

**SOME QUESTIONS RELATED TO
SYMMETRIC OPERATORS IN
HILBERT SPACES**

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Introduction

Let S be a symmetric operator or a symmetric linear relation with equal defect numbers (n, n) in a Hilbert space and assume that A is a selfadjoint extension of S in \mathfrak{H} . The reader, who is not familiar with linear relations, can look at S as a linear operator. With each pair (S, A) there is associated a holomorphic function $Q : \mathbb{C} \setminus \mathbb{R} \rightarrow L(\mathfrak{H})$, where \mathfrak{H} is an n -dimensional Hilbert space and $L(\mathfrak{H})$ denotes the space of all bounded linear operators in \mathfrak{H} .

This function Q is called a Q -function of (S, A) and belongs to a certain class $M(\mathfrak{H})$ which is defined in Definition 1.5. The function Q essentially describes (S, A) . This means, given a function $Q \in M(\mathfrak{H})$, we can construct a pair (S, A) such that Q is a Q -function of (S, A) , and if we assume some minimality condition on S , then the construction of (S, A) is unique up to unitary equivalence. Therefore, it is possible to formulate properties of (S, A) in terms of its Q -function.

The concept of Q -functions of a densely defined symmetric operator was introduced by Kreĭn and Langer. In this thesis we consider Q -functions from different points of view.

In Chapter 1 we give a short introduction to those topics in operator theory that will be used in this work. These are: linear relations, dissipative, symmetric, selfadjoint relations, resolvent set, Q -functions, integral representation of Q -functions, Kreĭn's formula, space triplets, spaces of boundary values. We also prove some results which are known, but which have not yet been stated explicitly (as far as it is known to the author).

Chapter 2 treats the following problem. Let two scalar valued Nevanlinna functions f and g be given, such that f is a Q -function of (S_f, A_f) and g is a Q -function of (S_g, A_g) . Construct a symmetric relation S and a selfadjoint extension A from the relations S_g, A_g, S_f, A_f such that $f \circ g$ is a Q -function of (S, A) . This problem is solved with help of the concept of tensor products of Hilbert spaces.

In Chapter 3 we start with a Q -function Q of (S, A) and numbers $a, b \in \mathbb{R}$, $a \neq 0$. It is proved that $aQ(\frac{z}{a})$ is a Q -function of (aS, aA) , that $Q(z + b)$ is a Q -function of $(S - b, A - b)$ and that $Q(-\frac{1}{z})$ is a Q -function of $(-S^{-1}, -A^{-1})$.

From this we obtain some results concerning special properties of S and A . In particular, we give a characterization of the Friedrichs and of the von Neumann extension of a semibounded symmetric relation S .

Chapter 4 is divided into three sections. In Section 4.1 we consider the following situation. Let \mathfrak{H}_1 and \mathfrak{H}_2 be Hilbert spaces, S_1 and S_2 symmetric closed linear relations on \mathfrak{H}_1 and \mathfrak{H}_2 , respectively. Denote by \mathfrak{H} and S the orthogonal sums $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and $S = S_1 \oplus S_2$. Clearly a selfadjoint extension \tilde{A} of S in a Hilbert space $\tilde{\mathfrak{H}} \supseteq \mathfrak{H}$ is also an extension of S_1 , thus induces a generalized resolvent $\tilde{P}_1(\tilde{A} - z)|_{\mathfrak{H}_1}$ of S_1 . Here \tilde{P}_1 is the orthogonal projection of $\tilde{\mathfrak{H}}$ onto \mathfrak{H}_1 .

It is well known (see e.g. [14]) that by Kreĭn's formula the family of all generalized resolvents of S_1 can be parametrized via operator valued parameter functions \mathcal{T} contained in a certain generalized Nevanlinna class. We determine those parameter functions \mathcal{T} , which correspond to a generalized resolvent induced by an extension \tilde{A} of S_1 which is also an extension of S .

Again by Kreĭn's formula, the extensions of S are parametrized by parameters \mathcal{N} . We establish a correspondence between the parameters \mathcal{N} and \mathcal{T} .

In Section 4.2 we apply these results to differential operators on a graph. More exactly, let $(\mathfrak{V}, \mathfrak{E})$ be a finite directed graph, where \mathfrak{V} is the set of all vertices v_1, \dots, v_m , and \mathfrak{E} is the set of all edges e_1, \dots, e_n . Assume that every vertex is the vertex of an edge, and suppose that to each edge e_j an interval $[a_j, b_j]$ is assigned ($j = 1, \dots, n$). We set $\hat{a}_j = v$ ($\hat{b}_j = v'$), where v is the starting (v' is the terminating) vertex of e_j . On each such interval $[a_j, b_j]$ a symmetric and regular second order differential operator $l_j = -Dp_jD + q_j$ is given.

Let A be a selfadjoint extension of $\bigoplus_{j=1}^n L_j$ in $\bigoplus_{j=1}^n L^2[a_j, b_j]$, where L_j is the minimal realization of l_j on $L^2[a_j, b_j]$. We assume that A satisfies some special conditions depending on the graph (cf. Definition 4.7).

Clearly, $(A - z)^{-1}(g_j)_{j=1}^n = (f_j)_{j=1}^n$ is the solution of $l_j f_j - z f_j = g_j$, $j = 1, \dots, n$ and $BC((f_j)_{j=1}^n) = 0$, where BC is the selfadjoint boundary condition corresponding to A . Applying results of Section 4.1 we obtain z -dependent boundary conditions $BC_1(z)$ such that $P_1(A - z)^{-1}g_1 = f_1$ is the solution of $l_1 f_1 - z f_1 = g_1$ and $BC_1(z)(f_1) = 0$. In Section 4.3 we apply these results to some special graphs.

Chapter 5 also consists of three sections. Throughout this chapter let S be a closed symmetric relation with finite and equal defect numbers (n, n) , let A be a selfadjoint extension of S in \mathfrak{H} , and denote by Q the Q -function of (S, A) . In this situation Q is a $n \times n$ matrix valued Nevanlinna function.

In Section 5.1 we introduce two subclasses \mathbf{N}_0 and \mathbf{N}_1 of the class of all $n \times n$ matrix valued Nevanlinna functions and we characterize the fact that Q belongs

to \mathbf{N}_0 or to \mathbf{N}_1 by means of S and A .

Assume that Q belongs to \mathbf{N}_0 (\mathbf{N}_1). Then for all selfadjoint extensions B of S in \mathfrak{H} with exception of a, in a certain sense, small set (exceptional extensions, cf. Proposition 5.6) the Q -function of (S, B) also belongs to \mathbf{N}_0 (\mathbf{N}_1). Among those exceptional extensions there is one special extension, which we call the generalized Friedrichs extension.

In Section 5.2 the case $Q \in \mathbf{N}_0$ is analysed. If S is assumed to be an operator then A is also an operator. We show that selfadjoint extensions B of S whose Q -function does not belong to \mathbf{N}_0 are proper relations, and show how the size of the nonoperator part of B is determined by the Q -function of (S, B) .

In Section 5.3 we extend the results of Section 5.2 to the case $Q \in \mathbf{N}_1$. To do so we use the concept of space triplets introduced in Section 1.3. Moreover, we obtain $(|A|^{\frac{1}{2}}) = (|B|^{\frac{1}{2}})$ for all nonexceptional selfadjoint extensions B of S .

In the case that S is semibounded it is shown that the generalized Friedrichs extension of S (cf. Proposition 5.25) coincides with the Friedrichs extension.

Finally we give an example to obtain some negative results, which, in particular, stress the difference between the cases $Q \in \mathbf{N}_1$ and $Q \in \mathbf{N}_0$.

The Chapter 6 is devoted to the proof of the following result: Let A be a selfadjoint operator in a Hilbert space \mathfrak{H} . Then A is semibounded if and only if each one dimensional restriction S of A has a selfadjoint extension B in \mathfrak{H} such that the Q -function of (S, B) belongs to \mathbf{N}_1 .

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Chapter 1

Some Definitions and Basic Results

1.1 Linear Relations

In this chapter we recall some definitions and results which are often used in this work.

Let \mathcal{H}, \mathcal{K} be Hilbert spaces, and denote by (\cdot, \cdot) the inner product on \mathcal{H} and on \mathcal{K} . A linear subspace T of $\mathcal{H} \oplus \mathcal{K}$ is called a linear relation (cf. [1], [6]) in $\mathcal{H} \oplus \mathcal{K}$. We say that the linear relation T is closed if T is closed as a subset of $\mathcal{H} \oplus \mathcal{K}$. If W is a (closed) linear operator from a linear subspace of \mathcal{H} into \mathcal{K} then the graph of W is a (closed) linear relation. In the sequel we identify the operator R and its graph. If $\mathcal{H} = \mathcal{K}$ we say that T is a linear relation on \mathcal{H} .

Let \mathcal{H} be a Hilbert space, too. For a linear relation S in $\mathcal{H} \oplus \mathcal{H}$ and linear relations R, T in $\mathcal{H} \oplus \mathcal{H}$ and $\alpha \in \mathbb{C}$ we introduce the following notation

$$D(T) = \{u \in \mathcal{H} : \exists v \in \mathcal{H} : \{u; v\} \in T\}, \text{ the domain of } T,$$

$$R(T) = \{v \in \mathcal{H} : \exists u \in \mathcal{H} : \{u; v\} \in T\}, \text{ the range of } T,$$

$$T^{-1} = \{\{u; v\} \in \mathcal{H} \oplus \mathcal{H} : \{v; u\} \in T\}, \text{ the inverse relation of } T,$$

$$R + T = \{\{u; v + w\} : \{u; v\} \in R, \{u; w\} \in T\}, \text{ the sum of } R \text{ and } T,$$

$$TS = \{\{u; w\} \in \mathcal{H} \oplus \mathcal{H} : \exists v \in \mathcal{H} : \{u; v\} \in S, \{v; w\} \in T\},$$

the product of S and T ,

$$\alpha T = \{\{u; \alpha v\} : \{u; v\} \in T\},$$

$$T^* = \{\{u; v\} \in \mathcal{H} \oplus \mathcal{H} : (y, u) = (x, v) \text{ for all } \{x; y\} \in T\}, \text{ the adjoint of } T,$$

$$N(T) = \{u \in \mathcal{H} : \{u; 0\} \in T\}, \text{ the kernel of } T,$$

$$T(0) = \{v \in \mathcal{H} : \{0; v\} \in T\}, \text{ the multivalued part of } T,$$

$$T_\infty = \{\{0; v\} : \{0; v\} \in T\}, \text{ the infinity part of } T,$$

$$T_s = \{\{u; v\} \in T : v \perp T(0)\}, \text{ operator part of } T.$$

The sets (T) , (T) and $(T), T(0)$ are linear subspaces of \mathfrak{H} and \mathfrak{H} , respectively, and $T_\infty, T_s, R + T \subseteq \oplus$, $TS \in \oplus$ and $T^*, T^{-1} \subseteq \oplus$ are linear relations. We always have $(T^*)^* = \overline{T}$, $R^* + T^* \subseteq (R + T)^*$ and $S^*T^* \subseteq (TS)^*$.

If T is a closed linear relation and W is a bounded operator from \mathfrak{H} into \mathfrak{H} then the linear relation $T + W$ is also closed. Moreover, we have $(T + W)^* = T^* + W^*$. This relation also holds if T is not closed.

If $\mathfrak{H} = \mathfrak{H}$ and if I is the identity operator, then we put $T + \alpha = T + \alpha I$.

Clearly, T is an operator if and only if $T(0) = \{0\}$. The operator part T_s of T satisfies $T_s(0) = \{0\}$ and $(T_s) = (T)$. Moreover, (T) is dense in $T^*(0)^\perp$. The range of T_s is contained in $T(0)^\perp$ (cf. [6]). If T is closed then T_∞ and T_s are closed and we have $T = T_s \oplus T_\infty$. For proofs and further basic results see for example [2], [6], [1].

In the following we assume $\mathfrak{H} = \mathfrak{H}$.

Definition 1.1. A linear relation T on a Hilbert space \mathfrak{H} is dissipative, if

$$\frac{1}{2i}((v, u) - (u, v)) \geq 0 \quad \text{for } \{u; v\} \in T.$$

If there exists no proper dissipative extension of T , we say that T is maximal dissipative. A linear relation T is called symmetric (selfadjoint), if $S \subseteq S^*$ ($S = S^*$).

Clearly, a symmetric relation is dissipative and a selfadjoint relation is maximal dissipative. Using the Lemma of Zorn one sees that every dissipative relation has a maximal dissipative extension. Every maximal dissipative relation is closed and satisfies $T(0) = T^*(0)$ (cf. [2], Chapter 2).

Thus, in the decomposition $T = T_s \oplus T_\infty$, the relation T_s is a densely defined operator in $T(0)^\perp$ and a maximal dissipative relation on $T(0)^\perp$ (cf. [2], [6]). We can therefore write $T = (T_s P + I - P)P^{-1}$, where P is the orthogonal projection of \mathfrak{H} onto $T(0)^\perp$.

Denote by $L(\mathfrak{H})$ the space of all bounded linear operators on \mathfrak{H} . In analogy to the operator case the resolvent set for a linear relation is defined.

Definition 1.2. Let T be a linear relation on \mathfrak{H} . The set of regular points $\gamma(T)$ is the set of all $z \in \mathbb{C}$ such that $(T - z)^{-1}$ is a bounded operator from $(T - z)$ into \mathfrak{H} . The resolvent set $\rho(T)$ is the set of all $z \in \mathbb{C}$ such that $(T - z)^{-1} \in L(\mathfrak{H})$. Set $\tilde{\rho}(T) = \rho(T) \cup \{\infty\}$ if $T \in L(\mathfrak{H})$ and $\tilde{\rho}(T) = \rho(T)$, otherwise. Moreover, set $\sigma(T) = \mathbb{C} \setminus \rho(T)$ and $\tilde{\sigma}(T) = \mathbb{C} \cup \{\infty\} \setminus \tilde{\rho}(T)$.

The set $\gamma(T)$ contains $\rho(T)$ and both are open subsets of \mathbb{C} (cf. [7]). For a linear relation T the so called resolvent identity is valid:

$$(T - z)^{-1} - (T - \zeta)^{-1} = (z - \zeta)(T - z)^{-1}(T - \zeta)^{-1} \quad (1.1)$$

for all $z, \zeta \in \rho(T)$ (cf. [6]).

Let $\mathbb{C}^+(\mathbb{C}^-)$ denote the open upper (lower) halfplane in \mathbb{C} . If T is a (maximal) dissipative relation, then $\mathbb{C}^- \subseteq \gamma(T)$ ($\rho(T)$). Hence, for every closed symmetric (selfadjoint) relation S we have $\mathbb{C} \setminus \mathbb{R} \subseteq \gamma(S)$ ($\rho(S)$).

Definition 1.3. Let T be a linear relation on \mathfrak{H} and let $\mu \in \mathbb{C} \setminus \mathbb{R}$. Then we call the linear relation

$$C_\mu(T) = \{\{v - \mu u; v - \bar{\mu}u\} : \{u; v\} \in T\}$$

the Cayley transform of T , and we call

$$F_\mu(T) = \{\{v - u; \mu v - \bar{\mu}u\} : \{u; v\} \in T\}$$

the inverse Cayley transform of T .

One immediately sees, that $C_\mu \circ F_\mu(T) = T = F_\mu \circ C_\mu(T)$. Hence C_μ and F_μ are in fact inverses of each other.

The Cayley transform has the property $\tilde{\rho}(C_\mu(T)) = C_\mu(\tilde{\rho}(T))$, when $C_\mu(z) = \frac{z - \bar{\mu}}{z - \mu}$. Moreover, a linear relation T is (maximal) dissipative, if and only if $C_\mu(T)$ is (an everywhere defined) contraction from $(C_\mu(T))$ into \mathfrak{H} for any $\mu \in \mathbb{C}^-$, and a linear relation S is symmetric (selfadjoint), if and only if $C_\mu(S)$ is an isometry (a unitary operator) from $(C_\mu(S))$ into \mathfrak{H} for any $\mu \in \mathbb{C} \setminus \mathbb{R}$.

Now we define a class of linear relation valued functions on $\mathbb{C} \setminus \mathbb{R}$ which generalizes the concept of Nevanlinna functions. Recall that a Nevanlinna function is a holomorphic function f on $\mathbb{C} \setminus \mathbb{R}$ such that $f(\bar{z}) = \overline{f(z)}$ for $z \in \mathbb{C} \setminus \mathbb{R}$ holds, and such that $\text{Im}f(z) \geq 0$, $z \in \mathbb{C}^+$.

Definition 1.4. A function \mathcal{N} from $\mathbb{C} \setminus \mathbb{R}$ into the set of all linear relations on \mathfrak{H} belongs to $N()$ if

1. $\mathcal{N}(z)$ is maximal dissipative for $z \in \mathbb{C}^+$,
2. $\mathcal{N}(\bar{z}) = \mathcal{N}(z)^*$ for $z \in \mathbb{C} \setminus \mathbb{R}$,
3. for some $\mu \in \mathbb{C}^-$ the Cayley transform $C_\mu(\mathcal{N}(z))$ depends holomorphically on $z \in \mathbb{C}^+$.

We note that all constant selfadjoint relations on \mathfrak{H} belong to $N()$.

Assume $\mathcal{N} \in N()$. Then for $z \in \mathbb{C}^-$ the identity $\mathcal{N}(z) = \mathcal{N}(\bar{z})^*$ implies that $-\mathcal{N}(z)$ is maximal dissipative. The infinity part of $\mathcal{N}(z)$ turns out to be independent of $z \in \mathbb{C} \setminus \mathbb{R}$, $\mathcal{N}(z)_\infty = \mathcal{N}_\infty$ (cf. [2]). Hence we can decompose $\mathcal{N}(z)$ as

$\mathcal{N}_s(z) \oplus \mathcal{N}_\infty$, where $\mathcal{N}_s(z) = \mathcal{N}(z)_s$ is the operator part of $\mathcal{N}(z)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, and we have $\mathcal{N}_s \in N(\mathcal{N}(0)^\perp)$. If P denotes the orthogonal projection of onto $\mathcal{N}(0)^\perp$ the decomposition of \mathcal{N} can also be written as

$$\mathcal{N}(z) = (\mathcal{N}_s(z)P + I - P)P^{-1} \quad (1.2)$$

Also a certain subclass of $N()$ will be used in the sequel. For bounded selfadjoint operators $A, B \in L()$ we write $A \geq B$ if $(Au, u) \geq (Bu, u)$ for all $u \in .$ If $B \geq \beta I$ for some $\beta > 0$ write $B \gg 0$.

Definition 1.5. A function $Q : \mathbb{C} \setminus \mathbb{R} \rightarrow L()$ belongs to $M()$ if

1. $\text{Im}Q(z) \gg 0$ for all $z \in \mathbb{C}^+$,
2. $Q(\bar{z}) = Q(z)^*$ for $z \in \mathbb{C} \setminus \mathbb{R}$,
3. $Q(z)$ depends holomorphically on $z \in \mathbb{C} \setminus \mathbb{R}$.

The class $M()$ is a subclass of $N()$. For arbitrary $Q \in M()$ and $\mathcal{N} \in N()$ the relation $(Q(z) + \mathcal{N}(z))^{-1}$ belongs to $L()$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. (cf. [2], [14]).

If is one dimensional then $M()$ can be identified with the class of all Nevanlinna functions which are not a constant real number.

1.2 Symmetric Relations and their Selfadjoint Extensions

1.2.1 Q -Functions

Let S be a closed symmetric relation. For $z \in \gamma(S)$ put $z = (S - \bar{z})^\perp = (S^* - z)$, the so called defect spaces of S . It is proved in [16] and [6] that the Hilbert space dimension of z is of constant cardinality for all regular points z . As $\mathbb{C} \setminus \mathbb{R} \subseteq \gamma(S)$ we may define n_+ (n_-) as the Hilbert space dimension of z for $z \in \mathbb{C}^+$, (\mathbb{C}^-), and (n_+, n_-) are called the defect numbers of S . The relation S has a selfadjoint extension in if and only if $n_+ = n_-$ (cf. [16], [6]), that is “equal defect numbers”.

Assume in the following that the closed symmetric relation S has equal defect numbers and let A be a selfadjoint extension of S in . Such extensions which are selfadjoint in the same Hilbert space are called canonical extensions.

As $A \supseteq S$ we have $\rho(A) \subseteq \gamma(S)$. By the resolvent identity the operator $I + (z - \zeta)(A - z)^{-1}$ has the inverse $I + (\zeta - z)(A - \zeta)^{-1}$ and is therefore bijective for $z, \zeta \in \rho(A)$. Moreover, the selfadjointness of A shows that it maps ζ bijectively onto z .

Let \mathfrak{H} be a Hilbert space with the same Hilbert space dimension as \mathfrak{H}_z , $z \in \mathbb{C} \setminus \mathbb{R}$, let Γ_i be a bicontinuous linear mapping from \mathfrak{H} onto \mathfrak{H}_i , and set $\Gamma_z = (I + (z - i)(A - z)^{-1})\Gamma_i$ for $z \in \rho(A)$. Then Γ_z is a bicontinuous linear mapping from \mathfrak{H} onto \mathfrak{H}_z and satisfies

$$\Gamma_z = (I + (z - \zeta)(A - z)^{-1})\Gamma_\zeta \quad (1.3)$$

for all $z, \zeta \in \rho(A)$. As (1.3) shows, it is not essential to start with the definition of Γ_z at the point i .

A function Q from $\rho(A)$ into $L(\mathfrak{H}_z, \mathfrak{H}_z)$ is called a Q -function of the triple (S, A, Γ_i) if the relation

$$\frac{Q(z) - Q(\zeta)^*}{z - \bar{\zeta}} = \Gamma_\zeta^* \Gamma_z \quad (1.4)$$

holds for all $z, \zeta \in \rho(A)$. It is known (cf. [14]) that Q is determined by (1.4) up to an arbitrary bounded selfadjoint operator C in \mathfrak{H}_z and has the form:

$$Q(z) = C - i\Gamma_i^* \Gamma_i + (z + i)\Gamma_i^* \Gamma_z. \quad (1.5)$$

The choice of the mapping Γ_i does in fact not influence the structure of Q , therefore we will also call Q a Q -function of the pair (S, A) . It is shown in [14] that $Q|_{\mathbb{C} \setminus \mathbb{R}}$ belongs to $M()$.

If the defect numbers of S are equal to one and if $\mathfrak{H} = \mathbb{C}$, then Q can be identified with the Nevanlinna function $Q(z)(1)$. In this case we consider the so called defect vectors $\gamma(z) = \Gamma_z(1) \in \mathfrak{H}_z$. Then (1.3) holds if Γ_z and Γ_ζ are replaced by $\gamma(z)$ and $\gamma(\zeta)$, respectively, and (1.4) reads as

$$\frac{f(z) - \overline{f(\zeta)}}{z - \bar{\zeta}} = (\gamma(z), \gamma(\zeta)). \quad (1.6)$$

Conversely, let Q be a function belonging to $M()$. Then there exists a model $(\mathfrak{Q}, S^{\mathfrak{Q}}, A^{\mathfrak{Q}}, \Gamma_i^{\mathfrak{Q}})$, where \mathfrak{Q} is a Hilbert space, $S^{\mathfrak{Q}}$ is a symmetric relation, $A^{\mathfrak{Q}}$ is a selfadjoint extension of $S^{\mathfrak{Q}}$ in \mathfrak{Q} , and $\Gamma_i^{\mathfrak{Q}}$ is a bicontinuous mapping from \mathfrak{H} onto \mathfrak{H}_i , such that Q is the restriction to $\mathbb{C} \setminus \mathbb{R}$ of a Q -function of $(S^{\mathfrak{Q}}, A^{\mathfrak{Q}}, \Gamma_i^{\mathfrak{Q}})$. The model $(\mathfrak{Q}, S^{\mathfrak{Q}}, A^{\mathfrak{Q}}, \Gamma_i^{\mathfrak{Q}})$ is uniquely determined up to unitary equivalence, if $S^{\mathfrak{Q}}$ is chosen completely nonselfadjoint, i.e. if the closed linear span $\text{c.l.s.}(\bigcup_{z \in \mathbb{C} \setminus \mathbb{R}} \gamma(z))$ equals the whole space \mathfrak{Q} .

Given a model $(\mathfrak{Q}, S^{\mathfrak{Q}}, A^{\mathfrak{Q}}, \Gamma_i^{\mathfrak{Q}})$, we can decompose \mathfrak{Q} as $\mathfrak{n} \oplus (\mathfrak{Q} \ominus \mathfrak{n})$ where $\mathfrak{n} = \text{c.l.s.}(\bigcup_{z \in \mathbb{C} \setminus \mathbb{R}} \gamma(z))$. Then the relations $S^{\mathfrak{Q}}$ and $A^{\mathfrak{Q}}$ can also be decomposed as $S^{\mathfrak{Q}} = S_n \oplus A'$ and $A^{\mathfrak{Q}} = A_n \oplus A'$ where S_n is a symmetric and completely nonselfadjoint relation on \mathfrak{n} and where $A_n \supseteq S_n$ and A' are selfadjoint relations on \mathfrak{n} and $\mathfrak{Q} \ominus \mathfrak{n}$, respectively (cf. [14]). Thus every model $(\mathfrak{Q}, S^{\mathfrak{Q}}, A^{\mathfrak{Q}}, \Gamma_i^{\mathfrak{Q}})$ can be reduced to a model where the symmetric relation is completely nonselfadjoint.

Note that if S is completely nonselfadjoint then it is an operator with no eigenvalues.

It is well known that every function $Q \in M()$ has the integral representation:

$$Q(z) = C + Dz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\Sigma_t, \quad z \in \mathbb{C} \setminus \mathbb{R} \quad (1.7)$$

where $C, D \in L()$ such that C, D are selfadjoint and $D \geq 0$, and where Σ_t is an increasing ($\Sigma_t \leq \Sigma_\tau$ for $t \leq \tau$) family of bounded positive operators on such that

$$\int_{\mathbb{R}} \frac{1}{t^2+1} d(\Sigma_t x, x) < \infty$$

for all $x \in$.

The integral in (1.7) exists as a Riemann-Stieltjes integral in the strong operator topology on $L()$.

The following proposition establishes a connection between the integral representation of a Q -function and the spectral family of the corresponding selfadjoint relation (cf. [12]).

Proposition 1.6. *Let be a Hilbert space and let S be closed symmetric relation on with equal defect numbers. Assume that A is a selfadjoint extension of S in . Let Γ_i be a bicontinuous linear mapping from a Hilbert space onto i and let Q be a Q -function of (S, A, Γ_i) . Denote by P the orthogonal projection of onto $\overline{(A)} = A(0)^\perp$ and let $(E_t)_{t \in \mathbb{R}}$ be the spectral family of the operator part A_s of A in the Hilbert space $\overline{(A)}$.*

Then Q has an integral representation of the form (1.7) where

$$D = \Gamma_i^*(I - P)\Gamma_i, \quad d(\Sigma_t x, y) = (t^2 + 1) d(\Gamma_i^* E_t P \Gamma_i x, y) \quad (1.8)$$

for all $x, y \in$.

Proof. Since a Q -function is uniquely determined up to a selfadjoint operator C , it is enough to show that a function \hat{Q} of the form (1.7) under the assumptions (1.8) is a Q -function of (S, A, Γ_i) . Indeed, for $z, \zeta \in \mathbb{C} \setminus \mathbb{R}$ and $x, y \in$

$$\begin{aligned} & \left(\frac{\hat{Q}(z) - \hat{Q}(\zeta)^*}{z - \bar{\zeta}} x, y \right) = \\ & = ((I - P)\Gamma_i x, \Gamma_i y) + \int_{\mathbb{R}} \left(1 + \frac{z-i}{t-z} \right) \overline{\left(1 + \frac{\zeta-i}{t-\zeta} \right)} d(E_t P \Gamma_i x, P \Gamma_i y) = \\ & = ((I - P)\Gamma_i x, (I - P)\Gamma_i y) + \end{aligned}$$

$$\begin{aligned}
& +((I + (z - i)(A_s - z)^{-1})P\Gamma_i x, (I + (\zeta - i)(A_s - \zeta)^{-1})P\Gamma_i y) = \\
& = ((I + (z - i)(A - z)^{-1})x, (I + (\zeta - i)(A - \zeta)^{-1})y) = (\Gamma_z x, \Gamma_\zeta y).
\end{aligned}$$

The last equality holds because of $(A - z)^{-1}x = 0$ for $x \perp \overline{(A)}$. \square

1.2.2 Kreĩn's Formula

Let S be a symmetric relation with equal defect numbers on \mathfrak{H} . Up to now we considered selfadjoint extensions of S in \mathfrak{H} . Now let \tilde{A} be a selfadjoint relation on a Hilbert space $\tilde{\mathfrak{H}} \supseteq \mathfrak{H}$ such that $\tilde{A} \supseteq S$. Clearly, if $\tilde{\mathfrak{H}} = \mathfrak{H}$, then \tilde{A} is a canonical extension of S . Let \tilde{P} denote the orthogonal projection of $\tilde{\mathfrak{H}}$ onto \mathfrak{H} .

We call the operator function $R(z) = \tilde{P}(\tilde{A} - z)^{-1}|$, $z \in \rho(\tilde{A})$, a generalized resolvent of \tilde{A} in \mathfrak{H} . For $z \in \mathbb{C} \setminus \mathbb{R} \subseteq \rho(\tilde{A})$ let $T(z)$ be defined by

$$T(z) = R(z)^{-1} + z \quad (1.9)$$

This is the so called Strauss relation. It turns out that $T(z) \supseteq S$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ and that $-T \in N(\cdot)$. Conversely, given $-T \in N(\cdot)$, such that $T(z) \supseteq S$ for $z \in \mathbb{C} \setminus \mathbb{R}$ there exists a Hilbert space $\tilde{\mathfrak{H}} \supseteq \mathfrak{H}$ and a selfadjoint relation $\tilde{A} \supseteq S$ such that $(T(z) - z)^{-1} = \tilde{P}(\tilde{A} - z)^{-1}|$ (cf. [6]).

Let A be a fixed canonical selfadjoint extension of S and let Γ_z and $\Gamma_{\bar{z}}$ be defined as above. Then every generalized resolvent $\tilde{P}(\tilde{A} - z)^{-1}|$ can be written as

$$\tilde{P}(\tilde{A} - z)^{-1}| = (A - z)^{-1} - \Gamma_z(Q(z) + \mathcal{N}(z))^{-1}\Gamma_{\bar{z}}^* \quad (1.10)$$

for some $\mathcal{N} \in N(\cdot)$. Here Q is the Q -function of (S, A, Γ_i) . We call (1.10) Kreĩn's formula. Conversely, given $\mathcal{N} \in N(\cdot)$ then there exists a Hilbert space $\tilde{\mathfrak{H}} \supseteq \mathfrak{H}$ and a selfadjoint relation $\tilde{A} \supseteq S$ such that (1.10) holds (cf. [14]). If we assume that $\tilde{\Theta} = \text{c.l.s.}(\bigcup_{z \in \mathbb{C} \setminus \mathbb{R}} (I - \tilde{P})(\tilde{A} - z)^{-1}(\cdot))$, then \tilde{A} is uniquely determined by \mathcal{N} up to unitary equivalence. In (1.10) the selfadjoint relation \tilde{A} is a canonical extension of S if and only if \mathcal{N} is a constant selfadjoint relation on \mathfrak{H} .

The following proposition will present a more explicit version of (1.10) if $\mathcal{N}(z)$ is not an operator.

Proposition 1.7. *Let $\mathcal{N}(z) = (\mathcal{N}_s(z)P + I - P)P^{-1}$ be the decomposition of \mathcal{N} as in (1.2). Then we can write (1.10) as*

$$\tilde{P}(\tilde{A} - z)^{-1}| = (A - z)^{-1} - \Gamma_z P(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}\Gamma_{\bar{z}}^*, \quad (1.11)$$

where $(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1} \in L(\cdot)$. Moreover, we have

$$(Q(z) + \mathcal{N}(z))^{-1} = P(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1} = \quad (1.12)$$

$$\begin{aligned}
&= P(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}P = (PQ(z)|_{\mathcal{N}(0)^\perp} + \mathcal{N}_s(z))^{-1}P = \\
&= P(PQ(z)P + \mathcal{N}_s(z)P + I - P)^{-1}P.
\end{aligned}$$

Proof. Considering all terms as linear relations the following equalities hold:

$$\begin{aligned}
&(Q(z) + \mathcal{N}(z))^{-1} = \\
&= \{ \{ (Q(z)P + \mathcal{N}_s(z)P + I - P)u; Pu \} \in {}^2 : Pu \in (\mathcal{N}_s(z)) \} = \quad (1.13) \\
&= P(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}.
\end{aligned}$$

Since $(Q(z) + \mathcal{N}(z))^{-1}$ is a bounded operator defined on $\mathcal{N}(z)$, (1.13) implies that the relation $(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}$ has domain $\mathcal{N}(z)$. Assume $x \in (Q(z)P + \mathcal{N}_s(z)P + (I - P))$ then by (1.13) $Px = 0$ as $Q(z) + \mathcal{N}(z)$ has kernel $\{0\}$. This implies $(I - P)x = 0$, and hence $x = 0$. Therefore $(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}$ exists as an operator defined on $\mathcal{N}(z)$ and as $Q(z)P + \mathcal{N}_s(z)P + (I - P)$ is a closed linear relation we see that $(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1} \in L(\mathcal{N}(z))$.

Moreover, if we consider $Q(z)P + \mathcal{N}_s(z)P + (I - P)$ as a 2×2 operator block matrix with respect to the decomposition $\mathcal{N}(0)^\perp \oplus \mathcal{N}(0)$ of $\mathcal{N}(z)$, we see that this matrix has domain $(\mathcal{N}(z)) \oplus \mathcal{N}(0)$ and that it is lower triangular with the identity operator in the lower right corner. Since $Q(z)P + \mathcal{N}_s(z)P + (I - P)$ is invertible, its left upper corner $PQ(z)|_{\mathcal{N}(0)^\perp} + \mathcal{N}_s(z)$ is also invertible and $(PQ(z)|_{\mathcal{N}(0)^\perp} + \mathcal{N}_s(z))^{-1}$ equals to the left upper corner of the lower triangular operator block matrix $(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}$ with respect to the same decomposition $\mathcal{N}(0)^\perp \oplus \mathcal{N}(0)$ of $\mathcal{N}(z)$. Hence

$$\begin{aligned}
&(Q(z) + \mathcal{N}(z))^{-1} = P(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1} = \\
&= P(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}P = (PQ(z)|_{\mathcal{N}(0)^\perp} + \mathcal{N}_s(z))^{-1}P = \\
&= P(PQ(z)P + \mathcal{N}_s(z)P + I - P)^{-1}P.
\end{aligned}$$

□

Assume that \mathcal{N}' is a closed subspace of $\mathcal{N}(z)$ and denote by P the orthogonal projection of $\mathcal{N}(z)$ onto \mathcal{N}' . If we define $S' = \{ \{ u; v \} \in A : v + iu \perp \Gamma_i(\mathcal{N}') \}$ then S' is a symmetric extension of S and $PQ(z)|_{\mathcal{N}'}$ is the Q -function of $(S', A, \Gamma_i|_{\mathcal{N}'})$.

If S has defect numbers $(1, 1)$ and if we set $\mathcal{N}(z) = \mathbb{C}$, $f(z) = Q(z)(1)$ and $\gamma(z) = \Gamma_z(1)$, then Kreĭn's formula for canonical selfadjoint extensions reads as

$$(\tilde{A} - z)^{-1} = (A - z)^{-1} - \gamma(z) \frac{(\cdot, \gamma(\bar{z}))}{f(z) + t} \quad (1.14)$$

and it establishes a bijective correspondence between $t \in \mathbb{R} \cup \{\infty\}$ and all canonical selfadjoint extensions \tilde{A} of S . The parameter $t = \infty$ corresponds to $\tilde{A} = A$.

Assume that \tilde{A} is a canonical selfadjoint extension of S such that $\mathcal{N}(z) = M$ is the corresponding selfadjoint parameter in (1.10). Then a Q -function of (S, \tilde{A}, Γ_i) is given by (cf. [14])

$$\tilde{Q}(z) = Q(z) - (Q(z) - Q(i)^*)(Q(z) + M)^{-1}(Q(z) - Q(i)). \quad (1.15)$$

If we write M as $M = (M_s P + I - P)P^{-1}$ (cf. 1.2), where P is the orthogonal projection of \mathfrak{H} onto $M(0)^\perp$ then (1.15) reads as (cf. Proposition 1.7)

$$\tilde{Q}(z) = Q(z) - (Q(z) - Q(i)^*)P(Q(z)P + M_s + I - P)^{-1}(Q(z) - Q(i)). \quad (1.16)$$

If S has defect numbers $(1, 1)$ as above and if $t \in \mathbb{R}$ then (1.15) reads as (cf. [12])

$$\tilde{f}(z) = 2\operatorname{Re}f(i) + t - \frac{|f(i) + t|^2}{f(z) + t}. \quad (1.17)$$

1.3 Space Triplets

Let A be a selfadjoint operator on a Hilbert space \mathfrak{H} .

Definition 1.8. On the Hilbert space \mathfrak{H} we define a new inner product $(\cdot, \cdot)_{-1}$ by

$$(x, y)_{-1} = ((I + |A|)^{-1}x, y), \quad x, y \in \mathfrak{H}.$$

The completion of \mathfrak{H} with respect to the inner product $(\cdot, \cdot)_{-1}$ is denoted by ${}_{-1}\mathfrak{H}(A) = \mathfrak{H}_{-1}$. By $\|\cdot\|_{-1}$ we denote the norm induced by $(\cdot, \cdot)_{-1}$ on \mathfrak{H}_{-1} .

On $(|A|^{\frac{1}{2}}\mathfrak{H})$ we define the inner product $(\cdot, \cdot)_{+1}$ by

$$(x, y)_{+1} = (x, y) + (|A|^{\frac{1}{2}}x, |A|^{\frac{1}{2}}y), \quad x, y \in (|A|^{\frac{1}{2}}\mathfrak{H}).$$

The subspace $(|A|^{\frac{1}{2}}\mathfrak{H})$ provided with the norm $\|\cdot\|_{+1}$ is a Hilbert space, denoted by ${}_{+1}\mathfrak{H}(A) = \mathfrak{H}_{+1}$, which is isometrically isomorphic to the graph of $|A|^{\frac{1}{2}}$. By $\|\cdot\|_{+1}$ we denote the norm induced by $(\cdot, \cdot)_{+1}$ on \mathfrak{H}_{+1} .

Clearly, the Hilbert space \mathfrak{H}_{+1} is contained contractively in \mathfrak{H} and \mathfrak{H} is contained contractively in \mathfrak{H}_{-1} . We get the space triplet:

$$\mathfrak{H}_{+1} \subseteq \mathfrak{H} \subseteq \mathfrak{H}_{-1}.$$

Using the functional calculus for selfadjoint operators it is easy to see that ${}_{+1}$ densely contains the linear manifold (A) . Moreover, (A) and hence ${}_{+1}$ and ${}_{-1}$ are densely contained in ${}_{-1}$.

The mapping $I + |A|$ from $(A) \subseteq {}_{+1}$ onto $\subseteq {}_{-1}$ is an isometry, with respect to the new inner products.

Definition 1.9. We denote by $V_{+1}(A) = V_{+1}$ the unique extension of $I + |A|$ from ${}_{+1}$ onto ${}_{-1}$. Now we define a duality $[\cdot, \cdot]$ between the Hilbert spaces ${}_{+1}$ and ${}_{-1}$ expressed by

$$[x, y] = (V_{+1}x, y)_{-1} = (x, V_{+1}^{-1}y)_{+1}, \quad x \in {}_{+1}, \quad y \in {}_{-1}. \quad (1.18)$$

We set $[y, x] = \overline{[x, y]}$ when $x \in {}_{+1}$ and $y \in {}_{-1}$.

For $x \in {}_{+1}$, $y \in {}_{-1}$ we clearly have

$$[x, y] = (x, y). \quad (1.19)$$

It is easy to check that for $[\cdot, \cdot]$ the Cauchy-Schwarz inequality is valid:

$$|[x, y]| \leq \|x\|_{+1} \|y\|_{-1}, \quad x \in {}_{+1}, \quad y \in {}_{-1}. \quad (1.20)$$

Hence $[\cdot, \cdot]$ is continuous on ${}_{+1} \times {}_{-1}$.

Proposition 1.10. Consider A as a mapping from $(A) \subseteq {}_{+1}$ into ${}_{-1}$. Then A is contractive and hence, A can be extended by continuity to a contractive operator \hat{A} from ${}_{+1}$ into ${}_{-1}$.

Moreover, for complex z the mappings $A - z$ considered as mappings from $(A) \subseteq {}_{+1}$ into ${}_{-1}$ are bounded and can be extended to mappings $\hat{A} - zI$ from ${}_{+1}$ into ${}_{-1}$.

Proof. Let $(E_t)_{t \in \mathbb{R}}$ be the spectral family of A . For $x \in (A)$ we have

$$\begin{aligned} \|Ax\|_{-1}^2 &= \int_{\mathbb{R}} \frac{t^2}{|t| + 1} d(E_t x, x) \leq \\ &\leq \int_{\mathbb{R}} (|t| + 1) d(E_t x, x) = \|x\|_{+1}^2. \end{aligned}$$

Hence A is continuous and can be extended to ${}_{+1}$.

Also the identity operator on (A) can be uniquely extended to all of ${}_{+1}$. This extension is just the embedding from ${}_{+1}$ into ${}_{-1}$.

Clearly, with A and I also $A - z$ can be extended to ${}_{+1}$ and the extension equals to $\hat{A} - zI$. \square

Note that we can look at \hat{A} as the closure of A in ${}_{+1} \oplus {}_{-1}$.

Since A and I are symmetric in \mathfrak{H} , we can extend this symmetric behaviour to $\hat{A} - zI$ by the continuity of $[\cdot, \cdot]$. Hence for all $z \in \mathbb{C}$:

$$[(\hat{A} - zI)x, y] = [x, (\hat{A} - \bar{z}I)y], \quad x, y \in \mathfrak{H}_+. \quad (1.21)$$

Proposition 1.11. *For $z \in \rho(A)$ consider the operator $(A - z)^{-1}$ as a mappings from $\mathfrak{H} \subseteq \mathfrak{H}_{-1}$ into \mathfrak{H}_+ . Then $(A - z)^{-1}$ is bounded, and can therefore be extended by continuity to a bounded operator $\hat{R}(z)$ from \mathfrak{H}_{-1} into \mathfrak{H}_+ .*

Moreover,

$$(\hat{A} - zI)^{-1} = \hat{R}(z), \quad z \in \rho(A),$$

and hence $(\hat{A} - zI)$ is bijective for all $z \in \rho(A)$. Also a kind of resolvent identity holds,

$$(\hat{A} - zI)^{-1} - (\hat{A} - \zeta I)^{-1} = (z - \zeta)(A - zI)^{-1}(\hat{A} - \zeta I)^{-1}, \quad z, \zeta \in \rho(A), \quad (1.22)$$

where we consider $(A - zI)^{-1}$ as an operator from \mathfrak{H}_+ into \mathfrak{H}_+ which turns out to be continuous.

Proof. For $z \in \rho(A)$ and $x \in \mathfrak{H}$ we have

$$\begin{aligned} \|(A - z)^{-1}x\|_{+1}^2 &= \int_{\mathbb{R}} \frac{|t| + 1}{|t - z|^2} d(E_t x, x) \leq \\ &\leq C(z) \int_{\mathbb{R}} \frac{1}{|t| + 1} d(E_t x, x) = \|x\|_{-1}^2, \end{aligned}$$

where $C(z) > 0$ is chosen such that $(|t| + 1)^2 \leq C(z)|t - z|^2$ for all $t \in \sigma(A)$. Since $z \in \rho(A)$ this is always possible as one can easily verify. Hence $\hat{R}(z)$ exists.

Moreover,

$$\hat{R}(z)(\hat{A} - z)x = (A - z)^{-1}(A - z)x = x \text{ for } x \in (A)$$

and

$$(\hat{A} - z)\hat{R}(z)x = (A - z)(A - z)^{-1}x = x \text{ for } x \in \mathfrak{H}.$$

Since these sets are dense in \mathfrak{H}_+ and \mathfrak{H}_{-1} , respectively, we obtain $(\hat{A} - zI)^{-1} = \hat{R}(z)$.

Concerning (1.22), we first note that $(A - zI)^{-1}$ clearly maps \mathfrak{H}_+ into itself. The continuity of this operator follows from the fact that it equals to the composition of the contractive embedding of \mathfrak{H}_+ into \mathfrak{H}_{-1} and the bounded operator $(\hat{A} - zI)^{-1}$.

For $x \in \mathfrak{H}$ we have

$$\begin{aligned} (\hat{A} - zI)^{-1}x - (\hat{A} - \zeta I)^{-1}x &= (A - zI)^{-1}x - (A - \zeta I)^{-1}x = \\ &= (z - \zeta)(A - zI)^{-1}(A - \zeta I)^{-1}x = (z - \zeta)(A - zI)^{-1}(\hat{A} - \zeta I)^{-1}x. \end{aligned}$$

Hence (1.22) holds on \mathcal{H} , and since the operators on the left and on the right hand side of the equality sign in (1.22) are both bounded and since \mathcal{H} is dense in \mathcal{H}_- , Identity (1.22) is verified. \square

The following relation is an immediate consequence of (1.21) and Proposition 1.11,

$$[(\hat{A} - zI)^{-1}x, y] = [x, (\hat{A} - \bar{z}I)^{-1}y], \quad x, y \in \mathcal{H}_-. \quad (1.23)$$

Finally we introduce here the concept of adjointness with respect to $[\cdot, \cdot]$.

Definition 1.12. Let T be a linear relation in $\mathcal{H}_+ \times \mathcal{H}_-$. Then we define the adjoint subspace T^+ as

$$T^+ = \{\{h; k\} \in \mathcal{H}_+ \times \mathcal{H}_- : [g, h] = [f, k] \text{ for all } \{f; g\} \in T\}.$$

We call $S \subseteq \mathcal{H}_+ \times \mathcal{H}_-$ symmetric if $S \subseteq S^+$ and selfadjoint if $S = S^+$.

It is easy to verify that

$$V_{+1}^{-1}(T^+) = (V_{+1}^{-1}T)^*.$$

Hence,

$$(T^+)^+ = \bar{T},$$

$$(T + W)^+ = T^+ + W^+ \text{ where } W \text{ is a bounded operator from } \mathcal{H}_+ \text{ into } \mathcal{H}_-,$$

$$(\alpha T)^+ = \bar{\alpha}T^+ \text{ for } \alpha \in \mathbb{C} \text{ and}$$

the adjoint of a bounded operator is a bounded operator.

As in the usual concept of adjointness we have here

$$T^+(0) = (T)^{\perp[\cdot, \cdot]} = \{y \in \mathcal{H}_- : [x, y] = 0 \text{ for all } x \in (T)\}$$

and

$$(T) = (T)^{\perp[\cdot, \cdot]} = \{x \in \mathcal{H}_+ : [x, y] = 0 \text{ for all } y \in (T)\}.$$

Moreover, by (1.21) the operator \hat{A} and the embedding I from \mathcal{H}_+ into \mathcal{H}_- are selfadjoint.

Similarly, we define R^+ for linear relations R in $\mathcal{H}_- \oplus \mathcal{H}_+$. Clearly, $(T^{-1})^+ = (T^+)^{-1}$ for a linear relation T in $\mathcal{H}_+ \oplus \mathcal{H}_-$. Hence the same results as above hold.

Definition 1.13. Let Γ be a bounded mapping from a Hilbert space \mathcal{H} into \mathcal{H}_- . Then we denote by Γ^+ the bounded mapping from \mathcal{H}_+ into \mathcal{H} defined by

$$[x, \Gamma y] = (\Gamma^+ x, y)$$

for all $x \in \mathcal{H}_+, y \in \mathcal{H}$.

Similarly, let Γ' be a bounded mapping from a Hilbert space \mathfrak{H} into \mathfrak{H}_+^1 . Then we denote by $(\Gamma')^+$ the bounded mapping from \mathfrak{H}_-^1 into \mathfrak{H} defined by

$$[\Gamma'x, y] = (x, (\Gamma')^+y)$$

for all $x \in \mathfrak{H}, y \in \mathfrak{H}_-^1$.

Here Γ^+ is in fact a bounded mapping because $\Gamma^+ = (V_{+1}^{-1}\Gamma)^*$. Similarly, $(\Gamma')^+$ is bounded.

1.4 Spaces of Boundary Values

Following [5], we introduce the concept of spaces of boundary values (SBV). For more details, see [5].

Definition 1.14. Let \mathfrak{H} be a Hilbert space, and let S be a closed, densely defined, symmetric operator in \mathfrak{H} with equal defect numbers. A triplet $(\mathfrak{H}, \Pi^1, \Pi^2)$ where \mathfrak{H} is a Hilbert space and Π^1, Π^2 are linear mappings from the domain (S^*) of S^* into \mathfrak{H} is called a space of boundary values (SBV) for S^* , if

1. $(S^*f, g) - (f, S^*g) = (\Pi^1f, \Pi^2g) - (\Pi^2f, \Pi^1g)$ for all $f, g \in (S^*)$ and
2. the mapping $\Pi : f \mapsto \{\Pi^2f; -\Pi^1f\}$ from (S^*) to \mathfrak{H}^2 is surjective.

Set $A^j = S^*|_{(\Pi^j)}$, $j = 1, 2$. Then A^j is a selfadjoint extensions of S , for $j = 1, 2$ (cf. [5]).

Since $(S^*) = (A^2) \dot{+}_z$, $z \in \mathbb{C} \setminus \mathbb{R}$, the mapping $\Pi^2|_z$ from \mathfrak{H}_z to \mathfrak{H} is bijective. Therefore we can set $\Gamma_z = (\Pi^2|_z)^{-1}$. Moreover, let $M(z)$ be defined as $\Pi^1\Gamma_z$ for $z \in \mathbb{C} \setminus \mathbb{R}$. We call M the Weyl function of $(\mathfrak{H}, \Pi^1, \Pi^2)$. It is shown in [5], Lemma 1 that Γ_z satisfies (1.3) and that M is a Q -function of (S, A^2, Γ_i) .

If \tilde{A} is a canonical selfadjoint extension of S then $\Theta = \Pi((\tilde{A}))$ is a selfadjoint relation on \mathfrak{H} . Conversely, if Θ is a selfadjoint relation on \mathfrak{H} , the operator $\tilde{A} = S^*|_{\Pi^{-1}(\Theta)}$ is selfadjoint in \mathfrak{H} (cf. [5], Section 2).

The relation Θ is the parameter in Kreĭn's formula:

$$(\tilde{A} - z)^{-1} = (A^2 - z)^{-1} - \Gamma_z(M(z) + \Theta)^{-1}\Gamma_z^*. \quad (1.24)$$

Assume \tilde{A} is a selfadjoint extension of S in a larger space $\tilde{\mathfrak{H}}$ and let $T(z)$ be the Strauss relation defined in (1.9). Set $\mathcal{T}(z) = \Pi((T(z)))$. Then \mathcal{T} belongs to $N(\mathfrak{H})$ and the generalized resolvent of \tilde{A} can be written as

$$\tilde{P}(\tilde{A} - z)^{-1}|_{\mathfrak{H}} = (T(z) - z)^{-1} = \quad (1.25)$$

$$= (A^2 - z)^{-1} - \Gamma_z(M(z) + \mathcal{T}(z))^{-1}\Gamma_z^*.$$

Conversely, if $\mathcal{T} \in N()$, then $T(z) = S^*|_{\Pi^{-1}(\mathcal{T}(z))}$ satisfies $-T \in N()$ and $T(z) \supseteq S$ for $z \in \mathbb{C} \setminus \mathbb{R}$. As indicated after (1.9) there exists a selfadjoint extension \tilde{A} of S such that (1.9) and therefore (1.25) hold. For $g \in \mathcal{T}(z)$ this means, that $f = (T(z) - z)^{-1}g$ is the solution of $S^*f - zf = g$, $\Pi(f) \in \mathcal{T}(z)$.

Chapter 2

A Model for the Composition of two Nevanlinna Functions

Every Nevanlinna function f has an integral representation of the form (cf. [16])

$$f(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t), \quad (2.1)$$

where $a, b \in \mathbb{R}$, $b \geq 0$, and μ is a positive measure on \mathbb{R} satisfying

$$\int_{\mathbb{R}} \frac{1}{t^2+1} d\mu(t) < \infty.$$

Using this integral representation, a simple model for the nonconstant Nevanlinna function f is obtained as follows (cf. [12]).

If $b = 0$ take $f = L_{\mu}^2$, A^f the operator of multiplication with the independent variable: $A^f(\varphi(t)) = t\varphi(t)$ with domain $(A^f) = \{\varphi(t) \in L_{\mu}^2 : t\varphi(t) \in L_{\mu}^2\}$, S^f the restriction of A^f to the set $(S^f) = \{\varphi \in (A^f) : \int_{\mathbb{R}} \varphi(t)d\mu(t) = 0\}$, and $\gamma^f(i) = \frac{1}{t-i}$.

If $b > 0$ we take $H^f = L_{\mu}^2 \oplus \mathbb{C}$ with the inner product on the second component defined by $(\alpha, \beta) = b\alpha\bar{\beta}$ for $\alpha, \beta \in \mathbb{C}$, A^f and S^f the linear relations

$$A^f = \left\{ \left\{ \begin{pmatrix} \varphi(t) \\ 0 \end{pmatrix}; \begin{pmatrix} t\varphi(t) \\ \alpha \end{pmatrix} \right\} \in (L_{\mu}^2 \oplus \mathbb{C})^2 : t\varphi(t) \in L_{\mu}^2, \alpha \in \mathbb{C} \right\}$$

$$S^f = \left\{ \left\{ \begin{pmatrix} \varphi(t) \\ 0 \end{pmatrix}; \begin{pmatrix} t\varphi(t) \\ \alpha \end{pmatrix} \right\} \in A^f : b\alpha + \int_{\mathbb{R}} \varphi(t)d\mu(t) = 0 \right\}$$

in $L_{\mu}^2 \oplus \mathbb{C}$, and set $\gamma^f(i) = \left(\frac{1}{t-i}, 1 \right)$.

One easily calculates that in both cases the symmetric relation S^f has defect numbers $(1, 1)$, and that f is a Q -function of $(A^f, S^f, \gamma^f(i))$. Hence $(f, S^f, A^f, \gamma^f(i))$

is a model for f . In addition, S^f is completely nonselfadjoint.

By $\tilde{\mu}$ we denote the measure on $\mathbb{R} \cup \{\infty\}$ such that $\tilde{\mu}(B) = \mu(B)$ for all Borel sets $B \subseteq \mathbb{R}$ and which satisfies $\tilde{\mu}(\{\infty\}) = b$. If $b = 0$, then $L_{\tilde{\mu}}^2$ can be identified with L_{μ}^2 , if $b > 0$, then $L_{\tilde{\mu}}^2 = L_{\mu}^2 \oplus \mathbb{C}$, where the scalar product on the second component is again defined by $(\alpha, \beta) = b\alpha\bar{\beta}$ for $\alpha, \beta \in \mathbb{C}$. In particular $L_{\tilde{\mu}}^2 = f$.

Assume now that f and g are two nonconstant Nevanlinna functions and that f has the integral representation (2.1). Let $(^f, S^f, A^f, \gamma^f(i))$ be the model for f constructed above, and let $(^g, S^g, A^g, \gamma^g(i))$ be any model for g such that S^g is completely nonselfadjoint. The latter property implies that g is separable as for any countable and dense subset X of $\mathbb{C} \setminus \mathbb{R}$ the set $\{\zeta\gamma^g(z) : \zeta, z \in X\}$ is dense in g . It is the aim of this section to build a model for $f \circ g$ on the tensor product of f and g .

For $t \in \mathbb{R} \cup \{\infty\}$ let A_t denote the selfadjoint extension of S^g in g , which is defined by Kreĭn's formula (1.14) with the parameter $-t$ instead of t :

$$(A_t - z)^{-1} = (A^g - z)^{-1} - \gamma^g(z) \frac{(\cdot, \gamma^g(\bar{z}))}{g(z) - t}. \quad (2.2)$$

As mentioned above $A_{\infty} = A^g$.

Now set $\gamma_t(z) = (I + (z - i)(A_t - z)^{-1})\gamma^g(i)$ for each $z \in \mathbb{C} \setminus \mathbb{R}$. Evidently $\gamma_t(i) = \gamma^g(i)$ for all $t \in \mathbb{R} \cup \{\infty\}$.

The relation (1.17) shows that for $t \in \mathbb{R}$ a Q -function of $(S^g, A_t, \gamma_t(i))$ is given by

$$\tilde{g}_t(z) = g(i) + \overline{g(i)} - t - \frac{|t - g(i)|^2}{g(z) - t}, \quad (2.3)$$

and since an arbitrary real constant can be added, a Q -function of $(S^g, A_t, \frac{1}{t - g(i)}\gamma_t(i))$ is given by

$$g_t(z) = \frac{1}{t - g(z)} - \frac{t}{t^2 + 1}.$$

To simplify the notation of the following results we set $g_{\infty}(z) = \tilde{g}_{\infty}(z) = g(z)$.

Definition 2.1. Let $L_{\tilde{\mu}}^2(^g)$ be the Hilbert space of all functions ϕ which map $\mathbb{R} \cup \{\infty\}$ into g such that ϕ is weakly measurable (that is, the function $t \mapsto (\phi(t), x)$ is measurable for all $x \in ^g$), and that $\int_{\mathbb{R} \cup \{\infty\}} \|\phi(t)\|^2 d\tilde{\mu}(t) < \infty$. The scalar product of two elements $\phi, \psi \in L_{\tilde{\mu}}^2(^g)$ is defined by $(\phi, \psi) = \int_{\mathbb{R} \cup \{\infty\}} (\phi(t), \psi(t)) d\tilde{\mu}(t)$ (cf. [21], pp. 383, 531).

We mention that the space $L_{\mu}^2(g)$ can be identified with the completion of $L_{\mu}^2 \otimes g$ with the tensor scalar product $\langle \varphi \otimes x, \omega \otimes y \rangle = (\int_{\mathbb{R} \cup \{\infty\}} \varphi(t) \overline{\omega(t)} d\tilde{\mu}(t))(x, y)$. A unitary mapping from this completion onto $L_{\mu}^2(g)$ can be established by extending the isometric mapping $(\varphi \otimes x) \mapsto \varphi x$ to the completion of $L_{\mu}^2 \otimes g$.

Now we construct a selfadjoint relation A on $L_{\mu}^2(g)$. To this end, define for each $z \in \mathbb{C} \setminus \mathbb{R}$ an operator R_z in $L_{\mu}^2(g)$ by

$$(R_z \phi)(t) = (A_t - z)^{-1} \phi(t), \quad t \in \mathbb{R} \cup \{\infty\}. \quad (2.4)$$

Proposition 2.2. *For $z \in \mathbb{C} \setminus \mathbb{R}$ the operator R_z maps $L_{\mu}^2(g)$ continuously into $L_{\mu}^2(g)$, for $z, \zeta \in \mathbb{C} \setminus \mathbb{R}$ the resolvent identity is satisfied, and R_z is the resolvent $(A - z)^{-1}$ of a selfadjoint relation A on $L_{\mu}^2(g)$.*

Proof. Let $\phi \in L_{\mu}^2(g)$. Then by (2.2) the relation (2.4) can be written as

$$(R_z \phi)(t) = (A^g - z)^{-1} \phi(t) - \gamma^g(z) \frac{(\phi(t), \gamma^g(\bar{z}))}{g(z) - t}. \quad (2.5)$$

This relation is valid also for $t = \infty$ if we set $\frac{1}{g(z) - \infty} = 0$.

The first term on the right hand side of (2.5) belongs to $L_{\mu}^2(g)$ since the function $t \mapsto ((A^g - z)^{-1} \phi(t), x) = (\phi(t), (A^g - \bar{z})^{-1} x)$ is measurable on $\mathbb{R} \cup \{\infty\}$ for all $x \in g$, and since

$$\int_{\mathbb{R} \cup \{\infty\}} \|(A^g - z)^{-1} \phi(t)\|^2 d\tilde{\mu}(t) \leq \frac{1}{(\operatorname{Im} z)^2} \int_{\mathbb{R} \cup \{\infty\}} \|\phi(t)\|^2 d\tilde{\mu}(t). \quad (2.6)$$

The second term on the right hand side of (2.5) belongs to $L_{\mu}^2(g)$ since the function $t \mapsto (\gamma^g(z) \frac{(\phi(t), \gamma^g(\bar{z}))}{g(z) - t}, x) = \frac{(\phi(t), \gamma^g(\bar{z}))}{g(z) - t} (\gamma^g(z), x)$ is measurable for all $x \in g$, and since

$$\begin{aligned} & \int_{\mathbb{R} \cup \{\infty\}} \left\| \gamma^g(z) \frac{(\phi(t), \gamma^g(\bar{z}))}{g(z) - t} \right\|^2 d\tilde{\mu}(t) \leq \\ & \leq \|\gamma^g(z)\|^2 \|\gamma^g(\bar{z})\|^2 \frac{1}{(\operatorname{Im} g(z))^2} \int_{\mathbb{R} \cup \{\infty\}} \|\phi(t)\|^2 d\tilde{\mu}(t). \end{aligned} \quad (2.7)$$

The relations (2.6) and (2.7) also show that R_z is bounded if $z \in \mathbb{C} \setminus \mathbb{R}$.

In order to verify the resolvent identity, we write for $t \in \mathbb{R} \cup \{\infty\}$

$$(R_z \phi - R_{\zeta} \phi)(t) = (A_t - z)^{-1} \phi(t) - (A_t - \zeta)^{-1} \phi(t) =$$

$$= (z - \zeta)(A_t - z)^{-1}(A_t - \zeta)^{-1}\phi(t) = (R_z R_\zeta \phi)(t)$$

for $z, \zeta \in \mathbb{C} \setminus \mathbb{R}$. The relation $R_z^* = R_{\bar{z}}$ is shown similarly.

Finally, [6], Theorem 4.1 implies the existence of a selfadjoint relation A on $L_\mu^2(g)$ such that R_z is the resolvent of A . \square

In the following we write a.e. for almost everywhere.

Proposition 2.3. *The relation A from Proposition 2.2 on $L_\mu^2(g)$ can be written as*

$$A = \{ \{ \phi; \psi \} \in (L_\mu^2(g))^2 : \{ \phi(t); \psi(t) \} \in A_t \text{ } \tilde{\mu}\text{-a.e. on } \mathbb{R} \cup \{ \infty \} \}.$$

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$ be fixed. Assume $\{ \phi(t); \psi(t) \} \in A_t$ $\tilde{\mu}$ -a.e. on $\mathbb{R} \cup \{ \infty \}$. Then $\{ \psi(t) - z\phi(t); \phi(t) \} \in (A_t - z)^{-1}$ $\tilde{\mu}$ -a.e. on $\mathbb{R} \cup \{ \infty \}$. Hence $R_z(\psi - z\phi)(t) = \phi(t)$ $\tilde{\mu}$ -a.e. on $\mathbb{R} \cup \{ \infty \}$. As two elements of $L_\mu^2(g)$ coincide if their representatives coincide $\tilde{\mu}$ -a.e. on $\mathbb{R} \cup \{ \infty \}$, we have $R_z(\psi - z\phi) = \phi$, and therefore $\{ \phi; \psi \} \in A$.

Conversely, $\{ \phi; \psi \} \in A$ implies $R_z(\psi - z\phi) = \phi$ and hence $R_z(\psi - z\phi)(t) = \phi(t)$ $\tilde{\mu}$ -a.e. on $\mathbb{R} \cup \{ \infty \}$. Therefore $\{ \phi(t); \psi(t) \} \in A_t$ $\tilde{\mu}$ -a.e. on $\mathbb{R} \cup \{ \infty \}$. \square

Denote by χ the following function from $\mathbb{R} \cup \{ \infty \}$ into g :

$$\chi(t) = \begin{cases} \frac{1}{t-g(i)}\gamma^g(i), & t \in \mathbb{R} \\ \gamma^g(i), & t = \infty \end{cases} \quad (2.8)$$

Since $\frac{1}{t-g(i)} \in L_\mu^2$, it is easy to see that $\chi \in L_\mu^2(g)$.

Let $S \subseteq A$ be defined as follows: $S = \{ \{ \phi; \psi \} \in A : (\psi + i\phi, \chi) = 0 \}$. Hence a pair $\{ \phi; \psi \} \in A$ belongs to S if and only if $\int_{\mathbb{R} \cup \{ \infty \}} (\psi(t) + i\phi(t), \chi(t)) d\tilde{\mu}(t) = 0$. Clearly S is a symmetric relation, and $(S + i)^\perp$ is the linear span of χ . We set $\gamma(i) = \chi$ and $\gamma(z) = (I + (z - i)(A - z)^{-1})\gamma(i)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Since g_t is a Q -function of $(S^g, A_t, \frac{1}{t-g(i)}\gamma^g(i))$ we have

$$\begin{aligned} (\gamma(z), \gamma(\zeta)) &= ((I + (z - i)(A - z)^{-1})\chi, (I + (\zeta - i)(A - \zeta)^{-1})\chi) = \\ &= \int_{\mathbb{R} \cup \{ \infty \}} \left(\frac{1}{t-g(i)} (I + (z - i)(A_t - z)^{-1})\gamma^g(i), \right. \\ &\quad \left. \frac{1}{t-g(i)} (I + (\zeta - i)(A_t - \zeta)^{-1})\gamma^g(i) \right) d\tilde{\mu}(t) = \end{aligned} \quad (2.9)$$

$$= \int_{\mathbb{R} \cup \{\infty\}} \left(\frac{1}{t - g(i)} \gamma_t(z), \frac{1}{t - g(i)} \gamma_t(\zeta) \right) d\tilde{\mu}(t) = \int_{\mathbb{R} \cup \{\infty\}} \frac{g_t(z) - \overline{g_t(\zeta)}}{z - \bar{\zeta}} d\tilde{\mu}(t)$$

for $z, \zeta \in \mathbb{C} \setminus \mathbb{R}$. Therefore with $q(z) = a + \int_{\mathbb{R} \cup \{\infty\}} g_t(z) d\tilde{\mu}(t)$ the relation (2.9) becomes $(\gamma(z), \gamma(\zeta)) = \frac{q(z) - \overline{q(\zeta)}}{z - \bar{\zeta}}$.

Moreover,

$$a + \int_{\mathbb{R} \cup \{\infty\}} g_t(z) d\tilde{\mu}(t) = a + bg(z) + \int_{\mathbb{R}} \left(\frac{1}{t - g(z)} - \frac{t}{t^2 + 1} \right) d\mu(t) = f(g(z)) = (f \circ g)(z).$$

These considerations imply the following Theorem.

Theorem 2.4. *Let f and g be nonconstant Nevanlinna functions, where f has the integral representation (2.1), and let $(g, S^g, A^g, \gamma^g(i))$ be a model for g , with a symmetric relation S^g being completely nonselfadjoint.*

Assume that $L_{\tilde{\mu}}^2(g)$ is the Hilbert space as in Definition 2.1, and that A is the selfadjoint relation on $L_{\tilde{\mu}}^2(g)$, which is defined through its resolvent in (2.4). Furthermore, let $\gamma(i) = \chi$ be the element of $L_{\tilde{\mu}}^2(g)$ defined by (2.8), and set $S = \{ \{ \phi; \psi \} \in A : (\psi + i\phi, \chi) = 0 \}$. Then $f \circ g$ is a Q function of $(S, A, \gamma(i))$.

We mention that the relation S is in general not completely nonselfadjoint.

Now we consider this model in special cases:

If $g(z) = z$, then $f \circ g = f$, and the model $(L_{\tilde{\mu}}^2(g), S, A, \gamma(i))$ coincides with the multiplication operator model (f, S^f, A^f, γ^f) mentioned at the beginning of this section.

If $f(z) = z$, then the model $(L_{\tilde{\mu}}^2(g), S, A, \gamma(i))$ is the model $(g, S^g, A^g, \gamma^g(i))$ for g .

More generally, if f is any rational Nevanlinna function, then it is of the form

$$f(z) = c + dz + \sum_{j=1}^k \frac{m_j}{t_j - z},$$

where $c, t_1, \dots, t_k \in \mathbb{R}$; $d \geq 0$; $m_1, \dots, m_k > 0$. Comparing this representation with (2.1), we get $a = c + \sum_{j=1}^k \frac{m_j t_j}{t_j^2 + 1}$, $b = d$ and $\mu = \sum_{j=1}^k m_j \delta_{t_j}$, where δ_t is the

Dirac measure at $t \in \mathbb{R}$: $\delta_t(M) = 0$ if $t \notin M \subseteq \mathbb{R}$ and $\delta_t(M) = 1$ if $t \in M \subseteq \mathbb{R}$. The function $f \circ g$ can be written as

$$(f \circ g)(z) = c + dg(z) + \sum_{j=1}^k \frac{m_j}{t_j - g(z)}.$$

We set $t_{k+1} = \infty$, $m_{k+1} = d$, $\beta_1 = \frac{1}{t_1 - g(i)}$, \dots , $\beta_k = \frac{1}{t_k - g(i)}$, $\beta_{k+1} = 1$, $l = k + 1$ if $d > 0$, and $l = k$ if $d = 0$.

The space $L_{\mu}^2(g)$ can be identified with $(g)^l$ with the inner product $\langle \cdot, \cdot \rangle$ defined as $\langle (x_j)_{j=1}^l, (y_j)_{j=1}^l \rangle = \sum_{j=1}^l m_j(x_j, y_j)$ for $(x_j)_{j=1}^l, (y_j)_{j=1}^l \in (g)^l$. Proposition 2.3 shows that $A = \bigoplus_{j=1}^l A_{t_j}$. Since χ belongs to $(i)^l \subseteq (g)^l$, the symmetric relation S is an extension of $(S^g)^l$. The latter symmetric relation has defect numbers (l, l) .

If we set $\Gamma_i((\alpha_j)_{j=1}^l) = (\alpha_j \gamma^g(i))_{j=1}^l$ and $Q(z) = \text{diag}(m_1 \tilde{g}_{t_1}(z), \dots, m_l \tilde{g}_{t_l}(z)) \in \mathbb{C}^{l \times l}$ (see (2.3)), then Γ_i is a bijective linear mapping from \mathbb{C}^l onto $(i)^l = ((S^g)^l + i)^\perp$ and $Q(z)$ is a Q -function of $((S^g)^l, A, \Gamma_i)$. Furthermore, $\chi = \gamma(i) = \Gamma_i((\beta_j)_{j=1}^l)$. Hence the function $(Q(z)(\beta_j)_{j=1}^l, (\beta_j)_{j=1}^l)$ is a Q -function of $(S, A, \gamma(i))$, and equals $(f \circ g)(z)$ up to an additive real constant.

Chapter 3

Q -Functions of Fractional Linearly Transformed Relations

Let \mathfrak{H} be a Hilbert space and let S be a symmetric relation on \mathfrak{H} with equal defect numbers. Further, let A be a fixed selfadjoint extension of S . Let \mathfrak{H}_z and Γ_z be defined as in (1.3) and assume that $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$.

Since $S \subseteq A$, we have $aS \subseteq aA$, $S - b \subseteq A - b$ and $-S^{-1} \subseteq -A^{-1}$. It is easy to see that relations aS , $S - b$ and $-S^{-1}$ are symmetric, and that the relations aA , $A - b$ and $-A^{-1}$ are selfadjoint. We remark that $\gamma(aS) = a\gamma(S)$, $\gamma(S - b) = \gamma(S) - b$ and $\gamma(-S^{-1}) \setminus \{0\} = -\frac{1}{\gamma(S) \setminus \{0\}}$ and that $\tilde{\rho}(aA) = a\tilde{\rho}(A)$, $\tilde{\rho}(A - b) = \tilde{\rho}(A) - b$ and $\tilde{\rho}(-A^{-1}) = -\frac{1}{\tilde{\rho}(A)}$ (cf. [2], Section 2.2.1). Set

$$\begin{aligned}\Pi_z &= (I + (z - i)(aA - z)^{-1})\Gamma_{\frac{z}{a}}, \quad z \in a\rho(A), \\ \Sigma_z &= (I + (z - i)((A - b) - z)^{-1})\Gamma_{i+b}, \quad z \in \rho(A) - b \\ \Delta_z &= (I + (z - i)(-A^{-1} - z)^{-1})\Gamma_i, \quad z \in \rho(-A^{-1}).\end{aligned}$$

Now define the functions

$$F(z) = aQ\left(\frac{z}{a}\right), \quad z \in a\rho(A), \quad G(z) = Q(z + b), \quad z \in \rho(A) - b$$

and

$$H(z) = Q\left(-\frac{1}{z}\right), \quad z \in \rho(-A^{-1}) \setminus \{0\}.$$

Proposition 3.1. *Let $\mathfrak{H}_z = (S - \bar{z})^\perp$ be the defect spaces of S . Then the defect spaces of aS , $S - b$ and $-S^{-1}$ are*

$$\begin{aligned}(aS - \bar{z})^\perp &= \frac{z}{a}, \quad z \in a\gamma(S), \\ ((S - b) - \bar{z})^\perp &= z + b, \quad z \in \gamma(S) - b\end{aligned}$$

and

$$(-S^{-1} - \bar{z})^\perp = -\frac{1}{\bar{z}}, \quad z \in \gamma(S) \setminus \{0\}.$$

Moreover,

$$\begin{aligned} (\Pi_z) &= (aS - \bar{z})^\perp, \quad \Pi_z = \Gamma_{\frac{z}{a}}, \quad z \in a\rho(A) \\ (\Sigma_z) &= ((S - b) - \bar{z})^\perp, \quad \Sigma_z = \Gamma_{z+b}, \quad z \in \rho(A) - b \end{aligned}$$

and

$$\begin{aligned} (\Delta_z) &= (-S^{-1} - \bar{z})^\perp, \quad z \in \rho(-A^{-1}), \\ \Delta_z &= \frac{i}{z} \Gamma_{-\frac{1}{\bar{z}}}, \quad z \in \rho(-A^{-1}) \setminus \{0\}. \end{aligned}$$

The functions F, G, H are Q -functions of $(aS, aA, \Gamma_{\frac{i}{a}})$, $(S - b, A - b, \Gamma_{i+b})$ and $(-S^{-1}, -A^{-1}, \Gamma_i)$, respectively.

Proof. Since $(aS - \bar{z}) = (S - \frac{\bar{z}}{a})$, $((S - b) - \bar{z}) = (S - \overline{(z + b)})$ and since $-S^{-1} - \bar{z} = \{\{g; -f - \bar{z}g\} : \{f; g\} \in S\}$ and $S - \frac{1}{\bar{z}} = \{\{f; g + \frac{1}{\bar{z}}f\} : \{f; g\} \in S\}$, the first three relations hold.

These relations applied with $z = i$, the definition of $\Pi_z, \Sigma_z, \Delta_z$ and the comment before (1.3) imply that Π_z, Δ_z and Δ_z map bicontinuously onto $(aS - \bar{z})^\perp$ for $z \in a\rho(A)$, onto $((S - b) - \bar{z})^\perp$ for $z \in \rho(A) - b$ and onto $(-S^{-1} - \bar{z})^\perp$ for $z \in \rho(-A^{-1})$, respectively.

Moreover, by (1.3),

$$\begin{aligned} \Pi_z &= (I + (z - i)(aA - z)^{-1})\Gamma_{\frac{i}{a}} = (I + (\frac{z}{a} - \frac{i}{a})(A - \frac{z}{a})^{-1})\Gamma_{\frac{i}{a}} = \Gamma_{\frac{z}{a}}, \\ \Sigma_z &= (I + (z - i)((A - b) - z)^{-1})\Gamma_{i+b} = \\ &= (I + ((z + b) - (i + b))(A - (z + b))^{-1})\Gamma_{i+b} = \Gamma_{z+b}. \end{aligned}$$

The identity $-\zeta(-A - \zeta)^{-1} - \frac{1}{\zeta}(-A^{-1} - \frac{1}{\zeta})^{-1} = I$ (cf. [3]) gives

$$\begin{aligned} \Delta_z &= I + (z - i)(-A^{-1} - z)^{-1}\Gamma_i = I - \frac{z - i}{z}(I + \frac{1}{z}(-A - \frac{1}{z})^{-1})\Gamma_i = \\ &= \frac{i}{z}(I + (-\frac{1}{z} - i)(A + \frac{1}{z})^{-1})\Gamma_i = \frac{i}{z}\Gamma_{-\frac{1}{\bar{z}}}. \end{aligned}$$

The assertion about the Q -functions is now a consequence of the relations

$$\frac{aQ(\frac{z}{a}) - aQ(\frac{\zeta}{a})^*}{z - \bar{\zeta}} = \frac{Q(\frac{z}{a}) - Q(\frac{\zeta}{a})^*}{\frac{z}{a} - \frac{\bar{\zeta}}{a}} = \Gamma_{\frac{\zeta}{a}}^* \Gamma_{\frac{z}{a}} = \Pi_{\zeta}^* \Pi_z,$$

$$\frac{Q(z+b) - Q(\zeta+b)^*}{z - \bar{\zeta}} = \frac{Q(z+b) - Q(\zeta+b)^*}{(z+b) - (\zeta+b)} = \Gamma_{\zeta+b}^* \Gamma_{z+b} = \Sigma_{\zeta}^* \Sigma_z,$$

$$\frac{Q(-\frac{1}{z}) - Q(-\frac{1}{\zeta})^*}{z - \bar{\zeta}} = \frac{1}{z\bar{\zeta}} \frac{Q(-\frac{1}{z}) - Q(-\frac{1}{\zeta})^*}{-\frac{1}{z} + \frac{1}{\zeta}} = \frac{i\bar{i}}{z\bar{\zeta}} \Gamma_{-\frac{1}{\zeta}}^* \Gamma_{-\frac{1}{z}} = \Delta_{\zeta}^* \Delta_z.$$

□

The following corollary is an immediate consequence of Proposition 3.1.

Corollary 3.2. *Let $a \in \mathbb{R} \setminus \{0\}$, $b, c \in \mathbb{R}$ then $aQ(-\frac{1}{az+ac} + \frac{b}{a})$ is a Q -function of $(-aS - b)^{-1} - c$, $-(aA - b)^{-1} - c$, $\frac{i}{i+c} \Gamma_{-\frac{1}{ai+ac} + \frac{b}{a}}$. In particular, $-Q(\frac{1}{z})$ is a Q -function of $(S^{-1}, A^{-1}, \Gamma_{-i})$.*

In the rest of this chapter we set $P(z) = -Q(\frac{1}{z})$ which is a Q -function of (S^{-1}, A^{-1}) by the previous Corollary.

Corollary 3.3. *Suppose that S is completely nonselfadjoint. Then $(A) = \{0\}$ if and only if $\lim_{\eta \downarrow 0} \eta(Q(i\eta)x, x) = 0$ for all $x \in \cdot$.*

Proof. The kernel of A is trivial if and only if A^{-1} is an operator. Since $= \text{c.l.s.}(\bigcup_{z \in \mathbb{C} \setminus \mathbb{R}} z) = \text{c.l.s.}(\bigcup_{z \in \mathbb{C} \setminus \mathbb{R}} z')$, with S also the relation S^{-1} is completely nonselfadjoint. Corollary 2.5 of [14] yields that A^{-1} is an operator if and only if $\lim_{\eta \uparrow \infty} \frac{1}{\eta}(P(i\eta)x, x) = 0$ for all $x \in \cdot$.

The Proposition 3.1 implies $\lim_{\eta \uparrow \infty} \frac{1}{\eta}(P(i\eta)x, x) = \lim_{\eta \uparrow \infty} \frac{1}{\eta}(-Q(\frac{1}{i\eta})x, x) = -\lim_{\eta \downarrow 0} \overline{\eta(Q(i\eta)x, x)}$. Therefore $\lim_{\eta \uparrow \infty} \frac{1}{\eta}(P(i\eta)x, x) = 0$ if and only if $\lim_{\eta \downarrow 0} \eta(Q(i\eta)x, x) = 0$. □

A Nevanlinna function f belongs by definition to the class \mathbf{N}_0 if and only if $\lim_{\eta \uparrow \infty} \eta \text{Im} f(i\eta) < \infty$ (cf. [15]).

Definition 3.4. We denote by \mathbf{N}_0^- the set of all Nevanlinna functions f which satisfy $\lim_{\eta \downarrow 0} \frac{1}{\eta} \text{Im} f(i\eta) < \infty$.

It is easy to show that $f(z) \in \mathbf{N}_0^-$ if and only if $-f(\frac{1}{z}) \in \mathbf{N}_0$.

Corollary 3.5. *The range (S) is dense in $\overline{(A)}$ if and only if for all $x \in \setminus \{0\}$ the Nevanlinna function $(Q(z)x, x)$ does not belong to \mathbf{N}_0^- .*

Proof. By [14], Corollary 2.6 (the assumption that S is completely nonselfadjoint is not necessary) the range $(S) = (S^{-1})$ is dense in $\overline{(A)} = \overline{(A^{-1})}$ if and only

if the Q -function P of (S^{-1}, A^{-1}) satisfies $(P(z)x, x) \notin \mathbf{N}_0$ for all $x \in \setminus \{0\}$, and this is equivalent to $(Q(z)x, x) \notin \mathbf{N}_0^-$ for all $x \in \setminus \{0\}$. \square

A Nevanlinna function f belongs by definition to the class \mathbf{N}_1 if and only if $\int_1^\infty \frac{1}{\eta} \operatorname{Im} f(i\eta) d\eta < \infty$ (cf. [12]).

Definition 3.6. We denote by \mathbf{N}_1^- the set of all Nevanlinna functions f which satisfy $\int_0^1 \frac{1}{\eta} \operatorname{Im} f(i\eta) d\eta < \infty$.

From the relation $\int_0^1 \frac{1}{\eta} \operatorname{Im} f(i\eta) d\eta = \int_1^\infty -\frac{1}{\eta} \operatorname{Im} f(\frac{1}{i\eta}) d\eta$ it immediately follows that $f(z) \in \mathbf{N}_1^-$ is equivalent to $-f(\frac{1}{z}) \in \mathbf{N}_1$.

For a closed nonnegative (hence symmetric) relation S ($(y, x) \geq 0$ for all $\{x, y\} \in S$) we denote by S_F the Friedrichs extension and by S_N the von Neumann extension of S (cf. [3]). The following proposition is a slight generalization of [4], Theorem 1.1, as we do not assume that S is a densely defined operator.

Proposition 3.7. *Suppose S that is a closed nonnegative relation on \cdot . If A is a selfadjoint extension of S in \cdot , then $A = S_F$ if and only if $(Q(z)x, x) \notin \mathbf{N}_1$ for all $x \in \setminus \{0\}$.*

Proof. First assume that A is nonnegative. Let \mathcal{H}_A be the Hilbert space $((A_s^{\frac{1}{2}}), ((A_s + I)^{\frac{1}{2}}, (A_s + I)^{\frac{1}{2}}))$ (cf. [3]), where A_s is the operator part of A .

Note that with $\cdot' = \overline{(A)}$ the space \mathcal{H}_A coincides with $\cdot'_{+1}(A_s)$ from Definition 1.8 if we start in that Definition with \cdot' and A_s instead of \cdot and A .

We have $S_F = A$ if and only if $\mathcal{H}_A = S_F$ (cf. [3] Section 3). From [3] Theorem 3 we see that $\mathcal{H}_A = S_F$ if and only if $\mathcal{H}_{-1} \cap (A_s^{\frac{1}{2}}) = \{0\}$. Since $(I + (i+1)(A-i)^{-1})$ maps \mathcal{H}_{-1} bicontinuously onto \mathcal{H}_i , the relation $\mathcal{H}_{-1} \cap (A_s^{\frac{1}{2}}) = \{0\}$ is equivalent to $\mathcal{H}_i \cap (A_s^{\frac{1}{2}}) = \{0\}$. With the Propositions 1.2 and 2.1 in [12] we find that $\Gamma_i(x) \in \mathcal{H}_i \cap (A_s^{\frac{1}{2}}) = \{0\}$ if and only if the Nevanlinna function $(Q(z)x, x)$ belongs to \mathbf{N}_1 . This proves the assertion for $A \geq 0$.

Finally we prove that $(Q(z)x, x) \notin \mathbf{N}_1$ for all $x \in \setminus \{0\}$ implies $A \geq 0$. Set $\cdot' = \overline{(A)}$ and let $\cdot'_{+1} = \cdot'_{+1}(A_s)$ and $\cdot'_{-1} = \cdot'_{-1}(A_s)$ be the Hilbert spaces defined in Definition 1.8 where we apply that Definition to \cdot' and A_s instead of \cdot and A .

For $\{u; v\} \in S$ and $w \in \cdot'_{+1}$ we get $[(\hat{A}_s + iI)u, w] = ((A_s + i)u, w) = (v + iu, w)$ since $w \perp A(0)$. This shows that $\{w \in \cdot'_{+1} : [(\hat{A}_s + iI)(S), w] = 0\} = \cdot'_{+1} \cap \mathcal{H}_i$.

From the assumption and from the Propositions 1.2 and 2.1 in [12] we find $\cdot'_{+1} \cap \mathcal{H}_i = \{0\}$. From the definition of $[\cdot, \cdot]$ in (1.18) it follows that $((\hat{A}_s + iI)(S))^{\perp(\cdot)_{-1}} =$

$\{0\}$. (Note that V_{+1} in (1.18) is bijective.) Observing the bicontinuity of $\hat{A}_s + iI$, we obtain the density of (S) in $'_{+1}$.

Since \hat{A}_s is continuous and (S) is dense, the relation $[\hat{A}_s u, u] = (v, u) \geq 0$ for $\{u; v\} \in S$ implies $[\hat{A}_s u, u] \geq 0$ for all $u \in '_{+1}$, and therefore $(v, u) = [\hat{A}_s u, u] \geq 0$ for all $\{u; v\} \in A$. \square

The following corollary is a generalization of [4], Theorem 1.3, as we assume neither that S is a densely defined operator nor that A is nonnegative.

Corollary 3.8. *Suppose that S is a closed nonnegative symmetric relation on \mathfrak{H} . If A is any selfadjoint extension of S in \mathfrak{H} , then $A = S_N$ if and only if $(Q(z)x, x) \notin \mathbf{N}_1^-$ for all $x \in \mathfrak{H} \setminus \{0\}$.*

Proof. Since $S_N = ((S^{-1})_F)^{-1}$, we have $A = S_N$ if and only if $A^{-1} = (S^{-1})_F$. Proposition 3.7 shows that this is equivalent to $(P(z)x, x) \notin \mathbf{N}_1$ for all $x \in \mathfrak{H} \setminus \{0\}$. By the remark after Definition 3.6 $(P(z)x, x) \notin \mathbf{N}_1$ is equivalent to $(Q(z)x, x) \notin \mathbf{N}_1^-$. \square

Corollary 3.9. *Suppose S is nonnegative, and let A be nonnegative selfadjoint extension of S . Then the Friedrichs extension S_F coincides with the von Neumann extension S_N if and only if $(Q(z)x, x) \notin (\mathbf{N}_1 \cup \mathbf{N}_1^-)$ for all $x \in \mathfrak{H} \setminus \{0\}$.*

Proof. If $S_N = S_F$, then since $S_N \leq A \leq S_F$ (cf. [3]), we have $A = S_F = S_N$. So Proposition 3.7 and Corollary 3.8 imply $(Q(z)x, x) \notin (\mathbf{N}_1 \cup \mathbf{N}_1^-)$ for all $x \in \mathfrak{H} \setminus \{0\}$.

Conversely, if $(Q(z)x, x) \notin (\mathbf{N}_1 \cup \mathbf{N}_1^-)$ for all $x \in \mathfrak{H} \setminus \{0\}$, Proposition 3.7 and Corollary 3.8 show that $A = S_F$ and $A = S_N$. \square

Chapter 4

Selfadjoint Extensions of the Orthogonal Sum of Symmetric Relations

4.1 The Parameter in Kreĭn's Formula

Let $\mathfrak{H}_1, \mathfrak{H}_2$ be two Hilbert spaces, and let S_1, S_2 be closed symmetric relations on \mathfrak{H}_1 and \mathfrak{H}_2 , respectively. Assume that S_1, S_2 have finite and equal defect numbers (n_1, n_1) and (n_2, n_2) , respectively. We fix canonical selfadjoint extensions A_1 of S_1 and A_2 of S_2 , i.e. A_1 and A_2 are selfadjoint in \mathfrak{H}_1 and \mathfrak{H}_2 , respectively.

For $j = 1, 2$ we denote by Γ_i^j a bijective linear mapping from \mathbb{C}^{n_j} onto $\mathfrak{H}_i^j = ((S_j + i))^\perp$. Set $\Gamma_z^j = (I + (z - i)(A_j - z)^{-1})\Gamma_i^j$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Then Γ_z^j is a bijective linear mapping from \mathbb{C}^{n_j} onto \mathfrak{H}_z^j which satisfies relation (1.3).

For $j = 1, 2$ let Q_j be the Q -function of (S_j, A_j, Γ_i^j) defined by the relation (1.4).

We set $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. By \tilde{P}_1, \tilde{P}_2 we denote the orthogonal projection of \mathfrak{H} onto \mathfrak{H}_1 or \mathfrak{H}_2 , respectively. Set $S = S_1 \oplus S_2$ ($S_1 \oplus S_2$ equals $\{ \{x_1; x_2\}; \{y_1; y_2\} \} \in \mathfrak{H}^2 : \{x_1; y_1\} \in S_1, \{x_2; y_2\} \in S_2$) and $A = A_1 \oplus A_2$. Then S is a closed symmetric relation on \mathfrak{H} with finite and equal defect numbers (n, n) , where $n = n_1 + n_2$.

Let Γ_z be defined as $\Gamma_z^1 \oplus \Gamma_z^2$ for $z \in \mathbb{C} \setminus \mathbb{R}$ ($\Gamma_z \{x; y\} = \{ \Gamma_z^1(x); \Gamma_z^2(y) \}$ for $\{x; y\} \in \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} = \mathbb{C}^n$). The mapping Γ_z is a bijective linear mapping from $\mathfrak{H} = \mathbb{C}^n$ onto $\mathfrak{H}_z = ((S - \bar{z}))^\perp = \mathfrak{H}_z^1 \oplus \mathfrak{H}_z^2$, and satisfies (1.3). Similarly, $Q(z) = Q_1(z) \oplus Q_2(z)$ satisfies (1.4), and hence $Q(z)$ is a Q -function of (S, A, Γ_i) .

By \mathfrak{H}_1 (\mathfrak{H}_2) we denote the subspace $\mathbb{C}^{n_1} \times \{0\}$ ($\{0\} \times \mathbb{C}^{n_2}$) of $\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} = \mathbb{C}^n$, and by P_1 (P_2) the orthogonal projections of $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$ onto \mathfrak{H}_1 (\mathfrak{H}_2). Then $\Gamma_z^j P_j = \tilde{P}_j \Gamma_z$, and with respect to the decomposition $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ of \mathbb{C}^n the 2×2 block matrix representation of $Q(z)$ has diagonal form $Q(z) = \text{diag}(Q_1(z), Q_2(z))$.

Our aim in this section is to describe all generalized resolvents on \mathfrak{H}_1 of selfadjoint relations \tilde{A} on a Hilbert space $\tilde{\mathfrak{H}}$, such that $\mathfrak{H}_1 \subseteq \tilde{\mathfrak{H}}$ and $S \subseteq \tilde{A}$. More precisely, let $\mathcal{N} \in N(\cdot)$ be the parameter in

$$\tilde{P}(\tilde{A} - z)^{-1}|_{\mathfrak{H}_1} = (A - z)^{-1} - \Gamma_z(Q(z) + \mathcal{N}(z))^{-1}\Gamma_z^*, \quad (4.1)$$

where \tilde{P} is the orthogonal projection of $\tilde{\mathfrak{H}}$ onto \mathfrak{H}_1 . Then we will determine $\mathcal{T} \in N(\mathfrak{H}_1)$ such that

$$\tilde{P}_1(\tilde{A} - z)^{-1}|_{\mathfrak{H}_1} = (A_1 - z)^{-1} - \Gamma_z^1(Q_1(z) + \mathcal{T}(z))^{-1}\Gamma_z^{1*} \quad (4.2)$$

holds for $z \in \mathbb{C} \setminus \mathbb{R}$, where \tilde{P}_1 also denotes the orthogonal projection of $\tilde{\mathfrak{H}}$ onto \mathfrak{H}_1 .

The following lemma is well known.

Lemma 4.1. *Let $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ be a 2×2 block matrix, such that $D_{11} \in \mathbb{C}^{k_1 \times k_1}$, $D_{12} \in \mathbb{C}^{k_1 \times k_2}$, $D_{21} \in \mathbb{C}^{k_2 \times k_1}$, $D_{22} \in \mathbb{C}^{k_2 \times k_2}$, where $k_1, k_2 \in \mathbb{N}$. Assume that D_{22} is invertible. Then D is invertible if and only if $R = D_{11} - D_{12}D_{22}^{-1}D_{21}$ is invertible, and we have*

$$D^{-1} = \begin{pmatrix} R^{-1} & -R^{-1}D_{12}D_{22}^{-1} \\ -D_{22}^{-1}D_{21}R^{-1} & D_{22}^{-1} + D_{22}^{-1}D_{21}R^{-1}D_{12}D_{22}^{-1} \end{pmatrix}.$$

We begin with the case that $\mathcal{N}(z)$ is an operator for $z \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 4.2. *Let \tilde{A} be a selfadjoint extension of S in $\tilde{\mathfrak{H}} \supseteq \mathfrak{H}$, and assume that $\mathcal{N} \in N(\cdot)$ is the parameter in (1.10), and that $\mathcal{N}(z)$ is an operator for $z \in \mathbb{C} \setminus \mathbb{R}$. With respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ of \mathbb{C}^n we write $\mathcal{N}(z)$ as a 2×2 block matrix*

$$\mathcal{N}(z) = \begin{pmatrix} \mathcal{N}_{11}(z) & \mathcal{N}_{12}(z) \\ \mathcal{N}_{21}(z) & \mathcal{N}_{22}(z) \end{pmatrix},$$

where $\mathcal{N}_{11}(z) \in \mathbb{C}^{n_1 \times n_1}$, $\mathcal{N}_{12}(z) \in \mathbb{C}^{n_1 \times n_2}$, $\mathcal{N}_{21}(z) \in \mathbb{C}^{n_2 \times n_1}$, $\mathcal{N}_{22}(z) \in \mathbb{C}^{n_2 \times n_2}$.

The generalized resolvent $\tilde{P}_1(\tilde{A} - z)^{-1}|_{\mathfrak{H}_1}$ can be written in the form (4.2) where

$$\begin{aligned} \mathcal{T}(z) &= \mathcal{N}_{11}(z) - \mathcal{N}_{12}(z)(Q_2(z) + \mathcal{N}_{22}(z))^{-1}\mathcal{N}_{21}(z) = \\ &= (P_1(Q_2(z)P_2 + \mathcal{N}(z) + \alpha P_1)^{-1}|_{\mathfrak{H}_1})^{-1} - \alpha I \end{aligned} \quad (4.3)$$

is a holomorphic matrix function belonging to $N(\mathfrak{H}_1)$. The second equality holds for all $\alpha \in \mathbb{C}$ whose modulus is sufficiently large and for those which are nonreal.

Proof. Since

$$\tilde{P}_1(\tilde{A} - z)^{-1}|_{\mathfrak{H}_1} = (A_1 - z)^{-1} - \Gamma_z^1 P_1(Q(z) + \mathcal{N}(z))^{-1}|_{\mathfrak{H}_1} \Gamma_z^{1*},$$

we write $Q(z) + \mathcal{N}(z)$ as a 2×2 block matrix

$$Q(z) + \mathcal{N}(z) = \begin{pmatrix} Q_1(z) + \mathcal{N}_{11}(z) & \mathcal{N}_{12}(z) \\ \mathcal{N}_{21}(z) & Q_2(z) + \mathcal{N}_{22}(z) \end{pmatrix}$$

Now apply Lemma 4.1 to obtain $P_1(Q(z) + \mathcal{N}(z))^{-1}|_1 = (Q_1(z) + \mathcal{T}(z))^{-1}$. The assumption of the lemma is satisfied since $\text{Im}z \text{Im}(Q_2(z) + \mathcal{N}_{22}(z)) > 0$ implies the invertibility of $Q_2(z) + \mathcal{N}_{22}(z)$.

Another application of Lemma 4.1 shows that $Q_2(z)P_2 + \mathcal{N}(z) + \alpha P_1$ is invertible in \mathbb{C}^n if and only if $\mathcal{T}(z) + \alpha I$ is invertible in \mathbb{C} . This always happens if $|\alpha|$ is sufficiently large or if $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Then $\mathcal{T}(z) = (P_1(Q_2(z)P_2 + \mathcal{N} + \alpha P_1)^{-1}|_1)^{-1} - \alpha I$. From this second representation it is easy to see that $\mathcal{T} \in N(\mathbb{C})$. \square

Lemma 4.3. *Let P be an orthogonal projection in \mathbb{C}^n , and let $\{e_1, \dots, e_n\}$ be an orthogonal basis of \mathbb{C}^n . Assume that P maps the linear span $\text{l.s.}\{e_1, \dots, e_r\}$ of $\{e_1, \dots, e_r\}$ bijectively onto $P(\mathbb{C}^n)$. Then $I - P$ maps $\text{l.s.}\{e_{r+1}, \dots, e_n\}$ bijectively onto $(I - P)(\mathbb{C}^n)$.*

Proof. The assumption implies $(P) \cap \text{l.s.}\{e_1, \dots, e_r\} = \{0\}$. Since the dimension of (P) equals $n - r$, we have $(P) \dot{+} \text{l.s.}\{e_1, \dots, e_r\} = \mathbb{C}^n$.

Passing to the orthogonal complements we get $(I - P) \cap \text{l.s.}\{e_{r+1}, \dots, e_n\} = \{0\}$. Hence $I - P$ maps $\text{l.s.}\{e_{r+1}, \dots, e_n\}$ injectively into $(I - P)(\mathbb{C}^n)$. A comparison of the dimensions finishes the proof. \square

Now consider the case that $\mathcal{N}(z)$ is a proper relation for one and hence for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 4.4. *Let \tilde{A} be a selfadjoint extension of S in $\tilde{\mathfrak{D}}$ and assume that $\mathcal{N} \in N(\tilde{\mathfrak{D}})$ is the parameter in (1.10). We write $\mathcal{N}(z)$ in the form (1.2):*

$$\mathcal{N}(z) = (\mathcal{N}_s(z)P + I - P)P^{-1},$$

where $\mathcal{N}_s(z) = \mathcal{N}(z)_s$ is the operator part of $\mathcal{N}(z)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ and where P denotes the orthogonal projection of \mathbb{C}^n onto $\mathcal{N}(0)^\perp$. Let Q be the orthogonal projection of \mathbb{C}^n onto the range of P_1P , and let Q' be the orthogonal projection of \mathbb{C}^n onto $P(\mathbb{C}^n) \cap \mathfrak{D}_2$.

We write P , $\mathcal{N}_s(z)P$ and $Q(z)$ as 4×4 block matrices with respect to the decomposition $Q(\mathfrak{D}_1) \oplus (I - Q)(\mathfrak{D}_1) \oplus (I - Q')(\mathfrak{D}_2) \oplus Q'(\mathfrak{D}_2)$ of \mathbb{C}^n :

$$P = \begin{pmatrix} P_{11} & 0 & P_{13} & 0 \\ 0 & 0 & 0 & 0 \\ P_{13}^* & 0 & P_{33} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad (4.4)$$

$$\mathcal{N}_s(z)P = \begin{pmatrix} N_{11}(z) & 0 & N_{13}(z) & N_{14}(z) \\ 0 & 0 & 0 & 0 \\ N_{31}(z) & 0 & N_{33}(z) & N_{34}(z) \\ N_{41}(z) & 0 & N_{43}(z) & N_{44}(z) \end{pmatrix}, \quad (4.5)$$

$$Q(z) = \begin{pmatrix} Q_{11}^1(z) & Q_{12}^1(z) & 0 & 0 \\ Q_{21}^1(z) & Q_{22}^1(z) & 0 & 0 \\ 0 & 0 & Q_{33}^2(z) & Q_{34}^2(z) \\ 0 & 0 & Q_{43}^2(z) & Q_{44}^2(z) \end{pmatrix}, \quad (4.6)$$

where P_{11} is invertible.

Then the generalized resolvent $\tilde{P}_1(\tilde{A} - z)^{-1}|_1$ can be written in the form (4.2) where $\mathcal{T}(z) = \mathcal{R}(z) \oplus \{\{0; (I - Q)x\} \in (1)^2 : x \in 1\} = (\mathcal{R}(z)Q + (I - Q))Q^{-1} \in N(1)$, and where the operator valued function \mathcal{R} belongs to $N(Q(\cdot))$ and has the form

$$\begin{aligned} \mathcal{R}(z) = & P_{11}^{-1}(P_{13}Q_{33}^2(z)P_{13}^* + N_{11}(z) - \\ & -(P_{13}Q_{34}^2(z) + N_{14}(z))(Q_{44}^2(z) + N_{44}(z))^{-1}(Q_{43}^2(z)P_{13}^* + N_{41}(z)))P_{11}^{-1} \end{aligned} \quad (4.7)$$

If we choose $\alpha \in \mathbb{C}$ such that $\mathcal{R}(z) + \alpha I$ is invertible (which is in particular the case if $|\alpha|$ is sufficiently large or if $\alpha \in \mathbb{C} \setminus \mathbb{R}$), then $\mathcal{R}(z)$ can also be written as

$$\begin{aligned} \mathcal{R}(z) = & P_{11}^{-1}((Q((Q + Q')(PQ_2(z)P_2P + N + \alpha Q)(Q + Q') + \\ & + I - (Q + Q'))^{-1}|_{Q(1)})^{-1} - \alpha I)P_{11}^{-1}. \end{aligned} \quad (4.8)$$

If $Q = 0$, then the above relation for $\mathcal{T}(z)$ reduces to

$$\mathcal{T}(z) = \{\{0; x\} \in (1)^2 : x \in 1\}.$$

If $Q' = 0$ the relation (4.7) reads as

$$\mathcal{R}(z) = P_{11}^{-1}(P_{13}Q_{33}^2(z)P_{13}^* + N_{11}(z))P_{11}^{-1}. \quad (4.9)$$

If P commutes with P_1 which is the case if e.g. $Q' = P_2$ then

$$\mathcal{R}(z) = N_{11}(z) - N_{14}(z)(Q_{44}^2(z) + N_{44}(z))^{-1}N_{41}(z). \quad (4.10)$$

Proof. From (1.11) we obtain

$$\tilde{P}_1(\tilde{A} - z)^{-1}|_1 = (A_1 - z)^{-1} - \Gamma_z^1 P_1 P (Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}|_1 \Gamma_z^{1*}.$$

If $Q = 0$, which is equivalent to $P_1 P = 0$, then $\tilde{P}_1(\tilde{A} - z)^{-1}|_1 = (A_1 - z)^{-1}$ and hence $\tilde{P}_1(\tilde{A} - z)^{-1}|_1$ can be written in the form (4.2) where $\mathcal{T}(z) = \{\{0; x\} \in$

$(1)^2 : x \in 1\}$.

Otherwise, (1.12) implies

$$\begin{aligned} P(Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1} &= \\ P(PQ(z)P + \mathcal{N}_s(z)P + I - P)^{-1}P. \end{aligned} \quad (4.11)$$

Now we write P and $\mathcal{N}_s(z)P$ as 4×4 block matrices with respect to the decomposition $Q(1) \oplus (I - Q)(1) \oplus (I - Q')(2) \oplus Q'(2)$ of \mathbb{C}^n . Since the range of P_1P equals the range of P_1PP_1 , since by the definition of Q the subspace $(I - Q)(1)$ is contained in the kernel of P and since $P|_{Q'(2)} = I$, it is easy to see that P_{11} is invertible in $Q(\mathbb{C}^n)$ and that P has the form (4.4). Since $\mathcal{N}_s(z)P$ commutes with P , $\mathcal{N}_s(z)P$ can be written as (4.5).

We will construct a unitary matrix $V \in \mathbb{C}^{d \times d}$ such that the column vectors of V either belong to the range of P or to the range of $I - P$, and such that for all $j = 1, \dots, d$ those column vectors of V between the first and the j -th column which belong to the range of P span exactly the same subspace of the range of P as the first j column vectors of P .

Such a V can be obtained with the Gram-Schmidt orthogonalization process. In the j -th step of the construction of V we apply the orthogonalization process to the j -th column of P or $I - P$ depending on whether the j -th column of P is linearly independent from the already constructed $j - 1$ columns of V or not. It follows from Lemma 4.3, that we end up with a unitary matrix, and this matrix has the indicated properties.

The matrix V^*PV is diagonal matrix where for $j = 1, \dots, d$ it has a 1 in the position (j, j) if and only if the j -th column vector of V belongs to the range of P , and zero everywhere else. It is easy to see that the matrix PV is lower triangular. From the properties of P it follows that we can write V , V^*PV and PV in the following forms (with respect to the decomposition $Q(1) \oplus (I - Q)(1) \oplus (I - Q')(2) \oplus Q'(2)$ of \mathbb{C}^n)

$$V = \begin{pmatrix} V_{11} & 0 & V_{13} & 0 \\ 0 & I & 0 & 0 \\ V_{31} & 0 & V_{33} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad V^*PV = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D_{33} & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

$$PV = \begin{pmatrix} C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_{31} & 0 & C_{33} & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

where C_{11} is invertible, since P_{11} is invertible.

Let $x \in (I - Q')_{(2)}$. Then $C_{33}x = PVx \in (I - Q')_{(2)} \subseteq {}_2$. Hence $C_{33}x \in P(\mathbb{C}^n) \cap {}_2 = Q'_{(2)}$. This means that $C_{33}x \in Q'_{(2)} \cap (I - Q')_{(2)} = \{0\}$. Therefore $C_{33} = 0$ and $D_{33} = 0$ and we have

$$V^* \mathcal{N}_s(z) PV = \begin{pmatrix} N'_{11}(z) & 0 & 0 & N'_{14}(z) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ N'_{41}(z) & 0 & 0 & N_{44}(z) \end{pmatrix}.$$

The relation (4.11) and the above considerations imply

$$\begin{aligned} & P_1 P (Q(z)P + \mathcal{N}_s(z)P + (I - P))^{-1}|_1 = \\ & P_1 PV (V^* P Q(z) PV + V^* \mathcal{N}_s(z) PV + I - V^* PV)^{-1} (PV)^*|_1 = \\ & QPVQ (Q(PV)^* Q Q_1(z) QPVQ + (PV)^* Q_2(z) P_2 PV + \\ & + V^* \mathcal{N}_s(z) PV + I - V^* PV)^{-1} Q(PV)^* Q|_1 = C_{11} R(z) C_{11}^*, \end{aligned}$$

where $R(z)$ is the left upper entry of the block matrix

$$\begin{pmatrix} C_{11}^* Q_{11}^1(z) C_{11} + C_{31}^* Q_{33}^2(z) C_{31} + N'_{11}(z) & 0 & 0 & C_{31}^* Q_{34}^2(z) + N'_{14}(z) \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ Q_{43}^2(z) C_{31} + N'_{41}(z) & 0 & 0 & Q_{44}^2(z) + N_{44}(z) \end{pmatrix}^{-1}$$

Now we apply Lemma 4.1 to this matrix, which is possible, since $\text{Im} Q_{44}^2(z) > 0$ implies that the 3×3 submatrix in the lower right corner is invertible. This gives

$$\begin{aligned} & QPVQ (Q(PV)^* Q Q_1(z) QPVQ + (PV)^* Q_2(z) P_2 PV + \\ & V^* \mathcal{N}_s(z) PV + I - V^* PV)^{-1} QPVQ|_1 = \\ & (Q_{11}^1(z) + \mathcal{R}(z))^{-1}|_1, \end{aligned}$$

where the operator $\mathcal{R}(z)$ acts on $Q(\mathbb{C}^n)$ and has the form

$$\begin{aligned} \mathcal{R}(z) &= C_{11}^{-*} C_{31}^* Q_{33}^2(z) C_{31} C_{11}^{-1} + C_{11}^{-*} N'_{11}(z) C_{11}^{-1} - \\ & - C_{11}^{-*} (C_{31}^* Q_{34}^2(z) + N'_{14}(z)) (Q_{44}^2(z) + N_{44}(z))^{-1} (Q_{43}^2(z) C_{31} + N'_{41}(z)) C_{11}^{-1}. \end{aligned}$$

Since P_{11} is invertible we obtain by the construction of V

$$\begin{pmatrix} V_{11} \\ 0 \\ V_{31} \\ 0 \end{pmatrix} = \begin{pmatrix} P_{11} \\ 0 \\ P_{13}^* \\ 0 \end{pmatrix} W,$$

where W is an invertible $\dim(Q(1)) \times \dim(Q(1))$ matrix. As $PP = P$ this yields

$$\begin{aligned} C_{31}C_{11}^{-1} &= \begin{pmatrix} P_{13}^* & 0 & P_{33} & 0 \end{pmatrix} \begin{pmatrix} P_{11} \\ 0 \\ P_{13}^* \\ 0 \end{pmatrix} \left(\begin{pmatrix} P_{11} & 0 & P_{13} & 0 \end{pmatrix} \begin{pmatrix} P_{11} \\ 0 \\ P_{13}^* \\ 0 \end{pmatrix} \right)^{-1} \\ &= P_{13}^* P_{11}^{-1}. \end{aligned}$$

Keeping in mind that P commutes with $\mathcal{N}_s(z)P$, we get with similar methods that $C_{11}^{-*}N'_{11}(z)C_{11}^{-1} = P_{11}^{-*}N_{11}(z)P_{11}^{-1}$ and $C_{11}^{-*}N'_{14}(z) = P_{11}^{-*}N_{14}(z)$, $N'_{41}(z)C_{11}^{-1} = N_{41}(z)P_{11}^{-1}$. As $P_{11} = P_{11}^*$, this immediately shows that $\mathcal{R}(z)$ has the form (4.7). An application of Lemma 4.1 shows that (4.8) holds.

Identity (4.8) easily implies $\mathcal{R} \in N(Q())$ and $\mathcal{T} \in N(1)$ is an obvious consequence of $\mathcal{R} \in N(Q())$.

If $Q' = 0$ then relation (4.9) follows from (4.7) since then the fourth component in the decomposition $Q(1) \oplus (I - Q)(1) \oplus (I - Q')(2) \oplus Q'(2)$ of \mathbb{C}^n has dimension zero.

In the case when P commutes with P_1 one easily checks that in the above 4×4 block matrix representation of P the entry P_{13} is zero and $P_{11} = I$. Hence this assumption reduces (4.7) to (4.10). □

4.2 Differential Operators on Graphs

In this section we introduce the concept of differential operators on finite graphs and we indicate how to apply the results of the previous section to such differential operators.

Let (\cdot, \cdot) be a finite directed graph, where \cdot is the set of all vertices v_1, \dots, v_m , and \cdot is the set of all edges e_1, \dots, e_n . Assume that every vertex is the vertex of an edge, and suppose to each edge e_j an interval $[a_j, b_j]$ is assigned ($j = 1, \dots, n$). We set $\hat{a}_j = v$ ($\hat{b}_j = v$), if v is the starting (terminating) vertex of e_j .

For each $j = 1, \dots, n$ let $l_j = -Dp_jD + q_j$ be a symmetric differential operator on $[a_j, b_j]$ which is regular, i.e. q_j and $\frac{1}{p_j}$ are absolutely summable functions on $[a_j, b_j]$. For each function h on $[a_j, b_j]$, such that $l_j(h)$ is defined, we set

$$h^{[1]} = p_j h', \quad h^{[2]} = l_j(h) = q_j h - (p_j h)'$$

Let j be the set of all $h \in L^2[a_j, b_j]$ such that $l_j(h)$ is defined and contained in $L^2[a_j, b_j]$, and set $j_0 = \{h \in j : h(a_j) = h(b_j) = h^{[1]}(a_j) = h^{[1]}(b_j) = 0\}$.

We denote by L_j the minimal realization of l_j in $L^2[a_j, b_j]$, which is defined as $l_j|_{j_0}$. Then L_j is a densely defined, symmetric operator with defect numbers $(2, 2)$. The maximal realization of l_j in $L^2[a_j, b_j]$ is defined as $l_j|_j$, and it equals the adjoint L_j^* of L_j (cf. [20]).

Let $\mathcal{H} = \bigoplus_{j=1}^n L^2[a_j, b_j]$, and define the operator $L = \bigoplus_{j=1}^n L_j$ in \mathcal{H} . It is easy to see that L is closed, densely defined, symmetric and has defect numbers $(2n, 2n)$. Moreover, the adjoint operator L^* of L in \mathcal{H} is $\bigoplus_{j=1}^n L_j^*$. Hence the domains of L and L^* satisfy the relation $(L) = \bigoplus_{j=1}^n (L_j)$ and $(L^*) = \bigoplus_{j=1}^n (L_j^*)$.

In the sequel, if $f \in \mathcal{H}$, we denote by f_j the projection of f onto the j -th component. Hence $f = (f_j)_{j=1}^n$.

Definition 4.5. For $j = 1, \dots, n$ define the following linear mappings from (L_j^*) to \mathbb{C}^2 :

$$\Pi_j^1(f) = \begin{pmatrix} f^{[1]}(a_j) \\ -f^{[1]}(b_j) \end{pmatrix}, \quad \Pi_j^2(f) = \begin{pmatrix} f(a_j) \\ f(b_j) \end{pmatrix}.$$

Set $\Pi_j(f) = \{\Pi_j^2(f); -\Pi_j^1(f)\} \in \mathbb{C}^2 \oplus \mathbb{C}^2 = \mathbb{C}^4$ for $f \in (L_j^*)$ and define $\Delta_z^j = (\Pi_j^2|_{(L_j - \bar{z})^\perp})^{-1}$ and $M_j(z) = \Pi_j^1 \Delta_z^j$ for $z \in \mathbb{C} \setminus \mathbb{R}$.

For each $j = 1, \dots, n$ the triplet $(\mathbb{C}^2, \Pi_j^1, \Pi_j^2)$ is a SBV for L_j^* . Indeed, the first condition of Definition 1.14 is satisfied by

$$\begin{aligned} (L_j^* f, g) - (f, L_j^* g) &= [f_j, g_j](b_j) - [f_j, g_j](a_j) = \\ &= f_j(b_j) \bar{g}_j^{[1]}(b_j) - f_j^{[1]}(b_j) \bar{g}_j(b_j) - f_j(a_j) \bar{g}_j^{[1]}(a_j) + f_j^{[1]}(a_j) \bar{g}_j(a_j) = \\ &= (\Pi_j^1(f), \Pi_j^2(g)) - (\Pi_j^2(f), \Pi_j^1(g)), \end{aligned}$$

where $[f_j, g_j](z) = f_j(z) \bar{g}_j^{[1]}(z) - f_j^{[1]}(z) \bar{g}_j(z)$ for $z \in [a_j, b_j]$ (cf. [20] Kapitel V). For the second condition of Definition 1.14 see [20] Kapitel V, 17, 3.

The selfadjoint operators $A_j^1 = L_j^*|_{(\Pi_j^1)}$ and $A_j^2 = L_j^*|_{(\Pi_j^2)}$ are selfadjoint extensions of L_j whose domains are determined by the boundary conditions $f^{[1]}(a_j) = f^{[1]}(b_j) = 0$ and $f(a_j) = f(b_j) = 0$, respectively. The function M_j is the Weyl function of $(\mathbb{C}^2, \Pi_j^1, \Pi_j^2)$.

In the following definition we introduce a SBV for L^* .

Definition 4.6. For $f = (f_j)_{j=1}^n \in (L^*)$ set

$$\Pi^1(f) = (\Pi_1^1(f_1)^T, \dots, \Pi_n^1(f_n)^T)^T$$

$$\Pi^2(f) = (\Pi_1^2(f_1)^T, \dots, \Pi_n^2(f_n)^T)^T.$$

Then $\Pi^1(f)$ and $\Pi^2(f)$ are linear mappings from (L^*) into \mathbb{C}^{2n} .

Set $\Pi(f) = \{\Pi^2(f); -\Pi^1(f)\} \in \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} = \mathbb{C}^{4n}$ for $f \in (L^*)$, and define $\Delta_z = (\Pi^2|_{(L-\bar{z})^\perp})^{-1}$ and $M(z) = \Pi^1 \Delta_z$ for $z \in \mathbb{C} \setminus \mathbb{R}$.

As before, the conditions of Definition 1.14 are satisfied. Hence $(\mathbb{C}^{2n}, \Pi^1, \Pi^2)$ is a SBV for L^* and $M(z)$ is the Weyl function of $(\mathbb{C}^{2n}, \Pi^1, \Pi^2)$.

With $A^1 = L^*|_{(\Pi^1)}$, $A^2 = L^*|_{(\Pi^2)}$ it is easy to show that

$$A^1 = \bigoplus_{i=1}^n A_i^1, \quad A^2 = \bigoplus_{i=1}^n A_i^2,$$

$$\Delta_z = \Delta_z^1 \oplus \dots \oplus \Delta_z^n, \quad M(z) = M_1(z) \oplus \dots \oplus M_n(z)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. This means that an element f of (L^*) belongs to (A^1) if and only if $f_1^{[1]}(a_1) = f_1^{[1]}(b_1) = \dots = f_n^{[1]}(a_n) = f_n^{[1]}(b_n) = 0$, and to (A^2) if and only if $f_1(a_1) = f_1(b_1) = \dots = f_n(a_n) = f_n(b_n) = 0$.

We provide $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with the indefinite hermitian scalar product

$$\langle \{v; w\}, \{x; y\} \rangle = \frac{1}{i}((w, x) - (v, y)), \quad \{v; w\}, \{x; y\} \in \mathbb{C}^{2n} \oplus \mathbb{C}^{2n},$$

and obtain a Kreĩn space with $2n$ positive and $2n$ negative squares.

A subspace Θ of $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ is called neutral if its orthogonal complement in $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with respect to $\langle \cdot, \cdot \rangle$ contains Θ and it is called maximal neutral if it is neutral and maximal with respect to this property. A subspace Θ of $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ is called hypermaximal neutral if its orthogonal complement in $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with respect to $\langle \cdot, \cdot \rangle$ coincides with Θ (cf. [17]). As $\langle \cdot, \cdot \rangle$ has $2n$ positive and $2n$ negative squares, a subspace of $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ is maximal neutral if and only if it is neutral and has dimension $2n$. Moreover, a subspace is hypermaximal neutral if and only if it is maximal neutral.

The definition of selfadjointness shows that $\Theta \subseteq \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ is a selfadjoint relation on $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ if and only if it is hypermaximal neutral with respect to $\langle \cdot, \cdot \rangle$.

From Section 1.4 we know that an extension \tilde{A} of L with $(\tilde{A}) \subseteq (L^*)$ is selfadjoint in if and only if $\Pi((\tilde{A}))$ is a selfadjoint relation on \cdot . By what was said above this means that $\Pi((\tilde{A}))$ is a $2n$ dimensional neutral subspace of $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with respect to $\langle \cdot, \cdot \rangle$. Inserting the definition of Π , we see that the neutrality of $\Pi((\tilde{A}))$ is equivalent to the boundary conditions

$$\sum_{j=1}^n ([f_j, g_j](b_j) - [f_j, g_j](a_j)) = 0 \quad (4.12)$$

for all $f = (f_j)_{j=1}^n, g = (g_j)_{j=1}^n \in (\tilde{A})$.

Definition 4.7. Let \tilde{A} be a canonical selfadjoint extension of L . We say that \tilde{A} satisfies local boundary conditions with respect to (\cdot, \cdot) if (4.12) decomposes as

$$\sum_{j \in \{1, \dots, n\}, \hat{b}_j = v} [f_j, g_j](b_j) - \sum_{j \in \{1, \dots, n\}, \hat{a}_j = v} [f_j, g_j](a_j) = 0 \quad (4.13)$$

for all $v \in \cdot$.

In order to characterize this condition we define orthogonal projections on \mathbb{C}^{2n} . For $v \in \cdot$ set $P_v = \text{diag}(\delta_1, \dots, \delta_{2n})$, where

$$\delta_j = \begin{cases} 1, & j \text{ is odd and } \hat{a}_{\frac{j+1}{2}} = v \\ 1, & j \text{ is even and } \hat{b}_{\frac{j}{2}} = v \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, set $(P_v)^2 = P_v \oplus P_v$. This means $(P_v)^2 \{x; y\} = \{P_v x; P_v y\}$ for $\{x; y\} \in \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. Then $(P_v)^2$ is an orthogonal projection in $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$.

Proposition 4.8. *A selfadjoint extension \tilde{A} of L satisfies local boundary conditions with respect to (\cdot, \cdot) if and only if $(P_v)^2 \Pi((\tilde{A}))$ is a neutral subspace of $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with respect to $\langle \cdot, \cdot \rangle$ for all $v \in \cdot$. In this case $\Pi((\tilde{A}))$ can be written as*

$$\Pi((\tilde{A})) = \bigoplus_{k=1}^m (P_{v_k})^2 \Pi((\tilde{A})).$$

Proof. The first assertion is a consequence of the relation

$$i \langle (P_v)^2 \Pi(f), (P_v)^2 \Pi(g) \rangle = \sum_{j \in \{1, \dots, n\}, \hat{b}_j = v} [f_j, g_j](b_j) - \sum_{j \in \{1, \dots, n\}, \hat{a}_j = v} [f_j, g_j](a_j).$$

Assume that $(P_v)^2\Pi((\tilde{A}))$ is a neutral subspace of $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with respect to $\langle \cdot, \cdot \rangle$ for all $v \in \cdot$. Since $\sum_{k=1}^m (P_{v_k})^2$ is the identity on $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$, we have $\Pi((\tilde{A})) \subseteq \bigoplus_{k=1}^m (P_{v_k})^2 \Pi((\tilde{A}))$.

It is easy to see that the ranges of the projections $(P_v)^2$ are orthogonal to each other with respect to the canonical scalar product and with respect to $\langle \cdot, \cdot \rangle$. Hence, the assumption implies that $\bigoplus_{k=1}^m (P_{v_k})^2 \Pi((\tilde{A}))$ is neutral with respect to $\langle \cdot, \cdot \rangle$. But since \tilde{A} is selfadjoint, $\Pi((\tilde{A}))$ is a maximal neutral subspace, and hence $\Pi((\tilde{A})) = \bigoplus_{k=1}^m (P_{v_k})^2 \Pi((\tilde{A}))$. \square

Suppose again, that \tilde{A} is again a canonical selfadjoint extension of L and write $\Theta = \Pi((\tilde{A}))$ as $\Theta = NP^{-1}$ (cf. (1.2)). Then the following proposition gives a necessary and sufficient condition in terms of N and P for \tilde{A} to satisfy local boundary conditions. Note that we do not exclude the case $P = 0$, which happens if Θ is an operator.

Proposition 4.9. *Let \tilde{A} be a canonical selfadjoint extension of L , and write $\Theta = \Pi((\tilde{A}))$ as NP^{-1} , where P is the orthogonal projection of \mathbb{C}^{2n} onto $\Theta(0)^\perp$ and $N = (I - P) + \Theta_s P$. Then \tilde{A} satisfies local boundary conditions with respect to (\cdot) if and only if P and N commute with P_v for all $v \in \cdot$.*

Proof. First assume that \tilde{A} satisfies local boundary conditions with respect to (\cdot) . From Proposition 4.8 we know that

$$\Theta = \bigoplus_{k=1}^m (P_{v_k})^2 \Theta. \quad (4.14)$$

This fact together with $(P_v)^2(0; x) = (0; P_v x)$ for $x \in \mathbb{C}^{2n}$ and $v \in \cdot$ implies that

$$\Theta_\infty = \bigoplus_{k=1}^m (P_{v_k})^2 \Theta_\infty \quad (4.15)$$

and hence $\Theta(0) = \bigoplus_{k=1}^m P_{v_k} \Theta(0)$. An easy consideration now gives $PP_v = P_v PP_v = (P_v PP_v)^* = (PP_v)^* = P_v P$ for all $v \in \cdot$.

From (4.14), (4.15) and $\Theta = \Theta_s \oplus \Theta_\infty$, we conclude $\Theta_s = \bigoplus_{k=1}^m (P_{v_k})^2 \Theta_s$.

If $\{P_v f; g\} \in \Theta_s$ then $g = P_v g$. Indeed, $\{P_v f; g\} \in \Theta_s$ implies $(P_v)^2 \{P_v f; g\} = \{P_v f; P_v g\} \in \Theta_s$ (cf. 4.14) and, since Θ_s is an operator, we get $g = P_v g$ for arbitrary $v \in \cdot$. This implies $NP_v = (I - P)P_v + \Theta_s PP_v = P_v(I - P)P_v + P_v \Theta_s PP_v =$

$P_v N P_v$. As N is selfadjoint, we get $P_v N = (N P_v)^* = (P_v N P_v)^* = P_v N P_v$. This proves that N commutes with all P_v .

The converse statement easily follows from $\Theta = \{\{P_f; Nf\} : f \in \mathbb{C}^{2n}\}$. Indeed, take an arbitrary $v \in \mathbb{C}^{2n}$ then the assumption implies

$$(P_v)^2(\Theta) = \{\{P_v P_f; P_v Nf\} : f \in \mathbb{C}^{2n}\} = \{\{P_f; Nf\} : f \in P_v(\mathbb{C}^{2n})\} \subseteq \Theta.$$

Hence $(P_v)^2(\Theta)$ is neutral with respect to $\langle \cdot, \cdot \rangle$. Finally, Proposition 4.8 shows that \tilde{A} satisfies local boundary conditions with respect to (\cdot, \cdot) . \square

We remark that a matrix $M = (m_{k,l})_{k,l=1}^{2n}$ commutes with all P_v , $v \in \mathbb{C}^{2n}$ if and only if $m_{k,l} \neq 0$ implies

$$\begin{aligned} \hat{b}_{\frac{k}{2}} &= \hat{b}_{\frac{l}{2}} \text{ for even } k \text{ and even } l, \\ \hat{a}_{\frac{k+1}{2}} &= \hat{b}_{\frac{l}{2}} \text{ for odd } k \text{ and even } l, \\ \hat{b}_{\frac{k}{2}} &= \hat{a}_{\frac{l+1}{2}} \text{ for even } k \text{ and odd } l, \\ \hat{a}_{\frac{k+1}{2}} &= \hat{a}_{\frac{l+1}{2}} \text{ for odd } k \text{ and odd } l. \end{aligned}$$

In order to apply the results of Section 4.1 we set:

$$\begin{aligned} \mathbb{1} &= L^2[a_1, b_1], \quad \mathbb{2} = \bigoplus_{j=2}^n L^2[a_j, b_j] \\ S_1 &= L_1, \quad S_2 = \bigoplus_{j=2}^n L_j, \quad A_1 = A_1^2, \quad A_2 = \bigoplus_{j=2}^n A_j^2, \\ \Gamma_z^1 &= \Delta_z^1, \quad \Gamma_z^2 = \Delta_z^2 \oplus \dots \oplus \Delta_z^n, \\ Q_1(z) &= M_1(z), \quad Q_2(z) = M_2(z) \oplus \dots \oplus M_n(z). \end{aligned}$$

Then, clearly, $\mathbb{1} = \mathbb{1} \oplus \mathbb{2}$, $S = S_1 \oplus S_2 = L$, $A = A_1 \oplus A_2 = A^2$, $\Gamma_z = \Gamma_z^1 \oplus \Gamma_z^2 = \Delta_z$, $Q(z) = Q_1(z) \oplus Q_2(z) = M(z)$. The operators S_1 and S_2 have defect numbers $(2, 2)$ and $(2n-2, 2n-2)$, respectively. Moreover, Γ_z^1 maps \mathbb{C}^2 bijectively onto $(S_1 - \bar{z})^\perp$, and Γ_z^2 maps \mathbb{C}^{2n-2} bijectively onto $(S_2 - \bar{z})^\perp$, and both satisfy (1.3). The functions $Q_1(z)$ and $Q_2(z)$ are Q -functions of (S_1, A_1, Γ_i^1) and (S_2, A_2, Γ_i^2) , respectively.

Assume now that \tilde{A} is a selfadjoint extension of S satisfying local boundary conditions with respect to (\cdot, \cdot) . We know from (1.24) that with $\Theta = \Pi((\tilde{A}))$

$$(\tilde{A} - z)^{-1} = (A - z)^{-1} - \Gamma_z(M(z) + \Theta)^{-1} \Gamma_z^*.$$

In the following we indicate how the assumption that \tilde{A} satisfies local boundary conditions affects the calculation of $\mathcal{T} \in N(\mathbb{C}^2)$ in (cf. Section 4.1)

$$\tilde{P}_1(\tilde{A} - z)^{-1}|_1 = (A_1 - z)^{-1} - \gamma_1(z)(Q_1(z) + \mathcal{T}(z))^{-1} \gamma_1(\bar{z})^*. \quad (4.16)$$

Once we have calculated \mathcal{T} , we know from (1.25) that $\tilde{P}_1(\tilde{A} - z)^{-1}g = f$ is the solution of the z -dependent boundary value problem

$$L_1^* f - z f = g, \quad \Pi^1(f) \in \mathcal{T}(z).$$

In Section 4.3 we will give some examples.

In general one can say that the assumption on \tilde{A} simplifies the calculation of \mathcal{T} . If \mathcal{G} has a special structure, we can say more.

Definition 4.10. Let $v_1, v_2 \in \mathcal{V}$. A function P from $\{1, \dots, p\}$, where $p \in \mathbb{N}$, into \mathcal{V} is called a path in $(\mathcal{G}, \mathcal{E})$ connecting v_1 and v_2 if v_1 is the starting or the terminating vertex of $P(1)$, if v_2 is the starting or the terminating vertex of $P(p)$, and if $P(k)$ and $P(k+1)$ have at least one vertex in common for all $k = 1, \dots, p-1$.

We define the relation \sim on \mathcal{V} by $v_1 \sim v_2$ if and only if $v_1 = v_2$, or there exists a path connecting v_1 and v_2 . This relation is an equivalence relation, and we call the equivalence classes of \sim connected components.

We say that an edge e is a separating edge if in the graph $(\mathcal{G}, \mathcal{E} \setminus \{e\})$ the number of the connected components is greater than the number of the connected components in $(\mathcal{G}, \mathcal{E})$. For $v_1, v_2 \in \mathcal{V}$ we define $v_1 \sim_e v_2$ if and only if there exists a path connecting v_1 and v_2 in the graph $(\mathcal{G}, \mathcal{E} \setminus \{e\})$.

A vertex v is called a loop if the starting and the terminating vertex of e coincide.

Note that if e is a separating edge, then in the graph $(\mathcal{G}, \mathcal{E} \setminus \{e\})$ the number of the connected components equals the number of the connected components in $(\mathcal{G}, \mathcal{E})$ plus one. Clearly, if e is a separating edge, it is impossible that e is a loop.

Proposition 4.11. Let $e_1 = [a_1, b_1]$ be a separating edge of $(\mathcal{G}, \mathcal{E})$. Assume that \tilde{A} is a canonical selfadjoint extension of S satisfying local boundary conditions with respect to $(\mathcal{G}, \mathcal{E})$, and suppose that $\Pi((\tilde{A})) = \Theta$ is an operator. Then $\mathcal{T} \in N(\mathbb{C}^2)$ in (4.16) is of diagonal form.

Proof. Let $P' = \text{diag}(\delta_1, \dots, \delta_{2n})$ be an orthogonal projection in \mathbb{C}^{2n} , where $\delta_k = 1$ if $\hat{a}_{\frac{k+1}{2}} \sim_{e_1} \hat{a}_1$ for odd k , and if $\hat{b}_{\frac{k}{2}} \sim_{e_1} \hat{a}_1$ for even k . By the assumption on the graph $\hat{a}_1 \not\sim_{e_1} \hat{b}_1$, and hence $\delta_2 = 0$.

Since $v_k = v_l$ implies $v_k \sim_{e_1} v_l$ for arbitrary $l, k = 1, \dots, m$, we have $P'P_v = P_vP' = P_v$ for all $v \in \mathcal{V}$ with $v \sim_{e_1} \hat{a}_1$ and $P'P_v = P_vP' = 0$ for all $v \in \mathcal{V}$ with $v \not\sim_{e_1} \hat{a}_1$.

Because of $\sum_{k=1}^m P_v = I$, we get $P' = \sum_{k=1; v_k \sim_{e_1} \hat{a}_1}^m P_v P'$. Thus, if a matrix commutes with all P_v , $v \in \mathcal{V}$, then it commutes with P' . By Proposition 4.9 Θ commutes with all P_v . So P' also commutes with Θ and clearly with P_1 and P_2 .

Since $\hat{a}_k \sim_{e_1} \hat{b}_k$ for $k = 2, \dots, n$, the fact that $Q_2(z)$ is a block diagonal matrix

with 2×2 entries easily shows that P' commutes with $Q_2(z)P_2$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

This implies that $(Q_2(z)P_2 + \Theta + \alpha P_1)^{-1}$ commutes with P' (cf. 4.3). Since $\delta_2 = 0$, the entries $(1, 2)$ and $(2, 1)$ of the matrix $(Q_2(z)P_2 + \Theta + \alpha P_1)^{-1}$ are zero. Hence $P_1(Q_2(z)P_2 + \Theta + \alpha P_1)^{-1}|_1$ and therefore $(P_1(Q_2(z)P_2 + \Theta + \alpha P_1)^{-1}|_1 - \alpha I)^{-1}$ is of diagonal form. \square

Lets come to the case, when $\Pi(\tilde{A})$ is a proper relation.

Proposition 4.12. *Assume that in the graph $(,)$ we have $\hat{a}_1 \neq \hat{b}_1$, and assume that \tilde{A} is a canonical selfadjoint extension of S satisfying local boundary conditions with respect to $(,)$, such that $\Theta = \Pi((B))$ is a proper relation. We represent Θ as NP^{-1} , where P is the orthogonal projection of \mathbb{C}^{2n} onto $\Theta(0)^\perp$ and $N = (I - P) + \Theta_s P$.*

Then with the same notation as in Proposition 4.4 there are four possibilities for $Q(\mathbb{C}^{2n})$:

$$\begin{aligned} Q(\mathbb{C}^{2n}) &= \{0\}, \quad Q(\mathbb{C}^{2n}) = {}_1 = P_1(\mathbb{C}^{2n}), \\ Q(\mathbb{C}^{2n}) &= \text{l.s.}\{1\}, \quad Q(\mathbb{C}^{2n}) = \text{l.s.}\{2\}. \end{aligned}$$

Here $1, \dots, 2n$ denotes the canonical basis of \mathbb{C}^{2n} .

Proof. By Proposition 4.9 we know that $P = (p_{i,j})_{i,j=1}^{2n}$ commutes with all P_v . Hence, if $p_{1,2} = \bar{p}_{2,1}$ were not equal zero, then \hat{a}_1 and \hat{b}_1 would be the same vertex. But this contradicts the assumption of this proposition.

Hence, if look at $P_1 P P_1|_1$ as a mapping on 1 , we have

$$P_1 P P_1|_1 = \begin{pmatrix} p_{1,1} & 0 \\ 0 & p_{2,2} \end{pmatrix}.$$

This means that the range of $P_1 P P_1$ has one of these four mentioned forms. Since the range of $P_1 P P_1 = (P_1 P)(P_1 P)^*$ equals the range of $P_1 P$ and thus by definition the range of Q , the proposition is proved. \square

If e_1 is a separating edge, we can formulate a proposition similar to Proposition 4.11.

Proposition 4.13. *Let $e_1 = [a_1, b_1]$ be a separating edge of $(,)$. Assume that \tilde{A} is a canonical selfadjoint extension of S satisfying local boundary conditions with respect to $(,)$, and suppose that $\Pi(\tilde{A}) = \Theta$ is a proper relation.*

We represent Θ as NP^{-1} , where P is the orthogonal projection of \mathbb{C}^{2n} onto $\Theta(0)^\perp$ and $N = (I - P) + \Theta_s P$. With the same notation as in Proposition 4.4 we also assume that $Q(\mathbb{C}^{2n}) = {}_1 = P_1(\mathbb{C}^{2n})$.

Then $\mathcal{T}(z)$ is an operator of diagonal form for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Let P' be defined as in the proof of Proposition 4.11. The range of Q' is by definition $P(\mathbb{C}^{2n}) \cap \mathfrak{L}_2$. If $x \in P(\mathbb{C}^{2n}) \cap \mathfrak{L}_2$, then, since P and P_2 commute with P_v for all $v \in \mathfrak{L}$, the vector $P_v x$ again belongs to $P(\mathbb{C}^{2n}) \cap \mathfrak{L}_2$. This implies $P_v Q' = Q' P_v Q'$ and hence $P_v Q' = Q' P_v$ for all $v \in \mathfrak{L}$. Similarly, as $Q(\mathbb{C}^{2n}) = P_1 P(\mathbb{C}^{2n})$, the projectors P_v , $v \in \mathfrak{L}$ commute with Q . By the proof of Proposition 4.11 Q' and Q commute with P' .

Using (4.8) the rest of the proof is similar to the final part of the proof of Proposition 4.11. □

4.3 Examples

In this section we use the notation of Section 4.1 and Section 4.2.

Let $(\mathfrak{L}, \mathfrak{E})$ be a graph with $\mathfrak{L} = \{v_1, v_2, v_3, v_4\}$, $\mathfrak{E} = \{e_1, e_2, e_3\}$ of the form

The terms $\hat{a}_j, \hat{b}_j, j = 1, 2, 3$ come from the intervals $[a_j, b_j]$ which are assigned to $e_j, j = 1, 2, 3$.

For $j = 1, 2, 3$ denote by $M_j(z) = \begin{pmatrix} m_{11}^j(z) & m_{12}^j(z) \\ m_{21}^j(z) & m_{22}^j(z) \end{pmatrix}$ the Weyl function of $(\mathbb{C}^2, \Pi_j^1, \Pi_j^2)$.

Let \tilde{A} be the restriction of L^* defined by: $h \in (\tilde{A})$ if and only if $h \in (L^*)$ and

$$\begin{array}{rcl} -h(a_1) & & = -h^{[1]}(a_1) \\ h(b_1) & -2h(b_2) & = h^{[1]}(b_1) \\ & -h(a_2) & = -h^{[1]}(a_2) \\ -2h(b_1) & 3h(b_2) & = h^{[1]}(b_2) \\ & -h(a_3) & = -h^{[1]}(a_3) \\ & -ih(b_1) & = h^{[1]}(b_3) \end{array} .$$

It is easy to see that \tilde{A} is a selfadjoint extension of L satisfying local boundary conditions with respect to $(\mathfrak{L}, \mathfrak{E})$. Moreover, $\Pi((\tilde{A})) = \Theta$ is a selfadjoint operator of the form

$$\Theta = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & i \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -i & 0 & 0 & 0 & 1 \end{pmatrix} .$$

From Proposition 4.2 we get

$$\begin{aligned} \mathcal{T}(z) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & i \end{pmatrix} \times \\ &= \begin{pmatrix} m_{1,1}^2(z) - 1 & m_{1,2}^2(z) & 0 & 0 \\ m_{2,1}^2(z) & m_{2,2}^2(z) + 3 & 0 & 0 \\ 0 & 0 & m_{1,1}^3(z) - 1 & m_{1,2}^3(z) \\ 0 & 0 & m_{2,1}^3(z) & m_{2,2}^3(z) + 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -2 \\ 0 & 0 \\ 0 & -i \end{pmatrix} = \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 - \frac{4m_{11}^2(z) - 4}{(m_{11}^2(z) - 1)(m_{22}^2(z) + 3) - m_{12}^2(z)m_{21}^2(z)} - \frac{m_{11}^3(z) - 1}{(m_{11}^3(z) - 1)(m_{22}^3(z) + 1) - m_{12}^3(z)m_{21}^3(z)} \end{pmatrix}. \end{aligned}$$

Hence $\tilde{P}_1(\tilde{A} - z)^{-1}g = f$ is the solution of

$$\begin{aligned} L_1^* f - z f &= g, \quad f(a_1) = f^{[1]}(a_1), \\ f(b_1) \left(1 - \frac{4m_{11}^2(z) - 4}{(m_{11}^2(z) - 1)(m_{22}^2(z) + 3) - m_{12}^2(z)m_{21}^2(z)} - \right. \\ &\quad \left. - \frac{m_{11}^3(z) - 1}{(m_{11}^3(z) - 1)(m_{22}^3(z) + 1) - m_{12}^3(z)m_{21}^3(z)} \right) = f^{[1]}(b_1). \end{aligned}$$

On the same graph let \tilde{A} be the restriction of L^* defined by: $h \in (\tilde{A})$ if and only if $h \in (L^*)$ and

$$\begin{aligned} h(a_2) &= 0, \quad 16h(b_1) + 12ih(b_2) = 0, \quad 2h(a_1) = -h^{[1]}(a_1), \quad h(a_3) = h^{[1]}(a_3), \\ -12h(b_1) + 16ih(b_2) + 625h(b_3) &= 625h^{[1]}(b_3), \\ 57(3h(b_1) - 4ih(b_2)) - 4h(b_3) &= 75h^{[1]}(b_1) - 100ih^{[1]}(b_2). \end{aligned}$$

The operator \tilde{A} is a selfadjoint extension of L satisfying local boundary conditions with respect to (\cdot, \cdot) . Moreover, we have $\Pi((\tilde{A})) = \Theta = NP^{-1}$ (cf. 1.2) where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{9}{25} & 0 & \frac{-12i}{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12i}{25} & 0 & \frac{16}{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$NP = \frac{1}{625} \begin{pmatrix} 1250 & 0 & 0 & 0 & 0 & 0 \\ 0 & 513 & 0 & -684i & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 684i & 0 & 912 & 0 & -16i \\ 0 & 0 & 0 & 0 & -625 & 0 \\ 0 & -12 & 0 & 16i & 0 & 625 \end{pmatrix}.$$

Then $Q = P_1$,

$$Q'P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and applying Proposition 4.4 we get

$$\begin{aligned} \mathcal{T}(z) = \mathcal{R}(z) &= \begin{pmatrix} 2 & 0 \\ 0 & \frac{19}{3} + \frac{16}{9}m_{22}^2(z) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -\frac{4}{75} \end{pmatrix} \times \\ & \begin{pmatrix} m_{11}^3(z) - 1 & m_{12}^3(z) \\ m_{21}^3(z) & m_{22}^3(z) + 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{4}{75} \end{pmatrix} = \\ & \begin{pmatrix} 2 & 0 \\ 0 & \frac{19}{3} + \frac{16m_{22}^2(z)}{9} - \frac{16(m_{11}^3(z)-1)}{5625((m_{11}^3(z)-1)(m_{22}^3(z)+1)-m_{12}^3(z)m_{21}^3(z))} \end{pmatrix}. \end{aligned}$$

Hence $\tilde{P}_1(\tilde{A} - z)^{-1}g = f$ is the solution of

$$L_1^* f - zf = g, \quad 2f(a_1) = -f^{[1]}(a_1),$$

$$f(b_1) \left(\frac{19}{3} + \frac{16m_{22}^2(z)}{9} - \frac{16(m_{11}^3(z) - 1)}{5625((m_{11}^3(z) - 1)(m_{22}^3(z) + 1) - m_{12}^3(z)m_{21}^3(z))} \right) = f^{[1]}(b_1).$$

In the following three examples we consider in some sense natural boundary conditions for \tilde{A} and we will see how the structure of the graph influences the form of the z -dependent boundary values.

Let (\cdot) be the graph $(= \{v_1, \dots, v_{n+1}\}, = \{e_1, \dots, e_n\}, n > 2)$:

The terms $\hat{a}_j, \hat{b}_j, j = 1, \dots, n$ come from the intervals $[a_j, b_j]$ which are assigned to $e_j, j = 1, \dots, n$.

For $j = 1, \dots, n$ denote by $M_j(z) = \begin{pmatrix} m_{11}^j(z) & m_{12}^j(z) \\ m_{21}^j(z) & m_{22}^j(z) \end{pmatrix}$ the Weyl function of $(\mathbb{C}^2, \Pi_j^1, \Pi_j^2), j = 1, \dots, n$.

Let \tilde{A} be the restriction of L^* defined by:
 $h \in (\tilde{A})$ if and only if $h \in (L^*)$ and

$$h(a_1) = h(b_n) = 0, \quad h(b_j) = h(a_{j+1}), h^{[1]}(b_j) = h^{[1]}(a_{j+1}), \quad j = 1, \dots, n-1.$$

It is easy to see that \tilde{A} is a selfadjoint extension of L satisfying local boundary conditions with respect to (\cdot, \cdot) . Moreover, we have $\Pi((\tilde{A})) = \Theta = NP^{-1}$ (cf. 1.2) where

$$P = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $NP = 0$.

$$\text{Then } QP_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Q' = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and by (4.4)

$$\mathcal{R}(z) = m_{11}^2(z) - \frac{m_{12}^2(z)m_{21}^2(z)}{m_{22}^2(z)}.$$

Hence $\tilde{P}_1(\tilde{A} - z)^{-1}g = f$ is the solution of

$$L_1^* f - zf = g, \quad f(a_1) = 0, \quad f^{[1]}(b_1) = f(b_1)\mathcal{R}(z).$$

Here $\mathcal{R}(z)$ depends only on $M_2(z)$. This is caused by the fact that $N_{44}(z) = 0$, which is in general not true for selfadjoint extensions of L satisfying local boundary conditions with respect to $(,)$.

Now assume that $(,)$ is a graph of the form $(= \{v_1, \dots, v_{n+1}\}, = \{e_1, \dots, e_n\})$:

Let \tilde{A} be the restriction of L^* defined by:
 $h \in (\tilde{A})$ if and only if $h \in (L^*)$ and

$$h(a_1) = 0, \quad h(b_j) = 0, \quad j = 2, \dots, n,$$

$$h(b_1) = h(a_j), \quad j = 2, \dots, n, \quad h^{[1]}(b_1) = \sum_{j=2}^n h^{[1]}(a_j).$$

The operator \tilde{A} is a selfadjoint extension of L satisfying local boundary conditions with respect to $(,)$. Moreover, we have $\Pi((\tilde{A})) = \Theta = NP^{-1}$ (cf. 1.2) where

$$P = \frac{1}{n} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & \dots & 1 & 0 \end{pmatrix}$$

and $NP = 0$. Here the dots stand for $n - 3$ times the sequence 1 0.

Then $QP_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $Q' = 0$ and

$$\mathcal{R}(z) = \sum_{j=2}^n m_{11}^j(z).$$

Hence $\tilde{P}_1(\tilde{A} - z)^{-1}g = f$ is the solution of

$$L_1^* f - zf = g, \quad f(a_1) = 0, \quad f^{[1]}(b_1) = f(b_1)\mathcal{R}(z).$$

Let $(,)$ be a graph of the form $(= \{v_1, v_2\}, = \{e_1, \dots, e_n\})$:

Let \tilde{A} be the restriction of L^* defined by:

$h \in (\tilde{A})$ if and only if $h \in (L^*)$ and

$$h(a_1) = h(a_j), h(b_1) = h(b_j) \quad j = 2, \dots, n,$$

$$\sum_{j=1}^n h^{[1]}(a_j) = \sum_{j=1}^n h^{[1]}(b_j) = 0.$$

We see that \tilde{A} is a selfadjoint extension of L satisfying local boundary conditions with respect to (\cdot, \cdot) . Moreover, we have $\Pi((\tilde{A})) = \Theta = NP^{-1}$ (cf. 1.2) where

$$P = \frac{1}{n} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} (1 \ 0 \ \dots \ 1 \ 0) + \frac{1}{n} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix} (0 \ 1 \ \dots \ 0 \ 1)$$

and $NP = 0$.

Then $Q = P_1$, $Q' = 0$ and

$$\mathcal{T}(z) = \mathcal{R}(z) = \sum_{j=2}^n M_j(z).$$

Hence $\tilde{P}_1(\tilde{A} - z)^{-1}g = f$ is the solution of

$$L_1^* f - zf = g, \quad -f^{[1]}(a_1) = f(a_1) \sum_{j=2}^n m_{11}^j(z) + f(b_1) \sum_{j=2}^n m_{12}^j(z),$$

$$f^{[1]}(b_1) = f(a_1) \sum_{j=2}^n m_{21}^j(z) + f(b_1) \sum_{j=2}^n m_{22}^j(z).$$

Chapter 5

A Generalized Friedrichs Extension

5.1 Matrix Nevanlinna Functions, the Classes \mathbf{N}_0 and \mathbf{N}_1

Let $n \in \mathbb{N}, n > 0$ and let $Q \in M(\mathbb{C}^n)$. Then Q is called a strict $n \times n$ matrix Nevanlinna function. It is elementary to see that for each element $h \in \mathbb{C}^n$ the (scalar valued) Nevanlinna function $(Q(\cdot)h, h)$ is a nonconstant Nevanlinna function.

The function Q has an integral representation (cf. 1.7)

$$Q(z) = C + Dz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\Sigma(t), \quad (5.1)$$

with $n \times n$ matrices $C = C^*$, $D \geq 0$, and with a nondecreasing $n \times n$ matrix function $\Sigma(t) \geq 0$ defined on \mathbb{R} which satisfies

$$\int_{\mathbb{R}} \frac{d\Sigma(t)}{t^2+1} < \infty. \quad (5.2)$$

As $\mathbb{C}^{n \times n}$ is finite dimensional these integrals (5.2) and (5.1) exist as Riemann-Stieltjes integrals in the uniform operator topology on $L(\mathbb{C}^n)$.

It follows from the integral representation (5.1) that

$$\operatorname{Re}Q(i\eta) = C + \int_{\mathbb{R}} \left(\frac{t}{t^2+\eta^2} - \frac{t}{t^2+1} \right) d\Sigma(t). \quad (5.3)$$

Hence, we see that

$$C = \operatorname{Re}Q(i), \quad \lim_{\eta \rightarrow \infty} \frac{\operatorname{Re}Q(i\eta)}{\eta} = 0. \quad (5.4)$$

It also follows from the integral representation (5.1) that

$$\operatorname{Im}Q(i\eta) = \eta D + \int_{\mathbb{R}} \frac{\eta}{t^2 + \eta^2} d\Sigma(t). \quad (5.5)$$

Hence, we have

$$\frac{\operatorname{Im}Q(i\eta)}{\eta} = D + \int_{\mathbb{R}} \frac{1}{t^2 + \eta^2} d\Sigma(t), \quad (5.6)$$

so that $\operatorname{Im}\frac{Q(i\eta)}{\eta} \geq 0$ is nonincreasing for $\eta > 0$ and that (cf. 5.4)

$$D = \lim_{\eta \rightarrow \infty} \frac{\operatorname{Im}Q(i\eta)}{\eta} = \lim_{\eta \rightarrow \infty} \frac{Q(i\eta)}{i\eta}. \quad (5.7)$$

Moreover, it follows from (5.5) that $\eta\operatorname{Im}Q(i\eta)$ is nondecreasing for $\eta > 0$ and that

$$\sup_{\eta > 0} \eta\operatorname{Im}Q(i\eta) = \lim_{\eta \rightarrow \infty} \eta\operatorname{Im}Q(i\eta) (\leq \infty).$$

This means that this limit either exists or the norm of $\eta\operatorname{Im}Q(i\eta)$ tends to infinity.

If $D = 0$, then

$$\sup_{\eta > 0} \eta\operatorname{Im}Q(i\eta) = \int_{\mathbb{R}} d\Sigma(t). \quad (5.8)$$

It is possible that both sides are infinite. Note that, since $\operatorname{Im}Q(z) \gg 0$ the measure $d\Sigma(t)$ must satisfy

$$\int_{\mathbb{R}} d\Sigma(t) \geq \eta\operatorname{Im}Q(i\eta) \gg 0, \quad \eta > 0. \quad (5.9)$$

Definition 5.1. An $n \times n$ matrix Nevanlinna function $Q \in M(\mathbb{C}^n)$ belongs to the subclass \mathbf{N}_1 of $M(\mathbb{C}^n)$ if for all $h \in \mathbb{C}^n$ the Nevanlinna function $(Q(\cdot)h, h)$ belongs to the class \mathbf{N}_1 (cf. Chapter 3), i.e.

$$\int_1^\infty \frac{\operatorname{Im}(Q(i\eta)h, h)}{\eta} d\eta < \infty.$$

Similarly, a function $Q \in M(\mathbb{C}^n)$ belongs to the subclass \mathbf{N}_0 if for all $h \in \mathbb{C}^n$ the Nevanlinna function $(Q(\cdot)h, h)$ belongs to the class \mathbf{N}_0 (cf. Chapter 3), i.e.

$$\sup_{\eta > 0} \eta\operatorname{Im}(Q(i\eta)h, h) < \infty.$$

Clearly, the class \mathbf{N}_0 is contained in \mathbf{N}_1 .

Lemma 5.2. *Let Q be an $n \times n$ matrix Nevanlinna function. Then $Q(z)$ belongs to the class \mathbf{N}_1 if and only if*

$$\int_1^\infty \frac{\|\operatorname{Im}Q(i\eta)\|}{\eta} d\eta < \infty, \quad (5.10)$$

and $Q(z)$ belongs to the class \mathbf{N}_0 if and only if

$$\sup_{\eta>0} \eta \|\operatorname{Im}Q(i\eta)\| < \infty. \quad (5.11)$$

Proof. Clearly, the conditions (5.10) and (5.11) are sufficient for the function Q to belong to \mathbf{N}_1 or \mathbf{N}_0 , respectively.

To prove the necessity note first that with a certain constant C depending on the norm $\|\cdot\|$

$$\begin{aligned} \|\operatorname{Im}Q(i\eta)\| &\leq C \max_{k,l=1,\dots,n} |(\operatorname{Im}Q(i\eta)_{k,l})| \leq \\ &\leq C \max_{k=1,\dots,n} (\operatorname{Im}Q(i\eta)_{k,k}), \quad \eta > 0, \end{aligned} \quad (5.12)$$

where $1, \dots, n$ denotes the canonical basis in \mathbb{C}^n .

Hence, if $Q \in \mathbf{N}_1$ then by definition $\int_1^\infty \operatorname{Im} \frac{Q_{kk}(i\eta)}{\eta} d\eta < \infty$ for $k = 1, \dots, n$ so that (5.10) follows. The relation (5.11) can be proved similarly. \square

Proposition 5.3. *Let Q be an $n \times n$ matrix Nevanlinna function. Then $Q \in \mathbf{N}_1$ if and only if the matrix D and the operator function $(\Sigma_t)_{t \in \mathbb{R}}$ in the integral representation (5.1) of Q satisfy*

$$D = 0, \quad \int_{\mathbb{R}} \frac{d\Sigma(t)}{|t|+1} < \infty. \quad (5.13)$$

In this case we can write $Q(z)$ as

$$Q(z) = G + \int_{\mathbb{R}} \frac{d\Sigma(t)}{t-z}, \quad (5.14)$$

where

$$G = \lim_{\eta \rightarrow \infty} Q(i\eta) = \lim_{\eta \rightarrow \infty} \operatorname{Re}Q(i\eta). \quad (5.15)$$

Moreover, $Q \in \mathbf{N}_0$ if and only if the matrix D and the operator function $(\Sigma_t)_{t \in \mathbb{R}}$ in the integral representation (5.1) of Q satisfy

$$D = 0, \quad \int_{\mathbb{R}} d\Sigma(t) < \infty. \quad (5.16)$$

In this case $Q(z)$ has the integral representation (5.14).

Proof. For the scalar case ($n = 1$) the proof of the necessary and sufficient conditions follows from Propositions 1.2 and 1.3 in [12]. The general case is an easy consequence of the scalar case if one always keeps in mind that the norm of a positive matrix can be estimated from above by its diagonal elements (cf. 5.12).

Assume now that $Q \in \mathbf{N}_1$ (in particular, this is the case if $Q \in \mathbf{N}_0$) and that Q has the integral representation (5.1). The following integral exists by (5.13)

$$G = C - \int_{\mathbb{R}} \frac{t}{t^2 + 1} d\Sigma(t). \quad (5.17)$$

Then we immediately see that the integral representation (5.1) reduces to (5.14). \square

As indicated in Section 1.2.1 each function $Q \in M(\mathbb{C}^n)$ can be seen as a Q -function of a triple (S, A, Γ_i) where S (A) is a symmetric (selfadjoint) relation on a Hilbert space such that $S \subseteq A$ and Γ_i is a bijective linear mapping from \mathbb{C}^n onto i . Let $\Gamma_z, z \in \rho(A)$ be defined as in (1.3). We saw in Section 1.2.1 that it is always possible to choose S as an operator.

Definition 5.4. Let S be a closed symmetric relation with finite defect numbers (n, n) . We say that S is of category \mathbf{N}_1 (\mathbf{N}_0) if S has a canonical selfadjoint extension A such that the Q -function of (S, A) belongs to \mathbf{N}_1 (\mathbf{N}_0).

Recall that $|A_s|$ denotes the modulus of the operator part A_s of A . The following Proposition gives in particular an operator theoretic interpretation of the fact that Q belongs to \mathbf{N}_1 or to \mathbf{N}_0 (cf. [12]).

Proposition 5.5. *Let Q be a Q -function of (S, A, Γ_i) and let (5.1) be its integral representation. Then:*

(i) *The kernel (D) equals $\{h \in \mathbb{C}^n : \lim_{\eta \rightarrow \infty} \operatorname{Im} \frac{(Q(i\eta)h, h)}{\eta} = 0\}$ and $h \in (D)$ if and only if $\Gamma_z h \in \overline{(A)}$, for some, and hence, for all $z \in \rho(A)$.*

If S is an operator, then A is an operator if and only if $(D) = \{0\}$.

(ii) *For an element $h \in \mathbb{C}^n$ we have $(Q(\cdot)h, h) \in \mathbf{N}_1$ if and only if*

$$\Gamma_z h \in (|A_s|^{\frac{1}{2}})$$

for some, and hence, for all $z \in \rho(A)$. The set of all $h \in \mathbb{C}^n$ such that $(Q(\cdot)h, h) \in \mathbf{N}_1$ is a linear subspace of \mathbb{C}^n .

(iii) *For an element $h \in \mathbb{C}^n$ we have $(Q(\cdot)h, h) \in \mathbf{N}_0$ if and only if*

$$\Gamma_z h \in (A) = (A_s)$$

for some, and hence for all $z \in \rho(A)$. The set of all $h \in \mathbb{C}^n$ such that $(Q(\cdot)h, h) \in \mathbf{N}_0$ is a linear subspace of \mathbb{C}^n .

Proof. First note that by (1.3) $\Gamma_z h \in \overline{(A)}$, $\Gamma_z h \in (|A_s|^{\frac{1}{2}})$, $\Gamma_z h \in (A) = (A_s)$ for one $z \in \rho(A)$ implies $\Gamma_z h \in \overline{(A)}$, $\Gamma_z h \in (|A_s|^{\frac{1}{2}})$, $\Gamma_z h \in (A) = (A_s)$ for all $z \in \rho(A)$.

The proofs of the first parts of item (i), (ii) and (iii) follow from (5.7), Proposition 1.6 and Proposition 5.3.

Assume that S is an operator. If A is an operator then $\overline{(A)} =$ and hence by the first part of (i) we have $(D) = \{0\}$. Conversely, by the von Neumann formula and by the first part of (i) $(D) = \{0\}$ implies $(S^*) \subseteq \overline{(A)}$. Since $\overline{(S^*)} = S(0)^\perp =$, we get $\overline{(A)} =$, and hence $A(0) = (A)^\perp = \{0\}$.

The set of all $h \in \mathbb{C}^n$ such that $(Q(\cdot)h, h) \in \mathbf{N}_1(\mathbf{N}_0)$ is just the linear subspace $\Gamma_i^{-1}(i \cap (|A_s|^{\frac{1}{2}})) \cap \Gamma_i^{-1}(i \cap (A))$ of \mathbb{C}^n . \square

Proposition 5.5 also shows that the fact that (S, A, Γ_i) has a Q -function which belongs to $\mathbf{N}_1(\mathbf{N}_0)$ does not depend on Γ_i .

In the sequel we will consider the other canonical selfadjoint extensions of S in \mathbb{C}^n . Let M be a selfadjoint relation on \mathbb{C}^n , and denote by A_M the canonical selfadjoint extension of S defined in (1.10) with the parameter $\mathcal{N}(z) = M$, $z \in \mathbb{C} \setminus \mathbb{R}$. By Q_M we denote the Q -function of (S, A_M, Γ_i) defined in (1.16). Moreover, we call M exceptional if $GP + M_sP + I - P$ is not invertible, where $G = \lim_{\eta \rightarrow \infty} Q(i\eta)$ (cf. 5.17).

Proposition 5.6. *Let $Q(z)$ belong to \mathbf{N}_1 or to \mathbf{N}_0 , respectively, and let $M = (M_sP + I - P)P^{-1}$ (cf. 1.2) be a selfadjoint relation on \mathbb{C}^n . Then the Q -function Q_M of (S, A_M, Γ_i) belongs to \mathbf{N}_1 or to \mathbf{N}_0 , respectively, if and only if M is not exceptional.*

Proof. Using (1.16) we can write

$$Q_M(z) = Q(z) + T(z) + E(z), \quad (5.18)$$

where $T(z)$ and $E(z)$ are given by

$$T(z) = -(Q(z) - Q(i))^*K(z), \quad E(z) = (Q(z) - Q(z)^*)K(z), \quad (5.19)$$

where $K(z) = P(Q(z)P + M_sP + I - P)^{-1}(Q(z) - Q(i))$.

For $\eta \in [1, \infty)$ we obtain

$$\frac{\|\operatorname{Im}Q_M(i\eta)\|}{\eta} \leq \frac{\|\operatorname{Im}Q(i\eta)\|}{\eta} + \frac{\|\operatorname{Im}T(i\eta)\|}{\eta} + \frac{\|\operatorname{Im}E(i\eta)\|}{\eta}. \quad (5.20)$$

It follows from Proposition 1.7 and from $M^* = M$ that

$$(P(Q(z)P + M_sP + I - P)^{-1})^* = P(Q(z)^*P + M_sP + I - P)^{-1}.$$

From this we easily get

$$\frac{\operatorname{Im}T(i\eta)}{\eta} = K(i\eta)^* \frac{\operatorname{Im}Q(i\eta)}{\eta} K(i\eta).$$

If M is not exceptional then $\|K(i\eta)\|$ is uniformly bounded for $\eta \in [1, \infty)$. Hence, the second and the third terms in (5.20) are integrable on $[1, \infty)$. The first term is clearly integrable. Thus, it follows from Lemma 5.2 that $Q_M \in \mathbf{N}_1$.

A similar argument in the case $Q \in \mathbf{N}_0$ shows that if M is not exceptional then $Q_M \in \mathbf{N}_0$.

Now let M be exceptional. From (1.16) we obtain

$$Q_M(z)P = Q(i)P +$$

$$(Q(i)^*P + M_sP + I - P)[I - (Q(z)P + M_sP + I - P)^{-1}(Q(i)P + M_sP + I - P)]$$

and, further, since M is exceptional we have $\lim_{\eta \rightarrow \infty} \|(Q(i\eta)P + M_sP + I - P)^{-1}\| = \infty$ and as $(Q(\pm i)P + M_sP + I - P)$ is invertible this implies

$$\lim_{\eta \rightarrow \infty} \|Q_M(i\eta)P\| = \infty,$$

which shows that Q_M does not have a finite limit and hence cannot belong to \mathbf{N}_1 . \square

In case $n = 1$ there is only one exceptional parameter:

$$t = -\gamma = -\lim_{\eta \rightarrow \infty} Q(i\eta)$$

(cf. 1.14). Thus, we can state the following Corollary (cf. [12]), which shows that $A_{-\gamma}$ does not depend on the selfadjoint extension A we have started with. In the general case ($n \geq 1$) the extension A_{-G} does also not depend on A , as we will see later.

Corollary 5.7. *Assume that S has defect numbers $(1, 1)$. Then there are two possibilities:*

- (i) *For all canonical selfadjoint extensions A of S the Q -function of (S, A) does not belong to \mathbf{N}_1 . In this case S is not of category \mathbf{N}_1 .*
- (ii) *For all but one canonical selfadjoint extensions A of S the Q -function of (S, A) belongs to \mathbf{N}_1 , and there is one canonical selfadjoint extension A' of S such that the Q -function Q' of (S, A') does not belong to \mathbf{N}_1 and satisfies*

$$\lim_{\eta \rightarrow \infty} |Q'(i\eta)| = \infty.$$

In this case S is of category \mathbf{N}_1 .

The same statement holds with \mathbf{N}_0 instead of \mathbf{N}_1 .

Using the same notation as in Proposition 5.6 the next Lemma shows that $(GP + M_sP + I - P)$ is a measurement for how much A_M and A_{-G} have in common.

Lemma 5.8. *For all selfadjoint relations M on \mathbb{C}^n we have*

$$\begin{aligned} \dim(A_M \cap A_{-G})/S &= \dim(M \cap (-G)) = \\ &= \dim(GP + M_sP + I - P) = \dim(PG + M_sP + I - P) = \\ &= \dim(PGP + M_sP + I - P), \end{aligned} \quad (5.21)$$

so that A_M is exceptional if and only if $A_M \cap A_{-G}$ is a proper symmetric extension of S .

Proof. It easily follows from (1.10) that

$$\dim(A_M \cap A_{-G}) = \dim(M \cap (-G)).$$

Moreover, as $-G$ is an operator there is a linear mapping $T : (M \cap (-G)) \rightarrow (I - P)(\mathbb{C}^n)$ such that

$$\begin{aligned} M \cap (-G) &= \\ &= \{\{x; -Gx\} \in (\mathbb{C}^n)^2 : Px = x, -Gx = M_sx + (I - P)y \text{ for some } y \in \mathbb{C}^n\} = \\ &= \{\{x; -Gx\} \in (\mathbb{C}^n)^2 : Px = x, -Gx = M_sx + T(x)\} = \\ &= \{\{x; -Gx\} \in (\mathbb{C}^n)^2 : \\ &Px = x, GP(x + T(x)) + M_sP(x + T(x)) + (I - P)(x + T(x)) = 0\}. \end{aligned} \quad (5.22)$$

Hence, $x \mapsto x + T(x)$ is an injective mapping from $(M \cap (-G))$ into $(GP + M_sP + I - P)$. It is also surjective. Indeed, if $y \in (GP + M_sP + I - P)$ then $\{Py; -GPy\} \in M \cap (-G)$ by (5.22). Hence,

$$\dim(M \cap (-G)) = \dim(GP + M_sP + I - P).$$

Finally, looking at the operators as 2×2 block matrices with respect to the decomposition $\mathbb{C}^n = P(\mathbb{C}^n) \oplus (I - P)(\mathbb{C}^n)$ of \mathbb{C}^n we easily see that $(PGP + M_sP + I - P) = (PG + M_sP + I - P) \subseteq P(\mathbb{C}^n)$ and that

$$(GP + M_sP + I - P) = \{x - (I - P)GPx : x \in (PGP + M_sP + I - P)\}.$$

□

5.2 The \mathbf{N}_0 Case

In this section we consider the special case that the symmetric relation S is of category \mathbf{N}_0 , i.e. that there is a canonical selfadjoint extension A of S such that the Q -function of (S, A) belongs to \mathbf{N}_0 .

Lemma 5.9. *Let S be a closed symmetric relation with finite and equal defect numbers (n, n) and let A be a canonical selfadjoint extension of S . Then*

$$(A - z)^{-1}S^*(0) = {}_z \cap (A), \quad z \in \rho(A), \quad (5.23)$$

and

$$\dim({}_z \cap (A)) \text{ is independent of } z. \quad (5.24)$$

In particular, if A is an operator, then

$$S^*(0) = \{(A - z)h : h \in {}_z \cap (A)\}, \quad z \in \rho(A). \quad (5.25)$$

Proof. First note that ${}_z = (S^* - z)$. Now let $x \in S^*(0)$, that is $\{0; x\} \in S^*$. It follows from the inclusion

$$\{(A - z)^{-1}x; (I + z(A - z)^{-1})x\} \in A \subset S^*,$$

that $\{(A - z)^{-1}x; z(A - z)^{-1}x\} \in S^*$. Hence, $(A - z)^{-1}x \in (A) \cap (S^* - z)$.

Conversely, let $x \in (A) \cap (S^* - z)$. Then $\{x; y\} \in A \subset S^*$ and $\{x; zx\} \in S^*$, which shows that $\{0, y - zx\} \in S^*$ and $x = (A - z)^{-1}(y - zx) \in (A - z)^{-1}S^*(0)$.

Relation (5.24) follows from Proposition 5.5 (iii), and (5.25) is an easy consequence of (5.23). \square

Corollary 5.10. *Let S be a closed symmetric operator with finite defect numbers (n, n) . Then S is of category \mathbf{N}_0 if and only if $S^*(0) = (S)^\perp$ has dimension n .*

Proof. Assume that S is of category \mathbf{N}_0 , and let A be a canonical selfadjoint extension of S such that the Q -function of (S, A) belongs to \mathbf{N}_0 . It follows from Propositions 5.3 and 5.5 and (5.25) that $S^*(0)$ has dimension n .

Conversely, assume that $\dim S^*(0) = n$. It is well known that S has a canonical selfadjoint operator extension A . It follows from (5.25) that ${}_i \subseteq (A)$, and by Proposition 5.5 the Q -function of (S, A) belongs to \mathbf{N}_0 . \square

Proposition 5.11. *Let S be a closed symmetric operator with finite defect numbers (n, n) , and let A be a canonical selfadjoint extension of S such that the Q -function of (S, A) belongs to \mathbf{N}_0 . Set*

$$\Gamma = (A - i)\Gamma_i.$$

Then $\Gamma = (A - z)\Gamma_z$ for all $z \in \rho(A)$ and Γ maps \mathbb{C}^n bijectively onto $S^*(0)$. Moreover, the function

$$Q(z) = \Gamma^*(A - z)^{-1}\Gamma = \Gamma^*\Gamma_z$$

is a Q -function of (S, A, Γ_i) and

$$\lim_{\eta \rightarrow \infty} Q(i\eta) = 0.$$

Proof. From Propositions 5.3 and 5.5 we see that A is an operator and that Γ is well defined.

The relation $\Gamma = (A - z)\Gamma_z$ for all $z \in \rho(A)$ is a consequence of (1.3), and it immediately follows from Lemma 5.9 that Γ maps \mathbb{C}^n bijectively onto $S^*(0)$. Further, the relation

$$\begin{aligned} \frac{Q(z) - Q(\bar{\zeta})^*}{z - \bar{\zeta}} &= \frac{1}{z - \bar{\zeta}} \Gamma^*((A - z)^{-1} - (A - \bar{\zeta})^{-1})\Gamma = \\ &= \Gamma^*(A - \bar{\zeta})^{-1}(A - z)^{-1}\Gamma = \Gamma_{\bar{\zeta}}^*\Gamma_z \end{aligned}$$

shows that Q is a Q -function of (S, A, Γ_i) .

Using the functional calculus for selfadjoint operators the last identity in Proposition 5.11 follows from the Bounded Convergence Theorem. \square

Let S, A and Γ be as in Proposition 5.11 and assume that Q is a Q -function of (S, A, Γ_i) . Let G, A_M etc. be defined as in Proposition 5.6. It follows from Proposition 5.11 and the uniqueness of the Q -function up to a selfadjoint matrix that $Q(z) = G + \Gamma^*(A - z)^{-1}\Gamma$. The following proposition is a refinement of Proposition 5.6 in the \mathbf{N}_0 case.

Proposition 5.12. *Let $M = (M_s P + I - P)P^{-1}$ be a selfadjoint relation on \mathbb{C}^n . Then*

$$(A_M)^\perp = A_M(0) = \Gamma (PG + M_s P + I - P).$$

Hence,

$$\begin{aligned} \dim A_M(0) &= \dim (PG + M_s P + I - P) = \dim (GP + M_s P + I - P) = \\ &= \dim(M \cap (-G)) = \dim(A_M \cap A_{-G})/S. \end{aligned}$$

In particular, $A_{-G} = S \dot{+} (\{0\} \times S^*(0))$.

Proof. By (1.10), Proposition 1.7 and Proposition 5.11

$$\begin{aligned} ((A_M - z)^{-1}\Gamma_{\bar{z}}x, \Gamma y) &= ((A - z)^{-1}\Gamma_{\bar{z}}x - \Gamma_z(Q(z) + M)^{-1}\Gamma_{\bar{z}}^*\Gamma_{\bar{z}}x, \Gamma y) = \\ &= (\Gamma_{\bar{z}}^*\Gamma_{\bar{z}}x, y) - ((Q(z) - G)(Q(z) + M)^{-1}\Gamma_{\bar{z}}^*\Gamma_{\bar{z}}x, y) = \end{aligned}$$

$$\begin{aligned}
&= ((I - (Q(z)P - GP)(Q(z)P + M_sP + I - P)^{-1})\Gamma_{\bar{z}}^*\Gamma_{\bar{z}}x, y) = \\
&= ((GP + M_sP + I - P)(Q(z)P + M_sP + I - P)^{-1}\Gamma_{\bar{z}}^*\Gamma_{\bar{z}}x, y)
\end{aligned}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$. Note that

$$(Q(z)P + M_sP + I - P)^{-1}\Gamma_{\bar{z}}^*\Gamma_{\bar{z}}$$

is a bijective mapping in \mathbb{C}^n .

Now we can determine the dimension of $A_M(0) = (A_M)^\perp$. Indeed, since $A_M(0) \subseteq S^*(0) = (S)^\perp$ and $= (S - z) \oplus_{\bar{z}}$ we have

$$\begin{aligned}
A_M(0) &= \{\Gamma y : y \in \mathbb{C}^n, \Gamma y \perp (A_M - z)^{-1}\} = \\
&= \{\Gamma y : y \in \mathbb{C}^n, \Gamma y \perp (A_M - z)^{-1}\Gamma_{\bar{z}}(\mathbb{C}^n)\} = \\
&= \{\Gamma y : y \in (PG + M_sP + I - P)\} = \Gamma (PG + M_sP + I - P).
\end{aligned}$$

In particular, for $M = -G$ we find $A_{-G}(0) = S^*(0)$. Hence $A_{-G} \supseteq S \dot{+} (\{0\} \times S^*(0))$. But, as $S \dot{+} (\{0\} \times S^*(0))$ is an n -dimensional extension of S , we in fact have $A_{-G} = S \dot{+} (\{0\} \times S^*(0))$. Referring to Lemma 5.8 the rest of the proof is clear. \square

Proposition 5.12 shows that A_{-G} does not depend on the selfadjoint extension A we start with, and we call A_{-G} the generalized Friedrichs extension of S . We will justify this notation in the next Section.

Corollary 5.13. *Assume that S is an operator, and let Q_M be the Q -function of (S, A_M, Γ_i) defined in (1.15). Then the dimension of the space of all $h \in \mathbb{C}^n$ such that $(Q_M(\cdot)h, h) \in \mathbf{N}_0$ is $n - \dim (PG + M_sP + I - P)$.*

If $(Q_M(\cdot)h, h) \notin \mathbf{N}_0$ for $h \in \mathbb{C}^n$, then

$$\lim_{\eta \rightarrow \infty} \frac{(\operatorname{Im} Q_M(i\eta)h, h)}{\eta} > 0.$$

Proof. From Proposition 5.5 we know that $(Q_M(\cdot)h, h) \in \mathbf{N}_0$ if and only if $\Gamma_i h \in (A_M)$. By Lemma 5.9 we have ${}_i \cap (A_M) = (A_M - i)^{-1}S^*(0)$. As $A_M(0) = (A_M - i)^{-1} \subseteq S^*(0)$ has dimension $\dim (PG + M_sP + I - P)$, the space ${}_i \cap (A_M)$ has dimension $n - \dim (PG + M_sP + I - P)$.

To prove the remaining statement, note first that as $\dim(A/S) = n$ we have $\dim((A)/(S)) = n$. Since $Q \in \mathbf{N}_0$ we get ${}_i \subseteq (A)$. Now ${}_i \cap (S) = \{0\}$ as otherwise we would have for an element x in this intersection $(x, Sx) = i(x, x)$, which is impossible. Hence $(A) = (S) \dot{+} {}_i$ and as $S^*(0) = (S)^\perp$ has dimension n we get $= \overline{(S) \dot{+} {}_i}$.

Then also $= \overline{(A_M) \dot{+} {}_i} = \overline{(A_M)} \dot{+} {}_i$ for any subspace ${}_i$ of ${}_i$ such that $\dot{+}({}_i \cap (A_M)) = {}_i$. A comparison of the dimensions shows that $\overline{(A_M)} \cap {}_i = \{0\}$.

Hence for a $h \in \mathbb{C}^n$ such that $(Q_M(\cdot)h, h) \notin \mathbf{N}_0$, which means $\Gamma_i h \notin i \cap (A_M)$ we have $\Gamma_i h \notin \overline{(A_M)}$, and by Proposition 5.5 we get the desired result. \square

Corollary 5.14. *For all selfadjoint relations M on \mathbb{C}^n we have*

$$(A_M) = (S)\dot{+}(i \cap (A_M)) \subseteq (A),$$

and $\overline{(A_M)}\dot{+}$ for any subspace of i such that $\dot{+}(i \cap (A_M)) = i$. Exactly for nonexceptional M the selfadjoint extension A_M of S satisfies

$$(A_M) = (A).$$

In this case A_M is an operator.

Proof. By Proposition 5.5 we have $i \subseteq D(A)$. It follows from $(A_M) = (A_M - i)^{-1}$ and (1.10) that $(A_M) \subseteq (A)$. From the proof of the previous corollary we know that $(A) = (S)\dot{+}i$. Hence, $(A_M) = (S)\dot{+}(i \cap (A_M))$.

We also know from the proof of the previous corollary that $\overline{(A_M)}\dot{+}$ for any subspace of i such that $\dot{+}(i \cap (A_M)) = i$. For such spaces we have $\overline{(A_M)}\dot{+} = \cdot$.

By symmetry we obtain that $(A_M) = (A)$ for nonexceptional M , and that A_M is an operator if and only if M is nonexceptional follows for example from Proposition 5.12. \square

Corollary 5.15. *Let S' be a symmetric extension of S with defect numbers (m, m) such that $0 < m \leq n$. Then for all selfadjoint extensions A' of S' with the exception of one there is a vector $h \in \mathbb{C}^m$ such that $(Q'(z)h, h) \in \mathbf{N}_0$ where Q' is the Q -function of (S', A') .*

Proof. It is easy to see that

$$\dim(S'/S) = \dim((S')/(S)) + \dim S'(0) = n - m < n.$$

Assume now that for two canonical selfadjoint extensions A' and B' there is no vector $h \in \mathbb{C}^m$ such that $(Q'(z)h, h) \in \mathbf{N}_0$ where Q' is the Q -function of (S', A') , (of (S', B')).

By Corollary 5.13 and Proposition 5.5 the defect space $'_i = (S' + i)^\perp$ has nothing in common with $\overline{(A')}$ or with $\overline{(B')}$. Hence, by Lemma 5.9 we have $A'(0) = (S')^*(0) = B'(0)$, and therefore $A' \supseteq S' + (\{0\} \times (S')^*(0))$ and $B' \supseteq S' + (\{0\} \times (S')^*(0))$. As

$$\begin{aligned} \dim((S' + (\{0\} \times (S')^*(0)))/S) &= \dim(S'/S) + \dim(S')^*(0) - \dim S'(0) = \\ &= \dim((S')/(S)) + \dim(S')^*(0) = \end{aligned}$$

$$= \dim(\overline{(S')/(S)}) + \dim(S')^*(0) = \dim(S^*(0)) = n,$$

we obtain that $S' + (\{0\} \times (S')^*(0))$ is selfadjoint, and hence $A' = B'$. Here $\dim((S')/(S)) = \dim(\overline{(S')/(S)})$ easily follows from the previous corollary.

From Lemma 5.9 and Proposition 5.5 we obtain that there is no $h \in \mathbb{C}^n$ such that $(Q'(\cdot)h, h) \in \mathbf{N}_0$ where Q' denotes the Q -function of $(S', S' + (\{0\} \times (S')^*(0)))$. \square

5.3 The \mathbf{N}_1 Case and Space Triplets

Assume now that S is a closed symmetric operator with finite defect numbers (n, n) and let A be a canonical selfadjoint extension of S such that the Q -function of (S, A) belongs to \mathbf{N}_1 . From Propositions 5.3 and 5.5 we know that A is an operator.

In the sequel we consider the spaces ${}_{+1, -1}$ and the operators $\hat{A}, \hat{A} - zI, (\hat{A} - zI)^{-1}$ from Definition 1.8.

First we show that these spaces do not depend on the extension A if we assume that the Q -function of (S, A) belongs to \mathbf{N}_1 .

Theorem 5.16. *Assume that the Q -function Q of (S, A, Γ_i) belongs to \mathbf{N}_1 and let $G = \lim_{\eta \rightarrow \infty} Q(i\eta)$. Then for all nonexceptional M*

$$(|A_M|^{\frac{1}{2}}) = (|A|^{\frac{1}{2}}), \quad (5.26)$$

and the graph norms of $|A_M|^{\frac{1}{2}}$ and of $|A|^{\frac{1}{2}}$ induced on $(|A|^{\frac{1}{2}})$ are equivalent.

Conversely, if (5.26) holds then M is nonexceptional.

Proof. In Proposition 5.6 it was proved that M is nonexceptional if and only if the Q -function of (S, A_M) belongs to \mathbf{N}_1 . This shows that whether a selfadjoint extension B of S is exceptional or not does not depend on G . Hence, we can assume $G = 0$. So, let M be nonexceptional. Then $M_s P + I - P$ is invertible.

First note that ${}_i \subseteq (|A|^{\frac{1}{2}})$ and ${}_i \subseteq (|A_M|^{\frac{1}{2}})$ by Propositions 5.6 and 5.5. If we show that $(S + i) \cap (|A|^{\frac{1}{2}}) \subseteq (|A_M|^{\frac{1}{2}})$ then we also have

$$(|A|^{\frac{1}{2}}) \subseteq (|A_M|^{\frac{1}{2}}).$$

Indeed, since ${}_i \subseteq (|A|^{\frac{1}{2}})$ we have $(|A|^{\frac{1}{2}}) = {}_i + ((S + i) \cap (|A|^{\frac{1}{2}}))$.

Assume that $x \in (S + i) \cap (|A|^{\frac{1}{2}})$. We define a symmetric restriction S' of S as

$$S' = \{\{y; Sy\} : (S + i)(y) \perp x\}.$$

Then the defect space $'_i = (S' + i)^\perp$ of S' is given by $'_i = {}_i \oplus \text{l.s.}\{x\}$. We define a bijective linear mapping from \mathbb{C}^{n+1} onto $'_i$ by

$$\Gamma'_i(\xi_j)_{j=1, \dots, n+1} = \Gamma_i(\xi_j)_{j=1, \dots, n} + \xi_{n+1}x$$

for $(\xi_j)_{j=1,\dots,n+1} \in \mathbb{C}^{n+1}$. We denote by Q' the Q -function of (S', A, Γ'_i) . As $\Gamma'_i \subseteq (|A|^{\frac{1}{2}})$ Proposition 5.5 shows that Q' belongs to \mathbf{N}_1 . Assume that $G' = \lim_{\eta \rightarrow \infty} Q(i\eta) = 0$. By the remark after Proposition 1.7 we find that

$$\Gamma_i = \Gamma'_i|_{\mathbb{C}^n}, \quad P_{n+1}Q'(z)|_{\mathbb{C}^n} = Q(z), \quad z \in \rho(A), \quad (5.27)$$

where P_{n+1} is the orthogonal projection of \mathbb{C}^{n+1} onto \mathbb{C}^n projecting a vector onto its first n components.

By (1.10) there is a selfadjoint relation $M' = (M'_s P' + I - P)P'^{-1}$ on \mathbb{C}^{n+1} such that

$$(A_M + i)^{-1} = (A + i)^{-1} - \Gamma'_{-i}(Q'(-i) + M')^{-1}(\Gamma'_i)^*.$$

Clearly,

$$(A_M + i)^{-1} = (A + i)^{-1} - \Gamma_{-i}(Q(-i) + M)^{-1}\Gamma_i^*,$$

hence by (5.27)

$$\Gamma'_{-i}(Q'(-i) + M')^{-1}(\Gamma'_i)^* = \Gamma'_{-i}|_{\mathbb{C}^n}(Q(-i) + M)^{-1}P_{n+1}(\Gamma'_i)^*$$

and by the injectivity of Γ'_z we find

$$(Q'(-i) + M')^{-1} = (Q(-i) + M)^{-1}P_{n+1}.$$

In particular, the kernels of both sides coincide (cf. 1.12). Hence $\text{l.s.}\{e_{n+1}\} \oplus M(0) = M'(0)$, which means that $P' = PP_{n+1}$. Here e_1, \dots, e_{n+1} denotes the canonical basis in \mathbb{C}^{n+1} .

Using Proposition 1.7 and (5.27) we get

$$\begin{aligned} (P'Q'(-i)P' + M'_s P' + I - P')^{-1}P' &= (PQ(-i)P + M_s P + I - P)^{-1}PP_{n+1} = \\ &= (P'Q'(-i)P' + M_s P' + I - P')^{-1}P'. \end{aligned}$$

Note that the identity mapping in the last term of this relation is the identity on \mathbb{C}^{n+1} and the identity mapping in the middle term is the identity on \mathbb{C}^n . Now we also obtain $M_s P' = M'_s P'$.

This means that

$$M'_s P' + I - P' = (M_s P + I - P)P_{n+1} + I - P_{n+1}$$

is invertible and hence, M' is nonexceptional. By Proposition 5.6 the Q -function of (S', A_M, Γ'_i) belongs to \mathbf{N}_1 , and Proposition 5.5 shows that $\Gamma'_i(e_{n+1}) = x$ belongs to $(|A_M|^{\frac{1}{2}})$.

The converse inclusion $(|A_M|^{\frac{1}{2}}) \subseteq (|A|^{\frac{1}{2}})$ follows from the fact that both A and A_M have a Q -function belonging to \mathbf{N}_1 so that their roles above can be interchanged.

Assume conversely that (5.26) holds. Then $i \subseteq (|A|^{\frac{1}{2}}) = (|A_M|^{\frac{1}{2}})$. By Proposition 5.5 we find that the Q -function of (S, A_M) belongs to \mathbf{N}_1 and by Proposition 5.6 M is nonexceptional.

The statement about the norms follows from the closed graph theorem. \square

Corollary 5.17. *Let M be nonexceptional. Then the spaces ${}_{+1}(A)$ and ${}_{+1}(A_M)$ coincide as vector spaces and their norms are equivalent.*

There exists a bicontinuous linear mapping W from ${}_{-1}(A)$ onto ${}_{-1}(A_M)$ such that $Wx = x$ for all $x \in \cdot$. Hence, we can identify ${}_{-1}(A)$ and ${}_{-1}(A_M)$ as vector spaces and their norms are equivalent.

Proof. The first part follows immediately from Theorem 5.16.

Consider now the unitary mappings $V_{+1}(A) : {}_{+1}(A) \rightarrow {}_{-1}(A)$ and $V_{+1}(A_M) : {}_{+1}(A_M) \rightarrow {}_{-1}(A_M)$ from Definition 1.9.

The spaces ${}_{+1}(A)$ and ${}_{+1}(A_M)$ coincide as vector spaces with $(|A|^{\frac{1}{2}}) = (|A_M|^{\frac{1}{2}})$ and their norms are equivalent. Thus there is a selfadjoint bicontinuous linear mapping G on ${}_{+1}(A)$ such that $(x, y)_{+1(A_M)} = (Gx, y)_{+1(A)}$, $x, y \in {}_{+1}(A)$.

For $x \in {}_{+1}(A)$ and $y \in \cdot$ we have

$$\begin{aligned} (x, V_{+1}(A)^{-1}y)_{+1(A)} &= [x, y] = \\ &= (x, V_{+1}(A_M)^{-1}y)_{+1(A_M)} = (x, GV_{+1}(A_M)^{-1}y)_{+1(A)}. \end{aligned}$$

Therefore $V_{+1}(A)^{-1}y = GV_{+1}(A_M)^{-1}y$, and hence, $V_{+1}(A)GV_{+1}(A_M)^{-1}y = y$. Now $W^{-1} = V_{+1}(A)GV_{+1}(A_M)^{-1}$ clearly maps ${}_{-1}(A_M)$ bicontinuously onto ${}_{-1}(A)$. \square

We denote by \hat{S} the closure of S in ${}_{+1} \oplus {}_{-1}$. As S is a restriction of A , \hat{S} is a bounded operator from the closed subspace $(\hat{S}) \subseteq {}_{+1}$ into ${}_{-1}$ and $\hat{S} = \hat{A}|_{(\hat{S})}$.

Lemma 5.18. *The codimension of (\hat{S}) equals n and*

$$(\hat{S} - zI)^{\perp[\cdot, \cdot]} = \{x \in {}_{+1} : [x, y] = 0 \text{ for all } y \in (\hat{S} - zI)\} = {}_z, \quad z \in \rho(A).$$

Moreover,

$$(\hat{S} - zI)\dot{+}(\hat{S})^{\perp[\cdot, \cdot]} = {}_{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and $\hat{A} - zI$ maps ${}_z$ bijectively onto $(\hat{S})^{\perp[\cdot, \cdot]}$.

Proof. Let Q be the Q -function of (S, A, Γ_i) . As $Q \in \mathbf{N}_1$, z is contained in ${}_{+1}(= (|A|^{\frac{1}{2}}))$.

As by Proposition 1.11 the operator $\hat{A} - zI$ maps ${}_{+1}$ bijectively onto ${}_{-1}$ for $z \in \rho(A)$ the codimension of (\hat{S}) equals the codimension of $(\hat{S} - zI)$. Since V_{+1}

is also bijective, it follows from the definition of $[\cdot, \cdot]$ in 1.9 that the codimension of $(\hat{S} - zI)$ equals $\dim (\hat{S} - zI)^{\perp[\cdot, \cdot]}$.

As $(S) \subseteq_{+1}$ is dense in (\hat{S}) , the bijectivity of $\hat{A} - zI$ yields the density of $(S - z) \subseteq_{-1}$ in $(\hat{S} - zI)$. Hence

$$(\hat{S} - zI)^{\perp[\cdot, \cdot]} = (S - z)^{\perp[\cdot, \cdot]},$$

and

$$\begin{aligned} (S - z)^{\perp[\cdot, \cdot]} &= \{x \in {}_{+1} : [x, y] = 0 \text{ for all } y \in (S - z)\} = \\ &= \{x \in {}_{+1} : (x, y) = 0 \text{ for all } y \in (S - z)\} = {}_{\bar{z}} \cap {}_{+1} = {}_{\bar{z}}. \end{aligned}$$

So, the codimension of (\hat{S}) equals n , and by a similar argumentation as above the set $(\hat{S})^{\perp[\cdot, \cdot]} \subseteq_{-1}$ is also of dimension n .

To prove

$$(\hat{S} - zI) \dot{+} (\hat{S})^{\perp[\cdot, \cdot]} = {}_{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

it is enough to show that $(\hat{S} - zI) \cap (\hat{S})^{\perp[\cdot, \cdot]} = \{0\}$, because $\text{codim} (\hat{S} - zI) = n$ and $\dim (\hat{S})^{\perp[\cdot, \cdot]} = n$.

Assume that y belongs to this intersection. Then $(\hat{A} - zI)x = y$ for some $x \in (\hat{S})$, and hence $0 = [y, x] = [\hat{A}x, x] - z[x, x]$. But by (1.21) $[\hat{A}x, x]$ is real and $z[x, x]$ is real only for $x = 0$. Hence $x = 0$ and therefore $y = 0$.

It remains to show that $\hat{A} - zI$ maps ${}_z$ bijectively onto $(\hat{S})^{\perp[\cdot, \cdot]}$. This is a consequence of the density of $(S) \subseteq_{+1}$ in (\hat{S}) , the bijectivity of $\hat{A} - zI$ and the following identity for $x \in (S)$ and $h \in {}_{+1}$

$$[(\hat{A} - zI)h, x] = [h, (S - \bar{z})x] = (h, (S - \bar{z})x).$$

□

Corollary 5.19. *The operator \hat{S} is symmetric and its adjoint is given by*

$$\hat{S}^* = \hat{A} \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]}). \quad (5.28)$$

Proof. Clearly, the right side of (5.28) is contained in \hat{S}^* . Moreover, by the remark after Definition 1.12 we have $\hat{S}^*(0) = (\hat{S})^{\perp[\cdot, \cdot]}$.

Let $\{x; y\} \in \hat{S}^*$. Then $\{x; \hat{A}x\} \in \hat{A} \subseteq \hat{S}^*$. Hence $\{0; y - \hat{A}x\} \in S^*$ and $\{x; y\} \in A \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]})$. □

The following is the analog proposition to 5.11 in the \mathbf{N}_1 case.

Proposition 5.20. *Let Γ_i be as in Section 1.2.1, and set*

$$\Gamma = (\hat{A} - iI)\Gamma_i.$$

Then $\Gamma = (\hat{A} - zI)\Gamma_z$ for all $z \in \rho(A)$ and Γ maps \mathbb{C}^n bijectively onto $\hat{S}^*(0)$. Moreover, the function (cf. Definition 1.13)

$$Q(z) = \Gamma^+(\hat{A} - zI)^{-1}\Gamma = \Gamma^+\Gamma_z$$

is a Q -function of (S, A, Γ_i) and

$$\lim_{\eta \rightarrow \infty} Q(i\eta) = 0.$$

Proof. By Lemma 5.18 and (5.28) Γ maps \mathbb{C}^n bijectively onto $\hat{S}^*(0)$.

For $x \in (A)$ we have for all $z \in \rho(A)$

$$(\hat{A} - zI)(I + (z - i)(A - z)^{-1})x = (A - i)x = (\hat{A} - iI)x.$$

By continuity this relation extends to

$$(\hat{A} - zI)(I + (z - i)(A - z)^{-1}) = (\hat{A} - iI),$$

where we consider I and $(A - z)^{-1}$ as bounded mappings from $_{+1}$ into itself. Hence $\Gamma = (\hat{A} - zI)\Gamma_z$.

As for all $x \in , y \in \mathbb{C}^n$ the relation

$$\begin{aligned} (\Gamma^+(A - \bar{\zeta})^{-1}x, y) &= [(A - \bar{\zeta})^{-1}x, \Gamma y] = \\ &= [(A - \bar{\zeta})^{-1}x, (\hat{A} - iI)\Gamma_i y] = [(A + i)(A - \bar{\zeta})^{-1}x, \Gamma_i y] = \\ &= ((A + i)(A - \bar{\zeta})^{-1}x, \Gamma_i y) = (\Gamma_\zeta^* x, y) \end{aligned}$$

holds (cf. 1.21), we have

$$\Gamma^+(A - \bar{\zeta})^{-1} = \Gamma_\zeta^*. \quad (5.29)$$

This shows

$$\begin{aligned} \frac{Q(z) - Q(\zeta)^*}{z - \bar{\zeta}} &= \frac{1}{z - \bar{\zeta}} \Gamma^+((\hat{A} - zI)^{-1} - (\hat{A} - \bar{\zeta}I)^{-1})\Gamma = \\ &= \Gamma^+(A - \bar{\zeta})^{-1}(\hat{A} - zI)^{-1}\Gamma = \Gamma_\zeta^*\Gamma_z \end{aligned}$$

(cf. 1.22). Hence Q is a Q -function of (S, A, Γ_i) .

Finally, to prove the last identity in Proposition 5.20 we note first that for $x, y \in (A)$

$$[(\hat{A} + iI)x, y] = ((A + i)x, y) = \int_{\mathbb{R}} (t + i)d(E_t x, y). \quad (5.30)$$

Since,

$$\left| \int_{\mathbb{R}} (t + i)d(E_t x, y) \right|^2 \leq \left| \int_{\mathbb{R}} (|t| + 1)d(E_t x, x) \right| \left| \int_{\mathbb{R}} (|t| + 1)d(E_t y, y) \right|$$

(5.30) extends by continuity to all $x, y \in {}_{+1}$. Hence,

$$\begin{aligned} (Q(i\eta)x, y) &= [(\hat{A} + iI)\Gamma_{i\eta}x, \Gamma_i y] = \\ &= \int_{\mathbb{R}} (t + i) d(E_t \Gamma_{i\eta}x, \Gamma_i y) = \int_{\mathbb{R}} \frac{t^2 + 1}{t - i\eta} d(E_t \Gamma_i x, \Gamma_i y), \quad x, y \in \mathbb{C}^n \end{aligned}$$

tends to zero by the Bounded Convergence Theorem. \square

Let Γ be as in Proposition 5.20 and assume that Q is a Q -function of (S, A, Γ_i) . Let G, A_M etc. be defined as in Proposition 5.6. It follows from Proposition 5.20 and the uniqueness of the Q -function up to a selfadjoint matrix that $Q(z) = G + \Gamma^+(\hat{A} - z)^{-1}\Gamma$. Now we show that there is a bijective correspondence between all selfadjoint extensions of S and all selfadjoint extensions of \hat{S} .

Proposition 5.21. *Let M be a selfadjoint relation on \mathbb{C}^n . Then the closure \hat{A}_M of A_M in ${}_{+1} \oplus {}_{-1}$ is a selfadjoint relation in ${}_{+1} \oplus {}_{-1}$, which extends \hat{S} , and the inverse relation of $(\hat{A}_M - zI)$, where I is the embedding of ${}_{+1}$ into ${}_{-1}$, is a bounded operator given by (cf. Definition 1.13)*

$$(\hat{A}_M - zI)^{-1} = (\hat{A} - zI)^{-1} - \Gamma_z(Q(z) + \mathcal{N}(z))^{-1}\Gamma_{\bar{z}}^+, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (5.31)$$

Moreover, $A_M = \hat{A}_M \cap ({}_{+1} \oplus)$.

Conversely, let \hat{B} be a selfadjoint relation in ${}_{+1} \oplus {}_{-1}$ such that $\hat{S} \subseteq \hat{B}$. Then $B = \hat{B} \cap ({}_{+1} \oplus)$ is a canonical selfadjoint extension of S , and hence, $B = A_M$ for some selfadjoint relation M on \mathbb{C}^n . Moreover, the closure of B equals \hat{B} .

Proof. As $[\cdot, \cdot]$ is continuous on ${}_{+1} \oplus {}_{-1}$ the relation

$$[A_M x, y] = (A_M x, y) = (x, A_M y) = [x, A_M y]$$

extends to the closure \hat{A}_M of A_M . Hence \hat{A}_M is a symmetric relation.

Moreover, the closure $(\overline{A_M - zI})^{-1}$ of $(A_M - zI)^{-1}$ in ${}_{-1} \oplus {}_{+1}$ equals $(\hat{A}_M - zI)^{-1}$. Now $(A_M - zI)^{-1}$ is a bounded mapping from ${}_{-1}$ to ${}_{+1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Indeed, we have for $x \in \subseteq_{-1}$

$$(A_M - zI)^{-1}x = (\hat{A} - zI)^{-1}x - \Gamma_z(Q(z) + \mathcal{N}(z))^{-1}\Gamma_{\bar{z}}^+ x, \quad z \in \rho(A), \quad (5.32)$$

because

$$(\Gamma_{\bar{z}}^+ x, y) = [x, \Gamma_{\bar{z}} y] = (x, \Gamma_{\bar{z}} y) = (\Gamma_{\bar{z}}^* x, y)$$

for all $y \in \mathbb{C}^n$ implies $\Gamma_{\bar{z}}^+ x = \Gamma_{\bar{z}}^* x$.

Hence, $(\hat{A}_M - zI)^{-1}$ is a bounded operator of ${}_{-1}$ into ${}_{+1}$, and (5.32) extends to (5.31). Since

$$((\hat{A}_M - zI)^{-1})^+ = (\hat{A}_M^+ - \bar{z}I)^{-1} \supseteq (\hat{A}_M - \bar{z}I)^{-1}, \quad (5.33)$$

and since both sides of (5.33) are bounded operators from $_{-1}$ into $_{+1}$ (see the remark before Definition 1.13), we find $((\hat{A}_M - zI)^{-1})^+ = (\hat{A}_M - \bar{z}I)^{-1}$ and therefore $A_M^+ = A_M$. Clearly, $\hat{A}_M \supseteq \hat{S}$.

Consider now $\hat{A}_M \cap ({}_{+1}\oplus)$. As $[\cdot, \cdot]$ restricted to ${}_{+1}\oplus$ is just (\cdot, \cdot) , $\hat{A}_M \cap ({}_{+1}\oplus)$ is a symmetric extension of A_M . Since A_M is selfadjoint, we have $A_M = \hat{A}_M \cap ({}_{+1}\oplus)$.

Conversely, assume that \hat{B} is a selfadjoint relation in ${}_{+1}\oplus {}_{-1}$ such that $\hat{S} \subseteq \hat{B}$. Reasoning as above we see that $B = \hat{B} \cap ({}_{+1}\oplus)$ is a symmetric relation in \cdot , and as $S \subseteq \hat{S} \subseteq \hat{B}$ the relation B contains S .

Consider $\hat{B} - iI$ and $\hat{B} + iI$. Assume that an element of the form $\{x; iIx\}$ belongs to \hat{B} . Here I is again the embedding of ${}_{+1}$ into ${}_{-1}$. Then by the symmetry of \hat{B} we find that $[iIx, x]$ is real. But $[Ix, x] = (x, x)$ is also real. Thus, $x = 0$, and hence, $\hat{B} - iI$ has no kernel. Similarly, $\hat{B} + iI$ has no kernel.

As $\overline{(\hat{B} - iI)} = (\hat{B} + iI)^{\perp[\cdot, \cdot]}$, we see that $\hat{B} - iI$ has a dense range. But, since $(\hat{B} - iI)$ contains the closed space $(\hat{S} - iI)$, which has finite codimension by Lemma 5.18, $(\hat{B} - iI) = {}_{-1}$. Similarly, $(\hat{B} + iI) = {}_{-1}$. This shows that $(B - i) = (\hat{B} - iI) \cap \cdot = {}_{-1}$ and $(B + i) = \cdot$. In other words, B is a selfadjoint relation.

Finally, as \hat{B} is closed in ${}_{+1} \oplus {}_{-1}$, the closure \overline{B} of B is contained in \hat{B} . But by the first part of Proposition 5.21 \overline{B} is also selfadjoint on ${}_{+1} \oplus {}_{-1}$. Hence $\hat{B} = (\hat{B})^+ \subseteq \overline{B}^+ = \overline{B} \subseteq \hat{B}$. \square

The following proposition is a refinement of Proposition 5.6 and the analogue of Proposition 5.12 in the \mathbf{N}_1 case.

Proposition 5.22. *Let $M = (M_s P + I - P)P^{-1}$ be a selfadjoint relation on \mathbb{C}^n . Then*

$$(\hat{A}_M)^{\perp[\cdot, \cdot]} = \Gamma (PG + M_s P + I - P).$$

Hence,

$$\begin{aligned} \dim \hat{A}_M(0) &= \dim (GP + M_s P + I - P) = \dim (PG + M_s P + I - P) = \\ &= \dim(M \cap (-G)) = \dim(A_M \cap A_{-G})/S. \end{aligned}$$

In particular, $\hat{A}_{-G} = \hat{S} \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]})$.

Proof. For $x, y \in \mathbb{C}^n$ we have

$$\begin{aligned} (\Gamma_{\bar{z}}^+ \Gamma x, y) &= [\Gamma x, \Gamma_{\bar{z}} y] = \\ &= (x, \Gamma^+ \Gamma_{\bar{z}} y) = (x, (Q(\bar{z}) - G)y) = ((Q(z) - G)x, y). \end{aligned}$$

Hence $Q(z) - G = \Gamma_{\bar{z}}^+ \Gamma$. Therefore by Lemma 5.18 and Proposition 5.21

$$[(\hat{A}_M - zI)^{-1} \Gamma x, \Gamma y] = [(\hat{A} - zI)^{-1} \Gamma x - \Gamma_z (Q(z) + M)^{-1} \Gamma_{\bar{z}}^+ \Gamma x, \Gamma y] =$$

$$\begin{aligned}
&= (((Q(z) - G) - (Q(z) - G)(Q(z) + M)^{-1}(Q(z) - G))x, y) = \\
&= (((Q(z) - G) - (Q(z)P - GP)(Q(z)P + M_sP + I - P)^{-1}(Q(z) - G))x, y) = \\
&= ((GP + M_sP + I - P)(Q(z)P + M_sP + I - P)^{-1}(Q(z) - G)x, y)
\end{aligned}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$ and $x, y \in \mathbb{C}^n$. Note here that

$$(Q(z)P + M_sP + I - P)^{-1}(Q(z) - G)$$

is a bijective mapping in \mathbb{C}^n .

Now we can determine the dimension of $\hat{A}_M(0) = (\hat{A}_M)^{\perp[\cdot, \cdot]}$. Indeed, since $\hat{A}_M(0) \subseteq \hat{S}^*(0)$ and ${}_{-1} = (\hat{S} - zI) \dot{+} (\hat{S})^{\perp[\cdot, \cdot]}$ for $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$\begin{aligned}
(\hat{A}_M)^{\perp[\cdot, \cdot]} &= \{\Gamma y : y \in \mathbb{C}^n, \Gamma y \perp_{[\cdot, \cdot]} (\hat{A}_M - zI)^{-1} {}_{-1}\} = \\
&= \{\Gamma y : y \in \mathbb{C}^n, \Gamma y \perp_{[\cdot, \cdot]} (\hat{A}_M - zI)^{-1} \Gamma(\mathbb{C}^n)\} = \\
&= \{\Gamma y : y \in (PG + M_sP + I - P)\} = \Gamma(PG + M_sP + I - P).
\end{aligned}$$

In particular, for $M = -G$ we find $\hat{A}_{-G}(0) = S^*(0)$. Hence $\hat{A}_{-G} \supseteq \hat{S} \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]})$.

Assume now that $\{x; y\} \in (\hat{S} \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]}))^+$. Then $x \in ((\hat{S})^{\perp[\cdot, \cdot]})^{\perp[\cdot, \cdot]} = (\hat{S})$ and since $[u, y - \hat{S}x] = 0$ for $u \in (\hat{S})$ we find $y - \hat{S}x \in (\hat{S})^{\perp[\cdot, \cdot]}$. Therefore $\{x; y\} \in \hat{S} \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]})$, and hence, $\hat{S} \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]})$ is selfadjoint. Thus $\hat{A}_{-G} = \hat{S} \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]})$.

Referring to Lemma 5.8 the rest of the proof is clear. \square

Corollary 5.23. *The closure of S^* in ${}_{+1} \oplus {}_{-1}$ equals \hat{S}^+ , and $\hat{S}^+ \cap ({}_{+1} \oplus) = S^*$.*

Proof. Let $\{x; y\} \in S^*$. Then $[u, y] = (u, y) = (Su, x) = [\hat{S}u, x]$ for all $u \in (S)$. Since $\overline{(S)} = (\hat{S})$, we obtain $\{x; y\} \in \hat{S}^+$. Hence, $\overline{S^*} \subseteq \hat{S}^+$.

Conversely, the closure of S^* contains the closure \hat{A}_{-G} of A_{-G} and the closure \hat{A} of A . Hence by Proposition 5.22 and by (5.28) $\hat{S}^+ = \hat{A} \dot{+} (\{0\} \times (\hat{S})^{\perp[\cdot, \cdot]}) \subseteq \overline{S^*}$.

As $[\cdot, \cdot]$ restricted to ${}_{+1} \oplus$ is just (\cdot, \cdot) , we find $\hat{S}^+ \cap ({}_{+1} \oplus) \subseteq S^*$. The other inclusion follows from the first part of the proof. \square

It follows from Corollary 5.17 that for a canonical selfadjoint extension B of S the relation \hat{B} does not depend on A . Since, by Proposition 5.22 A_{-G} is the only canonical selfadjoint extension of S with $(\hat{A}_{-G}) = (\hat{S})$, A_{-G} does not depend on the selfadjoint extension A we started with, if we assume that the Q -function of (S, A) belongs to \mathbf{N}_1 . We call A_{-G} the generalized Friedrichs extension of S . This name will be justified in Proposition 5.25.

Proposition 5.24. *The generalized Friedrichs extension equals*

$$S_{GF} = \{\{h; k\} \in S^* : h \in (\hat{S})\}. \quad (5.34)$$

It is the only selfadjoint extension A' of S with the property $(A') \subseteq (\hat{S})$.

Proof. As $A_{-G} \subseteq S^*$ and $(A_{-G}) \subseteq (\hat{S})$, we have $A_{-G} \subseteq S_{GF}$. But by (5.28) and Corollary 5.23 S_{GF} is a symmetric relation. Hence $A_{-G} = S_{GF}$. \square

Proposition 5.25. *Let S be closed symmetric operator with defect numbers (n, n) and assume that S is semibounded.*

Then S is of category \mathbf{N}_1 , and the generalized Friedrichs extension coincides with the Friedrichs extension.

Proof. We show that there is a selfadjoint extension whose Q -function belongs to \mathbf{N}_1 . It is easy to see that we can assume that S has a nonnegative lower bound.

Let $_{S_F}$ be the Hilbert space $_{S_F} \subseteq$ obtained as the completion of (S) with respect to the inner product $((S + I)\cdot, \cdot) = (\cdot, \cdot)_+$. From [3] we see that $_{S_F}$ is the domain of the Friedrichs extension. It is elementary to show that $((S + I)x, y) = (x, y)_+$ for all $x \in (S)$, $y \in _{S_F}$ (cf. [3], [19]).

The space $_{S_F}$ is contractively embedded in $,$ and it does not contain any element from $_{-1} = (S + 1)^\perp$. Indeed, if $x \in _{-1} \cap _{S_F}$ then $(x, y)_+ = (x, (S + I)y) = 0$ for all $y \in (S)$, and as (S) is dense in $_{S_F}$, we have $x = 0$.

Extend $_{S_F}$ to $\tilde{=} _{S_F} \oplus _{-1}$, where $_{-1} = (S + 1)^\perp$. We denote the inner product on $\tilde{}$ again by $(\cdot, \cdot)_+$. Note that $(x, y)_+ = ((S + I)x, y)$ for all $x \in (S)$ and $y \in \tilde{}$.

Since $_{S_F} \cap _{-1} = \{0\}$, the space $\tilde{}$ is embedded in $,$ and the embedding has a norm less than or equal to $\sqrt{2}$. Moreover, as $x \perp (S) + _{-1}$ implies $x = (S + I)y$ for some $y \in (S)$ and $0 = (x, y) = ((S + I)y, y) \geq (y, y)$ yields $y = 0$, we obtain that $\tilde{}$ is dense in $.$

By the representation theorem (cf. [19]) there is a selfadjoint operator $A \geq (\frac{1}{\sqrt{2}} - 1)I$ on $\tilde{}$ such that $((A + I)^{\frac{1}{2}}) = \tilde{}$ and $(x, y)_+ = ((A + I)^{\frac{1}{2}}x, (A + I)^{\frac{1}{2}}y)$.

Let $x \in (S)$. Since $((S + I)x, y) = (x, y)_+ = (x, (A + I)y)$ for all $y \in (A)$, and since $A^* = A$, we find $x \in (A)$ and $Sx = Ax$, and hence $S \subseteq A$.

As $(I + (i + 1)(A - i)^{-1})$ maps $_{-1}$ bicontinuously onto $_i$ we find that $_{-1} \subseteq ((A + I)^{\frac{1}{2}})$ is equivalent with $_i \subseteq ((A + I)^{\frac{1}{2}})$, and as $((A + I)^{\frac{1}{2}}) = (|A|^{\frac{1}{2}})$, Proposition 5.5 shows that the Q -function of (S, A) belongs to \mathbf{N}_1 .

Finally, as $((A + I)^{\frac{1}{2}}) = (|A|^{\frac{1}{2}})$ the Hilbert spaces $\tilde{}$ and $_{+1}$ coincide as vector spaces, and their norms are equivalent. Hence, the closure (\hat{S}) of (S) in $_{+1}$ equals $_{S_F}$, and by Proposition 5.24 we obtain $S_F = S_{GF}$. \square

It follows from [3] that $_{S_F}$ from the preceding proof equals $_{+1}(S_F)$.

In the rest of this section we will give an example which yields some negative results. In particular, we show that there is no analogue to Corollary 5.15 in the \mathbf{N}_1 case.

Consider the operator iD in $L^2[-1, 1]$ with maximal domain

$$(iD) = \{ \{f; if'\} \in L^2[-1, 1] : f \text{ is absolutely continuous, } f' \in L^2[-1, 1] \},$$

and let S be defined by

$$S = \{f \in (iD) : f(-1) = f(1) = 0\}.$$

Then S is a densely defined symmetric operator with defect numbers $(1, 1)$. The selfadjoint extensions of S are parametrized by

$$A(\tau) = \left\{ \{f; if'\} L^2[-1, 1] : f(1) = \theta f(-1), \theta = -\frac{1 - 2i\tau}{1 + 2i\tau} \right\}, \quad \tau \in \mathbb{R} \cup \{\infty\}.$$

It can be shown that the defect elements $\gamma(z)$ and the Q -function q of $(S, A(0), \gamma(i))$ are given by

$$\gamma(z) = \frac{2e^{-izt}}{e^{iz+e^{-iz}}} \quad q(z) = 2 \tan z.$$

Now $\lim_{\eta \rightarrow \infty} q(i\eta) = 2i$, which implies that for every $\tau \in \mathbb{R} \cup \{\infty\}$ the Q -function of $(S, A(\tau))$ does not belong to \mathbf{N}_1 or equivalently $\gamma(z) \notin (|A(\tau)|^{\frac{1}{2}})$ (cf. Proposition 5.3, Corollary 5.7, Proposition 5.5).

Now we have for $\tau_1 \neq \tau_2$

$$(A(\tau_1)) \not\subseteq (|A(\tau_2)|^{\frac{1}{2}}), \quad (5.35)$$

since otherwise, $(S^*) = (A(\tau_1)) + D(A(\tau_2)) \subseteq (|A(\tau_2)|^{\frac{1}{2}})$ and in particular $\gamma(z) \in (|A(\tau_2)|^{\frac{1}{2}})$. The identity (5.35) clearly yields $(|A(\tau_1)|^{\frac{1}{2}}) \not\subseteq (|A(\tau_2)|^{\frac{1}{2}})$ for $\tau_1 \neq \tau_2$. The spectrum of $A(\tau)$ and the corresponding eigenfunctions are given by

$$\begin{aligned} \{\lambda_n(\tau) = \beta(\tau) + n\pi : n \in \mathbb{Z}\}, \quad \theta(\tau) = e^{-2i\beta(\tau)}, \\ v_n^\tau(t) = e^{-i\lambda_n(\tau)t}. \end{aligned}$$

Set $v_n = v_n^0$ and define

$${}_1 = \text{c.l.s.}\{v_n : n \geq 0\}, \quad {}_2 = \text{c.l.s.}\{v_n : n < 0\}.$$

Decompose A and $\gamma(z)$ accordingly:

$$A = A_1 \oplus A_2, \quad A_1 = A|_{{}_1}, \quad A_2 = A|_{{}_2},$$

$$\gamma(z) = \gamma_1(z) \oplus \gamma_2(z).$$

It is easy to check that

$$(v_n, \gamma(i)) = \frac{2i(-1)^{n+1}}{1 - (\frac{1}{2}\pi + n\pi)}.$$

Since,

$$(|A_1|^{\frac{1}{2}}) = \{f \in L^2[-1, 1] : \sum_{n=0}^{\infty} |\lambda_n(0)| |(f, v_n)| < \infty\},$$

$$(|A_2|^{\frac{1}{2}}) = \{f \in L^2[-1, 1] : \sum_{n=1}^{\infty} |\lambda_{-n}(0)| |(f, v_n)| < \infty\},$$

and,

$$\sum_{n=1}^{\infty} \left| \frac{1}{2}\pi + n\pi \right| \frac{1}{|1 - (\frac{1}{2}\pi + n\pi)|} = \infty,$$

we conclude that $\gamma_j(i) \notin (|A_j|^{\frac{1}{2}})$, $j = 1, 2$. Hence, the symmetric restrictions

$$S_j = \{f; A(0)f\} : ((A_j + i)f, \gamma_j(i)) = 0\}$$

are densely defined and the corresponding Q -functions satisfy $Q_j \notin \mathbf{N}_1$. In particular, A_j is the Friedrichs extension of S_j , $j = 1, 2$ (cf. 3.7). We consider the operators $S_0 = S_1 \oplus S_2 \subseteq S \subseteq A = A(0)$, and assume that B_j is another self-adjoint extension of S_j in \mathcal{H}_j such that $0 \in \rho(B_j)$, $B_1 \geq 0$, and $B_2 \leq 0$. Since $B_j \neq A_j$, we conclude by Proposition 5.25 and Corollary 5.7 that $Q_j(z) \in \mathbf{N}_1$. Then S_0 and $B = B_1 \oplus B_2$ have a 2×2 matrix Q -function Q belonging to \mathbf{N}_1 . For B we have

$${}_{+1}(B) = (|B|^{\frac{1}{2}}) = (|B_1|^{\frac{1}{2}}) \oplus (|B_2|^{\frac{1}{2}}) = {}_{+1}(B_1) \oplus {}_{+1}(B_2).$$

This space contains (\hat{S}_0) with codimension 2 by Lemma 5.18. Since A_j is the Friedrichs extension of S_j , $(\hat{A}_j) = (\hat{S}_j)$ has codimension 1 in ${}_{+1}(B_j)$ (cf. Proposition 5.25, Lemma 5.18), and hence, the codimension of $(\hat{A}) = (\hat{S}_0)$ equals 2 in ${}_{+1}(B)$. This means that A is the generalized Friedrichs extension of S_0 . Moreover, by the remark after Proposition 5.25 and by the proof of this proposition we also have ${}_{+1}(A_j) = (\hat{A}_j) \subseteq {}_{+1}(B_j)$. Hence, ${}_{+1}(A) = (\hat{A}) \subseteq {}_{+1}(B)$.

Now let A' be any canonical selfadjoint extension of S different from A . Then $A' = B_M$ for some selfadjoint relation M on \mathbb{C}^2 such that $\dim(M \cap (-G)) = 1$, where $\lim_{\eta \rightarrow \infty} Q(i\eta) = G$. Hence, A' is also an exceptional extension of S_0 by Proposition 5.22. Moreover, as $\dim(M \cap (-G)) = 1$ it is easy to see that there is a selfadjoint relation N on \mathbb{C}^2 such that $\dim(N \cap (-G)) = 0$ and $\dim(N \cap M) = 1$. Hence B_N is selfadjoint extension of S_0 such that $S' = B_N \cap A'$ is a proper symmetric extension of S_0 and that B_N is not an exceptional extension of S_0 .

Therefore Q -function of (S', B_N) belongs to \mathbf{N}_1 . Since ${}_{+1}(B) = (|\tilde{B}_N|^{\frac{1}{2}})$ (cf. Theorem 5.16), and since $(\hat{A}') = (\hat{S}')$ by Proposition 5.22, the operator A' is the generalized Friedrichs extension of S' .

Now we have

$$(|A'|^{\frac{1}{2}}) \not\subseteq {}_{+1}(B) = {}_{+1}(B_N).$$

Indeed, if this inclusion would hold, then

$$(|A|^{\frac{1}{2}}) = {}_{+1}(A) = (\hat{A}) = (\hat{S}_0) \subseteq (\hat{A}') \subseteq (|A'|^{\frac{1}{2}}),$$

contradicting (5.35).

These considerations yield the following two results.

Corollary 5.26. *Let S be a symmetric operator with defect numbers $(1, 1)$ and let A be a selfadjoint extension of S . If the Q -function of (S, A) belongs to \mathbf{N}_1 , then for the generalized Friedrichs extension S_{GF} of S it is not in general true that $(|S_{GF}|^{\frac{1}{2}}) \subseteq (|A|^{\frac{1}{2}})$.*

Proof. Take $S = S'$ and $A = B_N$ from the above example. □

Corollary 5.27. *There is a symmetric operator S with defect numbers $(2, 2)$ and a selfadjoint extension A of S with a (2×2) matrix Q -function $Q \in \mathbf{N}_1$ and a symmetric extension S' of S with defect numbers $(1, 1)$ such that the scalar Q -functions of (S', A') , where A' runs through all canonical selfadjoint extensions of S' , do not belong \mathbf{N}_1 . Moreover, S and A can be selected in such a way that $(|S_{GF}|^{\frac{1}{2}}) = (\hat{S}) \subseteq (|A|^{\frac{1}{2}})$, where S_{GF} is the generalized Friedrichs extension of S .*

Proof. Take $S = S_0$, $A = B$ and $S' = S$ from the above example. □

Corollary 5.27 indicates a difference between the classes \mathbf{N}_0 and \mathbf{N}_1 (cf. Corollary 5.15).

Chapter 6

A Characterization of Semiboundedness

In this chapter we will give a necessary and sufficient condition in terms of Q -functions for a selfadjoint operator to be semibounded.

6.1 Some Auxiliary Results

If $d\mu$ is a finite measure and if h is a nonnegative, bounded measurable function on \mathbb{R} , then $hd\mu$ is a finite measure and, hence, the function

$$Q^{hd\mu}(z) = \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) (t^2+1)h(t)d\mu(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (6.1)$$

is well-defined and is clearly a Nevanlinna function. Moreover, it does not belong to \mathbf{N}_1 if and only if the function h satisfies

$$\int_{\mathbb{R}} |t|h(t) d\mu(t) = \infty, \quad (6.2)$$

which is certainly true if

$$\int_0^{\infty} |t|h(t) d\mu(t) = \infty, \quad \text{and} \quad \int_{-\infty}^0 |t|h(t) d\mu(t) = \infty. \quad (6.3)$$

In this section we consider finite measures $d\mu$ which satisfy the additional conditions:

$$\int_0^{\infty} |t| d\mu(t) = \infty, \quad \int_{-\infty}^0 |t| d\mu(t) = \infty. \quad (6.4)$$

These conditions guarantee the existence of a nonnegative, bounded measurable function h on \mathbb{R} for which the corresponding function $Q^{hd\mu}(z)$ in (6.1) has some special limiting properties.

Lemma 6.1. *Let $d\mu$ be a finite measure which satisfies (6.4). Then there exists a measurable function h on \mathbb{R} with $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$, such that (6.3) holds and such that $\lim_{y \rightarrow \infty} \text{Im} Q^{hd\mu}(iy) = 0$.*

Proof. Define the nonnegative function $\iota(t, y)$ by

$$\iota(t, y) = y \frac{t^2 + 1}{t^2 + y^2}, \quad y > 0, \quad t \in \mathbb{R}.$$

Clearly, we have for $y \geq 1$,

$$\iota(t, y) \leq y, \quad t \in \mathbb{R}, \quad (6.5)$$

and

$$\iota(t, y) \leq t^2 + 1, \quad \lim_{y \rightarrow \infty} \iota(t, y) = 0, \quad t \in \mathbb{R}. \quad (6.6)$$

Note that for all $y \in \mathbb{R}$

$$\iota(t, y) \leq |t|, \quad \text{if } |t| \geq 1. \quad (6.7)$$

We construct the function h inductively. Let $u_1 = 1$. Due to (6.4) there exists $v_1 > u_1$ such that

$$m_1^- = \int_{[-v_1, -u_1]} |t| d\mu(t) \geq 1, \quad m_1^+ = \int_{[u_1, v_1]} |t| d\mu(t) \geq 1. \quad (6.8)$$

Define the function h_1 by

$$h_1(t) = \frac{1}{m_1^-} 1_{[-v_1, -u_1]} + \frac{1}{m_1^+} 1_{[u_1, v_1]}, \quad (6.9)$$

so that h_1 has its support in $[-v_1, -u_1] \cup [u_1, v_1]$, and $|h_1(t)| \leq 1$, $t \in \mathbb{R}$.

Assume that the function h_{n-1} has been defined, with support contained in

$$\sum_{k=1}^{n-1} ([-v_k, -u_k] \cup [u_k, v_k]) \quad (6.10)$$

(disjoint union). Since the measure $d\mu$ is finite, we can choose $u_n > v_{n-1}$ so large that

$$\left(\int_{(-\infty, -u_n]} + \int_{[u_n, \infty)} \right) \iota(t, y) d\mu(t) \leq \frac{1}{n}, \quad \text{for all } 1 \leq y \leq u_{n-1}, \quad (6.11)$$

due to (6.5), and such that

$$\int_{[-v_{n-1}, v_{n-1}]} \iota(t, y) d\mu(t) \leq \frac{1}{n}, \quad \text{for all } y \geq u_n, \quad (6.12)$$

due to (6.6) and dominated convergence. Moreover, there exists $v_n > u_n$ such that

$$m_n^- = \int_{[-v_n, -u_n]} |t| d\mu(t) \geq 1, \quad m_n^+ = \int_{[u_n, v_n]} |t| d\mu(t) \geq 1, \quad (6.13)$$

due to (6.4). Define

$$h_n = h_{n-1} + \frac{1}{nm_n^-} 1_{[-v_n, -u_n]} + \frac{1}{nm_n^+} 1_{[u_n, v_n]}, \quad (6.14)$$

so that the supports of the summands are disjoint. Clearly (u_n) is an increasing sequence, with $u_n \rightarrow \infty$ as $n \rightarrow \infty$, on account of (6.13). The inductive argument shows that the function $h(t) = \lim_{n \rightarrow \infty} h_n(t)$ is measurable and $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$. It follows from (6.13) and (6.14) that

$$\int_0^\infty |t| h(t) d\mu(t) = \int_{-\infty}^0 |t| h(t) d\mu(t) = \sum_{k=1}^\infty \frac{1}{k} = \infty,$$

so that (6.3) is satisfied.

Finally, we prove the limiting behaviour of $Q^{hd\mu}(z)$. Let $y \geq 1$ and assume that $y \in [u_n, u_{n+1}]$ (with $n > 1$). Then

$$\begin{aligned} \operatorname{Im} Q^{hd\mu}(iy) &= \int_{\mathbb{R}} \iota(t, y) h(t) d\mu(t) \\ &= \int_{[-v_{n-1}, v_{n-1}]} \iota(t, y) h(t) d\mu(t) \\ &\quad + \left(\int_{[-v_{n+1}, -u_n]} + \int_{[u_n, v_{n+1}]} \right) \iota(t, y) h(t) d\mu(t) \\ &\quad + \left(\int_{(-\infty, -u_{n+2})} + \int_{[u_{n+2}, \infty)} \right) \iota(t, y) h(t) d\mu(t). \end{aligned}$$

Due to (6.12) we have

$$\int_{[-v_{n-1}, v_{n-1}]} \iota(t, y) h(t) d\mu(t) \leq \int_{[-v_{n-1}, v_{n-1}]} \iota(t, y) d\mu(t) \leq \frac{1}{n}.$$

It follows from (6.7), that

$$\begin{aligned} &\left(\int_{[-v_{n+1}, -u_n]} + \int_{[u_n, v_{n+1}]} \right) \iota(t, y) h(t) d\mu(t) \\ &\leq \left(\int_{[-v_{n+1}, -u_n]} + \int_{[u_n, v_{n+1}]} \right) |t| h(t) d\mu(t) = \frac{2}{n} + \frac{2}{n+1}, \end{aligned}$$

where the last equality follows from (6.13) and (6.14). Finally, due to (6.11) we have

$$\begin{aligned} & \left(\int_{(-\infty, -u_{n+2}]} + \int_{[u_{n+2}, \infty)} \right) \iota(t, y) h(t) d\mu(t) \\ & \leq \left(\int_{(-\infty, -u_{n+2}]} + \int_{[u_{n+2}, \infty)} \right) \iota(t, y) d\mu(t) \leq \frac{1}{n+2}. \end{aligned}$$

Hence for the values $y \in [u_n, u_{n+1}]$ we obtain the estimate

$$\operatorname{Im} Q^{hd\mu}(iy) \leq \frac{3}{n} + \frac{2}{n+1} + \frac{1}{n+2}.$$

Letting $n \rightarrow \infty$ so that $y \rightarrow \infty$ we conclude that $\lim_{y \rightarrow \infty} \operatorname{Im} Q^{hd\mu}(iy) = 0$. \square

In the above lemma we constructed a Nevanlinna function whose imaginary part has a specific limiting behaviour. Similarly, we can construct a Nevanlinna function whose real part has a specific limiting behaviour. However, the arguments have to be suitably modified as the reasoning is a little more subtle.

Lemma 6.2. *Let $d\mu$ be a finite measure which satisfies (6.4). Then there exists a measurable function h on \mathbb{R} with $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$, such that (6.3) holds and such that $\lim_{y \rightarrow \infty} \operatorname{Re} Q^{hd\mu}(iy) = 0$.*

Proof. Define the function $\rho(t, y)$ by

$$\rho(t, y) = |t| \frac{(y^2 - 1)}{t^2 + y^2}, \quad y > 0, \quad t \in \mathbb{R}.$$

If $y \geq 1$ this function is nonnegative and, moreover

$$\rho(t, y) \leq \frac{y^2 - 1}{2y}, \quad t \in \mathbb{R}, \quad (6.15)$$

(where the upper bound itself is increasing with y) and

$$|t| - \rho(t, y) = |t| \frac{t^2 + 1}{t^2 + y^2} \leq |t|, \quad \lim_{y \rightarrow \infty} (|t| - \rho(t, y)) = 0, \quad t \in \mathbb{R}. \quad (6.16)$$

Note that

$$\rho(t, y) \leq |t|, \quad y \geq 1. \quad (6.17)$$

Now we construct the function h . Let $u_1 = 1$. Due to (6.4) there exists $v_1 > u_1$ such that (6.8) holds. Define the function h_1 by (6.9), so that h_1 has its support in $[-v_1, -u_1] \cup [u_1, v_1]$, and $|h_1(t)| \leq 1$, $t \in \mathbb{R}$.

Assume that the function h_{n-1} has been defined, with support contained in the set given by the disjoint union (6.10). Since the measure $d\mu$ is finite, we can choose $u_n > v_{n-1}$ so large that

$$\left(\int_{(-\infty, -u_n]} + \int_{[u_n, \infty)} \right) \rho(t, y) d\mu(t) \leq \frac{1}{n}, \quad \text{for all } 1 \leq y \leq u_{n-1}, \quad (6.18)$$

due to (6.15), and such that

$$\int_{[-v_{n-1}, v_{n-1}]} (|t| - \rho(t, y)) d\mu(t) \leq \frac{1}{n}, \quad \text{for all } y \geq u_n. \quad (6.19)$$

due to (6.16) and dominated convergence. Moreover, due to (6.4), there exists $v_n > u_n$, such that (6.13) holds. Define the function h_n by (6.14) (so that the supports of the summands are disjoint). Clearly, (u_n) is a monotonically increasing sequence and $u_n \rightarrow \infty$ as $n \rightarrow \infty$, cf. (6.13). Moreover, the function $h(t) = \lim_{n \rightarrow \infty} h_n(t)$ is measurable and $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$. It follows from (6.13) and (6.14) that for all $n \in \mathbb{N}$:

$$\int_{[0, v_n]} |t| h(t) d\mu(t) = \sum_{k=1}^n \frac{1}{k} = \int_{[-v_n, 0]} |t| h(t) d\mu(t). \quad (6.20)$$

In particular,

$$\int_{[0, \infty)} |t| h(t) d\mu(t) = \int_{(-\infty, 0]} |t| h(t) d\mu(t) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

so that (6.3) is satisfied.

Finally, we prove the announced limiting behaviour of $Q^{hd\mu}(z)$. Let $y \geq 1$ and assume that $y \in [u_n, u_{n+1}]$. Then,

$$\begin{aligned} \operatorname{Re} Q^{hd\mu}(iy) &= \int_{\mathbb{R}} t \frac{1-y^2}{(t^2+y^2)} h(t) d\mu(t) \\ &= \int_{[0, v_{n-1}]} (|t| - \rho(t, y)) h(t) d\mu(t) - \int_{[-v_{n-1}, 0]} (|t| - \rho(t, y)) h(t) d\mu(t) \\ &\quad - \int_{[0, v_{n-1}]} |t| h(t) d\mu(t) + \int_{[-v_{n-1}, 0]} |t| h(t) d\mu(t) \\ &\quad - \int_{[u_n, \infty)} \rho(t, y) h(t) d\mu(t) + \int_{(-\infty, -u_n]} \rho(t, y) h(t) d\mu(t). \end{aligned}$$

Since the integrands of the summands are nonnegative we obtain by using (6.20)

$$|\operatorname{Re} Q^{hd\mu}(iy)| \leq \left(\int_{[-v_{n-1}, 0]} + \int_{[0, v_{n-1}]} \right) (|t| - \rho(t, y)) h(t) d\mu(t)$$

$$\begin{aligned}
& + \left(\int_{[-v_{n+1}, -u_n]} + \int_{[u_n, v_{n+1}]} \right) \rho(t, y) h(t) d\mu(t) \\
& + \left(\int_{(-\infty, -u_{n+2}]} + \int_{[u_{n+2}, \infty)} \right) \rho(t, y) h(t) d\mu(t).
\end{aligned}$$

We obtain from (6.19) that

$$\left(\int_{[-v_{n-1}, 0]} + \int_{[0, v_{n-1}]} \right) (|t| - \rho(t, y)) h(t) d\mu(t) \leq \int_{[-v_{n-1}, v_{n-1}]} (|t| - \rho(t, y)) d\mu(t) \leq \frac{1}{n}.$$

Due to (6.17)

$$\begin{aligned}
& \left(\int_{[-v_{n+1}, -u_n]} + \int_{[u_n, v_{n+1}]} \right) \rho(t, y) h(t) d\mu(t) \\
& \leq \left(\int_{[-v_{n+1}, -u_n]} + \int_{[u_n, v_{n+1}]} \right) |t| h(t) d\mu(t) = \frac{2}{n} + \frac{2}{n+1},
\end{aligned}$$

where the last equality follows from (6.13) and (6.14). The relation (6.18) implies that

$$\begin{aligned}
& \left(\int_{(-\infty, -u_{n+2}]} + \int_{[u_{n+2}, \infty)} \right) \rho(t, y) h(t) d\mu(t) \\
& \leq \left(\int_{(-\infty, -u_{n+2}]} + \int_{[u_{n+2}, \infty)} \right) \rho(t, y) d\mu(t) \leq \frac{1}{n+2}.
\end{aligned}$$

Hence we obtain

$$|\operatorname{Re} Q^{hd\mu}(iy)| \leq \frac{3}{n} + \frac{2}{n+1} + \frac{1}{n+2}, \quad \text{for } y \in [u_n, u_{n+1}].$$

Letting $n \rightarrow \infty$ so that $y \rightarrow \infty$ we conclude that

$$\lim_{y \rightarrow \infty} \operatorname{Re} Q^{hd\mu}(iy) = 0.$$

□

Proposition 6.3. *Let $d\mu$ be a finite measure which satisfies (6.4). Then there exists a measurable function with $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$, such that (6.3) holds and such that $\lim_{y \rightarrow \infty} Q^{hd\mu}(iy) = 0$.*

Proof. Since $d\mu$ satisfies the assumptions of Lemma 6.1, there exists a measurable function h_1 on \mathbb{R} with $0 \leq h_1(t) \leq 1$, $t \in \mathbb{R}$, such that

$$\lim_{y \rightarrow \infty} \operatorname{Im} Q^{h_1 d\mu}(iy) = 0, \quad \int_0^\infty |t| h_1(t) d\mu(t) = \infty, \quad \int_{-\infty}^0 |t| h_1(t) d\mu(t) = \infty.$$

The measure $h_1 d\mu$ satisfies the assumptions of Lemma 6.2 and hence there exists a measurable function h_2 on \mathbb{R} with $0 \leq h_2(t) \leq 1$, $t \in \mathbb{R}$, such that

$$\lim_{y \rightarrow \infty} \operatorname{Re} Q^{h_2 h_1 d\mu}(iy) = 0, \quad \int_0^\infty |t| h_2(t) h_1(t) d\mu(t) = \infty, \quad \int_{-\infty}^0 |t| h_2(t) h_1(t) d\mu(t) = \infty.$$

Since $y \frac{t^2+1}{t^2+y^2} > 0$ for $y > 0$ and since $h_1(t)h_2(t) \leq h_1(t)$ we see that

$$\operatorname{Im} Q^{h_2 h_1 d\mu}(iy) \leq \operatorname{Im} Q^{h_1 d\mu}(iy),$$

for all $y > 0$. We conclude that $Q^{h_2 h_1 d\mu}(iy) \rightarrow 0$ as $y \rightarrow \infty$. \square

6.2 A Characterization of Semiboundedness

In this section we characterize those selfadjoint operators which are semibounded or bounded in terms of their symmetric one-dimensional restrictions. We first state a result concerning selfadjoint operators which are not semibounded.

First we note the following. Let A be a selfadjoint relation on a Hilbert space and $\varphi \in \cdot$. Set $\gamma(z) = (I + (z - i)(A - z)^{-1})\varphi$. Then it is easy to see that the function

$$Q(z) = c - i(\varphi, \varphi) + (z + i)(\gamma(z), \varphi) \quad (6.21)$$

(cf. 1.5) is a Q -function of $(S, A, \gamma(i))$, where

$$S = \{\{x; y\} \in A : (y + ix, \varphi) = 0\} \quad (6.22)$$

Proposition 6.4. *Let A be a selfadjoint operator and assume that A is not semibounded. Then there exists an element $\varphi \in \cdot \setminus (|A|^{\frac{1}{2}})$, such that for the Q -function Q defined in (6.21) of $(S, A, \gamma(i))$, where $\gamma(i) = \varphi$ and S is defined in (6.22), there is a finite limit $\lim_{y \rightarrow \infty} Q(iy) \in \mathbb{R}$.*

Proof. Let $(E_t)_{t \in \mathbb{R}}$ be the spectral family of A . Since A is not semibounded, the operators $AE_{(-\infty, 0)}$ and $AE_{[0, \infty)}$ are both unbounded, and therefore $(-AE_{(-\infty, 0)})^{\frac{1}{2}}$ and $(AE_{[0, \infty)})^{\frac{1}{2}}$ are unbounded. Hence there exists an element $\chi \in \cdot$ such that

$$\int_0^\infty |t| d(E_t \chi, \chi) = \infty, \quad \int_{-\infty}^0 |t| d(E_t \chi, \chi) = \infty. \quad (6.23)$$

The Hilbert space \cdot , defined by

$$= \overline{\operatorname{l.s.}\{(A - i)(A - z)^{-1}\chi : z \in \mathbb{C} \setminus \mathbb{R}\}},$$

reduces the selfadjoint operator A . Since $\chi \in \mathcal{D}(A)$, it follows that the restriction of A to $\mathcal{D}(A)$ is not semibounded. Therefore it suffices to prove Proposition 6.4 for the restriction of A to $\mathcal{D}(A)$. We define the measure $d\sigma$ by

$$d\sigma(t) = (t^2 + 1)d(E_t\chi, \chi),$$

so that

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2 + 1} < \infty.$$

We associate with $d\sigma$ the Hilbert space L^2_σ . Clearly, there is a unitary operator so that the triple

$$(\mathcal{H}, A|_{\mathcal{D}(A)}, \chi)$$

may be identified with the triple

$$\left(L^2_\sigma, .t, \frac{1}{t-i} \right),$$

where $.t$ is the multiplication operator on L^2_σ . For any $\varphi \in L^2_\sigma$ the corresponding Q -function with $c = 0$ in (6.21) has the form

$$Q(z) = \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) (t^2+1)|\varphi(t)|^2 d\sigma(t). \quad (6.24)$$

Let $d\mu$ be the finite measure defined by $d\mu(t) = d(E(t)\chi, \chi)$, so that $d\sigma(t) = (t^2 + 1)d\mu(t)$. Condition (6.23) is equivalent to (6.4). According to Proposition 6.3 there is a nonnegative measurable function h , such that $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$ with the indicated properties. Define the function $\tilde{\varphi}(t)$ by

$$\tilde{\varphi}(t) = \sqrt{\frac{h(t)}{t^2 + 1}}.$$

Then the identity

$$\int_{\mathbb{R}} |\tilde{\varphi}(t)|^2 d\sigma(t) = \int_{\mathbb{R}} h(t) d\mu(t),$$

implies that $\tilde{\varphi} \in L^2_\sigma$. Let $\varphi \in \mathcal{D}(A)$ be the element which corresponds to $\tilde{\varphi} \in L^2_\sigma$. Then for the function $Q(z)$ associated with $\varphi(t)$ in (6.21), we find

$$Q(z) = Q^{h d\mu}(z).$$

The identity (6.3) shows

$$\int_{\mathbb{R}} |t| |\tilde{\varphi}(t)|^2 d\sigma(t) = \int_{\mathbb{R}} |t| h(t) d\mu(t) = \infty.$$

This implies $\tilde{\varphi} \in L^2_\sigma \setminus (|t|^{\frac{1}{2}})$, so that $\varphi \in \mathcal{D}(A) \setminus (|A|^{\frac{1}{2}})$. Moreover, if $y \rightarrow \infty$, $Q(iy)$ converges to a real number. \square

Theorem 6.5. *Let A be a selfadjoint operator. Then A is semibounded if and only if each one-dimensional symmetric restriction S of A is of category \mathbf{N}_1 .*

Proof. Let A be a semibounded selfadjoint operator. Since each symmetric restriction S of A is also semibounded, it is clear that S is of category \mathbf{N}_1 (cf. Proposition 5.25).

For the converse we have to show that A is semibounded if each one-dimensional symmetric restriction S of A is of category \mathbf{N}_1 . Assume that A is not semibounded. According to Proposition 6.4 there exists an element $\varphi \in \setminus (|A|^{\frac{1}{2}})$ and the Q -function Q of (S, A) satisfies $\lim_{y \rightarrow \infty} Q(iy) \in \mathbb{R}$. In particular, this implies that Q in (6.21) does not belong \mathbf{N}_1 . If S is of category \mathbf{N}_1 , this means that A is the exceptional extension of S , so that $Q(iy) \rightarrow \infty$ as $y \rightarrow \infty$ (cf. Corollary 5.7). This contradicts $\lim_{y \rightarrow \infty} Q(iy) \in \mathbb{R}$. We conclude that S is not of category \mathbf{N}_1 . \square

Theorem 6.6. *Let A be a selfadjoint operator. Then A is bounded if and only if each one-dimensional symmetric restriction S of A is of category \mathbf{N}_0 .*

Proof. Let A be bounded. Then each φ in (6.22) belongs to (A) , so that Q in (6.21) belongs to \mathbf{N}_0 (cf. Proposition 5.5). Hence, S is of category \mathbf{N}_0 .

Conversely, let S be of category \mathbf{N}_0 . Hence, the Q -function $Q(z)$ in (6.21) of S and A belongs to \mathbf{N}_0 (as it clearly is not exceptional by Proposition 5.12, since A is an operator). Equivalently, $\varphi \in (A)$. Therefore $(A) = (A)$, from which it follows that A is bounded. \square

Bibliography

- [1] R. ARENS, *Operational calculus of linear relations*, Pacific J. Math., **11** (1961) 9-23
- [2] P. BRUINSMA, *Interpolation problems for Schur and Nevanlinna Pairs*, Dissertation, Rijksuniversiteit Groningen
- [3] E.A. CODDINGTON, H.S.V. DE SNOO, *Positive selfadjoint extensions of positive symmetric subspaces*, Mathematische Zeitschrift, **159** (1978) 203-214
- [4] B. ČURĀUS, *Definitizable extensions of positive symmetric operators in a Kreĭn space*, Integral Equations and Operator Theory, **12** (1989) 615-631
- [5] V.A. DERKACH, M.M. MALAMUD, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal., **95** (1991) no.1
- [6] A. DIJKSMA, H.S.V. DE SNOO, *Selfadjoint extensions of symmetric subspaces*, Pacific J. Math., **54** (1974) 71-99
- [7] A. DIJKSMA, H.S.V. DE SNOO, *Symmetric and Selfadjoint Relations in Kreĭn Subspaces. I*, -Operators in indefinite metric spaces, scattering theory and other topics (Bucarest, 1985). Operator Theory: Advances and Applications 24. Birkhuser Verlag, Basel-Boston, 1987, 144-166.
- [8] A. DIJKSMA, H.S.V. DE SNOO, *Symmetric and Selfadjoint Relations in Kreĭn Subspaces. II*, Annales Academiae Scientiarum Fennicae, Series A. I. Mathematica **12** (1987) 199-216
- [9] S. HASSI, M. KALTENBACK, H.S.V. DE SNOO, *Triples of Hilbert spaces and Friedrichs extensions associated with the subclass \mathbf{N}_1 of Nevanlinna functions*, in preparation
- [10] S. HASSI, M. KALTENBACK, H.S.V. DE SNOO, *Matrix Nevanlinna functions of class \mathbf{N}_1* , in preparation

- [11] S. HASSI, M. KALTENBÄCK, H.S.V. DE SNOO, *A characterization of semi-bounded selfadjoint operators*, in preparation
- [12] S. HASSI, H. LANGER, H.S.V. DE SNOO, *Selfadjoint extensions for a class of symmetric operators with defect numbers (1,1)*, Proc. Oper. Theory Conf., 1 (1995) 115-145
- [13] M. Kaltenbäck, *Model spaces for compositions of Q -functions*, in preparation
- [14] H. LANGER, B. TEXTORIUS, *On generalized resolvents and Q -functions of symmetric linear relations (subspaces) in Hilbert space*, Pacific J. Math., **72** (1977) 135-165
- [15] S. KAC, M.G. KREĀN, *R -Functions – Analytic functions mapping the upper halfplane into itself*, Amer. Math. Soc. Transl., **103** (1974) 1-18
- [16] N.I. ACHIESER, I.M. GLASSMANN, *Theorie der linearen Operatoren im Hilbertraum*, Akademie Verlag, Berlin 1981
- [17] J. BOGNAR, *Indefinite Inner Product Spaces*, Ergebnisse der Mathematik und ihre Grenzgebiete, Band 78, Springer-Verlag, 1974
- [18] W.F. DONOGHUE, JR. *Monotone Matrix Functions and Analytic Continuation*, Springer-Verlag, Berlin-Heidelberg-New York 1974
- [19] T. KATO, *Perturbation Theory for Linear Operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag, 1966
- [20] M.A. NEUMARK, *Lineare Differentialoperatoren*, Mathematische Monographien, Band XI, Akademie-Verlag, Berlin 1960
- [21] M.A. NEUMARK, *Normierte Algebren*, Deutscher Verlag der Wissenschaften, Berlin 1990
- [22] K. YOSIDA, *Functional Analysis*, Die Grundlehren der mathematischen Wissenschaften, Band 123, Springer-Verlag, 1964