]\!]},$$

$$\operatorname{dom} R \dot{+} \mathcal{E} = c\ell(\operatorname{dom} L) \ \ \textit{and} \ \ \operatorname{ran} R \dot{+} \mathcal{F} = c\ell(\operatorname{ran} L), \tag{2.5.6}$$ II.42po

and

$$c\ell(\mathcal{L}) = \operatorname{graph} R \dot{+} (\{0\} \times \mathcal{F}) \dot{+} (\mathcal{E} \times \{0\}). \tag{2.5.7}$$ II.43

Proof. The validity of (2.5.5) for all $(x; y), (x'; y') \in c\ell(\mathcal{L})$ is an immediate consequence of the continuity of the scalar products and the projections from $\mathcal{A} \times \mathcal{B}$ onto $\mathcal{A}$ and onto $\mathcal{B}$.

Since closed subspaces of almost Pontryagin spaces are almost Pontryagin spaces and since $\mathcal{L} \subseteq c\ell(\mathcal{L}) \subseteq c\ell(\operatorname{dom} L) \times c\ell(\operatorname{ran} L)$, for the remaining assertions we can replace $\mathcal{A}$ by $c\ell(\operatorname{dom} \mathcal{L})$ and $\mathcal{B}$ by $c\ell(\operatorname{ran} \mathcal{L})$ in order to ensure that $\operatorname{dom} \mathcal{L}$ ($\operatorname{ran} \mathcal{L}$) is densely contained in $\mathcal{A}$ ($\mathcal{B}$).

Take Hilbert subspaces $(C, [.,.])$ of $(\mathcal{A}, [.,.], O)$ and $(\mathcal{D}, [\![.,.]\!])$ of $(\mathcal{B}, [\![.,.]\!], O)$ of finite codimension; see Theorem 2.5.2. Since $C \times \mathcal{D}$ is contained in $\mathcal{A} \times \mathcal{B}$ with finite codimension, for

$$\mathcal{L}' := \mathcal{L} \cap (C \times \mathcal{D})$$

we have $c\ell(\mathcal{L}') = c\ell(\mathcal{L}) \cap (C \times \mathcal{D})$, where $c\ell(\mathcal{L}')$ is contained in $c\ell(\mathcal{L})$ with finite codimension; see Lemma 2.8.2.

$\mathcal{L}'$ is the graph of an injective, linear and isometric map $L' : \operatorname{dom} \mathcal{L}' \to \operatorname{ran} \mathcal{L}'$. In fact, for $(x; y), (x'; y') \in \mathcal{L}'$ we conclude from (2.5.5)

$$x = x' \Leftrightarrow [x - x', x - x'] = 0 \Leftrightarrow [\![y - y', y - y']\!] = 0 \Leftrightarrow y = y'.$$

It is a standard result in Functional Analysis (see) that $L' : \operatorname{dom} \mathcal{L}' \to \operatorname{ran} \mathcal{L}'$ admits an isometric, linear, continous and bijective extension $S : c\ell(\operatorname{dom} \mathcal{L}') \to c\ell(\operatorname{ran} \mathcal{L}')$. Since S is continous, its graph is closed. Since S extends L', we have $c\ell(\mathcal{L}') = c\ell(\operatorname{graph} L') \subseteq \operatorname{graph} S$. Again using that S is a continuous extension of L', the fact that $\operatorname{dom} \mathcal{L}'$ is dense in $c\ell(\operatorname{dom} \mathcal{L}')$ implies $\operatorname{graph} S \subseteq c\ell(\operatorname{graph} L') = c\ell(\mathcal{L}')$. Thus, Missing Reference

$$\operatorname{graph} S = c\ell(\mathcal{L}') \ (\subseteq C \times \mathcal{D}).$$

Now we consider the subspaces $\mathcal{E}$ and $\mathcal{F}$ defined by

$$c\ell(\mathcal{L}) \cap (\{0\} \times \mathcal{B}) =: \{0\} \times \mathcal{F} \ \ \text{and} \ \ c\ell(\mathcal{L}) \cap (\mathcal{A} \times \{0\}) =: \mathcal{E} \times \{0\}.$$

Due to (2.5.5) we have $0 = [x, 0] = [\![y, z]\!]$ for all $(0; z) \in c\ell(\mathcal{L}) \cap (\{0\} \times \mathcal{B})$ and arbitrary $(x; y) \in \mathcal{L}$. Hence, $z \ [\![\perp]\!] \ \operatorname{ran} \mathcal{L}$, and by $\operatorname{ran} \mathcal{L}$'s density, $z \in \mathcal{B}^{[\![\circ]\!]}$. Thus,

$$\mathcal{F} \subseteq \mathcal{B}^{[\![\circ]\!]}, \ \ \mathcal{E} \subseteq \mathcal{A}^{[\circ]},$$

where the second inclusion is seen in the same way. From $\mathcal{A}^{[\circ]} \cap C = \{0\}$ and $\mathcal{B}^{[\![\circ]\!]} \cap \mathcal{D} = \{0\}$ we conclude that

$$c\ell(\mathcal{L}') \dot{+} (\{0\} \times \mathcal{F}) \dot{+} (\mathcal{E} \times \{0\})$$

is in fact a direct sum. With $c\ell(\mathcal{L}')$ also this subspace is contained in $c\ell(\mathcal{L})$ with finite codimension. Hence, we find a finite dimensional extension $\mathcal{R}$ of $c\ell(\mathcal{L}')$ such that

$$c\ell(\mathcal{L}) = \mathcal{R} \dotplus (\{0\} \times \mathcal{F}) \dotplus (\mathcal{E} \times \{0\}) = \mathcal{R} \dotplus \Big(c\ell(\mathcal{L}) \cap (\mathcal{A} \times \{0\}) \Big) \dotplus \Big(c\ell(\mathcal{L}) \cap (\{0\} \times \mathcal{B}) \Big).$$

This sum being direct implies that $\mathcal{R}$ is the graph of an injective linear operator $R : \operatorname{dom} R \to \operatorname{ran} R$ which extends S. Since $\operatorname{graph} R = \mathcal{R}$ ($\operatorname{dom} R$, $\operatorname{ran} R$) is a finite dimensional extension of $\operatorname{graph} S = c\ell(\mathcal{L}')$ ($\operatorname{dom} S$, $\operatorname{ran} S$), all these spaces are closed; see Lemma 2.8.3. By the closed graph Theorem (see), R and its inverse are continuous linear mappings. From $\operatorname{graph} R = \mathcal{R} \subseteq c\ell(\mathcal{L})$ we conclude that R is isometric.

Missing Reference

We have $\operatorname{dom} R \cap \mathcal{E} = \{0\}$, since $(x; y) \in \mathcal{R}$ and $(x; 0) \in c\ell(\mathcal{L})$ yields $(0; y) \in c\ell(\mathcal{L})$, and in turn $(x; y) \in \mathcal{R} \cap \Big((c\ell(\mathcal{L}) \cap (\{0\} \times \mathcal{B})) \dotplus (c\ell(\mathcal{L}) \cap (\mathcal{A} \times \{0\})) \Big) = \{(0; 0)\}$. Similarly, $\operatorname{ran} R \cap \mathcal{F} = \{0\}$. Finally, as a finite dimensional extension of $\operatorname{dom} R$ ($\operatorname{ran} R$) also $\operatorname{dom} R \dotplus \mathcal{E}$ ($\operatorname{ran} R \dotplus \mathcal{F}$) is closed. On the other hand this space contains the dense subspace $\operatorname{dom} \mathcal{L}$ ($\operatorname{ran} \mathcal{L}$). Hence, $\operatorname{dom} R \dotplus \mathcal{E} = \mathcal{A}$ and $\operatorname{ran} R \dotplus \mathcal{F} = \mathcal{B}$. ❑

A comparison of (2.5.6) and (2.5.7) immediately gives the following corollary.

II.41post1. **2.5.15 Corollary.** *With the notation and assumptions of Theorem* 2.5.14 *we have*

$$\operatorname{dom} c\ell(\mathcal{L}) = c\ell(\operatorname{dom} \mathcal{L}) \quad \textit{and} \quad \operatorname{ran} c\ell(\mathcal{L}) = c\ell(\operatorname{ran} \mathcal{L})\,. \tag{2.5.8}$$ II.42a

In particular, $\operatorname{dom} \mathcal{L}$ *and* $\operatorname{ran} \mathcal{L}$ *are closed, if* $\mathcal{L}$ *is closed.*

II.41post2. **2.5.16 Corollary.** *With the notation and assumptions of Theorem* 2.5.14 *assume in addition that* $c\ell(\operatorname{ran} L)$ *is nondegenerated. Then* $c\ell(\mathcal{L})$ *is the graph of an isometric, continuous, linear and surjective mapping* $T : c\ell(\operatorname{dom} L) \to c\ell(\operatorname{ran} L)$.

If $c\ell(\operatorname{ran} L)$ *and* $c\ell(\operatorname{dom} L)$ *are nondegenerated, then* T *is bijective and bi-continuous.*

Proof. If $c\ell(\operatorname{ran} L)$ is nondegenerated, then we have $\mathcal{F} \subseteq c\ell(\operatorname{ran} L)^{[\![\circ]\!]} = \{0\}$ in Theorem 2.5.14. Hence,

$$c\ell(\mathcal{L}) = \operatorname{graph} R \dotplus (\mathcal{E} \times \{0\})\,, \tag{2.5.9}$$ udo58

with $\operatorname{ran} R = c\ell(\operatorname{ran} L)$ and a closed $\operatorname{dom} R$ such that $c\ell(\operatorname{dom} L) = \operatorname{dom} R \dotplus \mathcal{E}$. According to Lemma 2.8.1, $\operatorname{dom} R \dotplus \mathcal{E}$ is algebracically and topologically isomorphic to $\operatorname{dom} R \times \mathcal{E}$. Consequently, $T : \operatorname{dom} R \dotplus \mathcal{E} \to c\ell(\operatorname{ran} \mathcal{L})$ defined by $T(x + y) := Rx,\ x \in \operatorname{dom} R,\ y \in \mathcal{E}$, is linear and continuous such that $\operatorname{ran} T = \operatorname{ran} R = c\ell(\operatorname{ran} L)$. Its graph, obviously, coincides with (2.5.9). (2.5.5) for all $(x; y), (x'; y') \in c\ell(\mathcal{L})$, finally yields the fact that T is isometric.

If $c\ell(\operatorname{ran} L)$ and $c\ell(\operatorname{dom} L)$ are nondegenerated, then also $\mathcal{E} \subseteq c\ell(\operatorname{ran} L)^{[\circ]} = \{0\}$. Hence, $R = T$, and bijectivity and bi-continuity follow from Theorem 2.5.14. ❑

2.6 Completions

section2.6

It is a basic and standard fact that a positive definite scalar product space $\mathcal{L}$ has an essentially unique Hilbert space completion. By this we mean that there exists a

Hilbert space which contains $\mathcal{L}$ as a dense subspace, and that this Hilbert space is unique up to isomorphisms which leave $\mathcal{L}$ pointwise fixed. When passing to the indefinite setting, the corresponding fact is not true in general. A scalar product space neither necessarily possesses a completion, cf. Example 2.6.5, nor has a unique completion, cf. Example 2.6.11. In view of this it becomes important to see that scalar product spaces with finite negative index behave well concerning completions.

First of all, let us make precise what means "completion" and "essentially the same" when considering two completions.

II.60. **2.6.1 Definition.** Let $(\mathcal{L}, [.,.])$ be a scalar product space. We call a tuple $\langle \iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O})\rangle$ a *completion* of $(\mathcal{L}, [.,.])$, if

(1) $(\mathcal{A}, [\![.,.]\!], \mathcal{O})$ is a Gram space.

(2) ι is a linear and isometric map of $\mathcal{L}$ onto a dense subspace of $\mathcal{A}$.

If $\langle \iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O})\rangle$ is a completion of $(\mathcal{L}, [.,.])$ with $(\mathcal{A}, [\![.,.]\!], \mathcal{O})$ being a Krein (almost Pontryagin, Pontryagin, or Hilbert) space, then we speak of a *Krein* (*almost Pontryagin-*, *Pontryagin-*, or *Hilbert-*) *space completion* of $(\mathcal{L}, [.,.])$. ◊

II.60post. *2.6.2 Remark.* Notice that in Definition 2.6.1 we do not assume ι to be injective. However, if $(\mathcal{L}, [.,.])$ is nondegenerated, then it always is, because then $\ker \iota \subseteq \mathcal{L}^{[\circ]} = \{0\}$; see (1.2.7). ◊

II.66. **2.6.3 Definition.** Let $(\mathcal{L}, [.,.])$ be a scalar product space, and let $\langle \iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O})\rangle$ and $\langle \iota', (\mathcal{A}', [\![.,.]\!]', \mathcal{O}')\rangle$ be two completions of $\mathcal{L}$. We say that these two completions are isomorphic, if there exists a linear, bijective, isometric, and bi-continuous operator T from $\mathcal{A}$ onto $\mathcal{A}'$, such that $T \circ \iota = \iota'$.

$$\begin{tikzcd} & \mathcal{L} \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\ \mathcal{A} \arrow[rr, "T"'] & & \mathcal{A}' \end{tikzcd}$$

◊

It is immediate that isomorphy of completions is an equivalence relation on the set of all completions of a given scalar product space.

isorem. *2.6.4 Remark.* In Definition 2.6.3 the assumption that T is isometric, is not necessary. In fact, $T \circ \iota = \iota'$ yields

$$[\![\iota x, \iota y]\!] = [x, y] = [\![\iota' x, \iota y']\!]' = [\![T(\iota x), T(\iota y)]\!]', \ x, y \in \mathcal{L}.$$

By continuity the fact that $\iota\mathcal{L}$ is dense in $\mathcal{A}$ yields the isometry of T.

Also note, that by the open mapping theorem, in the above definition it would be enough to assume that $T : \mathcal{A} \to \mathcal{A}'$ is linear, bijective, isometric, and continuous, because then the continuity of T^{-1} automatically follows. ◊

Let us provide an example of a scalar product which does not have a completion. In fact, we already met such a space earlier.

II.61. *2.6.5 Example.* We start with a general observation. Let $(\mathcal{L}, [.,.])$ be a nondegenerated scalar product space, and assume that $\mathcal{L}$ has a completion, say $\langle \iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O})\rangle$. Choose a compatible Hilbert space scalar product $(\!|.,.|\!)$ on $\mathcal{A}$, and define

$$(x,y) := (\!|\iota x, \iota y|\!), \quad x,y \in \mathcal{L}\,.$$

Then $(.,.)$ is a positive definite scalar product on $\mathcal{L}$. As mentioned in Remark 2.6.2, ι is injective. Since $[\![.,.]\!]$ is continuous w.r.t. $(.,.)$, we find $C \geq 0$ with

$$\big|[x,y]\big| = \big|[\![\iota x, \iota y]\!]\big| \leq C(\!|\iota x, \iota y|\!) = C(x,y), \quad x,y \in \mathcal{L}\,.$$

Hence, $[.,.]$ is continuous w.r.t. $(.,.)$.

Consider the scalar product space $(\mathcal{L}, [.,.])$ defined in Example 2.1.2. This space is obviously nondegenerated, however, its scalar product is not continuous w.r.t. any norm. From the above we conclude that it does not possess a completion. ◊

In the next statement we provide a large class of scalar product spaces which do have a completion.

II.62. **2.6.6 Proposition.** *Let $(\mathcal{L}, [.,.])$ be a scalar product space, and assume that $\mathcal{L}$ can be decomposed as a direct and orthogonal sum*

$$\mathcal{L} = \mathcal{L}_+ [\dot{+}] \mathcal{L}_- [\dot{+}] \mathcal{L}^{[\circ]} \tag{2.6.1}$$ II.63

with $\mathcal{L}_+$ positive definite and $\mathcal{L}_-$ negative definite. Then $\mathcal{L}$ has a Krein space completion.

Proof. Choose Hilbert space completions $\langle \iota_+, (\mathcal{H}_+, (.,.)_+)\rangle$ and $\langle \iota_-, (\mathcal{H}_-, (.,.)_-)\rangle$ of $(\mathcal{L}_+, [.,.]|_{\mathcal{L}_+ \times \mathcal{L}_+})$ and $(\mathcal{L}_-, -[.,.]|_{\mathcal{L}_- \times \mathcal{L}_-})$, respectively. Then the space $\mathcal{A} := \mathcal{H}_+ \times \mathcal{H}_-$ endowed with the scalar product

$$[\![(x_+;x_-),(y_+;y_-)]\!] := (x_+,y_+)_+ - (x_-,y_-)_-, \quad (x_+;x_-),(y_+;y_-)\mathcal{H}_+ \times \mathcal{H}_-\,,$$

and the product topology is a Krein space; cf. Theorem 2.3.3. Moreover, a scalar product compatible with its Krein space topology is given as

$$((x_+;x_-),(y_+;y_-)) := (x_+,y_+)_+ - (x_-,y_-)_-, \quad (x_+;x_-),(y_+;y_-)\mathcal{H}_+ \times \mathcal{H}_-\,.$$

Set

$$\iota : \begin{cases} \mathcal{L}_+ [\dot{+}] \mathcal{L}_- [\dot{+}] \mathcal{L}^{[\circ]} & \rightarrow & \mathcal{A} \\ x_+ + x_- + x_0 & \mapsto & (\iota_+ x_+; \iota_- x_-) \end{cases}$$

Then, for each two elements $x_+ + x_- + x_0, y_+ + y_- + y_0 \in \mathcal{L}$, we have

$$\begin{aligned} [\![\iota(x_+ + x_- + x_0), \iota(y_+ + y_- + y_0)]\!] &= (\iota_+ x_+, \iota_+ y_+)_+ - (\iota_- x_-, \iota_- y_-)_- = \\ = [x_+,y_+] + [x_-,y_-] &= [x_+ + x_-, y_+ + y_-] = [x_+ + x_- + x_0, y_+ + y_- + y_0]\,, \end{aligned}$$

i.e., ι is isometric. Since $\mathcal{A}$ carries the product topology and

$$\operatorname{ran}\iota = \operatorname{ran}\iota_+ + \operatorname{ran}\iota_-\,,$$

the range of ι is dense in $\mathcal{A}$. ❑

As a consequence of continuity passing to completions preserves positive and negative index of the space.

II.64. **2.6.7 Lemma.** *Let* $(\mathcal{L}, [.,.])$ *be a scalar product space, and let* $\langle \iota, (\mathcal{A}, [\![.,.]\!], O)\rangle$ *be a completion of* $\mathcal{L}$*. Then*

$$\operatorname{ind}_+(\mathcal{L}, [.,.]) = \operatorname{ind}_+(\mathcal{A}, [\![.,.]\!]), \quad \operatorname{ind}_-(\mathcal{L}, [.,.]) = \operatorname{ind}_-(\mathcal{A}, [\![.,.]\!]) .$$

Proof. By Proposition 1.6.7 we have

$$\operatorname{ind}_+(\mathcal{L}, [.,.]) = \operatorname{ind}_+(\iota\mathcal{L}, [\![.,.]\!]), \quad \operatorname{ind}_-(\mathcal{L}, [.,.]) = \operatorname{ind}_-(\iota\mathcal{L}, [\![.,.]\!]) .$$

Since $\iota\mathcal{L}$ is dense in $\mathcal{A}$, the assertion follows from Proposition 2.1.4. ❑

The condition $\operatorname{ind}_-(\mathcal{L}, [.,.]) < \infty$ suffices, to be sure that there are completions.

II.65. **2.6.8 Proposition.** *Let* $(\mathcal{L}, [.,.])$ *be a scalar product space. Then the following are equivalent.*

(1) $\operatorname{ind}_-(\mathcal{L}, [.,.]) < \infty$.

(2) $(\mathcal{L}, [.,.])$ *has a Pontryagin space completion.*

(3) $(\mathcal{L}, [.,.])$ *has an almost Pontryagin space completion.*

Proof. Assume that $\operatorname{ind}_- \mathcal{L} < \infty$. Then $\mathcal{L}$ can be decomposed as in (2.6.1), cf. Proposition 1.6.2. Thus there exists a Krein space completion $\mathcal{A}$ of $\mathcal{L}$. However, $\operatorname{ind}_- \mathcal{A} = \operatorname{ind}_- \mathcal{L} < \infty$, and hence $\mathcal{A}$ is in fact a Pontryagin space.

Since each Pontryagin space is also an almost Pontryagin space, the implication "(2) ⇒ (3)" is trivial. Finally, assume that $\mathcal{L}$ has an almost Pontryagin space completion, say $\mathcal{A}$. Then $\operatorname{ind}_- \mathcal{L} = \operatorname{ind}_- \mathcal{A} < \infty$. ❑

It turns out that a space $\mathcal{L}$ with finite negative index always has nonisomorphic almost Pontryagin space completions. We defer a detailed treatment to Section 2.7*; right now let us only mention that nonuniqueness originates from presence of degeneracy as one can guess after a glance at the following result.

II.68. **2.6.9 Proposition.** *Let* $(\mathcal{V}, [.,.])$ *be a scalar product space with* $\operatorname{ind}_- \mathcal{V} < \infty$*. Then each two Pontryagin space completions of* $\mathcal{V}$ *are isomorphic.*

Proof. Let $\langle \iota_1, (\mathcal{A}_1, [\![.,.]\!], O)\rangle$ and $\langle \iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, O_2)\rangle$ be two completions of $(\mathcal{V}, [.,.])$. Consider

$$\mathcal{L} := \{(\iota_1 x; \iota_2 x) : x \in \mathcal{V}\} \subseteq \mathcal{A}_1 \times \mathcal{A}_2 .$$

Then $\mathcal{L}$ is a linear subspace and we have

$$[\![\iota_1 x, \iota_1 y]\!] = [x, y] = [\![\iota_2 x, \iota_2 y]\!], \quad x, y \in \mathcal{V} .$$

Moreover,

$$\operatorname{dom} \mathcal{L} := \{x \in \mathcal{A}_1 : \exists x_2 \in \mathcal{A}_2 : (x; x_2) \in \mathcal{L}\} = \iota_1(\mathcal{V}) ,$$
$$\operatorname{ran} \mathcal{L} := \{x_2 \in \mathcal{A}_2 : \exists x \in \mathcal{A}_1 : (x; x_2) \in \mathcal{L}\} = \iota_2(\mathcal{V}) ,$$

and hence $c\ell(\operatorname{dom}\mathcal{L}) = \mathcal{A}_1$ and $c\ell(\operatorname{ran}\mathcal{L}) = \mathcal{A}_2$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are nondegenerated, by Corollary 2.5.16 we obtain a linear, isometric, bijective, and bi-continuous map $T : \mathcal{A}_1 \to \mathcal{A}_2$ with graph $T = c\ell(\mathcal{L})$. ❑

II.56.

2.6.10 Example. Let q be a function which is meromorphic in the open upper half plane $\mathbb{C}^+$, and denote its domain of homolorphy by $\rho(q)$. For $z_1, \dots, z_n \in \rho(q)$ set

$$\mathbb{P}_{z_1,\dots,z_n} := \left(\frac{q(z_i) - \overline{q(z_j)}}{z_i - \overline{z_j}}\right)_{i,j=1}^{n},$$

and denote by $\operatorname{ind}_- \mathbb{P}_{z_1,\dots,z_n}$ the number of negative eigenvalues of this matrix counted according to their multiplicities; cf. Example 1.6.5.

We say that the function q belongs to the *indefinite Nevanlinna class* $\mathcal{N}_{<\infty}$, if

$$\sup_{n\in\mathbb{N},\ z_1,\dots,z_n\in\rho(q)} \operatorname{ind}_- \mathbb{P}_{z_1,\dots,z_n} < \infty .$$

With each function $q \in \mathcal{N}_{<\infty}$ a Pontryagin space $\mathcal{A}_q$ is naturally associated. We will see that this space reflects a big amount of the properties of q, cf. . For $M = \rho(q)$ and

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$$K(\zeta, \eta) = \frac{q(\eta) - \overline{q(\zeta)}}{\eta - \overline{\zeta}}, \quad \zeta, \eta \in \rho(q),$$

consider the space $\mathcal{F}(M) = \mathcal{F}(\rho(q))$ provided with the scalar product

$$[f, g]_q := \sum_{\zeta,\eta\in M} \overline{g(\zeta)} \cdot K(\zeta, \eta) \cdot f(\eta), \quad f, g \in \mathcal{F}(\rho(q))$$

as in Example 1.1.4. The condition that $q \in \mathcal{N}_{<\infty}$ implies $\operatorname{ind}_-(\mathcal{F}(\rho(q)), [.,.]_q) < \infty$; see Example 1.6.9. In fact,

$$\operatorname{ind}_-(\mathcal{F}(\rho(q)), [.,.]_q) = \sup_{n\in\mathbb{N},\ z_1,\dots,z_n\in\rho(q)} \operatorname{ind}_- \mathbb{P}_{z_1,\dots,z_n} .$$

Hence, there exists an, up to isomorphism, unique Pontryagin space completion $\mathcal{A}_q$ of $\mathcal{F}(\rho(q))$. ◊

It is interesting to see an example of a scalar product space which has two non isomorphic Krein space completions.

2.6.11 Example. Let $\mathcal{V}$ be the space all komplex *double-finite two-sided sequences*

Ich habs mal two-sided sequences genannt, weil bouble seuqence schon vergeben!

$$\mathcal{V} := \{(\alpha_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : \ \alpha_j \in \mathbb{C}, \exists N \in \mathbb{N} : \alpha_j = 0, |j| > N\},$$

and define a scalar product on $\mathcal{V}$ as

$$[(\alpha_j)_{j\in\mathbb{Z}}, (\beta_j)_{j\in\mathbb{Z}}] := \sum_{j\geq 0} \alpha_j \cdot \overline{\beta_j} - \sum_{j<0} \alpha_j \cdot \overline{\beta_j}, \quad (\alpha_j)_{j\in\mathbb{Z}}, (\beta_j)_{j\in\mathbb{Z}} \in \mathcal{V}.$$

This expression is well-defined since each of the sums on the right side contains only finitely many nonzero summands.

Set $e_n := (\delta_{nj})_{j\in\mathbb{Z}}$, and consider the subspaces

$$\mathcal{V}_+ := \operatorname{span}\{e_n : n \geq 0\}, \quad \mathcal{V}_- := \operatorname{span}\{e_n : n < 0\}.$$

Then $\mathcal{V}_+$ is positive definite and $\mathcal{V}_-$ is negative definite. Moreover, $\mathcal{V}_+[\perp]\mathcal{V}_-$ and $\mathcal{V}_+ \dot{+} \mathcal{V}_- = \mathcal{V}$. Let $\langle \iota, (\mathcal{A}, [\![.,.]\!])\rangle$ be the Krein space completion of $\mathcal{V}$ as constructed in the proof of Proposition 2.6.6 starting from the decomposition $\mathcal{V} = \mathcal{V}_+[\dot{+}]\mathcal{V}_-$. Moreover, denote by $(.,.)$ the compatible Hilbert space scalar product on $\mathcal{A}$ specified in this construction.

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ defined as

$$a_n := \iota(\frac{e_n}{\sqrt{n}}), \quad n \in \mathbb{N}.$$

Then $a_n \in \iota(\mathcal{V}_+)$, and hence

$$(a_n, a_n) = (\iota(\frac{e_n}{\sqrt{n}}), \iota(\frac{e_n}{\sqrt{n}}))_+ = [\frac{e_n}{\sqrt{n}}, \frac{e_n}{\sqrt{n}}] = \frac{1}{n}.$$

Thus $\lim_{n\to\infty} a_n = 0$.

Set $f_n := e_n + \frac{|n|}{|n|+1} e_{-n}$, $n \in \mathbb{Z}$, and consider the subspaces

$$\mathcal{V}'_+ := \operatorname{span}\{f_n : n \geq 0\}, \quad \mathcal{V}'_- := \operatorname{span}\{f_n : n < 0\}.$$

A computation gives

$$f_n - \frac{|n|}{|n|+1} f_{-n} = \frac{2|n|+1}{(|n|+1)^2} e_n, \quad n \in \mathbb{Z},$$

and we conclude that $\mathcal{V}'_+ + \mathcal{V}'_- = \mathcal{V}$. Another computation gives

$$[f_n, f_m] = \begin{cases} \frac{2|n|+1}{(|n|+1)^2}, & n = m \geq 0 \\ -\frac{2|n|+1}{(|n|+1)^2}, & n = m < 0 \\ 0 \quad , & n \neq m \end{cases}$$

and we conclude that $\mathcal{V}'_+$ is positive definite, $\mathcal{V}'_-$ is negative definite, and $\mathcal{V}'_+[\perp]\mathcal{V}'_-$. Consequently, $\mathcal{V}'_+ \cap \mathcal{V}'_- = \{0\}$, and in turn $\mathcal{V} = \mathcal{V}'_+[\dot{+}]\mathcal{V}'_-$. Let $\langle \iota', (\mathcal{A}', [\![.,.]\!]')\rangle$ be the Krein space completion constructed from this decomposition as in the proof of Proposition 2.6.6, and denote by $(.,.)'$ the compatible Hilbert space scalar product specified in this construction.

Consider the sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathcal{A}'$ defined as

$$b_n := \iota'(\frac{e_n}{\sqrt{n}}), \quad n \in \mathbb{N}.$$

We can write

$$b_n = \frac{(n+1)^2}{(2n+1)\sqrt{n}} \cdot \Big[\iota'(f_n) - \frac{n}{n+1}\iota'(f_{-n})\Big].$$

Since $\iota'(f_n) \in \iota'(\mathcal{V}'_+)$ and $\iota'(f_{-n}) \in \iota'(\mathcal{V}'_-)$, we have

$$\begin{aligned} (b_n, b_n)' &= \Big[\frac{(n+1)^2}{(2n+1)\sqrt{n}}\Big]^2 \cdot \Big[(\iota'_+ f_n, \iota'_+ f_n)_+ + \frac{n^2}{(n+1)^2}(\iota'_-(f_{-n}), \iota'_-(f_{-n}))_-\Big] = \\ &= \Big[\frac{(n+1)^2}{(2n+1)\sqrt{n}}\Big]^2 \cdot \Big[[f_n, f_n] - \frac{n^2}{(n+1)^2}[f_{-n}, f_{-n}]\Big] = \\ &= \Big[\frac{(n+1)^2}{(2n+1)\sqrt{n}}\Big]^2 \cdot \frac{2n+1}{(n+1)^2} \cdot \Big[1 + \frac{n^2}{(n+1)^2}\Big]. \end{aligned}$$

Thus $\lim_{n\to\infty}(b_n, b_n)' = 1$.

Assume now that $\langle\iota, (\mathcal{A}, [\![.,.]\!])\rangle$ and $\langle\iota', (\mathcal{A}', [\![.,.]\!]')\rangle$ were isomorphic, and let T be as in Definition 2.6.3. Then

$$b_n = \iota'(\frac{e_n}{\sqrt{n}}) = (T\circ\iota)(\frac{e_n}{\sqrt{n}}) = T(a_n), \quad n \in \mathbb{N}\,.$$

Since T is continuous, we would arrive at the contradiction $\lim_{n\to\infty}(b_n, b_n)' = 0$. ◊

2.7 *Almost Pontryagin Space Completions

section2.7

In this section we give a detailed description of the totality of all almost Pontryagin space completions of a given scalar product space $\mathcal{V}$. It turns out that these completions are in a one-to-one correspondence with certain subspaces of the *algebraic dual space* $\mathcal{V}^*$ of $\mathcal{V}$, i.e., of

$$\mathcal{V}^* := \{f \in \mathbb{C}^{\mathcal{V}} : \ f \text{ is linear } \}\,.$$

In order to formulate the definition of the map which turns out to establish this connection, let us recall the concept of the *algebraic dual map*. For two linear spaces $\mathcal{V}$ and $\mathcal{W}$ and a linear map $f : \mathcal{V} \to \mathcal{W}$ this is the linear mapping defined by

$$f^* : \begin{cases} \mathcal{W}^* & \to & \mathcal{V}^* \\ \varphi & \mapsto & \varphi\circ f \end{cases}$$

In particular, for a completion $(\iota, (\mathcal{A}, [\![.,.]\!], O))$ of a scalar product space $(\mathcal{V}, [.,.])$ we have $\iota^* : \mathcal{A}^* \to \mathcal{V}^*$. Since the topological dual space $(\mathcal{A}, O)'$ of $\mathcal{A}$ is contained in $\mathcal{A}^*$ the following definition makes sense.

II.12. **2.7.1 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space, let $(\iota, (\mathcal{A}, [\![.,.]\!], O))$ be a completion of $\mathcal{V}$. Then we set

$$\Psi(\iota, (\mathcal{A}, [\![.,.]\!], O)) := \iota^*((\mathcal{A}, O)')\,.$$

◊

We will apply the usual abuse of language, and write $\Psi(\iota, \mathcal{A})$ if no confusion concerning inner product and topology can occur.

II.12post. *2.7.2 Remark.* If $\iota^*(\varphi_1) = \iota^*(\varphi_2)$ for $\varphi_1, \varphi_2 \in (\mathcal{A}, O)'$, then $\varphi_1(\iota(x)) = \varphi_2(\iota(x))$ for all $x \in \mathcal{V}$. Due to the density of $\iota(\mathcal{V})$ in $\mathcal{A}$ this yields $\varphi_1 = \varphi_2$. Thus, ι^* acts injectively on $(\mathcal{A}, O)'$. ◊

The correspondence given by Ψ will be seen to respect a certain order structure.

II.13. **2.7.3 Definition.** Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, O_1))$ and $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, O_2))$ be two completions of $\mathcal{V}$. Then we write

$$(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, O_1)) \le (\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, O_2))\,,$$

if there exists a linear, isometric, surjective, and continuous map $\varpi : \mathcal{A}_2 \to \mathcal{A}_1$, such that $\varpi\circ\iota_2 = \iota_1$.

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] & \\ \mathcal{A}_1 & & \mathcal{A}_2 \arrow[ll, two heads, "\varpi"'] \end{tikzcd}$$

◊

For short we will use $(\iota_1, \mathcal{A}_1) \le (\iota_2, \mathcal{A}_2)$ for $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, O_1)) \le (\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, O_2))$ if no confusion can occur.

II.20. **2.7.4 Lemma.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space. The relation "$\le$" defined between completions of $\mathcal{V}$ is reflexive and transitive. Moreover,*

$$(\iota_1, \mathcal{A}_1) \le (\iota_2, \mathcal{A}_2) \text{ and } (\iota_2, \mathcal{A}_2) \le (\iota_1, \mathcal{A}_1), \tag{2.7.1}$$ tiokgf

if and only if $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, O_1))$ and $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, O_2))$ are isomorphic.

Proof. Reflexivity and transitivity is obvious. Assume that (2.7.1) holds true, and let $\varpi : \mathcal{A}_2 \to \mathcal{A}_1$ and $\vartheta : \mathcal{A}_1 \to \mathcal{A}_2$ be linear, isometric, surjective, and continuous maps, with

$$\varpi \circ \iota_2 = \iota_1, \quad \vartheta \circ \iota_1 = \iota_2 .$$

Then, for each $x \in \mathcal{V}$,

$$(\varpi \circ \vartheta)(\iota_1 x) = \varpi(\iota_2 x) = \iota_1 x, \quad (\vartheta \circ \varpi)(\iota_2 x) = \vartheta(\iota_1 x) = \iota_2 x .$$

Hence, $(\varpi \circ \vartheta)|_{\iota_1 \mathcal{V}} = \mathrm{id}_{\iota_1 \mathcal{V}}$ and $(\vartheta \circ \varpi)|_{\iota_2 \mathcal{V}} = \mathrm{id}_{\iota_2 \mathcal{V}}$. Since $\iota_1 \mathcal{V}$ is dense in $\mathcal{A}_1$, $\iota_2 \mathcal{V}$ is dense in $\mathcal{A}_2$, and ϖ and ϑ are both continuous, we get

$$\varpi \circ \vartheta = \mathrm{id}_{\mathcal{A}_1}, \quad \vartheta \circ \varpi = \mathrm{id}_{\mathcal{A}_2} .$$

This shows that ϖ and ϑ are mutually inverse bijections, and hence bi-continuous. Thus, $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, O_1))$ and $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, O_2))$ are isomorphic. The converse is clear. ❑

II.16pre1. **2.7.5 Lemma.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space, and let $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, O_1))$ and $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, O_2))$ be two completions of $\mathcal{V}$. Then $(\iota_1, \mathcal{A}_1) \le (\iota_2, \mathcal{A}_2)$ implies $\Psi(\iota_1, \mathcal{A}_1) \subseteq \Psi(\iota_2, \mathcal{A}_2)$.*

Proof. By definition we have $\varpi \circ \iota_2 = \iota_1$ for some linear, isometric, surjective, and continuous map $\varpi : \mathcal{A}_2 \to \mathcal{A}_1$. This gives

$$\iota_1^* = (\varpi \circ \iota_2)^* = \iota_2^* \circ \varpi^* ,$$

and in turn

$$\Psi(\iota_1, \mathcal{A}_1) = \iota_1^*(\mathcal{A}_1') = \iota_2^* \circ \varpi^*(\mathcal{A}_1') \subseteq \iota_2^*(\mathcal{A}_2') = \Psi(\iota_2, \mathcal{A}_2) .$$

❑

In the following example we shall show, how to construct from a given completion another completion, which is larger w.r.t. $\le$.

construex. *2.7.6 Example.* Let $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, O_1))$ be a completion of $\mathcal{V}$, and let $\mathcal{L}$ be a subspace of $\mathcal{V}^*$ which contains $\Psi(\iota_1, \mathcal{A}_1)$ with finite codimension $n \in \mathbb{N}$. We shall construct a completion $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, O_2))$ such that $(\iota_1, \mathcal{A}_1) \le (\iota_2, \mathcal{A}_2)$ and $\Psi(\iota_2, \mathcal{A}_2) = \mathcal{L}$.

Choose a linearly independent set $\{\varphi_1, \dots, \varphi_n\}$ with

$$\mathcal{L} = \Psi(\iota_1, \mathcal{A}_1) \dotplus \operatorname{span}\{\varphi_1, \dots, \varphi_n\} .$$

Let $(.,.)_1$ be a compatible scalar product on $\mathcal{A}_1$, and set

$$\begin{aligned}
&\mathcal{A}_2 := \mathcal{A}_1 \times \mathbb{C}^n\,,\\
&[\![(x;\alpha_1;\ldots;\alpha_n),(y;\beta_1;\ldots;\beta_n)]\!]_2 := [\![x,y]\!]_1\,,\\
&\Big((x;\alpha_1;\ldots;\alpha_n),(y;\beta_1;\ldots;\beta_n)\Big)_2 := (x,y)_1 + \sum_{i=1}^n \alpha_i\overline{\beta_i}\,,\\
&\iota_2 z := (\iota_1 z;\varphi_1(z);\ldots;\varphi_n(z)),\quad z\in\mathcal{V}\\
&\varpi(x;\alpha_1;\ldots;\alpha_n) := x\,.
\end{aligned}$$

$(.,.)_2$ is a Hilbert space scalar product on $\mathcal{A}_2$, and the topology $\mathcal{O}_2$ induced by $(.,.)_2$ is the product topology of $\mathcal{O}_1$ and the euclidean topology on $\mathbb{C}^n$. Obviously, $[\![.,.]\!]_2$ is continuous w.r.t. $\mathcal{O}_2$. Thus, $(\mathcal{A}_2,[\![.,.]\!]_2,\mathcal{O}_2)$ is a Gram space.

Moreover, the mapping $\varpi:\mathcal{A}_2\to\mathcal{A}_1$ is isometric and surjective, and the mapping $\iota_2:\mathcal{V}\to\mathcal{A}_2$ is isometric and satisfies $\iota_2\circ\varpi=\iota_1$. Let us show that $\iota_2(\mathcal{V})$ is dense in $\mathcal{A}_2$. For this assume that $(y;\beta_1;\ldots;\beta_n)\in(\operatorname{ran}\iota_2)^{(\perp)_2}$. Then

$$0=\Big((\iota_1 z;\varphi_1(z);\ldots;\varphi_n(z)),(y;\beta_1;\ldots;\beta_n)\Big)_2 = (\iota_1 z,y)_1+\sum_{i=1}^n\varphi_i(z)\overline{\beta_i},\quad z\in\mathcal{V}\,, \tag{2.7.2}$$ II.26

and hence

$$\sum_{i=1}^n\overline{\beta_i}\varphi_i(z) = -(\iota_1 z,y)_1 = \iota_1^*((.,y)_1)\,(z),\quad z\in\mathcal{V}\,.$$

Consequently, $\sum_{i=1}^n\overline{\beta_i}\varphi_i\in\Psi(\iota_1,\mathcal{A}_1)$. By our choice of the functionals φ_i, we get $\beta_1=\ldots=\beta_n=0$. In turn, (2.7.2) implies $y\in(\operatorname{ran}\iota_1)^{(\perp)_1}$, and hence $y=0$. Thus, $(\iota_2,(\mathcal{A}_2,[\![.,.]\!]_2,\mathcal{O}_2))$ is a completion of $\mathcal{V}$ such that $(\iota_1,\mathcal{A}_1)\preceq(\iota_2,\mathcal{A}_2)$.

For the verification of $\Psi(\iota_2,\mathcal{A}_2)=\mathcal{L}$ note that from Lemma 2.7.5 we immediately get $\Psi(\iota_1,\mathcal{A}_1)\subseteq\Psi(\iota_2,\mathcal{A}_2)$. If we take $(y;\beta_1;\ldots;\beta_n)\in\mathcal{A}_2$ with $y=0$ and $\beta_j=\delta_{ij}$, then

$$\iota_2^*((z,(y;\beta_1;\ldots;\beta_n))_2)(z)=\varphi_i(z),\ z\in\mathcal{V}\,.$$

Hence, $\varphi_i\in\Psi(\iota_2,\mathcal{A}_2)$ for $i=1,\ldots,n$. Thus, $\mathcal{L}\subseteq\Psi(\iota_2,\mathcal{A}_2)$.

Missing Reference

On the other hand, by the Riesz-Fischer Theorem (see) any element of $\Psi(\iota_2,\mathcal{A}_2)$ is of the form $\varphi(z)=(\iota_2 z,(y;\beta_1;\ldots;\beta_n))_2,\ z\in\mathcal{V}$, for some $(y;\beta_1;\ldots;\beta_n)\in\mathcal{A}_2$, i.e.,

$$\varphi(z)=(\iota_1 z,y)_1+\sum_{i=1}^n\varphi_i(z)\overline{\beta_i}=\iota_1^*(y)(z)+\sum_{i=1}^n\varphi_i(z)\overline{\beta_i}\,.$$

Therefore, $\varphi\in\Psi(\iota_1,\mathcal{A}_1)+\{\varphi_1,\ldots,\varphi_n\}=\mathcal{L}$, and we have also shown that $\Psi(\iota_2,\mathcal{A}_2)\subseteq\mathcal{L}$.

Finally note that if $(\mathcal{V},[.,.])$ admits almost Pontryagin space completions, i.e., $\operatorname{ind}_-(\mathcal{V},[.,.])<\infty$ (see Proposition 2.6.8), then $(\mathcal{A}_1,[\![.,.]\!]_1,\mathcal{O}_1)$ is an almost Pontryagin space completion if and only if $(\mathcal{A}_2,[\![.,.]\!]_2,\mathcal{O}_2)$ is an almost Pontryagin space completion. This is a consequence of Lemma 2.5.8 together with the fact that $\dim\ker\varpi=n<\infty$. ◊

In Proposition 2.7.7 we will see that $\Psi(\iota_1,\mathcal{A}_1)=\Psi(\iota_2,\mathcal{A}_2)$ imlpies that $(\iota_1,\mathcal{A}_1)$ and $(\iota_2,\mathcal{A}_2)$ are isomorphic. In order to motivate the verification of this result, let us pause and revisit the proof of Proposition 2.6.9, where we showed that each two Pontryagin space completions are isomorphic. The relation $T \circ \iota_1 = \iota_2$ means nothing else but

$$\operatorname{graph} T \supseteq \{(\iota_1 x; \iota_2 x) : x \in \mathcal{V}\}.$$

In the proof of Proposition 2.6.9 we applied Corollary 2.5.16 with the subspace on the right side, and obtained T as an extension by continuity; the source of continuity being isometry. This is not possible in the present situation. Indeed for degenerated $\mathcal{A}_2$ not even the hypothesis of the first part of Corollary 2.5.16 is fullfilled. In the present situation, the source of continuity shall be the assumed relation $\Psi(\iota_1, \mathcal{A}_1) = \Psi(\iota_2, \mathcal{A}_2)$.

II.16pre2. **2.7.7 Proposition.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space. Two completions $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, \mathcal{O}_1))$ and $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, \mathcal{O}_2))$ of $\mathcal{V}$ are isomorphic if and only if $\Psi(\iota_1, \mathcal{A}_1) = \Psi(\iota_2, \mathcal{A}_2)$.*

Proof. By Lemma 2.7.4 and Lemma 2.7.5 it remains to show that $\Psi(\iota_1, \mathcal{A}_1) = \Psi(\iota_2, \mathcal{A}_2)$ implies that $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ are isomorphic. For this consider the linear subspace (again we anticipate the language of linear relations)

$$R := \{(\psi; \varphi) \in \mathcal{A}_1' \times \mathcal{A}_2' : \iota_1^* \psi = \iota_2^* \varphi\} \tag{2.7.3}$$ Rdef

of the Banach space $\mathcal{A}_1' \times \mathcal{A}_2'$. R is closed, because if $(\psi_n; \varphi_n)$, $n \in \mathbb{N}$, is a sequence in R with

$$\lim_{n\to\infty} (\psi_n; \varphi_n) = (\psi; \varphi) \in \mathcal{A}_1' \times \mathcal{A}_2',$$

then

$$(\iota_1^* \psi)(x) = \psi(\iota_1 x) = \lim_{n\to\infty} \psi_n(\iota_1 x) = \lim_{n\to\infty} \varphi_n(\iota_2 x) = \varphi(\iota_2 x) = (\iota_2^* \varphi)(x), \quad x \in \mathcal{V},$$

i.e., $(\psi; \varphi) \in R$.

We saw in Remark 2.7.2 that the restrictions $\iota_1^*|_{\mathcal{A}_1'}$ and $\iota_2^*|_{\mathcal{A}_2'}$ act injectively. Hence, $(0; \varphi) \in R$ implies $\varphi = 0$ and $(\psi; 0) \in R$ implies $\psi = 0$, and R turns out to be the graph of an injective mapping again denoted by R. Since the subspace defining R is linear, this map is linear. Its domain and range are given by

$$\begin{aligned} \operatorname{dom} R &= \{\psi \in \mathcal{A}_1' : \exists \varphi \in \mathcal{A}_2', \iota_1^* \psi = \iota_2^* \varphi\}, \\ \operatorname{ran} R &= \{\varphi \in \mathcal{A}_2' : \exists \psi \in \mathcal{A}_1', \iota_1^* \psi = \iota_2^* \varphi\}. \end{aligned}$$

By our assumption $\iota_1^*(\mathcal{A}_1) = \Psi(\iota_1, \mathcal{A}_1) = \Psi(\iota_2, \mathcal{A}_2) = \iota_2^*(\mathcal{A}_2)$, thus $\operatorname{dom} R = \mathcal{A}_1'$ and $\operatorname{ran} R = \mathcal{A}_2'$. The closed graph Theorem (see) yields the by-continuity of the bijection $R : \mathcal{A}_1' \to \mathcal{A}_2'$. Missing Reference

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ carry Hilbert space topologies, they are reflexiv, i.e., their bidual spaces $\mathcal{A}_1''$ and $\mathcal{A}_2''$ can be identified canonicallly with $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. In turn, the conjugate mapping $R' : \mathcal{A}_2'' \to \mathcal{A}_1''$ of R can be viewed as a bijective, bi-continuous, linear mapping $R' : \mathcal{A}_2 \to \mathcal{A}_1$ such that (see) Missing Reference

$$\psi(R' y) = (R\psi)(y), \quad \text{for all } \; y \in \mathcal{A}_2, \; \psi \in \mathcal{A}_1'.$$

Since by definition $R\psi$ is the element from $\mathcal{A}_2'$ satisfying $(R\psi)(\iota_2 x) = \psi(\iota_1 x)$ for all $x \in \mathcal{V}$, we obtain

$$\psi(R'(\iota_2 x)) = (R\psi)(\iota_2 x) = \psi(\iota_1 x), \quad \text{for all } \; x \in \mathcal{V}, \; \psi \in \mathcal{A}_1'.$$

Due to $\iota_2(\mathcal{V})$'s density we get $R' \circ \iota_2 = \iota_1$. Setting $T := R'$ we obtain a bijective, bi-continuous, linear mapping from $\mathcal{A}_2$ onto $\mathcal{A}_1$ satisfying $T \circ \iota_2 = \iota_1$. As we saw in Remark 2.6.4, T is automatically isometric. Therefore, the completions $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, \mathcal{O}_1))$ and $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, \mathcal{O}_2))$ of $\mathcal{V}$ are isomorphic. ❑

II.16pre2allg. *2.7.8 Remark.* If we only have $\Psi(\iota_1, \mathcal{A}_1) \subseteq \Psi(\iota_2, \mathcal{A}_2)$ for two completions $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, \mathcal{O}_1))$ and $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, \mathcal{O}_2))$ of $\mathcal{V}$, the proof of Proposition 2.7.7 also works up to a certain extent.

In fact, we can again consider the subspace R of $\mathcal{A}_1' \times \mathcal{A}_2'$ as in (2.7.3). As above, R is closed, and is the graph of an injective linear mapping. Our assumption $\Psi(\iota_1, \mathcal{A}_1) \subseteq \Psi(\iota_2, \mathcal{A}_2)$ now gives us information only about the domain of R. In fact, $\operatorname{dom} R = \mathcal{A}_1'$. Thus, $R : \mathcal{A}_1' \to \mathcal{A}_2'$ is a continous, linear operator, and therefore has a conjugate operator $T := R' : \mathcal{A}_2 \to \mathcal{A}_1$, which is continuous, linear and satisfies $T \circ \iota_2 = \iota_1$. The latter fact also yields the density of $\operatorname{ran} T$. Applying the arguments in Remark 2.6.4 we see that T is isometric. ◊

II.16pre2almos. **2.7.9 Corollary.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space with $\operatorname{ind}_- \mathcal{V} < \infty$, and let $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, \mathcal{O}_1))$ and $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, \mathcal{O}_2))$ be almost Pontryagin space completions of $\mathcal{V}$. Then $(\iota_1, \mathcal{A}_1) \leq (\iota_2, \mathcal{A}_2)$ if and only if $\Psi(\iota_1, \mathcal{A}_1) \subseteq \Psi(\iota_2, \mathcal{A}_2)$.*

Proof. By Lemma 2.7.5 it remains to show that $\Psi(\iota_1, \mathcal{A}_1) \subseteq \Psi(\iota_2, \mathcal{A}_2)$ implies that $(\iota_1, \mathcal{A}_1) \leq (\iota_2, \mathcal{A}_2)$.

As mentioned in Remark 2.7.8 there is exists continous, linear and isometric $T : \mathcal{A}_2 \to \mathcal{A}_1$ satisfying $T \circ \iota_2 = \iota_1$. We also know that $\operatorname{ran} T$ is dense in $\mathcal{A}_1$. Applying Corollary 2.5.15 to the graph of T, which is closed, we see that $\operatorname{ran} T$ is closed, and hence coincides with $\mathcal{A}_1$. Thus, $(\iota_1, \mathcal{A}_1) \leq (\iota_2, \mathcal{A}_2)$; cf. Definition 2.7.3. ❑

For all almost Pontryagin space completions of a given space $\mathcal{V}$ we can be even more precise.

II.16. **2.7.10 Theorem.** *Let $(\mathcal{V}, [.,.])$ be a scalar product space with $\operatorname{ind}_- \mathcal{V} < \infty$. Then the following hold true.*

(1) *The set*

$$\Big\{ \Psi(\iota, \mathcal{A}) \ : \ (\iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O})) \text{ almost Pontryagin space completion of } \mathcal{V} \Big\} \tag{2.7.4}$$ minimbeschr

of subspaces of $\mathcal{V}^*$ has a minimum $\mathcal{V}^\curlywedge$ w.r.t. "$\subseteq$". In fact, $\mathcal{V}^\curlywedge = \Psi(\iota, \mathcal{A})$ for any Pontryagin space completion $(\iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O}))$ of $\mathcal{V}$, and

$$\mathcal{V}^\curlywedge = \iota\Big(\{[\![.,y]\!] : y \in \mathcal{A}\}\Big), \tag{2.7.5}$$ almostbeschr

for any almost Pontryagin space completion $(\iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O}))$ of $\mathcal{V}$.

(2) *Let $(\iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O}))$ be an almost Pontryagin space completion of $\mathcal{V}$. Then $\Psi(\iota, \mathcal{A})$ contains $\mathcal{V}^\curlywedge$, and*

$$\dim \Psi(\iota, \mathcal{A}) / \mathcal{V}^\curlywedge = \operatorname{ind}_0(\mathcal{A}) . \tag{2.7.6}$$ II.19

(3) *The map Ψ induces an order isomorphism of the set of all isomorphy classes of almost Pontryagin space completions of $\mathcal{V}$ onto the set of all linear subspaces of $\mathcal{V}^*$ which contain $\mathcal{V}^\curlywedge$ with finite codimension (ordered by inclusion).*

Proof.

(1) Let $(\iota, (\mathcal{A}, [\![.,.]\!]))$ be an almost Pontryagin space completion of $\mathcal{V}$. From (2.3.1) we known that $\mathcal{A}' = \{[\![.,y]\!] : y \in \mathcal{A}\}$. Therefore,

$$\Psi(\iota, \mathcal{A}) = \iota^*\{[\![.,y]\!] : y \in \mathcal{A}\}.$$

Since, according to Proposition 2.6.9, each two Pontryagin space completions of $\mathcal{V}$ are isomorphic, this space does not depend on the particular choice of $\mathcal{A}$; see Proposition 2.7.7. We set $\mathcal{V}^\curlywedge := \Psi(\iota, \mathcal{A})$ for the moment, and shall prove (2.7.5) and the fact that $\mathcal{V}^\curlywedge$ is the minimum of (2.7.4).

For an almost Pontryagin space completion $(\iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O}))$ of $\mathcal{V}$ consider the Pontryagin space $(\mathcal{A}/\mathcal{A}^{[\![\circ]\!]}, [\![.,.]\!]_{\mathcal{A}^{[\![\circ]\!]}})$; cf. Corollary 2.5.9. Denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^{[\![\circ]\!]}$ the canonical projection. Then π is linear, isometric, surjective, and continuous. In particular, π maps dense subsets of $\mathcal{A}$ onto dense subsets of $\mathcal{A}/\mathcal{A}^{[\![\circ]\!]}$. Thus $(\pi \circ \iota, (\mathcal{A}/\mathcal{A}^{[\![\circ]\!]}, [\![.,.]\!]))$ is a Pontryagin space completion of $\mathcal{V}$. Here $[\![.,.]\!]$ denotes the scalar product on $\mathcal{A}/\mathcal{A}^{[\![\circ]\!]}$ defined as in (2.2.4).

For $y \in \mathcal{A}$ and $x \in \mathcal{V}$ we have

$$\iota^*([\![.,y]\!])(x) = [\![\iota x, y]\!] = [\![(\pi \circ \iota)(x), \pi(y)]\!] = (\pi \circ \iota)^*([\![.,\pi y]\!])(x),$$

and hence

$$\iota^*\Big(\{[\![.,y]\!] : y \in \mathcal{A}\}\Big) = (\pi \circ \iota)^*\Big(\{[\![.,z]\!] : z \in \mathcal{A}/\mathcal{A}^{[\![\circ]\!]}\}\Big) = \mathcal{V}^\curlywedge.$$

Finally, from $\{[\![.,y]\!] : y \in \mathcal{A}\} \subseteq \mathcal{A}'$ we conclude that $\mathcal{V}^\curlywedge$ is indeed the minimum of (2.7.4).

(2) Let $(\iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O}))$ be an almost Pontryagin space completion of $\mathcal{V}$. We know that $\mathcal{A}'$ contains $\{[\![.,y]\!] : y \in \mathcal{A}\}$ with codimension $\operatorname{ind}_0 \mathcal{A}$, cf. Lemma 2.5.10. Since by Remark 2.7.2, $\iota^*|_{\mathcal{A}'}$ is injective, $\iota^*(\mathcal{A}') = \dim \Psi(\iota, \mathcal{A})$ contains $\mathcal{V}^\curlywedge$ with codimension $\operatorname{ind}_0 \mathcal{A}$.

(3) By Proposition 2.7.7 the mapping Ψ maps the set of all isomorphy classes of almost Pontryagin space completions of $\mathcal{V}$ injectively into the set of all subspaces of $\mathcal{V}^*$, where, according to Corollary 2.7.9, $(\iota_1, \mathcal{A}_1) \preceq (\iota_2, \mathcal{A}_2)$ if and only if $\Psi(\iota_1, \mathcal{A}_1) \subseteq \Psi(\iota_2, \mathcal{A}_2)$.

In (2) we saw that Ψ actually maps into the set of all linear subspaces of $\mathcal{V}^*$ which contain $\mathcal{V}^\curlywedge$ with finite codimension. It remains to show that for any subspace $\mathcal{L}$ of $\mathcal{V}^*$, which contains $\mathcal{V}^\curlywedge$ with finite codimension there exists an almost Pontryagin space completion $(\iota, (\mathcal{A}, [\![.,.]\!], \mathcal{O}))$ such that $\Psi(\iota, \mathcal{A}) = \mathcal{L}$.

For this we simply apply the construction in Example 2.7.6 to any Pontryagin space completion $(\iota_1, (\mathcal{A}_1, [\![.,.]\!]_1, \mathcal{O}_1))$ of $\mathcal{V}$ and to our subspace $\mathcal{L}$ of $\mathcal{V}^*$, which is possible since $\mathcal{V}^\curlywedge = \Psi(\iota_1, \mathcal{A}_1)$. The resulting almost Pontryagin space completion $(\iota_2, (\mathcal{A}_2, [\![.,.]\!]_2, \mathcal{O}_2))$ satisfies $\Psi(\iota_2, \mathcal{A}_2) = \mathcal{L}$.

❑

II.39. *2.7.11 Example.* We revisit the Paley-Wiener spaces $\mathcal{PW}_a$, cf. Example 2.5.6. Consider the scalar product space

$$\mathcal{V} := \bigcup_{0\leq a<1} \mathcal{PW}_a ,$$

$$[f,g] := \int_{\mathbb{R}} f(\eta)\overline{g(\eta)}\,d\eta - \pi f(0)\overline{g(0)}, \quad f,g\in\mathcal{V}. \tag{2.7.7}$$ II.58

For each $a\in[0,1)$ the space $(\mathcal{PW}_a,[.,.])$ is a Hilbert space. In particular, $(\mathcal{V},[.,.])$ is positive definite.

Consider the space $\mathcal{PW}_1$ endowed with the inner product defined by the formula (2.7.7) and with the Hilbert space topology induced by the usual $L^2(\mathbb{R})$-scalar product $(.,.)$. Then $(\mathcal{PW}_1,[.,.],\mathcal{O})$ is an almost Pontryagin space with

$$\operatorname{ind}_-(\mathcal{PW}_1,[.,.]) = 0, \quad \operatorname{ind}_0(\mathcal{PW}_1,[.,.]) = 1 .$$

Denote by ι the set-theoretic inclusion map of $\mathcal{V}$ into $\mathcal{PW}_1$. Since

$$\overline{\bigcup_{0\leq a<1} \{f\in L^2(\mathbb{R}) : \operatorname{supp} f\subseteq[-a,a]\}} = \{f\in L^2(\mathbb{R}) : \operatorname{supp} f\subseteq[-1,1]\},$$

and the Fouriertransform is unitary, $\mathcal{V}$ is dense in $\mathcal{PW}_1$. Hence,

$$(\iota,(\mathcal{PW}_1,[.,.],\mathcal{O})) \tag{2.7.8}$$ II.59

is an almost Pontryagin space completion of $(\mathcal{V},[.,.])$.

Of course, since $(\mathcal{V},[.,.])$ is positive definite, it also has a Hilbert space completion. However, the completion (2.7.8) is much more natural. Indeed, the space $\mathcal{V}$ is a space of functions defined on $\mathbb{C}$, and therefore for each $\eta\in\mathbb{C}$ we naturally have the point evaluation functionals $\chi_\eta : f\mapsto f(\eta)$, $f\in\mathcal{V}$. These are continuous w.r.t. the topology $\mathcal{O}|_{\mathcal{V}}$; the completion (2.7.8) is again a space of functions and point evaluation is continuous.

Let us show that for $\eta\notin\pi\mathbb{Z}\setminus\{0\}$ the point evaluation functional χ_η is not continuous w.r.t. the topology induced on $\mathcal{V}$ by a Hilbert space completion. To this end, we compute those elements $h_a(\eta,.)\in\mathcal{PW}_a$, $a<1,\eta\in\mathbb{C}$, with

$$f(\eta) = [f,h_a(\eta,.)],\ f\in\mathcal{PW}_a, \quad a<1,\eta\in\mathbb{C}. \tag{2.7.9}$$ II.69

The Gram operator G_a of $[.,.]|_{\mathcal{PW}_a\times\mathcal{PW}_a}$ w.r.t. $(.,.)|_{\mathcal{PW}_a\times\mathcal{PW}_a}$ is given as (notation $k_a(\eta,.)$ as in Example 2.5.6)

$$G_a = I - \pi(.,k_a(\eta,.))k_a(\eta,.) .$$

A computation shows that for each $a<1$ (actually for each $a\neq 1$) the element

$$h_a(\eta,.) := k_a(\eta,.) + \frac{1}{1-a}\frac{\sin a\eta}{\eta}k_a(0,.)$$

satisfies $G_a h_a(\eta,.) = k_a(\eta,.)$. Hence, (2.7.9) holds. We have

$$\alpha_a(\eta) := \big[h_a(\eta,.),h_a(\eta,.)\big] = h_a(\eta,\eta) = k_a(\eta,\eta) + \frac{1}{1-a}\frac{\sin a\eta}{\eta}k_a(0,\eta) = \\ = \frac{\sinh(2a\operatorname{Im}\eta)}{2\pi\operatorname{Im}\eta} + \frac{1}{1-a}\frac{1}{\pi}\Big(\frac{\sin a\eta}{\eta}\Big)^2 ,$$

and see that

$$\big|\chi_\eta(h_a(\eta,.))\big| = \sqrt{\alpha_(\eta)}\cdot([h_a(\eta,.),h_a(\eta,.)])^{\frac{1}{2}}, \quad a<1,\eta\in\mathbb{C}\,.$$

Let $(\iota,(\mathcal{B},(\!(.,.)\!)))$ be a Hilbert space completion of $(\mathcal{V},[.,.])$. All elements $h_a(\eta,.)$ belong to $\mathcal{V}$. If $\eta\notin\pi\mathbb{Z}\setminus\{0\}$, then $\lim_{a\nearrow 1}\alpha_a(\eta)=\infty$, and hence the functional χ_η is not bounded w.r.t. $(\!(.,.)\!)$.

Interestingly, if $\eta\in\pi\mathbb{Z}\setminus\{0\}$, then χ_η is bounded w.r.t. $(\!(.,.)\!)$: Let $\eta\in\pi\mathbb{Z}\setminus\{0\}$, then $\alpha_a(\eta)$ remains bounded. Hence, for an appropriate sequence $a_n\nearrow 1$, the limit $h_1(\eta,.):=\lim_{n\to\infty}h_{a_n}(\eta,.)$ exists in the weak topology of $\mathcal{B}$. Let $f\in\mathcal{V}$, and choose $a_0<1$ with $f\in\mathcal{P}W_{a_0}$. Then

$$f(\eta)=[f,h_a(\eta,.)],\quad a\in[a_0,1)\,,$$

and hence

$$f(\eta)=\lim_{n\to\infty}[f,h_{a_n}(\eta,.)]=(\!(f,h_1(\eta,.))\!)\,.$$

Let us finally compute $\Psi(\iota,\mathcal{P}W_1)$. We certainly have

$$\mathcal{V}^\curlywedge\subseteq\mathcal{V}^\curlywedge+\operatorname{span}\{\chi_\eta:\eta\in\mathbb{C}\}\subseteq\Psi(\iota,\mathcal{P}W_1)\,.$$

Since $\chi_\eta\notin\mathcal{V}^\curlywedge$ for $\eta\notin\pi\mathbb{Z}\setminus\{0\}$, the first inclusion is proper. However,

$$\dim\Psi(\iota,\mathcal{P}W_1)/\mathcal{V}^\curlywedge=\operatorname{ind}_0(\mathcal{P}W_1,[.,.])=1\,,$$

and hence in the second inclusion equality must hold. ◊

II.71. *2.7.12 Remark.* It is interesting to review Example 2.7.11 from a slightly different perspective. Say, we are given a scalar product space $(\mathcal{V},[.,.])$ and a family $\mathfrak{F}$ of linear functionals on $\mathcal{V}$ (in the concrete example Example 2.7.11 this would be the family of all point evaluation maps). Now we want to complete $(\mathcal{V},[.,.])$ in such a way that functionals from $\mathcal{F}$ become well-defined continuous functionals on the completion. To this end, consider the space

$$\mathcal{L}:=\mathcal{V}^\curlywedge+\mathcal{F}\subseteq\mathcal{V}^*\,.$$

Provided that $\dim\mathcal{L}/\mathcal{V}^\curlywedge<\infty$, we find an almost Pontryagin space completion $(\iota,(\mathcal{A},[\![.,.]\!],\mathcal{O}))$ such that $\Psi(\iota,\mathcal{A})=\mathcal{L}$. This is then the minimal completion to which functionals in $\mathcal{F}$ extend continuously. Thereby, of course, we understand a functional $\psi\in\mathcal{A}'$ with $\iota^*\psi=\varphi$ as an "extension" of φ. ◊

2.8 To become part of the Appendix

In den Appendix

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append1. **2.8.1 Lemma.** *Let $(\mathcal{A},\|.\|)$ be a normed space. If $\mathcal{B}$ is a finite dimensional subspace of $\mathcal{A}$, then $\mathcal{B}$ is closed. Moreover, it is complemented, which means that that for some closed subspace C of $\mathcal{A}$ we have $\mathcal{A}=\mathcal{B}\dot{+}C$.*

Moreover, for any such closed subspace C the linear bijection $(b;c)\mapsto b+c$ from $\mathcal{B}\times C$ onto $\mathcal{A}$ is bi-continuous.

append2.

2.8.2 Lemma. *Let $(\mathcal{A}, \|.\|)$ be a normed space, and let C be a closed subspace of finite codimension. If $\mathcal{D}$ is any linear subspace of $\mathcal{A}$, then*

$$c\ell(\mathcal{D} \cap C) = c\ell(\mathcal{D}) \cap C \quad \text{and} \quad c\ell(\mathcal{D}) = c\ell(\mathcal{D} \cap C) + \mathcal{D},$$

where $c\ell(\mathcal{D} \cap C)$ is of finite codimension in $c\ell(\mathcal{D})$. In particular, $\mathcal{D} \cap C$ is dense in C, if $\mathcal{D}$ is dense in $\mathcal{A}$.

Proof. Let $b_i,\ i \in I$, be an algebraic basis of $\mathcal{D} \cap C$, and continue this basis to a basis $b_i,\ i \in J$, of $\mathcal{D}$, where $J \supseteq I$. Clearly, the linearly independent vectors $b_i,\ i \in J \setminus I$, do not belong to C. By the assumption on the codimension of C, $J \setminus I$ is finite. Let $\mathcal{B}$ be a finite dimensional subspace of $\mathcal{A}$ containing $b_i,\ i \in J \setminus I$, such that $\mathcal{A} = \mathcal{B} \dot{+} C$. Finally, due to Lemma 2.8.1 we have

$$c\ell(\mathcal{D}) = c\ell(\mathcal{D} \cap C) \dot{+} c\ell(\operatorname{span}\{b_i : i \in J \setminus I\}) = c\ell(\mathcal{D} \cap C) \dot{+} \operatorname{span}\{b_i : i \in J \setminus I\}.$$

❑

append3.

2.8.3 Lemma. *Let $(\mathcal{A}, \|.\|)$ be a normed space. If $\mathcal{B}$ is a finite dimensional subspace of $\mathcal{A}$ and $\mathcal{N}$ is a closed subspace of $\mathcal{A}$, then also $\mathcal{B} + \mathcal{N}$ is a closed subspace of $\mathcal{A}$.*

Chapter 3

Linear Relations

chapter_EOT

3.1 Operators as linear relations

An elegant way to study not necessarily everywhere defined operators or even multivalued operators between spaces $\mathcal{X}$ and $\mathcal{X}$ and their closures is to consider their graphs as a linear subspace of $\mathcal{X} \times \mathcal{Y}$. This leads to the concept of linear relations.

linreldef. **3.1.1 Definition.** Given two vector spaces $\mathcal{X}$ and $\mathcal{Y}$ we call any linear subspace T of $\mathcal{X} \times \mathcal{Y}$ a *linear Relation* between $\mathcal{X}$ and $\mathcal{Y}$. For such a linear subspace T we call

- $\operatorname{dom} T = \{x \in \mathcal{X} \;:\; \exists y \in \mathcal{Y} : (x; y) \in T\}$ the *domain* of T,
- $\operatorname{ran} T = \{y \in \mathcal{Y} \;:\; \exists x \in \mathcal{X} : (x; y) \in T\}$ the *range* of T,
- $\operatorname{mul} T = \{y \in \mathcal{Y} \;:\; (0; y) \in T\}$ the *multivalued part* of T,
- $\ker T = \{x \in \mathcal{X} \;:\; (x; 0) \in T\}$ the *kernel* of T.

Finally, we denote by $I_{\mathcal{X}}$ the identity operator on $\mathcal{X}$, i.e. $I_{\mathcal{X}} = \{(x; x) : x \in \mathcal{X}\}$. If it is clear from the context, what space is under consideration, we will write I for short.

◊

It is easy to check that all these subsets are in fact linear subspaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively. The size of the subspace $\operatorname{mul} T$ measures how far away a linear relation is from being an operator.

linrel3. **3.1.2 Lemma.** *Let T be a linear relation between vector spaces $\mathcal{X}$ and $\mathcal{Y}$. For any $(x; y) \in T$ one has*

$$\{z \in \mathcal{Y} :\; (x; z) \in T\} = y + \operatorname{mul}(T)\,.$$

In particular, T is the graph of a not necessarily everywhere defined linear operator if and only if $\operatorname{mul} T = \{0\}$.

Ⓟ Preliminary version Mon 02 June 2014 17:08

Proof. If $(x; y)$ and $(x; z)$ belong to T, then also $(x; z) - (x; y) = (0; z - y)$ does. Hence, $z - y \in \operatorname{mul}(T)$. Conversely, $a \in \operatorname{mul} T$ implies $(0; a) \in T$ and further $(x; y + a) = (x; y) + (0; a) \in T$. ❑

As in the operator case we can define operations on the class of linear relations.

linrel5. **3.1.3 Definition.** Let $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be vector spaces, $S, T \subseteq \mathcal{X} \times \mathcal{Y}$, $Q \subseteq \mathcal{W} \times \mathcal{X}$ be linear relations and let $\alpha \in \mathbb{C}$. Then we define

- $S + T := \{(x; y) \in \mathcal{X} \times \mathcal{Y} : \exists\, u, v \in \mathcal{Y} : y = u + v,\ (x; u) \in S,\ (x; v) \in T\}$
- $\alpha T := \{(x; \alpha y) \in \mathcal{X} \times \mathcal{Y} : (x; y) \in T\}$
- $T^{-1} := \{(y; x) \in \mathcal{Y} \times \mathcal{X} : (x; y) \in T\}$
- $SQ := \{(w; y) \in \mathcal{W} \times \mathcal{Y} : \exists\, x \in \mathcal{X} : (w; x) \in Q, (x; y) \in S\}$

◊

bemsumdi. *3.1.4 Remark.* In order to distinquish the sum $S + T$ as defined in Definition 3.1.3 and the sum S and T as linear subspaces of $\mathcal{X} \times \mathcal{Y}$ we will write $S \boxplus T$ for the sum S and T as linear subspaces. ◊

It is easy to verify that with S, T, Q also $S + T$, αT, T^{-1} and SQ are linear relations. Also the proofs of the assertions in Lemma 3.1.5 are straight forward.

linrelopeig. **3.1.5 Lemma.** *Let $R, S, T \subseteq \mathcal{X} \times \mathcal{Y}$, $Q \subseteq \mathcal{W} \times \mathcal{X}$, $U \subseteq \mathcal{Y} \times \mathcal{Z}$ be linear relations and $\alpha, \beta \in \mathbb{C}$. Then the following assertions hold true.*

(*i*) *'+' is associative. More exactly,*

$$R + (S + T) = (R + S) + T = R + S + T := \tag{3.1.1}$$ assocplus

$$\{(x; y) \in \mathcal{X} \times \mathcal{Y} : \exists\, u, v, w \in \mathcal{Y} : y = u + v + w,\ (x; u) \in R,\ (x; v) \in S,\ (x; w) \in T\}.$$

(*ii*) *'+' is kommutative, i.e. $S + T = T + S$.*

(*iii*) $\alpha(S + T) = (\alpha S) + (\alpha T)$.

(*iv*) $\operatorname{dom}(S + T) = \operatorname{dom} S \cap \operatorname{dom} T$, $\operatorname{mul}(S + T) = \operatorname{mul} S + \operatorname{mul} T$, $\operatorname{ran}(S + T) \subseteq \operatorname{ran} S + \operatorname{ran} T$ *and* $\ker S \cap \ker T \subseteq \ker(S + T)$.

(*v*) *The relational product is associative, i.e.*

$$U(SQ) = (US)Q = USQ := \tag{3.1.2}$$ assocmal

$$\{(w; z) \in \mathcal{W} \times \mathcal{Z} : \exists\, x \in \mathcal{X}, y \in Y : (w; x) \in Q,\ (x; y) \in S,\ (y; z) \in U\}.$$

sex (*vi*) $(\alpha I_{\mathcal{Y}})S = \alpha S$ *and* $I_{\mathcal{Y}} S = S = S I_{\mathcal{X}}$.

(*vii*) $(\alpha\beta)S = \alpha(\beta S)$.

(*viii*) $\alpha(SQ) = (\alpha S)Q$. *If $\alpha \neq 0$, then also $\alpha(SQ) = S(\alpha Q)$.*

(*ix*) $\operatorname{dom}(SQ) \subseteq \operatorname{dom} Q$, $\operatorname{ran}(SQ) \subseteq \operatorname{ran} S$ *and* $\operatorname{mul} S \subseteq \operatorname{mul}(SQ)$, $\ker Q \subseteq \ker(SQ)$.

(*x*) $(SQ)^{-1} = Q^{-1}S^{-1}$.

(*xi*) $(S^{-1})^{-1} = S$.

(*xii*) $\operatorname{dom} S^{-1} = \operatorname{ran} S$ *and* $\ker S^{-1} = \operatorname{mul} S$.

Proof. In order to get a feeling for linear relations, let us verify for example $\alpha(SQ) = (\alpha S)Q$ and $\alpha(SQ) = S(\alpha Q)$ if $\alpha \neq 0$. By (vi), $\alpha(SQ) = (\alpha S)Q$ follows from (3.1.2).

For $\alpha \neq 0$ the inclusion $(w; y) \in \alpha(SQ)$ is equivalent to $\alpha(w; \frac{1}{\alpha}y) = (\alpha w; y) \in SQ$. Hence, $(w; y) \in \alpha(SQ)$ if and only if $(\alpha w; x) \in Q$ and $(x; y) \in S$ for some $x \in \mathcal{X}$. Since $(\alpha w; x) \in Q$ is the same as $(w; x) \in \alpha Q$, we showed the equivalence of $(w; y) \in \alpha(SQ)$ and $(w; y) \in S(\alpha Q)$. ❑

counterex1. *3.1.6 Example.* If $\alpha = 0$, then $\alpha(SQ) = S(\alpha Q)$ is no longer true for arbitrary linear relations S and Q. In fact, if $\operatorname{mul} S \supsetneq \{0\}$, then $\operatorname{mul} S(0Q) \supseteq \operatorname{mul} S \supsetneq \{0\}$, whereas $0(SQ) \subseteq \mathcal{W} \times \{0\}$, and hence $\operatorname{mul} 0(SQ) = \{0\}$. ◊

linrel6. *3.1.7 Remark.* If R, S, T are the graphs of operators, then it is easy to see that αT is the graph of the mapping $x \mapsto \alpha T x$ defined for all $x \in \operatorname{dom}(\alpha T) = \operatorname{dom} T$, that $S + T$ is the graph of the mapping $x \mapsto Sx + Tx$ defined for all $x \in \operatorname{dom}(S + T) = \operatorname{dom} S \cap \operatorname{dom} T$, and that RS is the graph of the composition mapping $x \mapsto R(Sx)$ defined for all $\operatorname{dom} RS = \{x \in \operatorname{dom} S : Sx \in \operatorname{dom} R\}$.

If S is an operator and T a linear relation, then we can write

$$S + T := \{(x; y + Sx) \in \mathcal{X} \times \mathcal{Y} : \exists\, y \in \mathcal{Y} : (x; y) \in T,\ x \in \operatorname{dom} S\}\ .$$

In the case that $\mathcal{X} = \mathcal{Y}$ and that $S = \alpha I_{\mathcal{X}}$ with $\alpha \in \mathbb{C}$,

$$T + \alpha I_{\mathcal{X}} := \{(x; y + \alpha x) \in \mathcal{X} \times \mathcal{X} : \exists\, y \in \mathcal{X} : (x; y) \in T\}\ .$$

For this expression we also write $T + \alpha$ for short. ◊

In general the relational product is not distributive. However, the following inclusions hold true.

distribut. **3.1.8 Lemma.** *For linear relations $S, T \subseteq \mathcal{X} \times \mathcal{Y}$, $Q \subseteq \mathcal{W} \times \mathcal{X}$, $U \subseteq \mathcal{Y} \times \mathcal{Z}$ we have*

(*i*) $(S + T)Q \subseteq SQ + TQ$

(*ii*) $US + UT \subseteq U(S + T)$

Proof. $(w; y) \in (S + T)Q$ means $(w; x) \in Q$ and $(x; y) \in S + T$ for some $x \in \mathcal{X}$. The latter inclusion yields $(x; y_1) \in S$ and $(x; y_2) \in T$ for some $y_1, y_2 \in \mathcal{Y}$ with $y = y_1 + y_2$. By the definition of the relational product we get $(w; y_1) \in SU$ and $(w; y_2) \in TU$, and finally $(w; y) = (w; y_1 + y_2) \in SU + TU$.

For the second inclusion take $(x; z) \in US + UT$. Hence, $(x; z_1) \in US$ and $(x; z_2) \in UT$ for some $z_1, z_2 \in \mathcal{Z}$ with $z = z_1 + z_2$. In turn we get $(x; y_1) \in S$, $(y_1; z_1) \in U$ and $(x; y_2) \in T$, $(y_2; z_2) \in U$ for some $y_1, y_2 \in \mathcal{Y}$. Since U is a linear subspace and due to the definition of '+', we get $(y_1 + y_2; z_1 + z_2) = (y_1 + y_2; z) \in U$ and $(x; y_1 + y_2) \in S + T$. Finally, $(x; z) \in (S + T)U$. ❑

3.2 Transformations of linear Relations

Consider a linear relation $T \subseteq \mathcal{X}_1 \times \mathcal{Y}_1$. If $\mathcal{X}_2$ and $\mathcal{Y}_2$ are two more vectore spaces and if $\tau : \mathcal{X}_1 \times \mathcal{Y}_1 \to \mathcal{X}_2 \times \mathcal{Y}_2$ is a linear mapping, then obviously, $\tau(T)$ is a linear relation between $\mathcal{X}_2$ and $\mathcal{Y}_2$. Likewise, if $S \subseteq \mathcal{X}_2 \times \mathcal{Y}_2$ is a linear relation, then the inverse image $\tau^{-1}(S) = \{(x; y) \in \mathcal{X}_1 \times \mathcal{Y}_1 : \tau(x; y) \in S\}$ of S is a linear relation between $\mathcal{X}_1$ and $\mathcal{Y}_1$. We will see later on that expressions like $T + S$ and $(T - \alpha)^{-1}$ can be expressed as the images of T under certain transformations τ.

dze1. *3.2.1 Remark.* Let $\mathcal{X}_j, \mathcal{Y}_j,\ j = 1, 2$ be vector spaces. A lineare mapping

$$\tau : \mathcal{X}_1 \times \mathcal{Y}_1 \to \mathcal{X}_2 \times \mathcal{Y}_2$$

can be represented in block operator form

$$\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.2.1}$$ fzq12

where

$$A = \pi_{\mathcal{X}_2} \circ \tau \circ \iota_{\mathcal{X}_1} : \mathcal{X}_1 \to \mathcal{X}_2,\ B = \pi_{\mathcal{X}_2} \circ \tau \circ \iota_{\mathcal{Y}_1} : Y_1 \to \mathcal{X}_2,$$
$$C = \pi_{\mathcal{Y}_2} \circ \tau \circ \iota_{\mathcal{X}_1} : \mathcal{X}_1 \to \mathcal{Y}_2,\ D = \pi_{\mathcal{Y}_2} \circ \tau \circ \iota_{\mathcal{Y}_1} : \mathcal{Y}_1 \to \mathcal{Y}_2.$$

Here $\pi_{\mathcal{X}_2}$ and $\pi_{\mathcal{Y}_2}$ denote the projections from $\mathcal{X}_2 \times \mathcal{Y}_2$ onto $\mathcal{X}_2$ and $\mathcal{Y}_2$, respectively. $\iota_{\mathcal{X}_1} : \mathcal{X}_1 \to \mathcal{X}_1 \times \mathcal{Y}_1$ and $\iota_{\mathcal{Y}_1} : \mathcal{Y}_1 \to \mathcal{X}_1 \times \mathcal{Y}_1$ act such that $\iota_{\mathcal{X}_1}(x) = (x; 0)$ and $\iota_{\mathcal{Y}_1}(y) = (0; y)$.

Writting the elements of $\mathcal{X}_1 \times \mathcal{Y}_1$ and $\mathcal{X}_2 \times \mathcal{Y}_2$ as two vectors, i.e. $(x; y) = \binom{x}{y}$, we see that τ acts like a matrix vector multiplication:

$$\tau\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}.$$

◊

We are going to consider some examples of such thransformations, which shall be of interest later on.

transfbspmul. *3.2.2 Example.* Let $\mathcal{X}, \mathcal{Y}$ be vector spaces.

1. For $\alpha \in \mathbb{C}$ define the linear mapping $\mu_\alpha : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ by its block operator representation
$$\mu_\alpha = \begin{pmatrix} I_{\mathcal{X}} & 0 \\ 0 & \alpha I_{\mathcal{Y}} \end{pmatrix}.$$
The nice thing about this linear mapping is that the image $\mu_\alpha(T)$ of any linear relation $T \subseteq \mathcal{X} \times \mathcal{Y}$ is nothing else but αT.

   Moreover, for $\alpha \neq 0$ the transformation μ_α is bijective, where $(\mu_\alpha)^{-1} = \mu_{\frac{1}{\alpha}}$.

2. For a linear operator $B : \mathcal{X} \to \mathcal{Y}$ define $a_B : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ by
$$a_B = \begin{pmatrix} I_{\mathcal{X}} & 0 \\ B & I_{\mathcal{Y}} \end{pmatrix}.$$
If $T \subseteq \mathcal{X} \times \mathcal{Y}$ is a linear relation, then it is easy to check that $a_B(T) = T + B$. a_B is also bijective with $(a_B)^{-1} = a_{-B}$.

3. We define $\tau_{\mathcal{X}\leftrightarrows\mathcal{Y}} : \mathcal{X}\times\mathcal{Y} \to \mathcal{Y}\times\mathcal{X}$ by

$$\tau_{\mathcal{X}\leftrightarrows\mathcal{Y}} = \begin{pmatrix} 0 & I_{\mathcal{Y}} \\ I_{\mathcal{X}} & 0 \end{pmatrix}.$$

Then $\tau_{\mathcal{X}\leftrightarrows\mathcal{Y}}(T) = T^{-1}$ for any linear relation $T \subseteq \mathcal{X}\times\mathcal{Y}$. $\tau_{\mathcal{X}\leftrightarrows\mathcal{Y}}$ is bijective with $\tau_{\mathcal{Y}\leftrightarrows\mathcal{X}} : \mathcal{Y}\times\mathcal{X} \to \mathcal{X}\times\mathcal{Y}$ as its inverse.

◊

For transformations of diagonal form we have the following assertion.

diagtrofo.

3.2.3 Lemma. *Let $\tau : \mathcal{X}_1\times\mathcal{Y}_1 \to \mathcal{X}_2\times\mathcal{Y}_2$ be a linear mapping with a block operator representation of the form*

$$\tau = (A\times D) := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

where $A : \mathcal{X}_1 \to \mathcal{X}_2$ and $D : \mathcal{Y}_1 \to \mathcal{Y}_2$ are everywhere defined linear operators. Then we have

$$\tau(T) = DTA^{-1} \text{ and } \tau^{-1}(S) = D^{-1}SA$$

for linear all relations $T \subseteq \mathcal{X}_1\times\mathcal{Y}_1$ and $S \subseteq \mathcal{X}_2\times\mathcal{Y}_2$. Hereby, in DTA^{-1} and $D^{-1}SA$ the operators A and D have to be understood as linear relations $A \subseteq \mathcal{X}_1\times\mathcal{X}_2$ and $D \subseteq \mathcal{Y}_1\times\mathcal{Y}_2$.

Proof. By the definition of products of relations

$$\begin{aligned}\tau(T) &= \{(Ax; By) : (x;y)\in T\} = \\ &\{(u;v)\in\mathcal{X}_2\times\mathcal{Y}_2 : \exists x\in\mathcal{X}_1, y\in\mathcal{Y}_1,\ (x;u)\in A, (y;v)\in D, (x;y)\in T\} = DTA^{-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\tau^{-1}(S) &= \{(x;y)\in\mathcal{X}_1\times\mathcal{Y}_1 : (Ax; Dy)\in T\} = \\ &\{(x;y)\in\mathcal{X}_1\times\mathcal{Y}_1 : \exists u\in\mathcal{X}_2, v\in\mathcal{Y}_2,\ (x;u)\in A, (y;v)\in D,\ (u;v)\in T\} = D^{-1}TA.\end{aligned}$$

❑

domrangevinv.

3.2.4 Lemma. *With the notation from Lemma* 3.2.3 *we have*

$$\operatorname{dom}(A\times D)^{-1}(S) \subseteq A^{-1}(\operatorname{dom} S).$$

In case $\operatorname{ran} S \subseteq \operatorname{ran} D$ *equality prevails.*

Proof. $\operatorname{dom}(A\times D)^{-1}(S) \subseteq A^{-1}(\operatorname{dom} S)$ is straight forward.

If $\operatorname{ran}(S) \subseteq \operatorname{ran} D$ and if $x \in A^{-1}(\operatorname{dom} S)$, then $(Ax; v) \in S$ for some $v \in \operatorname{ran} S \subseteq \operatorname{ran} D$. Hence, $v = Ay$ for some $y \in \mathcal{Y}_2$ and, in turn, $(x;y) \in (A\times D)^{-1}(S)$. Thus, $x \in \operatorname{dom}(A\times D)^{-1}(S)$. ❑

fhwr5. **3.2.5 Corollary.** *With the same notation as in Lemma* 3.2.3 *and* $\alpha \in \mathbb{C}$ *we have* $\tau(\alpha T) = \alpha\tau(T)$ *and for* $\alpha \neq 0$ *also* $\tau^{-1}(\alpha S) = \alpha\tau^{-1}(S)$. *Moreover,* $(D \times A)(T^{-1}) = \tau(T)^{-1}$ *and* $(D \times A)^{-1}(S^{-1}) = \left(\tau^{-1}(S)\right)^{-1}$, *where* $\tau = (A \times D)$.

For linear relations $T_1, T_2 \subseteq \mathcal{X}_1 \times \mathcal{Y}_1$ *and* $S_1, S_2 \subseteq \mathcal{X}_2 \times \mathcal{Y}_2$ *we have* $\tau(T_1 + T_2) \subseteq \tau(T_1) + \tau(T_2)$ *and* $\tau^{-1}(S_1) + \tau^{-1}(S_2) \subseteq \tau^{-1}(S_1 + S_2)$.

Finally, if $\mathcal{X}_j = \mathcal{Y}_j$, $j = 1, 2$, *and if* $A = D$, *then* $\tau(T_1 T_2) \subseteq \tau(T_1)\tau(T_2)$ *and* $\tau^{-1}(S_1)\,\tau^{-1}(S_2) \subseteq \tau^{-1}(S_1 S_2)$ *for* $T_1, T_2 \subseteq \mathcal{X}_1 \times \mathcal{X}_1$ *and* $S_1, S_2 \subseteq \mathcal{X}_2 \times \mathcal{X}_2$.

Proof. The assertions on scalar multiplictions and inversions are easy to check.

For $(x; y) \in T_1 + T_2$ we have $(x; u) \in T_1, (x; y - u) \in T_2$ for some $u \in Y_1$. Consequently, $(Ax; Dy) = (Ax; Du) + (Ax; D(x - u)) \in \tau(T_1) + \tau(T_2)$. Thus, $\tau(T_1 + T_2) \subseteq \tau(T_1) + \tau(T_2)$.

For $(x; y) \in \tau^{-1}(S_1) + \tau^{-1}(S_2)$ we get $(Ax; Du) \in S_1$ and $(Ax; D(y - u)) \in S_2$ for some $u \in \mathcal{Y}_1$. Hence, $(Ax; Dy) \in S_1 + S_2$ which yields $(x; y) \in \tau^{-1}(S_1 + S_2)$.

Finally, by $I \subseteq A^{-1}A$

$$\tau(T_1T_2) = AT_1T_2A^{-1} \subseteq (AT_1A^{-1})\,(AT_2A^{-1}) = \tau(T_1)\tau(T_2)\,,$$

and by $AA^{-1} \subseteq I$

$$\tau^{-1}(S_1)\,\tau^{-1}(S_2) = (A^{-1}S_1A)\,(A^{-1}S_1A) \subseteq A^{-1}S_1S_2A = \tau^{-1}(S_1S_2)\,.$$

❑

comminv. **3.2.6 Corollary.** *Let* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ *be vector spaces, let* $A : \mathcal{X}_1 \to \mathcal{X}_2,\ D : \mathcal{Y}_1 \to \mathcal{Y}_2$ *be linear mappings and let* $T \subseteq \mathcal{X}_1 \times \mathcal{Y}_1$ *and* $S \subseteq \mathcal{X}_2 \times \mathcal{Y}_2$ *be linear relations. Then we have*

$$DT \subseteq SA \ \text{ if and only if } \ (A \times D)(T) \subseteq S\,.$$

Proof. Clearly, $(A \times D)(T) \subseteq S$ is equivalent to $T \subseteq (A \times D)^{-1}(S)$. If we apply D from the left in $T \subseteq (A \times D)^{-1}(S) = D^{-1}SA$, we get $DT \subseteq DD^{-1}SA \subseteq SA$, because $DD^{-1} = \{(Dy; Dy) : y \in \mathcal{Y}_1\} \subseteq I_{\mathcal{Y}_2}$. Conversely, applying D^{-1} from the left in $DT \subseteq SA$ yields $T \subseteq D^{-1}DT \subseteq D^{-1}SA = (A \times D)^{-1}(S)$, because $I_{\mathcal{Y}_1} \subseteq D^{-1}D$. ❑

comminvrem. *3.2.7 Remark.* In the special case that $\mathcal{X} := \mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2$, $A = D$ and $S = T$ by the previous assertion the commutativity relation $AS \subseteq SA$ is equivalent to the invariance property $(A \times A)(S) \subseteq S$.

By the way, if S is an everywhere defined operator, then $AS \subseteq SA$ automatically yields $AS = SA$, because $\operatorname{dom} AS = \mathcal{X}$ and $AS \subsetneq SA$ implied that SA were multivalued, i.e. $\operatorname{mul} SA \neq \{0\}$. ◊

2xanwe. **3.2.8 Lemma.** *If* $A, S : \mathcal{X} \to \mathcal{X}$ *are linear, everywhere defined operators such that* $AS = SA$, *then*

$$(A \times A)^{-1}(S) = S \boxplus (\ker A \times \ker A)\,.$$

Proof. $AS = SA$ can be expressed as $(A \times A)(S) \subseteq S$ or as $S \subseteq (A \times A)^{-1}(S)$. As obviously $(\ker A \times \ker A) \subseteq (A \times A)^{-1}(S)$, we have $S \boxplus (\ker A \times \ker A) \subseteq (A \times A)^{-1}(S)$.

If $(x; y)$ belongs to the right hand side, then $(x; Sx) \in S \subseteq (A \times A)^{-1}(S)$. Hence, $(0; y - Sx) \in (A \times A)^{-1}(S)$, and in turn $(0; A(y - Sx)) \in S$. From $\operatorname{mul} S = \{0\}$ we conclude $y - Sx \in \ker A$. Thus, $(x; y) = (x; Sx) + (0; y - Sx) \in S \boxplus (\ker A \times \ker A)$. ❑

Finally, we bring yet another type of linear transformations for linear relations, the so-called *Potapov-Ginzburg transform*; compare . Missing Reference

potapovginzb. *3.2.9 Example.* Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2$ be vector spaces. Denote by

$$\tau_{PG} : (\mathcal{M}_1 \times \mathcal{N}_1) \times (\mathcal{M}_2 \times \mathcal{N}_2) \to (\mathcal{M}_1 \times \mathcal{N}_2) \times (\mathcal{M}_2 \times \mathcal{N}_1)$$

the mapping defined by

$$\tau_{PG}((m_1; n_1); (m_2; n_2)) = ((m_1; n_2); (m_2; n_1)) .$$

Setting $\mathcal{X}_1 := \mathcal{M}_1 \times \mathcal{N}_1$, $\mathcal{Y}_1 := \mathcal{M}_2 \times \mathcal{N}_2$ and $\mathcal{X}_2 := \mathcal{M}_1 \times \mathcal{N}_2$, $\mathcal{Y}_2 := \mathcal{M}_2 \times \mathcal{N}_1$ the mapping $\tau_{PG} : \mathcal{X}_1 \times \mathcal{Y}_1 \to \mathcal{X}_2 \times \mathcal{Y}_2$ has the block matrix structure

$$\tau_{PG} = \begin{pmatrix} \begin{pmatrix} I_{\mathcal{M}_1} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{N}_2} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{N}_1} \end{pmatrix} & \begin{pmatrix} I_{\mathcal{M}_2} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

Clearly, τ_{PG} is a linear bijection. Its inverse is nothing else but the Potapov-Ginzburg transform from $(\mathcal{M}_1 \times \mathcal{N}_2) \times (\mathcal{M}_2 \times \mathcal{N}_1)$ onto $(\mathcal{M}_1 \times \mathcal{N}_1) \times (\mathcal{M}_2 \times \mathcal{N}_2)$. ◊

3.3 Möbius-Calculus for Linear Relations

In the present section we shall study very particular transformations from $\mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ which was studied for example in . Missing Reference; dijksma snoo

jur44. **3.3.1 Definition.** Let $\mathcal{X}$ be a vector space. For any $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{C}^{2\times 2}$ define $\tau_M : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ via its block structure

$$\tau_M = \begin{pmatrix} \delta I & \gamma I \\ \beta I & \delta I \end{pmatrix},$$

i.e.

$$\tau_M(x; y) = (\delta x + \gamma y; \beta x + \alpha y) \text{ for all } (x; y) \in \mathcal{X} \times \mathcal{X} .$$

◊

trabsp2. *3.3.2 Example.* For $M = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ with $\alpha \in \mathbb{C}$ we have $\tau_M(x; y) = \alpha(x; y)$, i.e. $\tau_{\alpha I_{\mathbb{C}^2}}$ is the identity operator $I_{\mathcal{X}\times\mathcal{X}}$ on $\mathcal{X} \times \mathcal{X}$ multiplied with α.

For $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we calculate $\tau_M(x; y) = (y; x) = \tau_{\mathcal{X}\leftrightarrows\mathcal{X}}(x; y)$, i.e. $\tau_M = \tau_{\mathcal{X}\leftrightarrows\mathcal{X}}$. Similarly, for $\alpha \neq 0$ we can verify (see Example 3.2.2)

$$\tau_{\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}} = \mu_\alpha, \ \tau_{\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}} = a_{\beta I_{\mathcal{X}}} .$$

◊

multmikonst. *3.3.3 Remark.* For $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{C}^{2\times 2}$ and $\lambda \in \mathbb{C}$ we have $\tau_{\lambda M}(x; y) = \lambda(\delta x + \gamma y; \beta x + \alpha y) = \lambda \tau_M(x; y)$, i.e. $\tau_{\lambda M} = \lambda \tau_M$. But this implies that for a linear relation $T \subseteq \mathcal{X} \times \mathcal{X}$ and for $\lambda \neq 0$ we have $\tau_M(T) = \tau_{\lambda M}(T)$. ◊

multipl. **3.3.4 Lemma.** *Let $\mathcal{X}$ be a vector space and matrices $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right), N = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$. Then $\tau_N \circ \tau_M = \tau_{NM}$. For invertible $M \in \mathbb{C}^{2\times 2}$ we have $(\tau_M)^{-1} = \tau_{M^{-1}}$.*

Proof. For $(x; y) = \binom{x}{y} \in \mathcal{X} \times \mathcal{X}$ we have

$$\tau_N \circ \tau_M \begin{pmatrix} x \\ y \end{pmatrix} = \tau_N \begin{pmatrix} \delta x + \gamma y \\ \beta x + \alpha y \end{pmatrix} = \begin{pmatrix} d(\delta x + \gamma y) + c(\beta x + \alpha y) \\ b(\delta x + \gamma y) + a(\beta x + \alpha y \end{pmatrix} =$$

$$\begin{pmatrix} (d\delta + c\beta)x + (d\gamma + c\alpha)y \\ (b\delta + a\beta)x + (b\gamma + a\alpha)y \end{pmatrix} = \tau_{NM} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For invertible $M \in \mathbb{C}^{2\times 2}$ we then have $\tau_{M^{-1}} \circ \tau_M = I_{\mathcal{X}\times\mathcal{X}} = \tau_M \circ \tau_{M^{-1}}$; see Example 3.3.2. Therefore, $(\tau_M)^{-1} = \tau_{M^{-1}}$. ❑

trabsp3. *3.3.5 Example.* For a linear relation T between $\mathcal{X}$ and $\mathcal{X}$ we obtain from Lemma 3.3.4

$$(\beta + \alpha T) = \tau_{\left(\begin{smallmatrix}1 & \beta\\ 0 & 1\end{smallmatrix}\right)} \circ \tau_{\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)}(T) = \tau_{\left(\begin{smallmatrix}\alpha & \beta\\ 0 & 1\end{smallmatrix}\right)}(T)\,, \tag{3.3.1}$$ hutr0

$$(T + \lambda)^{-1} = \tau_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)} \circ \tau_{\left(\begin{smallmatrix}1 & \lambda\\ 0 & 1\end{smallmatrix}\right)}(T) = \tau_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \lambda\\ 0 & 1\end{smallmatrix}\right)}(T) = \tau_{\left(\begin{smallmatrix}0 & 1\\ 1 & \lambda\end{smallmatrix}\right)}(T)\,, \tag{3.3.2}$$ hutr1

$$\beta + \alpha(T + \lambda)^{-1} = \tau_{\left(\begin{smallmatrix}1 & \beta\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & 1\\ 1 & \lambda\end{smallmatrix}\right)}(T) = \tau_{\left(\begin{smallmatrix}\beta & \alpha+\lambda\beta\\ 1 & \lambda\end{smallmatrix}\right)}(T)\,. \tag{3.3.3}$$ hutr3

◊

multcomp. **3.3.6 Lemma.** *Let $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right), N = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ and let T be as linear relation between $\mathcal{X}$ and $\mathcal{X}$. Assume that $(a, b) = (\gamma, \delta)$, i.e. $a = \gamma$ and $b = \delta$. Then the following equality for the relation product of $\tau_N(T)$ and $\tau_M(T)$ is valid:*

$$\begin{aligned} \tau_M(T)\,\tau_N(T) &= \tau_{\left(\begin{smallmatrix}\alpha & \beta\\ c & d\end{smallmatrix}\right)}(T) \boxplus (\{0\} \times \operatorname{mul} \tau_M(T)) \\ &= \tau_{\left(\begin{smallmatrix}\alpha & \beta\\ c & d\end{smallmatrix}\right)}(T) \boxplus (\ker \tau_N(T) \times \{0\})\,. \end{aligned}$$

Proof. The elements of $\tau_N(T)$ are of the form

$$(dx_1 + cy_1; bx_1 + ay_1), \quad (x_1; y_1) \in T\,, \tag{3.3.4}$$ ggt12

and due to $a = \gamma, b = \delta$ those of $\tau_M(T)$ are of the form

$$(bx_2 + ay_2; \beta x_2 + \alpha y_2), \quad (x_2; y_2) \in T\,. \tag{3.3.5}$$ ggt13

For $(x_1; y_1) = (x_2; y_2)$ we see that all elements of the form

$$(dx_1 + cy_1; \beta x_1 + \alpha y_1), \quad (x_1; y_1) \in T\,,$$

belong to $\tau_M(T)\,\tau_N(T)$. Therefore, $\tau_{\left(\begin{smallmatrix}\alpha & \beta\\ c & d\end{smallmatrix}\right)}(T) \subseteq \tau_M(T)\,\tau_N(T)$.

If $z \in \operatorname{mul} \tau_M(T)$, then $z = \beta x_2 + \alpha y_2$ for $(x_2; y_2) \in T$, where $bx_2 + ay_2 = 0$. With $(x_1; y_1) = (0; 0)$ we get from (3.3.4) and (3.3.5) that $(0; z) = (0; \beta x_2 + \alpha y_2) \in \tau_M(T) \cdot \tau_N(T)$. Thus,

$$\tau_{\left(\begin{smallmatrix}\alpha & \beta\\ c & d\end{smallmatrix}\right)}(T) \boxplus (\{0\} \times \operatorname{mul} \tau_M(T)) \subseteq \tau_M(T) \cdot \tau_N(T)\,.$$

If, conversely, $(k;m) \in \tau_M(T)\cdot\tau_N(T)$, i.e. $(k;l) \in \tau_N(T)$ and $(l;m) \in \tau_M(T)$ fopr some $l \in \mathcal{X}$, then because of (3.3.4) and (3.3.5) we get

$$(k;l) = (dx_1 + cy_1; bx_1 + ay_1), \quad (l;m) = (bx_2 + ay_2; \beta x_2 + \alpha y_2)$$

with $bx_1 + ay_1 = bx_2 + ay_2$ for certain $(x_1;y_1),(x_2;y_2) \in T$. Employing (3.3.5) again we obtain

$$(bx_2 + ay_2; \beta x_2 + \alpha y_2) - (bx_1 + ay_1; \beta x_1 + \alpha y_1) = (0; \beta(x_2 - x_1) + \alpha(y_2 - y_1)) \in \tau_M(T)\,,$$

and, hence,

$$(k;m) = (dx_1 + cy_1; \beta x_2 + \alpha y_2) =$$
$$(dx_1 + cy_1; \beta x_1 + \alpha y_1) + (0; \beta(x_2 - x_1) + \alpha(y_2 - y_1)) \in \tau_{\left(\begin{smallmatrix}\alpha & \beta\\ c & d\end{smallmatrix}\right)}(T) \boxplus (\{0\}\times \operatorname{mul}\tau_M(T))\,.$$

We verified $\tau_M(T)\,\tau_N(T) = \tau_{\left(\begin{smallmatrix}\alpha & \beta\\ c & d\end{smallmatrix}\right)}(T) \boxplus (\{0\}\times\operatorname{mul}\tau_M(T))$. The second equality can be shown similarly. ❑

Recall the concept of *Möbius transformation*; see . For an invertible $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ by Missing Reference

$$\phi_M(\lambda) := \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta},\ \lambda \in \mathbb{C}\cup\{\infty\}$$

a mapping ϕ_M from $\mathbb{C}\cup\{\infty\}$ onto itself is well-defined. Here $\phi_M(\infty)$ has to be interpreted as $\frac{\alpha}{\gamma}$ if $\gamma \neq 0$ and as ∞ if $\gamma = 0$. Moreover, if $\gamma \neq 0$, then $\phi_M(-\frac{\delta}{\gamma}) := \infty$.

$\phi_M : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ is, in fact, a bijection. If we equip $\mathbb{C}\cup\{\infty\}$ with the *chordale metric* (see), then ϕ_M is bi-continuous. Finally, it is elementary to check that for invertible $M, N \in \mathbb{C}^{2\times 2}$ and $\lambda \in \mathbb{C}\setminus\{0\}$ Missing Reference

$$\phi_{\lambda M} = \phi_M,\ \ \phi_{I_{\mathbb{C}^2}} = \mathrm{id}_{\mathbb{C}\cup\{\infty\}},\ \ \phi_M \circ \phi_N = \phi_{MN},\ \ \phi_M^{-1} = \phi_{M^{-1}}\,.$$

Formally, these properties of the Möbius transformatoion reminds us to the properties of τ_M; see Lemma 3.3.4. But there is also a less formal connection.

transfomoebconn. **3.3.7 Theorem.** *Let T be a linear relation between $\mathcal{X}$ and $\mathcal{X}$, and let $M \in C^{2\times 2}$ be invertible. For $\lambda \in \mathbb{C}\cup\{\infty\}$ we then have in the case that $\phi_M(\lambda) \neq \infty$*

$$(\tau_M(T) - \phi_M(\lambda))^{-1} = \begin{cases} t(T-\lambda)^{-1} + sI & \lambda \neq \infty \\ tT + sI & \lambda = \infty \end{cases} \tag{3.3.6}$$ dhzz3

for some $t, s \in \mathbb{C}$ with $t \neq 0$. In the case $\phi_M(\lambda) = \infty$ we have

$$\tau_M(T) = \begin{cases} t(T-\lambda)^{-1} + sI & \lambda \neq \infty \\ tT + sI & \lambda = \infty \end{cases} \tag{3.3.7}$$ dhzz4

for some $t, s \in \mathbb{C}$ with $t \neq 0$.

Proof. For $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ we have $M^{-1} = \frac{1}{\det M}\left(\begin{smallmatrix}\delta & -\beta\\ -\gamma & \alpha\end{smallmatrix}\right)$.

Assume first that $\phi_M(\lambda) \neq \infty$, i.e. $\lambda \neq \phi_M^{-1}(\infty) = -\frac{\delta}{\gamma}$, where $-\frac{\delta}{\gamma} = \infty$, if $\gamma = 0$. According to (3.3.2) we have

$$(\tau_M(T) - \phi_M(\lambda))^{-1} = \tau_{\left(\begin{smallmatrix}0 & 1\\ 1 & -\phi_M(\lambda)\end{smallmatrix}\right)M}(T) = \tau_{\left(\begin{smallmatrix}\gamma & \delta\\ \alpha-\phi_M(\lambda)\gamma & \beta-\phi_M(\lambda)\delta\end{smallmatrix}\right)}(T)\,. \tag{3.3.8}$$ 65zhzt43

If $\lambda \neq \infty$, then $\gamma\lambda + \delta \neq 0$ because otherwise $\phi_M(\lambda) = \infty$. As $\alpha - \phi_M(\lambda)\gamma = \frac{\det M}{\gamma\lambda+\delta}$ we get

$$(\tau_M(T) - \phi_M(\lambda))^{-1} = \tau_{\frac{\det M}{\gamma\lambda+\delta}\begin{pmatrix} \frac{\gamma(\gamma\lambda+\delta)}{\det M} & \frac{\delta(\gamma\lambda+\delta)}{\det M} \\ 1 & \frac{\beta\gamma\lambda-\alpha\delta\lambda}{\det M}\end{pmatrix}}(T) = \tau_{\begin{pmatrix} \frac{\gamma(\gamma\lambda+\delta)}{\det M} & \frac{\delta(\gamma\lambda+\delta)}{\det M} \\ 1 & -\lambda\end{pmatrix}}(T)\,,$$

Comparing with (3.3.3) we see that $(\tau_M(T) - \phi_M(\lambda))^{-1} = t(T-\lambda)^{-1} + sI$ with $s = \frac{\gamma(\gamma\lambda+\delta)}{\det M}$ and $t = \lambda s + \frac{\delta(\gamma\lambda+\delta)}{\det M} = \frac{(\gamma\lambda+\delta)^2}{\det M} \neq 0$.

If $\lambda = \infty$, then $(\infty \neq)\ \phi_M(\lambda) = \frac{\alpha}{\gamma}$. Consequently, $\gamma \neq 0$ and as $\beta - \phi_M(\lambda)\delta = \frac{\beta\gamma-\alpha\delta}{\gamma}$ we have

$$(\tau_M(T) - \phi_M(\lambda))^{-1} = \tau_{\begin{pmatrix} \gamma & \delta \\ 0 & \beta-\phi_M(\lambda)\delta\end{pmatrix}}(T) = \tau_{\begin{pmatrix} -\frac{\gamma^2}{\det M} & -\frac{\gamma\delta}{\det M} \\ 0 & 1\end{pmatrix}}(T)\,.$$

Comparing with (3.3.1), we get $(\tau_M(T) - \phi_M(\lambda))^{-1} = tT + sI$ with $t = -\frac{\gamma^2}{\det M} \neq 0$ and $s = -\frac{\gamma\delta}{\det M}$.

Now we come to the case $\phi_M(\lambda) = \infty$, which means that $\lambda = -\frac{\delta}{\gamma}$. If additionally $\lambda \neq \infty$, then $\gamma \neq 0$ and, hence,

$$\tau_M(T) = \tau_{\begin{pmatrix} \frac{\alpha}{\gamma} & \frac{\beta}{\gamma} \\ 1 & \frac{\delta}{\gamma}\end{pmatrix}}(T) = \tau_{\begin{pmatrix} s & t-rs \\ 1 & -r\end{pmatrix}}(T) = t(T-r)^{-1} + sI\,,$$

with $s = \frac{\alpha}{\gamma}$ und $t = rs + \frac{\beta}{\gamma} = -\frac{\det M}{\gamma^2} \neq 0$.

Finally, if $\phi_M(\lambda) = \infty = \lambda$, then $\gamma = 0$. From $\det M \neq 0$ we infer $\delta \neq 0$. Hence,

$$\tau_M(T) = \tau_{\begin{pmatrix} \frac{\alpha}{\delta} & \frac{\beta}{\delta} \\ 0 & 1\end{pmatrix}}(T) = \tau_{\begin{pmatrix} t & s \\ 0 & 1\end{pmatrix}}(T) = tT + sI\,,$$

where $s = \frac{\beta}{\delta}$ and $t = \frac{\alpha}{\delta} \neq 0$, because otherwise we would have $\det M = 0$. ❑

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3.3.8 Remark. In order unify the cases in Theorem 3.3.7 it is convenient to set

$$(R - \infty)^{-1} := R$$

for any linear relation $R \subseteq \mathcal{X} \times \mathcal{X}$. Then the asserted equalities in Theorem 3.3.7 can be written in one formula as

$$(\tau_M(T) - \phi_M(\lambda))^{-1} = t(T-\lambda)^{-1} + sI$$

for all $\lambda \in \mathbb{C} \cup \{\infty\}$ an certain $s, t \in \mathbb{C},\ t \neq 0$. Considering the respective domains / multivalued parts this equality implies

$$\operatorname{ran}(\tau_M(T) - \phi_M(\lambda)) = \operatorname{ran}(T-\lambda)\ \ \ker(\tau_M(T) - \phi_M(\lambda)) = \ker(T-\lambda)\,, \qquad (3.3.9)$$ dqwutf122

where $\operatorname{ran}(R-\infty)$ / $\ker(R-\infty)$ has to be interpreted as $\operatorname{dom}(R)$ / $\operatorname{mul}(R)$ for any linear relation $R \subseteq \mathcal{X} \times \mathcal{X}$. ◊

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3.3.9 Remark. Let us return to transformations of the kind we studied in Lemma 3.2.3 and their interplay with the Möbius type transformations. In fact, if $\mathcal{V}, \mathcal{W}$ are vector spaces, $A : \mathcal{V} \to \mathcal{W}$ is an everywhere defined linear operator, and if $M \in \mathbb{C}^{2\times 2}$, then for $(A \times A) \circ \tau_M : \mathcal{V} \times \mathcal{V} \to \mathcal{W} \times \mathcal{W}$ and $\tau_M \circ (A \times A) : \mathcal{V} \times \mathcal{V} \to \mathcal{W} \times \mathcal{W}$ we have

$$(A \times A) \circ \tau_M = (A \times A) \circ \begin{pmatrix} \delta I_{\mathcal{V}} & \gamma I_{\mathcal{V}} \\ \beta I_{\mathcal{V}} & \alpha I_{\mathcal{V}}\end{pmatrix} = \begin{pmatrix} \delta T & \gamma T \\ \beta T & \alpha T\end{pmatrix} =$$

$$\begin{pmatrix} \delta I_{\mathcal{W}} & \gamma I_{\mathcal{W}} \\ \beta I_{\mathcal{W}} & \alpha I_{\mathcal{W}}\end{pmatrix} \circ (A \times A) = \tau_M \circ (A \times A)\,.$$

From this one gets $(A \times A)(\tau_M(T)) = \tau_M((A \times A)(T))$ for any linear relation T on $\mathcal{V}$, and for invertible M and any linear relation S on $\mathcal{W}$ also

$$\tau_M\left((A \times A)^{-1}(S)\right) = ((A \times A) \circ \tau_{M^{-1}})^{-1}(S) = \\ (\tau_{M^{-1}} \circ (A \times A))^{-1}(T) = (A \times A)^{-1}(\tau_M(T)) \,. \quad (3.3.10)$$

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3.4 Linear Relations on normed Spaces

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3.4.1 Definition. Let $(\mathcal{X}, \|.\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|.\|_{\mathcal{Y}})$ be two normed spaces. We call a linear relation T between $\mathcal{X}$ and $\mathcal{Y}$ *closed* , if T is closed as a subset of $\mathcal{X} \times \mathcal{Y}$ where $\mathcal{X} \times \mathcal{Y}$ is equipped with the product topology. The *closure* $c\ell(T)$ of a linear relation T between $\mathcal{X}$ and $\mathcal{Y}$ is simply the closure of T as a subset of $\mathcal{X} \times \mathcal{Y}$. ◊

Here the product topology on $\mathcal{X} \times \mathcal{Y}$ is the croasest topology such that both projections from $\mathcal{X} \times \mathcal{Y}$ onto $\mathcal{X}$ and onto $\mathcal{Y}$ are continuous. It is a well known fact, that this topology coincides with the topology induced by the norm $\|(x; y)\| = \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ or by the norm $\|(x; y)\| = \max(\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}})$ on $\mathcal{X} \times \mathcal{Y}$; see . It is also a well-known fact that the closure of a linear subspace of a normed space is a linear subspace again. In particular, the closure of a linear relation is a linear relation.

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For a linear relation $T \subseteq \mathcal{X} \times \mathcal{Y}$ the kernel and the multivalued part can be written as

$$\ker T = \pi_{\mathcal{X}}(T \cap (\mathcal{X} \times \{0\})) \text{ and } \operatorname{mul} T = \pi_{\mathcal{Y}}(T \cap (\{0\} \times \mathcal{Y}))\,,$$

where $\pi_{\mathcal{X}}$ $(\pi_{\mathcal{Y}})$ denotes the projection from $\mathcal{X} \times \mathcal{Y}$ onto $\mathcal{X}$ $(\mathcal{Y})$. Since $\pi_{\mathcal{X}}|_{\mathcal{X}\times\{0\}} : \mathcal{X} \times \{0\} \to \mathcal{X}$ $(\pi_{\mathcal{Y}}|_{\{0\}\times\mathcal{Y}} : \{0\} \times \mathcal{Y} \to \mathcal{Y})$ constitutes a bi-continuous linear mapping we arrive at the following assertion.

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3.4.2 Lemma. *The kernel and the multivalued part of any closed linear relation are closed.*

Clearly, the domain and the range of linear relations are not closed,in general.

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3.4.3 Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, $\mathcal{M} \subseteq \mathcal{X}$ a linear subspace and let $B : \mathcal{M} \to \mathcal{Y}$ be a linear operator.*

(*i*) *If B is continuous and $\mathcal{M}$ is closed, then (the graph of) B is closed in $\mathcal{X} \times \mathcal{Y}$.*

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(*ii*) *If $\mathcal{Y}$ is a Banach space and if B is continuous, then the closure $c\ell(B)$ of (the graph of) B in $\mathcal{X} \times \mathcal{Y}$ coincides with the unique lineare and bounded continuation of B to $c\ell(M)$; see . In particular, the continuity and the closedness of B implies the closedness of $\mathcal{M}$.*

Missing Reference

(*iii*) *For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ the closedness of $\mathcal{M}$ and B implies the continuity of B.*

Proof.

(*i*) If a sequence $((x_n; Bx_n))_{n\in\mathbb{N}}$ in $B \subseteq \mathcal{X} \times \mathcal{Y}$ converges to $(x; y)$, then $x_n \to x$ and $Bx_n \to y$. For a closed $\mathcal{M}$ we conclude $x \in \mathcal{M} = \mathrm{dom}(B)$. Because of the assumed continuity of B we get $Bx_n \to Bx$ and by the uniqueness of limits $y = Bx$. Thus $(x; y)$ belongs to the graph of B.

(*ii*) Let $C : c\ell(\mathcal{M}) \to \mathcal{Y}$ be the unique linear and bounded continuation of B to the closed subspace $c\ell(\mathcal{M}) \subseteq \mathcal{X}$. (*i*) immediately yields $c\ell(B) \subseteq C$.

Conversely, if $x \in c\ell(\mathcal{M})$ and if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}$ with $x_n \to x$, then $Bx_n = Cx_n \to Cx$ and further $(x_n; Bx_n) \to (x; Cx)$, i.e. $(x; Cx) \in c\ell(B)$. Hence, also $c\ell(B) \supseteq C$ holds true.

Missing Reference

(*iii*) This assertion is an immediate consequence of the closed graph theorem; see .

❑

transmittop.

3.4.4 Remark. If $\mathcal{X}_j, \mathcal{Y}_j,\ j = 1, 2$ are normed spaces and a transformation $\tau : \mathcal{X}_1 \times \mathcal{Y}_1 \to \mathcal{X}_2 \times \mathcal{Y}_2$ as in Remark 3.2.1 is given, then τ is easily checked to be continuous, or equivalently bounded, if and only if all four operators A, B, C, D are bounded, where A, B, C, D are the block operator entries in (3.2.1). In this case $\tau^{-1}(S)$ is closed for any closed $S \subseteq \mathcal{X}_2 \times \mathcal{Y}_2$.

Moreover, if $\tau : \tau : \mathcal{X}_1 \times \mathcal{Y}_1 \to \mathcal{X}_2 \times \mathcal{Y}_2$ is bijective and bi-continuous, then $\tau(c\ell(T)) = c\ell(\tau(T))$ for any $S \subseteq \mathcal{X}_1 \times \mathcal{Y}_1$. In particular, T is closed if and only if $\tau(T)$ is closed. ◊

clobijbsp.

3.4.5 Example. The transformations μ_α for $\alpha \neq 0$, a_B for any bounded $B : \mathcal{X} \to \mathcal{Y}$ and $\tau_{\mathcal{X}\leftrightarrows\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y} \times \mathcal{X}$ as in Example 3.2.2 are all bijective and bi-continuous.

The diagonal transformation $\tau = (A \times D)$ from $\mathcal{X}_1 \times \mathcal{Y}_1$ to $\mathcal{X}_2 \times \mathcal{Y}_2$ as in Lemma 3.2.3 is bounded if and only if A and D are bounded, and they are bijective and bi-continuous if and only if A and D have these properties.

The Potapov-Ginzburg transform is always bijective and bi-continuous provided that the underlying spaces are normed.

Finally, for $M \in \mathbb{C}^{2\times2}$ also the transformation τ_M as in Definition 3.3.1 is continuous, if $\mathcal{X}$ is a normed space. It is bijective and bi-continuous, if M is a regular matrix. ◊

Combining Example 3.2.2, Example 3.4.5 and Remark 3.4.4 we get the following assertion.

linrel7.

3.4.6 Corollary. *If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, $\alpha \in \mathbb{C} \setminus \{0\}$ and $B : \mathcal{X} \to \mathcal{Y}$ is continuous, then $c\ell(\alpha T) = \alpha\, c\ell(T)$, $c\ell(T + B) = c\ell(T) + B$ and $c\ell(T^{-1}) = c\ell(T)^{-1}$ for any linear relation $T \subseteq \mathcal{X} \times \mathcal{Y}$. In particular, the closedness of T is equivalent to the closedness of any of the relations αT, $T + B$ and T^{-1}.*

Moreover, for any linear relation $T \subseteq \mathcal{X} \times \mathcal{X}$ and any regular $M \in \mathbb{C}^{2\times2}$ one has $c\ell(\tau_M(T)) = \tau_M(c\ell(T))$. In particular, the closedness of T is equivalent to the closedness of $\tau_M(T)$ for some $M \in \mathbb{C}^{2\times2}$.

3.5 Spectrum, resolvent set and points of regular type

In contrast to Definition 3.5.2 the following definition also makes sense in general vector spaces.

pointspecdef.

3.5.1 Definition. Let $\mathcal{V}$ be a vector space, and let $T \subseteq \mathcal{V} \times \mathcal{V}$ be a linear relation. Define the *point spectrum* by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \cup \{\infty\} : \ker(T - \lambda) \neq \{0\}\},$$

where $\ker(T - \infty) = \operatorname{mul} T$. The elements of $\sigma_p(T)$ are called *eigenvalues* of T, and for $\lambda \in \sigma_p(T)$ the vectors in $\ker(T - \lambda)$ are called *eigenvectors* of T corresponding to the eigenvalue λ.

More generally, for $\lambda \in \sigma_p(T)$ we call the elements from

$$E_\lambda(T) = \bigcup_{\nu \in \mathbb{N}} \ker(T - \lambda)^\nu$$

the *root vectors* of T, where $\ker(T - \infty)^\nu = \operatorname{mul} T^\nu$. ◊

Here $(T - \lambda)^\nu$ is $(T - \lambda)$ ν-times (linear relation) multiplied with itself. If $\ker(T - \lambda) = \{0\}$, then it is straight forward to see that $\ker(T - \lambda)^{(n)} = \{0\},\ n \in \mathbb{N}$. Hence, setting $E_\lambda(T) := \bigcup_{\nu \in \mathbb{N}} \ker(T - \lambda)^\nu$ for any $\lambda \in \mathbb{C} \cup \{\infty\}$ we have $E_\lambda(T) = \{0\}$ if and only if $\lambda \notin \sigma_p(T)$.

resdef.

3.5.2 Definition. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. By $B(\mathcal{X}, \mathcal{Y})$ we denote the space of all bounded linear mappings from $\mathcal{X}$ into $\mathcal{Y}$. In the case that $\mathcal{X} = \mathcal{Y}$ we write $B(\mathcal{X})$ for it.

For a linear relation $T \subseteq \mathcal{X} \times \mathcal{X}$ define

- the *set of points of regular type* by

$$r(T) = \left\{\lambda \in \mathbb{C} \cup \{\infty\} : \ (T - \lambda)^{-1} \in B(\operatorname{ran}(T - \lambda), X)\right\}.$$

- the *resolvent set* by

$$\rho(T) = \left\{\lambda \in \mathbb{C} \cup \{\infty\} : \ (T - \lambda)^{-1} \in B(X)\right\},$$

- the *spectrum* by

$$\sigma(T) = (\mathbb{C} \cup \{\infty\}) \backslash \rho(T),$$

where $(T - \infty)^{-1} := T$ and $\operatorname{ran}(T - \infty) = \operatorname{dom}(T)$. ◊

trivex.

3.5.3 Example. If $T = \{0\} \times \{0\}$, then $(T - \lambda)^{-1} = \{0\} \times \{0\}$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$. Hence, $r(T) = \mathbb{C} \cup \{\infty\}$.

Since in finite dimensional spaces all operators are continuous, we have $\sigma(T) = \sigma_p(T)$ in case that $\dim \mathcal{X} < \infty$. ◊

It is elementary to check that

$$\rho(T) \subseteq r(T) \subseteq (\mathbb{C} \cup \{\infty\}) \setminus \sigma_p(T).$$

Concerning the point ∞ note that $\infty \notin \sigma_p(T)$ just means that T is a not necessarily everywhere defined operator, that $\infty \in r(T)$ means that T is a not necessarily everywhere defined bounded operator and that $\infty \notin \sigma(T)$ (or equivalently $\infty \in \rho(T)$) just means that T is a bounded linear mapping defined on all of $\mathcal{X}$.

abschlussrel.

3.5.4 Lemma. *If $T \subseteq \mathcal{X} \times \mathcal{X}$ is a linear relation, then $r(T) = r(c\ell(T))$ and* $\operatorname{dom}(c\ell(T) - \lambda)^{-1} = c\ell(\operatorname{ran}(T - \lambda))$ *for any $\lambda \in r(T)$.*

Proof. By Corollary 3.4.6 we have $(c\ell(T) - \lambda)^{-1} = c\ell((T - \lambda)^{-1})$. If $\lambda \in r(c\ell(T))$, then $(c\ell(T) - \lambda)^{-1}$ and, hence, its linear subspace $(T - \lambda)^{-1}$ is a linear operator ($\operatorname{mul}(T - \lambda)^{-1} = \{0\}$) which is bounded, because the restriction of a bounded operator is bounded.

Conversely, $\lambda \in r(T)$ means that $(T - \lambda)^{-1}$ is the graph of a bounded linear operator. According to Proposition 3.4.3 its closure $(c\ell(T) - \lambda)^{-1}$ is nothings else but the unique linear and bounded continuation of the operator $(T - \lambda)^{-1}$ to $c\ell(\operatorname{dom}(T - \lambda)^{-1}) = c\ell(\operatorname{ran}(T - \lambda))$. ❑

In particular, we see that for a closed T the space $\operatorname{dom}(T - \lambda)^{-1} = \operatorname{ran}(T - \lambda)$ is closed for all $\lambda \in \mathbb{C} \cup \{\infty\}$. If we are working in Banach spaces, there is some kind of converse of this fact.

Indeed, if $R \subseteq \mathcal{X} \times \mathcal{Y}$ is a closed linear relation between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, then according to Proposition 3.4.3 the fact that R is the graph of a linear and continuous operator defined on a linear subspace $\mathcal{M}$ of $\mathcal{X}$ is equivalent to $\operatorname{mul} T = \{0\}$ and $c\ell(\mathcal{M}) = \mathcal{M}$. Applying this fact to $(T - \lambda)^{-1}$ – recall that $\operatorname{mul}(T - \lambda)^{-1} = \ker(T - \lambda)$– for a closed $T \subseteq \mathcal{X} \times \mathcal{X}$ yields

efrew.

3.5.5 Lemma. *If $\mathcal{X}$ is a Banach space and if $T \subseteq \mathcal{X} \times \mathcal{X}$ is a closed linear relation, then for any $\lambda \in \mathbb{C} \cup \{\infty\}$ we have*

$$\begin{aligned} \lambda \in r(T) &\Leftrightarrow \ker(T - \lambda) = \{0\} \ \ \textit{and} \ \ \operatorname{ran}(T - \lambda) = c\ell(\operatorname{ran}(T - \lambda))\,, \\ \lambda \in \rho(T) &\Leftrightarrow \ker(T - \lambda) = \{0\} \ \ \textit{and} \ \ \operatorname{ran}(T - \lambda) = X\,, \end{aligned}$$

where $\ker(T - \infty) = \operatorname{mul} T$.

spectrtrans.

3.5.6 Theorem. *Let $T \subseteq \mathcal{X} \times \mathcal{X}$ be a linear relation, and let $M \in \mathbb{C}^{2\times 2}$ be invertible. Then we have*

$$r(\tau_M(T)) = \phi_M(r(T)),\ \rho(\tau_M(T)) = \phi_M(\rho(T)), \sigma(\tau_M(T)) = \phi_M(\sigma(T))\,,$$

where ϕ_M denotes the Möbius transform related to M.

Proof. For $\lambda \in \mathbb{C} \cup \{\infty\}$ we obtain from Theorem 3.3.7 together with Remark 3.3.8 that $(\tau_M(T) - \phi_M(\lambda))^{-1} = t(T - \lambda)^{-1} + sI$ for certain $s, t \in \mathbb{C},\ t \neq 0$. In particular, $(\tau_M(T) - \phi_M(\lambda))^{-1}$ is a bounded (bounded and everywhere defined) operator if and only if $(T - \lambda)^{-1}$ is a bounded (bounded and everywhere defined) operator. This shows $r(\tau_M(T)) = \phi_M(r(T))$ ($\rho(\tau_M(T)) = \phi_M(\rho(T))$). Taking complements the fact that $\phi_M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is bijective yields $\sigma(\tau_M(T)) = \phi_M(\sigma(T))$. ❑

In order to get a similar result as Theorem 3.5.6 for the point spectrum, we need a little lemma.

mulvonpot.

3.5.7 Lemma. *Let $\mathcal{V}$ be a vector space, let $R \subseteq \mathcal{V} \times \mathcal{V}$ be a linear relation and let $t, s \in \mathbb{C},\ t \neq 0$. For any $\nu \in \mathbb{N}$ we have*

$$\operatorname{mul} R^\nu = \operatorname{mul}(tR + sI)^\nu\,.$$

Proof. It suffices to show that $\operatorname{mul} R^\nu \subseteq \operatorname{mul}(tR + sI)^\nu$, because equality then follows by applying this inclusion to the linear relation $tR + sI$ and the scalars $\frac{1}{t}, -\frac{s}{t}$.

Since mul of a linear relation is a linear subspace, the assertion for $\nu = 1$ easily follows from the definition of $tR + sI$. Assume the assertion is true for $\nu \in \mathbb{N}$. We will show that $\operatorname{mul} R^{\nu+1} \subseteq \operatorname{mul}(tR + sI)^{\nu+1}$.

$x \in \operatorname{mul} R^{\nu+1}$ menas that $(0; y) \in R^\nu$ and $(y; x) \in R$ for some vector y. By induction hypothesis $y \in \operatorname{mul}(tR + sI)^\nu$, i.e. $(0; y) \in (tR + sI)^\nu$. Moreover, $(y; tx + sy) \in (tR + sI)$. Hence, $(0; tx + sy) \in (tR + sI)^{\nu+1}$ or $tx + sy \in \operatorname{mul}(tR + sI)^{\nu+1}$. But from $y \in \operatorname{mul}(tR + sI)^\nu \subseteq \operatorname{mul}(tR + sI)^{\nu+1}$ we also get $x \in \operatorname{mul}(tR + sI)^{\nu+1}$. ❑

spectrtranspoint. **3.5.8 Theorem.** *Let $\mathcal{V}$ be a vector space, let $T \subseteq \mathcal{V} \times \mathcal{V}$ be a linear relation, and let $M \in \mathbb{C}^{2\times 2}$ be invertible. Then we have $\sigma_p(\tau_M(T)) = \phi_M(\sigma_p(T))$. Moreover, for any $\nu \in \mathbb{N}$ we have*

$$\ker(\tau_M(T) - \phi_M(\lambda))^\nu = \ker(T - \lambda)^\nu, \ \lambda \in \mathbb{C} \cup \{\infty\}.$$

In particular, $E_\lambda(T) = E_{\phi_M(\lambda)}(\tau_M(T))$, $\lambda \in \mathbb{C} \cup \{\infty\}$.

Proof. We know from Theorem 3.3.7 together with Remark 3.3.8 that $(\tau_M(T) - \phi_M(\lambda))^{-1} = t(T - \lambda)^{-1} + sI$. By Lemma 3.1.5 and Lemma 3.5.7 we get

$$\ker(\tau_M(T) - \phi_M(\lambda))^\nu = \operatorname{mul}\left((\tau_M(T) - \phi_M(\lambda))^{-1}\right)^\nu = \operatorname{mul}\left(t(T - \lambda)^{-1} + sI\right)^\nu = \operatorname{mul}\left((T - \lambda)^{-1}\right)^\nu = \ker(T - \lambda)^\nu .$$

❑

cortranpo. **3.5.9 Lemma.** *Let $\mathcal{V}, \mathcal{W}$ be vector spaces, $A : \mathcal{V} \to \mathcal{W}$ an everywhere defined linear operator, and let T be a linear relation on $\mathcal{W}$. Then we have*

$$\ker\left((A \times A)^{-1}(T) - \lambda\right) = A^{-1} \ker(T - \lambda),$$

for all $\lambda \in \mathbb{C} \cup \{\infty\}$.

In particular, $\sigma_p((A \times A)^{-1}(T)) \subseteq \sigma_p(T)$ if A is injective.

Proof. We have $y \in \operatorname{mul}(A \times A)^{-1}(T)$ if and only if $(0; y) \in (A \times A)^{-1}(T)$. This is the same as $(0; Ay) \in T$, or as $Ay \in \operatorname{mul} T$. Hence, $\operatorname{mul}(A \times A)^{-1}(T) = A^{-1} \operatorname{mul} T$. This proves the assertion for $\lambda = \infty$.

For the general case we set $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\lambda \end{smallmatrix}\right)$. Due to (3.3.10) we then have

$$\ker((A \times A)^{-1}(T) - \lambda) = \operatorname{mul} \tau_M((A \times A)^{-1}(T)) = \operatorname{mul}(A \times A)^{-1}\tau_M(T)) = A^{-1}(\operatorname{mul} \tau_M(T)) = A^{-1} \ker(T - \lambda) .$$

For injective A we have $\sigma_p((A \times A)^{-1}(T)) \subseteq \sigma_p(T)$ because $A^{-1} \ker(T - \lambda) \neq \{0\}$ implies $\ker(T - \lambda) \neq \{0\}$. ❑

On the resolvent set $\rho(T)$ of a linear relation $T \subseteq \mathcal{X} \times \mathcal{X}$ we can consider the so-called *resolvent function* $\lambda \mapsto (T - \lambda)^{-1}$ as a mapping from $\rho(T)$ into $B(\mathcal{X})$. For the rest of the present section let aus assume that $\mathcal{X}$ is a Banach space. Due to Lemma 3.5.4 this is no essential restriction.

resolgleich.

3.5.10 Theorem. *For $\lambda, \eta \in \rho(T) \cap \mathbb{C}$ the so-called* resolvent equality

$$(T-\lambda)^{-1} - (T-\eta)^{-1} = (\lambda - \eta)(T-\lambda)^{-1}(T-\eta)^{-1}, \tag{3.5.1}$$ dett3

holds true. In particular, $(T-\lambda)^{-1}$ and $(T-\eta)^{-1}$ commute.

Moreover, $\rho(T)$ is open as a subset of $\mathbb{C} \cup \{\infty\}$ equipped with the chordal metric. In fact, $\eta \in \rho(T) \cap \mathbb{C}$ implies $U_{\frac{1}{\|(T-\eta)^{-1}\|}}(\lambda) \subseteq \rho(T)$, and $\infty \in \rho(T)$ yields $(\mathbb{C} \cup \{\infty\}) \setminus K_{\|T\|}(0) \subseteq \rho(T)$.

Proof. By (3.3.2) we have $(T-\eta)^{-1} = \tau_{\begin{pmatrix} 0 & 1 \\ 1 & -\eta \end{pmatrix}}(T)$, which is an everywhere defined operator for $\eta \in \rho(T)$, i.e. $\operatorname{ran}(T-\eta) = \operatorname{dom}(T-\eta)^{-1} = \mathcal{X}$ and $\ker(T-\eta) = \operatorname{mul}(T-\eta)^{-1} = \{0\}$. As by (3.3.3)

$$(I + (\lambda-\eta)(T-\lambda)^{-1}) = \tau_{\begin{pmatrix} 1 & -\eta \\ 1 & -\lambda \end{pmatrix}}(T)$$

we conclude from Lemma 3.3.6

$$(T-\eta)^{-1}\,(I + (\lambda-\eta)(T-\lambda)^{-1}) = \tau_{\begin{pmatrix} 0 & 1 \\ 1 & -\eta \end{pmatrix}}(T)\,\tau_{\begin{pmatrix} 1 & -\eta \\ 1 & -\lambda \end{pmatrix}}(T) = \tau_{\begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}}(T) = (T-\lambda)^{-1}. \tag{3.5.2}$$ jju5

Since this equation only involves bounded and everywhere defined operators, we obtain (3.5.1).

If $\infty \in \rho(T)$, i.e. $T \in B(\mathcal{X})$, then for $|\eta| < \|T\|$ by considering the absolute convergent Neumann series

$$\sum_{n=0}^{\infty} \frac{1}{\eta^{n+1}} T^n,$$

whose limit in $B(\mathcal{X})$ turns out to be $(\eta - T)^{-1}$, we obtain $\{z \in \mathbb{C} \cup \{\infty\} : |\eta| > \|T\|\} \subseteq \rho(T)$. Thus, with ∞ also a neighbourhood of it belongs to $\rho(T)$.

For $\lambda \in \rho(T) \cap \mathbb{C}$ and $\eta \in \mathbb{C}$ satisfying $|\lambda-\eta| \cdot \|(T-\lambda)^{-1}\| < 1$ the absolute convergent Neumann series

$$\sum_{n=0}^{\infty} (\eta-\lambda)^n (T-\lambda)^{-n}$$

tends to the inverse of $I + (\lambda-\eta)(T-\lambda)^{-1}$ in $B(\mathcal{X})$. In particular, $(I + (\lambda-\eta)(T-\lambda)^{-1})^{-1}$ is an everywhere defined operator. Moreover, due to the associativity of the relational product $(T-\eta)^{-1}$ as a linear relation satisfies

$$(T-\eta)^{-1} = (T-\eta)^{-1} I = (T-\eta)^{-1}\,(I+(\lambda-\eta)(T-\lambda)^{-1})\,(I+(\lambda-\eta)(T-\lambda)^{-1})^{-1}. \tag{3.5.3}$$ jju6

Because of $\ker(I + (\lambda-\eta)(T-\lambda)^{-1}) = \{0\}$ we get from Lemma 3.3.6 again relation (3.5.2). Therefore, the right hand side of (3.5.3) is $(T-\lambda)^{-1} \cdot (I + (\lambda-\eta)(T-\lambda)^{-1})^{-1}$ and, hence, the product of two operators from $B(X)$, i.e. $(T-\eta)^{-1} \in B(X)$. ❑

holomorhresoleig.

Missing Lokal Reference

3.5.11 Remark. The mapping $\lambda \mapsto (T-\lambda)^{-1}$ as a function from the open subset $\rho(T) \cap \mathbb{C}$ of $\mathbb{C}$ into the Banach space $B(\mathcal{X})$ is holomorphic (see for the definition of holomorphy). Indeed, for $\lambda \in \rho(T)$ we get from (3.5.1) that

$$\lim_{\eta\to\lambda} \frac{(T-\lambda)^{-1} - (T-\eta)^{-1}}{\lambda - \eta} = (T-\lambda)^{-2}\,.$$

If $\infty \in \rho(T)$, then

$$(T-\eta)^{-1} = -\sum_{n=0}^{\infty} \frac{1}{\eta^{n+1}} T^n, \quad \text{for } |\eta| > \|T\|\,,$$

shows that $\lambda \mapsto (T - \frac{1}{\lambda})^{-1}$ is holomorphic on $U_{\frac{1}{\|T\|}}(0)$ with value 0 for $\lambda = 0$, which means that $\lambda \mapsto (T-\lambda)^{-1}$ is holomorphic on all of $\rho(T)$ with value zero for $\lambda = \infty$.

◊

In order to verify that also $r(T)$ is open in $\mathbb{C} \cup \{0\}$ we need the following result.

cvfeijupre.

3.5.12 Lemma. *Let $\mathcal{X}$ be a Banach space and let $\mathcal{M} \leq \mathcal{X}$ be a linear subspace. If $A : \mathcal{M} \to \mathcal{X}$ is a bounded lineare operator satisfying*

$$\|A\| \;\; (= \sup_{x\in\mathcal{M}, \|x\|\leq 1} \|Ax\|) \; < 1\,,$$

then $\ker(I+A) = \{0\}$ *and with* $\mathcal{N} := (I+A)(\mathcal{M})$

$$I + A : \mathcal{M} \to \mathcal{N}$$

is a bijective and bi-continuous mapping with $\|(I+A)^{-1}\| \leq \frac{1}{1-\|A\|}$. Moreover, $\mathcal{N}$ is closed subspace of $\mathcal{X}$ if and only if $\mathcal{M}$ is closed.

Proof. For $x \in \mathcal{M}$ we have

$$\|(I+A)x\| = \|x - (-Ax)\| \geq \|x\| - \|Ax\| \geq (1 - \|A\|) \cdot \|x\|\,.$$

From this relation we see that $(I+A)x = 0$ implies $x = 0$, i.e. $\ker(I+A) = \{0\}$ and, hence, $I + A : \mathcal{M} \to \mathcal{N}$ is linear and bijective. Moreover, for $y \in \mathcal{N}$ the above inequality yields

$$\|y\| = \|(I+A)(I+A)^{-1}y\| \geq (1 - \|A\|) \cdot \|(I+A)^{-1}y\|\,,$$

which proves the boundedness of $(I+A)^{-1} : \mathcal{N} \to \mathcal{M}$ with $\|(I+A)^{-1}\| \leq \frac{1}{1-\|A\|}$. The boundedness of $I + A : \mathcal{M} \to \mathcal{N}$ is clear anyhow.

Finally, the boundedness of $I + A : \mathcal{M} \to \mathcal{X}$ and the closedness of $\mathcal{M} \subseteq \mathcal{X}$ yield by Proposition 3.4.3 the closedness of the graph of $I + A$ as a subset of $\mathcal{X} \times \mathcal{X}$. According to Corollary 3.4.6 $(I+A)^{-1} \subseteq \mathcal{X} \times \mathcal{X}$ is closed, too. Employing Proposition 3.4.3 once more, we see that $\mathcal{N} = \operatorname{dom}(I+A)^{-1}$ is closed, because $(I+A)^{-1} : \mathcal{N} \to \mathcal{M} \subseteq \mathcal{X}$ is continuous. As these arguments can be reversed, also the closedness of $\mathcal{N}$ implies the same property for $\mathcal{M}$. ❏

punktregtyo.

3.5.13 Theorem. *For a linear relation $T \subseteq \mathcal{X} \times \mathcal{X}$ the set $r(T)$ is an open subset of $\mathbb{C} \cup \{\infty\}$.*

Proof. For $\lambda \in r(T) \cap \mathbb{C}$, $(T-\lambda)^{-1} : \operatorname{ran}(T-\lambda) \to \mathcal{X}$ is a bounded linear operator. For any $\eta \in \mathbb{C} \setminus \{\lambda\}$ with $|\eta - \lambda| < \frac{1}{\|(T-\lambda)^{-1}\|}$ we can apply Lemma 3.5.12 to $A = (\lambda-\eta)(T-\lambda)^{-1}$ and $\mathcal{M} = \operatorname{dom}(T-\lambda)^{-1}$ and get that

$$(I + (\lambda-\eta)(T-\lambda)^{-1}) : \operatorname{dom}(T-\lambda)^{-1} \to \operatorname{ran}(I + (\lambda-\eta)(T-\lambda)^{-1})$$

is a linear and bi-continuous bijection with an operator norm less or equal to $\frac{1}{1-|\lambda-\eta|\|(T-\lambda)^{-1}\|}$. Hereby

$$\operatorname{ran}(T-\lambda) = \operatorname{dom}(T-\lambda)^{-1} = \operatorname{dom}(I + (\lambda-\eta)(T-\lambda)^{-1}),$$

and as by (3.3.3)

$$(I + (\lambda-\eta)(T-\lambda)^{-1}) = \tau_{\left(\begin{smallmatrix} 1 & -\eta \\ 1 & -\lambda \end{smallmatrix}\right)}(T)$$

we also have

$$\begin{aligned} &\operatorname{ran}(I + (\lambda-\eta)(T-\lambda)^{-1}) = \\ &\quad = \operatorname{ran} \tau_{\left(\begin{smallmatrix} 1 & -\eta \\ 1 & -\lambda \end{smallmatrix}\right)}(T) = \{y - \eta x : (x;y) \in T\} = \operatorname{ran}(T-\eta) = \operatorname{dom}(T-\eta)^{-1}. \end{aligned} \tag{3.5.4}$$

firiuzr

Because of $\ker(I + (\lambda-\eta)(T-\lambda)^{-1}) = \{0\}$ we get from Lemma 3.3.6

$$(T-\eta)^{-1}\,(I + (\lambda-\eta)(T-\lambda)^{-1}) = \tau_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & -\eta \end{smallmatrix}\right)}(T)\,\tau_{\left(\begin{smallmatrix} 1 & -\eta \\ 1 & -\lambda \end{smallmatrix}\right)}(T) = \tau_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & -\lambda \end{smallmatrix}\right)}(T) = (T-\lambda)^{-1}$$

as an equality of linear relations. Together with (3.5.4) this equation gives

$$\begin{aligned} (T-\eta)^{-1} &= (T-\eta)^{-1}\, I_{\operatorname{dom}(T-\eta)^{-1}} = \\ &\quad (T-\eta)^{-1}\,(I + (\lambda-\eta)(T-\lambda)^{-1})\,(I + (\lambda-\eta)(T-\lambda)^{-1})^{-1} = \\ &\quad\quad (T-\lambda)^{-1}\,(I + (\lambda-\eta)(T-\lambda)^{-1}))^{-1}. \end{aligned} \tag{3.5.5}$$

inve23

is a bounded operator as a product of two bounded operators. Thus, we verified, that the open disc $\{\eta \in \mathbb{C} : |\eta - \lambda| < \frac{1}{\|(T-\lambda)^{-1}\|}\}$ is contained in $r(T)$.

Finally, if $\lambda = \infty \in r(T)$, then for $|\eta| > \|T\|$ we can apply Lemma 3.5.12 to $A = -\frac{1}{\eta}T$ and $\mathcal{M} = \operatorname{dom} T$ in order to see that

$$T - \eta = -\eta(I - \frac{1}{\eta}T) : \operatorname{dom} T \to \operatorname{ran}(T-\eta) \tag{3.5.6}$$

tzz34

is a linear and bi-continuous bijection. Thus, the neighbourhood $\{\eta \in \mathbb{C} \cup \{\infty\} : |\eta| > \|T\|\}$ of ∞ is contained in $r(T)$. ❏

locbeschr.

3.5.14 Remark. We'd like to use the previous proof in order to show that $\eta \mapsto \|(T-\eta)^{-1}\|$ is bounded on any compact subset of $r(T)$. In fact, from (3.5.5) together with the estimate $\|(I+A)^{-1}\| \le \frac{1}{1-\|A\|}$ from Lemma 3.5.12 we get

$$\|(T-\eta)^{-1}\| \le \frac{\|(T-\lambda)^{-1}\|}{1 - |\lambda-\eta|\,\|(T-\lambda)^{-1}\|}$$

for all η in the disc $U_{\frac{1}{\|(T-\lambda)^{-1}\|}}(\lambda) \subseteq r(T)$, where $\lambda \in r(T)$. Clearly, on any closed disc around λ with strictly smaller radius $\|(T-\eta)^{-1}\|$ is bounded. A compactness argument then shows that $\eta \mapsto \|(T-\eta)^{-1}\|$ is bounded on any compact subset of $r(T) \cap \mathbb{C}$.

Moreover, from (3.5.6) together with the estimate from Lemma 3.5.12 we conclude that $\|(T-\eta)^{-1}\| \le \frac{1}{|\eta|-\|T\|}$ for $\infty \in r(T)$ and $|\eta| > \|T\|$. In particular, $\|(T-\eta)^{-1}\| \to 0$ for $|\eta| \to \infty$. ◊

clopenrho. **3.5.15 Corollary.** *With the assumptptions and notation from Theorem* 3.5.13*, $\rho(T)$ is closed and open in $r(T)$ wrt. to the relative topology.*

Proof. For $\mu_0 \in r(T) \cap c\ell(\rho(T)) \cap \mathbb{C}$ let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\rho(T) \cap \mathbb{C}$ converging to μ_0. Fix $\lambda \in \rho(T)$. In the previous proof we saw that
$(I + (\lambda - \mu_n)(T - \lambda)^{-1}) = \tau_{\left(\begin{smallmatrix} 1 & -\mu_n \\ 1 & -\lambda \end{smallmatrix}\right)}(T)$ maps $\operatorname{ran}(T - \lambda) = \mathcal{X}$ bijectively onto $\operatorname{ran}(T - \mu_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly, $\lim_{n\to\infty}(I + (\lambda - \mu_n)(T - \lambda)^{-1}) = (I + (\lambda - \mu_0)(T - \lambda)^{-1})$ in $B(\mathcal{X})$ wrt. the operator norm. Moreover,

$$(I + (\lambda - \mu_n)(T - \lambda)^{-1})^{-1} = \tau_{\left(\begin{smallmatrix} 1 & -\lambda \\ 1 & -\mu_n \end{smallmatrix}\right)}(T) = (I + (\mu_n - \lambda)(T - \mu_n)^{-1}).$$

For $n \in \mathbb{N}$ this operator has domain $\operatorname{ran}(T - \mu_n) = \mathcal{X}$. Due to the resolvent identity (3.5.1) we get $(m, n \in \mathbb{N})$

$$\begin{aligned} &\|(I + (\mu_n - \lambda)(T - \mu_n)^{-1}) - (I + (\mu_m - \lambda)(T - \mu_m)^{-1})\| \leq \\ &\qquad |\mu_n - \mu_m| \left(\|(T - \mu_n)^{-1})\| + |\mu_m - \lambda| \, \|(T - \mu_n)^{-1}(T - \mu_m)^{-1}\|\right). \end{aligned}$$

Since $\{\mu_n : n \in \mathbb{N} \cup \{0\}\}$ is a compact subset of $r(T)$, we obtain from Remark 3.5.14, that this expressen is arbitrarily small for m, n sufficiently large, i.e.
$(I + (\mu_n - \lambda)(T - \mu_n)^{-1})$ is a Cauchy-sequence in $B(\mathcal{X})$. If $C \in B(\mathcal{X})$ denotes its limit, the continuity of composition in $B(\mathcal{X})$ wrt. the operator norm gives
$C(I + (\lambda - \mu_0)(T - \lambda)^{-1}) = I = (I + (\lambda - \mu_0)(T - \lambda)^{-1})C$, which yields
$\operatorname{ran}(T - \mu_0) = \operatorname{ran}(I + (\lambda - \mu_0)(T - \lambda)^{-1}) = \mathcal{X}$. Thus, $\mu_0 \in \rho(T)$.

If $\infty \in r(T) \cap c\ell(\rho(T))$, then $0 \in r(T^{-1}) \cap c\ell(\rho(T^{-1}))$ by Theorem 3.5.6. Hence, $0 \in \rho(T^{-1})$ or, equivalently, $\infty \in \rho(T)$. ❑

contrainhilb. *3.5.16 Example.* If T is a contractive linear relation a Banach space $\mathcal{X}$, i.e. $\|y\| \leq \|x\|$ for all $(x; y) \in T$, then T is an operator. Clearly, $\infty \in r(T)$. By Lemma 3.5.12 applied to $-\frac{1}{\lambda}T$ we see that all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ also belong to $r(T)$. Thus,
$(\mathbb{C} \cup \{\infty\}) \setminus c\ell(\mathbb{D}) \subseteq r(T)$.

If $\operatorname{dom} T = \mathcal{X}$ or $\operatorname{ran}(T - \lambda) = \mathcal{X}$ for at least one λ with $|\lambda| > 1$, then by Corollary 3.5.15 we even have $(\mathbb{C} \cup \{\infty\}) \setminus c\ell(\mathbb{D}) \subseteq \rho(T)$. ◊

unitspec. *3.5.17 Example.* Let us apply our results to not necessarily everywhere defined isometric operators V on a Hilbert space $(\mathcal{H}, (.,.))$, i.e. $V : \operatorname{dom} V \to \operatorname{ran} V$ with $\operatorname{dom} V, \operatorname{ran} V \subseteq \mathcal{H}$ and $(Vx, Vy) = (x, y),\ x, y \in \operatorname{dom} V$.

Clearly, $\infty \in r(V)$. By Lemma 3.5.12 the operator $(I - \frac{1}{\lambda}V)$ and hence $V - \lambda$ has a bounded inverse on its range for $|\lambda| > 0$. Hence, $(\mathbb{C} \cup \{\infty\}) \setminus (\mathbb{D} \cup \mathbb{T}) \subseteq r(V)$.

Since with V also V^{-1} is an isometric operator, we also get $(\mathbb{C} \cup \infty) \setminus (\mathbb{D} \cup \mathbb{T}) \subseteq r(V^{-1})$. If we apply Theorem 3.5.6 with $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with the convention $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$ we get $(\mathbb{C} \cup \infty) \setminus (\mathbb{D} \cup \mathbb{T}) \subseteq r(V^{-1}) = r(\tau_M(V)) = \phi_M(r(V)) = \{\frac{1}{\lambda} : \lambda \in r(V)\}$ and in turn

$$\left((\mathbb{C} \cup \infty) \setminus (\mathbb{D} \cup \mathbb{T})\right) \cup \mathbb{D} \subseteq r(V).$$

If $V : \operatorname{dom} V \to \operatorname{ran} V$ is isometric and in addition $\operatorname{dom} V = \mathcal{H}$, then $\infty \in \rho(V)$. Since by Corollary 3.5.15 $\rho(V)$ is a closed and open subset of $r(V)$, $\infty \in \rho(V)$ implies $(\mathbb{C} \cup \{\infty\}) \setminus (\mathbb{D} \cup \mathbb{T}) \subseteq \rho(V)$. Indeed, the contrary would given an accumulation point of $\rho(V)$ in $r(V) \setminus \rho(V)$. Similarly, if $V : \operatorname{dom} V \to \operatorname{ran} V$ is isometric and in addition $\operatorname{ran} V = \mathcal{H}$, then $0 \in \rho(V)$. Consequently, $\mathbb{D} \subseteq \rho(V)$.

For a unitary $V : \operatorname{dom} V \to \operatorname{ran} V$ – this means $\operatorname{dom} V = \mathcal{H} = \operatorname{ran} V$ and $(Vx, Vy) = (x, y),\ x, y \in \operatorname{dom} V$ – we then have

$$\Big((\mathbb{C} \cup \infty) \setminus (\mathbb{D} \cup \mathbb{T})\Big) \cup \mathbb{D} \subseteq \rho(V)\,. \tag{3.5.7}$$ rhovuni

Note also that if $V : \operatorname{dom} V \to \operatorname{ran} V$ is isometric and if $\rho(V) \cap \mathbb{D} \neq \emptyset$ and $\rho(V) \cap (\mathbb{C} \cup \infty) \setminus (\mathbb{D} \cup \mathbb{T}) \neq \emptyset$, then using Corollary 3.5.15 similar as above we derive (3.5.7). In particular, $0, \infty \in \rho(V)$, and therefore, V is unitary. ◊

shift. *3.5.18 Example.* Consider the Hilbert space $\mathcal{H} = \ell_2(\mathbb{N})$ provided with $((\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \xi_n \bar{\eta}_n$ and the operator $V : \mathcal{H} \to \mathcal{H}$ defined by $V(\xi_n)_{n\in\mathbb{N}} = (\xi_{n-1})_{n\in\mathbb{N}}$, where $\xi_0 := 0$. It is elementary to show that V is isometric. We have $\operatorname{dom} V = \mathcal{H}$ but $\operatorname{ran} V \neq \mathcal{H}$, because $(1, 0, 0, \dots) \neq \operatorname{ran} V$. By the deliberations in Example 3.5.17 we have $\rho(V) = (\mathbb{C} \cup \infty) \setminus (\mathbb{D} \cup \mathbb{T})$ and $r(V) = ((\mathbb{C} \cup \infty) \setminus (\mathbb{D} \cup \mathbb{T})) \cup \mathbb{D}$. ◊

3.6 Functional Calculus for rational functions

In this section $\mathcal{X}$ is always a fixed Banach space, and T is a fixed linear relation on $\mathcal{X}$. We assume that $\rho(T) \neq \emptyset$. By $C_{\rho(T)}(z)$ we denote the set of all rational functions with poles in $\rho(T)$ ($\subseteq \mathbb{C} \cup \{\infty\}$). Recall that ∞ is a pole of the rational function $s(z) = \frac{u(z)}{v(z)}$ if the polynomial $u(z)$ is of great degree than $v(z)$.

Moreover, also recall that by considering limits we can evaluate any rational $s(z)$ at any point $\zeta \in \mathbb{C} \cup \{\infty\}$ and get an element $s(\zeta) \in \mathbb{C} \cup \{\infty\}$. Clearly, ζ is a pole if and only if $s(\zeta) = \infty$.

By partial fractional decomposition any rational function $s(z)$ can be represented in the following way:

$$s(z) = p(z) + \sum_{k=1}^{m} \sum_{j=1}^{n(k)} \frac{c_{kj}}{(z - \alpha_k)^j}\,, \tag{3.6.1}$$ fracdec

where $s(z)$ is a polynomial, $c_{kj} \in \mathbb{C}$ with $c_{kn(k)} \neq 0$, and where $\alpha_1, \dots, \alpha_m$ are the finite poles of $s(z)$. Clearly, $s(z) \in C_{\rho(T)}(z)$ if and only if $\alpha_1, \dots, \alpha_m \in \rho(T)$ and $\deg p > 0$ only if $\infty \in \rho(T)$.

ratfufunkdef. **3.6.1 Definition.** For $s \in C_{\rho(T)}(z)$ given in the form (3.6.1) we set

$$s(T) := p(T) + \sum_{k=1}^{m} \sum_{j=1}^{n(k)} c_{kj}(T - \alpha_k)^j\,.$$

◊

As $p(z)$ is non-constant only for bouded and everywhere defined T, $s(T)$ is a well-defined element in $B(\mathcal{X})$.

ratfunceig. **3.6.2 Theorem.** *The mapping $s \mapsto s(T)$ is an algebra homomorphism from $C_{\rho(T)}(z)$ into $B(\mathcal{X})$. If $R \in B(\mathcal{X})$ satisfies $(R \times R)(T) \subseteq T$, then R commutes with all $s(T),\ s \in C_{\rho(T)}(z)$.*

Proof. The constant one-function together with the rational functions $\frac{1}{(z-\lambda)^j}$, $\lambda \in \rho(T) \cap \mathbb{C}$, $j \in \mathbb{N}$, together with z^j, $j \in \mathbb{N}$ in the case that $\infty \in \rho(T)$, form an algebraic basis of $C_{\rho(T)}(z)$. Since a linear mapping is uniquely determined by its action on a basis, $s \mapsto s(T)$ is indeed linear.

In order to show $(s_1 \cdot s_2)(T) = s_1(T)\, s_2(T)$ by linearity it is enough to check this relation for elements s_1, s_2 from the above mentioned basis. If s_1 or s_2 is the constant one-function, then this relation is clear. Also if $s_1(z) = \frac{1}{(z-\alpha)^i}, s_2(z) = \frac{1}{(z-\alpha)^j}$ for some $\alpha \in \rho(T)$ or if $s_1(z) = z^i$, $s_2 = z^j$ (in case that $\infty \in \rho(T)$), then $(s_1 \cdot s_2)(T) = s_1(T)\, s_2(T)$ trivially holds true.

Let $s_1(z) = \frac{1}{(z-\alpha)^i}$ and $s_2(z) = \frac{1}{(z-\beta)^j}$ for distinct $\alpha, \beta \in \rho(T)$ and $i, j \in \mathbb{N} \cup \{0\}$. For $i = 0$ or $j = 0$ the function s_1 or s_2 coincides with the constant one-function. Hence, $(s_1 \cdot s_2)(T) = s_1(T)\, s_2(T)$ holds true.

Suppose that $(s_1 \cdot s_2)(T) = s_1(T)\, s_2(T)$ is true for $i, j \in \mathbb{N} \cup \{0\}$ with $i + j < k$ for some $k \in \mathbb{N}$. Suppose $i + j = k$ and $i, j \in \mathbb{N}$. Then we have

$$(s_1 s_2)(z) = \frac{1}{z-\alpha}\frac{1}{z-\beta}\frac{1}{(z-\alpha)^{i-1}}\frac{1}{(z-\beta)^{j-1}} = \\ \frac{1}{\alpha-\beta}\left(\frac{1}{(z-\alpha)^i}\frac{1}{(z-\beta)^{j-1}} - \frac{1}{(z-\alpha)^{i-1}}\frac{1}{(z-\beta)^j}\right).$$

By linearity, by induction hypothesis and by (3.5.1)

$$(s_1 s_2)(T) = \frac{1}{\alpha-\beta}\left((T-\alpha)^{-i}(T-\beta)^{-j+1} - (T-\alpha)^{-i+1}(T-\beta)^{-j}\right) \\ = (T-\alpha)^{-1}(T-\beta)^{-1}\,(T-\alpha)^{-i+1}(T-\beta)^{-j+1} = s_1(T)\, s_2(T)\,.$$

In case that $\infty \in \rho(T)$ we use the identity $\frac{1}{(z-\alpha)^i} z^j = \frac{1}{(z-\alpha)^{i-1}}\left(1 + \frac{\alpha}{z-\alpha}\right) z^{j-1}$ in order to verify $(s_1 \cdot s_2)(T) = s_1(T)\, s_2(T)$ for $s_1(z) = \frac{1}{(z-\alpha)^i}$, $s_2(z) = z^j$ or for $s_1(z) = z^i$, $s_2(z) = \frac{1}{(z-\alpha)^j}$ by induction very similar to the above considerations. We omit the details.

Finally, $R \in B(\mathcal{X})$ with $(R \times R)(T) \subseteq T$ yields $(R \times R)(\tau_M(T)) \subseteq \tau_M(T)$ for all regular $M \in \mathbb{C}^{2\times 2}$; see Remark 3.3.9. By Remark 3.2.7 R commutes with $\tau_M(T)$ for M such that $\tau_M(T)$ is a bounded operator, and hence with $(T-\lambda)^{-1}$, $\lambda \in \rho(T)$; see (3.3.2). In turn, R commutes with with all $s(T)$, $s \in C_{\rho(T)}(z)$. ❑

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3.6.3 Remark. Our functional calculus is compatible with Möbius type transformations. In fact, if $N \in \mathbb{C}^{2\times 2}$ is regular such that the pole of ϕ_N belongs to $\rho(T)$, then it easily follows from (3.3.1) and (3.3.3), that $\tau_N(T) = \phi_N(T)$. Since $s \mapsto s(T)$ is compatible with multiplication, we even have

$$\tau_N(T)^j = \phi_N(T)^j,\ j \in \mathbb{N} \cup \{0\}\,, \tag{3.6.2}$$

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for all regular $N \in \mathbb{C}^{2\times 2}$ such that the pole of ϕ_N belongs to $\rho(T)$. ◊

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3.6.4 Lemma. *Let T be a linear relation on the Banach space $\mathcal{X}$ with non-empty resolvent set, and let $M \in \mathbb{C}^{2\times 2}$ be regular. Then $s \mapsto s \circ \phi_M$ constitutes an algebra isomorphism from $C_{\rho(\tau_M(T))}(z)$ onto $C_{\rho(T)}(z)$. Moreover,*

$$s \circ \phi_M(T) = s(\tau_M(T)),\ s \in C_{\rho(\tau_M(T))}(z)\,.$$

Proof. According to Theorem 3.5.6 we have $\rho(\tau_M(T)) = \phi_M(\rho(T))$. Hence, the bijection $\phi_M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ maps $\rho(T)$ onto $\rho(\tau_M(T))$. Consequently, the poles of the rational functin $s \circ \phi_M$ are contained in $\rho(T)$ if and only if those of s are contained in $\rho(\tau_M(T))$ for any rational $s(z)$. Obviously, $s \mapsto s \circ \phi_M$ is a homomorphism. Since $s \mapsto s \circ \phi_M^{-1}$ is its inverse, $s \mapsto s \circ \phi_M$ is an algebra isomorphism from $C_{\rho(\tau_M(T))}(z)$ onto $C_{\rho(T)}(z)$.

It is easy to verify that the functions of the form $\left(\phi_N(z)\right)^j$ where $j \in \mathbb{N} \cup \{0\}$ and $N \in \mathbb{C}^{2\times 2}$ is regular with the pole of ϕ_N belonging to $\rho(\tau_M(T))$ span $C_{\rho(\tau_M(T))}(z)$. Clearly, the images of these function under $s \mapsto s \circ \phi_M$ are $\left(\phi_N(z)\right)^j \circ \phi_M = \left(\phi_{NM}(z)\right)^j \in C_{\rho(T)}(z)$. By (3.6.2) and Lemma 3.3.4 we then have

$$\left(\phi_{NM}(z)\right)^j(T) = \tau_{NM}(T)^j = \tau_N(\tau_M(T))^j = \phi_N(\tau_M(T))\,.$$

Linearity finally gives $s \circ \phi_M(T) = s(\tau_M(T))$ for all $s \in C_{\rho(\tau_M(T))}(z)$. ❑

spectransrat. **3.6.5 Proposition.** *For any $s \in C_{\rho(T)}(z)$ we have $\sigma(s(T)) = s(\sigma(T))$.*

Proof. By Lemma 3.6.4 and Theorem 3.5.6 we have $\sigma(q(\tau_M(T))) = \sigma((q \circ \phi_M)(T))$ and $q(\sigma(\tau_M(T))) = q \circ \phi_M(\sigma(T))$ for $q \in C_{\rho(\tau_M(T))}(z)$. Since $q \circ \phi_M$ runs through all of $C_{\rho(T)}(z)$, in order to prove $\sigma(s(T)) = s(\sigma(T))$, it is enough to show that $\sigma(q(\tau_M(T))) = q(\sigma(\tau_M(T)))$ for an appropriate M. Choosing M so that $\tau_M(T) \in B(\mathcal{X})$ and replacing $\tau_M(T)$ by T we can, therefore, assume that $T \in B(\mathcal{X})$.

For a fixed $s \in C_{\rho(T)}(z)$ note that due to $s(T) \in B(\mathcal{X})$ we always have $\infty \notin \sigma(s(T))$ and due to $s \in C_{\rho(T)}(z)$ also $\infty \notin s(\sigma(T))$.

Suppose $\lambda \notin s(\sigma(T))$, $\lambda \neq \infty$. Since the equation $s(z) - \lambda = 0,\ z \in \mathbb{C} \cup \{0\}$ yields $z \in \rho(T)$, the rational $\frac{1}{s(z)-\lambda}$ belongs to $C_{\rho(T)}(z)$. By the above verified homomorphism property

$$(s(T) - \lambda)\frac{1}{s(z) - \lambda}(T) = \frac{1}{s(z) - \lambda}(T)(s(T) - \lambda) = \left((s(z) - \lambda)\ \frac{1}{s(z) - \lambda}\right)(T) = I\,.$$

Hence, $s(T) - \lambda$ is invertible, i.e. $\lambda \notin \sigma(s(T))$.

Now suppose $\infty \neq s(\zeta) \in s(\sigma(T))$ for some $\zeta \in \sigma(T)$. Due to our additional assumption $T \in B(\mathcal{X})$ we have $\zeta \neq \infty$. Consider the rational $q(z) := \frac{s(z)-s(\zeta)}{z-\zeta} \in C_{\rho(T)}(z)$.

If we had $s(\zeta) \notin \sigma(s(T))$, then by the homomorphism property and by our additional assumption $T \in B(\mathcal{X})$

$$\left[q(T)(s(T) - s(\zeta))^{-1}\right]\ (T - \zeta) = (T - \zeta)\ \left[q(T)(s(T) - s(\zeta))^{-1}\right] = \left[q(z)\ \frac{1}{s(z) - s(\zeta)}\ (z - \zeta)\right](T) = I\,.$$

Therefore, ζ also lies in $\rho(T)$, which contradicts $\zeta \in \sigma(T)$. ❑

ratrange. **3.6.6 Proposition.** *For $s \in C_{\rho(T)}(z)$ we have* $\ker(T - \lambda) \subseteq \ker(s(T) - s(\lambda))$ *and* $\operatorname{ran}(s(T) - s(\lambda)) \subseteq \operatorname{ran}(T - \lambda)$.

Proof. Assume first that $T \in B(\mathcal{X})$. In this case $\operatorname{ran}(T-\infty) = \operatorname{dom} T = \mathcal{X}$ and $\ker(T-\infty) = \operatorname{mul} T = \{0\}$, and the desired inclusions hold true. For $\lambda \in \mathbb{C}$ consider again the rational $q(z) := \frac{s(z)-s(\lambda)}{z-\lambda} \in C_{\rho(T)}(z)$. By the above verified homomorphism property we have

$$q(T)(T-\lambda) = (T-\lambda)q(T) = s(T) - s(\lambda)\,,$$

which immediately gives the desired inclusions.

For $T \notin B(\mathcal{X})$ choose a regular $M \in \mathbb{C}^{2\times 2}$ with $\tau_M(T) \in B(\mathcal{X})$ we employ Lemma 3.6.4 and (3.3.9) in order to obatin

$$\ker(T-\lambda) = \ker(\tau_M(T) - \phi_M(\lambda)) \subseteq \\ \ker\Big(s\circ\phi_{M^{-1}}(\tau_M(T)) - s\circ\phi_{M^{-1}}(\phi_M(\lambda))\Big) = \ker(s(T)-s(\lambda))\,.$$

The second inclusion is shown in the same way. ❑

3.7 Finite dimensional perturbations of linear relations

Especially in Pontryagin spaces linear relations can be seen as finite dimensional extensions of linear relations on Hilbert spaces. In the present section we want to study $r(T)$ and $\rho(T)$, when T is a finite dimensional extension of a relation S, where $r(S)$ and $\rho(S)$ is known.

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3.7.1 Remark. Recall the follwoing fact. Let $\mathcal{X}, \mathcal{Y}$ be normed spaces, $\mathcal{N}, \mathcal{M}$ be closed subspaces with $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{X}$ and with $\operatorname{codim}_{\mathcal{M}} \mathcal{N} < +\infty$, and let $A : \mathcal{N} \to \mathcal{Y}$ be linear and bounded. Then any linear operator extension $B : \mathcal{M} \to \mathcal{Y}$ of A is automatically bounded. This is an immediate consequence of the fact that $\mathcal{M} = \mathcal{N} \dot{+} \mathcal{L}$ for a finite dimensional $\mathcal{L}$, and hence, that there exists a bounded projection $P : \mathcal{M} \to \mathcal{N}$. In fact, we then can write B as $AP + B(I-P)$, where $B(I-P)$ is bounded due to $\dim \mathcal{L} < \infty$. ◊

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3.7.2 Lemma. *Let $(\mathcal{A}, (.,.))$ and $(\mathcal{B}, (.,.))$ be Hilbert spaces and let $f : G \to B(\mathcal{A}, \mathcal{B})$ be holomorphic for some region $G \subseteq \mathbb{C}$ such that $f(.)|_{\mathcal{M}} \equiv C$ for some closed subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension m and some constant $C \in B(\mathcal{M}, \mathcal{B})$ where $C : \mathcal{M} \to C(\mathcal{M})$ is bijective and bi-continuous.*

Then there exists some discrete subset D of G and some $k \in \{0, \dots, m\}$ such that $\dim\ker f(\lambda) = k$ for $\lambda \in G \setminus D$ and $\dim\ker f(\lambda) > k$ for $\lambda \in D$.

Moreover, for any $\mu \in G \setminus D$ there exists a holomorphic function $h : G \to B(\mathbb{C}^k, \mathcal{A})$ with $\operatorname{ran} h(\lambda) \subseteq \ker f(\lambda)$ for $\lambda \in G$ with equality for all $\lambda \in G \setminus D'$ with a certain discrete subset $D' \supseteq D$ of G with $\mu \notin D'$.

Proof. We write $f(\lambda)$ in the block structure

$$f(\lambda) = \begin{pmatrix} C & D(\lambda) \\ 0 & E(\lambda) \end{pmatrix}$$

according to the decompositions $\mathcal{A} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $\mathcal{B} = C(\mathcal{M}) \oplus C(\mathcal{M})^\perp$. Obviously, $x + y \in \mathcal{M} \oplus \mathcal{M}^\perp$ belongs to $\ker f(\lambda)$ if and only if $y \in \ker E(\lambda)$ and $x = -C^{-1}D(\lambda)y$.

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In particular, $\dim\ker f(\lambda) = \dim\ker E(\lambda)$. Set $k := \min_{\lambda\in G} \dim\ker f(\lambda) \in \{0,\dots,m\}$. By the rank theorem from Linear Algebra we have $\dim\operatorname{ran} E(\lambda) \leq m-k$ for all $\lambda \in G$.

If $k = m$, then $E(\lambda) = 0$ for all $\lambda \in G$ and chosing a basis $x_1,\dots,x_m$ of $\mathcal{M}^\perp$ the holomorphic $B(\mathbb{C}^m, \mathcal{A})$-valued mapping

$$h : \lambda \mapsto \Big((\xi_1,\dots,\xi_m)^T \mapsto (\sum_{j=1}^m \xi_j x_j) - C^{-1}D(\lambda)(\sum_{j=1}^m \xi_j x_j)\Big).$$

satisfies $\operatorname{ran} h(\lambda) = \ker f(\lambda)$.

For $k < m$ let $\mu \in G$ be any point such that $k = \dim\ker f(\mu) = \dim\ker E(\mu)$. Then we have $\dim\operatorname{ran} E(\mu) = m-k > 0$ by the rank theorem from Linear Algebra. If we denote by P the orthogonal projection from $\mathcal{B}$ onto $\operatorname{ran} E(\mu)$, then

$$M : \; G \ni \lambda \mapsto PE(\lambda) \in B(\mathcal{M}^\perp \cap \ker E(\mu)^\perp, \operatorname{ran} E(\mu))$$

is a holomorphic mapping. As $\dim \mathcal{M}^\perp \cap \ker E(\mu)^\perp = m-k$ and $\dim\operatorname{ran} E(\mu) = m-k$ we can interprete this mapping as an $(m-k)\times(m-k)$-matrix valued function whose value is invertible for $\lambda = \mu$, i.e $\det M(\mu) \neq 0$.

Since $\lambda \mapsto \det M(\lambda)$ is holomorphic, there is a discrete subsete D' of G such that $\det M(\lambda) \neq 0$ for $\lambda \in G \setminus D'$. This means that $m-k = \dim\operatorname{ran} PE(\lambda)$ and hence $m-k \leq \dim\operatorname{ran} E(\lambda) \leq m-k$ for $\lambda \in G\setminus D'$. Consequently, $\dim\ker E(\lambda) = \dim\ker PE(\lambda) = k$ and, hence, $\ker E(\lambda) = \ker PE(\lambda)$ for $\lambda \in G \setminus D'$.

Moreover, choosing a basis $x_1,\dots,x_k$ of $\ker E(\mu)$ for $\lambda \in G\setminus D'$ and $(\xi_1,\dots,\xi_k)^T \in \mathbb{C}^k$ the vector

$$g(\lambda)(\xi_1,\dots,\xi_k)^T := (\sum_{j=1}^k \xi_j x_j) - M(\lambda)^{-1}PE(\lambda)(\sum_{j=1}^k \xi_j x_j)$$

belongs to $\ker PE(\lambda)$. As $g(\lambda)$ is injective we have $\operatorname{ran} g(\lambda) = \ker PE(\lambda) = \ker E(\lambda)$. Since $\lambda \mapsto \det M(\lambda)\, M(\lambda)^{-1}$ has a holomorphic continuation to G, also $\lambda \mapsto \det M(\lambda)\, g(\lambda)$ has a holomorphic continuation to all of G. Hence, the mapping $h : G \to B(\mathbb{C}^k, \mathcal{A})$ defined by

$$h(\lambda)(\xi_1,\dots,\xi_m)^T := \det M(\lambda)\Big(g(\lambda)(\xi_1,\dots,\xi_k)^T - C^{-1}D(\lambda)\, g(\lambda)(\xi_1,\dots,\xi_k)^T\Big)$$

is also holomorphic and satisfies $\operatorname{ran} h(\lambda) = \ker f(\lambda)$ for $\lambda \in G\setminus D'$. By continuity we have $\operatorname{ran} h(\lambda) \subseteq \ker f(\lambda)$ for $\lambda \in D'$.

Of course, the equality $\dim\ker f(\lambda) = k$ may be true for a larger set $G \setminus D$, where $D \subseteq D'$ is necessarily discrete, too. ❑

For linear relation S, T on $\mathcal{A}$ such that $S \subseteq T$ we obviously have $r(T) \subseteq r(S)$, because if $(T-\lambda)^{-1}$ is the graph of a bounded linear operator, then also its restriction $(S-\lambda)^{-1}$ is the graph of a bounded linear operator. Concerning the converse inclusion the following result is valid.

`regtyp.`

3.7.3 Theorem. *Let S, T be linear relation on a Hilbert space $(\mathcal{A}, (.,.))$ such that $S \subseteq T$. If S is closed and has finite codimension in T and if $G \subseteq r(S)$ is open and connected, then either $G \subseteq \sigma_p(T)$ or $G \setminus D \subseteq r(T)$ and $D \subseteq \sigma_p(T)$ for some discrete $D \subseteq G$.*

In the first case that $G \subseteq \sigma_p(T)$ *there also existst a discrete subset* D *of* G *such that* $\dim\ker(T-\lambda) = k$ *for all* $\lambda \in G \setminus D$ *and* $\dim\ker(T-\lambda) > k$ *for* $\lambda \in D$.

Moreover in the first case, for any $\mu \in G$ *there exists an open neighbourhoud* $O(\mu) \subseteq G$ *of* μ *and a holomorphic mapping* $h : O(\mu) \to B(\mathbb{C}^k, T)$ *such that*

$$\operatorname{ran} h(\lambda) = \{(x; \lambda x) : x \in \ker(T-\lambda)\} \tag{3.7.1}$$

regtypeq

for any $\lambda \in O(\mu) \setminus \{\mu\}$*, where* $(x; \lambda x)$ *has to be interpreted as* $(0; x)$ *for* $\lambda = \infty$*. If* $\mu \in G \setminus D$*, then* (3.7.1) *also holds for* $\lambda = \mu$.

Proof. Obviously, $\mathbb{C} \ni \lambda \mapsto \psi_\lambda \in B(\mathcal{A} \times \mathcal{A}, \mathcal{A})$ is holomorphic, where $\psi_\lambda(x; y) = y - \lambda x$. By definition $\lambda \in r(S)$ just means that $\psi_\lambda|_S : S \to \operatorname{ran}(S-\lambda) = \psi_\lambda(S)$ is bijective and bi-continuous. Moreover, $\ker\psi_\lambda = \{(x; \lambda x) : x \in \ker(S-\lambda)\}$.

For $\mu \in G \cap \mathbb{C} \subseteq r(S)$ let $R(\mu) \supseteq S$ be such that $\mu \in \rho(R(\mu))$. For $R(\mu)$ take for example $(c\ell((S-\mu)^{-1})\,P)^{-1} + \mu$, where P is a bounded projection with range $c\ell \operatorname{ran}(S-\mu)$. With $\epsilon(\mu) > 0$ small enough we have $U_{\epsilon(\mu)}(\mu) \subseteq \rho(R(\mu)) \subseteq r(S)$ and, hence,

$$\psi_\lambda|_{R(\mu)} : R(\mu) \to \mathcal{A}$$

is a linear bijection for any $\lambda \in U_{\epsilon(\mu)}(\mu)$. With $\lambda \mapsto \psi_\lambda|_{R(\mu)} \in B(R(\mu), \mathcal{A})$ also $\lambda \mapsto \psi_\lambda|_{R(\mu)}^{-1} \in B(\mathcal{A}, R(\mu))$ is holomorphic . Consequently, for a closed $T \supseteq S$ the mapping

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$$\phi^\mu(\lambda) := \psi_\lambda|_{R(\mu)}^{-1} \circ \psi_\lambda|_T : T \to R(\mu)$$

is continuous and $U_{\epsilon(\mu)}(\mu) \ni \lambda \mapsto \phi^\mu(\lambda) \in B(T, R(\mu))$ is holomorphic. Obviously, $\phi^\mu(\lambda)|_S = \mathrm{id}_S$.

Hence, for a closed S such that $\operatorname{codim}_S T = m < \infty$ it follows from Lemma 3.7.2 that $\dim\ker(T-\lambda) = \dim\ker\psi_\lambda|_T = \dim\ker\phi^\mu(\lambda) = k_\mu$ for all $\lambda \in U_{\epsilon(\mu)}(\mu) \setminus D_\mu$ and $m \geq \dim\ker\psi_\lambda|_T = \dim\ker\phi^\mu(\lambda) > k_\mu$ for $\lambda \in D_\mu$, where D_μ is discrete in $U_{\epsilon(\mu)}(\mu)$ and $k_\mu \in \{0, \ldots, m\}$.

Thus, defining $f : G \cap \mathbb{C} \to \mathbb{R}$ by $f(\mu) = k_\mu$ we get continuous function with values in $\mathbb{N} \cup \{0\}$. Since $G \cap \mathbb{C}$ is connected, $f(G \cap \mathbb{C}) \subseteq \mathbb{N} \cup \{0\}$ is connected in $\mathbb{R}$ and, hence, constantly equal to $k \in \{0, \ldots, m\}$. The set $\{\lambda \in G : \dim\ker(T-\lambda) > k\}$ does not have an accumulation point μ in $G \cap \mathbb{C}$, because otherwise D_μ would have an accumulation point in $U_{\epsilon(\mu)}(\mu)$.

Applying the above arguments to $S^{-1} \subseteq T^{-1}$ by Theorem 3.5.6 and Theorem 3.5.8 we see that $\dim\ker(T-\lambda) = \dim(T^{-1} - \frac{1}{\lambda}) = l$ for all $\lambda \in G \setminus \{0\}$ up to a discrete subset of λ's. For such exceptional $\lambda \in G \setminus \{0\}$ we have $\dim\ker(T-\lambda) > l$.

Combining the previous two paragraphs showes that for a certain dicrete subset D of G we have

$$\dim\ker(T-\lambda) = k \quad \text{for all} \quad \lambda \in G \setminus D$$

and

$$m \geq \dim\ker(T-\lambda) > k \quad \text{for all} \quad \lambda \in D$$

for some $k \in \{0, \ldots, m\}$.

If $k = 0$, then $(T-\lambda)^{-1}$ is an operator extension of $(S-\lambda)^{-1}$ for any $\lambda \in G \setminus D$; see Remark 3.7.1. Thus, $G \setminus D \subseteq r(T)$. For $\lambda \in D$ the mapping $\dim\ker(T-\lambda) > 0$, i.e. $D \subseteq \sigma_p(T)$.

If $k \geq 1$, then $\ker(T-\lambda)$ has a non-trivial kernel for all $\lambda \in G$. This means that $G \subseteq \sigma_p(T)$.

Moreover, for $\mu \in G \cap \mathbb{C}$ consider the open neighbourhood $U_{\epsilon(\mu)}(\mu)$ of μ. According to the final assertion in Lemma 3.7.2 applied to $\phi^\mu(\lambda)$ (for the G in Lemma 3.7.2 take $U_{\epsilon(\mu)}(\mu)$ and for μ in Lemma 3.7.2 take the present μ if $\mu \in U_{\epsilon(\mu)}(\mu) \setminus D$ and take any other point from $U_{\epsilon(\mu)}(\mu) \setminus D$ if our μ belongs to D) there exists a holomorphic function

$$h : U_{\epsilon(\mu)}(\mu) \to B(\mathbb{C}^k, T)$$

such that $\operatorname{ran} h(\lambda) \subseteq \ker \phi^\mu(\lambda) = \ker \psi_\lambda|_T = \{(x; \lambda x) : x \in \ker(T-\lambda)\}$ for all $\lambda \in U_{\epsilon(\mu)}(\mu)$ with equality for $\lambda \notin D'$ for a certain discrete subset of $U_{\epsilon(\mu)}(\mu)$. Hence, we finde an open neighbourhood $O(\mu) \subseteq U_{\epsilon(\mu)}(\mu)$ of μ such that $O(\mu) \cap D'$ contains μ if $\mu \in D$ and such that $O(\mu) \cap D' = \emptyset$ if $\mu \in U_{\epsilon(\mu)}(\mu) \setminus D$.

Finally, for $\mu = \infty \in G$ in view of Theorem 3.5.6 and Theorem 3.5.8 we can use the same arguments just applied to S^{-1} and T^{-1} in order to settle this case. ❑

In order to have a grip on the root spaces, we bring

rootspaceprop.

3.7.4 Lemma. *Let T be a closed linear relation on a Hilbert space $(\mathcal{A}, (.,.))$, let $O \subseteq \mathbb{C}$ be open and let $h : O \to B(\mathbb{C}^k, T)$ be holomorphic such that $\operatorname{ran} h(\lambda) \subseteq \{(x; \lambda x) \in T : x \in \mathcal{A}\}$ for all $\lambda \in O$.*

Then for $g(\lambda) = \pi_1 \circ h(\lambda) : \mathbb{C}^k \to \mathcal{A}$ we have

$$(g^{(n)}(\lambda)x; ng^{(n-1)}(\lambda)x) \in T - \lambda\,,\ x \in \mathbb{C}^k\,, \tag{3.7.2}$$ rooteq1

for all $n \in \mathbb{N} \cup \{0\}$, where $ng^{(n-1)}(\lambda)x := 0$ for $n = 0$. In particular, $\ker(T-\lambda)^n \supseteq \operatorname{ran} g(\lambda) + \cdots + \operatorname{ran} g^{(n-1)}(\lambda)$ for all $n \in \mathbb{N}$.

For all $\lambda \in O$ with $\operatorname{ran} h(\lambda) = \{(x; \lambda x) \in T : x \in \mathcal{A}\}$ we even have

$$\ker(T-\lambda)^n = \operatorname{ran} g(\lambda) + \cdots + \operatorname{ran} g^{(n-1)}(\lambda) \tag{3.7.3}$$ rooteq2

for all $n \in \mathbb{N}$.

Proof. For $(y; z) \in T^*$ and $\lambda \in O$ we obtain

$$(g(\lambda)x, z) = (\lambda g(\lambda)x, y)\,.$$

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Taking the n-th derivations we get from the Leibniz formula (see)

$$(g^{(n)}(\lambda)x, z) = (\lambda g^{(n)}(\lambda)x + ng^{(n-1)}(\lambda)x, y)\,.$$

As this is true for any $(y; z) \in T^*$ we conclude that $(g^{(n)}(\lambda)x; ng^{(n-1)}(\lambda)x + \lambda g^{(n)}(\lambda)x) \in T$ and in turn that (3.7.2) holds true.

For $\lambda \in O$ with $\operatorname{ran} h(\lambda) = \{(x; \lambda x) \in T : x \in \mathcal{A}\}$ the relation (3.7.3) is clearly true for $n = 1$. Assume now that (3.7.3) holds true for $1, \dots, n$. We will show that it is true for $n+1$.

The inclusion $\ker(T-\lambda)^{n+1} \supseteq \operatorname{ran} g(\lambda) + \cdots + \operatorname{ran} g^{(n)}(\lambda)$ is a consequence of (3.7.2). Let $a \in \ker(T-\lambda)^{n+1}$. Then $(a; b) \in T - \lambda$ for some $b \in \ker(T-\lambda)^n$. By induction hypothesis $b \in \operatorname{ran} g(\lambda) + \cdots + \operatorname{ran} g^{(n-1)}(\lambda)$. Hence,

$$b = g(\lambda)x_1 + \cdots + g^{(n-1)}(\lambda)x_n$$

for some $x_1, \dots, x_n \in \mathbb{C}^k$. By (3.7.2) applied for $1, \dots, n$ we get

$$(g'(\lambda)x_1 + \cdots + \frac{1}{n}g^{(n)}(\lambda)x_n; g(\lambda)x_1 + \cdots + g^{(n-1)}(\lambda)x_n) \in T - \lambda \,,$$

and further

$$(g'(\lambda)x_1 + \cdots + \frac{1}{n}g^{(n)}(\lambda)x_n - a, 0) \in T - \lambda \,.$$

According to our assumption we have $\operatorname{ran} g(\lambda) = \ker(T - \lambda)$. Consequently, $g'(\lambda)x_1 + \cdots + \frac{1}{n}g^{(n)}(\lambda)x_n - a = g(\lambda)x_0$ and, hence,

$$a = -g(\lambda)x_0 + g'(\lambda)x_1 + \dots \frac{1}{n}g^{(n)}(\lambda)x_n \in \operatorname{ran} g(\lambda) + \cdots + \operatorname{ran} g^{(n)}(\lambda)$$

for some $x_0 \in \mathbb{C}^k$. ❑

rootspacecor.

3.7.5 Corollary. *With the notations from Theorem* 3.7.3 *assume that* $\dim\ker(T - \lambda) = k > 0$ *for all* $\lambda \in G \setminus D$. *Then we have* $c\ell(E_\lambda(T)) = c\ell(E_\eta(T))$ *for all* $n \in \mathbb{N}$ *and all* $\eta, \lambda \in G \setminus D$. *Moreover,* $c\ell(E_\lambda(T)) \subseteq c\ell(E_\eta(T))$ *for all* $n \in \mathbb{N}$, $\lambda \in G \setminus D$ *and all* $\eta \in G$.

Proof. Take a $\mu \in G \cap \mathbb{C}$ and let $O(\mu) \subseteq G \setminus D$ and $h : O(\mu) \to B(\mathbb{C}^k, T)$ be as in Theorem 3.7.3. For a sufficiently small $\epsilon > 0$ we have $U_{3\epsilon}(\mu) \subseteq O(\mu)$. If $\xi \in U_\epsilon(\mu)$, then due the the triangle inequality $\mu \in U_{2\epsilon}(\xi) \subseteq U_{3\epsilon}(\mu)$ and $\xi \in U_{2\epsilon}(\mu) \subseteq U_{3\epsilon}(\mu)$. Expanding $g(\lambda) := \pi_1 \circ h(\lambda)$ around μ and around ξ we get

$$g(\lambda) = \sum_{n=0}^{\infty}(\lambda - \mu)^n g^{(n)}(\mu), \ \lambda \in U_{2\epsilon}(\mu) \,,$$

and

$$g(\lambda) = \sum_{n=0}^{\infty}(\lambda - \xi)^n g^{(n)}(\xi), \ \lambda \in U_{2\epsilon}(\xi) \,.$$

In particular,

$$g(\xi) = \sum_{n=0}^{\infty}(\xi - \mu)^n g^{(n)}(\mu) \ \text{ and } \ g(\mu) = \sum_{n=0}^{\infty}(\mu - \xi)^n g^{(n)}(\xi) \,.$$

More generally, for $k \in \mathbb{N} \cup \{0\}$ we have

$$g^{(k)}(\xi) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}(\xi - \mu)^{n-k} g^{(n)}(\mu) \ \text{ and } \ g^{(k)}(\mu) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}(\mu - \xi)^{n-k} g^{(n)}(\xi) \,.$$

As $\operatorname{ran} h(\xi) = \{(x; \xi x) \in T : x \in \mathcal{A}\}$, using (3.7.3) and (3.7.2) we obtain from the first relation that $E_\xi(T) \subseteq c\ell(E_\mu(T))$. If $\mu \in G \setminus D$, then $E_\mu(T) \subseteq c\ell(E_\xi(T))$ by the second relation.

In particular, we showed that $c\ell(E_\xi(T)) \subseteq c\ell(E_\mu(T))$, and in the case that $\mu \in G \setminus D$,

$$c\ell(E_\xi(T)) = c\ell(E_\mu(T)) \tag{3.7.4}$$

gleirootglei

for all $\xi \in U_\epsilon(\mu)$.

In case that $\mu = \infty \in G$ we consider T^{-1} instead of T. By the above arguments in combination with Theorem 3.5.8 there exists a neighbourhood $U(\infty) \subseteq G$ of ∞ such that $c\ell(E_\xi(T)) \subseteq c\ell(E_\mu(T))$, and in the case that $\mu = \infty \in G \setminus D$, (3.7.4) for all $\xi \in U_\epsilon(\mu)$.

Now fix $\eta \in G \setminus D$ and define

$$A := \{\lambda \in G \setminus D : c\ell(E_\lambda(T)) = c\ell(E_\eta(T))\}, \quad B := \{\lambda \in G \setminus D : c\ell(E_\lambda(T)) \neq c\ell(E_\eta(T))\}.$$

Obviously, $\eta \in A \neq \emptyset$, $A \cup B = G \setminus D$ and $A \cap B = \emptyset$. According to (3.7.4) for all ξ in a certain sufficiently small neighbourhood of $\mu \in G \setminus D$, A and B are open. Since $G \setminus D$ is connected, we necessarily have $A = G \setminus D$. ❑

rootspaceendl.

3.7.6 Corollary. *If in the situation of Lemma* 3.7.4 $E_\mu(T)$ *is finite-dimensional for at least one* $\mu \in G$*, then we even have*

$$E_\lambda(T) \subseteq E_\eta(T)$$

for all $\lambda \in G \setminus D$ *and all* $\eta \in \mathbb{C} \cup \{\infty\}$.

Proof. By Corollary 3.7.5 the assumption implies that the linear space $\mathcal{B} := E_\lambda(T)$ does not depend on $\lambda \in G \setminus D$ and is also finite dimensional. Obviously, $\mathcal{B}$ contains all eigenvectors and, more generally, all root vectors corresponding to the eigenvalues $\lambda \in G \setminus D$. Hence, the eigenspaces and the root spaces of $B := T \cap (\mathcal{B} \times \mathcal{B})$ and those of T coincide for eigenvalues $\lambda \in G \setminus D$.

Moreover, B is a finite dimensional extension of the zero relation $\{0\} \times \{0\}$ in $\mathcal{B}$. Since $r(\{0\} \times \{0\}) = \mathbb{C} \cup \{\infty\}$ and since $\{0\} \neq \ker(T - \lambda) = \ker(B - \lambda)$ for $\lambda \in G \setminus D \neq \emptyset$ we can apply Theorem 3.7.3 and Corollary 3.7.5 to $\{0\} \times \{0\}$ and its extension B with $G := \mathbb{C} \cup \{\infty\}$ as the connected, open subset of $r(\{0\} \times \{0\})$.

According to Corollary 3.7.5 we obtain $E_\lambda(B) \subseteq E_\eta(B)$ for all $\eta \in \mathbb{C} \cup \{\infty\}$ and all $\lambda \in (\mathbb{C} \cup \{\infty\}) \setminus E$ for a discrete subset E of $\mathbb{C} \cup \{\infty\}$, i.e. E is finite. If $\lambda \in G \setminus (D \cup E)$, $E_\lambda(B)$ is nothing else but $\mathcal{B}$. $E_\eta(B)$ is always contained in $E_\eta(T)$. ❑

3.8 Adjoint linear Relations

In scalar product spaces it is possible to defined adjoint linear relations, which correspond to the adjoint of an operator in the Hilbert space case.

adjungdef.

3.8.1 Definition. Let $(\mathcal{V}, [.,.]_\mathcal{V})$, $(\mathcal{W}, [.,.]_\mathcal{W})$, be scalar product spaces, and let $T \subseteq \mathcal{V} \times \mathcal{W}$ be a linear relation. Then

$$T^{[*]} := \{(y; x) \in \mathcal{W} \times \mathcal{V} \ : \ [y, w]_\mathcal{W} = [x, v]_\mathcal{V} \ \text{ for all } \ (v; w) \in T\}$$

is called the *adjoint* relation to T.

As usual, when no confusion is possible we sometimes drop explit notation of the scalar product $[.,.]$ under consideration, and write T^* instead of $T^{[*]}$. ◊

adjungdarst. *3.8.2 Remark.* Clearly, providing $\mathcal{V} \times \mathcal{W}$ with the scalar product $[(a;b),(c;d)] := [a,c] + [b,d],\ (a;b),(c;d) \in \mathcal{W} \times \mathcal{V}$ (see Proposition 1.1.8) we have $(y;x) \in T^{[*]}$ if and only if

$$[y,w] - [x,v] = [(y;x),(w;-v)] = 0,\ \forall\, (v;w) \in T\ . \qquad (3.8.1)$$ adjung1

$\{(w;-v) : (u;v) \in T\}$ is nothing elese, but the image of T under the transformation $\mu_{-1} \circ \tau_{\mathcal{V} \leftrightarrows \mathcal{W}} : \mathcal{V} \times \mathcal{W} \to \mathcal{W} \times \mathcal{V},\ (v;w) \mapsto (w;-v)$; see Example 3.2.2. Therefore, (3.8.1) shows

$$T^{[*]} = (\mu_{-1} \circ \tau_{\mathcal{V} \leftrightarrows \mathcal{W}}(T))^{[\perp]} = (-T^{-1})^{[\perp]}\,, \qquad (3.8.2)$$ adjung2

where the orthogonal complement on the right hand side is taken in $\mathcal{W} \times \mathcal{V}$ wrt. $[.,.]$. Since $(y;x) \in T^{[*]}$ can also be characterized by $[x,v] - [y,w] = [(x;-y),(v;w)] = 0,\ \forall\, (v;w) \in T$, we also have

$$T^{[*]} = \tau_{\mathcal{V} \leftrightarrows \mathcal{W}} \circ \mu_{-1}(T^{[\perp]}) = (-T^{[\perp]})^{-1}\,, \qquad (3.8.3)$$ adjung3

where the orthogonal complement on the right hand side is taken in $\mathcal{V} \times \mathcal{W}$ wrt. to the sum scalar product.

Finally, the characterization of $(y;x) \in T^{[*]}$ via (3.8.1). Can be seen from a slightly different point of view. In fact, defining the scalar product $\langle (x;y),(v;w) \rangle := [x,v] - [y,w] = [\mu_{-1}(x;y),(v;w)]$ on $\mathcal{V} \times \mathcal{W}$ (3.8.1) or (3.8.3) show that

$$T^{[*]} = \tau_{\mathcal{V} \leftrightarrows \mathcal{W}}(T^{\langle \perp \rangle}) = (T^{\langle \perp \rangle})^{-1}\,. \qquad (3.8.4)$$ adjung5

◊

invkomp. **3.8.3 Lemma.** *We have $(T^{-1})^{[*]} = (T^{[*]})^{-1}$, $(\alpha S)^{[*]} = (\bar{\alpha})S^{[*]}$ $S^{[*]}Q^{[*]} \subseteq (QS)^{[*]}$ and $S^{[*]} + T^{[*]} \subseteq (S+T)^{[*]}$ for linear relations $Q \subseteq \mathcal{U} \times \mathcal{V},\ S,T \subseteq \mathcal{V} \times \mathcal{W}$ and a scalar $\alpha \in \mathbb{C} \setminus \{0\}$.*

Proof. Clearly, $(x;y) \in (T^{-1})^{[*]}$ if and only if $[x,v] = [y,w]$ for all $(v;w) \in T^{-1}$, or $[x,v] = [y,w]$ for all $(v;w) \in T$, which is equivalent to $(y;x) \in T^{[*]}$. Also $(\alpha S)^{[*]} = (\bar{\alpha})S^{[*]}$ can be checked in such a straight forward manner.

$(a;c) \in S^{[*]}Q^{[*]}$ yields $(a;b) \in Q^{[*]}$ and $(b;c) \in S^{[*]}$ for some $b \in \mathcal{V}$. For any $(u;w) \in QS$, i.e. $(u;v) \in S, (v;w) \in Q$ for some $v \in \mathcal{V}$, we have

$$[c,u] = [b,v] = [a,w]\,,$$

which shows $(a;c) \in (QS)^{[*]}$.

Applying $S^{[*]}Q^{[*]} \subseteq (QS)^{[*]}$ to $Q = \alpha I, S$ and to $Q = \frac{1}{\alpha}I, \alpha S$ gives $(\alpha S)^{[*]} = (\bar{\alpha})S^{[*]}$.

For $(b;c) \in S^{[*]} + T^{[*]}$ we have $(b;d) \in S^{[*]}$ and $(b;c-d) \in T^{[*]}$ for some $d \in \mathcal{V}$. For any $(v;w) \in S+T$, i.e. $(v;x) \in S$ and $(v;w-x) \in T$ for some $x \in \mathcal{W}$, we get

$$[b,w] = [b,x] + [b,w-x] = [d,v] + [c-d,v] = [c,v]\,,$$

and therefore, $(b,c) \in (S+T)^{[*]}$. ❑

adungeig1. **3.8.4 Lemma.** *Let $(\mathcal{A},[.,.])$ and $(\mathcal{B},[.,.])$ be scalar product spaces both provided with a norm such that on $\mathcal{A}$ and on $\mathcal{B}$ the respective scalar product $[.,.]$ is continuous with respect to the respective norm. Moreover, let $T \subseteq \mathcal{A} \times \mathcal{B}$ be a linear relation.*

Then the adjoint $T^{[*]} \subseteq \mathcal{B} \times \mathcal{A}$ is a closed linear relation. Moreover, $T^{[*]}$ coincides with the adjoint of the closure of T.

Proof. Because of (3.8.2) the assertion immediately follows from Lemma 2.1.3 in combination with that fact that $\mu_{-1}(T) \circ \tau_{\mathcal{V} \leftrightarrows \mathcal{W}}$ is a bi-continuous bijection from $\mathcal{A} \times \mathcal{B}$ onto $\mathcal{B} \times \mathcal{A}$; see Example 3.4.5. ❑

In Krein spaces we can say much more about the properties of the adjoint linear relation as in general scalar product spaces.

adjunggleich.

3.8.5 Remark. Let $(\mathcal{K}_j, (.,.))$ be Hilbert spaces, and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be an everywhere defined, bounded linear operator viewed as a linear relation by identifying T with its graph. Let $T^{(*)}$ be the adjoint of T in the sense of Definition 3.8.1 and just in the present remark let $T^{(+)}$ be the adjoint of T in the classical sense of, i.e. $(Tu, y) = (u, T^{(+)}y),\ u \in \mathcal{H}_1, y \in \mathcal{H}_2$; see . We are going to verify that $T^{(*)} = T^{(+)}$.

Missing Reference

From $(Tu, y) = (u, T^{(+)}y),\ u \in \mathcal{H}_1$ for all $u \in \mathcal{H}_1$ we obtain $(y; T^{(*)}y) \in T^{(*)}$ for all $y \in \mathcal{H}_2$ and, hence, $T^{(+)} \subseteq T^{(*)}$ identifying $T^{(+)}$ with its graph. Conversely, if $(y; x) \in T^{(*)}$, then $(u, T^{(+)}y) = (Tu, y) = (u, x)$ for all $u \in \mathcal{H}_1$. Hence, $x = T^{(+)}y$ or $(y; x) \in T^{(+)}$ again identifying $T^{(+)}$ with its graph. ◊

adungeigkrein.

3.8.6 Proposition. *Let $(\mathcal{A}, [.,.]_{\mathcal{A}})$ and $(\mathcal{B}, [.,.]_{\mathcal{B}})$ be Krein spaces, and let $T \subseteq \mathcal{A} \times \mathcal{B}$ be a linear relation. Then we have:*

(*i*) $(T^{[*]})^{[*]} = c\ell(T)$.

(*ii*) $\operatorname{mul} T^{[*]} = (\operatorname{dom} T)^{[\perp]}$ *and* $\ker T^{[*]} = (\operatorname{ran} T)^{[\perp]}$.

(*iii*) *If $(.,.)_{\mathcal{A}}$ is a compatible Hilbert space scalar product on $\mathcal{A}$ and $(.,.)_{\mathcal{B}}$ is a compatible Hilbert space scalar product on $\mathcal{B}$ (see Definition 2.2.1 and Remark 2.3.4) and if $G_{\mathcal{A}}$ and $G_{\mathcal{B}}$ denote the respective Gram operators, then the adjoint $T^{(*)}$ of T with respect to $(\mathcal{A}, (.,.)_{\mathcal{A}}), (\mathcal{B}, (.,.)_{\mathcal{B}})$ and the adjoint $T^{[*]}$ of T with respect to $(\mathcal{A}, [.,.]_{\mathcal{A}}), (\mathcal{B}, [.,.]_{\mathcal{B}})$ are related as follows:*

$$T^{[*]} = G_{\mathcal{A}}^{-1} T^{(*)} G_{\mathcal{B}} .$$

(*iv*) *If $B : \mathcal{A} \to \mathcal{B}$ is an everywhere defined, bounded linear operator, then so is $B^{[*]}$. Moreover,*

$$(B^{[*]}M)^{[\perp]_{\mathcal{A}}} = B^{-1}(M^{[\perp]_{\mathcal{B}}}) \tag{3.8.5}$$ fllab

for any subset $M \subseteq \mathcal{B}$.

(*v*) *If $B : \mathcal{A} \to \mathcal{B}$ is an everywhere defined, bounded linear operator, then* $(T + B)^{[*]} = T^{[*]} + B^{[*]}$.

Proof.

(*i*) According to (3.8.2), (3.8.3), Lemma 2.3.8 and by the fact, that $\tau_{\mathcal{B} \leftrightarrows \mathcal{A}} \circ \mu_{-1}$ is a bi-continuous bijection, we have

$$(T^{[*]})^{[*]} = \tau_{\mathcal{B} \leftrightarrows \mathcal{A}} \circ \mu_{-1}\Big(\big((\mu_{-1} \circ \tau_{\mathcal{A} \leftrightarrows \mathcal{B}}(T))^{[\perp]}\big)^{[\perp]}\Big) = \\ \tau_{\mathcal{B} \leftrightarrows \mathcal{A}} \circ \mu_{-1}\Big(c\ell(\mu_{-1} \circ \tau_{\mathcal{A} \leftrightarrows \mathcal{B}}(T))\Big) = c\ell(T) .$$

(*ii*) For $x \in \mathcal{A}$ we have $y \in \operatorname{mul} T^{[*]}$ if and only if $(0; x) \in T^{[*]}$. The latter fact means that $[x, u] = [0, v]$ for all $(u; v) \in T$, which is clearly equivalent to $x \in (\operatorname{dom} T)^{[\perp]}$. $\ker T^{[*]} = (\operatorname{ran} T)^{[\perp]}$ is proved in the same way.

(*iii*) For $(y; x) \in \mathcal{B} \times \mathcal{A}$ we have

$$(y; x) \in T^{[*]} \Leftrightarrow (G_{\mathcal{B}} y, w)_{\mathcal{B}} = (G_{\mathcal{A}} x, u)_{\mathcal{A}},\ (u; w) \in T \Leftrightarrow (G_{\mathcal{B}} y; G_{\mathcal{A}} x) \in T^{(*)},$$

which gives $T^{[*]} = G_{\mathcal{A}}^{-1} T^{(*)} G_{\mathcal{B}}$; see Lemma 3.2.3.

(*iv*) According to Remark 3.8.5 $B^{[*]} = G_{\mathcal{A}}^{-1} B^{(*)} G_{\mathcal{B}}$ is everywhere defined and bounded. For $x \in \mathcal{A}$ we have

$$x \in (B^{[*]} M)^{[\perp]_{\mathcal{A}}} \Leftrightarrow [x, B^{[*]} b]_{\mathcal{A}},\ b \in M \Leftrightarrow [Bx, b]_{\mathcal{B}},\ b \in M \Leftrightarrow x \in B^{-1}(M^{[\perp]_{\mathcal{B}}}).$$

(*v*) Apply $S^{[*]} + T^{[*]} \subseteq (S+T)^{[*]}$ from Lemma 3.8.3 gives
$B^{[*]} + T^{[*]} \subseteq (B+T)^{[*]} = (B+T)^{[*]} + (-B)^{[*]} + B^{[*]} \subseteq T^{[*]} + B^{[*]}$.

❑

In the previous proof we used that $B^{[*]} - B^{[*]} = 0$, where 0 stands for the zero-mapping defined on all of $\mathcal{B}$, i.e. for $\mathcal{B} \times \{0\}$. Note that for relations $S \subseteq \mathcal{B} \times \mathcal{A}$ with $\operatorname{mul} S \neq \{0\}$ or with $\operatorname{dom} T \neq \mathcal{B}$, $T - T \neq \mathcal{B} \times \{0\}$.

adjungeig.

3.8.7 Lemma. *For $j = 1, 2$ let $(\mathcal{A}_j, [.,.])$ and $(\mathcal{B}_j, [.,.])$ be Krein spaces. Let $A : \mathcal{A}_1 \to \mathcal{A}_2$ and $D : \mathcal{B}_1 \to \mathcal{B}_2$ be bounded and linear. For linear relation $T \subseteq \mathcal{A}_2 \times \mathcal{B}_2$ we have (see Lemma 3.2.3)*

$$\Big((A^{[*]} \times D^{[*]})(T)\Big)^{[*]} = (D \times A)^{-1}(T^{[*]}).$$

In particular, $(D \times A)^{-1}(T^{[*]})^{[*]}$ is the closure of $(A^{[*]} \times D^{[*]})(T)$.

Proof. Firstly, it is easily checked that $A \times D : \mathcal{A}_1 \times \mathcal{B}_1 \to \mathcal{A}_2 \times \mathcal{B}_2$ has $A^{[*]} \times D^{[*]}$ is its adjoint when the respective product spaces are equipped with the sum scalar product; see Lemma 2.3.9.

Applying (3.8.5) to $A \times D$ and T yields

$$\Big((A^{[*]} \times D^{[*]})(T)\Big)^{[\perp]} = (A \times D)^{-1}(T^{[\perp]}).$$

By (3.8.3) and Corollary 3.2.5 we obtain

$$\Big((A^{[*]} \times D^{[*]})(T)\Big)^{[*]} = \Big(-\Big((A^{[*]} \times D^{[*]})(T)\Big)^{[\perp]}\Big)^{-1} =$$
$$\Big(-(A \times D)^{-1}(T^{[\perp]})\Big)^{-1} = (D \times A)^{-1}\Big(\big(-T^{[\perp]}\big)^{-1}\Big) = (D \times A)^{-1}(T^{[*]}).$$

The final assertion follows from Proposition 3.8.6. ❑

tauundadj.

3.8.8 Lemma. *Let $(\mathcal{V}, [.,.]_{\mathcal{V}})$ be a scalar product space, and let $T \subseteq \mathcal{V} \times \mathcal{V}$ be a linear relation. For a regular $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{C}^{2\times 2}$ we then have (see Definition 3.3.1)*

$$(\tau_M(T))^{[*]} = \tau_{\bar{M}}(T^{[*]}),$$

where $\bar{M} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$.

Proof. The elements from $\tau_{\bar{M}}(T^{[*]})$ are of the form $(\bar{\delta}u + \bar{\gamma}v; \bar{\beta}u + \bar{\alpha}v)$ where $(u; v) \in T^{[*]}$, and those from $\tau_M(T)$ look like $(\delta x + \gamma y; \beta x + \alpha y)$ with $(x; y) \in T$. For such elements we calculate

$$[\delta x + \gamma y, \bar{\beta}u + \bar{\alpha}v] = \delta\beta[x, u] + \delta\alpha[x, v] + \gamma\beta[y, u] + \gamma\alpha[y, v] =$$
$$\delta\beta[x, u] + \delta\alpha[y, u] + \gamma\beta[x, v] + \gamma\alpha[y, v] = [\beta x + \alpha y, \bar{\delta}u + \bar{\gamma}v],$$

and conclude that $\tau_{\bar{M}}(T^{[*]}) \subseteq \tau_M(T)^{[*]}$. This fact applied to $\tau_M(T)^{[*]}$ and M^{-1} yields

$$\tau_{\overline{M^{-1}}}(\tau_M(T)^{[*]}) \subseteq \tau_{M^{-1}}(\tau_M(T))^{[*]} = T^{[*]}.$$

Applying $\tau_{\bar{M}}$ on both sides and keeping in mind that $(\bar{M})^{-1} = \overline{M^{-1}}$ also gives $\tau_M(T)^{[*]} \subseteq \tau_{\bar{M}}(T^{[*]})$. ❑

For $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\lambda \end{smallmatrix}\right)$ we have $\tau_M(T) = (T - \lambda)^{-1}$; see Example 3.3.5. Hence, Applying Lemma 3.8.8 gives

$$((T - \lambda)^{-1})^{[*]} = (T^{[*]} - \bar{\lambda})^{-1}. \tag{3.8.6}$$ resuadj

rootsporth. **3.8.9 Corollary.** *Let $(\mathcal{V}, [.,.]_{\mathcal{V}})$ be a scalar product space, and let $T \subseteq \mathcal{V} \times \mathcal{V}$ be a linear relation. For $\mu, \lambda \in \mathbb{C} \cup \{\infty\}$ with $\bar{\mu} \neq \lambda$ we have $E_\lambda(T)[\perp]E_\mu(T^{[*]})$.*

Proof. Let $M \in \mathbb{C}^{2\times 2}$ ve regular such that $\phi_M(\lambda) = \infty$ and $\phi_M(\bar{\mu}) = 0$. By Theorem 3.5.8 we have $(r, s \in \mathbb{N})$

$$\operatorname{mul} \tau_M(T)^r = \ker(\tau_M(T) - \phi_M(\lambda))^r = \ker(T - \lambda)^r$$

and by $\overline{\phi_M(\bar{\mu})} = \phi_{\bar{M}}(\mu)$ together with Lemma 3.8.8 also

$$\ker(\tau_M(T)^{[*]})^s = \ker(\tau_M(T)^{[*]} - \overline{\phi_M(\bar{\mu})})^s = \ker(\tau_{\bar{M}}(T^{[*]}) - \bar{\phi}_{\bar{M}}(\mu))^s = \ker(T^{[*]} - \mu)^s.$$

Therefore, in order to verify the assertion, it is enough to show that $\operatorname{mul} R^r[\perp]\ker(R^{[*]})^s$ for any $r, s \in \mathbb{N}$ and any linear relation $R \subseteq \mathcal{V} \times \mathcal{V}$. Setting $\operatorname{mul} R^0 := \{0\} =: \ker(R^{[*]})^0$ we shall now prove $\operatorname{mul} R^r[\perp]\ker(R^{[*]})^s$, $r, s \in \mathbb{N} \cup \{0\}$ by induction on $\min(r, s)$. For $\min(r, s) = 0$ the assertion is clear.

For $\min(r, s) \geq 0$ and $x \in \operatorname{mul} R^r$, $y \in \ker(R^{[*]})^s$ we have $(x'; x) \in R$ and $(y; y') \in R^{[*]}$ with some $x' \in \operatorname{mul} R^{r-1}$ and $y' \in \ker(R^{[*]})^{s-1}$. Hence, $[x, y] = [x', y'] = 0$ by induction hypothesis. ❑

resadjeig. **3.8.10 Corollary.** *For a Krein space $(\mathcal{A}, [.,.]_{\mathcal{A}})$ and a linear relation $T \subseteq \mathcal{A} \times \mathcal{A}$ we have $\rho(T^{[*]}) = \overline{\rho(T)}$, $\sigma(T^{[*]}) = \overline{\sigma(T)}$.*

Proof. $\rho(T^{[*]}) = \overline{\rho(T)}$ and consequently $\sigma(T^{[*]}) = \overline{\sigma(T)}$ follow from (3.8.6) and the fact, that the adjoint of an everywhere defined and bounded operator is also everywhere defined and bounded; see Proposition 3.8.6. ❑

Another corollary deals with the functional calculus introduced in Definition 3.6.1. For that corollary we define $s^{\#}(z) = \overline{s(\bar{z})}$ for any rational function s. Note that for $s(z) = \phi_M(z)$ we have $s^{\#}(z) = \phi_{\bar{M}}(z)$.

adjfunccal.

3.8.11 Corollary. *Let T be a linear relation on a Krein space $(\mathcal{A}, [.,.]_{\mathcal{A}})$ with non-empty resolvent set. Then $s \mapsto s^{\#}$ constitutes an conjugate linear isomorphism from $C_{\rho(T)}(z)$ onto $C_{\rho(T^{[*]})}(z)$ which is compatible with multiplication. Moreover,*

$$s^{\#}(T^{[*]}) = s(T)^{[*]}, \ s \in C_{\rho(T)}(z) .$$

Proof. $s \mapsto s^{\#}$ is easily checked to be a conjugate linear bijection on the space of all rational functions, which is compatible with multiplication. Moreover, the poles of $s^{\#}$ are contained in $\rho(T^{[*]}) = \overline{\rho(T)}$ if and only if those of s are contained in $\rho(T)$. Therefore, $s \mapsto s^{\#}$ maps $C_{\rho(T)}(z)$ onto $C_{\rho(T^{[*]})}(z)$.

Finally, since the functions of the form $\phi_N(z)^j$ for $j \in \mathbb{N} \cup \{0\}$ and regular $N \in \mathbb{C}^{2\times 2}$ such that the pole of ϕ_N is contained in $\rho(T)$ span $C_{\rho(T)}(z)$ and since (see (3.6.2) and Lemma 3.8.8)

$$\left(\phi_N(T)^j\right)^{[*]} = \left(\tau_N(T)^{[*]}\right)^j = \left(\tau_{\bar{N}}(T^{[*]})\right)^j = \phi_N^{\#}(T^{[*]}) \, , ,$$

we get $s^{\#}(T^{[*]}) = s(T)^{[*]}$ for all $s \in C_{\rho(T)}(z)$. ❑

3.9 Special types of linear relations and their connection

uncontrisodef.

3.9.1 Definition. Let $(\mathcal{V}, [.,.]_{\mathcal{V}}), (\mathcal{W}, [.,.]_{\mathcal{W}})$, be scalar product spaces, and let $T \subseteq \mathcal{V} \times \mathcal{W}$ be a linear relation.

The linear relation T is called *isometric* (*unitary*) if $T^{-1} \subseteq T^{[*]}$ ($T^{-1} = T^{[*]}$). T is called *contractive*, if

$$[y, y] \leq [x, x] \ \text{ for all } \ (x; y) \in T .$$

If we have $\mathcal{V} = \mathcal{W}$ and $T \subseteq T^{[*]}$ ($T = T^{[*]}$), then we call T *symmetric* (*selfadjoint*). T is called *dissipative* if

$$\operatorname{Im}[y, x] \geq 0 \ \text{ for all } \ (x; y) \in T .$$

◊

isometrunit.

3.9.2 Remark. If for an isometric T we have $\operatorname{ran} T = \mathcal{W}$ and $(\operatorname{dom} T)^{[\perp]} = \{0\}$, then T is unitary. In fact, $T^{-1} \subsetneq T^{[*]}$ and $\operatorname{dom} T^{-1} = \mathcal{W}$ would give $(\operatorname{dom} T)^{[\perp]} = \operatorname{mul} T^{[*]} \neq \{0\}$.

In Krein spaces the condition $(\operatorname{dom} T)^{[\perp]} = \{0\}$ just means that $\operatorname{dom} T$ is dense; see Lemma 2.3.8. ◊

isometriccontr.

3.9.3 Remark. Isometric linear relations between scalar product spaces are extremal contractive linear relations. Indeed, being isometric means $T^{-1} \subseteq T^{[*]}$, or equivalently,

$$[y, b] = [x, a] \ \text{ for all } \ (x; y), (a; b) \in T .$$

Due to the polar identity (1.1.1) this is the same as $[y, y] = [x, x]$ for all $(x; y) \in T$. In particular, T is isometric if and only if T and T^{-1} are contractive. This, is in turn, equivalent to T^{-1} being isometric.

Similarly, $T \subseteq \mathcal{V} \times \mathcal{V}$ is symmetric if and only if

$$[y, a] = [x, b] \text{ for all } (x; y), (a; b) \in T ,$$

Again due to the polar identity this time applied to $\langle (x; y), (a; b) \rangle := [y, a] - [x, b]$ the symmetry of T can also be characterized by $[y, x] = [y, x]$ for all $(x; y) \in T$. In particular, T is symmetric if and only of T and $-T$ are dissipative. This, is in turn, equivalent to $-T$ being symmetric. ◊

In order to somewhat unify these characterizations consider for $(x; y) \in \mathcal{V} \times \mathcal{V}$ the 2×2-matrix

$$[(x; y)] := \begin{pmatrix} [y, y] & [y, x] \\ [x, y] & [x, x] \end{pmatrix} .$$

The contractivity of a linear relation $T \subseteq \mathcal{V} \times \mathcal{V}$ can then be characterized by ($e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

$$e_2^T [(x; y)] e_2 - e_1^T [(x; y)] e_1 \geq 0, \ (x; y) \in T ,$$

whereas $T \subseteq \mathcal{V} \times \mathcal{V}$ being dissipative means

$$(-i)(e_1^T [(x; y)] e_2 - e_2^T [(x; y)] e_1) \geq 0, \ (x; y) \in T .$$

An equality sign instead of $\geq$ here means that the respective relation is isometric or symmetric.

A straight forward calculation shows that for $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ and any $(x; y) \in \mathcal{V} \times \mathcal{V}$

$$[\tau_M(x; y)] = M \, [(x; y)] \, M^* . \tag{3.9.1}$$ numwerttrans

Consequently,

$$\begin{aligned} e_2^T [\tau_M(x; y)] e_2 - e_1^T [\tau_M(x; y)] e_1 = \\ (|\gamma|^2 - |\alpha|^2) e_1^T [(x; y)] e_1 + (|\delta|^2 - |\beta|^2) e_2^T [(x; y)] e_2 + \\ (\delta\bar{\gamma} - \beta\bar{\alpha}) e_2^T [(x; y)] e_1 + (\bar{\delta}\gamma - \bar{\beta}\alpha) e_1^T [(x; y)] e_2 . \end{aligned} \tag{3.9.2}$$ numwerttrans2

and

$$\begin{aligned} (e_1^T [\tau_M(x; y)] e_2 - e_2^T [\tau_M(x; y)] e_1) = \\ (\alpha\bar{\gamma} - \bar{\alpha}\gamma) e_1^T [(x; y)] e_1 + (\beta\bar{\delta} - \bar{\beta}\delta) e_2^T [(x; y)] e_2 + \\ (\bar{\gamma}\beta - \bar{\alpha}\delta) e_2^T [(x; y)] e_1 + (\alpha\bar{\delta} - \gamma\bar{\beta}) e_1^T [(x; y)] e_2 . \end{aligned} \tag{3.9.3}$$ numwerttrans3

Using these equalities we can consider special matrices M. But before recall that for a regular M the Möbius transform ϕ_M maps $\mathbb{C}^+$ bijectively onto itself if a certain scalar multiple of M belongs to $\mathbb{R}^{2\times 2}$ and has positive determinant, whereas ϕ_M maps $\mathbb{D}$ bijectively onto itself if it is of the from $\zeta\left(\begin{smallmatrix} 1 & -w \\ -\bar{w} & 1 \end{smallmatrix}\right)$ for some $\zeta \in \mathbb{T}, w \in \mathbb{D}$; see .

Missing Reference

contrtrans. **3.9.4 Theorem.** *Let $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ be regular and let $T \subseteq \mathcal{V} \times \mathcal{V}$ be a linear relation. If ϕ_M maps $\mathbb{D}$ ($\mathbb{C}^+$) bijectively onto itself, then T is contractive (dissipative), if and only if $\tau_M(T)$ is contractive (dissipative).*

If ϕ_M maps $\mathbb{D}$ bijectively onto $\mathbb{D}$ or onto $(\mathbb{C}\cup\{\infty\})\setminus c\ell(\mathbb{D})$, then T is isometric (unitary) if and only if $\tau_M(T)$ is isometric (unitary).

If ϕ_M maps $\mathbb{C}^+$ bijectively onto $\mathbb{C}^+$ or onto $\mathbb{C}^-$, then T is symmetric (selfadjoint) if and only if $\tau_M(T)$ is symmetric (selfadjoint).

Proof. Assume first that ϕ_M maps $\mathbb{D}$ bijectively onto itself. Then we can assume that $M = \left(\begin{smallmatrix} 1 & -w \\ -\bar{w} & 1 \end{smallmatrix}\right)$ for some $w \in \mathbb{D}$. (3.9.2) then gives

$$e_2^T[\tau_M(x;y)]e_2 - e_1^T[\tau_M(x;y)]e_1 = (1-|w|^2)\left(e_2^T[(x;y)]e_2 - e_1^T[(x;y)]e_1\right).$$

Thus T is contractive (isometric) if and only if $\tau_M(T)$ is.

Similarly, if $\phi_M(\mathbb{C}^+) = \mathbb{C}^+$, then we can assume that $M \in \mathbb{R}^{2\times 2}$ with $\det M > 0$. By (3.9.3) we have

$$(-i)(e_1^T[\tau_M(x;y)]e_2 - e_2^T[\tau_M(x;y)]e_1) = (-i)(\det M)(e_1^T[(x;y)]e_2 - e_2^T[(x;y)]e_1).$$

We see that T is dissipative (symmetric) if and only if $\tau_M(T)$ is.

If ϕ_M maps $\mathbb{D}$ bijectively onto $\mathbb{D}$ or onto $(\mathbb{C}\cup\{\infty\})\setminus c\ell(\mathbb{D})$, then then we can assume that $M = \left(\begin{smallmatrix} 1 & -w \\ -\bar{w} & 1 \end{smallmatrix}\right)$ or that $M = \left(\begin{smallmatrix} -\bar{w} & 1 \\ 1 & -w \end{smallmatrix}\right)$. Hence, $\bar{M} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)M\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. According to Lemma 3.8.8 we have

$$\tau_M(T)^{[*]} = \tau_{\bar{M}}(T^{[*]}) = \tau_M((T^{[*]})^{-1})^{-1}.$$

Therefore, T being isometric (unitary) implies that τ_M is isometric (unitary). As $(\tau_M)^{-1} = \tau_{M^{-1}}$ with $\overline{M^{-1}} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)M^{-1}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ also the converse holds ture.

If $\phi_M(\mathbb{C}^+) = \mathbb{C}^+$ or $\phi_M(\mathbb{C}^+) = \mathbb{C}^-$, then we can assume that $M \in \mathbb{R}^{2\times 2}$. According to Lemma 3.8.8 we then have $\tau_M(T)^{[*]} = \tau_M(T^{[*]})$. Therefore, T being symmetric (selfadjoint) implies that τ_M is symmetric (selfadjoint). As $(\tau_M)^{-1} = \tau_{M^{-1}}$ with a regular $M^{-1} \in \mathbb{R}^{2\times 2}$ also the converse holds ture. ❑

We can also consider a regular $M \in \mathbb{C}^{2\times 2}$ such that ϕ_M maps $\mathbb{C}^+$ bijectively onto $\mathbb{D}$. It is easy to check that the M appearing in the following definition has this property.

cayleydef. **3.9.5 Definition.** For $\mu \in \mathbb{C}^+$ and $M = \left(\begin{smallmatrix} 1 & -\mu \\ 1 & -\bar{\mu} \end{smallmatrix}\right)$ the transformation $C_\mu := \tau_M$ is called the *Cayley transform* (with base μ). Its inverse $\mathcal{F}_\mu := C_\mu^{-1}$ is called the *Inverse-Cayley transform.* ◊

According to Lemma 3.3.4 we have $\mathcal{F}_\mu = \tau_{M^{-1}}$, where $M = \left(\begin{smallmatrix} 1 & -\mu \\ 1 & -\bar{\mu} \end{smallmatrix}\right)$ and hence $M^{-1} = \frac{1}{\bar{\mu}-\mu}\left(\begin{smallmatrix} \bar{\mu} & -\mu \\ 1 & -1 \end{smallmatrix}\right)$. If we apply (3.9.2) to the current M, we get

$$e_2^T[\tau_M(x;y)]e_2 - e_1^T[\tau_M(x;y)]e_1 = (2\operatorname{Im}\mu)(-i)\left(e_1^T[(x;y)]e_2 - e_2^T[(x;y)]e_1\right). \quad (3.9.4)$$ cayleyequ

cayleytrans. **3.9.6 Theorem.** *Let $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ be regular and let $T \subseteq \mathcal{V}\times\mathcal{V}$ be a linear relation. If ϕ_M maps $\mathbb{C}^+$ bijectively onto $\mathbb{D}$, then T is dissipative, if and only if $\tau_M(T)$ is contractive. In particular, T is dissipative (symmetric,selfadjoint), if and only if $C_\mu(T)$ is contractive (isometric,unitary).*

If ϕ_N maps $\mathbb{D}$ bijectively onto $\mathbb{C}^+$, then T is contractive (isometric), if and only if $\tau_N(T)$ is dissipative (symmetric). In particular, T is contractive (isometric, unitary), if and only if $\mathcal{F}_\mu(T)$ is dissipative (symmetric,selfadjoint).

Proof. For $M = \left(\begin{smallmatrix} 1 & -\mu \\ 1 & -\bar{\mu} \end{smallmatrix}\right)$ the Möbius transformation ϕ_M maps $\mathbb{C}^+$ bijectively onto $\mathbb{D}$. In this case we immediately get from (3.9.4) that T is dissipative, if and only if $\tau_M(T) = C_\mu(T)$ is contractive. By Lemma 3.8.8 we have ($\bar{M} = \left(\begin{smallmatrix} 1 & -\bar{\mu} \\ 1 & -\mu \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) M$)

$$\tau_M(T)^{[*]} = \tau_{\bar{M}}(T^{[*]}) = (\tau_M(T^{[*]}))^{-1} .$$

Therefore, T is symmetric (selfadjoint) if and only if $\tau_M(T)$ is isometric (unitary). Substituting T by $\mathcal{F}_\mu(T)$ we see that T is contractive (isometric, unitary) if and only if $\mathcal{F}_\mu(T)$ is dissipative (symmetric, selfadjoint).

For a general M mapping $\mathbb{C}^+$ bijectively onto $\mathbb{D}$, clearly, $\phi^{-1}_{\left(\begin{smallmatrix} 1 & -\mu \\ 1 & -\bar{\mu} \end{smallmatrix}\right)} \circ \phi_M$ maps $\mathbb{C}^+$ bijectively onto $\mathbb{C}^+$. According to Theorem 3.9.4 $\tau_{\left(\begin{smallmatrix} 1 & -\mu \\ 1 & -\bar{\mu} \end{smallmatrix}\right)^{-1} M}(T) = \mathcal{F}_\mu(\tau_M(T))$ is dissipative (symmetric, selfadjoint), if and only if T has this property. By the first paragraph this is equivalent to $\tau_M(T)$ being contractive (isometric, unitary). Finally, for a ϕ_N mapping $\mathbb{D}$ bijectively onto $\mathbb{C}^+$ we substitute $\tau_N(T)$ for T and get for $M = N^{-1}$ that $T = \tau_M(\tau_N(T))$ is contractive (isometric, unitary) if and only if $\tau_N(T)$ is dissipative (symmetric, selfadjoint). ❑

secvsymm.

3.9.7 Example. We can use this result in order to obtain the well-known shape of $r(T)$ and $\rho(T)$ for a symmetric or selfadjoint relation on a Hilbert space $(\mathcal{H}, (.,.))$. In fact, for a symmetric T by Theorem 3.9.6 the relation $V := C_\mu(T)$ is an isometric relation on $\mathcal{H}$. Being on a Hilbert (see the forthcoming Remark 3.11.2) implies that $V : \operatorname{dom} V \to \operatorname{ran} V$ is in fact an isometric operator.

Therefore, by the considerations in Example 3.5.17 and by Theorem 3.5.6, $r(T)$ contains $\mathbb{C}^+ \cup \mathbb{C}^-$. For selfadjoint T we have $\rho(T) \supseteq \mathbb{C}^+ \cup \mathbb{C}^-$. Moreover, a symmetric relation is selfadjoint if only $\rho(T) \cap \mathbb{C}^+ \neq \emptyset$ and $\rho(T) \cap \mathbb{C}^- \neq \emptyset$.

Since for a closed relation S by Proposition 3.4.3 the space $\operatorname{ran}(S - \lambda) = \operatorname{dom}(S - \lambda)^{-1}$ is closed for any $\lambda \in r(S)$ and since $\operatorname{ran}(S - \lambda)^{(\perp)} = \ker(S^{(*)} - \bar{\lambda})$, for a closed symmetric relation S its selfadjointness is equivalent to

$$\mathbb{C}^+ \not\subseteq \sigma_p(S^{(*)}) \text{ and } \mathbb{C}^- \not\subseteq \sigma_p(S^{(*)}) ,$$

and it is also equivalent to

$$\sigma_p(S^{(*)}) \subseteq \mathbb{R} \cup \{\infty\} ,$$

◊

Concerning selfadjoint and unitary relations in Krein spaces we state an easy corollary of Corollary 3.8.10. Note that in general Krein spaces the spectrum of selfadjoint or unitary relations can be empty.

specvunsa.

3.9.8 Corollary. *Let $(\mathcal{A}, [.,.])$ be a Krein space and let $T \subseteq \mathcal{A} \times \mathcal{A}$ be a linear relation. If T is selfadjoint, then $\sigma(T)$ symmetric wrt. the real line, i.e. $\lambda \in \sigma(T) \Leftrightarrow \bar{\lambda} \in \sigma(T)$. If T is unitary then $\sigma(T)$ is symmetric wrt. the unique circle, i.e. $\lambda \in \sigma(T) \Leftrightarrow \frac{1}{\bar{\lambda}} \in \sigma(T)$.*

Proof. The assertion about selfadjoint T is clear from Corollary 3.8.10, and for unitary T we have $\sigma(T) = \sigma((T^{-1})^{[*]}) = \overline{\sigma(T^{-1})} = \{\frac{1}{\bar{\lambda}} : \lambda \in \sigma(T)\}$; see Theorem 3.5.6. ❑

3.10 The Potapov-Ginzburg transform

Recall from Example 3.2.9 the definition of the Potapov-Ginzburg tranform:

$$\tau_{PG} : (\mathcal{M}_1 \times \mathcal{N}_1) \times (\mathcal{M}_2 \times \mathcal{N}_2) \to (\mathcal{M}_1 \times \mathcal{N}_2) \times (\mathcal{M}_2 \times \mathcal{N}_1)\,,$$

$$\tau_{PG}((m_1; n_1); (m_2; n_2)) = ((m_1; n_2); (m_2; n_1))\,.$$

If all the vector spaces are provided with a scalar product, then this transform has an interesting property.

potapoveig.

3.10.1 Lemma. *Let* $(\mathcal{M}_j, [.,.])$, $j = 1, 2$, $(\mathcal{N}_j, [.,.])$, $j = 1, 2$, *be scalar product spaces. For* $j = 1, 2$ *we provide* $\mathcal{M}_j \times \mathcal{N}_j$ *with the scalar product* $[(x; y), (a; b)] = [x, y]_{\mathcal{M}_j} + [a, b]_{\mathcal{N}_j}$ *and for distinct* $i, j \in \{1, 2\}$ *we provide* $\mathcal{M}_i \times \mathcal{N}_j$ *with the scalar product* $\langle (x; y), (a; b)\rangle = [x, y]_{\mathcal{M}_i} - [a, b]_{\mathcal{N}_j}$*; see Proposition* 1.1.8. *Then for* $((a_1; b_1); (a_2; b_2)), ((m_1; n_1); (m_2; n_2)) \in (\mathcal{M}_1 \times \mathcal{N}_1) \times (\mathcal{M}_2 \times \mathcal{N}_2)$ *we have*

$$\begin{aligned} &\langle \tau_{PG}((a_1; b_1); (a_2; b_2))_1, \tau_{PG}((m_1; n_1); (m_2; n_2))_1\rangle - \\ &\qquad \langle \tau_{PG}((a_1; b_1); (a_2; b_2))_2, \tau_{PG}((m_1; n_1); (m_2; n_2))_2\rangle = \\ &\qquad [(a_1; b_1), (m_1; n_1)] - [(a_2; b_2), (m_2; n_2)]\,, \end{aligned} \tag{3.10.1}$$

poteigeq

where $\tau_{PG}((a_1; b_1); (a_2; b_2))_j$, $j = 1, 2$, *is the j'th entry of* $\tau_{PG}((a_1; b_1); (a_2; b_2))$, *i.e.* $\tau_{PG}((a_1; b_1); (a_2; b_2))_1 = (m_1; n_2)$, $\tau_{PG}((a_1; b_1); (a_2; b_2))_2 = (m_2; n_1)$.

Proof. The left hand side of the asserted equality is

$$\langle (a_1; b_2), (m_1; n_2)\rangle - \langle (a_2; b_1), (m_2; n_1)\rangle = [a_1, m_1] - [b_2, n_2] - [a_2, m_2] + [b_1, n_1]\,,$$

and the right hand side equals

$$[(a_1; b_1), (m_1; n_1)] - [(a_2; b_2), (m_2; n_2)] = [a_1, m_1] + [b_1, n_1] - ([a_2, m_2] + [b_2, n_2])\,.$$

❑

The Potapov-Ginzburg transform is a useful tool, in order to handle contractive relations.

potapcontr.

3.10.2 Proposition. *With the notation from Lemma* 3.10.1 *we have* $\tau_{PG}(T)^{\langle * \rangle} = \tau_{PG}(T^{[*]})$ *for any linear relation* T *between* $\mathcal{M}_1 \times \mathcal{N}_1$ *and* $\mathcal{M}_2 \times \mathcal{N}_2$*. The transform* τ_{PG} *on the right hand side of this equation is the Potapov-Ginzburg transform from* $(\mathcal{M}_2 \times \mathcal{N}_2) \times (\mathcal{M}_1 \times \mathcal{N}_1)$ *onto* $(\mathcal{M}_2 \times \mathcal{N}_1) \times (\mathcal{M}_1 \times \mathcal{N}_2)$.

Moreover, τ_{PG} *constitutes a linear bijection, which maps all contractive (isometric) linear relations between* $\mathcal{M}_1 \times \mathcal{N}_1$ *and* $\mathcal{M}_2 \times \mathcal{N}_2$ *both provided with* $[.,.]$ *bijectively onto all contractive (isometric) linear relations between* $\mathcal{M}_1 \times \mathcal{N}_2$ *and* $\mathcal{M}_2 \times \mathcal{N}_1$ *both provided with* $\langle .,.\rangle$.

Proof. To check $\tau_{PG}(T)^{\langle * \rangle} = \tau_{PG}(T^{[*]})$ let $((a_2; b_2); (a_1; b_1)) \in (\mathcal{M}_2 \times \mathcal{N}_2) \times (\mathcal{M}_1 \times \mathcal{N}_1)$. Then

$$((a_2; b_1); (a_1; b_2)) \in \tau_{PG}(T^{[*]}) \Leftrightarrow ((a_2; b_2); (a_1; b_1)) \in T^{[*]} \Leftrightarrow$$

$$\underbrace{[(a_2;b_2),(m_2;n_2)]}_{=[a_2,m_2]+[b_2,n_2]} = \underbrace{[(a_1;b_1),(m_1;n_1)]}_{=[a_1,m_1]+[b_1,n_1]} \quad \text{for all } ((m_1;n_1);(m_2;n_2)) \in T \;\Leftrightarrow$$

$$\underbrace{[(a_2;b_1),(m_2;n_1)]}_{=[a_2,m_2]-[b_1,n_1]} = \underbrace{[(a_1;b_2),(m_1;n_2)]}_{=[a_1,m_1]-[b_2,n_2]} \quad \text{for all } ((m_1;n_1);(m_2;n_2)) \in T \;\Leftrightarrow$$

$$((a_2;b_1);(a_1;b_2)) \in \tau_{PG}(T)^{\langle * \rangle}.$$

$T \subseteq (\mathcal{M}_1 \times \mathcal{N}_1) \times (\mathcal{M}_2 \times \mathcal{N}_2)$ being contractive (isometric) just means that the expression on the right hand side of (3.10.1) is ≥ 0 ($= 0$). According to (3.10.1) this happens if and only if the left hand side is ≥ 0 ($= 0$), which means that $\tau_{PG}(T)$ is contractive. ❑

pottrop. **3.10.3 Lemma.** *Let $\mathcal{M}_j$, $j = 1,2$, $\mathcal{N}_j$, $j = 1,2$, be vector spaces, let $T \subseteq (\mathcal{M}_1 \times \mathcal{N}_1) \times (\mathcal{M}_2 \times \mathcal{N}_2)$ be a linear relation. Assume that there is a linear bijection $S : \mathcal{N}_1 \to \mathcal{N}_2$ such that $S \subseteq T$. Then we have*

$$\operatorname{dom} \tau_{PG}(T) = (\operatorname{dom} T \cap \mathcal{M}_1) \times \mathcal{N}_2, \quad \textit{and}$$
$$\operatorname{dom} T = (\operatorname{dom} \tau_{PG}(T) \cap \mathcal{M}_1) \times \mathcal{N}_1. \tag{3.10.2}$$ ttz33

Moreover, T is an operator if and only if $\tau_{PG}(T)$ is an operator. In this case we have

$$T = \big(P_{\mathcal{M}_2\times\mathcal{N}_1,\mathcal{M}_2}\tau_{PG}(T)P_{\mathcal{M}_1\times\mathcal{N}_1,\mathcal{M}_1} + S\,(P_{\mathcal{M}_1\times\mathcal{N}_1,\mathcal{N}_1} - P_{\mathcal{M}_2\times\mathcal{N}_1,\mathcal{N}_1}\tau_{PG}(T)P_{\mathcal{M}_1\times\mathcal{N}_1,\mathcal{M}_1})\big)|_{\operatorname{dom} T}, \tag{3.10.3}$$ ttz34

where $P_{\mathcal{M}_j\times\mathcal{N}_k,\mathcal{M}_j}$ ($P_{\mathcal{M}_j\times\mathcal{N}_k,\mathcal{N}_k}$) denotes to projection from $\mathcal{M}_j \times \mathcal{N}_k$ onto the first (second) component.

If the involved spaces carry a norm such that S is bi-continuous, then T is a bounded operator if and only if $\tau_{PG}(T)$ is a bounded operator.

Proof. For $m_j \in \mathcal{M}_j$ and $n_j \in \mathcal{N}_j$ for $j = 1,2$ by $S \subseteq T$ we have

$$((m_1;n_1);(m_2;n_2)) \in T \Leftrightarrow$$
$$((m_1;n_1 - S^{-1}n_2);(m_2;0)) = ((m_1;n_1);(m_2;n_2)) - ((0;S^{-1}n_2);(0;n_2)) \in T \Leftrightarrow$$
$$((m_1;0);(m_2;n_1 - S^{-1}n_2)) \in \tau_{PG}(T). \tag{3.10.4}$$ ttz35

Thus, $\operatorname{dom} \tau_{PG}(T) \cap \mathcal{M}_1 = P_{\mathcal{M}_1\times\mathcal{N}_1,\mathcal{M}_1} \operatorname{dom} T$. Because of $\mathcal{N}_1 \subseteq \operatorname{dom} S \subseteq \operatorname{dom} T$ the latter coincides with $\operatorname{dom} T \cap \mathcal{M}_1$, and $\operatorname{dom} T = (\operatorname{dom} T \cap \mathcal{M}_1) \times \mathcal{N}_1$. From $\mathcal{N}_2 \subseteq \operatorname{ran} S \subseteq \operatorname{dom} \tau_{PG}(T)$ we get $\operatorname{dom} \tau_{PG}(T) = (\operatorname{dom} \tau_{PG}(T) \cap \mathcal{M}_1) \times \mathcal{N}_2$. This show the first relation in (3.10.2). The second follows from this one and the fact that $\tau_{PG}(\tau_{PG}(T)) = T$.

If $\tau_{PG}(T)$ is an operator, then (3.10.4) is the same as $P_{\mathcal{M}_1\times\mathcal{N}_1,\mathcal{M}_1}(m_1;n_1) = (m_1;0) \in \operatorname{dom} \tau_{PG}(T)$ and $\tau_{PG}(T)(m_1;0) = (m_2;n_1 - S^{-1}n_2)$ or – the latter written component wise –

$$m_2 = P_{\mathcal{M}_2\times\mathcal{N}_1,\mathcal{M}_2}\tau_{PG}(T)P_{\mathcal{M}_1\times\mathcal{N}_1,\mathcal{M}_1}(m_1;n_1),$$

$$n_2 = S\,(P_{\mathcal{M}_1\times\mathcal{N}_1,\mathcal{N}_1} - P_{\mathcal{M}_2\times\mathcal{N}_1,\mathcal{N}_1}\tau_{PG}(T)P_{\mathcal{M}_1\times\mathcal{N}_1,\mathcal{M}_1})(m_1;n_1).$$

Consequently, $(0;y) \in T$ implies $y = 0$, i.e. $T : \operatorname{dom} T \to \mathcal{M}_2 \times \mathcal{N}_2$ is an operator, and (3.10.3) holds true. This representation of T also shows that the boundedness of $\tau_{PG}(T)$ implies the boundedness of T.

The converse again follows from the fact that $\tau_{PG}(\tau_{PG}(T)) = T$. ❑

3.11 More on contractive relations

Brauchen wir folgendes Resultat?

isomant.

3.11.1 Lemma. *For a a contractive linear relation T between scalar product spaces $(\mathcal{V}, [.,.])$ and $(\mathcal{W}, [.,.])$ we have*

$$T^{-1} \cap T^{[*]} = \{(y; x) : [y, y] = [x, x]\},$$

and this linear relation is isometric.

Proof. First note that T being contractive means that T is a positive semidefinite subspace of $\mathcal{V} \times \mathcal{W}$ provided with the Krein space scalar product $\langle (a; b), (c; d) \rangle = [a, c] - [b, d]$. By Corollary 1.4.6 we have

$$T \cap T^{\langle\perp\rangle} = T^{\langle\circ\rangle} = \{(x; y) \in T : \langle (x; y), (x; y) \rangle = 0\} = \{(x; y) \in T : [y, y] = [x, x]\}.$$

It is easy to check that $T^{\langle\perp\rangle} = \{(c; d) \in \mathcal{V} \times \mathcal{W} : [(c; -d); (a; b)] = 0, \ (a; b) \in T\} = \mu_{-1}(T^{[\perp]})$. Hence, according to (3.8.3) we have

$$T^{-1} \cap T^{[*]} = \tau_{\mathcal{V}\leftrightarrows\mathcal{W}}(T \cap \mu_{-1}(T^{[\perp]})) = \tau_{\mathcal{V}\leftrightarrows\mathcal{W}}(T^{\langle\circ\rangle}) = \{(y; x) \in T : [y, y] = [x, x]\}.$$

Finally, in Remark 3.9.3 we saw that $\{(y; x) \in T : [y, y] = [x, x]\}$ is isometric. ❑

In the case of positive definiteness contractive (isometric) relation are well-known objects.

posdefcontr.

3.11.2 Remark. If the scalar products of $(\mathcal{V}, [.,.])$ and $(\mathcal{W}, [.,.])$ are both positive definite – this is clearly the case for Hilbert spaces – and if T is a contractive linear relation between $\mathcal{V}$ and $\mathcal{W}$, then $[y, y] \leq [x, x]$ for all $(x; y) \in T$ implies $\operatorname{mul} T = \{0\}$, i.e. $T : \operatorname{dom} T \to \mathcal{W}$ is an operator, which is bounded with operator norm ≤ 1 wrt. to the norms induced by the products $[.,.]$ on $\mathcal{V}$ and $\mathcal{W}$.

If only the scalar product space $(\mathcal{W}, [.,.])$ is positive definite and if $(\mathcal{V}, [.,.])$ is in fact a Gram space $(\mathcal{V}, [.,.], \mathcal{O})$, then also all contractive linear relation between $\mathcal{V}$ and $\mathcal{W}$ are continuous operators, when $\mathcal{V}$ is provided with $\mathcal{O}$ and $\mathcal{W}$ is provided with the norm induced by $[.,.]$. Choosing a compatible Hilbert space scalar product $(.,.)$ and denoting the Gram operator of $[.,.]$ wrt. $(.,.)$ by G this immediately follows from

$$[y, y] \leq [x, x] = (Gx, x) \leq \|G\| \, (x, x), \quad (x; y) \in T.$$

If $(\mathcal{V}, [.,.])$ and $(\mathcal{W}, [.,.])$ are both of finite index of negativity and both of finite index of nullity, then we can write $\mathcal{V} := \mathcal{V}_+[\dot{+}]\mathcal{V}_-[\dot{+}]\mathcal{V}^{[\circ]}$ and $\mathcal{W} := \mathcal{W}_+[\dot{+}]\mathcal{W}_-[\dot{+}]\mathcal{W}^{[\circ]}$ where the respective second and third components are finite dimensional; see (1.6.1).

If T is a linear relation between $\mathcal{V}$ and $\mathcal{W}$, then $T_+ := T \cap (\mathcal{V}_+ \times \mathcal{W}_+)$ is a linear relation between the positive definite spaces $\mathcal{V}_+$ and $\mathcal{W}_+$. Hence, for contractive T we have $T = T_+ \dot{+} R$, where $T_+ : \operatorname{dom} T \ (\subseteq \mathcal{V}_+) \to \mathcal{W}_+$ is a bounded linear operator and where $\dim R \leq \operatorname{ind}_- \mathcal{V} + \operatorname{ind}_- \mathcal{W} + \operatorname{ind}_0 \mathcal{V} + \operatorname{ind}_0 \mathcal{W} \ (< \infty)$. ◊

The following result is taken from .

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3.11.3 Lemma. *Let $(\mathcal{A}, [.,.], O)$ be a Gram space, and let $B : \mathcal{A} \to \mathcal{A}$ be an everywhere defined, bounded, linear and contractive operator, i.e. $[Bx, Bx] \leq [x, x],\ x \in \mathcal{A}$. If the spectrum of B is contained in $\mathbb{D}$, then $[.,.]$ is positive semidefinite.*

Die nun folgenden zusaetzlichen Aussagen sind vll unnoetig

In case that in addition $[Bx, Bx] < [x, x],\ x \in \mathcal{A} \setminus \{0\}$, $[.,.]$ is positive definite.

In case that in addition $(\mathcal{A}, [.,.])$ is a Krein space or in case that $[Bx, Bx] + \delta(x, x) \leq [x, x],\ x \in \mathcal{A}$ for some $\delta > 0$ and some compatible Hilbert space product $(.,.)$, $[.,.]$ is indeed a Hilbert space scalar product compatible with O.

Proof. Let G be the Gram operator of $[.,.]$ with respect to a compatible Hilbert space scalar product $(.,.)$ on $\mathcal{A}$. The contractivity of B means that $(GBx, Bx) \leq (Gx, x),\ x \in \mathcal{A}$, or equivalently $G - B^{(*)}GB \geq 0$, i.e. $G - B^{(*)}GB$ is a positive operator. By induction on m one easily checks that

$$G - (B^{m+1})^{(*)}GB^{m+1} = \sum_{j=0}^{m} (B^j)^{(*)}\big(G - B^{(*)}GB\big)B^j .$$

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By the condition on the spectrum the spectral radius $r(B)$ (see) is less than one. Since $r(B)$ coincides with $\lim_{m\to\infty} \|B^m\|^{\frac{1}{m}}$, we get $\|B^m\| \leq \left(\frac{r(B)+1}{2}\right)^m$ for all sufficiently large m. Hence, $\lim_{m\to\infty} B^m = 0$ and in turn

$$G = \sum_{j=0}^{\infty} (B^j)^{(*)}\big(G - B^{(*)}GB\big)B^j .$$

Finally, it is easy to check that the sum of a series with positive operator summands is a positive operator. This shows that $[,.,]$ is positive semidefinite.

The stronger assumption $[Bx, Bx] < [x, x],\ x \in \mathcal{A} \setminus \{0\}$ yields $G - B^{(*)}GB > 0$, and further $G > 0$, which is equivalent to the positive definiteness of $[.,.]$. The even stronger condition $[Bx, Bx] + \delta(x, x) \leq [x, x],\ x \in \mathcal{A}$ for some $\delta > 0$ gives $G - B^{(*)}GB \geq \delta I$. Hence, $G \geq \delta I$, which means that $[.,.]$ is in fact a Hilbert space scalar product compatible with O; see 56. We get the same conclusion if we assume that $(\mathcal{A}, [.,.])$ is a Krein space, because $G \geq 0$ and $0 \notin \sigma(G)$ also yields $G \geq \delta I$ for some $\delta > 0$; see Proposition 2.2.12, (3). ❑

Concerning spectral properties of contractive relations we have

contrev.

3.11.4 Theorem. *Let $T \subseteq \mathcal{V} \times \mathcal{V}$ be a contractive linear relation, where $(\mathcal{V}, [.,.])$ is a scalar product space. Moreover, let $\mathcal{L}_{<1}$ ($\mathcal{L}_{>1}$) be the subspace of $\mathcal{V}$ spanned by all root vectors of T corresponding to eigenvalues with modulus smaller than one (greater than one possibly including ∞), i.e.*

$$\mathcal{L}_{<1} = \operatorname{span} \bigcup_{\lambda\in\mathbb{D}} E_\lambda(T) ,$$

$$\mathcal{L}_{>1} = \operatorname{span} \bigcup_{\lambda\in(\mathbb{C}\cup\{\infty\})\setminus c\ell(\mathbb{D})} E_\lambda(T) .$$

Then $\mathcal{L}_{<1}$ ($\mathcal{L}_{>1}$) is positive (negative) semidefinite, and $\dim \mathcal{L}_{<1} \leq \operatorname{ind}_+(\mathcal{V}, [.,.]) + \operatorname{ind}_0(\mathcal{V}, [.,.])$ *(*$\dim \mathcal{L}_{>1} \leq \operatorname{ind}_-(\mathcal{V}, [.,.]) + \operatorname{ind}_0(\mathcal{V}, [.,.])$*).*

Proof. Let $x_j \in \ker(T-\mu_j)^{\nu(j)}$ for $j = 1,\dots,k$ and $\mu_1,\dots,\mu_k \in \sigma_p(T)\cap\mathbb{D}$. Choose $\nu(1),\dots,\nu(k) \in \mathbb{N}$ minimal with $\in \ker(T-\mu_j)^{\nu(j)}$. Let $y_j^0 = 0$,
$y_j^l \in \ker(T-\mu_j)^l$, $l = 1,\dots,\nu(j)$, such that $y_j^{\nu(j)} = x_j$ and $(y_j^l; y_j^{l-1}) \in (T-\mu_j)$. Set $m := \sum_{j=1}^k \nu(j)$

Define $\psi : \mathbb{C}^m \cong \times_{j=1}^k \mathbb{C}^{\nu(j)} \to \mathcal{V}$ by

$$\psi\Big(((\xi_j^l)_{l=1}^{\nu(j)})_{j=1}^k\Big) = \sum_{j=1}^{k}\sum_{l=1}^{\nu(j)} \xi_j^l y_j^l .$$

Define the linear operator $B := \mathbb{C}^m \to \mathbb{C}^m$ by $B(((\xi_j^l)_{l=1}^{\nu(j)})_{j=1}^k) = ((\xi_j^{l-1} + \mu_j\xi_j^l)_{l=1}^{\nu(j)})_{j=1}^k$, where $\xi_j^0 := 0$.

It is easy to check that $(\psi\times\psi)(B) \subseteq T$. Providing $\mathbb{C}^m$ with the scalar product $\langle x, y\rangle := -[\psi(x), \psi(y)]$ we get for $x \in B$ that

$$\langle Bx, Bx\rangle = -[\psi(Bx), \psi(Bx)] \le -[\psi(x), \psi(x)] = \langle x, x\rangle ,$$

since $(\psi(Bx); \psi(x))$ belongs to our contractive T. Therefore, B is a contractive operator on $(\mathbb{C}^m, \langle .,.\rangle)$ with eigenvalues $\mu_1,\dots,\mu_k \in \mathbb{D}$.

From Lemma 3.11.3 we infer that $\langle .,.\rangle$ is positive semidefinite. $x_j \in \psi(\mathbb{C}^m)$ yields the negative semidefiniteness of $[.,.]$ on the span of the vectors $x_1,\dots,x_k$. Since any element form $\mathcal{L}_{>1}$ is contained in this span for an appropriate choice of $x_1,\dots,x_k$, $\mathcal{L}_{>1}$ is negative semidefinite.

The inequality $\dim\mathcal{L}_{<1} \le \mathrm{ind}_+(\mathcal{V},[.,.]) + \mathrm{ind}_0(\mathcal{V},[.,.])$ follows from Corollary 1.6.6.

The assertion for $\mathcal{L}_{>1}$ follows, if we consider the contractive relation T^{-1} in the scalar product space $(\mathcal{V}, -[.,.])$ and recall from Theorem 3.5.8 that
$\ker(T^{-1}-\lambda)^\nu = \ker(T-\frac{1}{\lambda})^\nu$ for all $\lambda \in \mathbb{C}\cup\{\infty\}$ and all $\nu \in \mathbb{N}$. ❑

Using the ideas from the last paragraph of Remark 3.11.2 we obatin the following assertion on the resolvent set of contractive operators.

`contropspec.`

3.11.5 Corollary. *Let $(\mathcal{A},[.,.])$ be a Pontryagin space, and let $T : \mathcal{A}\to\mathcal{A}$ be an everywhere defined, bounded linear and contractive operator. Then we have $\sigma(T)\setminus c\ell(\mathbb{D}) = \sigma_p(T)\setminus c\ell(\mathbb{D})$, and this set contains at most $\mathrm{ind}_-(\mathcal{A},[.,.])$ points.*

Proof. As noted in Remark 3.11.2 $T \supseteq S$ with finite codimension, where S is a contractive relation on a Hilbert space. Hence, $r(S) \supseteq (\mathbb{C}\cup\{\infty\})\setminus c\ell(\mathbb{D})$; see Example 3.5.16.

By Theorem 3.7.3 either $(\mathbb{C}\cup\{\infty\})\setminus c\ell(\mathbb{D}) \subseteq \sigma_p(T)$ or
$((\mathbb{C}\cup\{\infty\})\setminus c\ell(\mathbb{D}))\setminus D \subseteq r(T)$, $D \subseteq \sigma_p(T)$, where D is discrete in $(\mathbb{C}\cup\{\infty\})\setminus c\ell(\mathbb{D})$.

As $\infty \in \rho(T) \subseteq r(T)$ the second assertion is true, and by Corollary 3.5.15 we even have $((\mathbb{C}\cup\{\infty\})\setminus c\ell(\mathbb{D}))\setminus D \subseteq \rho(T)$ with $D \subseteq \sigma_p(T)$. Due to Theorem 3.11.4 and the well-known fact, that eigenspaces of operators are linearly independent, D contains at most $\mathrm{ind}_-(\mathcal{A},[.,.])$ points. ❑

The proof of the following result stems from deliberation in .

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3.11.6 Lemma. *Let $\mathcal{A}$ be a Pontryagin space with* $\operatorname{ind}_- \mathcal{A} > 0$*, and let $T : \mathcal{A} \to \mathcal{A}$ be a bounded linear contraction. Then T has always an eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{D}$ and a corresponding eigenvector $x \neq 0$ such that $[x, x] \leq 0$.*

Proof. Take a fixed fundamental decomposition $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-$ and denote by $(.,.)$ the corresponding Hilbert space scalar product.

For $\delta \in (0, 1)$ define $S_\delta : \mathcal{A} \to \mathcal{A}$ by $S_\delta(x_+ + x_-) := \sqrt{1-\delta}\, x_+ + \sqrt{1+\delta}\, x_-$. Clearly, S_δ is a bi-continuous linear bijection with $\|S_\delta\| \leq 2$. The operator TS_δ satisfies

$$[TS_\delta x, TS_\delta x] \leq [S_\delta x, S_\delta x] = (1-\delta)[x_+, x_+] + (1+\delta)[x_-, x_-] = [x, x] - \delta(x, x)\,.$$

As $|[x, x]| \leq (x, x)$ we have $[TS_\delta x, TS_\delta x] \leq (1-\delta)[x, x]$. Hence, $\frac{1}{\sqrt{1-\delta}} TS_\delta$ is also contractive. By Corollary 3.11.5 we then get

$$\sigma(TS_\delta) \subseteq K_{\sqrt{1-\delta}}(0) \dot\cup M\,,$$

where M is a discrete subset of $\mathbb{C} \setminus K_{\sqrt{1-\delta}}(0)$ consisting only of eigenvalues. If M where empty, then Lemma 3.11.3 would imply the positive definiteness of $[.,.]$, which contradicts our assumption $\operatorname{ind}_- \mathcal{A} > 0$. By Theorem 3.11.4 applied to $\frac{1}{\sqrt{1-\delta}} TS_\delta$ the eigenvectors of TS_δ corresponding to the eigenvalues $\lambda \in M$ are non-positive.

Applying this to a zero sequence $(\delta_n)_{n\in\mathbb{N}}$ from $(0, 1)$ gives rise to eigenvalues λ_n with $|\lambda_n| \in (\sqrt{1-\delta}, r(TS_\delta)] \subseteq (\sqrt{1-\delta}, 2\|T\|]$ and a corresponding eigenvectors x_n with $[x_n, x_n] \leq 0$. We can normalize x_n such that $(x_{n-}, x_{n-}) = 1$. Clearly, then $(x_{n+}, x_{n+}) \leq 1$. In terms of linear relations we have $(x_n; \lambda_n x_n) \in TS_\delta$ or equivalently

$$\Big(\frac{1}{\sqrt{1-\delta_n}} x_{n+} + \frac{1}{\sqrt{1+\delta_n}} x_{n-}; \lambda_n(x_{n+} + x_{n-})\Big) \in T\,. \tag{3.11.1}$$

folgeigvec

Since $\big((x_{n+}; x_{n-}; \lambda_n)\big)_{n\in\mathbb{N}}$ is bounded in the Hilbert space $\mathcal{A}_+ \times \mathcal{A}_- \times \mathbb{C}$ and since closed balls are weakly compact (see) there, it has at least one accumulation point $(x_+; x_-; \lambda)$ wrt. the weak topology. Hence, x_+ is a weak accumulation point of $\{x_{n+}\}$, which satisfies $(x_+, x_+) \leq 1$ by the weak closedness of the unit ball in $\mathcal{A}_+$. By $\dim \mathcal{A}_- \times \mathbb{C} < \infty$ the vector x_- and the scalar λ are norm accumulation points of $\{x_{n-}\}$ and $\{\lambda_n\}$, respectively; see . In particular, $|\lambda| \geq 1$ and $(x_-, x_-) = 1$.

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Since T is closed, and hence, weakly closed (see), (3.11.1) yields $(x; \lambda x) \in T$ with $x = x_+ + x_- \neq 0$ and $[x, x] = (x_+, x_+) - (x_-, x_-) \leq 0$. ❑

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elemcontprop.

3.11.7 Lemma. *Let T be a contractive linear relation between scalar product spaces $(\mathcal{V}, [.,.])$ and $(\mathcal{W}, [.,.])$. If $\mathcal{N}_1$ is a negative linear subspace of* $\operatorname{dom} T$ *of dimension n, if $\{x_1, \dots, x_n\}$ is a basis of $\mathcal{N}_1$ and if $y_1, \dots, y_n \in \mathcal{W}$ are such that $(x_k; y_k) \in T$ for $k = 1, \dots, n$, then $\mathcal{N}_2 :=$* $\operatorname{span}\{y_1, \dots, y_n\}$ *is a negative subspace of dimension n.*

Moreover, the mapping $S : \mathcal{N}_1 \to \mathcal{N}_2$ defined by $S(x_j) = y_j$ for $j = 1, \dots, n$, is a linear bijection whose graph is contained in T.

Proof. For $y \in \mathcal{N}_2 \setminus \{0\}$ we have $y = \sum_{j=1}^n \lambda_j y_j$, where $\lambda_j \neq 0$ for at least one j. Setting $x := \sum_{j=1}^n \lambda_j x_j$ we get $(x; y) = \sum_{j=1}^n \lambda_j (x_j; y_j) \in T$. Hence, $[y, y]_2 \leq [x, x]_1 < 0$. We conclude that $y \neq 0$ and, therefore, $\dim \mathcal{N}_2 = n$. Moreover, $\mathcal{N}_2$ is a negative subspace. The statement about S is obvious by construction. ❑

indungl.

3.11.8 Corollary. *For a contractive linear relation T between scalar product spaces $(\mathcal{V}, [.,.])$ and $(\mathcal{W}, [.,.])$ we have* $\operatorname{ind}_- \operatorname{ran} T \geq \operatorname{ind}_- \operatorname{dom} T$.

conadj.

3.11.9 Theorem. *Let $(\mathcal{P}_1, [.,.])$ and $(\mathcal{P}_2, [.,.])$ be Pontryagin spaces satisfying* $\operatorname{ind}_- \mathcal{P}_1 = \operatorname{ind}_- \mathcal{P}_2$. *If $T : \mathcal{P}_1 \to \mathcal{P}_2$ is a contractive linear operator, i.e.* $[Tx, Tx] \leq [x, x]$ *for all $x \in \mathcal{P}_1$, then the adjoint operator $T^{[*]} : \mathcal{P}_2 \to \mathcal{P}_1$ (see Proposition* 3.8.6*) is a contraction, too.*

Proof. Let $\mathcal{N}_1$ be a maximal negative subspace of $\mathcal{P}_1$. By Lemma 3.11.7 the space $\mathcal{N}_2 := T(\mathcal{N}_1)$ is a negative subspace of $\mathcal{P}_2$ with $\dim \mathcal{N}_1 = \dim \mathcal{N}_2$. By our assumption $\operatorname{ind}_- \mathcal{P}_1 = \operatorname{ind}_- \mathcal{P}_2$ the space $\mathcal{N}_2$ is maximal.

Setting $\mathcal{M}_1 := \mathcal{N}_1^{[\perp]}$ within $\mathcal{P}_1$ and $\mathcal{M}_2 := \mathcal{N}_2^{[\perp]}$ within $\mathcal{P}_2$ we get two Hilbert spaces; see Corollary 2.4.8. Applying Proposition 3.10.2 we see that $R = \tau_{PG}(T)$ is a contractive linear relation between $(\mathcal{M}_1 \times \mathcal{N}_2)$ provided with $\langle (m_1; n_2), (a_1, b_2) \rangle := [m_1, a_1] - [n_2, b_2]$ and $(\mathcal{M}_2 \times \mathcal{N}_1)$ provided with $\langle (m_2; n_1), (a_2, b_1) \rangle = [m_2, a_2] - [n_1, b_1]$.

By (3.10.2) $\operatorname{dom} T = \mathcal{P}_1$ implies $\operatorname{dom} R = (\mathcal{M}_1 \times \mathcal{N}_2)$. Since $-[.,.]$ is positive definite on $\mathcal{N}_1$ and $\mathcal{N}_2$, these two product spaces are Hilbert spaces. As mentioned in Remark 3.11.2 $R : \mathcal{M}_1 \times \mathcal{N}_2 \to \mathcal{M}_2 \times \mathcal{N}_1$ is then an everywhere defined bounded linear operator with $\|R\| \leq 1$.

Its adjoint $R^{[*]}$ is in fact a Hilbert space adjoint. Therefore, $R^{[*]} : \mathcal{M}_2 \times \mathcal{N}_1 \to \mathcal{M}_1 \times \mathcal{N}_2$ is a linear and bounded operator with $\|R^{[*]}\| = \|R\| \leq 1$, i.e. $R^{[*]}$ is a contraction. But by Proposition 3.10.2 $R^{[*]} = \tau_{PG}(T^{[*]})$ is a contraction if and only if $T^{[*]}$ is. ❑

dirichletadj.

3.11.10 Example. Eventuell Beispiel aus rosenblum ravnyak mit verallgem dirichlet spaces. ◊

Oder ins Kapitel ueber RKPS? oder RKPS Kapitel als Kapitel 3.

The following assertion is a generalization of Theorem 2.5.14.

contlinrekl.

3.11.11 Theorem. *Let T be a contractive linear relation between Pontryagin spaces $(\mathcal{P}_1, [.,.])$ and $(\mathcal{P}_2, [.,.])$. If* $\operatorname{ind}_- \operatorname{ran} T = \operatorname{ind}_- \operatorname{dom} T$ *and if* $c\ell(\operatorname{ran}(T))$ *is non-degenerated, then T is (the graph of) a continuous, linear and contractive operator* $T : \operatorname{dom} T \to \mathcal{P}_2$.

The closure of T in $\mathcal{P}_1 \times \mathcal{P}_2$ is a linear, bounded and contractive $\overline{T} : c\ell(T) = c\ell(\operatorname{dom} T) \to \mathcal{P}_2$.

Proof. As we assume that $c\ell(\operatorname{ran}(T))$ is a Pontryagin subspace of $\mathcal{P}_2$, we can view T as a linear relation between $\mathcal{P}_1$ and $c\ell(\operatorname{ran}(T))$. Obviously, the fact that $T : \operatorname{dom} T \to \mathcal{P}_2$ is a continuous operator, and the fact, that $T : \operatorname{dom} T \to c\ell(\operatorname{ran}(T))$ is a continuous operator, are equivalent. Therefore, we can assume that $\mathcal{P}_2 = c\ell(\operatorname{ran}(T))$.

Let $\mathcal{N}_1$ be a maximal negative subspace of $\operatorname{dom} T$. By Lemma 3.11.7 there is a linear bijection S $(\subseteq T)$ from $\mathcal{N}_1$ onto a negative subspace $\mathcal{N}_2$ of $\mathcal{P}_2$. By Proposition 2.1.4

$$\operatorname{ind}_- \mathcal{P}_2 = \operatorname{ind}_- c\ell(\operatorname{ran} T) = \operatorname{ind}_- \operatorname{ran} T = \operatorname{ind}_- \operatorname{dom} T \,.$$

Hence, $\mathcal{N}_2$ is maximal negative definite.

Setting $\mathcal{M}_1 := \mathcal{N}_1^{[\perp]}$ within $\mathcal{P}_1$ and $\mathcal{M}_2 := \mathcal{N}_2^{[\perp]}$ within $\mathcal{P}_2$ due to Lemma 2.4.7 and Proposition 2.4.6 we get two Pontryagin spaces where $\mathcal{M}_2$ is in fact a Hilbert space;

see Corollary 2.4.8. Applying Proposition 3.10.2, $\tau_{PG}(T)$ is a contractive linear relation between $(\mathcal{M}_1 \times \mathcal{N}_2)$ and $(\mathcal{M}_2 \times \mathcal{N}_1)$.

Since the latter space is a Hilbert space and the former is a Pontryagin space, and hence a Gram space, we saw in Remark 3.11.2 that then $R := \tau_{PG}(T) : \operatorname{dom} R \ (\subseteq \mathcal{M}_1 \times \mathcal{N}_2) \ \to \mathcal{M}_2 \times \mathcal{N}_1$ is a bounded operator with norm ≤ 1. According to Lemma 3.10.3 this is the case if and only if T is a bounded linear operator. The final assertion is an immediate consequence of the following Proposition 3.4.3. ❑

contlinreklcor.

3.11.12 Corollary. *Let T be a contractive linear relation between Pontryagin spaces $(\mathcal{P}_1, [.,.])$ and $(\mathcal{P}_2, [.,.])$. If $\operatorname{ind}_-(\mathcal{P}_1, [.,.]) = \operatorname{ind}_-(\mathcal{P}_2, [.,.])$ and $\operatorname{dom} T$ is dense, then the closure of T in $\mathcal{P}_1 \times \mathcal{P}_2$ is a linear, bounded and contractive $c\ell(T) : c\ell(T) = \mathcal{P}_1 \to \mathcal{P}_2$.*

Proof. By Proposition 2.1.4 we have $\operatorname{ind}_- \operatorname{dom} T = \operatorname{ind}_- \mathcal{P}_1 = \operatorname{ind}_- \mathcal{P}_2$, and according to Corollary 3.11.8 $\operatorname{ind}_- \mathcal{P}_2 \geq \operatorname{ind}_- \operatorname{ran} T \geq \operatorname{ind}_- \operatorname{dom} T$. Hence, $\operatorname{ind}_- \operatorname{ran} T = \operatorname{ind}_- \operatorname{dom} T$. Moreover, by (1.6.3) applied to the sum of a $\operatorname{ind}_- \mathcal{P}_2$ dimensional negative definite subspace of $c\ell(\operatorname{ran}(T))$ and of $c\ell(\operatorname{ran}(T))^{[\circ]}$ shows that $c\ell(\operatorname{ran}(T))^{[\circ]} = \{0\}$. Thus, we can apply Theorem 3.11.11. ❑

defpolynom.

3.11.13 Theorem. *Let $T : \mathcal{A} \to \mathcal{A}$ be a bounded contractive operator on the Pontryagin space $(\mathcal{A}, [.,.])$. Then there exists a polynomial $p(z) \in \mathbb{C}[z]$ of degree less or equal to $\operatorname{ind}_-(\mathcal{A}, [.,.])$ such that*

$$[p(T)x, p(T)x] \geq 0, \quad \text{for all} \quad x \in \mathcal{A} \, .$$

Proof. We prove the assertion by induction on $\operatorname{ind}_-(\mathcal{A}, [.,.])$. For $\operatorname{ind}_-(\mathcal{A}, [.,.]) = 0$, clearly, $p(z) = 1$ satisfies $[p(T)x, p(T)x] \geq 0$.

Assume the assertion is true for any Pontryagin space with index of negativity less or equal to $\kappa \in \mathbb{N} \cup \{0\}$, and suppose $ind_-(\mathcal{A}, [.,.]) = \kappa + 1$. From Lemma 3.11.6 and Theorem 3.11.9 we know that the contractive $T^{[*]}$ has an eigenvalue $\bar{\lambda} \in \mathbb{C} \setminus \mathbb{D}$ and a corresponding eigenvector $x \neq 0$ with $[x, x] \leq 0$. Hence, $\operatorname{ind}_- x^{[\perp]} = \kappa$; see Lemma 2.4.7 and Proposition 1.6.7 in case that $[x, x] < 0$ and Theorem 2.4.10 and Proposition 1.6.7 in case that $[x, x] = 0$. For $y \in \mathcal{A}$ we then have

Oder sollen wir bei den Pontryaginraeumen ein resultat dazugeben, wo der negative index eines orthogonalen komplements explizit ausgerechnet wird?

$$[(T - \lambda)y, x] = [y, (T^{[*]} - \bar{\lambda})x] = 0 \, .$$

We derive $(T - \lambda)(\mathcal{A}) \subseteq x^{[\perp]}$. Hence, $\langle x, y \rangle := [(T - \lambda)x, (T - \lambda)y]$ defines a scalar product on $\mathcal{A}$ with $\operatorname{ind}_-(\mathcal{A}, \langle .,. \rangle) \leq \kappa$. Since

$$\langle Tx, Ty \rangle = [(T-\lambda)Tx, (T-\lambda)Tx] = [T(T-\lambda)x, T(T-\lambda)x] \leq [(T-\lambda)x, (T-\lambda)x] = \langle x, y \rangle \, ,$$

T is contractive wrt. $\langle .,. \rangle$. By Proposition 2.6.8 $(\mathcal{A}, \langle .,. \rangle)$ has a Pontryagin space completion $(\mathcal{B}, \langle .,. \rangle)$, i.e. there is an $\langle .,. \rangle$-isometric mapping $\iota : \mathcal{A} \to \mathcal{B}$ with dense range. Clearly, $(\iota \times \iota)(T)$ is a contractive linear relation with dense domain $\iota(\mathcal{A})$ in $\mathcal{B}$. By Corollary 3.11.12 the closure R of $(\iota \times \iota)(T)$ is a continuous contraction defined on $\mathcal{B}$. Moreover, $R \circ \iota = \iota \circ T$ (see Remark 3.2.7), and hence $R^k \circ \iota = \iota \circ T^k, \ k \in \mathbb{N} \cup \{0\}$.

By induction hypothesis there exists a polynomial $q(z)$ of degree less or equal to κ such that $\langle q(R)z, q(R)z \rangle \geq 0, \ z \in \mathcal{B}$. For $p(z) := (z - \lambda)q(z)$ and $x \in \mathcal{A}$ we then have

$$[p(T)x, p(T)x] = \langle q(T)x, q(T)x \rangle = \langle \iota q(T)x, \iota q(T)x \rangle = \langle q(R)\iota x, q(R)\iota x \rangle \geq 0 \, .$$

❑

3.12 Moving linear relations

In this section we consider two Krein spaces $(\mathcal{A}, [.,.])$ and $(\mathcal{B}, [.,.])$ which are linked by a bounded linear mapping $A : \mathcal{A} \to \mathcal{B}$. We shall study linear relations of the form $(A \times A)^{-1}(S)$ on $\mathcal{A}$, where S is a linear relation on $\mathcal{B}$; see Lemma 3.2.3. Mostly, $(\mathcal{A}, [.,.])$ is indeed a Hilbert space so that we can study $(A \times A)^{-1}(S)$ instead of S in the more familiar Hilbert space setting.

kommmitransf.

3.12.1 Proposition. *Let $A : \mathcal{A} \to \mathcal{B}$ be a bounded and linear mapping between the Krein spaces $\mathcal{A}$ and $\mathcal{B}$. If S is a closed linear relation on $\mathcal{B}$, which satisfies*

$$(AA^{[*]} \times AA^{[*]})(S^{[*]}) \subseteq S\,,$$

then $(A \times A)^{-1}(S)^{[*]}$ is the closure of $(A^{[*]} \times A^{[*]})(S^{[*]})$, and it is a symmetric linear relation on $\mathcal{A}$.

In the special case that A is injective, that $\mathcal{A}$ is a Hilbert space and that $\mathbb{C} \setminus \sigma_p(S)$ contains points from $\mathbb{C}^+$ and from $\mathbb{C}^-$, the relation $(A \times A)^{-1}(S)$ is selfadjoint.

Proof. The assumption $(AA^{[*]} \times AA^{[*]})(S^{[*]}) = (A \times A)\,(A^{[*]} \times A^{[*]})(S^{[*]}) \subseteq S$ implies $(A^{[*]} \times A^{[*]})(S^{[*]}) \subseteq (A \times A)^{-1}(S)$. Thus, also the closure $(A \times A)^{-1}(S)^{[*]}$ of $(A^{[*]} \times A^{[*]})(S^{[*]})$ (see Lemma 3.8.7) is contained in the closed $(A \times A)^{-1}(S)$; see Remark 3.4.4. Hence, $(A \times A)^{-1}(S)^{[*]}$ is symmetric.

If $\mathcal{A}$ is a Hilbert space, then $(A \times A)^{-1}(S)^{[*]}$ not being a selfadjoint relation on $\mathcal{A}$ by what was said at the end of Example 3.9.7 implies that the point spectrum of its adjoint $(A \times A)^{-1}(S)$ contains all points from the upper halfplane or all points from the lower halfplane. But due to Lemma 3.5.9 we have $\sigma_p((A \times A)^{-1}(S)) \subseteq \sigma_p(S)$. Hence, $(A \times A)^{-1}(S)^{[*]}$ must be selfadjoint if $\mathbb{C} \setminus \sigma_p(S)$ contains points from $\mathbb{C}^+$ and from $\mathbb{C}^-$. ❑

kommmitransfrem.

3.12.2 Remark. We can extract a little more information from the previous proof in the case of $\mathcal{A}$ being a Hilbert space and A being injective. For the adjoint of the symmetric relation $(A \times A)^{-1}(S)^{[*]}$ by Lemma 3.5.9 we have $\ker((A \times A)^{-1}(S) - \lambda) = A^{-1} \ker(S - \lambda)$, and hence, $\dim\ker((A \times A)^{-1}(S) - \lambda) \leq \dim\ker(S - \lambda)$ for any $\lambda \in \mathbb{C} \cup \{\infty\}$. ◊

Vll Detailierter??? eventuell bem ueber def indeizes ?? Vll Bsp. von diff operator?

wuzuab.

3.12.3 Lemma. *Let $(\mathcal{H}, (.,.))$ be a Hilbert space and let $A, C \in B(\mathcal{H})$ such that C and AC are selfadjoint and such that $C \geq 0$. Then we have $|(ACx, x)| \leq \|A\|\,(Cx, x),\ x \in \mathcal{H}$.*

Proof. Using the functional calculus (see) for the selfadjoint operator C we see that $C + \epsilon$ is boundedly invertible for any $\epsilon > 0$, and $C(C + \epsilon)^{-1}$ has norm $\sup_{t\in\sigma(C)} \frac{t}{t+\epsilon} = \frac{\|C\|}{\|C\|+\epsilon}$.

Folgendes Lemma vll in den Appendix!

Missing Referenz.....oder in den Anhang

Since for the spectral radius we have $\mathrm{spr}(FG) = \mathrm{spr}(GF)$ for all bounded operators F, G,

$$\mathrm{spr}((C + \epsilon)^{-\frac{1}{2}} AC(C + \epsilon)^{-\frac{1}{2}}) = \mathrm{spr}(AC(C + \epsilon)^{-1}) \leq \|A\|\,\frac{\|C\|}{\|C\| + \epsilon}\,.$$

For selfadjoint operators spectral radius and norm coincide. Hence, due to the Cauchy-Schwarz inequality

$$|(ACx,x)| = |((C+\epsilon)^{-\frac{1}{2}}AC(C+\epsilon)^{-\frac{1}{2}}\,(C+\epsilon)^{\frac{1}{2}}x,(C+\epsilon)^{\frac{1}{2}}x)| \leq$$
$$\|(C+\epsilon)^{-\frac{1}{2}}AC(C+\epsilon)^{-\frac{1}{2}}\|\,\|(C+\epsilon)^{\frac{1}{2}}x\|^2 \leq \|A\|\,\frac{\|C\|}{\|C\|+\epsilon}((C+\epsilon)x,x)\,.$$

The desired inequality follows for $\epsilon \searrow 0$. ❑

Folgendes Lemma vll in den Appendix! Welcher Beweis?

selbadjbeschr.

3.12.4 Lemma. *Let $(\mathcal{H},(.,.))$ be a Hilbert space, $c \in [0,+\infty)$ and let B be a selfadjoint operator. If $|(Bx,x)| \leq c(x,x)$ for $x \in \operatorname{dom} B$, then B is bounded with $\|B\| \leq c$.*

Missing Reference; Dritschel u. Rovnyak

The ideas in the subsequent lemma are take from .

contranspo.

3.12.5 Lemma. *With the notation and assumptions from Proposition 3.12.1 additionally suppose that A is injective, that $\mathcal{A}$ is a Hilbert space and that $S:\mathcal{B}\to\mathcal{B}$ is bounded. Then $(A\times A)^{-1}(S)$ is a bounded linear and selfadjoint operator on $\mathcal{A}$ with*

$$\|\,(A\times A)^{-1}(S)\,\| \leq \|S\|\,. \tag{3.12.1}$$

contranspoeq

Here $\|.\|$ on the right is the operator norm with respect to any Hilbert space scalar product $(.,.)$ on $\mathcal{B}$ compatible with $[.,.]$.

Proof. Lemma 3.5.12 implies $\mathbb{C}\setminus K_{\|S\|}(0) \subseteq r(S) \subseteq (\mathbb{C}\cup\{\infty\})\setminus\sigma_p(S)$. In particular, $\mathbb{C}\setminus\sigma_p(S)$ contains points from $\mathbb{C}^+$ and from $\mathbb{C}^-$.

By Lemma 3.5.12 the relation $(A\times A)^{-1}(S)$ is selfadjoint and coincides with the closure of $(A^{[*]}\times A^{[*]})(S^{[*]})$. According to Lemma 3.5.9 applied with $\lambda=\infty$, $(A\times A)^{-1}(S)$ is indeed an operator.

By Corollary 3.2.6 the assumption $(AA^{[*]}\times AA^{[*]})(S^{[*]}) \subseteq S$ is equivalent to $AA^{[*]}S^{[*]} \subseteq SAA^{[*]}$. Since the expressions on the left and on the right of this inclusions are everywhere defined linear operators, in fact equality prevails. Denoting by G the Gram operator of $[.,.]$ with respect to $(.,.)$ we get $AA^{(*)}S^{(*)}G = AA^{[*]}S^{[*]} = SAA^{[*]} = SAA^{(*)}G$, where $A^{(*)}$ is the Hilbert space adjoint of $A:(\mathcal{A},[.,.])\to(\mathcal{B},(.,.))$. Consequently, $(SAA^{(*)})^{(*)} = AA^{(*)}S^{(*)} = SAA^{(*)}$ is selfadjoint.

For $(x;y)\in(A^{[*]}\times A^{[*]})(S^{[*]}) \subseteq (A\times A)^{-1}(S)$ we have $x = A^{[*]}u$ for some $u\in\mathcal{B}$. We conclude that $(AA^{[*]}u;Ay)\in S$ or $S(AA^{[*]}u) = Ay$, and hence

$$|[y,x]| = |[y,A^{[*]}u]| = |[Ay,u]| = |[SAA^{[*]}u,u]| = |(SAA^{(*)}Gu,Gu)|\,.$$

By Lemma 3.12.3 this expressions is less or equal to

$$\|S\|\,(AA^{(*)}Gu,Gu) = \|S\|\,[AA^{[*]}u,u] = \|S\|\,[x,x]\,.$$

$(A^{[*]}\times A^{[*]})(S^{[*]})$ being dense in $(A\times A)^{-1}(S)$ implies $|[y,x]| \leq \|S\|\,[x,x]$ for all $(x;y)\in(A\times A)^{-1}(S)$. Therefore, according to Lemma 3.12.4, $(A\times A)^{-1}(S)$ is a bounded and selfadjoint operator with norm less or equal to $\|S\|$. ❑

thetadefeig.

3.12.6 Theorem. *Let $A : \mathcal{A} \to \mathcal{B}$ be a bounded and injective linear mapping from the Hilbert space $(\mathcal{A}, [.,.])$ into the Krein space $(\mathcal{B}, [.,.])$. Then*

$$\Theta : C \mapsto (A \times A)^{-1}(C)$$

constitutes a $*$-algebra homomorphisms from $(AA^{[*]})' \subseteq B(\mathcal{B})$ of all bounded linear operators on $\mathcal{B}$, that commute with $AA^{[*]}$, into $(A^{[*]}A)' \subseteq B(\mathcal{A})$ with $\|\Theta\| \leq 1$. Hereby, $\Theta(I) = I$ and $\Theta(AA^{[*]}) = A^{[*]}A$, and

$$\ker \Theta = \{C \in (AA^{[*]})' : \operatorname{ran} C \subseteq \ker A^{[*]}\}.$$

Moreover, $(A^{[*]} \times A^{[*]})(C)$ is densely contained in $\Theta(C)$ for all $C \in (AA^{[*]})'$, and we have $A^{[*]}C = \Theta(C)A^{[*]}$. Finally,

$$\Xi : D \mapsto ADA^{[*]}$$

constitutes an injectiv, bounded linear mapping from $(A^{[*]}A)'$ ($\subseteq B(\mathcal{A})$) into $(AA^{[*]})'$ ($\subseteq B(\mathcal{B})$), which satisfies ($C, \in (AA^{[*]})',\ D, D_1, D_2 \in (A^{[*]}A)'$)

$$\Xi(D^{[*]}) = \Xi(D)^{[*]},\ \Xi(D\,\Theta(C)) = \Xi(D)C,$$
$$\Xi(D_1 D_2\, A^{[*]}A) = \Xi(D_1)\,\Xi(D_2),\ \Xi \circ \Theta(C) = AA^{[*]}\, C = C\, AA^{[*]},$$

and $\Xi(D)$ commutes with all operators from $(AA^{[*]})'$ if D commutes with all operators from $(A^{[*]}A)'$.

Proof. First of all it is easy to check that $(AA^{[*]})' \subseteq B(\mathcal{B})$ and $(A^{[*]}A)' \subseteq B(\mathcal{A})$ are closed Banach-$*$ algebras when provided with ${}^{[*]}$. For any seladjoint $C \in (AA^{[*]})'$ we have $(AA^{[*]} \times AA^{[*]})(C^{[*]}) = (AA^{[*]} \times AA^{[*]})(C) \subseteq C$ due to Remark 3.2.7. Thus, we can apply Lemma 3.12.5 and see that $(A \times A)^{-1}(C)$ is a bounded seladjoint linear mapping on $\mathcal{A}$ containing $(A^{[*]} \times A^{[*]})(C)$ densely. Hence,

$$(A^{[*]}A \times A^{[*]}A)\,(A \times A)^{-1}(C) \subseteq (A^{[*]} \times A^{[*]})(C) \subseteq (A \times A)^{-1}(C).$$

Again by Remark 3.2.7 this means $(A \times A)^{-1}(C) \in (A^{[*]}A)'$. Clearly, $(A \times A)^{-1}(I) = A^{-1}A = I$ and $(A \times A)^{-1}(AA^{[*]}) = A^{-1}AA^{[*]}A = A^{[*]}A$; see Lemma 3.2.3..

For a not necessarily selfadjoint $C \in (AA^{[*]})'$ we also have $C^{[*]} \in (AA^{[*]})'$, and in turn

$$\operatorname{Re} C = \frac{C + C^{[*]}}{2},\ \ \operatorname{Im} C = \frac{C - C^{[*]}}{2i} \in (AA^{[*]})'.$$

Consequently, $(A \times A)^{-1}(\operatorname{Re} C)$ and $(A \times A)^{-1}(\operatorname{Im} C)$ are selfadjoint elements from $(A^{[*]}A)'$. Moreover, by Corollary 3.2.5

$$(A \times A)^{-1}(C) = (A \times A)^{-1}(\operatorname{Re} C + i \operatorname{Im} C) \supseteq (A \times A)^{-1}(\operatorname{Re} C) + i(A \times A)^{-1}(\operatorname{Im} C),$$

$$(A \times A)^{-1}(C^{[*]}) = (A \times A)^{-1}(\operatorname{Re} C - i \operatorname{Im} C) \supseteq (A \times A)^{-1}(\operatorname{Re} C) - i(A \times A)^{-1}(\operatorname{Im} C),$$

where the right hand sides have domain $\mathcal{A}$ and the left hand sides are operators; see Lemma 3.12.5. Consequently, equalties prevail, and we obtain $(A \times A)^{-1}(C) \in (A^{[*]}A)'$ and $(A \times A)^{-1}(C^{[*]}) = (A \times A)^{-1}(C)^{[*]}$. Therefore, $\Theta : (AA^{[*]})' \to (A^{[*]}A)'$ is well-defined and respects $^{[*]}$. Using Corollary 3.2.5 two more times shows that Θ is in fact linear and multiplicative. Using (3.12.1) we get for all $x \in \mathcal{A}$, $[x, x] = 1$,

$$[\Theta(C)x, \Theta(C)x] = [\Theta(C^{[*]}C)x, x] \leq \|\Theta(C^{[*]}C)\| \leq \|C^*C\| \leq \|C\|^2 ,$$

and conclude that Θ is a contraction. From Lemma 3.8.7 we infer

$$\left((A^{[*]} \times A^{[*]})(C)\right)^{[*]} = (A \times A)^{-1}(C^{[*]}) = (A \times A)^{-1}(C)^{[*]}$$

showing that $(A^{[*]} \times A^{[*]})(C)$ is densely contained in $(A \times A)^{-1}(C)$. In particular, $(A \times A)^{-1}(C) = \Theta(C) = 0$ is equivalent to the fact that $(a; b) \in (A^{[*]} \times A^{[*]})$ always implies $b = 0$, i.e., $A^{[*]}y = 0$ for all $(x; y) \in C$. This just means that $\operatorname{ran} C$ is contained in $\ker A^{[*]}$.

We have $[AA^{[*]}Cu, v] = [A^{[*]}Cu, A^{[*]}v] = [\Theta(C)A^{[*]}u, A^{[*]}v]$ for any $u, v \in \mathcal{B}$ because of $(A^{[*]}u; A^{[*]}Cu) \in \Theta(C)$. From this equality we obtain $AA^{[*]}C = A\Theta(C)A^{[*]}$ which, in turn, implies $A^{[*]}C = \Theta(C)A^{[*]}$ by A's injectivity.

$\Xi : D \mapsto ADA^{[*]}$ is clearly linear, bounded by $\|A\|^2$ and satisfies $\Xi(D)^{[*]} = \Xi(D^{[*]})$. Its injectivity follows from A's injectivity and from $c\ell(\operatorname{ran} A^{[*]}) = \ker A^{[\perp]} = \mathcal{A}$.

For $D \in (A^{[*]}A)'$ we have $\Xi(D)\, AA^{[*]} = ADA^{[*]}\, AA^{[*]} = A\, A^{[*]}ADA^{[*]} = AA^{[*]}\, \Xi(D)$, i.e. $\Xi(D) \in (AA^{[*]})'$. For $C \in (AA^{[*]})', D \in (A^{[*]}A)'$ we have $\Xi(D\Theta(C)) = AD\Theta(C)A^{[*]} = ADA^{[*]}C = \Xi(D)C$, and $D_1, D_2 \in (A^{[*]}A)'$ yields

$$\Xi(D_1D_2\, A^{[*]}A) = AD_1D_2A^{[*]}AA^{[*]} = AD_1\, A^{[*]}A\, D_2A^{[*]} = \Xi(D_1)\, \Xi(D_2) ,$$

and $A^{[*]}C = \Theta(C)A^{[*]}$ implies $\Xi \circ \Theta(C) = A\Theta(C)A^{[*]} = AA^{[*]}\, C = C\, AA^{[*]}$.

Finally, assume that D commutes with all operators from $(A^{[*]}A)'$, and let $C \in (AA^{[*]})'$. Then $A^{-1}CA = \Theta(C) \in (A^{[*]}A)'$. Hence,

$$\Xi(D)C = \Xi(D\, \Theta(C)) = \Xi(\Theta(C)D) = AA^{-1}CADA^{[*]} \subseteq CADA^{[*]} = C\Xi(D) .$$

The fact, that here only everywhere defined operators are involved, gives $\Xi(D)C = C\Xi(D)$. ❑

thetarem. *3.12.7 Remark.* The fact that Θ from Theorem 3.12.6 is an algebra homomorphism yields $\rho(\Theta(C)) \supseteq \rho(C)$. ◊

thetaremretour. *3.12.8 Remark.* For $C \in (AA^{[*]})'$ we can apply Lemma 3.2.8, and obtain

$$(A^{[*]} \times A^{[*]})^{-1}\Theta(C) = (AA^{[*]} \times AA^{[*]})^{-1}(C) = C \boxplus (\ker AA^{[*]} \times \ker AA^{[*]}) ,$$

where $\ker AA^{[*]} = \ker A^{[*]}$ by A's injectivity. ◊

thetcor. **3.12.9 Corollary.** *Let $A : \mathcal{A} \to \mathcal{B}$ be a bounded and injective linear mapping from the Hilbert space $(\mathcal{A}, [.,.])$ into the Krein space $(\mathcal{B}, [.,.])$. If S is a linear relation on $\mathcal{B}$ satisfying $\rho(S) \neq \emptyset$ and $(AA^{[*]} \times AA^{[*]})(S) \subseteq S$ (or equivalently $AA^{[*]}\, S \subseteq S\, AA^{[*]}$; see Remark* 3.2.7*), then $\Theta(S) := (A \times A)^{-1}(S)$ satisfies $\rho(\Theta(S)) \supseteq \rho(S)\ (\neq \emptyset)$ and $(A^{[*]}A \times A^{[*]}A)(\Theta(S)) \subseteq \Theta(S)$.*

Moreover, we have $s(S) \in (AA^{[*]})'$ for any rational $s \in C_{\rho(S)}(z)$ and $s(\Theta(S)) = \Theta(s(S))\ \in (A^{[*]A})'$.

Finally, $\rho(S^{[*]}) \neq \emptyset$, $(AA^{[*]} \times AA^{[*]})(S^{[*]}) \subseteq S^{[*]}$ and $\Theta(S^{[*]}) = \Theta(S)^{[*]}$.

Proof. Take a regular $M \in \mathbb{C}^{2\times 2}$ such that $\tau_M(S) \in B(\mathcal{B})$. Since τ_M commutes with $AA^{[*]} \times AA^{[*]}$ (see Remark 3.3.9), we have $(AA^{[*]} \times AA^{[*]})(\tau_M(S)) \subseteq \tau_M(S)$ which by Remark 3.2.7 means that $\tau_M(S) \in (AA^{[*]})'$. According to Theorem 3.12.6 we then have $\Theta(\tau_M(S)) \in (A^{[*]}A)'$, or equivalently

$$(A^{[*]}A \times A^{[*]}A)(\Theta(\tau_M(S))) \subseteq \Theta(\tau_M(S)) . \tag{3.12.2}$$ tzhw5

By (3.3.10) we have $\Theta(\tau_M(S))) = (A \times A)^{-1}(\tau_M(S)) = \tau_M((A \times A)^{-1}(S))$. Therefore, applying τ_M to (3.12.2) gives $(A^{[*]}A \times A^{[*]}A)((A \times A)^{-1}(S)) \subseteq (A \times A)^{-1}(S)$. $\rho(\Theta(S)) \supseteq \rho(S)$ follows from

$$\phi_M(\rho((A \times A)^{-1}(S))) = \rho(\tau_M((A \times A)^{-1}(S))) = \rho(\Theta(\tau_M(S))) \supseteq \rho(\tau_M(S)) = \phi_M(\rho(S))$$

By Lemma 3.6.4 for any rational $s \in C_{\rho(S)}(z) \subseteq C_{\rho(\Theta(S))}(z)$ we have $s \circ \phi_{M^{-1}} \in C_{\rho(\tau_M(S))}(z)$ and $(s \circ \phi_{M^{-1}})(\tau_M(S)) = s(S)$. Since $\tau_M(S) \in (AA^{[*]})'$ and since Θ in Theorem 3.12.6 is a homomorphism, we get

$$\Theta(s(S))\Theta((s \circ \phi_{M^{-1}})(\tau_M(S))) = (s \circ \phi_{M^{-1}})(\Theta(\tau_M(S))) = \\ (s \circ \phi_{M^{-1}})(\tau_M((A \times A)^{-1}(S))) = s((A \times A)^{-1}(S)) .$$

The assertion about $S^{[*]}$ follows from $\tau_{\bar{M}}(S^{[*]}) = \tau_M(S)^{[*]} \in (AA^{[*]})'$ (see Lemma 3.8.8) and the fact that $\Theta(\tau_M(S)^{[*]}) = \Theta(\tau_M(S))^{[*]}$; see Theorem 3.12.6. ❑

3.13 Definitizable linear relations

definitzdef. **3.13.1 Definition.** Let $(\mathcal{B}, [.,.])$ be a Krein space. We call a linear relation C on $\mathcal{B}$ satisfying $\rho(C) \neq \emptyset$, *definitizable* , if $[q(C)x, x] \geq 0,\ x \in \mathcal{B}$, for some rational $q \in C_{\rho(C)}(z)$. Any rational $q \in C_{\rho(C)}(z)$ satisfying this condition is called *definitizing rational function* for C. ◊

zdtzh. *3.13.2 Remark.* With the notation from Definition 3.13.1 $\langle .,. \rangle := [q(C).,.]$ defines a positive semidefinite hermitian sesquilinear from on $\mathcal{B}$. Let $(\mathcal{A}, \langle .,. \rangle)$ be the Hilbert space completion of $(\mathcal{B}/\mathcal{B}^{\langle\circ\rangle})$, and denote by $\iota : \mathcal{B} \to \mathcal{A}$ the canonical embedding. Because of $\langle \iota x, \iota x \rangle = [q(C)x, x] \leq \|q(C)\|\,(x, x)$, ι is bounded. Here $(.,.)$ is a compatible Hilbert space scalar product on $\mathcal{B}$ as in Lemma 2.3.6. Hereby, clearly $\mathcal{A} = \{0\}$ if and only if $q(C) = 0$.

For $q(C) \neq 0$ let $A : \mathcal{A} \to \mathcal{B}$ be the adjoint of $\iota : (\mathcal{B}, [.,.]) \to (\mathcal{A}, \langle .,. \rangle)$, i.e. $A = \iota^{[*]}$. By definition the range of $\iota = A^{[*]}$ is dense, and therefore, A is injective. Moreover, due to

$$[AA^{[*]}x, y] = \langle A^{[*]}x, A^{[*]}y \rangle = \langle x, y \rangle = [q(C)x, y],\ x, y \in \mathcal{B} ,$$

we have $AA^{[*]} = q(C)$. For a regular $M \in \mathbb{C}^{2\times 2}$ with $\tau_M(C) \in B(\mathcal{B})$ by Remark 3.2.7 $\tau_M(C)q(C) = q(C)\tau_M(C)$ just means $(AA^{[*]} \times AA^{[*]})\tau_M(C) \subseteq \tau_M(C)$. Applying $\tau_{M^{-1}}$ gives $(AA^{[*]} \times AA^{[*]})C \subseteq C$; cf. Remark 3.3.9.

Thus, we can apply Corollary 3.12.9 and Theorem 3.12.6 to $S := C$. In particular, $\Theta(q(C)) = \Theta(AA^{[*]}) = A^{[*]}A$. Moreover, if C is selfadjoint, then the same is true for $\Theta(C)$. ◊

positopspecpost.

3.13.3 Theorem. *Let* $(\mathcal{B}, [.,.])$ *be a Krein space and let* C *be a definitizable linear relation on* $\mathcal{B}$ *with* $\rho(C) \neq \emptyset$. *If* q *is are definitizing rational function for* C *and* $p := sq$ $(\in C_{\rho(C)}(z))$ *for some* $s \in C_{\rho(C)}(z)$, *then*

$$\sigma(\Theta(C)) \subseteq \sigma(C) \subseteq p^{-1}\Big(p(\sigma(\Theta(C))) \cup \{0\}\Big),$$

where $\sigma(\Theta(C))$ *has to be interpreted as* $\emptyset$ *for* $q(C) = 0$. *Here* Θ *is the mapping as in Corollary* 3.12.9 *applied in the setting of Remark* 3.13.2 *on base of the definitizing rational function* q.

Proof. The left inclusion immediately follows from Corollary 3.12.9. For $p(C) = 0$, which is in particular the case if $q(C) = 0$, the assertion immediately follows from the spectral mapping theorem, Proposition 3.6.5. Hence, we can assume that $s(C)q(C) = p(C) \neq 0$ and hence $q(C) \neq 0$.

Assume that $\lambda \notin p^{-1}(p(\sigma(\Theta(C))) \cup \{0\})$. Then $p(\lambda) \neq 0$ and $p(\lambda) \notin p(\sigma(\Theta(C))) = \sigma(\Theta(p(C)))$; see Proposition 3.6.5 and Corollary 3.12.9.

In particular, for $M = \left(\begin{smallmatrix} 1 & 0 \\ 1 & -p(\lambda) \end{smallmatrix}\right)$

$$\Theta(\tau_M(p(C))) = \tau_M(\Theta(p(C))) = \\ I + \lambda(\Theta(p(C)) - p(\lambda))^{-1} = \Theta(p(C))(\Theta(p(C)) - p(\lambda))^{-1} \quad (3.13.1)$$ hwr567

is an everywhere defined and bounded linear operator on $\mathcal{A}$ whose range is contained in $\operatorname{ran}\Theta(p(C))$. According to Remark 3.12.8 we then have

$$(A^{[*]} \times A^{[*]})^{-1}\Theta(\tau_M(p(C))) = \tau_M(p(C)) \boxplus (\ker AA^{[*]} \times \ker AA^{[*]}).$$

From (3.13.1) and $p(C) = q(C)s(C) = A^{[*]}A\ s(C)$ we derive $\operatorname{ran}\tau_M(\Theta(p(C))) \subseteq \operatorname{ran}\Theta(p(C)) \subseteq \operatorname{ran}A^{[*]}A \subseteq \operatorname{ran}A^{[*]}$. Therefore, by Lemma 3.2.4

$$\operatorname{dom}(A^{[*]} \times A^{[*]})^{-1}\Theta(\tau_M(p(C))) = (A^{[*]})^{-1}\operatorname{dom}\tau_M(\Theta(p(C))) = \mathcal{B},$$

and in turn we get
$\mathcal{B} = \operatorname{dom}\tau_M(p(C)) \boxplus (\ker AA^{[*]} \times \ker AA^{[*]}) = \operatorname{ran}(p(C) - p(\lambda)) + \ker AA^{[*]}$.
$\ker AA^{[*]} = \ker p(C) \subseteq \operatorname{ran}(p(C) - p(\lambda))$ for $p(\lambda) \neq 0$ yields $\operatorname{ran}(p(C) - p(\lambda)) = \mathcal{B}$. By Proposition 3.6.6 we derive $\operatorname{ran}(C - \lambda) = \mathcal{B}$.

Finally, $0 \neq x \in \ker(C - \lambda) \subseteq \ker(p(C) - p(\lambda))$ (see Proposition 3.6.6) yields $(A^{[*]}x; A^{[*]}p(\lambda)x) \in (A^{[*]} \times A^{[*]})p(C) \subseteq \Theta(p(C))$; see Theorem 3.12.6. Since the intersection of $\ker A^{[*]} = \ker AA^{[*]} = \ker q(C)$ with $\ker(p(C) - p(\lambda)) = \ker(s(C)q(C) - p(\lambda))$ is just the zero vector, we get the contradiction $0 \neq A^{[*]}x \in \ker(\Theta(p(C)) - p(\lambda)) = \{0\}$. Thus we showed $\lambda \in \rho(C)$. ❑

If for bounded C the assumptions from the previous theorem are satisfied for $p(z) = q(z) = z$, then $\Theta(C) = \Theta(s(C)) = \Theta(AA^{[*]}) = A^{[*]}A$ (see Theorem 3.12.6) is selfadjoint in a Hilbert space. Hence, $\sigma(\Theta(C)) \subseteq \mathbb{R}$, and we obtain the following

positopspec.

3.13.4 Corollary. *Let* $(\mathcal{B}, [.,.])$ *be a Krein space and assume that* $C : \mathcal{B} \to \mathcal{B}$ *is a bounded, linear operator, such that* $[Cx, x] \geq 0$ *for all* $x \in \mathcal{B}$, *i.e.,* C *is* positive. *Then* $\sigma(C) \subseteq \mathbb{R}$.

defspecpro.

3.13.5 Corollary. *With the assumptions and notation from Theorem* 3.13.3 *assume in addition that C is selfadjoint. Then $\sigma(C) \subseteq \mathbb{R} \cup \{\infty\} \cup \{z \in \mathbb{C} : q(z) = 0\}$, where $\sigma(C) \cap \{z \in \mathbb{C} : q(z) = 0\}$ is symmetric with respect to $\mathbb{R}$. Moreover,*

$$\sigma(\Theta(C)) \subseteq \sigma(C) \subseteq \sigma(\Theta(C)) \cup \{z \in \mathbb{C} : q(z) = 0\}.$$

Here Θ is the mapping as in Corollary 3.12.9 *applied in the setting of Remark* 3.13.2 *on base of the definitizing rational function q.*

Proof. By assumption $q(C)$ is a positive and bounded operator. Hence, by Corollary 3.13.4 $q(\sigma(C)) = \sigma(q(C)) \subseteq \mathbb{R}$.

For any $\infty \neq \mu \in \rho(C) = \overline{\rho(C)}$ (see Corollary 3.8.10) consider $s_\mu(z) := \frac{1}{z-\mu}$, $s_\mu^\#(z) = \frac{1}{z-\bar{\mu}} \in C_{\rho(C)}(z)$. As $C = C^{[*]}$ we get from Theorem 3.6.2 and Corollary 3.8.11

$$[(qs_\mu s_\mu^\#)(C)x, x] = [q(C)\, s_\mu(C)x, s_\mu(C)x] \geq 0, \;\; x \in \mathcal{B}.$$

Hence, we also have $(qs_\mu s_\mu^\#)(\sigma(C)) = \sigma((qs_\mu s_\mu^\#)(C)) \subseteq \mathbb{R}$.

Assume that $z \in \sigma(C) \setminus \mathbb{R}$ and $q(z) \neq 0$. From $q(z) \in \mathbb{R}$ we conclude

$$\frac{1}{(z-\mu)(z-\bar{\mu})} q(z) - \overline{\frac{1}{(z-\mu)(z-\bar{\mu})} q(z)} = \\ \frac{q(z)}{|z-\mu|^2|z-\bar{\mu}|^2}\left(\bar{z}^2 - z^2 - 2\operatorname{Re}\mu(\bar{z}-z)\right) = \frac{2\,q(z)}{|z-\mu|^2|z-\bar{\mu}|^2}(\bar{z}-z)(\operatorname{Re} z - \operatorname{Re}\mu).$$

For $\operatorname{Re}\mu \neq \operatorname{Re} z$ this term does not vanish, i.e., $(qs_\mu s_\mu^\#)(z) \notin \mathbb{R}$. Since $\rho(C)$ is open, this can always be achieved by perturbing $\operatorname{Re}\mu$ a litte, and we obtain the contradiction to $(qs_\mu s_\mu^\#)(\sigma(C)) \subseteq \mathbb{R}$.

For $w \in \sigma(C) \cap \{z \in \mathbb{C} : q(z) = 0\}$ from $C = C^{[*]}$ we infer $\bar{w} \in \sigma(C) \subseteq \mathbb{R} \cup \{\infty\} \cup \{z \in \mathbb{C} : q(z) = 0\}$; see Corollary 3.8.10. Hence, $\bar{w} \in \sigma(C) \cap \{z \in \mathbb{C} : q(z) = 0\}$.

Finally, assume that $\lambda \notin \sigma(\Theta(C)) \cup \{z \in \mathbb{C} : q(z) = 0\}$ but $\lambda \in \sigma(C)$. Hence, $\lambda \in \mathbb{R} \cup \{\infty\}$. Let $U(\lambda)$ be a compact neighbourhood of λ with $U(\lambda) = \overline{U(\lambda)}$ and $U(\lambda) \cap (\sigma(\Theta(C)) \cup \{z \in \mathbb{C} : q(z) = 0\}) = \emptyset$. Since $q(\sigma(\Theta(C))) = \sigma(q(\Theta(C)))$ is bounded in $\mathbb{C}$, the same is true for

$$\bigcup_{\mu \in U(\lambda)} (qs_\mu s_\mu^\#)(\sigma(\Theta(C))) \subseteq \frac{1}{\sigma(\Theta(C)) - U(\lambda)} \cdot \frac{1}{\sigma(\Theta(C)) - U(\lambda)} \cdot q(\sigma(\Theta(C))).$$

On the other hand, for any sequence $\mu_n \in U(\lambda) \setminus (\mathbb{R} \cup \{\infty\}) \subseteq \rho(C)$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} \mu_n = \lambda$ we have

$$(qs_{\mu_n} s_{\mu_n}^\#)(\sigma(C)) \ni (qs_\mu s_\mu^\#)(\lambda) = \frac{q(\lambda)}{|\lambda - \mu_n|^2} \to +\infty,$$

which contradicts the boundedness of

$$\bigcup_{n\in\mathbb{N}} (qs_{\mu_n} s_{\mu_n}^\#)(\sigma(C)) \subseteq \bigcup_{n\in\mathbb{N}} (qs_{\mu_n} s_{\mu_n}^\#)(\sigma(\Theta(C))) \cup \{0\} \subseteq \bigcup_{\mu \in U(\lambda)} (qs_\mu s_\mu^\#)(\sigma(\Theta(C))) \cup \{0\},$$

where the first inclusion follows from Theorem 3.13.3. ❑

definitzreal. *3.13.6 Remark.* According to Corollary 3.13.5 the zeros of q, that lie in $\sigma(C)$, are symmetric with respect to $\mathbb{R}$. If we consider $s := q + q^{\#}$, then $s^{\#} = s$ and $s(C) = q(C) + q^{\#}(C) = q(C) + q(C)^{[*]} = 2q(C)$. Hence, with q also $s := q + q^{\#}$ is a definitizing rational function. The latter is real, i.e. $s^{\#} = s$. ◊

pontrunit. *3.13.7 Example.* If $(\mathcal{B}, [.,.])$ is a Pontryagin space and $U : \mathcal{B} \to \mathcal{B}$ is a unitary bounded linear operator, then U is definitizable. To see this recall Theorem 3.11.13. Thus, we have $[p(U)^{[*]}p(U)x, x] = [p(U)x, p(U)x] \geq 0$ for all $x \in \mathcal{B}$ and a certain polynomial $p \in \mathbb{C}[z]$. By Corollary 3.8.11 we have $p(U)^{[*]}p(U) = p^{\#}(U^{[*]}p(U) = p^{\#}(U^{-1})p(U) = q(U)$, where $q(z) := p^{\#}(\frac{1}{z})\, p(z)$ is a rational functions with poles at most in $\{0, \infty\} \subseteq \rho(U)$; see Corollary 3.9.8. Thus, $q \in C_{\rho(U)}(z)$. ◊

pontrsead. *3.13.8 Example.* If S is a selfadjoint linear relation on the Pontryagin space $(\mathcal{B}, [.,.])$ with $\rho(S) \neq \emptyset$, then taking $\mu \in \rho(S) = \overline{\rho(S)}$ with strictly positive imaginary part we know from Theorem 3.9.6 that the Cayley transform $C_\mu(S)$ is unitary. Choosing M as in Definition 3.9.5 we obtain from Theorem 3.5.6 that $\phi_M(\mu) = 0$ and $\phi_M(\bar{\mu}) = \infty$ belong to $\rho(C_\mu(S))$. As we saw in Example 3.13.7 $q(C_\mu(S))$ is positive for some $q \in C_{\rho(C_\mu(S))}(z)$. Since by Lemma 3.6.4 $q \circ \phi_M(S) = q(C_\mu(S))$ with $q \circ \phi_M \in C_{\rho(S)}(z)$, S is definitizable. ◊

Similar arguments as in Example 3.13.8 show that all unitary linear relations in a Pontryagin space with non-empty resolvent set are definitizable.

Gdefundeig. **3.13.9 Lemma.** *Let $(\mathcal{B}, [.,.])$ be a Krein space, assume that C is a selfadjoint and definitizable linear relation on $\mathcal{B}$, and fix some definitizing $q \in C_{\rho(C)}(z)$ with $q(C) \neq 0$. We denote by F the spectral measure of the selfadjoint linear relation $\Theta(C)$ on $\mathcal{A}$, where the Hilbert space $\mathcal{A}$ and the mapping Θ is as in Remark* 3.13.2. *For any bounded and Borel measurable function $g : \sigma(\Theta(C)) \to \mathbb{C}$ we define*

$$G(g) := A \int_{\sigma(\Theta(C))} g \, dF \, A^{[*]} \; (\in B(\mathcal{B}))\,.$$

Then G depends linearly on g and $G(g)$ commutes with all $R \in B(\mathcal{B})$ which satisfy $(R \times R)(C) \subseteq C$. $G(g) = 0$ if and only if $\int_{\sigma(\Theta(C))} g \, dF = 0$. Moreover,

$$G(g)^{[*]} = G(\bar{g})\,, \qquad (3.13.2)$$ Geig0

$$G(s|_{\sigma(\Theta(C))}) = s(C)\, q(C)\,, \qquad (3.13.3)$$ Geig1

$$G(s|_{\sigma(\Theta(C))} \cdot g) = G(g)\, s(C) = s(C)\, G(g)\,, \qquad (3.13.4)$$ Geig2

and

$$G(g)\, G(h) = G(q|_{\sigma(\Theta(C))} \cdot g \cdot h)\,, \qquad (3.13.5)$$ Geig3

for any $s \in C_{\rho(C)}(z)$ and any bounded and Borel measurable function $g, h : \sigma(\Theta(C)) \to \mathbb{C}$.

Proof. Recall that $A^{[*]} A = \Theta(q(C)) = q(\Theta(C))$ and $q(C) = AA^{[*]}$. Moreover, for $s \in C_{\rho(C)}(z)$ we have (see Corollary 3.12.9)

$$\int_{\sigma(\Theta(C))} s \, dF = s(\Theta(C)) = \Theta(s(C)) = (A \times A)^{-1}(s(C)) = A^{-1}s(C)A \in (A^{[*]}A)'\,,$$

Since $\int_{\sigma(\Theta(C))} g\,dF$ commutes with $q(\Theta(C))$, we have $\int_{\sigma(\Theta(C))} g\,dF \in (A^{[*]}A)'$. Hence, $G(g)$ is nothing else but $\Xi(\int_{\sigma(\Theta(C))} g\,dF)$, where Ξ is as in Theorem 3.12.6. Hence, $G(g) = 0$ if and only if $\int_{\sigma(\Theta(C))} g\,dF = 0$.

By the properties of the functional calculus for selfadjoint linear relations in Hilbert spaces (see) linearity and (3.13.2) follow from Ξ's linearity and its compatibility with taking adjoint. Moreover, (3.13.3) follows from $\Xi(\Theta(s(C))) = s(C)\,AA^{[*]}$, the first equality of (3.13.4) follows from $\Xi(D\Theta(s(C))) = \Xi(D)s(C)$, and (3.13.5) from $\Xi(DH\,A^{[*]}A) = \Xi(D)\Xi(H)$ with $D = \int g\,dF,\ H = \int h\,dF$.

Missing Reference

If R satisfies $(R \times R)(C) \subseteq C$, then by Theorem 3.6.2 R commutes with $q(C) = AA^{[*]}$, i.e. $R \in (AA^{[*]})'$. By Theorem 3.12.6 R commutes with $G(g) = \Xi(\int_{\sigma(\Theta(C))} g\,dF)$ since $\int_{\sigma(\Theta(C))} g\,dF$ commutes with all operators of the from $(\Theta(C) - \lambda)^{-1},\ \lambda \in \rho(\Theta(C))$, and hence with $(A^{[*]}A)' = (q(\Theta(C)))'$. Finally, for $R = s(C)$ we get the second equality in (3.13.4). ❑

3.14 Functional Calculus for seladjoint definitizable relations

We start this section with a little algebra.

muldef.

3.14.1 Remark. Consider the ring $\mathbb{C}[z]$ of polynomials with complex coefficients. Its elements shall be written in the form $p(z) = \sum_{n\in\mathbb{N}_0} a_n z^n$, where only finitely many a_n do not vanish. By $\mathbb{C}[z]_{<m}$ we denote the m-dimensional subspace of all polynomials with degree less than m, and we denote by $\pi_m : \mathbb{C}[z] \to \mathbb{C}[z]_{<m}$ the projection $\sum_{n\in\mathbb{N}_0} a_n z^n \mapsto \sum_{n=0}^{m-1} a_n z^n$.

BESSER und schner machen!!!!!

As $(pq)(z) = \sum_{n\in\mathbb{N}_0} (\sum_{k=0}^{n} a_k b_{n-k}) z^n$ for two polynomials $p(z) = \sum_{n\in\mathbb{N}_0} a_n z^n,\ q(z) = \sum_{n\in\mathbb{N}_0} b_n z^n$ the n-th coefficient of pq only depends on $a_0, \dots, a_n$ and $b_0, \dots, b_n$. Therefore, by $\pi_m(p) \cdot \pi_m(p) := \pi_m(pq)$ a mapping $\cdot : \mathbb{C}[z]_{<m} \times \mathbb{C}[z]_{<m} \to \mathbb{C}[z]_{<m}$ is well-defined.

Since multiplication on $\mathbb{C}[z]$ is bilinear and associative, the same is true for $\cdot : \mathbb{C}[z]_{<m} \times \mathbb{C}[z]_{<m} \to \mathbb{C}[z]_{<m}$. Moreover, the constant polynomial 1 is the multiplicative neutral element.

More generally, if $p(z) = \sum_{n\geq j} a_n z^n \in \mathbb{C}[z]_{\geq j}$ (polynomials of the form $z^j \cdot r(z)$) and $q = \sum_{n\geq 0} a_n \in \mathbb{C}[z]$, then also $\sum_{n\geq j} c_n z^n := (pq)(z) \in \mathbb{C}[z]_{\geq j}$, where $c_n = \sum_{k=j}^{n} a_k b_{n-k}$ for $n \geq j$. Hence, for $m > j$ the coefficients $c_j, \dots, c_{m-1}$ only depend on $a_j, \dots, a_{m-1}$ and $b_0, \dots, b_{m-1-j}$.

Therefore, also by $\pi_m(p) \cdot \pi_{m-j}(p) := \pi_m(pq)$ a bilinear mapping $\cdot : (\mathbb{C}[z]_{<m} \cap \mathbb{C}[z]_{\geq j}) \times \mathbb{C}[z]_{<m-j} \to \mathbb{C}[z]_{<m} \cap \mathbb{C}[z]_{\geq j}$ is well-defined, and this mapping is associative in the sense that $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ for $p \in \mathbb{C}[z]_{<m} \cap \mathbb{C}[z]_{\geq j}, q, r \in \mathbb{C}[z]_{<m-j}$ and that $p \cdot (q \cdot r) = q \cdot (\pi_{m-j})(p) \cdot r)$ for $p \in \mathbb{C}[z]_{<m}, q \in \mathbb{C}[z]_{<m} \cap \mathbb{C}[z]_{\geq j}$ and $r \in \mathbb{C}[z]_{<m-j}$.

Since for any $m \in \mathbb{N}$ by $\sum_{n=0}^{m-1} a_n z^n \mapsto (a_0, \dots, a_{m-1})$ the spaces $\mathbb{C}[z]_{<m}$ can be identified with $\mathbb{C}^m$, we thus defined a multiplication $\cdot$ on $\mathbb{C}^m$, such that $\mathbb{C}^m$ provided with the componentwise scalar multiplication and addition and with $\cdot$ constitutes an

algebra with unit, where the unit is given by $(1, 0, \dots, 0)$. Moreover, for $j < m$ we also defined a bilinear $\cdot : (\{0\} \times \mathbb{C}^{m-j}) \times \mathbb{C}^{m-j} \to \mathbb{C}^m$, where $\{0\} \times \mathbb{C}^{m-j} \subseteq \mathbb{C}^m$. ◊

We fix now a seladjoint definitizable linear relations C on a Krein space $(\mathcal{B}, [.,.])$ with $\rho(C) \neq \emptyset$. By Remark 3.13.6 there is a real definitizing rational function $q \in C_{\rho(C)}(z)$, i.e. $q^\# = q$. Let's fix this q, and denote by $\mathfrak{d} : \sigma(C) \to \mathbb{N}_0$ the function, which assigns to $w \in \sigma(C)$ q's degrees of zero at w. $q^\# = q$ then implies $\mathfrak{d}(w) = \mathfrak{d}(\bar{w}),\ w \in \sigma(C)$.

intbarfunc. **3.14.2 Definition.** By $\mathcal{M}(q, C)$ we denote the set of functions $f : \sigma(C) \to \dot{\bigcup}_{m\in\mathbb{N}} \mathbb{C}^m$ such that $f(\lambda) \in \mathbb{C}^{\mathfrak{d}(\lambda)+1}$, and provide $\mathcal{M}(q, C)$ pointwise with scalar multiplication, addition $+$ and multiplication $\cdot$, where the operations on $\mathbb{C}^{\mathfrak{d}(\lambda)+1}$ are given as in Remark 3.14.1.

For $f \in \mathcal{M}(q, C)$ we define $f^\# \in \mathcal{M}(q, C)$ by $f^\#(\lambda) = \overline{f(\bar{\lambda})},\ \lambda \in \sigma(C)$, where this conjugation is ment componentwise.

By $\mathcal{M}_0(q, C)$ we denote the set of all function $f \in \mathcal{M}(q, C)$ such that for all $w \in \sigma(C)$ all entries of $f(w)$ with the possible exception of the last one vanish, or equivalently that $f(w) \in \{0\} \times \mathbb{C}^1$ $(\subseteq \mathbb{C}^{\mathfrak{d}(w)+1})$ for all w with $\mathfrak{d}(w) > 0$.

If $g : \sigma(C) \to \mathbb{C}$, i.e. $g \in \mathbb{C}^{\sigma(C)}$ and $f \in \mathcal{M}_0(q, C)$, we define also $g \cdot f \in \mathcal{M}(q, C)$ pointwise, where the multiplication on the components is the multiplication $\cdot : \underbrace{(\{0\} \times \mathbb{C}^1)}_{\subseteq \mathbb{C}^{\mathfrak{d}(w)+1}} \times \mathbb{C}^1 \to \mathbb{C}^{\mathfrak{d}(w)+1}$ as defined in Remark 3.14.1. ◊

With these operations $\mathcal{M}(q, C)$ becomes an $*$-algebra, $\mathcal{M}_0(q, C)$ is an ideal in $\mathcal{M}(q, C)$, and $\cdot : \mathcal{M}_0(q, C) \times \mathbb{C}^{\sigma(C)} \to \mathcal{M}_0(q, C)$ is also bilinear.

einbett. *3.14.3 Remark.* For rational functions $s \in C_{\rho(C)}(z)$ we define $\iota_q(s) \in \mathcal{M}(q, C)$ by $\iota_q(s)(\lambda) := s(\lambda)$ for $\mathfrak{d}(\lambda) = 0$ and by $\iota_q(s)(\lambda) := (s^{(0)}(\lambda), \dots, s^{(\mathfrak{d}(\lambda))}(\lambda))$ otherwise. By the Leibniz rule $\iota_q : C_{\rho(C)}(z) \to \mathcal{M}(q, C)$ is an algebra homomorphism. Moreover, $\iota_q(s^\#) = \iota_q(s)^\#$.

Clearly, in the same way it is also possible to define $\iota_q(f)$ for any f which is holomorphic on an open superset of $\sigma(C)$.

Note also that for all $\lambda \in \sigma(C)$ exactly the last entry of $\iota_q(q)(\lambda) \in \mathbb{C}^{\mathfrak{d}(w)+1}$ does not vanish because λ is a zero of degree exactly $\mathfrak{d}(w)$. In particular, $\iota_q(q) \in \mathcal{M}_0(q, C)$. ◊

einbett2. **3.14.4 Lemma.** *For any $f \in \mathcal{M}(q, C)$ there exists a $s \in C_{\rho(C)}(z)$ such that $f - \iota_q(s) \in \mathcal{M}_0(q, C)$.*

Proof. For $\mu \in \rho(C)$ and $m := \sum_{w\in\sigma(C)\cap q^{-1}\{0\}} \mathfrak{d}(w)$ consider the linear subspace

$$\mathcal{R} := \{\frac{p(z)}{(z-\mu)^{m-1}} : p \in \mathbb{C}[z], p \text{ is of degree } < m\}$$

of $C_{\rho(C)}(z)$. If $s(z) = \frac{p(z)}{(z-\mu)^{m-1}} \in \mathcal{R}$ satisfies $\iota_q(s) \in \mathcal{M}_0(q, C)$, then s and in turn p has zeros at all $w \in \sigma(C) \cap q^{-1}\{0\}$ with multiplicity at least $\mathfrak{d}(w)$. For $p(z) \not\equiv 0$ this gives the contradiction that $p(z)$ is of degree greater or equal to m.

Therefore, the linear mapping $s \mapsto (\pi_{\mathfrak{d}(w)}\iota_q(s)(w))_{w\in\sigma(C)\cap q^{-1}\{0\}}$ from $\mathcal{R}$ into $\prod_{w\in\sigma(C)\cap q^{-1}\{0\}} \mathbb{C}^{\mathfrak{d}(w)}$ is injective. Here $\pi_{\mathfrak{d}(w)} : \mathbb{C}^{\mathfrak{d}(w)+1} \to \mathbb{C}^{\mathfrak{d}(w)}$ denotes the projection onto the first $\mathfrak{d}(w)$ entries. Since the dimension of both spaces is m, this mapping is also onto.

In particular, for any $f \in \mathcal{M}(q, C)$ we find a $s \in C_{\rho(C)}(z)$ such that $\pi_{\mathfrak{d}(w)}\iota_q(s)(w) = \pi_{\mathfrak{d}(w)}f(w)$ for all $w \in \sigma(C) \cap q^{-1}\{0\}$. This just means that $f - \iota_q(s) \in \mathcal{M}_0(q, C)$. ❏

Fform. **3.14.5 Proposition.** *Any function $f \in \mathcal{M}(q, C)$ admits a decomposition of the form*

$$f = \iota_q(s) + g \cdot \iota_q(q)\,,$$

where $s \in C_{\rho(C)}(z)$ and $g : \sigma(C) \to \mathbb{C}$.

Proof. By Lemma 3.14.4 there exists a rational $s \in C_{\rho(C)}(z)$ such that $h := f - \iota_q(s) \in \mathcal{M}_0(q, C)$.

We define $g : \sigma(C) \to \mathbb{C}$ for $\lambda \in \sigma(C) \setminus q^{-1}\{0\}$ by $g(\lambda) := \frac{h(\lambda)}{q(\lambda)}$.

For $\lambda \in \sigma(C) \setminus q^{-1}\{0\}$ we have $\iota_q(q)(\lambda)_{\mathfrak{d}(w)} = q^{(\mathfrak{d}(w))}(\lambda) \neq 0$ since λ is a zero of q of degree exactly $\mathfrak{d}(\lambda)$. Hence, we can define

$$g(\lambda) := \frac{h(\lambda)_{\mathfrak{d}(\lambda)}}{\iota_q(q)(\lambda)_{\mathfrak{d}(\lambda)}}\,.$$

It is then easy to verify that $f = \iota_q(s) + g \cdot \iota_q(q)$. ❏

zerlgeig. **3.14.6 Lemma.** *Let $f \in \mathcal{M}(q, C)$ satisfy $f = \iota_q(s) + g \cdot \iota_q(q)$ with $s \in C_{\rho(C)}(z)$ and $g : \sigma(C) \to \mathbb{C}$. Then the Borel measurability of g is equivalent to the Borel measurability of $f|_{\sigma(C)\setminus q^{-1}\{0\}}$. Moreover, g is bounded if and only if $f|_{\sigma(C)\setminus q^{-1}\{0\}}$ is bounded and*

$$\frac{1}{(\lambda - w)^{\mathfrak{d}(w)}}\left(f(\lambda) - \sum_{j=0}^{\mathfrak{d}(w)-1} \frac{f(w)_j}{j!}(\lambda - w)^j\right),\ \lambda \neq w\,, \qquad (3.14.1)$$ fn8qw3

is bounded on a certain neighbourhood of w for any non-isolated $w \in \sigma(C) \cap q^{-1}\{0\}$, i.e. $w \in c\ell(\sigma(C) \setminus \{w\})$.

Proof. Any $r \in C_{\rho(C)}(z)$ is continuous on the compact set $\sigma(C)$. Hence, $\iota_q(r)|_{\sigma(C)\setminus q^{-1}\{0\}} = r|_{\sigma(C)\setminus q^{-1}\{0\}}$ is measurable and bounded.

Since q does not vanish on $\sigma(C) \setminus q^{-1}\{0\}$, it follows from $f|_{\sigma(C)\setminus q^{-1}\{0\}} = s|_{\sigma(C)\setminus q^{-1}\{0\}} + g|_{\sigma(C)\setminus q^{-1}\{0\}}\, q|_{\sigma(C)\setminus q^{-1}\{0\}}$ that $f|_{\sigma(C)\setminus q^{-1}\{0\}}$ is measurable if and only if $g|_{\sigma(C)\setminus q^{-1}\{0\}}$ has this property. As $q^{-1}\{0\}$ is finite, and hence a Borel set, this is equivalent to the Borel measurability of g.

Concerning boundedness note first that $f - \iota_q(s) \in \mathcal{M}_0(q, C)$. Hence, $f(w)_j = s^{(j)}(w),\ j = 0, \ldots, \mathfrak{d}(w) - 1$ for any $w \in q^{-1}\{0\}$. s being holomorphic and using Taylor expansion around w gives

$$\lim_{\lambda \to w} \frac{1}{(\lambda - w)^{\mathfrak{d}(w)}}\left(s(\lambda) - \sum_{j=0}^{\mathfrak{d}(w)-1} \frac{f(w)_j}{j!}(\lambda - w)^j\right) = \frac{s^{(\mathfrak{d}(w))}(w)}{\mathfrak{d}(w)!}\,.$$

Hence, this expression is bounded on a neighbourhood of w. Since q has a zero of degree exactly $\mathfrak{d}(w)$ at w, the same is true for $\frac{q(\lambda)}{(\lambda-w)^{\mathfrak{d}(w)}}$ and $\frac{(\lambda-w)^{\mathfrak{d}(w)}}{q(\lambda)}$.

For any non-isolated $w \in \sigma(C) \cap q^{-1}\{0\}$ and for $\lambda \notin q^{-1}\{0\}$ we have

$$\frac{q(\lambda)}{(\lambda - w)^{\mathfrak{d}(w)}} g(\lambda) = \frac{f(\lambda) - \iota_q(s)(\lambda)}{(\lambda - w)^{\mathfrak{d}(w)}} =$$

$$\frac{1}{(\lambda-w)^{\mathfrak{d}(w)}}\left(f(\lambda)-\sum_{j=0}^{\mathfrak{d}(w)-1}\frac{f(w)_j}{j!}(\lambda-w)^j\right)+\frac{1}{(\lambda-w)^{\mathfrak{d}(w)}}\left(\sum_{j=0}^{\mathfrak{d}(w)-1}\frac{f(w)_j}{j!}(\lambda-w)^j-s(\lambda)\right).$$

Hence, g is bounded on a certain neighbourhood $U(w)$ of w if and only if the same is true for the expression in (3.14.1). Note that this also implies the boundedness of f on $U(w)$.

If we choose $U(w)=\{w\}\cup((\mathbb{C}\cup\{\infty\})\setminus\sigma(C))$ for isolated $w\in\sigma(C)\cap q^{-1}\{0\}$, then for $\lambda\in\sigma(C)\setminus\bigcup_{w\in\sigma(C)\cap q^{-1}\{0\}}U(w)$ we have $f(\lambda)-s(\lambda)=g(\lambda)q(\lambda)$.
$0<\inf_{\lambda\in\sigma(C)\setminus\bigcup_w U(w)}|q|\leq\sup_{\lambda\in\sigma(C)\setminus\bigcup_w U(w)}|q|+\infty$ shows that $f-s$ is bounded on $\sigma(C)\setminus\bigcup_w U(w)$ if and only if g is bounded there. Since s is bounded on $\sigma(C)$, this is also equivalent to the fact that f is bounded on $\sigma(C)\setminus\bigcup_w U(w)$. ❑

zerlgeigrem.

3.14.7 Remark. Note that if for $f\in\mathcal{M}(q,C)$ (3.14.1) is bounded on a certain neighbourhood U of w for a non-isolated $w\in\sigma(C)\cap q^{-1}\{0\}$, then the entries $f(w)_0,\dots,f(w)_{\mathfrak{d}(w)-1}$ are uniquely determined by the values of f on $U\setminus\{w\}$. ◊

FdefklM.

3.14.8 Definition. By $\mathfrak{F}(q,C)$ we denote the set of all $f\in\mathfrak{F}(q,C)$ which satisfy the measurability and the boundedness condition from Lemma 3.14.6. ◊

mcFeig.

3.14.9 Lemma. *$\mathfrak{F}(q,C)$ is a $*$-subalgebra of $\mathfrak{M}(q,C)$.*

Proof. Since (3.14.1) is linear in f, $\mathfrak{F}(q,C)$ is a linear subspace. Considering (3.14.1) also shows that $\mathfrak{F}(q,C)^{\#}=\mathfrak{F}(q,C)$.

For $f_1,f_2\in\mathfrak{F}(q,C)$ choose $s_1,s_2\in C_{\rho(C)}(z)$ and $g_1,g_2:\sigma(C)\to\mathbb{C}$ such that $f_j=\iota_q(s_j)+g_j\cdot\iota_q(q),\ j=1,2$. Then

$$f_1\cdot f_2=\iota_q(s_1s_2)+g\cdot\iota_q(q),$$

where $g=g_1g_2q+s_1g_2+s_2g_1$ is bounded and measurable. ❑

calculwohldef.

3.14.10 Lemma. *Assume that for $f\in\mathcal{M}(q,C)$ we have $f=\iota_q(s_1)+g_1\cdot\iota_q(q)=\iota_q(s_2)+g_2\cdot\iota_q(q)$ for $s_1,s_2\in C_{\rho(C)}(z)$ and bounded, measurable $g_1,g_2:\sigma(C)\to\mathbb{C}$. Then with the notation from Lemma* 3.13.9 *we have*

$$s_1(C)+G(g_1|_{\sigma(\Theta(C))})=s_2(C)+G(g_2|_{\sigma(\Theta(C))}),$$

where we set $G(g)=0$ in case that $q(C)=0$.

Proof. We have to show that $\iota_q(s_1-s_2)=(g_2-g_1)\cdot\iota_q(q)$ implies $(s_1-s_2)(C)=G(g_2|_{\sigma(\Theta(C))}-g_1|_{\sigma(\Theta(C))})$, or equivalently, that $\iota_q(s)=g\cdot\iota_q(q)$ implies $s(C)=G(g|_{\sigma(\Theta(C))})$ for $s\in C_{\rho(C)}(z)$ and bounded, measurable $g:\sigma(C)\to\mathbb{C}$..

Clearly, $\iota_q(s)=g\cdot\iota_q(q)$ yields $g(\lambda)=\frac{s(\lambda)}{q(\lambda)},\ \lambda\in\sigma(C)\setminus q^{-1}\{0\}$.

For $w\in\sigma(C)\cap q^{-1}\{0\}$ we get
$s^{(0)}(w)=\iota_q(s)(w)_0=0,\dots,s^{(\mathfrak{d}(w)-1)}(w)=\iota_q(s)(w)_{\mathfrak{d}(w)-1}=0$ and $s^{(\mathfrak{d}(w))}(w)=\iota_q(s)(w)_{\mathfrak{d}(w)}=q^{(\mathfrak{d}(w))}(w)g(w)$. Since also $q^{(0)}(w)=0,\dots,q^{(\mathfrak{d}(w)-1)}(w)=0$, we obtain

$$g(w)=\frac{s^{(\mathfrak{d}(w))}(w)}{q^{(\mathfrak{d}(w))}(w)}=\lim_{\lambda\to w}\frac{s(\lambda)}{q(\lambda)}.$$

Hence, any $w\in\sigma(C)\cap q^{-1}\{0\}$ is a zero of s with degree greater or equal to the degree of zero of w for q, and g is the rational function $\frac{s}{q}$ restricted to $\sigma(C)$. If $q(C)=0$, then

$s(C) = q(C)\,\frac{s}{q}(C) = 0$ by Proposition 3.6.5. For $q(C) \neq 0$ we get from (3.13.3) and Theorem 3.6.2

$$G(g|_{\sigma(\Theta(C))}) = \frac{s}{q}(C)\,q(C) = s(C)\,.$$

❑

caldef. **3.14.11 Definition.** Let C be a seladjoint definitizable linear relations on the Krein space $(\mathcal{B}, [.,.])$ with $\rho(C) \neq \emptyset$. Moreover, let $q \in C_{\rho(C)}(z)$ be a real definitizing rational function for C. Then we define

$$E : \mathcal{F}(q, C) \to B(\mathcal{B})$$

by $E(f) := s(C) + G(g|_{\sigma(\Theta(C))})$, where f is decomposed as $\iota_q(s) + g \cdot \iota_q(q)$. ◊

By Lemma 3.14.10 is well-defined.

Eeig. **3.14.12 Theorem.** *The mapping E is a $*$-homomorphism from $\mathcal{F}(q, C)$ into $B(\mathcal{B})$ which continues the calculus for rational functions, i.e. $E(\iota_q(s)) = s(C)$ for $s \in C_{\rho(C)}(z)$. Moreover, $E(f)$ commutes with all $R \in B(\mathcal{B})$ such that $(R \times R)(C) \subseteq C$.*

Proof. $E(\iota_q(s)) = s(C)$ for $s \in C_{\rho(C)}(z)$ immediately follows from Definition 3.14.11 and the fact that $E(f)$ is well-defined.

For $\lambda \in \mathbb{C}$ and $f_1, f_2 \in \mathcal{F}(q, C)$ with $f_j = \iota_q(s_j) + g_j \cdot \iota_q(q),\ j = 1, 2$, we have $f_1 + \lambda f_2 = \iota_q(s_1 + \lambda s_2) + (g_1 + \lambda g_2) \cdot \iota_q(q)$, and in turn

$$\begin{aligned} E(f_1 + \lambda f_2) = (s_1 + \lambda s_2)(C) + G((g_1 + \lambda g_2)|_{\sigma(\Theta(C))}) = \\ s_1(C) + G(g_1|_{\sigma(\Theta(C))}) + \lambda s_2(C) + \lambda G(g_2|_{\sigma(\Theta(C))}) = E(f_1) + \lambda E(f_2)\,. \end{aligned}$$

Moreover, $f_1 \cdot f_2 = \iota_q(s_1 s_2) + g \cdot \iota_q(q)$, where $g = g_1 g_2 q + s_1 g_2 + s_2 g_1$. Hence,

$$\begin{aligned} E(f_1 \cdot f_2) = (s_1 s_2)(C) + G(g|_{\sigma(\Theta(C))}) = \\ s_1(C) s_2(C) + G(g_1|_{\sigma(\Theta(C))}) G(g_2|_{\sigma(\Theta(C))}) + \\ s_1(C) G(g_2|_{\sigma(\Theta(C))}) + s_2(C) G(g_1|_{\sigma(\Theta(C))}) = E(f_1)\, E(f_2) \end{aligned}$$

by Theorem 3.6.2 and Lemma 3.13.9. From $f_1^\# = \iota_q(s_1^\#) + g_1^\# \cdot \iota_q(q)$ we conclude (see Corollary 3.8.11 and Lemma 3.13.9)

$$E(f_1^\#) = s_1^\#(C) + G(\overline{g_1|_{\sigma(\Theta(C))}}) = s_1(C)^{[*]} + G(g_1|_{\sigma(\Theta(C))})^{[*]} = E(f_1)^{[*]}\,.$$

By Theorem 3.6.2 any $R \in B(\mathcal{B})$ with $(R \times R)(C) \subseteq C$ commutes with all $s(C),\ s \in C_{\rho(C)}(z)$ Theorem 3.6.2. Together with Lemma 3.13.9 we then see that R commutes with $E(f)$. ❑

Chapter 4

Reproducing kernel spaces

chapter_RKS

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We give a systematic treatment of reproducing kernel spaces of Hilbert space valued functions, and corresponding kernel functions. Special attention is paid to spaces of analytic functions; we discuss several particular classes of kernels induced by analytic functions in some detail (Nevanlinna kernel, a J-matrix kernel, de Branges kernel). Moreover, we study some general constructions which can be carried out with kernel functions, among them, sums of kernels. With exception of §1 and §5*, we focus on the Pontryagin space setting.

§1.	Kernel functions	153
§2.	Constructions with hermitian kernels	165
§3.	Analytic kernels	169
§4.	Some classes of kernels associated with analytic functions	172
§5.	*Reproducing kernel almost Pontryagin spaces	185

4.1 Kernel functions

section5.1

We have already met (scalar) hermitian kernels, and scalar product spaces of (scalar valued) functions generated by such. Remember the sequence Examples 1.1.4, 1.1.7, and 1.1.11 and its continuations. Now we develop this topic further. We are also slightly more general, and consider Hilbert space valued functions instead of scalar valued ones. The principles, however, remain the same.

III.1. **4.1.1 Definition.** Let M be a nonempty set, and let $(\mathcal{H}, (.,.))$ be a Hilbert space. We call a function $K : M \times M \to \mathcal{B}(\mathcal{H})$ an *$\mathcal{H}$-valued hermitian kernel on M*, if

$$K(\eta, \zeta)^* = K(\zeta, \eta), \quad \zeta, \eta \in M. \tag{4.1.1}$$ III.2

Here, $.^*$ denotes the adjoint in the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$. ◊

If $\mathcal{H} = \mathbb{C}$ (endowed with the euclidean scalar product), we can consider K as a function with values in $\mathbb{C}$ in the usual way. Namely, by identifying $T \in \mathcal{B}(\mathbb{C})$ with $T(1) \in \mathbb{C}$. In this situation, we speak of a *scalar hermitian kernel*. Similarly, if $\mathcal{H} = \mathbb{C}^n$, we may think of K as a $n \times n$-matrix valued function. Then we speak of a *$n \times n$-matrix valued hermitian kernel*.

Hermitian kernels arise from, and give rise to, spaces whose elements are functions.

III.5. **4.1.2 Definition.** Let M be a nonempty set, and let $(\mathcal{H}, (.,.))$ be a Hilbert space. We call a triple $(\mathcal{A}, [.,.], \mathcal{O})$ a *reproducing kernel Gram space of $\mathcal{H}$-valued functions on M*, if

(1) $(\mathcal{A}, [.,.], \mathcal{O})$ is a Gram space.

(2) The elements of $\mathcal{A}$ are $\mathcal{H}$-valued functions on M, and the linear operations of $\mathcal{A}$ are given by pointwise addition and scalar multiplication.

(3) For each $\eta \in M$ the *point evaluation functional*

$$\chi_\eta : \begin{cases} \mathcal{A} & \to & \mathcal{H} \\ g & \mapsto & g(\eta) \end{cases}$$

is continuous (where $\mathcal{H}$ is endowed with its Hilbert space topology).

We speak of a *reproducing kernel Krein (almost Pontryagin, Pontryagin, or Hilbert) space of $\mathcal{H}$-valued functions on M*, if in addition $(\mathcal{A}, [.,.], \mathcal{O})$ is a Krein (almost Pontryagin, Pontryagin, or Hilbert) space. ◊

III.14. *4.1.3 Remark.* In Definition 4.1.2, (3), we specified that continuity is understood w.r.t. the norm topology $\mathcal{T}_{(.,.)}$ of $\mathcal{H}$. It follows from the closed graph theorem that we may have replaced this by the (a priori weaker) requirement that χ_η is continuous w.r.t. the weak topology $\mathcal{T}$ of $\mathcal{H}$. To see this, note that $\mathcal{O}$-to-$\mathcal{T}$–continuity of χ_η implies that $\operatorname{graph}\chi_\eta \subseteq \mathcal{A} \times \mathcal{H}$ is closed w.r.t. $\mathcal{O} \times \mathcal{T}$. Since $\mathcal{T}$ is coarser than the Hilbert space topology $\mathcal{T}_{(.,.)}$ of $\mathcal{H}$, it follows that $\operatorname{graph}\chi_\eta$ is also closed w.r.t. $\mathcal{O} \times \mathcal{T}_{(.,.)}$. ◊

As we saw in Example 2.2.6 the topology of a Gram space $(\mathcal{A}, [.,.], \mathcal{O})$ need not be uniquely determined by its scalar product alone. For reproducing kernel Gram spaces, however, it is unique. This is an immediate consequence of Lemma 2.1.6.

III.9. **4.1.4 Corollary.** *Let $(\mathcal{A}, [.,.])$ be a scalar product space of $\mathcal{H}$-valued function on M (with linear operations given pointwise). Then there exists at most one Hilbert space topology on $\mathcal{A}$ which turns $\mathcal{A}$ into a reproducing kernel Gram space.*

Proof. The family $\{(\chi_\eta(.), a) : a \in \mathcal{H}, \eta \in M\}$ is point separating. Hence, we may apply Lemma 2.1.6 ❑

Due to this fact, we may drop explicit notation of the topology O, and speak of a reproducing kernel Gram space $(\mathcal{A}, [.,.])$.

III.6. **4.1.5 Proposition.** *Let M be a nonempty set, let $(\mathcal{H}, (.,.))$ be a Hilbert space, and let $(\mathcal{A}, [.,.])$ be a Krein space whose elements are $\mathcal{H}$-valued functions on M (with linear operations defined pointwise).*

(1) *If $(\mathcal{A}, [.,.])$ is a reproducing kernel Krein space, then there exists a unique function $K : M \times M \to \mathcal{H}^{\mathcal{H}}$, such that*

$$\begin{gathered} K(\eta, .)a \in \mathcal{A}, \quad a \in \mathcal{H}, \eta \in M, \\ (\chi_\eta(f), a) = [f, K(\eta, .)a], \quad f \in \mathcal{A}, a \in \mathcal{H}, \eta \in M\,. \end{gathered} \tag{4.1.2}$$ III.17

This function K is an $\mathcal{H}$-valued hermitian kernel on M. It is called the reproducing kernel *of the reproducing kernel Krein space $(\mathcal{A}, [.,.])$.*

(2) *If there exists a function $K : M \times M \to \mathcal{H}^{\mathcal{H}}$ with (4.1.2), then $(\mathcal{A}, [.,.])$ is a reproducing kernel Krein space.*

Proof. Since $\mathcal{A}$ is a Krein space, its dual space is equal to the collection of all functionals $[., y]$, $y \in \mathcal{A}$. Hence, if point evaluations are continuous, there exist elements $k(\eta, a)$, $a \in \mathcal{H}$, $\eta \in M$, with

$$(\chi_\eta(f), a) = [f, k(\eta, a)], \quad f \in \mathcal{A}, a \in \mathcal{H}, \eta \in M\,.$$

Now define a map $K(\eta, \zeta) : \mathcal{H} \to \mathcal{H}$ as

$$K(\eta, \zeta)a := [k(\eta, a)](\zeta), \quad a \in \mathcal{H}, \eta, \zeta \in M\,,$$

then (4.1.2) holds. Since the scalar product $[.,.]$ is nondegenerated, the function K is uniquely determined by (4.1.2).

If $a, b \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, then (using the fact that K is unique)

$$K(\eta, \zeta)(a + b) = K(\eta, \zeta)a + K(\eta, \zeta)b, \quad K(\eta, \zeta)(\alpha a) = \alpha K(\eta, \zeta)a\,.$$

Moreover, we have

$$\begin{aligned} (K(\eta, \zeta)a, b) = [K(\eta, .)a, K(\zeta, .)b] = \overline{[K(\zeta, .)b, K(\eta, .)a]} = \\ = \overline{(K(\zeta, \eta)b, a)} = (a, K(\zeta, \eta)b), \quad a, b \in \mathcal{H}, \eta, \zeta \in M\,, \end{aligned}$$

i.e., $K(\eta, \zeta)^* = K(\zeta, \eta)$. It remains to show that each operator $K(\eta, \zeta)$ is bounded. Choose a compatible scalar product $(.,.)_\mathcal{A}$ of $\mathcal{A}$ (with corresponding norm $\|.\|_\mathcal{A}$), and denote by $\|\chi_\eta\|$ the operator norm of χ_η w.r.t. $\|.\|_\mathcal{A}$ and the norm of $\mathcal{H}$, and let G be the Gram operator of $[.,.]$ w.r.t. $(.,.)_\mathcal{A}$. Then we have

$$\begin{aligned} \left|(f, K(\eta, .)a)_\mathcal{A}\right| = \left|[G^{-1}f, K(\eta, .)a]\right| = \left|(\chi_\eta(G^{-1}f), a)\right| \leq \|\chi_\eta\| \cdot \|G^{-1}f\|_\mathcal{A} \cdot \|a\|, \\ f \in \mathcal{A}, a \in \mathcal{H}\,. \end{aligned}$$

Hence, the family $\{K(\eta, .)a : \|a\| \leq 1\}$, considered as a subset of the dual space of the Hilbert space $(\mathcal{A}, (.,.)_\mathcal{A})$, is pointwise bounded. The principle of uniform boundedness implies

$$C := \sup\{\|K(\eta, .)a\|_\mathcal{A} : \|a\| \leq 1\} < \infty\,. \tag{4.1.3}$$ III.16

It follows that (with some appropriate $C' > 0$)

$$\big|(K(\eta,\zeta)a,b)\big| = \big|[K(\eta,.)a, K(\zeta,.)b]\big| \leq C' \cdot C^2 \cdot \|a\| \cdot \|b\|, \quad a,b \in \mathcal{H}\,.$$

Thus $K(\eta,\zeta)$ is bounded, in fact, $\|K(\eta,\zeta)\| \leq C'C^2$.

For the converse part stated in item (2) it is enough to remember Remark 4.1.3. ❑

As a first example, let us revisit Example 2.5.6.

III.7. *4.1.6 Example.* Let $a > 0$ and consider the Paley-Wiener space $\mathcal{P}W_a$ as introduced and studied in Example 2.5.6. There we have already shown that $\mathcal{P}W_a$ is a Hilbert space whose elements are entire functions, and that all point evaluation functionals are continuous. Moreover, we have computed the reproducing kernel of $\mathcal{P}W_a$. With the present notation,

$$K(\eta,\zeta) = \begin{cases} \frac{\sin a(\zeta-\overline{\eta})}{\pi(\zeta-\overline{\eta})}, & \zeta \neq \overline{\eta} \\ \frac{a}{\pi}, & \zeta = \overline{\eta}\,. \end{cases}$$

◊

Also our second example origins from function theory.

III.8. *4.1.7 Example.* Denote by $\mathbb{D}$ the open unit disk, and let $\mathbb{H}(\mathbb{D})$ be the set of all functions defined and analytic in $\mathbb{D}$.

The *Hardy-space on the unit disk* is the linear space

$$H^2(\mathbb{D}) := \Big\{ f \in \mathbb{H}(\mathbb{D}) : \sum_{n=0}^{\infty} \Big|\frac{f^{(n)}(0)}{n!}\Big|^2 < \infty \Big\}.$$

Observe that $H^2(\mathbb{D})$ can be naturally identified with $\ell^2(\mathbb{N}_0)$. Namely, consider the map $\varphi : H^2(\mathbb{D}) \to \ell^2(\mathbb{N}_0)$ which is defined as

$$\varphi(f) := \Big(\frac{f^{(n)}(0)}{n!}\Big)_{n=0}^{\infty}, \quad f \in H^2(\mathbb{D})\,.$$

Then, clearly, φ is linear and injective. Assume that a sequence $(\alpha_n)_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$ is given. Then $(\alpha_n)_{n=0}^{\infty}$ is in particular bounded, and hence the radius of convergence of the power series

$$f(\zeta) := \sum_{n=0}^{\infty} \alpha_n \zeta^n$$

is at least 1. Thus the function f may be considered as an element of $\mathbb{H}(\mathbb{D})$. Clearly,

$$f \in H^2(\mathbb{D}) \quad \text{and} \quad \varphi(f) = (\alpha_n)_{n=0}^{\infty}\,.$$

We see that φ is also surjective.

Let $(.,.)$ be the scalar product on $H^2(\mathbb{D})$ defined by requiring that φ is isometric (where $\ell^2(\mathbb{N}_0)$ is endowed with its usual scalar product). Then $(H^2(\mathbb{D}),(.,.))$ is a Hilbert space. Let $\eta \in \mathbb{D}$, then

$$\chi_\eta(f) = f(\eta) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \eta^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \overline{(\overline{\eta}^n)} =$$

$$= \Big(f, \varphi^{-1}((\overline{\eta}^n)_{n=0}^{\infty})\Big), \quad f \in H^2(\mathbb{D})\,.$$

This shows that each point evaluation functional χ_η is continuous, and that the reproducing kernel of $H^2(\mathbb{D})$ is given as

$$K(\eta,\zeta) = [\varphi^{-1}((\overline{\eta}^n)_{n=0}^\infty)](\zeta) = \sum_{n=0}^\infty \overline{\eta}^n \zeta^n = \frac{1}{1-\zeta\overline{\eta}}, \quad \zeta,\eta \in M\,.$$

It is interesting to observe that the scalar product of $H^2(\mathbb{D})$, which we defined above somewhat artificially by pulling back the scalar product of $\ell^2(\mathbb{N}_0)$, has an intrinsic function theoretic meaning. To see this, let us compute the integral $\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\,dt$ for $0<r<1$ (exchanging integration and summations in the below computation is justified by uniform convergence):

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\,dt &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}r^n e^{int}\Big)\overline{\Big(\sum_{m=0}^\infty \frac{g^{(m)}(0)}{m!}r^m e^{imt}\Big)}\,dt = \\
&= \sum_{n,m=0}^\infty \frac{f^{(n)}(0)}{n!}\overline{\Big(\frac{g^{(m)}(0)}{m!}\Big)}r^{n+m}\underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t}\,dt}_{=\delta_{nm}} = \\
&= \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}\overline{\Big(\frac{g^{(n)}(0)}{n!}\Big)}r^{2n}\,. \qquad (4.1.4)
\end{aligned}$$

III.71

By the Schwarz inequality in $\ell^2(\mathbb{N}_0)$, the sequence $\Big(\frac{f^{(n)}(0)}{n!}\overline{(\frac{g^{(n)}(0)}{n!})}\Big)_{n=0}^\infty$ belongs to $\ell^1(\mathbb{N}_0)$. Hence, we may apply the bounded convergence theorem to obtain the (in some sense more intrinsic) formula

$$(f,g) = (\varphi(f),\varphi(g)) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}\overline{\Big(\frac{g^{(n)}(0)}{n!}\Big)} = \lim_{r\uparrow 1}\frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\,dt\,.$$

Also the formula

$$f(\eta) = \Big(f(\zeta), \frac{1}{1-\zeta\overline{\eta}}\Big), \quad \eta\in\mathbb{D} \qquad (4.1.5)$$

III.3

which we proved above, has an intrinsic function theoretic explanation. In fact, it is just (a limiting case of) the Cauchy integral formula. To see this, let $\eta\in\mathbb{D}$ be given. For each $r\in(0,1)$ the point $r^2\eta$ lies in the open disk with radius r centered at the origin, and hence we obtain

$$\begin{aligned}
f(r^2\eta) &= \frac{1}{2\pi i}\oint_{|\zeta|=r}\frac{f(\zeta)}{\zeta - r^2\eta}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\frac{1}{re^{it}-r^2\eta}re^{it}\,dt = \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\frac{1}{1-r\eta e^{-it}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\overline{\Big(\frac{1}{1-re^{it}\overline{\eta}}\Big)}\,dt\,.
\end{aligned}$$

Passing to the limit $r\to 1$ we reobtain (4.1.5).

Let us observe explicitly that each function f which is analytic and bounded in $\mathbb{D}$ belongs to $H^2(\mathbb{D})$. This follows from (4.1.4). Namely, for each $r\in(0,1)$ we have

$$\sum_{n=0}^\infty \Big|\frac{f^{(n)}(0)}{n!}\Big|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^2\,dt \le [\sup_{\zeta\in\mathbb{D}}|f(\zeta)|]^2\,.$$

By the Fatou Lemma, thus

$$\sum_{n=0}^{\infty}\Big|\frac{f^{(n)}(0)}{n!}\Big|^2 \leq \liminf_{r\nearrow 1}\sum_{n=0}^{\infty}\Big|\frac{f^{(n)}(0)}{n!}\Big|^2 r^{2n} \leq [\sup_{\zeta\in\mathbb{D}}|f(\zeta)|]^2 .$$

◊

In general it is difficult to decide whether a given hermitian kernel is the reproducing kernel of some reproducing kernel Krein space. In the below Theorem 4.1.9 we give two characterisations of this property; but in practice this result is hard to apply. In order to formulate this result, we introduce one more notation.

III.72. **4.1.8 Definition.** Let M be a nonempty set, $(\mathcal{H},(.,.))$ a Hilbert space, and K an $\mathcal{H}$-valued hermitian kernel on M. Then we denote

$$\mathcal{F}(M,\mathcal{H}) := \Big\{f : M \to \mathcal{H} :\ \{\zeta\in M : f(\zeta)\neq 0\} \text{ is finite}\Big\},$$

$$[f,g]_K := \sum_{\zeta,\eta\in M}(K(\eta,\zeta)f(\zeta),g(\eta)),\quad f,g\in\mathcal{F}(M,\mathcal{H}) .$$

◊

Using (4.1.1), it is easy to check that $[.,.]_K$ is a scalar product on $\mathcal{F}(M,\mathcal{H})$. We set

$$\operatorname{ind}_+ K := \operatorname{ind}_+(\mathcal{F}(M,\mathcal{H}),[f,g]_K),\quad \operatorname{ind}_- K := \operatorname{ind}_-(\mathcal{F}(M,\mathcal{H}),[f,g]_K) ,$$

and speak of the *positive and negative, respectively, index of the kernel K.*

Remember that we already have discussed the space $\mathcal{F}(M,\mathbb{C})$ is some detail in Example 1.1.4, ?THM? **??**, and Example 1.6.9.

III.10. **4.1.9 Theorem.** *Let M be a nonempty set, let $(\mathcal{H},(.,.))$ a Hilbert space, and let K be an $\mathcal{H}$-valued hermitian kernel on M. Then the following are equivalent.*

(1) *There exists a reproducing kernel Krein space which has K as its reproducing kernel.*

(2) *The scalar product space $\mathcal{F}(M,\mathcal{H})$ generated by K has a Krein space completion.*

(3) *There exists a Krein space $\mathcal{D}$ and a map $V : M \to \mathcal{B}(\mathcal{H},\mathcal{D})$, such that*

$$K(\eta,\zeta) = V(\zeta)^*V(\eta),\ \zeta,\eta\in M,\qquad \operatorname{cls}\Big(\bigcup_{\eta\in M}\operatorname{ran} V(\eta)\Big) = \mathcal{D} .$$

A pair $(\mathcal{D},V)$ with the properties stated in (3) *is called a* Kolmogoroff decomposition of K.

Proof. We are going to show "(1) ⇒ (2) ⇒ (3) ⇒ (1)". First, one notation. Let $\delta_{\eta,a}\in\mathcal{F}(M,\mathcal{H})$ be the function defined as

$$\delta_{\eta,a}(\zeta) := \begin{cases} a , & \zeta=\eta \\ 0 , & \text{otherwise}\end{cases}$$

Then, clearly,

$$\delta_{\eta,a+b} = \delta_{\eta,a} + \delta_{\eta,b},\ \delta_{\eta,\alpha a} = \alpha\delta_{\eta,a}, \quad a,b \in \mathcal{H}, \alpha \in \mathbb{C}, \tag{4.1.6}$$ III.19

$$\mathcal{F}(M,\mathcal{H}) = \operatorname{span}\{\delta_{\eta,a} : a \in \mathcal{H}, \eta \in M\}. \tag{4.1.7}$$ III.20

For the proof of "(1) ⇒ (2)", assume that $\mathcal{D}$ is a reproducing kernel Krein space with kernel K. Consider the linear relation

$$\iota := \operatorname{span}\{(\delta_{\eta,a}; K(\eta,.)a) : a \in \mathcal{H}, \eta \in M\} \subseteq \mathcal{F}(M,\mathcal{H}) \times \mathcal{D}.$$

By the definition of $[.,.]_{\mathcal{D}}$, this relation is isometric. Its range equals $\operatorname{span}\{K(\eta,.)a : a \in \mathcal{H}, \eta \in M\}$, and hence $(\operatorname{ran}\iota)^{[\perp]_K} = \{0\}$. This implies that $\overline{\operatorname{ran}\iota} = \mathcal{D}$, and hence is nondegenerated. We thus may apply Theorem 2.5.14, and conclude that ι is (the graph of) an isometric map of $\mathcal{F}(M,\mathcal{H})$ onto a dense subspace of $\mathcal{D}$. Thus $(\iota,\mathcal{D})$ is a Krein space completion of $\mathcal{F}(M,\mathcal{H})$.

Next, for "(2) ⇒ (3)", assume that a Krein space completion $(\iota,\mathcal{D})$ of $\mathcal{F}(M,\mathcal{H})$ is given. Define, for each $\eta \in M$,

$$V(\eta) : \begin{cases} \mathcal{H} & \to & \mathcal{D} \\ a & \mapsto & \iota(\delta_{\eta,a}) \end{cases}$$

Due to (4.1.6), $V(\eta)$ is linear. Moreover, we have

$$\begin{aligned} &[V(\eta)a, V(\zeta)b]_{\mathcal{D}} = [\iota(\delta_{\eta,a}), \iota(\delta_{\zeta,b})]_{\mathcal{D}} = \\ &\qquad = [\delta_{\eta,a}, \delta_{\zeta,b}]_K = (K(\eta,\zeta)a, b), \quad a,b \in \mathcal{H}, \eta,\zeta \in M. \end{aligned} \tag{4.1.8}$$ III.27

From this it follows that $V(\eta)$ has closed graph: Assume that $a_n \to a$ and $V(\eta)a_n \to c$. Using that $K(\eta,\zeta)$ is bounded, we obtain

$$\begin{aligned} [c, \iota(\delta_{\zeta,b})]_{\mathcal{D}} &= \lim_{n\to\infty} [V(\eta)a_n, \iota(\delta_{\zeta,b})]_{\mathcal{D}} = \lim_{n\to\infty} (K(\eta,\zeta)a_n, b) = \\ &= (K(\eta,\zeta)a, b) = [V(\eta)a, \iota(\delta_{\eta,b})]_{\mathcal{D}}. \end{aligned}$$

By (4.1.7) and the fact that $\operatorname{ran}\iota$ is dense in $\mathcal{D}$, it follows that $c = V(\eta)a$. Thus, indeed, V is a map of M into $\mathcal{B}(\mathcal{H},\mathcal{D})$. The above computation (4.1.8) now implies that

$$[V(\zeta)^*V(\eta)a, b]_{\mathcal{H}} = [V(\eta)a, V(\zeta)b]_{\mathcal{D}} = (K(\eta,\zeta)a, b),$$

i.e. $V(\zeta)^*V(\eta) = K(\eta,\zeta)$. Finally, we have

$$\operatorname{cls}\Big(\bigcup_{\eta\in M} \operatorname{ran} V(\eta)\Big) = \operatorname{cls}\iota(\{\delta_{\eta,a} : a \in \mathcal{H}, \eta \in M\}) = \overline{\operatorname{ran}\iota} = \mathcal{D}.$$

Finally, assume that a Kolmogoroff decomposition consisting of a Krein space $\mathcal{D}$ and a map $V : M \to \mathcal{B}(\mathcal{H},\mathcal{D})$ is given. Consider the map

$$\Phi : \begin{cases} \mathcal{D} & \to & \mathcal{H}^M \\ x & \mapsto & (\zeta \mapsto V(\zeta)^*x) \end{cases}$$

Clearly, Φ is linear (linear operations on $\mathcal{H}^M$ are defined pointwise). Moreover,

$$\ker\Phi = \bigcap_{\zeta\in M} \ker[V(\zeta)^*] = \bigcap_{\zeta\in M} [\operatorname{ran} V(\zeta)]^{[\perp]} = \Big[\operatorname{span}\bigcup_{\zeta\in M} \operatorname{ran} V(\zeta)\Big]^{[\perp]} = \{0\},$$

i.e., Φ is surjective. Let $\Psi : \operatorname{ran}\Phi \to \mathcal{D}$ be the inverse of Φ, and let $[\![.,.]\!]$ be the scalar product defined on $\operatorname{ran}\Phi$ by requiring that Ψ is isometric. Then $(\operatorname{ran}\Phi, [\![.,.]\!])$ is Krein space whose elements are $\mathcal{H}$ valued functions on M. We have

$$\Phi(V(\eta)a) = V(.)^* V(\eta)a = K(\eta, .)a, \quad \eta \in M\,,$$

and hence the functions $K(\eta, .)a$ belong to $\operatorname{ran}\Phi$. Let $f \in \operatorname{ran}\Phi$, and write $f = \Phi(x)$ with some $x \in \mathcal{D}$. Then

$$\begin{aligned}[\![f, K(\eta, .)a]\!] = [\![\Phi(x), \Phi(V(\eta)a)]\!] = [x, V(\eta)a]_{\mathcal{D}} = \\ = (V(\eta)^* x, a)_{\mathcal{H}} = ([\Phi(x)](\eta), a)_{\mathcal{H}} = (f(\eta), a)_{\mathcal{H}}\,.\end{aligned}$$

This shows that $(\operatorname{ran}\Phi, [\![.,.]\!])$ is a reproducing kernel Krein space with kernel K. ❑

The proof of Theorem 4.1.9 not only gives existence, but a somewhat more refined information.

III.21. *4.1.10 Remark.* Assume that one (and hence each) of Theorem 4.1.9, (1)–(3), holds. Then:

(1) Each reproducing kernel Krein space with kernel K is a Krein space completion of $\mathcal{F}(M, \mathcal{H})$.

(2) For each Krein space completion $\mathcal{D}$ of $\mathcal{F}(M, \mathcal{H})$ there exists a Kolmogoroff decomposition with the space $\mathcal{D}$ as auxiliary space.

(3) For each Kolmogoroff decomposition $(\mathcal{D}, V)$ there exists a reproducing kernel Krein space with kernel K which is isometrically isomorphic to $\mathcal{D}$.

◊

As usual, matters are easier to handle in the Pontryagin space case.

III.11. **4.1.11 Theorem.** *Let M be a nonempty set, let $(\mathcal{H}, (.,.))$ a Hilbert space, and let K be an $\mathcal{H}$-valued hermitian kernel on M. Then the following are equivalent.*

(1) *There exists a reproducing kernel Pontryagin space which has K as its reproducing kernel.*

(2) *We have* $\operatorname{ind}_- \mathcal{F}(M, \mathcal{H}) < \infty$.

(3) *There exists a Pontryagin space $\mathcal{D}$ and a map $V : M \to \mathcal{B}(\mathcal{H}, \mathcal{D})$, such that*

$$K(\eta, \zeta) = V(\eta)^* V(\zeta), \quad \zeta, \eta \in M\,.$$

If one (and hence each) of these conditions holds, the reproducing kernel Pontryagin space $\mathcal{D}$ with reproducing kernel K is unique, and $\operatorname{ind}_- \mathcal{D} = \operatorname{ind}_- \mathcal{F}(M, \mathcal{H})$. *We denote this space by $\mathcal{K}(K)$ and speak of the* reproducing kernel Pontryagin space generated by K.

Proof. Again we show "(1) ⇒ (2) ⇒ (3) ⇒ (1)". If there exists a reproducing kernel Pontryagin space $\mathcal{A}$ with kernel K, then $\mathcal{A}$ is a Pontryagin space completion of $\mathcal{F}(M,\mathcal{H})$. It follows that $\operatorname{ind}_-\mathcal{F}(M,\mathcal{H})<\infty$. If $\operatorname{ind}_-\mathcal{F}(M,\mathcal{H})<\infty$, then $\mathcal{F}(M,\mathcal{H})$ has a Pontryagin space completion. Thus we find a Kolmogoroff decomposition of K with a Pontryagin space as auxiliary space, in particular (3) holds.

Assume (3). Set

$$\mathcal{B}:=\operatorname{cls}\Big(\bigcup_{\zeta\in M}\operatorname{ran}V(\zeta)\Big),\quad \mathcal{A}:=\mathcal{B}/\mathcal{B}^{[\circ]},$$

Then $\mathcal{A}$ is a Pontryagin space. Let $\pi:\mathcal{B}\to\mathcal{A}$ be the canonical projection, and consider the map

$$V_1(\zeta):=\pi\circ V(\zeta):\mathcal{H}\to\mathcal{A},\quad \zeta\in M.$$

We have, for each $x\in\mathcal{B}, a\in\mathcal{H}, \zeta\in M$,

$$(V_1(\zeta)^*\pi(x),a)_{\mathcal{H}}=[\pi(x),(\pi\circ V(\zeta))a]_{\mathcal{A}}=[x,V(\zeta)a]_{\mathcal{D}}=(V(\zeta)^*x,a)_{\mathcal{H}}.$$

Hence $V_1(\zeta)^*\circ\pi=V(\zeta)^*|_{\mathcal{B}}$, and it follows that

$$V_1(\zeta)^*V_1(\eta)=[V_1(\zeta)^*\circ\pi]\circ V(\eta)=V(\zeta)^*V(\eta)=K(\eta,\zeta).$$

Since π maps dense subsets of $\mathcal{B}$ to dense subsets of $\mathcal{A}$, the linear span $\operatorname{span}\bigcup_{\zeta\in M}\operatorname{ran}V_1(\zeta)$ is dense in $\mathcal{A}$. We conclude that $(\mathcal{A},V_1)$ forms a Kolmogoroff decomposition of K.

To show uniqueness, assume that $(\mathcal{A}_1,[.,.]_1)$ and $(\mathcal{A}_2,[.,.]_2)$ are reproducing kernel Pontryagin spaces which both have the reproducing kernel K. Set

$$\mathcal{B}:=\operatorname{span}\{K(\eta,.)a: a\in\mathcal{H},\eta\in M\}.$$

Then $\mathcal{B}$ is a dense subspace of $\mathcal{A}_1$ as well as of $\mathcal{A}_2$. Moreover, the scalar products $[.,.]_1$ and $[.,.]_2$ coincide on $\mathcal{B}$. This just says that the identity map

$$\operatorname{id}:\mathcal{B}\subseteq\mathcal{A}_1\longrightarrow\mathcal{B}\subseteq\mathcal{A}_2$$

is a linear and isometric bijection between dense subspaces of the Pontryagin spaces $\mathcal{A}_1$ and $\mathcal{A}_2$. It extends to a linear, isometric, and bicontinuous bijection φ of $\mathcal{A}_1$ onto $\mathcal{A}_2$, cf. Theorem 2.5.14. Since point evaluations are continuous in both spaces, $\mathcal{B}$ is dense, and

$$\chi_\eta\circ\varphi|_{\mathcal{B}}=\chi_\eta|_{\mathcal{B}},\quad \eta\in M,$$

it follows that $\chi_\eta|_{\mathcal{A}_2}\circ\varphi=\chi_\eta|_{\mathcal{A}_1}$, $\eta\in M$. This shows that φ acts as the identity map, and hence that $\mathcal{A}_1=\mathcal{A}_2$ and $[.,.]_1=[.,.]_2$.

Finally, since the reproducing kernel Pontryagin space $\mathcal{A}$ with kernel K is a completion of $\mathcal{F}(M,\mathcal{H})$, we must have $\operatorname{ind}_-\mathcal{A}=\operatorname{ind}_-\mathcal{F}(M,\mathcal{H})$, cf. Proposition 2.1.4. ❑

III.12. *4.1.12 Remark.* The positive and negative indices $\operatorname{ind}_\pm\mathcal{F}(M,\mathcal{H})$ can be expressed via certain matrices built from K. Namely, eigenvalues are counted according to their multiplicities,

$$\operatorname{ind}_+\mathcal{F}(M,\mathcal{H})=\sup_{\substack{n\in\mathbb{N},\ \xi_1,\dots,\xi_n\in M\\ \text{pairwise different}\\ a_1,\dots,a_n\in\mathcal{H}}}\#\Big\{\text{positive eigenvalues of }\Big((K(\xi_j,\xi_i)a_i,a_j)\Big)_{i,j=1}^n\Big\},$$

$$\operatorname{ind}_- \mathcal{F}(M,\mathcal{H}) = \sup_{\substack{n\in\mathbb{N},\ \xi_1,\dots,\xi_n\in M\\ \text{pairwise different}\\ a_1,\dots,a_n\in\mathcal{H}}} \#\Big\{\text{negative eigenvalues of } \Big((K(\xi_j,\xi_i)a_i,a_j)\Big)_{i,j=1}^n\Big\}.$$

To see this, apply Proposition 1.6.8 with the set

$$M := \{\delta_{\eta,a} : a\in\mathcal{H}, \eta\in M\},$$

cf. (4.1.7). ◊

We close this section with one general result which is technical in its nature but often useful.

III.4. **4.1.13 Proposition.** *Let $(\mathcal{A},[.,.],\mathcal{O})$ be an almost Pontryagin space, and let $\mathcal{F}$ be a point separating family of continuous linear functionals on $\mathcal{A}$.*

(1) *There exist $N\in\mathbb{N}_0$, linearly independent functionals $\phi_1,\dots,\phi_N\in\mathcal{F}$, and a number $\gamma_0>0$, such that for each $\gamma\geq\gamma_0$ the scalar product defined as*

$$(x,y)_+ := [x,y] + \gamma\sum_{j=1}^N \phi_j(x)\overline{\phi_j(y)},\quad x,y\in\mathcal{A}, \tag{4.1.9}$$ III.23

is a compatible Hilbert space scalar product on $\mathcal{A}$.

(2) *Let ϕ_j and γ_0 be as in (1), and assume that ϕ_j can be represented as $\phi_j=[.,b_j]$, $j=1,\dots,N$, with some $b_j\in\mathcal{A}$. Then there exists $\gamma_1\geq\gamma_0$, such that for each $\gamma\geq\gamma_1$ the following statement holds: There exist $\alpha_1,\dots,\alpha_N\in\mathbb{C}$, such that for each $a\in\mathcal{A}$ we have ($(.,.)_+$ defined using γ)*

$$[x,a] = (x,a_+)_+,\ x\in\mathcal{A},\quad \textit{with}\quad a_+ = a+\sum_{j=1}^N \alpha_j b_j. \tag{4.1.10}$$ III.25

Proof. If $\mathcal{A}$ is a Hilbert space, we can choose $N=0$ and all assertions are trivially fulfilled. Hence, assume throughout the following that $\mathcal{A}$ is not a Hilbert space.

Choose a compatible Hilbert space scalar product $(.,.)$ on $\mathcal{A}$, and let G be the Gram operator of $[.,.]$ w.r.t. $(.,.)$. Moreover, denote by E the spectral measure of G, and set

$$\mathcal{A}_{>0} := \operatorname{ran} E((0,\infty)),\quad \mathcal{A}_{\leq 0} := \operatorname{ran} E((-\infty,0]).$$

Then $d:=\dim\mathcal{A}_{\leq 0} = \operatorname{ind}_-\mathcal{A} + \operatorname{ind}_0\mathcal{A} < \infty$, and we find a $(.,.)$-orthonormal basis $a_1,\dots,a_d$ of $\mathcal{A}_{\leq 0}$ which consists of eigenvectors of G, say

$$Ga_i = \lambda_i a_i,\quad i=1,\dots,d.$$

The action of G on $\mathcal{A}_{\leq 0}$ is then given as

$$G|_{\mathcal{A}_{\leq 0}} = \sum_{i=1}^d \lambda_i(.,a_i)a_i.$$

Choose $\beta>\max_{i=1,\dots,d}|\lambda_i|$, and set $\delta_- := \beta - \max_{i=1,\dots,d}|\lambda_i|$. Denoting $T := \sum_{i=1}^d (.,a_i)a_i$, we thus have

$$(G+\beta T)|_{\mathcal{A}_{\leq 0}} \geq \delta_-\operatorname{id}|_{\mathcal{A}_{\leq 0}}.$$

Since the point 0 either belongs to the resolvent set of G or is an isolated eigenvalue, we can choose $\delta_+ > 0$ such that $(0, \delta_+) \subseteq \rho(G)$. Obviously, $\ker T = \mathcal{A}_{>0}$ and $\operatorname{ran} T = \mathcal{A}_{\leq 0}$, and it follows that

$$G + \beta T \geq \underbrace{\min\{\delta_+, \delta_-\}}_{=:\delta} \cdot \operatorname{id}|_{\mathcal{A}} . \tag{4.1.11}$$

III.22

For $\phi \in \mathcal{F}$ let $b(\phi) \in \mathcal{A}$ be the element which represents ϕ as $\phi = (., b(\phi))$. Since $\mathcal{F}$ is point separating, we have

$$\Big[\operatorname{span}\{b(\phi) : \phi \in \mathcal{F}\}\Big]^{(\perp)} = \bigcap_{\phi \in \mathcal{F}} \{b(\phi)\}^{(\perp)} = \bigcap_{\phi \in \mathcal{F}} \ker \phi = \{0\} .$$

This means that $\operatorname{span}\{b(\phi) : \phi \in \mathcal{F}\}$ is dense in $\mathcal{A}$. Choose elements $c_1, \dots, c_d$ in this linear span which are sufficiently close to $a_1, \dots, a_d$, namely such that ($\|.\|$ denotes the norm induced by $(., .)$)

$$\|c_i\| \leq 2, \ \|a_i - c_i\| \leq \frac{\delta}{6d\beta}, \quad i = 1, \dots, d .$$

Denoting $T_1 := \sum_{i=1}^d (., c_i) c_i$, we have

$$\|Tx - T_1 x\| \leq \sum_{i=1}^d \Big\|(x, a_i)(a_i - c_i) + (x, a_i - c_i) c_i\Big\| \leq$$
$$\leq d(1 + \max_{i=1,\dots,d} \|c_i\|) \max_{i=1,\dots,d} \|a_i - c_i\| \cdot \|x\| \leq \frac{\delta}{2\beta} \|x\|, \quad x \in \mathcal{A} .$$

Together with (4.1.11), this implies that

$$G + \beta T_1 \geq \frac{\delta}{2} \operatorname{id}|_{\mathcal{A}} .$$

Choose linearly independent functionals $\phi_1, \dots, \phi_N \in \mathcal{F}$, such that $c_i \in \operatorname{span}\{b(\phi_1), \dots, b(\phi_N)\}$, $i = 1, \dots, d$, and let α_{ij} be the coefficients with

$$c_i = \sum_{j=1}^N \alpha_{ij} b(\phi_j), \quad i = 1, \dots, d .$$

Then, for each $x \in \mathcal{A}$,

$$|(x, c_i)|^2 = \Big|\sum_{j=1}^N \overline{\alpha_{ij}} (x, b(\phi_j))\Big|^2 \leq \Big(\sum_{j=1}^N |\alpha_{ij}| \cdot |(x, b(\phi_j))|\Big)^2 \leq \Big(\sum_{j=1}^N |\alpha_{ij}|^2\Big) \sum_{j=1}^N |(x, b(\phi_j))|^2 .$$

Denoting $T_2 := \sum_{j=1}^N (., b(\phi_j)) b(\phi_j)$, thus

$$T_1 \leq d\Big(\sum_{i,j=1}^N |\alpha_{ij}|^2\Big) T_2 .$$

Set $\gamma_0 := \beta d\Big(\sum_{i,j=1}^N |\alpha_{ij}|^2\Big)$. Then, for each $\gamma \geq \gamma_0$,

$$G_+ := G + \gamma T_2 \geq G + \beta d\Big(\sum_{i,j=1}^N |\alpha_{ij}|^2\Big) T_2 \geq G + \beta T_1 \geq \frac{\delta}{2} \operatorname{id}|_{\mathcal{A}} .$$

Consider the scalar product $(.,.)_+$ defined by using G_+ as a Gram operator w.r.t. $(.,.)$, i.e.,

$$(x,y)_+ := (G_+x,y), \quad x,y \in \mathcal{A}.$$

This scalar product is a Hilbert space scalar product on $\mathcal{A}$ which is equivalent to $(.,.)$, and hence also compatible. From its definition, we compute

$$(x,y)_+ = (Gx,y) + \gamma(T_2x,y) = [x,y] + \gamma\sum_{j=1}^N (x, b(\phi_j))(b(\phi_j),y) = \\ = [x,y] + \gamma\sum_{j=1}^N \phi_j(x)\overline{\phi_j(y)}, \quad x,y \in \mathcal{A}.$$

This finishes the proof of (1).

For the proof of (2) we make the ansatz $a_+ := a + \sum_{k=1}^N \alpha_k b_k$ to satisfy the required equality $[x,a] = (x,a_+)_+$, $x \in \mathcal{A}$. By (4.1.9), we have

$$\begin{aligned}(x,a_+)_+ =&[x,a_+] + \gamma\sum_{j=1}^N \phi_j(x)\overline{\phi_j(a_+)} = \\ =&[x,a] + \sum_{k=1}^N \overline{\alpha_k}[x,b_k] + \gamma\sum_{j=1}^N [x,b_j]\overline{\Big(\phi_j(a) + \sum_{k=1}^N \alpha_k[b_k,b_j]\Big)} = \\ =&[x,a] + \gamma\sum_{j=1}^N \phi_j(x)\overline{\sum_{k=1}^N \Big((\frac{1}{\gamma}\delta_{jk} + [b_k,b_j])\alpha_k + \phi_j(a)\Big)}.\end{aligned}$$

If

$$\varepsilon \in \Big(0, \min\{|\lambda| : \lambda \text{ nonzero eigenvalue of } ([b_k,b_j])_{k,j=1}^N\}\Big),$$

the matrix $\varepsilon I + ([b_k,b_j])_{k,j=1}^N$ is invertible. Hence, for each sufficiently large γ, the choice of $\alpha_1,\dots,\alpha_N$ can be made such that

$$\sum_{k=1}^N \Big((\frac{1}{\gamma}\delta_{jk} + [b_k,b_j])\alpha_k + \phi_j(a)\Big) = 0, \quad j = 1,\dots,N.$$

❑

In the context of reproducing kernel spaces, the significance of this result is the following.

III.13. **4.1.14 Corollary.** *Let $(\mathcal{A},[.,.])$ be a reproducing kernel almost Pontryagin space of $\mathcal{H}$-valued functions on M, and let $M_0 \subseteq M$ be such that*

$$f \in \mathcal{A},\ f(\eta) = 0, \eta \in M_0 \qquad \Rightarrow \qquad f = 0.$$

Then there exist $N \in \mathbb{N}$, $\eta_1,\dots,\eta_N \in M_0$, $a_1,\dots,a_N \in \mathcal{H}$, and $\gamma_0 > 0$, such that for each $\gamma \geq \gamma_0$ the scalar product defined as

$$(f,g)_+ := [f,g] + \gamma\sum_{j=1}^N (f(\eta_j),a_j)_{\mathcal{H}}\overline{(g(\eta_j),a_j)_{\mathcal{H}}}, \quad f,g \in \mathcal{A}, \tag{4.1.12}$$ III.24

is a compatible Hilbert space scalar product on $\mathcal{A}$. The space $(\mathcal{A}, (.,.)_+)$ is a reproducing kernel Hilbert space.

Assume in addition that $(\mathcal{A}, [.,.])$ is a Pontryagin space, and let K be its reproducing kernel. Then there exists $\gamma_1 \geq \gamma_0$, such that for each $\gamma \geq \gamma_1$ the following statement holds: There exist $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$, such that the reproducing kernel K_+ of $(\mathcal{A}, (.,.)_+)$ is given as

$$K_+(\eta,\zeta)a = K(\eta,\zeta)a + \sum_{j=1}^{N} \alpha_j K(\eta_j,\zeta)a_j, \quad a \in \mathcal{H}, \zeta, \eta \in M\,. \tag{4.1.13}$$ III.26

Proof. Our present hypothesis just says that $\bigcap_{\eta \in M_0} \ker \chi_\eta = \{0\}$, i.e., the family $\mathcal{F} := \{(\chi_\eta(.), a)_{\mathcal{H}} : a \in \mathcal{H}, \eta \in M_0\}$ is point separating. Existence of N, η_j, a_j, and γ_0 now follows from Proposition 4.1.13, (1). The required relation (4.1.12) is nothing but (4.1.9).

If $\mathcal{A}$ is a Pontryagin space, the functionals $(\chi_\eta(.), a)_{\mathcal{H}}$ are represented as $\chi_\eta = [., K(\eta, .)a]$. Existence of γ_1 and α_j now follows from Proposition 4.1.13, (2). The required formula (4.1.13) for K_+ is relation (4.1.10) applied with $K(\eta, .)a$. ❑

4.2 Constructions with hermitian kernels

section5.2

In this section we study some constructions that can be carried out with reproducing kernels. Namely: composition and multiplication with functions, and sums of kernels. The first mentioned are elementary, whereas sums of kernels behave in a more involved way.

III.75. **4.2.1 Definition.** Let L and M be nonempty sets, $\mathcal{H}$ a Hilbert space, K an $\mathcal{H}$-valued hermitian kernel on M, and λ a function $\lambda : L \to M$. Then we denote

$$(K \boxdot \lambda)(\eta,\zeta) := K(\lambda(\eta), \lambda(\zeta)), \quad \eta, \zeta \in L\,,$$

and speak of the *composition of the kernel K with λ*. ◊

III.43. **4.2.2 Proposition.** *Let L and M be nonempty sets, $\mathcal{H}$ a Hilbert space, K an $\mathcal{H}$-valued hermitian kernel on M, and λ a function $\lambda : L \to M$.*

Then $K \boxdot \lambda$ is an $\mathcal{H}$-valued hermitian kernel on L. Composition with λ, i.e., the mapping rule $f \mapsto f \circ \lambda$, induces an isometric map of $\operatorname{span}\{K(\eta, .)a : a \in \mathcal{H}, \eta \in \lambda(L)\} \subseteq \mathcal{F}_K(M, \mathcal{H})$ *onto $\mathcal{F}_{K \boxdot \lambda}(L, \mathcal{H})$. In particular, we have* $\operatorname{ind}_-(K \boxdot \lambda) \leq \operatorname{ind}_- K$.

If $\operatorname{ind}_- K < \infty$*, then composition with λ induces a continuous and isometric map of* $\operatorname{cls}\{K(\eta, .)a : a \in \mathcal{H}, \eta \in \lambda(L)\} \subseteq \mathcal{K}(K)$ *onto $\mathcal{K}(K \boxdot \lambda)$.*

Proof. Clearly, we have

$$[(K \boxdot \lambda)(\eta,\zeta)]^* = K(\lambda(\eta), \lambda(\zeta))^* = K(\lambda(\zeta), \lambda(\eta)) = (K \boxdot \lambda)(\zeta,\eta), \quad \zeta, \eta \in L\,.$$

Thus, $K \boxdot \lambda$ is an $\mathcal{H}$-valued hermitian kernel on L.

Let $\eta \in \lambda(L)$ and $a \in \mathcal{H}$. Choose $\eta' \in L$ with $\eta = \lambda(\eta')$, then

$$[K(\eta, .)a \circ \lambda](\zeta) = K(\lambda(\eta'), \lambda(\zeta))a = (K \boxdot \lambda)(\eta', \zeta)a\,.$$

We see that the restriction to $\operatorname{span}\{K(\eta, .)a : a \in \mathcal{H}, \eta \in \lambda(L)\}$ of composition with λ maps into $\mathcal{F}_{K\boxdot\lambda}(L, \mathcal{H})$. Starting with $\eta' \in L$ and setting $\eta := \lambda(\eta')$, we see that the range of this map is all of $\mathcal{F}_{K\boxdot\lambda}(L, \mathcal{H})$. Isometry is established by computation. Let $a, b \in \mathcal{H}$, $\eta, \zeta \in \lambda(L)$, and choose $\eta', \zeta' \in L$ with $\eta = \lambda(\eta'), \zeta = \lambda(\zeta')$. Then

$$\begin{aligned}\big[K(\eta, .)a, K(\zeta, .)b\big]_K = \big(K(\eta, \zeta)a, b\big)_{\mathcal{H}} = \big((K \boxdot \lambda)(\eta', \zeta')a, b\big)_{\mathcal{H}} = \\ = \big[(K \boxdot \lambda)(\eta', .)a, (K \boxdot \lambda)(\zeta', .)b\big]_{K\boxdot\lambda} = \big[K(\eta, .)a \circ \lambda, K(\zeta, .)b \circ \lambda\big]_{K\boxdot\lambda}\,.\end{aligned}$$

Since composition with λ is isometric, we have

$$\begin{aligned}\operatorname{ind}_- K \boxdot \lambda = \operatorname{ind}_- \mathcal{F}_{K\boxdot\lambda}(L, \mathcal{H}) = \operatorname{ind}_- \operatorname{span}\{K(\eta, .)a : a \in \mathcal{H}, \eta \in \lambda(L)\} \leq \\ \leq \operatorname{ind}_- \mathcal{F}_K(M, \mathcal{H}) = \operatorname{ind}_- K\,.\end{aligned}$$

Assume now that $\operatorname{ind}_- K < \infty$. Then also $\operatorname{ind}_- K \boxdot \lambda < \infty$. Composition with λ is an isometric map between dense subspaces of
$\mathcal{A} := \operatorname{cls}\{K(\eta, .)a : a \in \mathcal{H}, \eta \in \lambda(L)\} \subseteq \mathcal{K}(K)$ and $\mathcal{K}(K \boxdot \lambda)$. Applying Theorem 2.5.14 provides us with a continuous, surjective, and isometric extension $\Phi : \mathcal{A} \to \mathcal{K}(K \boxdot \lambda)$ of $f \mapsto f \circ \lambda$. We have ($\chi_\zeta$ denoting point evaluation)

$$\chi_\zeta(\Phi(f)) = \chi_\zeta(f \circ \lambda) = \chi_{\lambda(\zeta)}(f), \quad f \in \operatorname{span}\{K(\eta, .)a : a \in \mathcal{H}, \eta \in \lambda(L)\}, \zeta \in L\,.$$

Since point evaluations in both spaces $\mathcal{K}(K)$ and $\mathcal{K}(K \boxdot \lambda)$ are continuous, it follows that

$$\chi_\zeta(\Phi(f)) = \chi_{\lambda(\zeta)}(f), \quad f \in \mathcal{A}, \zeta \in L\,.$$

This just says that $\Phi(f) = f \circ \lambda$, $f \in \mathcal{A}$. ❑

The particular case that $L \subseteq M$ and λ is the set-theoretic inclusion map occurs frequently and deserves seperate notation.

III.74. **4.2.3 Definition.** Let M be a nonempty set, $\mathcal{H}$ a Hilbert space, and K an $\mathcal{H}$-valued hermitian kernel on M. Let L be a nonempty subset of M, and let $\lambda : L \to M$ be the set-theoretic inclusion map. Then we denote

$$K|_L := K \boxdot \lambda\,,$$

and speak of the *restriction of the kernel K to L.* ◊

The second operation with hermitian kernels being under investigation is multiplication of a kernel with a function.

III.76. **4.2.4 Definition.** Let M be a nonempty set, $\mathcal{H}$ and $\mathcal{K}$ Hilbert spaces, K an $\mathcal{H}$-valued hermitian kernel on M, and h a function $h : M \to \mathcal{B}(\mathcal{H}, \mathcal{K})$. The we set

$$(h \boxdot K)(\eta, \zeta) := h(\zeta)K(\eta, \zeta)h(\eta)^*, \quad \eta, \zeta \in M\,,$$

and speak of the *multiplication of the kernel K with h.* ◊

III.45. **4.2.5 Proposition.** *Let M be a nonempty set, $\mathcal{H}$ and $\mathcal{K}$ Hilbert spaces, K an $\mathcal{H}$-valued hermitian kernel on M, and h a function $h : M \to \mathcal{B}(\mathcal{H}, \mathcal{K})$.*

Then $h \boxdot K$ is a $\mathcal{K}$-valued hermitian kernel on M. Pointwise composition with h, i.e., the mapping rule $f(.) \mapsto h(.)f(.)$, induces an isometric map of $\operatorname{span}\{K(\eta, .)h(\eta)^* a : a \in \mathcal{K}, \eta \in M\} \subseteq \mathcal{F}_K(M, \mathcal{H})$ *onto $\mathcal{F}_{h \bullet K}(L, \mathcal{K})$. In particular, we have* $\operatorname{ind}_-(h \boxdot K) \leq \operatorname{ind}_- K$.

If $\operatorname{ind}_- K < \infty$*, then pointwise composition with h induces a continuous and isometric map of* $\operatorname{cls}\{K(\eta, .)h(\eta)^* a : a \in \mathcal{K}, \eta \in M\} \subseteq \mathcal{K}(K)$ *onto $\mathcal{K}(h \boxdot K)$.*

Proof. Let $\eta, \zeta \in M$, then

$$[(h \boxdot K)(\eta, \zeta)]^* = [h(\eta)^* K(\eta, \zeta) h(\zeta)]^* = h(\zeta)^* K(\zeta, \eta) h(\eta) = (h \boxdot K)(\zeta, \eta) .$$

Thus, $h \boxdot K$ is a $\mathcal{K}$-valued hermitian kernel on M.

Let $a \in \mathcal{K}$ and $\eta \in M$. We have $h(.)[K(\eta, .)h(\eta)^* a] = (h \boxdot K)(\eta, .)a$, and hence pointwise composition with h maps $\operatorname{span}\{K(\eta, .)h(\eta)^* a : a \in \mathcal{K}, \eta \in M\}$ onto $\mathcal{F}_{h \boxdot K}(L, \mathcal{K})$. Isometry follows by computation. Let $a, a' \in \mathcal{K}$ and $\eta, \eta' \in M$, then

$$\begin{aligned}
[h(.)[K(\eta, .)h(\eta)^* a], & h(.)[K(\eta', .)h(\eta')^* a']]_{h \boxdot K} = \\
&= [(h \boxdot K)(\eta, .)a, (h \boxdot K)(\eta', .)a']_{h \boxdot K} = \\
&= ((h \boxdot K)(\eta, .)a, a')_{\mathcal{K}} = (h(\eta')K(\eta, \eta')h(\eta)^* a, a')_{\mathcal{K}} = \\
&= (K(\eta, \eta')h(\eta)^* a, h(\eta')^* a')_{\mathcal{H}} = [K(\eta, .)h(\eta)^* a, K(\eta', .)h(\eta')^* a']_K .
\end{aligned}$$

Since pointwise composition with h is isometric, we have

$$\begin{aligned}
\operatorname{ind}_- h \boxdot K = \operatorname{ind}_- \mathcal{F}_{h \boxdot K}(M, \mathcal{K}) = \operatorname{span}\{K(\eta, .)h(\eta)^* a : a \in \mathcal{K}, \eta \in M\} \leq \\
\leq \operatorname{ind}_- \mathcal{F}_K(M, \mathcal{H}) = \operatorname{ind}_- K .
\end{aligned}$$

Assume now that $\operatorname{ind}_- K < \infty$. Then also $\operatorname{ind}_- h \boxdot K < \infty$. Pointwise composition with h is an isometric map between dense subspaces of $\mathcal{A} := \operatorname{cls}\{K(\eta, .)h(\eta)^* a : a \in \mathcal{K}, \eta \in M\} \subseteq \mathcal{K}(K)$ and $\mathcal{K}(h \boxdot K)$. Applying Theorem 2.5.14 provides us with a continuous, surjective, and isometric extension $\Phi : \mathcal{A} \to \mathcal{K}(K \boxdot \lambda)$ of $f \mapsto hf$. We have (χ_ζ again denoting point evaluation)

$$\chi_\zeta(\Phi(f)) = \chi_\zeta(hf) = h(\zeta)\chi_\zeta(f), \quad f \in \operatorname{span}\{K(\eta, .)h(\eta)^* a : a \in \mathcal{K}, \eta \in M\}, \zeta \in M .$$

Since point evaluations in both spaces $\mathcal{K}(K)$ and $\mathcal{K}(h \boxdot K)$ are continuous and $h(\zeta)$ is bounded, it follows that

$$\chi_\zeta(\Phi(f)) = h(\zeta\chi_\zeta(f), \quad f \in \mathcal{A}, \zeta \in M .$$

This just says that $\Phi(f) = hf$, $f \in \mathcal{A}$. ❑

We turn to sums of kernels.

III.77. **4.2.6 Definition.** Let M be a nonempty set, $\mathcal{H}$ a Hilbert space, and let K_1, K_2 be $\mathcal{H}$-valued hermitian kernels on M. Then we denote

$$(K_1 + K_2)(\eta, \zeta) := K_1(\eta, \zeta) + K_2(\eta, \zeta), \quad \eta, \zeta \in M ,$$

and speak of the *pointwise sum of* K_1 *and* K_2. ◊

Clearly, when K_1 and K_2 are $\mathcal{H}$-valued hermitian kernels on a set M, then also their pointwise sum is an $\mathcal{H}$-valued hermitian kernel on M.

It is not difficult to see how negative indices behave.

III.68. **4.2.7 Lemma.** *Let M be a nonempty set, $\mathcal{H}$ a Hilbert space, and let K_1 and K_2 be $\mathcal{H}$-valued hermitian kernels on M. Moreover, let K be their pointwise sum. Then*

$$\operatorname{ind}_- K \leq \operatorname{ind}_- K_1 + \operatorname{ind}_- K_2 .$$

Proof. Denote by Ψ the map taking pointwise sums, i.e.,

$$\Psi : \begin{cases} \mathcal{H}^M \times \mathcal{H}^M & \to & \mathcal{H}^M \\ (f;g) & \mapsto & f+g \end{cases}$$

Then, by definition, $\Psi(K_1(\eta,.)a, K_2(\eta,.)a) = K(\eta,.)a$, $a \in \mathcal{H}$, $\eta \in M$. Moreover,

$$[K(\eta,.)a, K(\zeta,.)b]_K = (K(\eta,\zeta)a,b)_{\mathcal{H}} = (K_1(\eta,\zeta)a,b)_{\mathcal{H}} + (K_2(\eta,\zeta)a,b)_{\mathcal{H}} = \\ = [K_1(\eta,.)a, K_1(\zeta,.)b]_{K_1} + [K_2(\eta,.)a, K_2(\zeta,.)b]_{K_2} .$$

Hence, when the direct product $\mathcal{F}_{K_1}(M,\mathcal{H}) \times \mathcal{F}_{K_2}(M,\mathcal{H})$ is endowed with the sum scalar product, Ψ maps the space

$$\mathcal{L} := \operatorname{span}\{(K_1(\eta,.)a; K_2(\eta,.)a) \in \mathcal{F}_{K_1}(M,\mathcal{H}) \times \mathcal{F}_{K_2}(M,\mathcal{H}) : a \in \mathcal{H}, \eta \in M\} \\ \subseteq \mathcal{F}_{K_1}(M,\mathcal{H}) \times \mathcal{F}_{K_2}(M,\mathcal{H})$$

isometrically onto $\mathcal{F}_K(M,\mathcal{H})$. It follows that

$$\operatorname{ind}_- K_1 + \operatorname{ind}_- K_2 = \operatorname{ind}_- \mathcal{F}_{K_1}(M,\mathcal{H}) \times \mathcal{F}_{K_2}(M,\mathcal{H}) \geq \operatorname{ind}_- \mathcal{L} = \operatorname{ind}_- \mathcal{F}_K(M,\mathcal{H}) .$$

❑

The geometric relation between the reproducing kernel spaces generated by two kernels K_1 and K_2 and by the space generated by their sum, however, is not straightforward.

III.69. **4.2.8 Proposition.** *Let M be a nonempty set and $\mathcal{H}$ a Hilbert space. Let K_1 and K_2 be $\mathcal{H}$-valued hermitian kernels on M with* $\operatorname{ind}_- K_1, \operatorname{ind}_- K_2 < \infty$*, and denote by K their pointwise sum. Moreover, set*

$$C := \mathcal{K}(K_1) \cap \mathcal{K}(K_2), \qquad \mathcal{D} := \{(g;-g) \in \mathcal{H}^M \times \mathcal{H}^M : g \in C\},$$

$$\mathcal{B}_j := \mathcal{K}(K_j)[-]_j\mathcal{W},\ j = 1,2, \qquad \mathcal{B} := (\mathcal{K}(K_1) \times \mathcal{K}(K_2))[-]_+\mathcal{D},$$

where the orthogonal complement in the definition of $\mathcal{B}_j$ are understood within the scalar product $\mathcal{K}(K_j)$, and in the definition of $\mathcal{B}$ within the space $\mathcal{K}(K_1) \times \mathcal{K}(K_2)$ w.r.t. the sum scalar product. Finally, let $\Psi : \mathcal{H}^M \times \mathcal{H}^M \to \mathcal{H}^M$ be the map taking pointwise sums.

Then the following statements hold.

(1) $\Psi|_{\mathcal{B}}$ *maps $\mathcal{B}$ continuously, isometrically, and surjectively onto $\mathcal{K}(K)$.*

(2) *We have* $\ker \Psi|_{\mathcal{B}} = \mathcal{D}^{[\circ]_+}$, *and* $\Psi|_{\mathcal{B}}$ *maps closed subspaces of* $\mathcal{B}$ *onto closed subspaces of* $\mathcal{K}(K)$.

(3) *We have* $\mathcal{B}_1 + \mathcal{B}_2 \subseteq \mathcal{K}(K)$, *and each of* $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_1 + \mathcal{B}_2$ *is closed in* $\mathcal{K}(K)$. *Moreover,*

$$\mathcal{B}_1[\perp]\mathcal{B}_2, \qquad \mathcal{B}_1 \cap \mathcal{B}_2 = C^{[\circ]_1} \cap C^{[\circ]_2},$$

$$[f,g]_j = [f,g], \quad f,g \in \mathcal{B}_j,\ j = 1,2,$$

and the space $C^{[\circ]_1} + C^{[\circ]_2}$ *is* $[.,.]$*-neutral.*

(4) *Assume additionally that*

$$[f,g]_1 = -[f,g]_2, \quad f,g \in C. \tag{4.2.1}$$ III.70

Then $\mathcal{D}$ *is* $[.,.]_+$*-neutral, and (note that under the assumption* (4.2.1) *certainly* $C^{[\circ]_1} = C^{[\circ]_2}$*)*

$$\mathcal{B}_1 + \mathcal{B}_2 = \mathcal{K}(K)[-]C^{[\circ]_j}.$$

Proof. $\boxed{\text{insert proof}}$ ❑

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4.3 Analytic kernels

section5.3

In this section we study reproducing kernel Pontryagin spaces of analytic functions. The main result is the below Theorem 4.3.2, which shows that analyticity of the elements of a space is characterised by analyticity of its kernel.

III.73. **4.3.1 Definition.** Let Ω be an open subset of $\mathbb{C}$, $\mathcal{H}$ a Hilbert space, and $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{H})$ an $\mathcal{H}$-valued hermitian kernel on Ω. We say that K is an *analytic kernel*, if for each $\eta \in \Omega$ and $a \in \mathcal{H}$, the function

$$K(\eta,.)a : \left\{ \begin{array}{ccc} \Omega & \to & \mathcal{H} \\ \zeta & \mapsto & K(\eta,\zeta)a \end{array} \right.$$

is an analytic $\mathcal{H}$-valued function on Ω. ◊

III.28. **4.3.2 Theorem.** *Let* Ω *be an open subset of* $\mathbb{C}$, $\mathcal{H}$ *a Hilbert space, and* $\mathcal{A}$ *a reproducing kernel Pontryagin space of* $\mathcal{H}$*-valued functions on* Ω. *Moreover, let* $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{H})$ *be its reproducing kernel. Then the following are equivalent.*

(1) *All elements of* $\mathcal{A}$ *are analytic* $\mathcal{H}$*-valued functions on* Ω.

(2) K *is an analytic kernel.*

(3) *For each* $a \in \mathcal{H}$, *the function*

$$\overline{k_a} : \left\{ \begin{array}{ccc} \{\xi \in \mathbb{C} : \overline{\xi} \in \Omega\} & \to & \mathcal{A} \\ \xi & \mapsto & K(\overline{\xi},.)a \end{array} \right.$$

is an analytic $\mathcal{A}$*-valued function on* Ω.

Proof. The implication "(1) ⇒ (2)" is obvious, since for each $\eta \in \Omega$ and $a \in \mathcal{H}$ the function $K(\eta, .)a$ belongs to $\mathcal{A}$. The equivalence of (1) and (3) is easy to see: Due to the reproducing kernel property of K, we have

$$\frac{(f(\zeta), a)_{\mathcal{H}} - (f(\eta), a)_{\mathcal{H}}}{\zeta - \eta} = \frac{1}{\zeta - \eta}[f, k_a(\zeta) - k_a(\eta)] = \frac{1}{\zeta - \eta}[f, \overline{k_a}(\overline{\zeta}) - \overline{k_a}(\overline{\eta})] =$$
$$= \overline{\Big(\frac{[\overline{k_a}(\overline{\zeta}), f] - [\overline{k_a}(\overline{\eta}), f]}{\overline{\zeta} - \overline{\eta}}\Big)}, \quad \zeta, \eta \in \Omega, \zeta \neq \eta, a \in \mathcal{H}. \quad (4.3.1)$$ III.31

Hence, the function $(f(\zeta), a)_{\mathcal{H}}$ is differentiable at a point η, if and only if the function $[\overline{k_a}(\xi), f]$ is differentiable at $\overline{\eta}$.

The involved part is the the proof of "(2) ⇒ (1)". Assume that (2) holds. Choose a compatible Hilbert space scalar product $(.,.)_+$ of the form (4.1.12) with γ sufficiently large so that (4.1.13) applies. Since each function $K(\eta, .)a$, $\eta \in \Omega$, $a \in \mathcal{H}$, is analytic, so is each function $K_+(\eta, .)a$, $\eta \in \Omega$, $a \in \mathcal{H}$.

Set $\Omega' := \{\xi \in \mathbb{C} : \overline{\xi} \in \Omega\}$, and consider for $a, b \in \mathcal{H}$ the function

$$f_{a,b} : \begin{cases} \Omega' \times \Omega & \to & \mathbb{C} \\ (\eta, \zeta) & \to & (K_+(\overline{\eta}, \zeta)a, b)_{\mathcal{H}} \end{cases}$$

As we just observed, the function $f_{a,b}(\eta, \zeta)$ is, for each fixed $\eta \in \Omega'$, analytic in $\zeta \in \Omega$. Moreover, we have

$$\overline{f_{a,b}(\overline{\zeta}, \lambda)} = (b, K_+(\zeta, \lambda)a)_{\mathcal{H}} = (K_+(\lambda, \zeta)b, a)_{\mathcal{H}} = f_{b,a}(\overline{\lambda}, \zeta), \quad \zeta \in \Omega, \lambda \in \Omega.$$

Hence, for each fixed $\lambda \in \Omega'$, the function $f_{a,b}(\xi, \lambda)$ is analytic in $\xi \in \Omega'$. By Hartogs Theorem, cf. ?THM? **??**, the function $f_{a,b}$ is analytic as a function of two complex variables. In particular, thus, the function

$$f_{a,b}(\overline{\zeta}, \zeta) : \begin{cases} \Omega & \to & \mathbb{C} \\ \zeta & \mapsto & (K_+(\zeta, \zeta)a, b)_{\mathcal{H}} \end{cases}$$

is locally bounded. Applying twise the principle of uniform boundedness, yields that the function (here $\|.\|$ denotes the operator norm in $\mathcal{B}(\mathcal{H})$)

$$\begin{cases} \Omega & \to & [0, \infty) \\ \zeta & \mapsto & \|K_+(\zeta, \zeta)\| \end{cases}$$

is locally bounded.

We have (here $\|.\|_+$ denotes the norm induced by $(.,.)_+$)

$$\|K_+(\zeta, .)a\|_+^2 = \big(K_+(\zeta, .)a, K_+(\zeta, .)a\big)_+ = (K_+(\zeta, \zeta)a, a)_{\mathcal{H}} \leq$$
$$\leq \|K_+(\zeta, \zeta)\| \cdot \|a\|_{\mathcal{H}}^2, \quad \zeta \in \Omega, a \in \mathcal{H}.$$

Hence, for each $a \in \mathcal{H}$ and each compact subset L of Ω,

$$\sup_{\zeta \in L} \|K_+(\zeta, .)a\|_+ \leq \|a\|_{\mathcal{H}} \cdot \sup_{\zeta \in L} \|K_+(\zeta, \zeta)\|.$$

Now we obtain the estimate

$$\sup_{\zeta\in L}\big|(f(\zeta),a)_{\mathcal{H}}\big| = \sup_{\zeta\in L}\big|(f,K_+(\zeta,.)a)_+\big| \leq \|f\|_+\cdot\|a\|_{\mathcal{H}}^2\cdot\sup_{\zeta\in L}\|K_+(\zeta,\zeta)\|\,.$$

Passing to the supremum over $\|a\|_{\mathcal{H}}\leq 1$, yields that

$$\sup_{\zeta\in L}\|f(\zeta)\|_{\mathcal{H}} \leq \|f\|_+\cdot\sup_{\zeta\in L}\|K_+(\zeta,\zeta)\|,\quad L\subseteq\Omega \text{ compact}. \tag{4.3.2}$$ III.32

We conclude that convergence in the norm of $\mathcal{A}$ of a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n\in\mathcal{A}$, to an element $f\in\mathcal{A}$, implies that the functions f_n converge to f locally uniformly w.r.t. the norm of $\mathcal{H}$. Hence, $\mathcal{A}\cap\mathbb{H}(\Omega,\mathcal{H})$ is a closed linear subspace of $\mathcal{A}$.

By our hypothesis (2), we have $\{K(\eta,.)a:\eta\in\Omega,a\in\mathcal{H}\}\subseteq\mathcal{A}\cap\mathbb{H}(\Omega,\mathcal{H})$, and hence also $\operatorname{cls}\{K(\eta,.)a:\eta\in\Omega,a\in\mathcal{H}\}=\mathcal{A}$ where the closure understands w.r.t. the norm of $\mathcal{A}$. However,

$$\{K(\eta,.)a:\eta\in\Omega,a\in\mathcal{H}\}^{[\perp]} = \bigcap_{\eta\in\Omega,a\in\mathcal{H}}\ker(\chi_\eta(.),a)_{\mathcal{H}} = \{0\}\,,$$

and hence $\operatorname{cls}\{K(\eta,.)a:\eta\in\Omega,a\in\mathcal{H}\}=\mathcal{A}$. ❑

Let us point out the following properties of a reproducing kernel Pontryagin space of analytic functions which we have actually seen in the proof of Theorem 4.3.2.

III.29. **4.3.3 Corollary.** *Let Ω be an open subset of $\mathbb{C}$, $\mathcal{H}$ a Hilbert space, and $\mathcal{A}$ a reproducing kernel Pontryagin space of $\mathcal{H}$-valued functions on Ω. Then the following statements hold.*

(1) *Convergence in the norm of $\mathcal{A}$ implies locally uniform convergence w.r.t. the norm of $\mathcal{H}$.*

(2) *The unit ball of $\mathcal{A}$ is a normal family.*

(3) *For each $\eta\in\Omega$, $a\in\mathcal{H}$, and $n\in\mathbb{N}$, the function $[\frac{\partial^n}{\partial\xi^n}\overline{k_a}]\big|_{\xi=\overline{\eta}}$ belongs to $\mathcal{A}$ and*

$$\Big[f,[\tfrac{\partial^n}{\partial\xi^n}\overline{k_a}]\big|_{\xi=\overline{\eta}}\Big] = (f^{(n)}(\eta),a)_{\mathcal{H}},\quad f\in\mathcal{A}\,.$$

Proof. Item (1) was already stated in the proof of Theorem 4.3.2, as a consequence of (4.3.2). Item (2) is also an immediate consequence of (4.3.2). The case "$n=1$" of item (3) is immediate from (4.3.1), and the general case follows by straightforward induction on n. ❑

It is an interesting consequence of analyticity that the negative index is fully determined already on small subsets of Ω.

III.30. **4.3.4 Proposition.** *Let Ω be an open and connected subset of $\mathbb{C}$, $\mathcal{H}$ a Hilbert space, and K an analytic $\mathcal{H}$-valued hermitian kernel with* $\operatorname{ind}_- K<\infty$. *Moreover, let M be a subset of Ω which has an accumulation point in Ω. Then* $\operatorname{ind}_- K=\operatorname{ind}_- K|_M$.

Proof. Consider the reproducing kernel Pontryagin space $\mathcal{A}$ generated by the kernel K. All elements of $\mathcal{A}$ are analytic functions on Ω.

By the identity theorem, we have

$$\{K(\eta,.)a : \eta \in M, a \in \mathcal{H}\}^{[\perp]} = \bigcap_{\eta\in M, a\in\mathcal{H}} \ker(\chi_\eta(.), a)_{\mathcal{H}} = \{0\}. \qquad (4.3.3)$$ III.33

Hence $\mathcal{B} := \operatorname{span}\{K(\eta,.)a : \eta \in M, a \in \mathcal{H}\}$ is dense in $\mathcal{A}$. It follows that $\operatorname{ind}_- \mathcal{A} = \operatorname{ind}_- \mathcal{B}$.

By Proposition 4.2.2, the restriction map $f \mapsto f|_M$ maps $\mathcal{B}$ isometrically into $\mathcal{F}_{K|_M}(M, \mathcal{H})$. It follows that

$$\operatorname{ind}_- K = \operatorname{ind}_- \mathcal{A} = \operatorname{ind}_- \mathcal{B} \leq \operatorname{ind}_- \mathcal{F}_{K|_M}(M,\mathcal{H}) = \operatorname{ind}_- K|_M .$$

The reverse inequality "$\operatorname{ind}_- K \geq \operatorname{ind}_- K|_M$" holds by Proposition 4.2.2. ❑

The relation (4.3.3) has another immediate consequence. Namely, that a reproducing kernel Pontryagin space of $\mathcal{H}$-valued analytic functions is separabel provided $\mathcal{H}$ is. To see this notice that, in order to have the second equality in (4.3.3), it is enough to take the intersection over all a belonging to some dense subset of $\mathcal{H}$.

4.4 Some classes of kernels associated with analytic functions

section5.4

In this section we present some concrete types of reproducing kernel Pontryagin spaces of analytic functions. At the present stage, it is not our aim to go deeply into the theory of any of them. They rather serve as examples and provide what is necessary for the following chapters.

4.4.1 Analytic mappings between general disks

We call a subset Ω of $\mathbb{C}$ a general disk, if it either a nonempty open disk ($\neq \mathbb{C}$), or an open half-plane.

III.34. **4.4.1 Definition.** Let Ω_1 and Ω_2 be general disks. We denote by $\mathcal{S}(\Omega_1, \Omega_2)$ the set of all functions $f \in \mathbb{H}(\Omega_1)$ which map Ω_1 into $\overline{\Omega_2}$. If Ω_2 is a half-plane, we formally include the function $f(\zeta) = \infty$ into $\mathcal{S}(\Omega_1, \Omega_2)$. ◊

According to the historic development of the topic, the class $\mathcal{S}(\Omega_1, \Omega_2)$ is usually named in particular ways for particular cases of Ω_1 and Ω_2:

(1) The class $\mathcal{S}(\mathbb{D}, \mathbb{D})$ is called the *Schur-class*, and is denoted by $\mathcal{S}$.

(2) The class $\mathcal{S}(\mathbb{C}^+, \mathbb{C}^+)$ is called the *Nevanlinna-class*, and is denoted by $\mathcal{N}$.

(3) The class $\mathcal{S}(\mathbb{D}, -i\mathbb{C}^+)$ is called the *Caratheodory-class*, and is denoted by C.

III.35. *4.4.2 Remark.* A function $f \in \mathcal{S}(\Omega_1, \Omega_2)$ may assume a value on the boundary of Ω_2. However, since $f(\Omega_1)$ is open unless f is constant, this may happen only if f is constant. ◊

Each choice of general disks Ω_1 and Ω_2 gives rise to a certain type of hermitian kernels. This is based on the fact that a general disk can be described by a quadratic polynomial in ζ and $\overline{\zeta}$. The proof of this fact is elementary; we omit it.

III.36. **4.4.3 Lemma.** *Let $\Omega \subseteq \mathbb{C}$. Then Ω is a general disk, if and only if there exist $\alpha \leq 0$, $\beta \in \mathbb{C}$, and $\gamma \in \mathbb{R}$, with $|\beta|^2 > \alpha\gamma$, such that*

$$\Omega = \{\zeta \in \mathbb{C} : \alpha\zeta\overline{\zeta} + \beta\zeta + \overline{\beta\zeta} + \gamma > 0\}. \tag{4.4.1}$$ III.38

A general disk Ω is a half-plane, if and only if in one (equivalent, in each) representation of the form (4.4.1)*, the number α equals zero.*

If Ω is a general disk, the choice of α, β, γ can be made such that $\alpha \in \{0, -1\}$ and $|\beta| = 1$ if $\alpha = 0$. With these normalisations, the numbers α, β, γ are uniquely determined by Ω. ❑

III.39. **4.4.4 Definition.** Let Ω_1 and Ω_2 be general disks, and let $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ be numbers from the respective representations (4.4.1). For a function $f \in \mathcal{S}(\Omega_1, \Omega_2)$, we set

$$K^f_{\Omega_1,\Omega_2}(\eta, \zeta) := \frac{\alpha_2 f(\zeta)\overline{f(\eta)} + \beta_2 f(\zeta) + \overline{\beta_2 f(\eta)} + \gamma_2}{\alpha_1 f(\zeta)\overline{f(\eta)} + \beta_1 f(\zeta) + \overline{\beta_1 f(\eta)} + \gamma_1}, \quad \zeta, \eta \in \Omega_1 . \tag{4.4.2}$$ III.41

◊

Obviously, $K^f_{\Omega_1,\Omega_2}$ is a hermitian kernel.

The classes $\mathcal{S}(\Omega_1, \Omega_2)$ and the corresponding kernel types for different choices of Ω_1 and Ω_2 are closely related.

III.42. **4.4.5 Lemma.** *Let Ω_1, Ω_2 and Ω_1', Ω_2' be general disks, and let φ_1 and φ_2 be fractional linear transformations with $\varphi_i(\Omega_i) = \Omega_i'$, $i = 1, 2$. Then the mapping rule $\Lambda : f \mapsto \varphi_2 \circ f \circ \varphi_1^{-1}$ establishes a bijections of $\mathcal{S}(\Omega_1, \Omega_2)$ onto $\mathcal{S}(\Omega_1', \Omega_2')$. There exists a zerofree function $h \in \mathbb{H}(\Omega_1)$, such that*

$$K^{\Lambda f}_{\Omega_1',\Omega_2'} \star \varphi_1 = h \bullet K^f_{\Omega_1,\Omega_2} .$$

In particular, the mapping rule $g \mapsto h(g \circ \varphi^{-1})$ establishes a bijective isometry between $\mathcal{F}_{K^f_{\Omega_1,\Omega_2}}(\Omega_1)$ and $\mathcal{F}_{K^{\Lambda f}_{\Omega_1',\Omega_2'}}(\Omega_1')$.

Proof. insert proof ❑

The connection between the class $\mathcal{S}(\Omega_1, \Omega_2)$ and the corresponding kernel type $K^f_{\Omega_1,\Omega_2}$ is made as follows. This theorem is a basic result.

III.40. **4.4.6 Theorem.** *Let Ω_1 and Ω_2 be general disks, and let $f \in \mathbb{H}(\Omega_1)$. Then $f \in \mathcal{S}(\Omega_1, \Omega_2)$, if and only if the kernel $K^f_{\Omega_1,\Omega_2}$ is positive semidefinite.*

Due to Lemma 4.4.5 it is enough to consider the case that $\Omega_1 = \mathbb{D}$ and $\Omega_2 = -i\mathbb{C}^+$. The proof for this case is based on the *Riesz-Herglotz integral representation* of functions in the Caratheodory class. Since it is not our aim to go into harmonic function theory, we present a short proof based on the Cauchy-integral formula.

III.46. **4.4.7 Proposition.** *Let $f \in \mathcal{C}$. Then there exists a positive and finite Borel measure on the unit circle, such that*

$$f(\zeta) = i \operatorname{Im} f(0) + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta), \quad \zeta \in \mathbb{D}\,. \tag{4.4.3}$$ III.57

Proof. Fix $R \in (0,1)$, and set $f_R(\zeta) := f(R\zeta)$. Then $f_R \in \mathbb{H}(\frac{1}{R}\mathbb{D})$. We have

$$\frac{\xi + \zeta}{\xi - \zeta} = \frac{1 + \frac{\zeta}{\xi}}{1 - \frac{\zeta}{\xi}} = \frac{2}{1 - \frac{\zeta}{\xi}} - 1\,,$$

and hence ($\zeta = re^{i\theta} \in \mathbb{D}$)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\phi} + \zeta}{e^{i\phi} - \zeta} \operatorname{Re} f_R(e^{i\phi})\, d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\frac{2}{1 - \frac{re^{i\theta}}{e^{i\phi}}} - 1\Big) \frac{f_R(e^{i\phi}) + \overline{f_R(e^{i\phi})}}{2}\, d\phi =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_R(e^{i\phi})}{1 - \frac{re^{i\theta}}{e^{i\phi}}}\, d\phi + \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_R(e^{i\phi})}{1 - \frac{re^{-i\theta}}{e^{-i\phi}}}\, d\phi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re} f_R(e^{i\phi})\, d\phi\,.$$

We apply the Cauchy-integral formula and the calculus of residues to evaluate the integrals:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_R(e^{i\theta})}{1 - \frac{re^{i\theta}}{e^{i\phi}}}\, d\phi = \frac{1}{2\pi i}\oint\limits_{|\xi|=1} \frac{f_R(\xi)}{\xi - \zeta} d\xi = f_R(\zeta)\,,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_R(e^{i\phi})}{1 - \frac{re^{-i\theta}}{e^{-i\phi}}}\, d\phi = -\frac{1}{\zeta}\frac{1}{2\pi i}\oint\limits_{|\xi|=1} \frac{f_R(\xi)\xi^{-1}}{\xi - \frac{1}{\zeta}}\, d\xi = -\frac{1}{\zeta}\frac{f_R(0)}{-\frac{1}{\zeta}} = f_R(0), \quad \zeta \neq 0\,,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_R(e^{i\phi})\, d\phi = \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{f_R(\xi)}{\xi}\, d\xi = f_R(0)\,.$$

It follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\phi} + \zeta}{e^{i\phi} - \zeta} \operatorname{Re} f_R(e^{i\phi})\, d\phi = f_R(\zeta) + \overline{f_R(0)} - \operatorname{Re} f_R(0), \quad \zeta \neq 0\,,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\phi} + \zeta}{e^{i\phi} - \zeta} \operatorname{Re} f_R(e^{i\phi})\, d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re} f_R(e^{i\phi})\, d\phi = \operatorname{Re} f_R(0), \quad \zeta = 0\,.$$

Together, we thus have

$$f_R(\zeta) = i \operatorname{Im} f_R(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\phi} + \zeta}{e^{i\phi} - \zeta} \operatorname{Re} f_R(e^{i\phi})\, d\phi, \quad \zeta \in \mathbb{D}\,. \tag{4.4.4}$$ III.56

Now we let R increase to 1. For each fixed $\zeta \in \mathbb{D}$, clearly, $\lim_{R\uparrow 1} f_R(\zeta) = f(\zeta)$. The measure $\operatorname{Re} f_R(e^{i\phi})d\phi$ is positive. Hence, its total variation computes as $\operatorname{Re} f(0)$, and thus is independent of $R \in (0,1)$. By the Banach-Alaoglu Theorem, we may choose a sequence $R_n \in (0,1)$ which increases to 1, such that $\lim_{n\to\infty} \operatorname{Re} f_R(e^{i\phi})d\phi =: \nu$ exists in the w^*-topology. Passing to the limit also on the right hand side of (4.4.4), we obtain

$$f(\zeta) = i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_{(-\pi,\pi]} \frac{e^{i\phi} + \zeta}{e^{i\phi} - \zeta}\, d\nu, \quad \zeta \in \mathbb{D}\,.$$

It remains to define μ as the image measure of ν under the bijection $\phi \mapsto e^{i\phi}$ of $(-\pi,\pi]$ onto $\mathbb{T}$. ❑

Notice that each function of the form (4.4.3) clearly belongs to the class C.

Proof (of Theorem 4.4.6*).* $\boxed{\text{insert proof}}$ ❑

$\boxed{\text{Nevanlinna-function representation}}$

$\boxed{\texttt{III.48.}}$ *4.4.8 Remark.* Let $f \in \mathcal{S}(\Omega_1, \Omega_2)$. Clearly, the kernel $K^f_{\Omega_1,\Omega_2}$ is analytic in $\zeta \in \Omega_1$ for each fixed $\eta \in \Omega_1$. Hence, the reproducing kernel Hilbert space generated by $K^f_{\Omega_1,\Omega_2}$ consists of functions analytic in Ω_1. ◊

Now the following indefinite analogue of the classes $\mathcal{S}(\Omega_1, \Omega_2)$ comes naturally.

$\boxed{\texttt{III.47.}}$ **4.4.9 Definition.** Let Ω_1 and Ω_2 be general disks, let f be a function meromorphic in Ω_1, and denote by $\rho(f)$ its domain of analyticity in Ω_1. Then we write $f \in \mathcal{S}_{<\infty}(\Omega_1, \Omega_2)$, if the hermitian kernel $K^f_{\Omega_1,\Omega_2)}$ defined as in (4.4.2) for $\eta, \zeta \in \rho(f)$ has a finite number of negative squares.

If $f \in \mathcal{S}_{<\infty}(\Omega_1, \Omega_2)$, we denote $\operatorname{ind}_- f := \operatorname{ind}_- K^f_{\Omega_1,\Omega_2}$. Moreover, we set

$$\mathcal{S}_\kappa(\Omega_1, \Omega_2) := \{f \in \mathcal{S}_{<\infty}(\Omega_1, \Omega_2) : \operatorname{ind}_- f = \kappa\}, \quad \kappa \in \mathbb{N}_0\,.$$

◊

Let us point out explicitly that $\mathcal{S}_0(\Omega_1, \Omega_2) = \mathcal{S}(\Omega_1, \Omega_2)$, i.e., that the notion of $\mathcal{S}_{<\infty}(\Omega_1, \Omega_2)$ indeed naturally includes the notion of $\mathcal{S}(\Omega_1, \Omega_2)$. Thereby, the inclusion "$\supseteq$" is clear from Theorem 4.4.6. For the reverse inclusion, a short argument is necessary. Namely, that a function $f \in \mathcal{S}_0(\Omega_1, \Omega_2)$ is automatically analytic in Ω_1. To show this, assume without loss of generality that $\Omega_2 = \mathbb{D}$. This can be done due to Lemma 4.4.5. If $K^f_{\Omega_1,\mathbb{D}}$ is positive semidefinite on $\rho(f)$, in particular, we have $|f(\zeta)| \le 1$, $\zeta \in \rho(f)$. From this it follows at once that f cannot have any poles in Ω_1.

$\boxed{\mathcal{N}_\kappa \text{ etc...}}$

$\boxed{\text{extend by Schwarz-reflection}}$

4.4.2 Hilbert space valued Nevanlinna functions

4.4.3 De Branges' spaces of entire functions

Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. Then we denote by $f^\#$ the function

$$f^\# : \begin{cases} \{\zeta \in \mathbb{C} : \overline{\zeta} \in \Omega\} & \to & \mathbb{C} \\ \zeta & \mapsto & \overline{f(\overline{\zeta})} \end{cases} .$$

Note that, if Ω is open, the function f is analytic in Ω if and only of $f^\#$ is analytic in its domain.

III.49. **4.4.10 Definition.** Let e be an entire function. Then we write $e \in \mathcal{HB}_{<\infty}$, and term e an *indefinite Hermite-Biehler function*, if e and $e^\#$ have no common nonreal zeros, and the hermitian kernel

$$K_e(\eta, \zeta) := \frac{i}{2} \frac{e(\zeta)e^\#(\overline{\eta}) - e^\#(\zeta)e(\overline{\eta})}{\zeta - \overline{\eta}}, \quad \zeta, \eta \in \mathbb{C}^+ ,$$

has a finite number of negative squares.

If $e \in \mathcal{HB}_{<\infty}$, we denote $\operatorname{ind}_- e := \operatorname{ind}_- K_e$. Moreover, we set

$$\mathcal{HB}_\kappa := \{e \in \mathcal{HB}_{<\infty} : \operatorname{ind}_- e = \kappa\} .$$

$\Diamond$

III.50. **4.4.11 Lemma.** *Let $e \in \mathbb{H}(\mathbb{C})$, $\kappa \in \mathbb{N}_0$, and denote*

$$a := \frac{e + e^\#}{2}, \quad b := i \frac{e - e^\#}{2} .$$

Then the following are equivalent.

(1) *The function e belongs to the $\mathcal{HB}_\kappa$.*

(2) *The functions e and $e^\#$ have no common nonreal zeros and $\frac{e^\#}{e}|_{\mathbb{C}^+} \in \mathcal{S}_\kappa(\mathbb{C}^+, \mathbb{D})$.*

(3) *The functions e and $e^\#$ have no common nonreal zeros and $\frac{b}{a}|_{\mathbb{C}^+} \in \mathcal{N}_\kappa$.*

Proof. The kernel K_e can be rewritten as

$$K_e(\eta, \zeta) = \frac{i}{2} \frac{e(\zeta)e^\#(\overline{\eta}) - e^\#(\zeta)e(\overline{\eta})}{\zeta - \overline{\eta}} = \overline{\Big(\frac{e(\eta)}{\sqrt{2}}\Big)} \frac{1 - \Big(\frac{e^\#(\zeta)}{e(\zeta)}\Big)\overline{\Big(\frac{e^\#(\eta)}{e(\eta)}\Big)}}{\zeta - \overline{\eta}} \frac{e(\zeta)}{\sqrt{2}} .$$

This shows that $K_e = \frac{e}{\sqrt{2}} \bullet K^{\frac{e^\#}{e}}_{\mathbb{C}^+, \mathbb{D}}$, and hence that

$$\operatorname{ind}_- K_e = \operatorname{ind}_- K^{\frac{e^\#}{e}}_{\mathbb{C}^+, \mathbb{D}} .$$

The equivalence of (1) and (2) follows.

Unfolding the definitions of a and b gives

$$K_e(\eta,\zeta) = \frac{b(\zeta)a(\overline{\eta}) - a(\zeta)b(\overline{\eta})}{\zeta-\overline{\eta}} = \overline{a(\eta)}\frac{\frac{b(\zeta)}{a(\zeta)} - \overline{\left(\frac{b(\eta)}{a(\eta)}\right)}}{\zeta-\overline{\eta}}a(\zeta)\,,$$

i.e., $K_e = a \bullet K^{\frac{b}{a}}_{\mathbb{C}^+,\mathbb{C}^+}$. This gives "(1) ⇔ (3)". ❑

Using ?THM? **??**, we obtain that the class $\mathcal{HB}_0$ coincides with what is classically called the *Hermite-Biehler class* (or *de Branges-class*).

III.51. **4.4.12 Corollary.** *Let $e \in \mathbb{H}(\mathbb{C})$. Then $e \in \mathcal{HB}_0$ if and only if either e and $e^\#$ are linearly dependent, or*

$$|e(\overline{\zeta})| < |e(\zeta)|, \quad \zeta \in \mathbb{C}^+\,.$$

❑

Let $e \in \mathbb{H}(\mathbb{C})$. The kernel K_e has a continuation to an analytic kernel on all of $\mathbb{C}$ (which we again denote by K_e). Namely,

$$K_e(\eta,\zeta) := \begin{cases} \dfrac{i}{2}\dfrac{e(\zeta)e^\#(\overline{\eta}) - e^\#(\zeta)e(\overline{\eta})}{\zeta-\overline{\eta}} &, \quad \zeta\neq\overline{\eta} \\ \frac{i}{2}\big[e'(\zeta)e^\#(\zeta) - (e^\#)'(\zeta)e(\zeta)\big], & \zeta = \overline{\eta} \end{cases}\,.$$

Provided that $e \in \mathcal{HB}_{<\infty}$, thus, the kernel K_e generates a reproducing kernel Pontryagin space of entire functions. We denote this space as $(\mathcal{K}(e), [.,.]_e)$. Note that $\mathcal{K}(e) = \{0\}$, if and only if the kernel $K_e(\eta,\zeta)$ vanishes identically, which is the case if and only if e and $e^\#$ are linearly dependent.

It is a central result that spaces of the form $\mathcal{K}(e)$ can be characterised axiomatically.

III.52. **4.4.13 Theorem.** *Let $(\mathcal{A}, [.,.])$ be a reproducing kernel Pontryagin space of entire functions. Then there exists $e \in \mathcal{HB}_{<\infty}$ such that $(\mathcal{A}, [.,.]) = (\mathcal{K}(e), [.,.]_e)$, if and only if*

(1) *If $f \in \mathcal{A}$, $\eta \in \mathbb{C}\setminus\mathbb{R}$, and $f(\eta) = 0$, then $\frac{f(\zeta)}{\zeta-\eta} \in \mathcal{A}$.*

(2) *If $f, g \in \mathcal{A}$, $\eta \in \mathbb{C}\setminus\mathbb{R}$, and $f(\eta) = g(\overline{\eta}) = 0$, then*

$$\Big[\frac{\zeta-\overline{\eta}}{\zeta-\eta}f(\zeta), g(\zeta)\Big] = \Big[f(\zeta), \frac{\zeta-\eta}{\zeta-\overline{\eta}}g(\zeta)\Big]. \tag{4.4.5}$$ III.37

(3) *If $f \in \mathcal{A}$, then $f^\# \in \mathcal{A}$. We have*

$$[f,g] = [g^\#, f^\#], \quad f,g \in \mathcal{A}\,. \tag{4.4.6}$$ III.55

Note that, in conjunction with (1), the condition (4.4.5) is meaningful: Since $\frac{\zeta-\overline{\eta}}{\zeta-\eta} = 1 + (\eta-\overline{\eta})\frac{1}{\zeta-\eta}$, validity of (1) implies that $\frac{\zeta-\overline{\eta}}{\zeta-\eta}f(\zeta) \in \mathcal{A}$ if $\eta \in \mathbb{C}\setminus\mathbb{R}$ and $f(\eta) = g(\overline{\eta}) = 0$. The same argument applies with $\overline{\eta}$ and g in place of η and f.

In the proof of this theorem we use the following elementary consequence of analyticity.

III.58. **4.4.14 Lemma.** *Let Ω be a nonempty, open and connected subset of $\mathbb{C}$, set $\Omega' := \{\zeta \in \mathbb{C} : \overline{\zeta} \in \Omega\}$, and let $f : \Omega' \times \Omega \to \mathbb{C}$ be an analytic function. Moreover, let Ω_0 be a nonempty open subset of Ω.*

If $f(\overline{\zeta}, \zeta) = 0$, $\zeta \in \Omega_0$, then f vanishes identically.

Proof. Fix $\xi \in \Omega_0$. The function f admits a power series expansion in some neighbourhood of the point $(\overline{\xi}; \xi) \in \Omega' \times \Omega$. Say, we have

$$f(\eta, \zeta) = \sum_{i,j=0}^{\infty} \alpha_{ij}(\eta - \overline{\xi})^i (\zeta - \xi)^j, \quad |\eta - \overline{\xi}| < R, |\zeta - \xi| < R\,, \tag{4.4.7}$$ III.65

where $\alpha_{ij} \in \mathbb{C}$ and where $R > 0$ is chosen such that the polydisk $\{(\eta; \zeta) \in \mathbb{C} \times \mathbb{C} : |\eta - \overline{\xi}| < R, |\zeta - \xi| < R\}$ is contained in $\Omega_0' \times \Omega_0$ (here $\Omega_0' := \{\zeta \in \mathbb{C} : \overline{\zeta} \in \Omega_0\}$).

Assume that $\alpha_{ij} \neq 0$ for some $(i; j) \in \mathbb{N}_0 \times \mathbb{N}_0$, and set

$$n := \min\{i + j : i, j \in \mathbb{N}_0, \alpha_{ij} \neq 0\}\,.$$

Then we can write

$$\begin{aligned} f(\eta, \zeta) = (\eta - \overline{\xi})^n \Big(\sum_{i=n}^{\infty} \sum_{j=0}^{\infty} \alpha_{ij} (\eta - \overline{\xi})^{i-n} (\zeta - \xi)^j \Big) + \\ + \sum_{k=1}^{n} (\eta - \overline{\xi})^{n-k} (\zeta - \xi)^k \Big(\sum_{j=k}^{\infty} \alpha_{n-k,j} (\zeta - \xi)^{j-k} \Big), \quad |\eta - \overline{\xi}| < R, |\zeta - \xi| < R\,. \end{aligned} \tag{4.4.8}$$ III.59

For $\theta \in \mathbb{R}$ and $r \in (0, R)$, set $\zeta_r(\theta) := \xi + re^{i\theta}$. For all such values of θ and r, we have $\frac{1}{r^n} f(\overline{\zeta_r(\theta)}, \zeta_r(\theta)) = 0$. Passing to the limit $r \downarrow 0$ and using (4.4.8) gives

$$0 = e^{-in\theta}\alpha_{n0} + \sum_{k=1}^{n} e^{i(-n+2k)\theta} \alpha_{n-k,k}, \quad \theta \in \mathbb{R}\,.$$

Note here that the series (4.4.7) converges absolutely and locally uniformly.

Since the functions $e^{i(-n+2k)\theta}$, $k = 0, \ldots, n$, are linearly independent, it follows that

$$\alpha_{n0} = \alpha_{n-1,1} = \ldots = \alpha_{0n} = 0\,.$$

This contradicts our choice of n.

It follows that $\alpha_{ij} = 0$ for all $i, j \in \mathbb{N}_0$, i.e., the function f vanishes identically on the polydisk $\{(\eta; \zeta) \in \Omega' \times \Omega : |\eta - \overline{\xi}| < R, |\zeta - \xi| < R\}$. By the identity theorem it thus vanishes identically on $\Omega' \times \Omega$. ❑

Proof (of Theorem 4.4.13*).* In the first half of the proof we establish sufficiency. Assume that (1)–(3) holds; we have to construct $e \in \mathcal{HB}_{<\infty}$ such that $\mathcal{A} = \mathcal{K}(e)$. Equivalently, we have to find $e \in \mathcal{HB}_{<\infty}$ such that $K(\eta, \zeta) = K_e(\eta, \zeta)$, $\eta, \zeta \in \mathbb{C}$. Here, and in the following, we denote by $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ the reproducing kernel of $\mathcal{A}$.

If $\mathcal{A} = \{0\}$ there is nothing to proof. In fact, in this case, we have $\mathcal{A} = \mathcal{K}(e)$ whenever e is chosen such that e and $e^\#$ are linearly dependent (e.g., for $e := 1$). Hence, we may assume that $\mathcal{A} \neq \{0\}$. As a consequence, we know that the set $\{\eta \in \mathbb{C} : K(\eta, .) = 0\}$ has no finite accumulation point.

Before we define e, let us provide some simple consequences of the axioms (1)–(3).

- We show that, for no $\eta \in \mathbb{C} \setminus \mathbb{R}$, the function $K(\eta, .)$ vanishes identically. Choose $f \in \mathcal{A} \setminus \{0\}$, and denote by n the multiplicity of η as a zero of f. By (1), we have $g(\zeta) := (\zeta - \eta)^{-n} f(\zeta) \in \mathcal{A}$. Since $\chi_\eta(g) \neq 0$, it follows that $K(\eta, .) \neq 0$.
- We show that, for each $\eta \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$,

$$\Big[\frac{\alpha\zeta + \beta}{\zeta - \eta} f(\zeta), g(\zeta)\Big] = \Big[f(\zeta), \frac{\overline{\alpha}\zeta + \overline{\beta}}{\zeta - \overline{\eta}} g(\zeta)\Big], \quad f, g \in \mathcal{A},\ f(\eta) = g(\overline{\eta}) = 0\,. \quad (4.4.9)$$ III.66

  To see this, write

$$\frac{\alpha\zeta + \beta}{\zeta - \eta} = \frac{\alpha\eta + \beta}{\eta - \overline{\eta}} \frac{\zeta - \overline{\eta}}{\zeta - \eta} - \frac{\alpha\overline{\eta} + \beta}{\eta - \overline{\eta}}, \quad \frac{\overline{\alpha}\zeta + \overline{\beta}}{\zeta - \overline{\eta}} = \frac{\overline{\alpha}\overline{\eta} + \overline{\beta}}{\overline{\eta} - \eta} \frac{\zeta - \eta}{\zeta - \overline{\eta}} - \frac{\overline{\alpha}\eta + \overline{\beta}}{\overline{\eta} - \eta}\,.$$

  It follows from the argument brought after the statement of the theorem and condition (2) that $\frac{\alpha\zeta+\beta}{\zeta-\eta} f(\zeta), \frac{\overline{\alpha}\zeta+\overline{\beta}}{\zeta-\overline{\eta}} g(\zeta) \in \mathcal{A}$ and that (4.4.9) holds.
- We show that $K(\eta, .)^\# = K(\overline{\eta}, .)$, $\eta \in \mathbb{C}$. To see this, it is enough to use (3) to compute

$$\chi_{\overline{\eta}}(f) = f(\overline{\eta}) = \overline{f^\#(\eta)} = \overline{[f^\#, K(\eta, .)]} = [f, K(\eta, .)^\#], \quad f \in \mathcal{A}\,.$$

The function $(\eta; \zeta) \mapsto K(\overline{\eta}, \zeta)$ is analytic on $\mathbb{C}^- \times \mathbb{C}^+$. Hence, we may apply Lemma 4.4.14, and obtain a point $\xi \in \mathbb{C}^+$ with $K(\xi, \xi) \neq 0$. We have $K(\overline{\zeta}, \eta) = \overline{K(\eta, \overline{\zeta})} = K(\overline{\eta}, \zeta)$, $\eta, \zeta \in \mathbb{C}$. Setting $\eta = \overline{\xi}$, $\zeta = \xi$, gives $K(\overline{\xi}, \overline{\xi}) = K(\xi, \xi)$. By passing from ξ to $\overline{\xi}$ if necessary, we thus obtain a point

$$\xi \in \mathbb{C} \setminus \mathbb{R} \quad \text{with} \quad K(\xi, \xi) \operatorname{Im} \xi > 0\,.$$

Set (the required function $e(\zeta)$ will be a scalar multiple of this one)

$$e_0(\zeta) := (\zeta - \overline{\xi}) K(\xi, \zeta), \quad \zeta \in \mathbb{C}\,.$$

We have $e_0^\#(\zeta) = (\zeta - \xi) K(\overline{\xi}, \zeta)$, and hence can compute

$$\begin{aligned}
K_{e_0}(\eta, \zeta) &= \frac{e_0(\zeta) e_0^\#(\overline{\eta}) - e_0^\#(\zeta) e_0(\overline{\eta})}{\zeta - \overline{\eta}} = \\
&= \frac{1}{\zeta - \overline{\eta}} \Big[(\zeta - \overline{\xi}) K(\xi, \zeta) \cdot (\overline{\eta} - \xi) K(\overline{\xi}, \overline{\eta}) - (\zeta - \xi) K(\overline{\xi}, \zeta) \cdot (\overline{\eta} - \overline{\xi}) K(\xi, \overline{\eta})\Big] = \\
&= \frac{1}{\zeta - \overline{\eta}} \Big[(\zeta\overline{\eta} - \zeta\xi - \overline{\xi}\overline{\eta} + |\xi|^2) K(\xi, \zeta) K(\overline{\xi}, \overline{\eta}) - (\zeta\overline{\eta} - \zeta\overline{\xi} - \xi\overline{\eta} + |\xi|^2) K(\overline{\xi}, \zeta) K(\xi, \overline{\eta})\Big] = \\
&= (*)
\end{aligned}$$

Using

$$\frac{\zeta\overline{\eta}-\zeta\xi-\overline{\xi}\overline{\eta}+|\xi|^2}{\zeta-\overline{\eta}}=\frac{\zeta\overline{\eta}-\overline{\eta}(\overline{\xi}+\xi)+|\xi|^2}{\zeta-\overline{\eta}}-\xi\,,$$
$$\frac{\zeta\overline{\eta}-\zeta\overline{\xi}-\xi\overline{\eta}+|\xi|^2}{\zeta-\overline{\eta}}=\frac{\zeta\overline{\eta}-\overline{\eta}(\overline{\xi}+\xi)+|\xi|^2}{\zeta-\overline{\eta}}-\overline{\xi}\,,$$

we further rewrite

$$(*)=\frac{\zeta\overline{\eta}-\overline{\eta}(\overline{\xi}+\xi)+|\xi|^2}{\zeta-\overline{\eta}}\Big[K(\xi,\zeta)K(\overline{\xi},\overline{\eta})-K(\overline{\xi},\zeta)K(\xi,\overline{\eta})\Big]-$$
$$-\,\xi K(\xi,\zeta)K(\overline{\xi},\overline{\eta})-\overline{\xi}K(\overline{\xi},\zeta)K(\xi,\overline{\eta})\,.$$

It follows that, for each $f\in\mathcal{A}$ with $f(\eta)=0$,

$$\Big[f(\zeta),\frac{e_0(\zeta)e_0^\#(\overline{\eta})-e_0^\#(\zeta)e_0(\overline{\eta})}{\zeta-\overline{\eta}}\Big]=$$
$$=\Big[\frac{\zeta\eta-\eta(\xi+\overline{\xi})+|\xi|^2}{\zeta-\eta}f(\zeta),K(\xi,\zeta)K(\overline{\xi},\overline{\eta})-K(\overline{\xi},\zeta)K(\xi,\overline{\eta})\Big]-$$
$$-\,\overline{\xi}f(\xi)\overline{K(\overline{\xi},\overline{\eta})}+\xi f(\overline{\xi})\overline{K(\xi,\overline{\eta})}=$$
$$=\frac{\xi\eta-\eta(\xi+\overline{\xi})+|\xi|^2}{\xi-\eta}f(\xi)\overline{K(\overline{\xi},\overline{\eta})}-\frac{\overline{\xi}\eta-\eta(\xi+\overline{\xi})+|\xi|^2}{\overline{\xi}-\eta}f(\overline{\xi})\overline{K(\xi,\overline{\eta})}-$$
$$-\,\overline{\xi}f(\xi)\overline{K(\overline{\xi},\overline{\eta})}+\xi f(\overline{\xi})\overline{K(\xi,\overline{\eta})}=0\,,$$

i.e., $K_{e_0}(\eta,.)\in(\ker\chi_\eta)^{[\perp]}$. Since $\dim(\ker\chi_\eta)^{[\perp]}\in\{0,1\}$, we find $\gamma:\mathbb{C}\to\mathbb{C}$ such that

$$K_{e_0}(\eta,\zeta)=\gamma(\eta)K(\eta,\zeta),\quad \eta,\zeta\in\mathbb{C}\,.$$

Thereby, the number $\gamma(\eta)$ is uniquely determined if $\ker\chi_\eta\neq\mathcal{A}$ since then $(\ker\chi_\eta)^{[\perp]}=\operatorname{span}\{K(\eta,.)\}$, and arbitrary if $\ker\chi_\eta=\mathcal{A}$ since then $K_{e_0}(\eta,.)=K(\eta,.)=0$.

Since K_{e_0} and K are both hermitian, we have

$$\gamma(\eta)K(\eta,\zeta)=K_{e_0}(\eta,\zeta)=\overline{K_{e_0}(\zeta,\eta)}=\overline{\gamma(\zeta)K(\zeta,\eta)}=\overline{\gamma(\zeta)}K(\eta,\zeta),\quad \eta,\zeta\in\mathbb{C}\,.$$

Let $\eta_1,\eta_2\in\mathbb{C}$ be such that $\ker\chi_{\eta_i}\neq\mathcal{A}$, $i=1,2$. Then we may choose $\zeta\in\mathbb{C}$ with $K(\eta_i,\zeta)\neq 0$, $i=1,2$. It follows that

$$\gamma(\eta_1)=\overline{\gamma(\zeta)}=\gamma(\eta_2)\,.$$

We see that $\gamma(\eta)$ is constant on $\{\eta\in\mathbb{C}:\ker\chi_\eta\neq\mathcal{A}\}$. On the complement of this set the choice of $\gamma(\eta)$ is arbitrary, and we conclude that (with $\gamma:=\gamma(\eta)$, $\ker\chi_\eta\neq\mathcal{A}$)

$$K_{e_0}(\eta,\zeta)=\gamma K(\eta,\zeta),\quad \eta,\zeta\in\mathbb{C}\,.$$

Setting $\eta=\zeta=\xi$ in this this relation gives $\gamma=K(\xi,\xi)^{-1}K_{e_0}(\xi,\xi)$. Since

$$K_{e_0}(\xi,\xi)=\frac{i}{2}\frac{|e_0(\xi)|^2-|e_0^\#(\xi)|^2}{\xi-\overline{\xi}}=\frac{\big|(\xi-\overline{\xi})K(\xi,\xi)\big|^2}{4\operatorname{Im}\xi}=\frac{|\operatorname{Im}\xi|^2\cdot|K(\xi,\xi)|^2}{\operatorname{Im}\xi}\,,$$

and we made the choice of ξ such that $K(\xi,\xi)\operatorname{Im}\xi > 0$, it follows that

$$\gamma = |\operatorname{Im}\xi| \cdot |K(\xi,\xi)| > 0\,.$$

Now we define

$$e(\zeta) := \frac{1}{\sqrt{\gamma}} e_0(\zeta), \quad \zeta \in \mathbb{C}\,,$$

and obtain

$$K_e(\eta,\zeta) = \frac{1}{\gamma} K_{e_0}(\eta,\zeta) = K(\eta,\zeta), \quad \eta,\zeta \in \mathbb{C}\,.$$

Thus, we readily obtain that $\operatorname{ind}_- K_e < \infty$ and $\mathcal{K}(K_e) = \mathcal{A}$. Let $\eta \in \mathbb{C} \setminus \mathbb{R}$. Then $K_e(\overline{\eta}, .)$ does not vanish identically (since $K(\overline{\eta}, .)$ does not). If we had $e(\eta) = e^{\#}(\eta) = 0$, however, it would follow that $K_e(\overline{\eta},\zeta) = 0$, $\zeta \in \mathbb{C}$. We conclude that indeed $e \in \mathcal{HB}_{<\infty}$.

In the second half of the proof we deal with necessity. Assume that $e \in \mathcal{HB}_{<\infty}$, that e and $e^{\#}$ are not linearly dependent, and consider the space $\mathcal{A} := \mathcal{K}(e)$. As a first step towards the proof of properties (1) and (2), we show the identity

$$\Big(K_e(\eta,\zeta) - \frac{K_e(\xi,\zeta)K_e(\eta,\xi)}{K_e(\xi,\xi)}\Big)\frac{\zeta-\overline{\xi}}{\zeta-\xi} = \Big(K_e(\eta,\zeta) - \frac{K_e(\overline{\xi},\zeta)K_e(\eta,\overline{\xi})}{K_e(\overline{\xi},\overline{\xi})}\Big)\frac{\overline{\eta}-\overline{\xi}}{\overline{\eta}-\xi}\,,$$
$$\zeta \in \mathbb{C}, \quad \eta,\xi \in \mathbb{C},\ \xi \neq \overline{\eta},\ K(\xi,\xi) \neq 0\,. \qquad (4.4.10)$$ III.60

The proof is elementary, but somewhat tedious. The left side is an analytic function of ζ, and hence it is enough to check (4.4.10) for $\zeta \neq \xi$. We have

$$(\zeta-\overline{\eta})(\xi-\overline{\xi}) - (\zeta-\overline{\xi})(\xi-\overline{\eta}) + (\zeta-\xi)(\overline{\xi}-\overline{\eta}) = 0\,,$$

and hence

$$1 = \frac{(\zeta-\overline{\xi})(\xi-\overline{\eta}) - (\zeta-\xi)(\overline{\xi}-\overline{\eta})}{(\zeta-\overline{\eta})(\xi-\overline{\xi})}\,. \qquad (4.4.11)$$ III.61

Set

$$L(\eta,\zeta) := (\zeta-\overline{\eta})K_e(\eta,\zeta) = e(\zeta)e^{\#}(\overline{\eta}) - e^{\#}(\zeta)e(\overline{\eta})\,.$$

The following relation is checked by unfolding the definition of L:

$$L(\eta,\zeta)L(\xi,\xi) - L(\xi,\zeta)L(\eta,\xi) + L(\overline{\xi},\zeta)L(\eta,\overline{\xi}) = 0\,. \qquad (4.4.12)$$ III.62

Using (4.4.11), (4.4.12), and dividing by $(\zeta-\xi)(\xi-\overline{\eta})$ yields

$$\frac{L(\eta,\zeta)}{\zeta-\overline{\eta}}\frac{L(\xi,\xi)}{\xi-\overline{\xi}}\Big(\frac{\zeta-\overline{\xi}}{\zeta-\xi} - \frac{\overline{\xi}-\overline{\eta}}{\xi-\overline{\eta}}\Big) - \frac{\zeta-\overline{\xi}}{\zeta-\xi}\frac{L(\xi,\zeta)}{\zeta-\overline{\xi}}\frac{L(\eta,\xi)}{\xi-\overline{\eta}} + \frac{L(\overline{\xi},\zeta)}{\zeta-\xi}\frac{L(\eta,\overline{\xi})}{\overline{\xi}-\overline{\eta}}\frac{\overline{\xi}-\overline{\eta}}{\xi-\overline{\eta}} = 0$$

Since $K_e(\overline{\xi},\overline{\xi}) = K_e(\xi,\xi)$, the relation (4.4.10) follows.

Now fix $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $K_e(\xi,\xi) \neq 0$. Such a choice is possible by Lemma 4.4.14. Then $\ker\chi_\xi$ and $\ker\chi_{\overline{\xi}}$ are closed and nondegenerated subspaces of $\mathcal{A}$. We have the decompositions

$$\mathcal{A} = \ker\chi_\xi[+]\operatorname{span}\{K_e(\xi,.)\} = \ker\chi_{\overline{\xi}}[+]\operatorname{span}\{K_e(\overline{\xi},.)\}\,.$$

The orthogonal projections P of $\mathcal{A}$ onto $\ker\chi_\xi$ and Q of $\mathcal{A}$ onto $\ker\chi_{\overline{\xi}}$ act as

$$(Pf)(\zeta) = f(\zeta) - f(\xi)\frac{K_e(\xi,.)}{K_e(\xi,\xi)}, \quad (Qf)(\zeta) = f(\zeta) - f(\overline{\xi})\frac{K_e(\overline{\xi},.)}{K_e(\overline{\xi},\overline{\xi})}\,.$$

Set $\mathcal{L} := \operatorname{span}\{K(\eta, .) : \eta \in \mathbb{R}\}$. Then $\mathcal{L}$ is dense in $\mathcal{A}$, and hence $P(\mathcal{L})$ is dense in $\ker \chi_\xi$ and $Q(\mathcal{L})$ is dense in $\ker \chi_{\overline{\xi}}$. Consider the linear subspace

$$V_0 := \Big\{(f; g) \in \mathcal{A} \times \mathcal{A} : f \in \P(\mathcal{L}), g(\zeta) = \frac{\zeta - \overline{\xi}}{\zeta - \xi} f(\zeta)\Big\}.$$

Then, by (4.4.10),

$$V_0 = \operatorname{span}\Big\{\Big(PK_e(\eta, .); \frac{\overline{\xi} - \overline{\eta}}{\xi - \overline{\eta}} QK_e(\eta, .)\Big) : \eta \in \mathbb{R}\Big\}.$$

Moreover, $\operatorname{dom} V_0 = P(\mathcal{L})$ and $\operatorname{ran} V_0 = Q(\mathcal{L})$. Again using (4.4.10), we can check that V_0 is isometric:

$$\begin{aligned}
&\Big[\frac{\overline{\xi} - \overline{\eta}}{\xi - \overline{\eta}} QK_e(\eta, .), \frac{\overline{\xi} - \overline{\eta'}}{\xi - \overline{\eta'}} QK_e(\eta', .)\Big] = \frac{\overline{\xi} - \overline{\eta}}{\xi - \overline{\eta}} \frac{\xi - \eta'}{\overline{\xi} - \eta'} [QK_e(\eta, .), K_e(\eta', .)] = \\
&\quad = \frac{\overline{\xi} - \overline{\eta}}{\xi - \overline{\eta}} \frac{\xi - \eta'}{\overline{\xi} - \eta'} \Big(K_e(\eta, \eta') - \frac{K_e(\overline{\xi}, \eta') K_e(\eta, \overline{\xi})}{K_e(\overline{\xi}, \overline{\xi})}\Big) \overset{(4.4.10)}{=} \Big(K_e(\eta, \eta') - \frac{K_e(\xi, \eta') K_e(\eta, \xi)}{K_e(\xi, \xi)}\Big) = \\
&\quad = [PK_e(\eta, .), K_e(\eta', .)] = [PK_e(\eta, .), PK_e(\eta', .)].
\end{aligned}$$

Now we apply Theorem 2.5.14. This provides us with a linear, isometric, surjective and continuous map V of $\ker \chi_\xi$ onto $\ker \chi_{\overline{\xi}}$ with graph $V = \overline{V_0}$. Since point evaluations are continuous, V acts as

$$(Vf)(\zeta) = \frac{\zeta - \overline{\xi}}{\zeta - \xi} f(\zeta), \quad f \in \ker \chi_\xi.$$

Clearly, V is injective and $V^{-1} : \ker \chi_{\overline{\xi}} \to \ker \chi_\xi$ acts as

$$(V^{-1} f)(\zeta) = \frac{\zeta - \xi}{\zeta - \overline{\xi}} f(\zeta), \quad f \in \ker \chi_{\overline{\xi}}.$$

The required properties (1) and (2) readily follow for the base point ξ:

$$\frac{f(\zeta)}{\zeta - \xi} = \frac{1}{\xi - \overline{\xi}} \Big((Vf)(\zeta) - f(\zeta)\Big), \quad f \in \ker \chi_\xi,$$

in particular, $\frac{f(\zeta)}{\zeta - \xi} \in \mathcal{A}$. Moreover,

$$\Big[\frac{\zeta - \overline{\xi}}{\zeta - \xi} f(\zeta), g(\zeta)\Big] = [Vf, V(V^{-1} f)] = [f, V^{-1} g] = \Big[f(\zeta), \frac{\zeta - \xi}{\zeta - \overline{\xi}} g(\zeta)\Big],$$
$$f \in \ker \chi_\xi, \; g \in \ker \chi_{\overline{\xi}}.$$

In order to transfer this knowledge to other base points $\eta \in \mathbb{C} \setminus \mathbb{R}$, we proceed with an operator theoretic argument. For convenience, we reduce to the Hilbert space case. Since $\bigcap_{\eta \in \mathbb{R}} \ker \chi_\eta = \{0\}$, we may choose points $\eta_1, \dots, \eta_N \in \mathbb{R}$ and $\gamma > 0$ such that

$$(f, g)_+ := [f, g] + \gamma \sum_{j=1}^{N} f(\eta_j) \overline{g(\eta_j)}, \quad f, g \in \mathcal{A},$$

is a compatible Hilbert space scalar product, cf. Corollary 4.1.14. Since $\left|\frac{\zeta-\overline{\xi}}{\zeta-\xi}\right| = 1$ for $\zeta \in \mathbb{R}$, the operator V is also isometric w.r.t. $(.,.)_+$.

The inverse Cayley transform $S := (\xi V - \overline{\xi} I)(V - I)^{-1}$ is closed, symmetric, and

$$\operatorname{ran}(S-\xi) = \operatorname{dom} V = \ker \chi_\xi, \quad \operatorname{ran}(S-\overline{\xi}) = \operatorname{ran} V = \ker \chi_{\overline{\xi}}. \tag{4.4.13}$$ III.63

We have

$$\Big(\xi\frac{\zeta-\overline{\xi}}{\zeta-\xi} - \overline{\xi}\Big)\Big(\frac{\zeta-\overline{\xi}}{\zeta-\xi} - 1\Big)^{-1} = \zeta\,,$$

and hence $(Sf)(\zeta) = \zeta f(\zeta)$, $f \in \operatorname{dom} S$. Clearly, thus, $\operatorname{ran}(S-\eta) \subseteq \ker\chi_\eta$, $\eta \in \mathbb{C}$. For $\eta \in \mathbb{C}\setminus\mathbb{R}$ the spaces $\operatorname{ran}(S-\eta)$ are closed and $\dim\big[\mathcal{A}/\operatorname{ran}(S-\eta)\big]$ is constant on the half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$. In view of (4.4.13) we thus have $\dim\big[\mathcal{A}/\operatorname{ran}(S-\eta)\big] = 1$, $\eta \in \mathbb{C}\setminus\mathbb{R}$. It follows that

$$\operatorname{ran}(S-\eta) = \ker\chi_\eta, \quad \eta \in \mathbb{C}\setminus\mathbb{R}\,.$$

Denote by V_η the Cayley transform of S w.r.t. the base point η, i.e., $V_\eta := (S-\overline{\eta})(S-\eta)^{-1}$. Then V_η is isometric, acts as $(V_\eta f)(\zeta) = \frac{\zeta-\overline{\eta}}{\zeta-\eta} f(\zeta)$, and

$$\operatorname{dom} V_\eta = \operatorname{ran}(S-\eta) = \ker\chi_\eta, \quad \operatorname{ran} V_\eta = \operatorname{ran}(S-\overline{\eta}) = \ker\chi_{\overline{\eta}}\,.$$

It follows that (1) and (2) hold with the base point η.

It remains to show (3). The method is the same, but less difficult. Set $\mathcal{L} := \{K(\eta,.) : \eta \in \mathbb{C}\}$. Since, obviously, $K_e(\eta,.)^\# = K_e(\overline{\eta},.)$, $\eta \in \mathbb{C}$, the mapping rule $f \mapsto f^\#$ maps $\mathcal{L}$ onto itself. The map

$$T_0 : \begin{cases} \mathcal{L} & \to \mathcal{L} \\ f & \mapsto \quad f^\# \end{cases}$$

is conjugate linear. Since

$$\begin{aligned}\big[K_e(\eta,.)^\#, K_e(\eta',.)^\#\big] = \big[K_e(\overline{\eta},.), K_e(\overline{\eta'},.)\big] = K_e(\overline{\eta},\overline{\eta'}) = \\ = \overline{K_e(\eta,\eta')} = \big[K_e(\eta',.), K_e(\eta,.)\big], \quad \eta,\eta' \in \mathbb{C}\,,\end{aligned}$$

it satisfies

$$[T_0 f, T_0 g] = [g, f], \quad f, g \in \mathcal{L}\,.$$

The space $\mathcal{L}$ is dense in $\mathcal{K}(e)$. Let $\mathcal{B}$ be the linear space $\mathcal{K}(e)$ endowed with the conjugate linear operations and scalar product. Theorem 2.5.14 applied with the map T_0 considered as a map of $\mathcal{L} \subseteq \mathcal{K}(e)$ onto $\mathcal{L} \subseteq \mathcal{B}$, provides a conjugate linear continuous extension $T : \mathcal{K}(e) \to \mathcal{K}(e)$ of T_0 which again satisfies $[Tf, Tg] = [g, f]$, $f, g \in \mathcal{K}(e)$. Since point evaluations are continuous, this extension acts as $(Tf)(\zeta) = f^\#(\zeta)$. The required property (3) follows. ❑

III.53. **4.4.15 Definition.** We refer to $(\mathcal{A}, [.,.])$ as a *de Branges-Pontryagin space* (or *dB-Pontryagin space*, for short), if $(\mathcal{A}, [.,.])$ is a reproducing kernel Pontryagin space of entire functions and possesses the properties (1), (2), (3) of Theorem 4.4.13.

If $\mathcal{A}$ is a Hilbert space subject to these conditions, we speak of a *de Branges-Hilbert space* (or *dB-Hilbert space*, for short). ◊

By Theorem 4.4.13, each dB-Pontryagin space can be realized as $\mathcal{K}(e)$ with some $e \in \mathcal{H}B_{<\infty}$. As it is apparent from the construction in the proof of Theorem 4.4.13, this function e is not unique. However, what can be said is the following.

III.67. **4.4.16 Lemma.** *Assume that $e_1, e_2 \in \mathcal{H}B_{<\infty}$ with $\mathcal{K}(e_1) = \mathcal{K}(e_2)$, and there exists $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $e_1(\xi) = e_2(\xi) = 0$. Then $\frac{e_1}{e_2}$ is a unimodular constant.*

Proof. Observe that

$$K(\overline{\xi}, \zeta) = K_{e_i}(\overline{\xi}, \zeta) = e_i(\zeta) \cdot \frac{i}{2} \frac{e_i^\#(\xi)}{\zeta - \xi}, \; \zeta \in \mathbb{C}, \quad i = 1, 2 .$$

Hence, the quotient $\frac{e_1}{e_2}$ is constant. Evaluating $K(\xi, \xi)$, yields

$$K(\xi, \xi) = -\frac{i}{2} \frac{|e(\overline{\xi})|^2}{\xi - \overline{\xi}}, \quad i = 1, 2 ,$$

and hence this constant is unimodular. ❑

As a first example, let us again revisit the Paley-Wiener spaces $\mathcal{P}W_a$ which were introduced in Example 2.5.6 and further studied in Example 4.1.6.

III.54. *4.4.17 Example.* Let $a > 0$, and consider the Paley-Wiener space $\mathcal{P}W_a$. We already saw that $\mathcal{P}W_a$ is a reproducing kernel Hilbert space of entire functions. Moreover, we have computed its reproducing kernel; which turned out to be (for $\zeta = \overline{\eta}$ this formula is understood appropriately as a derivative)

$$K(\eta, \zeta) = \frac{\sin a(\zeta - \overline{\eta})}{\pi(\zeta - \overline{\eta}}, \quad \eta, \zeta \in \mathbb{C} .$$

As a short computation shows, we have $K(\eta, \zeta) = K_e(\eta, \zeta)$ with the function $e(\zeta) := \frac{1}{\sqrt{\pi}} e^{-ia\zeta}$. Clearly, this function belongs to the Hermite-Biehler class. We conclude that $\mathcal{P}W_a$ is a dB-Hilbert space.

It is interesting to directly check the axioms Theorem 4.4.13, (1), (2),(3), proceeding via our original definition of $\mathcal{P}W_a$ as the Fourier image of $L^2(-a, a)$. In fact, if $f \in L^2(-a, a)$, $\eta \in \mathbb{C} \setminus \mathbb{R}$, and $\hat{f}(\eta) = 0$, then we can compute $(F(\tau) := \int_{-a}^{\tau} f(\sigma) e^{-i\sigma\eta} \, d\sigma)$

$$\begin{aligned} \frac{\hat{f}(\zeta)}{\zeta - \eta} =& \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(\tau) \frac{e^{-i\tau\zeta}}{\zeta - \eta} \, d\tau = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(\tau) e^{-i\tau\eta} \cdot \frac{e^{-i\tau(\zeta-\eta)}}{\zeta - \eta} \, d\tau = \\ =& \frac{1}{\sqrt{2\pi}} \Big[F(\tau) \frac{e^{-i\tau(\zeta-\eta)}}{\zeta - \eta} \Big] \Big|_{\zeta=-a}^{a} - \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} F(\tau)(-i) e^{-i\tau(\zeta-\eta)} \, d\tau = \\ =& \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} iF(\tau) e^{i\tau\eta} \cdot e^{-i\tau\zeta} \, d\tau = [\widehat{iF(\tau) e^{i\tau\eta}}](\zeta) . \end{aligned}$$

This shows that $\frac{\hat{f}(\zeta)}{\zeta - \eta} \in \mathcal{P}W_a$. Moreover, we have

$$(\hat{f})^\#(\zeta) = \overline{\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(\tau) e^{-ia\overline{\zeta}} \, d\tau} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \overline{f(\tau)} e^{ia\zeta} \, d\tau = [\widehat{\overline{f(-\tau)}}](\zeta) ,$$

and hence $(\hat{f})^\# \in \mathcal{P}W_a$.

The Fourier transform is an isometry of $L^2(-a,a)$ into $L^2(\mathbb{R})$ and, by definition, an isometry of $L^2(-a,a)$ onto $\mathcal{P}W_a$. Hence, the scalar product of $\mathcal{P}W_a$ can be computed as an integral along the real line, namely as

$$(f,g) = \int_{-a}^{a} f(\tau)\overline{g(\tau)}\,d\tau, \quad f,g \in \mathcal{P}W_a\,.$$

From this the relations (4.4.5) and (4.4.6) follow immediately. ◊

example: bessel?

4.4.4 Entire J-inner matrix functions

4.5 *Reproducing kernel almost Pontryagin spaces

section5.5

Examples: adjoint of multiplication,composition

Index

$A \times D$, 99
$B(\mathcal{X})$, 107
$B(\mathcal{X}, \mathcal{Y})$, 107
$C_{\rho(T)}(z)$, 114
$I_{\mathcal{X}}$, 95
$K \boxdot \lambda$, 165
$T + \alpha$, 97
$[.,.]$, 1
$[.,.]$-orthogonal, 8
$\mathrm{ind}_+ G$, 30, 32
$\mathrm{ind}_- G$, 30, 32
$\mathrm{ind}_0 G$, 30
$\mathrm{ind}_+ \mathcal{V}$, 28
$\mathrm{ind}_0 \mathcal{L}$, 9
$\mathrm{ind}_0 \mathcal{V}$, 8
$\mathfrak{d}$, 148
$\mathcal{H}B_{<\infty}$, 176
$\mathbb{1}$, 54
C, 173
$\mathcal{K}(e)$, 177
$\mathcal{N}$, 172
$\mathcal{S}$, 172
$\mathcal{S}(\Omega_1, \Omega_2)$, 172
$h \bullet K$, 166
$\mathcal{M}(q, C)$, 148
$(\mathcal{A}, O)'$, 57
$E_{x,x}$, 54
$M[\perp]N$, 8
$\mathcal{P}W_a$, 72
$\mathbb{C}$, 1
$\mathbb{C}^+$, 84
$\mathbb{C}^n$, 1
$\mathbb{C}^{n\times m}$, 1
$\delta_{\{n\}}$, 69
δ_{kj}, 2
$\mathrm{graph}\,\varphi$, 78
$\lfloor . \rfloor$, 35
$\mathcal{D}_\nu$, 68
$\mathcal{L}[+]\ldots[+]\mathcal{L}_n$, 9
$\mathcal{L}[\dot{+}]\ldots[\dot{+}]\mathcal{L}_n$, 9
$\mathcal{L}\dot{+}\mathcal{M}$, 1
$\mathcal{L}^{[\perp]}$, 8
$\mathcal{N}_{<\infty}$, 84
$\mathcal{V}^*$, 86
$\mathcal{V}^{[\perp]}$, 8
$\preceq$, 86
$\mathrm{span}\, M$, 1
$x[\perp]y$, 8
A^*, 1
A^t, 1
G^*, 2
$H^2(\mathbb{D})$, 156
$K, _L$166
$K_1 + K_2$, 167
$M^{[\perp]}$, 8
$\mathbb{C}$, 1
$\mathbb{D}$, 156
$\mathbb{H}(\mathbb{D})$, 156
$\mathbb{N}$, 1
$\mathbb{R}$, 1
$\mathbb{Z}$, 1
$\mathrm{ind}_\pm K$, 158
$\ker \varphi$, 14
$\mathcal{F}(M)$, 3
$\mathcal{F}(M, \mathcal{H})$, 158
$\mathcal{K}(K)$, 160
$\mathrm{ran}\, \varphi$, 14
span, 9
δ_x, 34

adjoint, 122
    linear relation, 122
algebraic dual space, 86
almost Pontryagin space, 70
    completion, 81
    direct product, 74
    fundamental decomposition, 71
    reproducing kernel, 154
    subspace, 74
analytic kernel, 169
angular operator, 25

bases
  skewly linked, 41
basis
  orthonormal, 11
bounded operator
  postive, 144

canonical basis, 2
Caratheodory-class, 173
  integral representation, 174
Cayley transform, 129
chordale metric, 103
closed, 105
  linear relation, 105
closure, 105
  linear relation, 105
commutative diagram, 14
completion, 81
  almost Pontryagin space-, 81
  existence, 82, 83
  Hilbert space-, 81
  Krein space-, 81
  Pontryagin space-, 81
  pre-order, 86
  uniqueness, 83, 84
complex numbers, 1
composition of kernel, 165
conjugate linear in the second argument, 2
conjugate transpose, 1
contractive, 127
  linear relation, 127

dB-Hilbert space, 183
dB-Pontryagin space, 183
definite, 21
  subspace, 21
definitizable, 143
definitizing rational function, 143
degenerated, 8
de Branges-class, 177
de Branges-Hilbert space, 183
de Branges-Pontryagin space, 183
Dirac measure, 34, 69
direct sum, 1
Dirichlet space
  generalized, 69
dissipative, 127
  linear relation, 127
domain of a linear relation, 95
dual map, 86
dual space
  algebraic, 86
  topological, 57

eigenvalues, 107
eigenvectors, 107

factor scalar product, 6
finitely supported function, 3
Fouriertransform, 72
functional
  point evaluation, 73, 154
functions with finite support, 3
fundamental decomposition
  almost Pontryagin space, 71
  Krein space, 59

generalized Dirichlet space, 69
Gram
  matrix, 3
Gram operator, 50, 59
Gram space, 49
  direct product, 52
  factor, 52
  reproducing kernel, 154
  subspace, 51
Gram-Schmidt process, 12
graph of a map, 78

Hankel matrix, 35
Hardy-space
  on unit disk, 156
Hermite-Biehler class, 177
  indefinite, 176
hermitian, 1
hermitian kernel, 3
  Hilbert space valued, 154
  matrix valued, 154
  negative, 21
  positive, 21
  positive/negative index, 158
  scalar, 154
Hilbert space
  completion, 81
  reproducing kernel, 154
Hilbert space topology, 49
Hilbert space valued hermitian kernel, 154
hypermaximal neutral

subspace, 42

indefinite Hermite-Biehler class, 176
indefinite Nevanlinna class, 84
index of negativity, 28
- continuity, 48
- direct product, 31
- factor, 31
- subspace, 31

index of nullity, 8, 9
index of positivity, 28
- continuity, 48
- direct product, 31
- factor, 31
- subspace, 31

integers, 1
- positive, 1

Inverse-Cayley transform, 129
isometric, 127
- linear relation, 127

isometry, 4
- isotropic part, 14

isotropic part, 8
- direct product, 10
- factor, 14
- isometric map, 14
- subspace, 8

kernel, 14
kernel part of a linear relation, 95
Kolmogoroff decomposition, 158
Krein space, 57
- completion, 81
- direct product, 60
- fundamental decomposition, 59
- reproducing kernel, 154
- subspace, 60, 62
- topology, 58

Krein subspace, 61
- sum of, 63

Kronecker-Delta, 2

linear in the first argument, 1
linear Relation, 95
linear relation
- definitizable, 143
- eigenvalue of, 107
- eigenvectors of, 107
- root vector of, 107

linear span, 1

Möbius transformation, 103
map
- isometric, 4

matrix
- conjugate transpose, 1
- transpose, 1

matrix (n rows and m columns, 1
matrix valued hermitian kernel, 154
maximal negative definite
- subspace, 23

maximal negative semidefinite
- subspace, 23

maximal neutral
- subspace, 23

maximal positive definite
- subspace, 23

maximal positive semidefinite
- subspace, 23

multiplication of kernel, 166
multivalued part of a linear relation, 95

negative definite, 21
- subspace, 21

negative index
- hermitian kernel, 158

negative semidefinite, 21
- subspace, 21

neutral, 9, 21
- skewly linked subspaces, 38
- subspace, 21

Nevanlinna class
- indefinite, 84

Nevanlinna kernel, 4
Nevanlinna-class, 172
nondegenerated, 8
- subspace, 9

operator
- Gram-, 50, 59

orthocomplemented, 16
orthogonal, 8
orthogonal and direct sum, 9
orthogonal complement, 8
orthogonal projection, 16
orthogonal sum, 9
orthonormal basis, 11

Paley-Wiener space, 72
point evaluation functional, 73, 154
point separating, 48

point spectrum, 107
polar identity, 2
Pontryagin space, 63
  completion, 81
  direct product, 64
  generated by a reproducing kernel, 160
  reproducing kernel, 154
  subspace, 65
  topology, 64
positive definite, 21
  subspace, 21
positive integers, 1
positive semidefinite, 21
  subspace, 21
Potapov-Ginzburg transform, 101
projection, 16

range, 14
range of a linear relation, 95
rational function
  definitizing, 143
real numbers, 1
reproducing kernel, 155
  analytic, 169
  composition, 165
  Kolmogoroff decomposition, 158
  multiplication, 166
  pointwise sum, 167
  restriction, 166
reproducing kernel almost Pontryagin space, 154
reproducing kernel Gram space, 154
reproducing kernel Hilbert space, 154
reproducing kernel Krein space, 154
reproducing kernel Pontryagin space, 154
  generated by K, 160
resolvent equality, 110
resolvent function, 109
resolvent set, 107
restriction of kernel, 166
Riesz-Herglotz representation, 174
root vectors, 107

scalar hermitian kernel, 154
scalar product, 1
  compatible with $\mathcal{O}$, 49
  continuous
    w.r.t. $(.,.)$, 46
    w.r.t. $\mathcal{O}$, 46
    w.r.t. $\|.\|$, 46
  defined by requiring isometry, 6
  definite, 21
  factor-, 6
  negative definite, 21
  negative semidefinite, 21
  neutral, 9, 21
  positive definite, 21
  positive semidefinite, 21
  semidefinite, 21
  sum-, 5
scalar product space, 2
  degenerated, 8
  direct product, 5
  factor, 6
  generated by a hermitian kernel, 3
  index of negativity, 28
  index of nullity, 8, 9
  index of positivity, 28
  isotropic part of, 8
  nondegenerated, 8
  subspace, 5
Schur-class, 172
Schwarz inequality, 22
selfadjoint, 127
  linear relation, 127
semidefinite, 21
  subspace, 21
set of points of regular type, 107
skewly linked, 38
  bases, 41
space
  almost Pontryagin-, 70
  Gram-, 49
  Krein-, 57
  Pontryagin-, 63
spectrum, 107
subspace
  definite, 21
  hypermaximal neutral, 42
  isotropic part, 8
  maximal negative definite, 23
  maximal negative semidefinite, 23
  maximal neutral, 23
  maximal positive definite, 23
  maximal positive semidefinite, 23
  negative definite, 21
  negative semidefinite, 21
  neutral, 21
  nondegenerated, 9

orthocomplemented, 16
positive definite, 21
positive semidefinite, 21
semidefinite, 21
sum scalar product, 5
symmetric, 127
linear relation, 127

topological dual space, 57
topology
Hilbert space, 49
Krein space, 58
Pontryagin space, 64
transpose, 1
two-sided sequences
double-finite, 84
left-finite, 20

unit point mass, 34
unitary, 127
linear relation, 127

vector (n rows), 1