Diploma Thesis (reconstructed)

Bernhard Bodenstorfer

Some questions of the theory of perturbation of selfadjoint operators

Original 1996-02-23. Reconstructed 2010-12-09.

A note to the reader interested in the printed original version (all others may swiftly proceed to the next page):

The original printed paper version of the diploma thesis is publicly available at

- Vienna University of Technology Library (`http://www.ub.tuwien.ac.at/`)
- Austrian National Library (`http://www.onb.ac.at/`)

under the title "Some questions of the perturbations of selfadjoint operators".

The content of the following pages is the result of an attempt to reconstruct the diploma thesis for "Technische Mathematik" as it was accepted at the Vienna University of Technology in 1996. As the original printable files have subsequently been deleted, there are some minor differences between this version and the actually printed and submitted original version:

The reconstructed work is based on the original set of LaTeX-files, but the generation anew of the printable files has resulted in different page breaks and page numbers. (Note that the numbering of sections, propositions, formulae and the like has not changed.) Moreover, a few minor typographic errors have been corrected, none of which affected the meaningful essence of the results, however.

The upgraded version of the program **gnuplot** used slightly different defaults, in particular concerning the dimensions of some of the images.

Last but not least, all C-programs had to be ported to and executed on a different platform than what had been used for the orginal thesis. That is why the NAG-library subroutine `f02faf` was replaced by the LAPACK subroutine `dsyev` and the NAG-library subroutine `d01ahf` by the NETLIB (`http://www.netlib.org/`) implementation of `dqag`. (The text still mentions the originally used NAG-subroutines.) The results look essentially identical, but cleaner than the originals. The reason for the improvement is that in the process of porting the C-programs, the treatment of values close to the singularity in Cauchy principal value integrals has been slightly improved *en passant*.

Finally, the pre-release title of the thesis should be mentioned:

"On Compact Perturbations of Selfadjoint Linear Relations"

DIPLOMARBEIT

Some questions of the theory of perturbation of selfadjoint operators

ausgeführt am

Institut für Analysis, Technische Mathematik und
Versicherungsmathematik der Technischen Universität Wien

unter Anleitung von

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Contents

Introduction

If a selfadjoint linear operator A in a Hilbert-space is compactly changed, the spectrum may possess additional eigenvalues afterwards in comparison to the set $\sigma_p A$ of eigenvalues of A.

Beattie and Fox in their paper [3], study three kinds of change of A, all in the finite dimensional case: extension, modification, and restriction.

The present piece of work studies a one-dimensional extension, and the general modification by compact operators. Instead of "modification" the terminus "perturbation" is used here.

In Section 1, multiplication operators A on an $\mathcal{L}^2$-space are extended by one dimension. The eigenvalues of the extension $\tilde{A}$ are studied. The one-dimensional setting has the advantage of making a quick access possible even to eigenvalues which lie in the spectrum of the original operator. Therefore the intention of this section is to give an impression of the behaviour of these eigenvalues, when a real parameter is canged. This parameter specifies different extensions of the same multiplication operator.

In Section 2, a selfadjoint linear operator or relation A is perturbed by a compact selfadjoint operator T. The eigenvalues of $A + T$ which lie in the resolvent set ρA of A are considered. It is examined how many eigenvalues these may be and where they may occur. It is known from Weyl's Theorem that the essential spectra of A and $A + T$ are the same. Hence those eigenvalues can only have accumulation-points in the spectrum σA of A. By the methods proposed, a correspondence between the eigenvalues of $A+T$ in ρA and those of T is established which yields explicit bounds for the distance of these eigenvalues of $A+T$ from σA. The bounds are similar to those which can be obtained via a min-max principle, but they apply as well for eigenvalues of $A + T$ inside gaps $(a, b) \subseteq \rho A$ of A.

Section 3 contains a study of a special kind of perturbation, and Section 4 offers a set of particular numerical examples.

Conclusively, let me devote a word of thankfulness to Prof. Langer for his advice and indefatigable patience which he offered to me when this piece of work was carried out. It was as well him who succeeded in raising some financial support from the Fonds zur Förderung der Wissenschaftlichen Forschung.

I do not want to miss to thank my parents, too, who enabled and encouraged me to follow my various studies.

Vienna, February 23, 1996 Bernhard Bodenstorfer

1 One-dimensional extension of the multiplication operator $y \mapsto u \cdot y$

Definition 1.1 In the following, let $I \subseteq \mathbb{R}$ be a set of real numbers, and let μ be some positive measure on I. For $\mathcal{L}^p_\mu(I)$, $1 \le p \le \infty$, we shall write $\mathcal{L}^p_\mu$. If μ is the Lebesgue-measure ($\mu(t) = t$), then we only write $\mathcal{L}^p$ for $\mathcal{L}^p_\mu$.

Let a be a real number, let $u \in \mathcal{L}^\infty_\mu$ be real-valued, and let $v \in \mathcal{L}^2_\mu$ be complex-valued.

The number a defines the one-dimensional linear operator $z \mapsto az$ on $\mathbb{C}$, v the linear operator $\mathbb{C} \to \mathcal{L}^2_\mu : z \mapsto zv$, and u defines the multiplication operator $y \mapsto u \cdot y$ on $\mathcal{L}^2_\mu$. By v^* we denote the adjoint of v, that is $\mathcal{L}^2_\mu \to \mathbb{C} : y \mapsto \int_I y\overline{v} d\mu$.

Then we define the operator $\tilde{A}_1$ by

$$\tilde{A}_1 = \begin{pmatrix} a & v^* \\ v & u \end{pmatrix} \tag{1}$$

on $\mathbb{C} \oplus \mathcal{L}^2_\mu$ equipped with the inner product

$$\left\langle \begin{pmatrix} c_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} c_2 \\ y_2 \end{pmatrix} \right\rangle = c_1\overline{c_2} + \int_I y_1\overline{y_2} d\mu.$$

Obviously, $\tilde{A}_1$ is a selfadjoint bounded operator. It is a one-dimensional extension of the multiplication operator $y \mapsto u \cdot y$.

Proposition 1.1 *If for a number $z \in \boldsymbol{C}$ both*

$$\int_I \frac{|v|^2}{|u-z|^2} d\mu < \infty \tag{2}$$

and

$$z - a + \int_I \frac{|v|^2}{u - z} d\mu = 0, \tag{3}$$

hold[1]*, or if*[2]

$$\mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \boldsymbol{C}v \neq \{0\} \tag{4}$$

holds, z is eigenvalue of $\tilde{A}_1$. Conversely, for every eigenvalue z of $\tilde{A}_1$, both conditions (2) and (3), or the single condition (4) hold.

The condition that both (2) and (3) hold implies the existence of an eigenvector $\begin{pmatrix} 1 \\ y_1 \end{pmatrix} \in \boldsymbol{C} \oplus \mathcal{L}^2_\mu$.

[1]In case that for $t \in I$ it holds $u(t - z) = v(t) = 0$, the integrands $\frac{0}{0}$ have to be thought to be 0. This becomes clear from the proof below. Surely, if $u(t) = z$ but $v(t) \neq 0$ for all t out of a set of positive measure μ, the integrals in (2–3) diverge, and therefore both of these conditions are violated.

[2]Throughout this whole work, $u^{-1}(\{z\})$ means $\{t \in \mathcal{D}u : u(t) \in \{z\}\}$. This notation is used in the same sense for other sets than $\{z\}$ and other functions than u.

Proof: Let us write “for allmost all $t \in I$ with respect to the measure μ” in symbols as $\overset{\mu}{\forall}\, t$. Furthermore, let us agree upon the precedence rule $T_1 \wedge T_2 \vee T_3 = T_3 \vee T_1 \wedge T_2 = (T_1 \wedge T_2) \vee T_3$.

We fix $z \in \mathbb{C}$. Then a pair $\begin{pmatrix} c \\ y \end{pmatrix} \in \mathbb{C} \oplus \mathcal{L}^2_\mu$ is a nonzero eigenvector of $\tilde{A}_1$ if and only if the following equivalent expressions hold true:

- $\begin{pmatrix} c \\ y \end{pmatrix} \neq 0 \;\wedge\; (a-z)c + v^* y = 0 \wedge \overset{\mu}{\forall}\, t:\; cv(t) + (u(t)-z)y(t) = 0$

- $c = 0 \;\wedge\; y \neq 0 \;\wedge\; v^* y = 0 \wedge \overset{\mu}{\forall}\, t:\; (u(t) \neq z \;\Rightarrow\; y(t) = 0)$
  $\vee\; c \neq 0 \;\wedge\; (a-z)c + v^* y = 0 \wedge \overset{\mu}{\forall}\, t:\; \left(u(t) = z \;\wedge\; v(t) = 0 \;\vee\; u(t) \neq z \;\wedge\; y(t) = \dfrac{-cv(t)}{u(t)-z} \right)$

- $c = 0 \;\wedge\; y \neq 0 \;\wedge\; v^* y = 0 \;\wedge\; \mu\left(u^{-1}(\{z\}) \cap y^{-1}(\mathbb{C} \setminus \{0\}) \right) = 0$
  $\vee\; c \neq 0 \;\wedge\; (a-z)c + v^* y = 0$
  $\wedge\; \overset{\mu}{\forall}\, t:\; \left((u(t) = z \;\Rightarrow\; v(t) = 0) \;\wedge\; \left(u(t) \neq z \;\Rightarrow\; y(t) = \dfrac{-cv(t)}{u(t)-z} \right) \right)$

- $c = 0 \;\wedge\; y \in \mathcal{L}^2_\mu(u^{-1}(\{z\})) \setminus \{0\} \;\wedge\; v^* y = 0$
  $\vee\; c \neq 0 \;\wedge\; (a-z)c + v^* y = 0$
  $\wedge\; \overset{\mu}{\forall}\, t:\; (u(t) = z \;\Rightarrow\; v(t) = 0) \wedge \overset{\mu}{\forall}\, t:\; \left(u(t) \neq z \;\Rightarrow\; y(t) = \dfrac{-cv(t)}{u(t)-z} \right)$

Now we join $\exists y \in \mathcal{L}^2_\mu$ from left to all lines in thought and continue:

- $c = 0 \;\wedge\; \exists y \in \mathcal{L}^2_\mu(u^{-1}(\{z\})) \setminus \{0\} \cap \{v\}^\perp$
  $\vee\; c \neq 0 \wedge \overset{\mu}{\forall}\, t:\; (u(t) = z \;\Rightarrow\; v(t) = 0)$
  $\wedge\; \exists y \in \mathcal{L}^2_\mu:\; (a-z)c + v^* y = 0 \wedge \overset{\mu}{\forall}\, t:\; \left(u(t) \neq z \;\Rightarrow\; y(t) = \dfrac{-cv(t)}{u(t)-z} \right)$

- $c = 0 \;\wedge\; \mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v \neq \{0\}$
  $\vee\; c \neq 0 \wedge \overset{\mu}{\forall}\, t:\; (u(t) = z \;\Rightarrow\; v(t) = 0)$
  $\wedge\; \exists y_0 \in \mathcal{L}^2_\mu:\; (a-z)c + v^* y_0 = 0 \;\wedge\; y_0(u^{-1}(\{z\})) = \{0\}$
  $\wedge\; \overset{\mu}{\forall}\, t:\; \left(u(t) \neq z \;\Rightarrow\; y_0(t) = \dfrac{-cv(t)}{u(t)-z} \right)$

- $c = 0 \;\wedge\; \mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v \neq \{0\}$
  $\vee\; c \neq 0 \wedge \overset{\mu}{\forall}\, t:\; (u(t) = z \;\Rightarrow\; v(t) = 0)$
  $\wedge\; \exists y_0 \in \mathcal{L}^2_\mu:\; \forall t \in I:\; y_0(t) = \begin{cases} 0 & \ldots\; u(t) = z \\ \frac{-cv(t)}{u(t)-z} & \ldots\; u(t) \neq z \end{cases} \;\wedge\; (a-z)c + v^* y_0 = 0$

- $c = 0 \;\wedge\; \mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v \neq \{0\}$

  $$\vee\; c \neq 0 \wedge \mu\left(v^{-1}(\mathbb{C} \setminus \{0\}) \cap u^{-1}(\{z\}) \right) = 0 \tag{5}$$

  $$\wedge \int_{I \setminus u^{-1}(\{z\})} \frac{|v(t)|^2}{|u(t)-z|^2} d\mu(t) < \infty \;\wedge\; a - z + \int_{I \setminus u^{-1}(\{z\})} \frac{|v(t)|^2}{u(t)-z} d\mu(t) = 0$$

- $c = 0 \wedge \mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v \neq \{0\}$
  $\vee\ c \neq 0 \wedge \displaystyle\int_I \frac{|v(t)|^2}{|u(t)-z|^2} d\mu(t) < \infty \wedge a - z + \int_I \frac{|v(t)|^2}{u(t)-z} d\mu(t) = 0,$

where, in the last line, we have to interpret $\frac{0}{0} = 0$ for the case $u(t) - z = v(t) = 0$, confer to footnote 1.

The function y_0 is defined by $y_0(t) = \begin{cases} y(t) & \dots\ u(t) \neq z \\ 0 & \dots\ u(t) = z \end{cases}$. Note that then $v^* y_0 = v^* y$ if $\overset{\mu}{\forall} t:\ (u(t) = z \Rightarrow v(t) = 0)$. This fact has been used at the upmost occurence of y_0 in the above lines. ♡

Remark 1.1 Because of $\mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v = \ker((y \mapsto u \cdot y) - z) \cap \ker v^*$, condition (4) can be written as $\ker((y \mapsto u \cdot y) - z) \cap \ker v^* \neq \{0\}$. That is the reason why Proposition 1.1 can be viewed as a direct consequence of the more general and elegant Proposition 2.1 to follow later on.

Moreover, Proposition 1.1 reminds us of [1, Behauptung 1], which yields only a weaker statement however, because points z with $\mu\left(u^{-1}(\{z\})\right) > 0$ have to be excluded from the consideration there.

Remark 1.2 The properties of y and of y_0 imply $y - y_0 \in \mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v$. That is why $\dim\ker(\tilde{A}_1 - z) = 1 + \dim \mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v$ if for $z \in \mathbb{C}$ (2) and (3) hold. This can be interpreted in the following way: A one-dimensional extension of $y \mapsto u \cdot y$ can raise the multilpicity of z as an eigenvalue maximally by one.

Again, Proposition 2.1 together with Proposition 2.3 will show that this observation is no accident.

Remark 1.3 The eigenspace of $\tilde{A}_1$ for an eigenvalue z for which (4) holds includes the space

$$\mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v.$$

This space equals that eigenspace if (2) or (3) does not hold.

For $\mu(t) = t$, the space $\mathcal{L}^2_\mu(u^{-1}(\{z\}))$, and thus the space $\mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus \mathbb{C}\, v$ is either zero- or infinite dimensional. Hence, numbers z, for which (4) holds are then eigenvalues of infinite multiplicity.

Definition 1.2 The μ-essential range of a function u is

$$\mathcal{R}_{\mu ess} u = \left\{ z \in \mathbb{C} :\ \forall \varepsilon > 0 : \mu(u^{-1}(\{z' \in \mathbb{C} : |z' - z| < \varepsilon\})) > 0 \right\}.$$

With this notion we can formulate

Proposition 1.2 *$\sigma(y \mapsto u \cdot y) = \mathcal{R}_{\mu ess} u$.*

Proof: We define $M_\varepsilon = u^{-1}(\{z' \in \mathbb{C} : |z' - z| < \varepsilon\})$.

Let z be in $\mathcal{R}_{\mu ess} u$. Then, for $\varepsilon > 0$, we have $\mu(M_\varepsilon) > 0$. To avoid complications when $\mu(M_\varepsilon) = \infty$, chose a family of subsets $M_\varepsilon^1 \subseteq M_\varepsilon$ such that $\mu(M_\varepsilon^1) = \min(1, \mu(M_\varepsilon))$.

In order to show that z is in $\sigma(y \mapsto u \cdot y)$ we define a family of $\mathcal{L}^2_\mu$-vectors by

$$y_\varepsilon(t) = \begin{cases} \frac{1}{\sqrt{\mu(M_\varepsilon^1)}} & \ldots \ t \in M_\varepsilon^1 \\ 0 & \ldots \ t \notin M_\varepsilon^1 \end{cases} .$$

It holds

$$\|y_\varepsilon\|^2 = \int_{M_\varepsilon^1} \frac{1}{\mu(M_\varepsilon^1)} d\mu = 1,$$

but also

$$\|(u - z) y_\varepsilon\|^2 = \int_{M_\varepsilon^1} \frac{|u(t) - z|^2}{\mu(M_\varepsilon^1)} d\mu(t) \le \int_{M_\varepsilon^1} \frac{\varepsilon^2}{\mu(M_\varepsilon^1)} d\mu(t) = \varepsilon^2.$$

Hence, $y_\varepsilon \not\to 0$ but $(u - z) y_\varepsilon \to 0$ if $\varepsilon \to 0$. Consequently $z \in \sigma(y \mapsto u \cdot y)$.

Conversely, assume $z \notin \mathcal{R}_{\mu ess} u$. Then there is an $\varepsilon_0 > 0$ such that $\mu(M_{\varepsilon_0}) = 0$. So we have for all $y \in \mathcal{L}^2_\mu$:

$$\begin{aligned} \|(u - z) y\|^2 &= \int_I |u(t) - z|^2 |y(t)|^2 d\mu(t) = \int_{I \setminus M_{\varepsilon_0}} |u(t) - z|^2 |y(t)|^2 d\mu(t) \\ &\ge \int_{I \setminus M_{\varepsilon_0}} {\varepsilon_0}^2 |y(t)|^2 d\mu(t) = \int_I {\varepsilon_0}^2 |y(t)|^2 d\mu(t) = {\varepsilon_0}^2 \|y\|^2 . \end{aligned}$$

Thus, $z \in \rho(y \mapsto u \cdot y)$. ♡

Remark 1.4 Directly from Proposition 1.2, it can be seen that $\mathcal{R}_{\mu ess} u$ is a closed set.

Proposition 1.3 *$\sigma_{ess} \tilde{A}_1 = \sigma_{ess}(y \mapsto u \cdot y)$.*

Proof: The essential spectrum $\sigma_{ess} \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix} = \sigma_{ess}(y \mapsto u \cdot y)$ does not change under compact perturbation (Weyl's Theorem, see e.g. [RS, Theorem XIII.14]). ♡

1.1 Eigenvalues of $\tilde{A}_1(a)$ outside $\mathcal{R}_{\mu ess} u$

This section is devoted to the study of the movement of eigenvalues $z \notin \mathcal{R}_{\mu ess} u$ of the operator

$$\tilde{A}_1(a) = \begin{pmatrix} a & v^* \\ v & u \end{pmatrix}$$

depending on the real parameter a. The operator $\tilde{A}_1(a)$ is already familiar from the preceding section.

It should be noted that results obtained for $\tilde{A}_1(a)$ can easily be translated into similar results for the operator

$$\tilde{A}_{1v}(b) = \begin{pmatrix} a & \bar{b}v^* \\ bv & u \end{pmatrix}.$$

This operator describes a coupling problem, see Section 3.

Using the function

$$f(z) = \int_I \frac{|v|^2}{u-z} d\mu, \tag{6}$$

condition (3) can be written as

$$f(z) = a - z. \tag{7}$$

This function f is holomorphic outside $\mathcal{R}_{\mu ess}u$. Obviously, f is monotonous for z in any convex subset of $\mathbb{R}\setminus\mathcal{R}_{\mu ess}u$, especially for $z > \mathcal{R}_{\mu ess}u$ or $z < \mathcal{R}_{\mu ess}u$.

Proposition 1.4 *If $a > \mathcal{R}_{\mu ess}u$ holds, then there exists exactly one eigenvalue $z_+ > \mathcal{R}_{\mu ess}u$, and $z_+ \geq a$ holds.*

Additionally, one eigenvalue may exist in each convex subset of $\mathbb{R}\setminus\mathcal{R}_{\mu ess}u$. Further eigenvalues, if any exist, lie in $\mathcal{R}_{\mu ess}u$.

The case $a < \mathcal{R}_{\mu ess}u$ is wholly symmetric.

Proof: If $\mathcal{R}_{\mu ess}u < z < a$, then $z + f(z) < z < a$ holds, but $z + f(z) \to +\infty$ for $z \nearrow \infty$. Hence, by continuity of f on $\mathbb{R}\setminus\mathcal{R}_{\mu ess}u$, there is a number $z > \mathcal{R}_{\mu ess}u$ fulfilling $z + f(z) = a$. This is an eigenvalue of $\tilde{A}_1(a)$, because (2) trivially holds for $z \notin \mathcal{R}_{\mu ess}u$. It is the only one in $(\max \mathcal{R}_{\mu ess}u, +\infty)$, because $z \mapsto z - a + f(z)$ is strictly monotonous.

In the same way, monotony of this function is the reason that in no convex subset of $\mathbb{R}\setminus\mathcal{R}_{\mu ess}$ can be more than one number z such that (7) holds. ♡

Remark 1.5 The assertion of Proposition 1.4 can be adapted to the case $a = \max \mathcal{R}_{\mu ess}u$: First assume $f = 0$.[3] Then we trivially get that a itself is eigenvalue of $\tilde{A}_1(a)$, because $a + f(a) = a$ under these circumstances. In this case we have to admit $z_+ = \max \mathcal{R}_{\mu ess}$ as the desired eigenvalue of $\tilde{A}_1(a)$.

If $f \neq 0$, then it holds $f \leq -\varepsilon$ on (a, a') for some a' suitably near a, and for some $\varepsilon > 0$. Hence, $z - a + f(z) < 0$ for $a < z < a'$. On the other hand, $z - a + f(z) \to +\infty$ for $z \to +\infty$. Consequently, there is a $z > a$ such that $f(z) = a - z$.

Application of the Implicit function Theorem on (7) yields

Proposition 1.5 *All eigenvalues of $\tilde{A}_1(a)$ outside $\mathcal{R}_{\mu ess}u$ obey the differential equation*

$$\frac{dz}{da} = \frac{1}{1 + \int_I \frac{|v|^2}{(u-z)^2} d\mu}.$$

[3] This means $v = 0$ in $\mathcal{L}^2_\mu$.

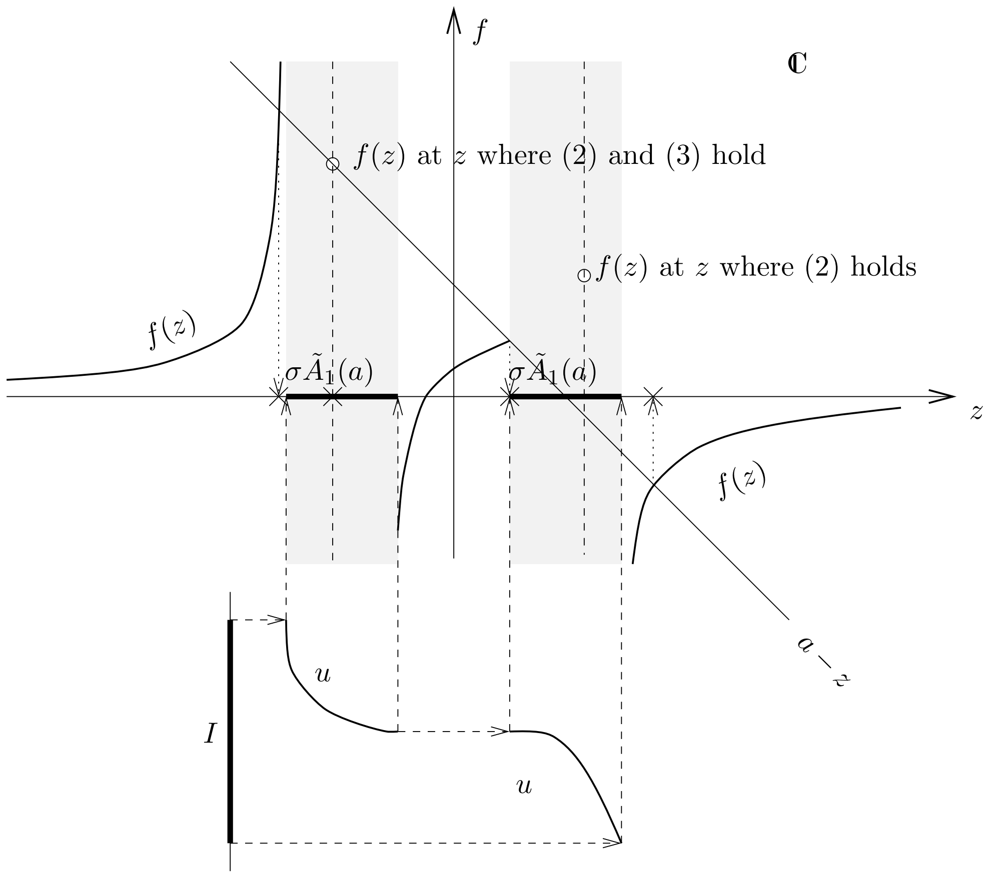

Figure 1: The meaning of (7): The eigenvalues of $A_1(a)$ are represented by little crosses on the z-axis.

An easy consequence[4] is

Proposition 1.6 *Isolated real eigenvalues of $\tilde{A}_1(a)$ vary monotonously with a.*

1.2 Eigenvalues of $\tilde{A}_1(a)$ on the border of $\mathcal{R}_{\mu ess}u$

The question arises what happens to formerly isolated eigenvalues when a moves into $\mathcal{R}_{\mu ess}u$. Let us consider the eigenvalue $z_-(a) < min\mathcal{R}_{\mu ess}u$ first, if a increases. The situation for z_+ and decreasing a is the same. Other eigenvalues outside $\mathcal{R}_{\mu ess}u$, behave similar (see Remark 1.7), but also those eigenvalues inside $\mathcal{R}_{\mu ess}u$, which can move in an interval (u_-, u_+), for which $v\left(u^{-1}((u_-, u_+))\right) = \{0\}$. The latter is plausible since the operator $y \mapsto u \cdot y$ can be decomposed into two parts then which act on orthogonal subspaces of $\mathcal{L}^2_\mu$, one including v. So the extension $\tilde{A}_1(a)$ leaves one of these two operators unperturbed, and the other has no spectrum in (u_-, u_+).

Proposition 1.7 *Set*

$$a_- = \min \mathcal{R}_{\mu ess}u + \int_I \frac{|v|^2}{u - \min \mathcal{R}_{\mu ess}u} d\mu \in I\!R \cup \{+\infty\}.$$

Then $\tilde{A}_1(a)$ has one eigenvalue $z_-(a) < \mathcal{R}_{\mu ess}u$ for $a < a_-$ and none for $a > a_-$. Further it holds

$$\lim_{a \nearrow a_-} z_-(a) = \min \mathcal{R}_{\mu ess}u. \tag{8}$$

If $a_- \in I\!R$, this means that the eigenvalue $z_-(a)$ has joined $\mathcal{R}_{\mu ess}u$ if $a > a_-$.

Proof: By the Fatou-Lemma,

$$a_- = \lim_{z \nearrow \min \mathcal{R}_{\mu ess}u} (z + f(z)). \tag{9}$$

Hence, $a_- \in \mathrm{IR}$ implies that (7) cannot be fulfilled by a number $z \leq \min \mathcal{R}_{\mu ess}u$ if $a > a_-$.

Equation (8) is a consequence of $z \mapsto z + f(z)$ being continuous, strictly monotonous, and thus bijective, together with (9). ♡

Remark 1.6 Note that v may vanish on some sets of positive μ-measure. Then Proposition 1.7 may hold trivially. For instance, if $v(u^{-1}((-\infty, u_+))) = \{0\}$, and $\min \mathcal{R}_{\mu ess}u < u_+$, the function z_- is even analytic for a near a_- since $z < u_+$ stays true there. See the notes at the beginning of this section.

Proposition 1.8 *Let a_- of Proposition 1.7 be finite and $\mu(v^{-1}(\{0\})) = 0$. Then the following are equivalent:*

(i) $\min \mathcal{R}_{\mu ess}u$ *is an eigenvalue of $\tilde{A}_1(a_-)$.*

[4] This could be seen as well from the monotony of $z \mapsto z - a + f(z)$.

(ii) $\frac{dz_-(a)}{da}(a)$ *remains bounded for* $a \nearrow a_-$.

Proof: Property (i) via Proposition 1.1 implies (2) which is equivalent to (ii) by Proposition 1.5 via Fatou-Lemma.

On the other hand, for $a = a_-$ (3) always holds if the integral converges. This convergence is implied by (ii), which is a consequence of (2) via Fatou-Lemma again. Hence (ii) also implies (i). ♡

Remark 1.7 Statements similar to Propositions 1.8 and 1.7 can be proved for all points z_∂ on the border of $\mathcal{R}_{\mu ess}u$ in quite the same manner. Then the values a_- have to be substituted by by $z_\partial + \int_I \frac{|v|^2}{u-z_\partial} d\mu \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Moreover, before the Fatou-Lemma can be applied, some integrals have to be split into two parts in order to get purely positive and purely negative integrands.

1.3 Eigenvalues of $\tilde{A}_1(a)$ in $\mathcal{R}_{\mu ess}u$

We continue the study of the operators $\tilde{A}_1(a)$ of the previous section. In this section, we restrict ourselves to the Lebesgue measure $\mu(t) = t$ on I and assume I to be an interval. These assumptions imply $\mathcal{R}_{\mu ess}u = \overline{u(I)}$ if u is continuous.

Proposition 1.9 *Let I be an interval and $\mu(t) = t$ determine the Lebesgue-measure on I. Further assume u to be differentiable and v to be continuous.*

Then

$$v(u^{-1}(\{z\})) = \{0\}$$

if z is an eigenvalue of $\tilde{A}_1(a)$ for which (2–3) hold.

Proof: Choose $t_z \in u^{-1}(\{z\}) \cap I$. Then there is a number $\varepsilon_0 > 0$ such that $(t_z - \varepsilon_0, t_z] \subseteq I$ or $[t_z, t_z + \varepsilon_0) \subset I$. Without loss of generaltiy we assume the latter case. The former can be explored analogously.

Since u is differentiable at t_z, we can estimate

$$|u(t) - z| \leq (1 + |u'(t_z)|)|t - t_z| \tag{10}$$

for $t \in I$ sufficiently near t_z.

Assume $v(t_z) \neq 0$. Then continuity of v tells us

$$|v(t)| \geq \frac{|v(t_z)|}{2} > 0 \tag{11}$$

for $t \in I$ sufficiently near t_z.

Now use (10) and (11) in condition (2). For sufficiently small $\varepsilon > 0$ this yields:

$$\int_I \frac{|v|^2}{|u-z|^2} dt \quad \geq \quad \frac{|v(t_z)|^2}{4(1 + |u'(t_z)|)^2} \int_{t_z}^{t_z+\varepsilon} |t - t_z|^{-2} dt = +\infty.$$

Hence, (2) is not fulfilled. This contradicts the converse assumption. Consequently $v(t_z) = 0$. $\heartsuit$

Remark 1.8 Note that the arguments in the proof of Proposition 1.9 can be used to show that not only (2) is violated if $v(t_z) \neq 0$, but $f(z)$ does not exist either. This follows from (10) and (11) by

$$\int_I \frac{|v|^2}{|u-z|} dt \quad \geq \quad \frac{|v(t_z)|^2}{4(1+|u'(t_z)|)} \int_{t_z}^{t_z+\varepsilon} |t-t_z|^{-1} dt = +\infty.$$

Remark 1.9 Remembering the proof of Proposition 2.7, you see that line (5) states that the measure μ of the set $v^{-1}(\mathbb{C} \setminus \{0\}) \cap u^{-1}(\{z\})$ is zero in the case $c \neq 0$. This case is just the case when (2–3) are valid. Proposition 1.9 strengthens this by asserting that that set is empty in fact, under additional assumptions on I, μ, u, and v.

The following is an immediate consequence of Proposition 1.9.

Proposition 1.10 *Under the assumptions of Proposition 1.9 it holds: If $v^{-1}(\{0\})$ is finite, then there are only finitely many pairs (a, z) such that (2–3) hold.*

From Proposition 1.10 can be seen that eigenvalues in $\mathcal{R}_{\mu ess} u$ are not stable in general. If (4) is not valid, we can even prove instability from suitable conditions. Under such conditions, those eigenvalues must disappear leaving no trace in $\sigma_p \tilde{A}_1(a)$ when a is changed.

The assumption that $v(t) = 0$ only at finitely many $t \in I$ is important here. This will be illustrated by Proposition 1.12, which uses some preliminaries.

Proposition 1.11 *For all sequences $(a_k)_{k \in \mathbb{Z}}$ of real numbers, there is a nonnegative, bounded function $v \in C^\infty((-1, +1))$ such that*

$$v^{-1}(\{0\}) \cap (-1, +1) = \{\tanh k : k \in \mathbb{Z}\}, \tag{12}$$

and for all $k \in \mathbb{Z}$

$$\int_{-1}^{1} \frac{|v(t)|^2}{|t - \tanh k|^2} dt < \infty \tag{13}$$

and

$$\int_{-1}^{1} \frac{|v(t)|^2}{t - \tanh k} dt = a_k \tag{14}$$

hold.

<u>Proof</u>: Note at the beginning: Exceptionally, throughout this proof we consider all linear spaces as linear spaces over the field $\mathbb{R}$ of real numbers! Especially the spaces of sequences are supposed to contain real-valued sequences only.

Let us remember the familiar $C^\infty(\mathbb{R})$ testing function $\varphi(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & \ldots\ -1 < t < 1 \\ 0 & \ldots\ \text{else} \end{cases}$ and its normalized integral $\psi(t) = \frac{\int_{-\infty}^{t} \varphi dt}{\int_{-\infty}^{+\infty} \varphi dt}$. From this, we define the function $w_{\varepsilon',\varepsilon^+}$ for $0 \le \varepsilon' \le \varepsilon^+ > 0$ by

$$w_{\varepsilon',\varepsilon^+}(t) = \begin{cases} 0 & \ldots\ t \le 0 \\ \psi\left(\frac{2t}{\varepsilon'} - 1\right) & \ldots\ 0 < t \le \varepsilon' \\ 1 & \ldots\ \varepsilon' < t \le \varepsilon^+ \\ 1 - \psi\left(\frac{2t}{\varepsilon^+} - 3\right) & \ldots\ \varepsilon^+ < t \le 2\varepsilon^+ \\ 0 & \ldots\ 2\varepsilon^+ < t \end{cases}.$$

Note, that $w_{\varepsilon',\varepsilon^+}$ is in $C^\infty(\mathbb{R})$ if $\varepsilon' > 0$.

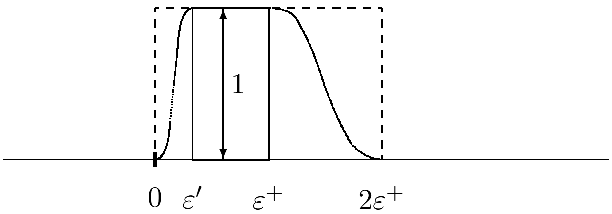

Figure 2: The function $w_{\varepsilon',\varepsilon^+}$, sketched.

We define $t_k = \tanh k$. Additionally we shall make use of a sequence of positive numbers b_k fulfilling

$$\sum_{k=-\infty}^{+\infty} b_k = \varepsilon < \frac{\int_0^2 \frac{\psi(t-1)}{t} dt}{2}. \tag{15}$$

Then there is a sequence of positive numbers

$$\varepsilon_k^+ < \frac{\min(t_{k+1} - t_k, t_k - t_{k-1})}{4} \tag{16}$$

such that

$$\int_{t_{k-1}}^{t_{k+1}} \frac{w_{\varepsilon_k',\varepsilon_k^+}(t - t_k) + w_{\varepsilon_k',\varepsilon_k^+}(t_k - t)}{|t - t_{k-1}|^2} dt \quad \le \quad b_k, \tag{17}$$

$$\int_{t_{k-1}}^{t_{k+1}} \frac{w_{\varepsilon_k',\varepsilon_k^+}(t - t_k) + w_{\varepsilon_k',\varepsilon_k^+}(t_k - t)}{|t - t_{k+1}|^2} dt \quad \le \quad b_k. \tag{18}$$

Let a sequence $(a_k')_{k\in\mathbb{Z}}$ be given.[5] Then there is a sequence of positive numbers ε_k' such that

$$|a_k'| \le \int_{t_{k-1}}^{t_{k+1}} \frac{w_{\varepsilon_k',\varepsilon_k^+}(t - t_k)}{t - t_k} dt. \tag{19}$$

[5]This will be the sum of the sequence $(a_k)_{k\in\mathbb{Z}}$ and a second, fix sequence $(a_{\perp k})_{k\in\mathbb{Z}}$.

For brevity, let us set $g_k^+(t) = w_{\varepsilon_k', \varepsilon_k^+}(t - t_k)$ and $g_k^-(t) = w_{\varepsilon_k', \varepsilon_k^+}(t_k - t)$. Note that (16) implies

$$\operatorname{supp} g_k^+ \subset \left[t_k, \frac{t_k + t_{k+1}}{2}\right), \qquad \operatorname{supp} g_k^- \subset \left(\frac{t_k + t_{k-1}}{2}, t_k\right]. \tag{20}$$

For any bounded (real valued) sequence $h = (h_k)_{k \in \mathbb{Z}}$ we define the nonnegative function g by

$$g_h(t) = \sum_{k=-\infty}^{+\infty} \left(\frac{|h_k| + h_k}{2} g_k^+(t) + \frac{|h_k| - h_k}{2} g_k^-(t) \right). \tag{21}$$

This may seem a bit queer, but it is necessary to decompose $(h_k)_{k \in \mathbb{Z}}$ according to the sign of its elements, since our aim is something similar to $v = \sqrt{g_h}$. Hence, g_h must be nonnegative and the signs of the numbers h_k must be encoded nonlinearly by the distinction between g_k^+ and g_k^-. See Figure 3. Note at this point that $g_k^-(t - t_k) = g^+(t_k - t)$ and thus

$$\int_{t_{k-1}}^{t_{k+1}} \frac{g_k^-(t)}{t - t_k} dt \quad = \quad - \int_{t_{k-1}}^{t_{k+1}} \frac{g_k^+(t)}{t - t_k} dt$$

hold.

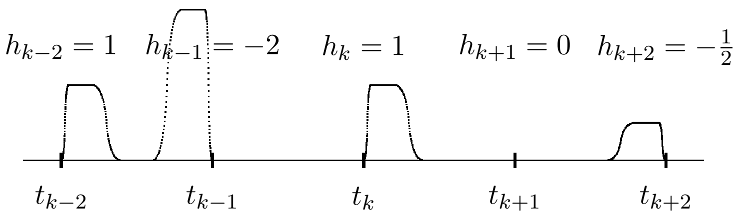

Figure 3: The shape of a typical function g_h, sketched.

From (20) follows that in the sum (21) for every t maximally one k yields a nonzero contribution to the sum. Consequently, $g_h \in C^\infty(I)$. Moreover, g_h is bounded by $\|h\|_\infty$, because the functions g_k^+ and g_k^- are bounded by 1, and only one of $g_k^+(t)$ and $g_k^-(t)$ may be nonzero for each specific number t.

So far we have defined a function g_h from a sequence h. Now from this, define two sequences Th and Ih setting

$$(Th)_k = \int_{-1}^{+1} \frac{g_h(t)}{t - t_k} dt \tag{22}$$

and

$$(Ih)_k = \int_{\frac{t_{k-1}+t_k}{2}}^{\frac{t_k+t_{k+1}}{2}} \frac{g_h(t)}{t - t_k} dt = \int_{-1}^{+1} \frac{\frac{|h_k|+h_k}{2} g_k^+(t) + \frac{|h_k|-h_k}{2} g_k^-(t)}{t - t_k} dt = h_k \int_{-1}^{+1} \frac{g_k^+(t)}{t - t_k} dt. \tag{23}$$

Obviously, the integral in (23) converges, because g_k^+ is a bounded differentiable function with $g_k^+(t_k) = 0$. Therefore we know

$$|g_k^+(t)| = O(t - t_k). \tag{24}$$

(Compare the argumentation of Proposition 1.14). If h is a bounded sequence, then the integral (22) converges as well for the same reason.

In this way, two operators T and I have been defined on $\ell^\infty_{\mathbb{R}}$. The operator I is linear, the operator T is not linear but stands in an almost linear relation to I, see (26). So it will turn out to be almost linear itself (see (30) in footnote 7).

But first, let us show that I is invertible on the space $\mathbb{R}^{\mathbb{Z}}$ of all real-valued sequences. This follows from (23) and that

$$\begin{aligned}\int_{-1}^{+1} \frac{g_k^+(t)}{t-t_k} dt &= \int_0^{2\varepsilon_k^+} \frac{g_k^+(t)}{t-t_k} dt \geq \int_0^{\varepsilon_k'} \frac{g_k^+(t)}{t-t_k} dt = \int_0^{\varepsilon_k'} \frac{\psi\left(\frac{2(t-t_k)}{\varepsilon_k'} - 1\right)}{t-t_k} dt \\ &= \left| \begin{array}{c} t' = \frac{2(t-t_k)}{\varepsilon_k'} \\ dt' = \frac{2}{\varepsilon_k'} \end{array} \right| = \int_0^2 \frac{\psi(t'-1)}{t'} dt' = C_{I^{-1}} > 0\end{aligned}$$

is bounded from zero by a positive constant. Hence, we can write for a sequence $(h_k')_{k\in\mathbb{Z}}$:

$$(I^{-1}h')_k = \frac{h_k}{\int_{-1}^{+1} \frac{g_k^+(t)}{t-t_k} dt}.$$

As the positive constant $C_{I^{-1}}$ does not depend on k, the operator I^{-1} acts continuously on the space $\ell^\infty_{\mathbb{R}} \subset \mathbb{R}^{\mathbb{Z}}$ of bounded real sequences. It holds

$$\left\| I^{-1} \right\|_\infty \leq \frac{1}{C_{I^{-1}}}. \tag{25}$$

Now we return to T in order to prove: [6]

$$\forall h, h' \in \ell^\infty_{\mathbb{R}} : \ \| T(h+h') - Th - Ih' \|_\infty \leq 2\varepsilon \| h' \|_\infty \tag{26}$$

From the basic inequality

$$\Big| |c \pm d| - |c| \Big| \leq |d|$$

we get that for all $k \in \mathbb{Z}$

$$\begin{aligned}\left| \frac{|h_k + h_k'| + (h_k + h_k')}{2} - \frac{|h_k| + h_k}{2} \right| &\leq |h_k'|, \\ \left| \frac{|h_k + h_k'| - (h_k + h_k')}{2} - \frac{|h_k| - h_k}{2} \right| &\leq |h_k'|\end{aligned}$$

hold. We use this to estimate

$$\left| (T(h+h'))_k - (Th)_k - (Ih')_k \right| = \left| \int_{-1}^{+1} \frac{g_{h+h'}(t) - g_h(t)}{t-t_k} dt - (Ih')_k \right|$$

[6] Note that neither of the three sequences $T(h+h')$, Th, and Ih' needs to be bounded, but (26) states that $T(h+h') - Th - Ih'$ is.

$$= \left| \underbrace{(I(h+h') - Ih - Ih')_k}_{0} + \int_{(-1,+1)\setminus\left[\frac{t_{k-1}+t_k}{2},\frac{t_k+t_{k+1}}{2}\right]} \frac{g_{h+h'}(t) - g_h(t)}{t-t_k} dt \right|$$

$$= \left| \int_{(-1,+1)\setminus\left[\frac{t_{k-1}+t_k}{2},\frac{t_k+t_{k+1}}{2}\right]} \sum_{j=-\infty}^{+\infty} \frac{\left(\frac{|h_j+h'_j|+(h_j+h'_j)}{2} - \frac{|h_j|+h_j}{2}\right) g_j^+(t) + \left(\frac{|h_j+h'_j|-(h_j+h'_j)}{2} - \frac{|h_j|-h_j}{2}\right) g_j^-(t)}{t-t_k} dt \right|$$

$$\leq \int_{(-1,+1)\setminus\left[\frac{t_{k-1}+t_k}{2},\frac{t_k+t_{k+1}}{2}\right]} \sum_{j=-\infty}^{+\infty} |h'_j| \frac{g_j^+(t) + g_j^-(t)}{|t-t_k|} dt$$

$$\leq 2\left\|h'\right\|_\infty \int_{(-1,+1)\setminus\left[\frac{t_{k-1}+t_k}{2},\frac{t_k+t_{k+1}}{2}\right]} \sum_{j=-\infty}^{+\infty} \frac{g_j^+(t) + g_j^-(t)}{|t-t_k|^2} dt$$

$$\leq 2\left\|h'\right\|_\infty \sum_{j\in\mathbb{Z}\setminus\{k\}} \int_{\frac{t_{j-1}+t_j}{2}}^{\frac{t_j+t_{j+1}}{2}} \frac{g_j^+(t) + g_j^-(t)}{|t-t_k|^2} dt \leq 2\left\|h'\right\|_\infty \sum_{j=-\infty}^{+\infty} b_j = 2\varepsilon \left\|h'\right\|_\infty .$$

In the last line remember the definitions of g_j^+ and g_j^- and (17–18). From these follows

$$\int_{\frac{t_{j-1}+t_j}{2}}^{\frac{t_j+t_{j+1}}{2}} \frac{g_j^+(t) + g_j^-(t)}{|t-t_k|^2} dt \leq b_j \tag{27}$$

for $j = k+1$ (from (17)), and for $j = k-1$ (from (18)) at first. A fortiori, the inequality (27) for arbitrary $j \neq k$ follows from these two cases. Well now, so we have proved (26).

Setting $h = 0$ in (26), we see that

$$\|T - I\|_\infty \leq 2\varepsilon. \tag{28}$$

Remembering (25), we can estimate

$$\left\|1 - I^{-1}T\right\|_\infty \leq \left\|I^{-1}\right\|_\infty \|T - I\|_\infty \leq \frac{2\varepsilon}{C_{I^{-1}}} \tag{29}$$

from (28). [7]

From (26) and (29) we see

$$\left\|I^{-1}T(h+h') - I^{-1}Th\right\|_\infty \leq \left\|0 + I^{-1}Ih'\right\|_\infty + \frac{2\varepsilon}{C_{I^{-1}}} \left\|h'\right\|_\infty = \left(1 + \frac{2\varepsilon}{C_{I^{-1}}}\right) \left\|h'\right\|_\infty . \tag{31}$$

Consequently, $I^{-1}T$ is continuous on $\ell^\infty_{\mathbb{R}}$.

[7] From (28), by the way, we can also get

$$\left\|T(h+h') - Th - Th'\right\|_\infty \leq \left\|T(h+h') - Th - Ih'\right\|_\infty + \|T - I\|_\infty \left\|h'\right\|_\infty \leq 4\varepsilon \left\|h'\right\|_\infty , \tag{30}$$

which states that T is "nearly linear".

Next we want to show that $I^{-1}T$ is an invertible operator on $\ell^\infty_{\mathbb{R}}$. The estimation (29) lets us think of Banach's Fixed Point Theorem, but alas, $I^{-1}T$ is not linear. However, a property such as (26)[8] suffices to be able to exploit the idea of this Fixed Point Theorem if ε is sufficiently small.

Fix a sequence $h^0 \in \ell^\infty_{\mathbb{R}}$. Then we set inductively

$$h^i = h^0 - I^{-1}T\left(\sum_{j=0}^{i-1} h^j\right). \tag{32}$$

The assertion is that for all $i \in \mathbb{N}$ it holds

$$\left\|h^i\right\|_\infty \leq \left(\frac{2\varepsilon}{C_{I^{-1}}}\right)^i \left\|h^0\right\|_\infty. \tag{33}$$

For $i = 0$, this is obvious. If (33) holds for $i > 0$, then

$$\begin{aligned}\left\|h^{i+1}\right\|_\infty &= \left\|h^0 - I^{-1}T\left(\sum_{j=0}^{i-1} h^j + h^i\right)\right\|_\infty \\ &\leq \left\|h^0 - I^{-1}T\left(\sum_{j=0}^{i-1} h^j\right) - I^{-1}Ih^i\right\|_\infty + 2\frac{\varepsilon}{C_{I^{-1}}}\left\|h^i\right\|_\infty = 0 + 2\frac{\varepsilon}{C_{I^{-1}}}\left\|h^i\right\|_\infty\end{aligned}$$

proves that it holds for $i+1$, too. Hence it is true for all $i \in \mathbb{N}$. Remember (15) to see from this that $h^\Sigma = \sum_{i=0}^{+\infty} h^i$ converges in $\ell^\infty_{\mathbb{R}}$. Equation (32) together with the continuity of $I^{-1}T$ (see (31)) tell us that $I^{-1}Th^\Sigma = h^0$. Hence $(I^{-1}T)^{-1} = T^{-1}I$ exists on $\ell^\infty_{\mathbb{R}}$.

Now we return to the initial problem to determine the function v that fulfills (13–14) for the sequence $(a_k)_{k\in\mathbb{Z}}$ given. Set

$$g_\perp(t) = \begin{cases} \varphi\left(\frac{2t-t_k-t_{k+1}}{t_{k+1}-t_k}\right) & \ldots\ t \in (t_k, t_{k+1}) \text{ for some } k \in \mathbb{Z} \\ 0 & \ldots\ \text{else.}\end{cases}$$

This defines a positive bounded function $g_\perp \in C^\infty((-1,1))$ such that ${g_\perp}^{-1}(\{0\}) \cap (-1,1) = \{t_k : k \in \mathbb{Z}\}$. We use this function in order to assure (12). Obviously, the integrals

$$a_{\perp k} = \int_{-1}^{+1} \frac{g_\perp(t)}{t-t_k}dt \tag{34}$$

converge because of the same reasons that assured the convergence of (23), see (24).

With this sequence $(a_{\perp k})_{k\in\mathbb{Z}}$, we set $a'_k = a_k - a_{\perp k}$ and construct the operator $T^{-1}I$ accordingly. From (19) we see that in any case $I^{-1}a' \in \ell^\infty_{\mathbb{R}}$. Thus we can define the sequence

$$h = T^{-1}II^{-1}a' \in \ell^\infty_{\mathbb{R}}.$$

With this sequence, we set $g = g_\perp + g_h$ and

$$v = \sqrt{g}.$$

[8] Property (30) would do as well.

Note here that $g_\perp \geq 0$ and $g_h \geq 0$, and that both functions are bounded.

First, it is to show that $v \in C^\infty((-1,+1))$ holds. From $g_\perp \in C^\infty$ and $g_h \in C^\infty$ it is clear that v is infinitely differentiable at points $t \in (-1,+1)$ where $g(t) = g_\perp(t) + g_h(t) > 0$. That is why we only have to consider those $t \in (-1,+1)$ where both of these functions vanish. These are exactly the points t_k for $k \in \mathbb{Z}$.

Near the points t_k, g can be written as a nonnegative linear combination of the four functions $t \mapsto \varphi\left(\frac{2t-t_{k-1}-t_k}{t_k-t_{k-1}}\right)$, $t \mapsto \varphi\left(\frac{2t-t_k-t_{k+1}}{t_{k+1}-t_k}\right)$, $t \mapsto \psi\left(\frac{2(t-t_k)}{\varepsilon'_k} - 1\right)$, and $t \mapsto \psi\left(\frac{2(t_k-t)}{\varepsilon'_k} - 1\right)$. The squareroot of such a linear combination is a C^∞-function around t_k.

The argument (see (24)) that showed the convergence of the integrals (23) and (34) above, can be modified: Because of $g \in C^\infty(I)$ and $g(t_k) = g'(t_k) = 0$, we know $g(t) = O(|t-t_k|^2)$ for t near t_k. Moreover, g is bounded on I. This yields (13). As a consequence of (13), the integral in (14) converges, too.

It remains to show (14) at last. But this is evident from

$$\int_{-1}^{+1} \frac{|v(t)|^2}{t-t_k} dt \quad = \quad \int_{-1}^{+1} \frac{g_\perp(t) + g_h(t)}{t-t_k} dt = a_{\perp k} + (Th)_k = a_{\perp k} + (TT^{-1}II^{-1}a')_k = a_k.$$

♡

Remark 1.10 It is in no way necessary to choose the essential singularities of the function g exactly at the values t_k. A type of functions $\breve{w}_{\varepsilon',\varepsilon^+}$, similar to the functions $w_{\varepsilon',\varepsilon^+}$, but equal to t^4 in a small neighbourhood of 0 will do as well, see Figure 4.

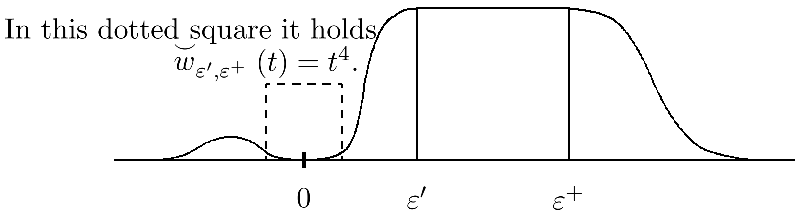

Figure 4: A suggested function $\breve{w}_{\varepsilon',\varepsilon^+}$, sketched.

From Proposition1.11 follows

Proposition 1.12 *There is a function $v \in C^\infty((-1,+1))$ such that*

$$\tilde{A}_1(a) = \begin{pmatrix} a & v^* \\ v & y \mapsto (t \mapsto t \cdot y(t)) \end{pmatrix}$$

has infinitely many eigenvalues in $(-1,+1)$ for each $a \in \mathbb{Q}$.

<u>Proof</u>: Enumerate $\mathbb{Q} \times \mathbb{N}$ by numbers $k \in \mathbb{Z}$. Then take the first component q_k of the elements (q_k, n_k) of this sequence. This yields an enumeration of $\mathbb{Q}$, where each number is attained infinitely often.

Then define $z_k = \tanh k$ and $a_k = q_k - z_k$. After that apply Proposition 1.11 with this sequence $(a_k)_{k\in\mathbb{Z}}$. So we get a function v such that (2) holds for all $z = z_k$.

Choose an arbitrary number $a \in \mathbb{Q}$. Then by Proposition 1.11, (7) with $z = z_k$ becomes $q_k - z_k = a - z_k$. This holds for infinitely many indices $k \in \mathbb{Z}$. Hence, $\tilde{A}_1(a)$ has infinitely many eigenvalues $z_k \in (-1, +1)$. ♡

Remark 1.11 From the idea of the construction in Proposition 1.12, it can be seen that Proposition 1.11 is optimal in the following sense: In any case only countably many values of f can be determined arbitrarily, since otherwise a selfadjoint operator $\tilde{A}_1(a)$ could be generated with an uncountable number of eigenvalues. This would contradict $\mathcal{L}^2$ to be separable because the eigenvectors of the selfadjoint operator $\tilde{A}_1(a)$ form an orthonormal set in $\mathcal{L}^2$.

A second obvious limitation to the occurence of eigenvalues in $\mathcal{R}_{\mu ess}$ is stated in

Proposition 1.13 *If $v^{-1}(\{0\})$ has empty inner domain, then the union $\bigcup_{a\in\mathbb{R}} \sigma_p \tilde{A}_1(a)$ lies nowhere dense in $u(I)$.*

Proof: If this were not the case, the continuous function v had to vanish on some open subinterval of I. ♡

1.4 Cauchy principal values

Throughout the present section we assume $\mu(t) = t$, and $u(t) = t$ and v to be differentiable.

Remark 1.12 To assume $u(t) = t$ is not as restrictive as it might seem. If u is a monotonous function, continuously differentiable with $u' > 0$ on I, then we can substitute variables. Just look at:

$$\int_I \frac{|v(t)|^2}{u(t) - z} dt \quad = \quad \left| \begin{array}{c} t'=u(t) \\ dt'=u'(t)dt \end{array} \right| = \int_{u(I)} \frac{\frac{|v(u^{-1}(t'))|^2}{u'(u^{-1}(t'))}}{(t' - z)} dt'.$$

Hence, we may use $I_{\text{new}} = u(I)$, $u_{\text{new}}(t) = t$, and $v_{\text{new}}(t) = \frac{v(u^{-1}(t))}{\sqrt{u'(u^{-1}(t))}}$. The meaning of the conditions (2) and (3) is preserved by this substitution, of course.

In Section 1.3 we have seen that eigenvalues of $\tilde{A}_1(a)$ in $\mathcal{R}_{\mu ess}$ appear and disappear suddenly if a is changed a bit. The question arises whether there is any kind of regularity to be seen at all. Well, in fact there is under the conditions of the present section. Since the characteristic function f defined by (6) is our main tool at hand to study eigenvalues of $\tilde{A}_1(a)$, it is not surprising that it might be of advantage to extend the domain of f as far as possible. A natural way to do so is to use Cauchy principal values:

$$f_{CP}(z) = (CP) \int_I \frac{|v|^2}{t - z} dt = \lim_{\eta \searrow 0} \int_{I\setminus(z-\eta, z+\eta)} \frac{|v|^2}{t - z} dt.$$

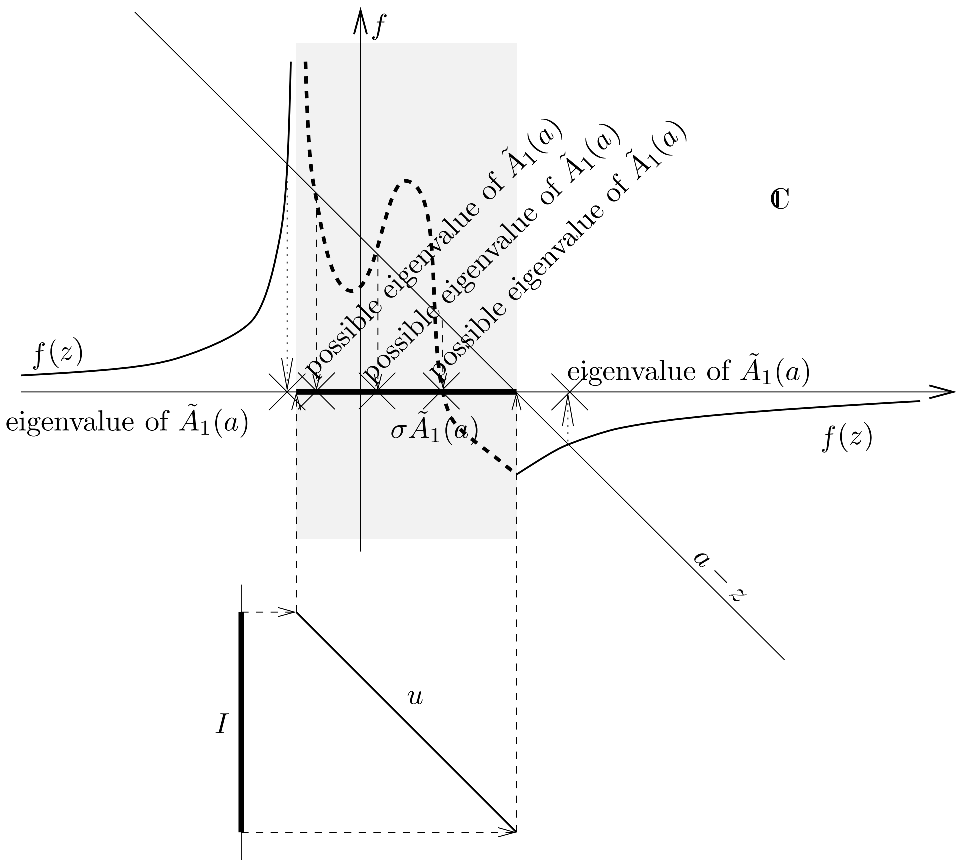

Figure 5: The extension of f to f_{CP} gives an idea where eigenvalues of $\tilde{A}_1(a)$ can possibly occur in dependence of a.

Since $f(z) = f_{CP}(z)$ if (2) holds, the criterion that both, (2) and (7) must hold can be substituted by (2) and $f_{CP}(z) = a - z$.

Remark 1.13 It is important to note that f_{CP} does not need to be monotonous at all. This can be seen from the example $I = [-1, +1]$ and $v = 1$. Then f_{CP}, defined by

$$f_{CP}(z) = (CP) \int_{-1}^{+1} \frac{dt}{t - z} = \ln \left| \frac{z - 1}{z + 1} \right|,$$

decreases on $(-1, +1)$.

By the way, the function f_{CP} sketched in Figure 5 is not monotonous.

That f_{CP} need not to be monotonous, makes it possible that there are several eigenvalues in $u(I)$ generated by the one-dimensional extension which $\tilde{A}_1(a)$ defines. Just look at Proposition 1.12 here.

Let us state where f is defined before f_{CP} is explored.

Proposition 1.14 *Let I be an interval, and $z \in I$. Then the following are equivalent:*

- $v(z) = 0$,
- *(2) holds,*
- *$f(z)$ is defined.*

Proof: Let $v(z) = 0$. Differentiability of v implies

$$|v(t)| = |v(t) - v(z)| \leq (1 + |v'(z)|)\, |t - z|$$

for $|t - z| \leq \varepsilon$. Consequently,

$$\int_I \frac{|v(t)|^2}{|t - z|^2} dt \leq 2\varepsilon \left(1 + (|v'(z)|)\right)^2 + \frac{1}{\varepsilon^2} \int_{I \cap [z-\varepsilon, z+\varepsilon]} |v(t)|^2 dt < +\infty$$

holds, because $v \in \mathcal{L}^2$. Hence, (2) is fulfilled.

That the existence of $f(z)$ is implied by (2) is clear. Just note $v \in \mathcal{L}^2$.

Let $f(z)$ be defined. From Remark 1.8 and $u(t) = t$ we know that $v(z) = 0$ must hold then. This step uses the assumption that I is an interval. ♡

Now we turn to f_{CP}. The following propositions tells us a kind of limitation to constructions such as they have been used in the proof of Proposition 1.11. As long as some minimal amount of smoothness of v is assumed, one cannot make f discontinuous. This follows from $f \subseteq f_{CP}$ and Propositions 1.15 and 1.16.

Proposition 1.15 *If $\mu(t) = u(t) = t$ and if v is differentiable, then the function f_{CP} extends f to the whole inner domain I^o of I.*

Proof: Choose a number $z \in I^o$. Then there is a number $\varepsilon > 0$ such that $[z-\varepsilon, z+\varepsilon] \subseteq I$. From

$$f_{CP}(z) = \int_{I\setminus[z-\varepsilon,z+\varepsilon]} \frac{|v|^2}{t-z}dt + \lim_{\eta\searrow 0}\int_{z+\eta}^{z+\varepsilon} \frac{|v(t)|^2 - |v(2z-t)|^2}{t-z}dt \tag{35}$$

can be seen the Cauchy principal value $f_{CP}(z)$ exists if and only if the limit term on the right side exists. But this is the case since the integrand $\frac{|v(t)|^2-|v(2z-t)|^2}{t-z}$ remains bounded because v and hence $|v|^2$ is differentiable. ♡

Proposition 1.16 *Let the assumptions of Proposition 1.15 be fulfilled. Then the following hold:*

If v is $k \geq 1$ times differentiable, then f_{CP} is $k-1$ times differentiable on I^o.

If v is $k \geq 1$ times continuously differentiable, then f_{CP} is $k-1$ times continuously differentiable on I^o.

Proof: Throughout this proof we shall abbreviate $g = |v|^2$. Of course, (continuous) differentiablility of v is inherited to g.

Let us assume that I is a compact interval, first: $I = [t_-, t_+]$.

Let g be continuously differentiable. We show that $z \mapsto \int_{t_-}^{t_+} \frac{g}{t-z}dt$ is continuous then. Choose $\varepsilon > 0$ and $z, z' \in I^o$. Let us assume $z' > z$. The case $z' < z$ is similar. Temporarily set $d = z' - z$. So we get

$$\begin{aligned}
|f_{CP}(z') - f_{CP}(z)| &= \left|(CP)\int_{t_-}^{t_+} \frac{g(t)}{t-z'}dt - (CP)\int_{t_-}^{t_+} \frac{g(t)}{t-z}dt\right| \\
&= \left|(CP)\int_{t_--d}^{t_+-d} \frac{g(t+d)}{t-z}dt - (CP)\int_{t_-}^{t_+} \frac{g(t)}{t-z}dt\right| \\
&\leq \underbrace{\left|\int_{t_--d}^{t_-} \frac{g(t+d)}{t-z}dt - \int_{t_+-d}^{t_+} \frac{g(t)}{t-z}dt\right|}_{E_1} \\
&\quad + \lim_{\eta\searrow 0}\left|\left(\int_{t_-}^{z-\eta} + \int_{z+\eta}^{t_+-d}\right) \frac{g(t+d)-g(t)}{t-z}dt\right|.
\end{aligned}$$

The term E_1 tends to 0 for $d \to 0$. Hence we suppose $|z'-z|$ so small that E_1 is less than $\frac{\varepsilon}{3}$. It remains to estimate the limit term. Integration by parts yields

$$\begin{aligned}
\left(\int_{t_-}^{z-\eta} + \int_{z+\eta}^{t_+-d}\right) \frac{g(t+d)-g(t)}{t-z}dt &= (g(t+d)-g(t))\ln|t-z|\left(\Big|_{t_-}^{z-\eta} + \Big|_{z+\eta}^{t_+-d}\right) \\
&\quad - \underbrace{\left(\int_{t_-}^{z-\eta} + \int_{z+\eta}^{t_+-d}\right)(g'(t+d)-g'(t))\ln|t-z|dt}_{E_3}
\end{aligned}$$

For $\eta \searrow 0$ the first term tends to

$$(g(t+d)-g(t))\ln|t-z|\Big|_{t_-}^{t_+-d}, \tag{36}$$

because

$$|g(z-\eta)-g(z+\eta)|\ln\eta \leq 2\left(1+|g'(z)|\right)\eta\ln\eta \to 0 \tag{37}$$

for $\eta \searrow 0$. Continuity of g yields $(g(t+d)-g(t))\ln|t-z|\Big|_{t_-}^{t_+-d} \to 0$ for $d \to 0$. So we admit only z' so near z that $\left|(g(t+d)-g(t))\ln|t-z|\Big|_{t_-}^{t_+-d}\right| < \frac{\varepsilon}{3}$.

It remains to estimate E_3. Since g' is continuous, it is uniformly continuous on $[t_-,t_+]$. Hence $|g'(t+d)-g'(t)|$ can be bounded uniformly on this interval for small d. If we assume $|d|=|z'-z|$ so small that

$$\|g'(t+d)-g'(t)\|_{\mathcal{L}^\infty([t_-,t_+-d])} < \frac{\varepsilon}{3}\int_{t_-}^{t_+-d} |\ln|t-z||\,dt,$$

we achieve $E_3 < \frac{\varepsilon}{3}$.

So we come to know $|f_{CP}(z')-f_{CP}(z)| < \varepsilon$ for z' sufficiently near z, and if I is compact.

Next, we show the assertion about higher derivatives of f_{CP} and their continuity. As the reader might expect, we use partial integration and an induction argument. Let g be two times differentiable. We continue to assume that I is a compact interval: $I=[t_-,t_+]$. Then

$$\begin{aligned}
\frac{\partial}{\partial z}(CP)\int_I \frac{g(t)}{t-z}dt &= \frac{\partial}{\partial z}\lim_{\eta\searrow 0}\left(\int_{t_-}^{z-\eta}+\int_{z+\eta}^{t_+}\right)\frac{g(t)}{t-z}dt \\
&= \frac{\partial}{\partial z}\lim_{\eta\searrow 0} g(t)\ln|t-z|\left(\Big|_{t_-}^{z-\eta}+\Big|_{z+\eta}^{t_+}\right) \\
&\quad -\frac{\partial}{\partial z}\lim_{\eta\searrow 0}\left(\int_{t_-}^{z-\eta}+\int_{z+\eta}^{t_+}\right) g'(t)\ln|t-z|dt \\
&= \frac{g(t)}{t-z}\Big|_{t_-}^{t_+} - \frac{\partial}{\partial z}\lim_{\eta\searrow 0}\left(\int_{t_-}^{z-\eta}+\int_{z+\eta}^{t_+}\right) g'(t)\ln|t-z|dt, \qquad (38)\\
&= \frac{g(t)}{t-z}\Big|_{t_-}^{t_+} - \lim_{\eta\searrow 0}\frac{\partial}{\partial z}\left(\int_{t_-}^{z-\eta}+\int_{z+\eta}^{t_+}\right) g'(t)\ln|t-z|dt \qquad (39)\\
&= \frac{g(t)}{t-z}\Big|_{t_-}^{t_+} - \lim_{\eta\searrow 0}\left(\int_{t_-}^{z-\eta}+\int_{z+\eta}^{t_+}\right) \frac{g'(t)}{t-z}dt \qquad (40)\\
&= \frac{g(t_+)}{t_+-z}-\frac{g(t_-)}{t_--z}-(CP)\int_{t_-}^{t_+}\frac{g'(t)}{t-z}dt. \qquad (41)
\end{aligned}$$

In (38), the argument (37) formerly used to obtain (36) has been applied once again.

To justify the exchange of differentiation and integration in (40), note that for $t\in[t_-,z-\eta]$ and $t\in[z+\eta,t_+]$ the expressions $g'(t)\ln|t-z|$ and $\frac{g'(t)}{t-z}$ determine bounded functions uniformly continuous in z. To justify the exchange of differentiation and the limit $\eta\searrow 0$ in (39), note that both, the η-parametrized family of functions $z\mapsto\left(\int_{t_-}^{z-\eta}+\int_{z+\eta}^{t_+}\right)g'(t)\ln|t-z|dt$ and that of their derivatives $z\mapsto\left(\int_{t_-}^{z-\eta}+\int_{z+\eta}^{t_+}\right)\frac{g'(t)}{t-z}dt$ converge uniformly with respect to z out of a suitable compact set. To see this, confer to (37), and the idea of (35) in the proof of Proposition 1.15, respectively. The same arguments can be used here to show that the above integrals in question can be represented by integrals over bounded functions.

Now, induction on k proves the assertion of the current proposition for the case of compact intervals I: If g is differentiable, Proposition 1.15 shows existence of f_{CP} on I^o. If g is even continuously differentiable, the first part of the current proof shows that f_{CP} is continuous on I^o.

Now we make the induction step: If we already know that $z \mapsto (CH) \int_I \frac{g}{t-z} dt$ is $k-1$ times (continuously) differentiable as soon as g is k times (continuously) differantiable, then (41) implies that f_{CP} is k times (continuously) differentiable if g' is k times (continuously) differentiable, or equivalently, if g is $k+1$ times (continuously) differentiable.

If I is not a compact interval, decompose $I = [t_-, t_+] \cup (I \setminus [t_-, t_+])$ with $t_- < z < t_+$. The integral $\int_{I \setminus [t_-, t_+]} \frac{g(t)}{t-z} dt$ exists and defines a function holomorphic in z. Hence, f_{CP} is as smooth as $z \mapsto (CP) \int_{t_-}^{t_+} \frac{g(t)}{t-z} dt$ is. Since the assertion of the current proposition has already been proved for compact intervals such as $[t_-, t_+]$, the assertion is true for general I. ♡

The following proposition tells us an important consequence of the preceding one.

Proposition 1.17 *Assume $v \in C^1(I)$. Let z be an accumulation point of $v^{-1}(\{0\})$.*

If z itself is an eigenvalue of $\tilde{A}_1(a)$ in the inner domain of I, then for every $\varepsilon > 0$, there are values a' and z', with $|a' - a| < \varepsilon$ and $|z' - z| < \varepsilon$, such that z' is eigenvalue of $\tilde{A}_1(a')$.

Proof: From Proposition 1.16 We know that f_{CP} is continuous in $z \in I^o$. Hence, for $z' \in \mathcal{D}f$, $a' = f(z')$ is near $a = f(z)$ if z' is near z. As z is assumed to be an accumulation point of $\mathcal{D}f$, we can choose z' such that both z' and a' are near z and a, respectively. ♡

Remark 1.14 The kind of stability which Proposition 1.17 tells about concerns accumulation-points of $v^{-1}(\{0\})$ only. Remember that Proposition 1.11 allowed to generate eigenvalues of $\tilde{A}_1(a)$ for arbitrary a in nothing but isolated points of $v^{-1}(\{0\})$. However, these points actually accumulate towards the end points of $I = (-1, +1)$. We have learned now that only this is possible because these end-points are not contained in I themselves. Otherwise, the smoothness $v \in C^1(I)$ would have to be sacrificed.

Summarizing we see, that even those eigenvalues of $\tilde{A}_1(a)$ which lie in $u(I)$ do obey a kind of stability. But as a-dependent eigenvalues can emerge only at points z, where (2) holds, this cannot be seen when nothing but $\sigma_p \tilde{A}_1(a)$ is considered.

1.5 Some notes on the n-dimensional extension of a multiplication operator

Many observations made for the one-dimensional extension of the multiplication operator $y \mapsto u \cdot y)$ can be proved for the n-dimensional extension defined by the matrix-operator

$$\tilde{A}_n = \begin{pmatrix} a_{11} & \dots & a_{1n} & v_1* \\ \vdots & \cdot & \vdots & \vdots \\ \overline{a}_{1n} & \dots & a_{nn} & v_n* \\ v_1 & \dots & v_n & u \end{pmatrix} = \left(\frac{A|V^*}{V|\ u} \right)$$

on $\mathbb{C}^n \oplus \mathcal{L}^2_\mu$.

Alas, theory becomes a bit more complicated than the thoughts for the one-dimensional extension in many cases. That is why you will find only some notes here. However, detailed analysis will be done for the perturbation problem in section 2. It turns out that extension and perturbation problems can be handled in a very similar manner. The methods to be presented in section 2 will be applicable to the present n-dimensional extension problem, as well: One just has to consider L of (42) instead of M defined by (60).

First, let us look at the conditions for z to be an eigenvalue of $\tilde{A}_n$. Let us assume $z \notin \mathcal{R}_{\mu ess} u$. Here, for instance [1, Behauptung 1] tells us that z is an eigenvalue if and only if

$$L(z) = z - A + V^*(u - z)^{-1} V \tag{42}$$

has a nontrivial kernel. This is the same as the condition $\det L(z) = 0$. This is the n-dimensional analogue of (3). Note that (2) is always valid for $z \notin \mathcal{R}_{\mu ess} u$.

Similar to Remark 1.3, for $\mu(t) = t$, numbers z with $\mu(u^{-1}(\{z\})) > 0$ are eigenvalues of infinite multiplicity of $\tilde{A}_n$. Like in Remark 1.3, this is, because $\mathcal{L}^2_\mu(u^{-1}(\{z\}))$ and hence $\mathcal{L}^2_\mu(u^{-1}(\{z\})) \ominus V\,\mathbb{C}^n$ is either zero- or infinite-dimensional.

2 Perturbed systems $A + T$

This section is devoted to the study of the change of spectra if a selfadjoint operator, or, more generally, linear relation A on some Hilbert-space $\mathcal{H}$ is changed into an operator $A+T$ by some compact selfadjoint operator T. The dimension of T will be called the rank of the perturbation.

Here, we shall restrict ourselves to the eigenvalues of $A+T$ which lie in ρA. One could say, the eigenvalues which emerged in ρA by the perturbation. The short treatise in Section 1 may give some clue for what can happen when σA becomes involved. But it is an excellent motivation for what we have to expect inside ρA.

Let us just take another look at the operator $\tilde{A}_1 = \begin{pmatrix} a & v^* \\ v & u \end{pmatrix}$ as it has been defined by (1). Let $I = [-1, 1]$ and u be a monotonously increasing function $I \to \mathbb{R}$, continuous on I with exception of 0, where it jumps from $u(0-)$ upwards to $u(0) = u(0+)$. Then $\mathcal{R}u$ consist of two disjoint intervals $[u(-1), u(0-))$ and $[u(0), u(1)]$. The matrix $\tilde{A}_1$ then defines an operator on $\mathbb{C} \oplus L^2$. Having set μ equal the Lebesgue measure, and $v = 1$, the function f defined by (6) can be written as

$$f(z) = \int_{-1}^{1} \frac{1}{u - z} dt.$$

This function is holomorphic on the whole of $\mathbb{C} \setminus \overline{\mathcal{R}u} = \mathbb{C} \setminus [u(-1), u(0-)] \cup [u(0+), u(1)]$ and increases strictly monotonously on each of the intervals $(-\infty, u(-1))$, $(u(0-), u(0+))$, and $(u(1), +\infty)$. Hence, condition (7), that is $f(z) = a - z$, will be fulfilled at no more than one z in each of these intervals. Since this condition characterizes eigenvalues outside $\overline{\mathcal{R}u}$, we get no more than one eigenvalue in each of these intervals.

However, the one-dimensional example above is an operator extension rather than a perturbation. Nevertheless, that is all right not only as a vague motivation: Later, we shall see that the very same kind of function f can be used in the context of the one-dimensional perturbation problem, too. Only the characteristic equation (7) will change slightly. It will become $f(z) = -a^{-1}$, see Example 2.1.

By the way, anticipating the characteristic equation to be of this form, one easily sees that one of the intervals $(-\infty, u(-1))$, and $(u(1), +\infty)$ will contain no eigenvalue of the operator $A + T$ at all.

In the following sections we shall see that even the cases of more general perturbations behave quite alike. Alas, we cannot use such simple scalar functions for higher rank, especially for infinite rank, of T. That is why some bulk of additional theory becomes necessary to be developed. Nevertheless, we shall see that monotony of certain operator functions will take a significant role comparable to that of f in the one-dimensional case.

First, we shall do some algebra, but then specialize to Hilbert-spaces and selfadjoint operators when things are expexted to become more interesting.

2.1 Operator- and relation-matrices and -pencils

Let A, B, C, and D be operators $A : \mathcal{H}_1 \to \mathcal{H}_1$, $B : \mathcal{H}_2 \to \mathcal{H}_1$, $C : \mathcal{H}_1 \to \mathcal{H}_2$, and $D : \mathcal{H}_2 \to \mathcal{H}_2$, or, more generally, linear relations $A \subseteq \mathcal{H}_1 \times \mathcal{H}_1$, $B \subseteq \mathcal{H}_2 \times \mathcal{H}_1$, $C \subseteq \mathcal{H}_1 \times \mathcal{H}_2$, and $D \subseteq \mathcal{H}_2 \times \mathcal{H}_2$.

We take a closer look at the operator-matrix (or relation-matrix in the general case of linear relations)

$$\tilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{43}$$

now. In the case of A, B, C, and D being operators, the operator-matrix $\tilde{A}$ defines the linear mapping $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \tilde{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}$ on the sum space $\mathcal{H}_1 \oplus \mathcal{H}_2$ of two linear spaces. If A, B, C, and D are general linear relations, $\tilde{A}$ defines a linear relation on $\mathcal{H}_1 \oplus \mathcal{H}_2$ by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \in \tilde{A}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow x' \in Ax + By \ \wedge \ y' \in Cx + Dy.$$

The operator (or relation) $\tilde{A}$ is defined on $\mathcal{D}\tilde{A} = (\mathcal{D}A \cap \mathcal{D}C) \oplus (\mathcal{D}B \cap \mathcal{D}D) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$.

Note the convention to write $x' \in Ax$ for $(x, x') \in A$ if A is a relation. That is, we interpret Ax as the set $\{x' : (x, x') \in A\}$. It must be said that this is a somewhat casual way of writing. The correct syntax is $(x, x') \in A$, or as well $x'Ax$. Nevertheless, $x' \in Ax$ should not lead to heavy confusion.

With $\tilde{A}$ we associate the linear operator or relation[9]

$$W = -A + BD^{-1}C. \tag{44}$$

Definition 2.1 If A is a linear relation on $\mathcal{H}$, then $\ker A = \{x \in \mathcal{H} : \ 0 \in Ax\}$.

With these preliminaries, we get the following characterisation of the invertibility of $\tilde{A}$:

Proposition 2.1 *Let $A \subseteq \mathcal{H}_1 \times \mathcal{H}_1$, $B \subseteq \mathcal{H}_2 \times \mathcal{H}_1$, $C \subseteq \mathcal{H}_1 \times \mathcal{H}_2$, and $D \subseteq \mathcal{H}_2 \times \mathcal{H}_2$ be linear relations.*

Then for the linear relation $\tilde{A} \subseteq (\mathcal{H}_1 \oplus \mathcal{H}_2) \times (\mathcal{H}_1 \oplus \mathcal{H}_2)$ defined by (43) and the linear relation $W \subseteq \mathcal{H}_1 \times \mathcal{H}_1$ defined by (44) it holds

$$\ker \tilde{A} = \begin{pmatrix} 1 \\ X \end{pmatrix} \ker W + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(D^{-1}C0 \cap B^{-1}A0 \right), \tag{45}$$

where X is a linear mapping with $\mathcal{D}X = \ker W$.

Proof: We have

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \ker \tilde{A} \Leftrightarrow 0 \in Ax + By \wedge 0 \in Cx + Dy. \tag{46}$$

Further, the equivalence

$$0 \in Cx + Dy \Leftrightarrow -Cx \cap Dy \neq \emptyset \Leftrightarrow y \in D^{-1}(-C)x$$

[9] If $\ker D \neq \{0\}$, W may be a relation even if A, B, C, and D are operators.

holds, and analogously $0 \in Ax + By \Leftrightarrow y \in B^{-1}(-A)x$. Together with (46) this yields

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \ker \tilde{A} \Leftrightarrow y \in B^{-1}(-A)x \cap D^{-1}(-C)x. \tag{47}$$

Now fix a space $\mathcal{H}'$ complementary to $B^{-1}A0 \cap D^{-1}C0$ in $\mathcal{H}_2$. As all relations are linear, it holds

$$B^{-1}(-A)x \cap D^{-1}(-C)x = \left(\mathcal{H}' \cap -B^{-1}Ax \cap -D^{-1}Cx\right) \dot{+} (B^{-1}A0 \cap D^{-1}C0).$$

Because of the choice of $\mathcal{H}'$, the set $\mathcal{H}' \cap -B^{-1}Ax \cap -D^{-1}Cx$ contains maximally one element. We call this element, if it exists, Xx. This uniquely defines the desired linear mapping X, which need not to be defined everywhere.

It remains to show that $\mathcal{D}X = \ker W$. This follows from

$$\begin{aligned} x \in \mathcal{D}X \quad &\Leftrightarrow \quad \exists y' \in \mathcal{H}' \cap -B^{-1}Ax \cap -D^{-1}Cx \Leftrightarrow \exists y \in -B^{-1}Ax \cap -D^{-1}Cx \\ &\Leftrightarrow \quad \exists y \in -D^{-1}Cx : \; 0 \in Ax + By \Leftrightarrow 0 \in Wx, \end{aligned}$$

where (46) has been applied once again but in the opposite direction than before. ♡

An immediate consequence of Proposition 2.1 is

Proposition 2.2 *Let $\tilde{A}$ be like in Proposition 2.1, and let $D^{-1}C$ be an operator. Then* $\ker \tilde{A}$ *is isomorphic to* $\ker W$*.*

Proof: Under the additional conditions made, $D^{-1}C0 = \{0\}$ holds. Hence, Proposition 2.1 yields $\ker \tilde{A} = \begin{pmatrix} 1 \\ X \end{pmatrix} \ker W$. Since $\begin{pmatrix} 1 \\ X \end{pmatrix}$ is invertible, it is the desired isomorphism. ♡

Remark 2.1 Note that we only have done some algebra so far. Thus, Propositions 2.1–2.2 could be translated to the more general context of group products.

Secondly, it is worthwile to write the term on the right side of (45) with the relation

$$\underline{X} = \left\{(x, y) : x \in \mathcal{H}_1 \wedge y \in Xx + (D^{-1}C0 \cap B^{-1}A0)\right\}$$

defined on $\ker \tilde{W}$:

$$\ker \tilde{A} = \begin{pmatrix} 1 \\ \underline{X} \end{pmatrix} \ker W.$$

A short moment of thought reveals that this relation $\underline{X}$ is nothing but

$$\underline{X} = \ker \tilde{A} \tag{48}$$

itself.[10] Seen from this viewpoint, Proposition 2.2 gives a condition under which $\ker \tilde{A} = \underline{X}$ is an operator.

[10]The remarkable fact is not that $\ker \tilde{A}$ can be parametrized by itself, but that the domain of the parametrisation equals $\ker W$! Note here that every linear subspace $\mathcal{S} \subseteq \mathcal{H}_1 \times \mathcal{H}_2$ can be parametrized as $\mathcal{S} = \begin{pmatrix} 1 \\ \mathcal{S} \end{pmatrix} \mathcal{D}\mathcal{S}$ by itself.

Let us finish these thoughts with a last step of generalisation: If we define $\underline{X}$ by (48) or, via (47) equivalently, by

$$\underline{X} = \left\{ (x,y) : \, y \in B^{-1}(-A)x \cap D^{-1}(-C)x \right\},$$

we do not use the linearity of A, B, C, and D at all. That is the reason why Proposition 2.1 appears to be a particular form of a very general statement about arbitrary relation-matrices on group products.

But we stay interested in spectral problems.

Definition 2.2 A number $z \in \mathbb{C}$ is said to be an eigenvalue of an operator-pencil H if and only if $\ker H(z) \neq \{0\}$. We then write $z \in \sigma_p H$. The multiplicity of $z \in \mathbb{C}$ as eigenvalue of H is $\dim \ker H(z)$.

From Proposition 2.2 we immediately get

Proposition 2.3 *Let C be an operator. Then a complex number $z \notin \sigma_p D$ is eigenvalue of $\tilde{A}$, defined by (43), if and only if it is eigenvalue of the pencil L given by $L(z) = z - A + B(D-z)^{-1}C$. The multiplicities of z as eigenvalue of $\tilde{A}$ and as eigenvalue of L are the same.*

Remark 2.2 This result can be compared to [1, Behauptung 1] and to a part of [3, Theorem 2.1]. Nevertheless, in [1] only bounded operators in Banach-spaces are dealt with, and [3] only considers operators on finite-dimensional linear spaces.

2.2 An algebraic criterion for eigenvalues of $A + T$

Let A be an operator (or linear relation), not necessarily defined everywhere, and let T be an operator, both A and T acting on some linear space $\mathcal{H}$.

First, we seek to represent the operator T by an invertible operator S. Therefore, we choose a space $\mathcal{H}'$, and operators $P : \mathcal{H} \to \mathcal{H}'$ and $Q : \mathcal{H}' \to \mathcal{H}$ such that

$$T \;=\; QSP \tag{49}$$

holds. This is possible

Proposition 2.4 *There is a linear space $\mathcal{H}'$, and there are linear mappings $P : \mathcal{H} \to \mathcal{H}'$, $Q : \mathcal{H}' \to \mathcal{H}$, and $S : \mathcal{H}' \to \mathcal{H}'$ such that S is invertible, and (49) holds.*

Proof: Just set $\mathcal{H}' = \mathcal{H}$, $S = Q = 1$, and $P = T$.[11] ♡

This is cheap. Therefore it is clear that (49) alone does not set up an all too tight relationship between T and S. As an additional assumption

$$1_{\mathcal{H}'} \;=\; PQ \tag{50}$$

may be imposed on a decomposition of T.

[11] To set $S = P = 1$ and $Q = T$ is as well possible.

Proposition 2.5 *If a decomposition* $(\mathcal{H}', Q, P, S)$ *of* T *satisfies (49) and (50),* $\ker P = \ker T$, $\ker Q = \{0\}$*, and the nonzero eigenvalues of* T *are exactly the eigenvalues of* S*.*

Proof: The mapping Q is injective because of (50). Since S is invertible, (49) yields $\ker P = \ker T$ from this.

Secondly, note $x = Qx' \in \mathcal{R}Q \Rightarrow QPx = QPQx' = Qx' = x$ in consequence of (50). Now consider a number $z \neq 0$.

$$Tx = zx \Leftrightarrow QSPx = zx \Leftrightarrow QSPx = zx \wedge zx \in \mathcal{R}Q \Leftrightarrow QSPx = QPzx = QzPx \Leftrightarrow SPx = zPx.$$

If $z \neq 0$, we know $x \notin \ker T = \ker P$. Hence nonzero eigenvalues of T are eigenvalues of S. Conversely, eigenvalues of S, which are nonzero since S is invertible, correspond to eigenvectors $x' = Px \neq 0$ of S. The above line of equivalences states that they are eigenvalues of T, too. ♡

It is possible to decompose T according to (49–50) if T has no principal vectors of the eigenvalue 0:

Proposition 2.6 *There is a linear space* $\mathcal{H}'$*, and there are linear mappings* $P : \mathcal{H} \to \mathcal{H}'$, $Q : \mathcal{H}' \to \mathcal{H}$*, and* $S : \mathcal{H}' \to \mathcal{H}'$ *such that* S *is invertible, and (49–50) hold if and only if* $\ker T = \ker T^2$.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "T"] \arrow[d, "P"'] & \mathcal{H} \\ \mathcal{H}' \arrow[r, "S"] & \mathcal{H}' \arrow[u, "Q"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[d, "P"'] & \mathcal{H} \arrow[l, "1_{\mathcal{H}}"'] \\ \mathcal{H}' & \mathcal{H}' \arrow[l, "1_{\mathcal{H}'}"'] \arrow[u, "Q"'] \end{tikzcd}$$

(49) (50)

Figure 6: The commutative diagrams that define $\mathcal{H}'$, P, Q, and S.

Proof: Let a decomposition $(\mathcal{H}', Q, P, S)$ of T with (49–50) be given. Then

$$\ker T^2 = \ker QSPQSP = \ker QS^2P = \ker P = \ker T$$

holds because of (50), and because S and Q are injective.

Conversely, assume $\ker T = \ker T^2$. Let $\mathcal{H}'$ be the factor space $\mathcal{H}' = \mathcal{H}/\ker T$. We define the canonical projection $P : \mathcal{H} \to \mathcal{H}' : x \mapsto x + \ker T$, and some arbitrary linear choice function $Q_0 : \mathcal{H}' \to \mathcal{H} : x' + \ker T \mapsto x \in x' + \ker T$. Such an operator Q_0 can always be obtained using Zorn's Lemma. Now set $S = PTQ_0$. Obviously, S does not depend on the choice of Q_0. Note that S is injective because $\ker T^2 = \ker T = \ker P$ and thus $\mathcal{R}T \cap \ker P = \{0\}$. Hence, we can define $Q_1 = TQ_0S^{-1}$. For this Q_1 it holds $PQ_1 = 1_{|\mathcal{D}S^{-1}}$, which means, that Q_1 is a linear choice function like Q_0. At last, we continue Q_1, again by Zorn's Lemma, onto the whole of $\mathcal{H}'$ to the linear choice function $Q : \mathcal{H}' \to \mathcal{H}$.

Now (50) holds, since Q is a choice function. And (49) follows from the construction of S and $Q_1 \subseteq Q$. Note here that already $T = Q_1SP$ holds. The operator extension $Q \supseteq Q_1$ has been introduced only to achieve (50). ♡

Remark 2.3 In the proof of Proposition 2.6, T need not be defined on the whole of $\mathcal{H}$. Hence this would not be the problem in Section 2.9 when operators T with compact resolvent will be studied in a Hilbert-space setting.

From the decomposition of the operator T, we get a most useful tool for the study of $A+T$:

Proposition 2.7 *Let $(\mathcal{H}', Q, P, S)$ be a decomposition for which (49) holds of the operator T.*

Then a complex number $z \notin \sigma_p A$ is eigenvalue of $A+T$ if and only if it is eigenvalue of the pencil M defined by

$$M(z) = S^{-1} + P(A-z)^{-1}Q. \tag{51}$$

Proof: Under the condition $z \notin \sigma_p A$, Proposition 2.2 yields

$$\begin{aligned} z \in \sigma_p(A+T) \quad &\Leftrightarrow \quad z \in \sigma_p(z \mapsto (A-z) + QSP) \Leftrightarrow z \in \sigma_p\left(z \mapsto \begin{pmatrix} z-A & Q \\ P & S^{-1} \end{pmatrix}\right) \\ &\Leftrightarrow \quad z \in \sigma_p(z \mapsto -S^{-1} - P(A-z)^{-1}Q) \Leftrightarrow z \in \sigma_p M, \end{aligned}$$

where in the third equivalence, we have used a symmetric version of Proposition 2.2 inverting the upper left operator $z - A$ instead of the lower right D. ♡

Remark 2.4 To apply Proposition 2.2 in the preceding proof, the facts that P, Q, S, and $(A-z)^{-1}$ are operators have been exploited.

2.3 The Hilbert-space case

In this section, we assume $\mathcal{H}$ to be a Hilbert-space. As in this context we have some topology at hand, we can now define

Definition 2.3 Let H be some operator-pencil defined on $\mathcal{D}H \subseteq \mathbb{C}$, where $H(z)$ need not be defined everywhere on $\mathcal{H}$. Then we define the resolvent set ρH to be the set of all $z_0 \in \mathcal{D}H$ such that $H(z_0)^{-1}$ is a bounded operator on $\mathcal{H}$. Note that for an operator (or a relation) A on $\mathcal{H}$ we get $\rho A = \rho(z \mapsto A - z)$.

The spectrum of H is defined to be $\sigma H = \mathbb{C} \setminus \rho H$.

Now we can formulate

Proposition 2.8 *Let the operator (or linear reltation) $\tilde{A}$, defined by (43), act on the sum space $\mathcal{H}_1 \oplus \mathcal{H}_2$ of two Hilbert-spaces. If D^{-1}, BD^{-1}, and $D^{-1}C$ are bounded operators*[12]*, then $\tilde{A}$ has a bounded inverse operator $\tilde{A}^{-1}$ if and only if the operator W defined by (44) has a bounded inverse operator W^{-1}.*

[12] If we speak of bounded operators, we mean bounded operators which are everywhere defined.

Proof: From Proposition 2.2, and under the assumption on $D^{-1}C$, we know that W^{-1} exists as an operator if and only if $\tilde{A}^{-1}$ exists, algebraically.

Thus, it remains to study the case when both, W^{-1} and $\tilde{A}^{-1}$, are operators: We have to show that W^{-1} is boundedly defined on $\mathcal{H}_1$ if and only if $\tilde{A}^{-1}$ is on $\mathcal{H}_1 \oplus \mathcal{H}_2$. We set

$$\tilde{A}_{-1} = \begin{pmatrix} -W^{-1} & W^{-1}BD^{-1} \\ D^{-1}CW^{-1} & D^{-1} - D^{-1}CW^{-1}BD^{-1} \end{pmatrix}, \tag{52}$$

and show that this is the inverse of $\tilde{A}$. Note that $\tilde{A}_{-1}$ is an operator in fact, under the assumptions which have been imposed on D^{-1}, BD^{-1}, and $D^{-1}C$. We have

$$\begin{aligned} \tilde{A}_{-1}\tilde{A} &= \begin{pmatrix} W^{-1}W & (-W^{-1}B + W^{-1}B)_{|\mathcal{D}D} \\ D^{-1}C(-W^{-1}W + 1) & 0 + D^{-1}D \end{pmatrix} = \begin{pmatrix} 1_{|\mathcal{D}A} & 0_{|\mathcal{D}D} \\ 0_{|\mathcal{D}A} & 1_{|\mathcal{D}D} \end{pmatrix} \\ &\supseteq 1_{|\mathcal{D}\tilde{A}}, \end{aligned} \tag{53}$$

and

$$\begin{aligned} \tilde{A}\tilde{A}_{-1} &= \begin{pmatrix} WW^{-1} & (-WW^{-1} + 1)BD^{-1} \\ (DD^{-1} - 1)CW^{-1} & (1 - DD^{-1})CW^{-1}BD^{-1} + DD^{-1} \end{pmatrix} \\ &\supseteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{|\mathcal{H}_1 \oplus \mathcal{H}_2}. \end{aligned} \tag{54}$$

In (53), we have used the boundedness of BD^{-1} and $D^{-1}C$. From (53) it follows $(x, y) \in \tilde{A} \Rightarrow (y, x) \in \tilde{A}_{-1}$, and (54) implies $(y, x) \in \tilde{A}_{-1} \Rightarrow (x, y) \in \tilde{A}$. Hence, $\tilde{A}^{-1} = \tilde{A}_{-1}$.

So obviously, boundedness of W^{-1} on $\mathcal{H}_1$ implies boundedness of $\tilde{A}^{-1}$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$, because under the assumptions over BD^{-1} and $D^{-1}C$, domain and boundedness of all entries in (52) only depend on the domain and the boundedness of W^{-1}. It is evident from (52) that boundedness of $\tilde{A}^{-1}$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$ implies boundedness of W^{-1} on $\mathcal{H}_1$. ♡

Remark 2.5 In [1], a method is given to construct resolvents of bounded operators written as block-matrices. This method can as well be extended to relations. Hence, the form of the operator $\tilde{A}_{-1}$ is no bold and lucky guess as it might seem.

The following statement about $\sigma(A + T)$ in the Hilbert-space[13] setting can be compared to the statement about $\sigma_p(A + T)$ of Proposition 2.7 in the purely algebraic context there.

Proposition 2.9 *Let $(\mathcal{H}', Q, P, S)$ be a decomposition of T for which (49) holds of the operator T, and let M be the pencil defined by (51).*

Then it holds

$$\sigma(A + T) \setminus \sigma A = \sigma M \setminus \sigma A$$

and

$$\sigma_p(A + T) \setminus \sigma_p A = \sigma_p M \setminus \sigma_p A.$$

[13] Banach-spaces will suffice.

Proof: The second statement is just the assertion of Proposition 2.7.

The first one can be derived by following the proof of Proposition 2.7 almost literally, only replacing σ_p by σ, and Proposition 2.2 by Proposition 2.8. ♡

Assuming $\mathcal{H}$ to be a Hilbert-space not only makes it possible to define the notion of the spectrum. If we assume T to be a normal operator, we can simplify the algebraic approach to decompose it according to (49–50), which has been proposed in Section 2.2. Zorn's Lemma, which has been used to prove Proposition 2.6, becomes superfluous in this case. Moreover, we can choose P and Q such that

$$P \;=\; Q^* \tag{55}$$

holds in addition to (49–50):

Proposition 2.10 *If T is a normal operator on $\mathcal{H}$, then there is a linear space $\mathcal{H}'$, and there are linear mappings $P : \mathcal{H} \to \mathcal{H}'$, $Q : \mathcal{H}' \to \mathcal{H}$, and $S : \mathcal{H}' \to \mathcal{H}'$ such that S is invertible, and (49–50), and (55) hold.*

Proof: We set[14]

$$\begin{aligned} \mathcal{H}' &= (\ker T)^\perp = \overline{\mathcal{R}T}, &(56)\\ Q &= 1_{\mathcal{H}|\mathcal{H}'}, &(57)\\ P &= Q^*, &(58)\\ S &= T_{|\mathcal{H}'}, &(59) \end{aligned}$$

and see that then all desired equations hold. Note that Q is the canonical embedding $\mathcal{H}' \to \mathcal{H}$, and so its adjoint Q^* is the orthogonal projection onto $\mathcal{H}' \subseteq \mathcal{H}$. ♡

We now assume for the whole rest of Section 2 that A and T are selfadjoint. Thus, so is S, and (51) becomes

$$M(z) = S^{-1} + Q^*(A - z)^{-1}Q, \tag{60}$$

which is a selfadjont pencil according to

Definition 2.4 An operator pencil H, holomorphic on $\mathcal{D}H \subseteq \mathbb{C}$, is called selfadjoint if $\forall z \in \mathcal{D}H \cap \mathbb{R} :\ H(z) = H(z)^*$ holds.

Remark 2.6 In [RS, Theorem XII.13] this condition for selfadjointness is used, which is the same as the condition $\forall z \in \mathcal{D}H :\ H(\overline{z}) = H(z)^*$ in [K, VII§3.1] in the most important case of analytic families of operators (see Definition 2.6 below).

By the way, for the particular pencil M given by (60), obviously $\forall z \in \rho A :\ M(\overline{z}) = M(z)^*$ is true.

[14]For $(\ker T)^\perp = \overline{\mathcal{R}T}$ note that T is bounded and normal. Hence

$$x \in \ker T \Leftrightarrow Tx = 0 \Leftrightarrow T^*Tx = 0 \Leftrightarrow TT^*x = 0 \Leftrightarrow T^{**}T^*x = 0 \Leftrightarrow T^*x = 0 \Leftrightarrow x \in \ker T^*$$

holds. The wellknown general equality $(\ker T^*)^\perp = \overline{\mathcal{R}T}$ concludes this thought.

Proposition 2.9 can be reformulated to say

A number $z \notin \sigma A$ is eigenvalue[15] of $A + T$ if and only if 0 is eigenvalue of $M(z)$.

This leads us to study the eigenvalues of the operators $M(z)$ and their dependence on the parameter z. The following section is devoted to this topic.

2.4 Analytic families of operators

Let us give some standard definitions for operator-valued functions to be called analytic:

Definition 2.5 ([K, VII§1]): A function H, defined on $\mathcal{D}H \subseteq \mathbb{C}$, which takes bounded operators there, is called an analytic family of (bounded) operators if for all $z \in \mathcal{D}H$ the differential quotient

$$\lim_{z' \to z} \frac{H(z') - H(z)}{z' - z} \tag{61}$$

does exist in the norm-topology of operators.

Remark 2.7 An application of the principle of uniform boundedness reveals that it is immaterial whether the limit (61) is taken in the norm-topology, the strong operator-topology, or the weak one (for some clue, see [K, VII§1.1]).

As the limit of bounded operators is bounded, the following is true:

Proposition 2.11 *If H is an analytic family of bounded operators around $z \in \mathbb{C}$, then $H'(z)$ is a bounded operator.*

The below definition extends Definition 2.5 to families of possibly unbounded operators:

Definition 2.6 ([RS, XII.2], also confer to [K, VII§1.2]): An operator-valued function H defined on some $\mathcal{D}H \subseteq \mathbb{C}$ is called an analytic family in the sense of Kato if for all $z \in \mathcal{D}H$ it holds $\rho H(z) \neq \emptyset$, and for each $z_0 \in \mathcal{D}H$ there is an $e_0 \in \rho H(z_0)$ such that $e_0 \in \rho H(z)$ for z near z_0 and $z \mapsto (H(z) - e_0)^{-1}$ is a bounded-analytic function of operators.

The notion of an analytic family in the sense of Kato is too general for our purposes. We want to compute reasonable derivatives such as (61) for the pencil M. That is why we use

Definition 2.7 An operator-pencil H defined on $\mathcal{D}H \subseteq \mathbb{C}$ is called an analytic family of type (A) if $H(z)$ is closed for $z \in \mathcal{D}H$, $\mathcal{D}H(z)$ is some set $\mathcal{D}_H \subseteq \mathcal{H}$ indepedent of z, and $z \mapsto H(z)x$ is analytic for all $x \in \mathcal{D}_H$.

Remark 2.8 In [K, VII§2.1] the above notion is defined, and a proof is given that analytic families of type (A) are analytic families in the sense of Kato.

[15] A similar formulation can be found to characterize $\sigma(A + T)$.

Remark 2.9 For holomorphic families of type (A), the derivative $H'(z)$ is defined by (61) where the limit has to be taken in the strong operator-topology now, since the expressions $\frac{H(z')-H(z)}{z'-z}$ are only defined on $\mathcal{D}_H$. However, H being of type (A) implies that $H'(z)$ is defined on all of $\mathcal{D}_H$.

Notwithstanding the closedness of $H(z)$, the operator $H'(z)$ is not closed in general.[16] That is the reason why the closed graph theorem[17] cannot be applied directly to show that the operator $H'(z)\,(H(z)-e)^{-1}$, defined everywhere on $\mathcal{H}$ if $e \in \rho H(z)$, is continuous. Nevertheless, it is true, tells Proposition 2.13, which will use

Proposition 2.12 *Let a sequence of bounded operators X_k converge in the strong operator topology to an operator X defined on the whole of $\mathcal{H}$.*

Then X is bounded.

Proof: For every vector $x \in \mathcal{H}$ it holds $\lim_{k\to\infty}(X_k - X)x = 0$. Thus, for every $x \in \mathcal{H}$ there is a constant C_x such that $\forall k \in \mathbb{N} : \|(X_k - X)x\| \leq C_x$. The principle of uniform boundedness yields $\forall k \in \mathbb{N} : \|X_k - X\| \leq C$ for a constant C. That is why X is bounded. ♡

Proposition 2.13 *If H is an analytic family of type (A) around some $z \in \mathcal{D}H$ and $e \in \rho H(z)$ then*

$$H'(z)\,(H(z)-e)^{-1}$$

is a bounded operator. Moreover, this expression defines a holomorphic family of bounded operators with respect to the parameter $e \in \rho H(z)$.

Proof: From $H'(z)\,(H(z)-e)^{-1}x = \lim_{z'\to z}\frac{H(z')-H(z)}{z'-z}\,(H(z)-e)^{-1}x$ and the fact $\mathcal{D}H'(z) = \mathcal{D}H(z) = \mathcal{R}\,(H(z)-e)^{-1}$ the assertion follows via Proposition 2.12. ♡

After these technical sidesteps we return to the main thread and cite an important theorem:

Proposition 2.14 *Let H be a selfadjoint analytic family of type (A) defined for z in an neighbourhood of an interval $(a,b) \subseteq \mathbb{R}$. Furthermore, let $H(z)$ have compact resolvent.*

Then all eigenvalues of $H(z)$ can be represented by functions which are holomorphic on (a,b). More precisely, there is a sequence of scalar-valued functions e_ι and a sequence of vector-valued functions x_ι, all holomorphic on (a,b), such that for $z \in (a,b)$, the $e_\iota(z)$ represent all the repeated eigenvalues of $H(z)$ and the $x_\iota(z)$ form a complete orthonormal family of the associated eigenvectors of $H(z)$.

For a proof see [K, VII§3, Theorem 3.9]. This statement can be derived by the methods applied to prove [RS, Theorem XII.13], too.

[16] In the case of analytic families of bounded operators, Proposition 2.11 assured closedness of $H'(z)$.

[17] See e.g. [K, III§5.4].

Proposition 2.15 *Let H be an analytic family of type (A), $z \in \mathcal{D}H$ and Γ a positively oriented closed Jordan-curve in $\rho H(z)$. Then for the derivative of the pencil of eigenprojections*[18]

$$P_\Gamma(z) = \frac{1}{2\pi i} \oint_\Gamma (H(z) - e)^{-1} \, de$$

it holds:

$$\mathcal{R}P'_\Gamma(z) \subseteq \mathcal{D}H(z)$$

Proof: As a Jordan-curve, the curve Γ is compact. Since H is an analytic family in the sense of Kato (confer to Remark 2.8), Γ thus lies in $\rho H(z')$ for z' near z, too. Hence, $z' \mapsto (H(z') - e)^{-1}$ is analytic near z. Consequently, so is $P_\Gamma(z)$. From

$$\begin{aligned} \frac{d\,(H(z) - e)^{-1}}{dz}(z) &= \lim_{z' \to z} \frac{(H(z') - e)^{-1} - (H(z) - e)^{-1}}{z' - z} \\ &= \lim_{z' \to z} \frac{(H(z) - e)^{-1} (H(z) - H(z')) (H(z') - e)^{-1}}{z' - z} \\ &= -(H(z) - e)^{-1} H'(z) (H(z) - e)^{-1} \end{aligned}$$

we get

$$\begin{aligned} H(z)P'_\Gamma(z) &= -\frac{1}{2\pi i} H(z) \oint_\Gamma (H(z) - e)^{-1} H'(z) (H(z) - e)^{-1} \, de & (62) \\ &= -\frac{1}{2\pi i} \oint_\Gamma H(z) (H(z) - e)^{-1} H'(z) (H(z) - e)^{-1} \, de \\ &= -\frac{1}{2\pi i} \oint_\Gamma \left(e + (H(z) - e)^{-1}\right) H'(z) (H(z) - e)^{-1} \, de. & (63) \end{aligned}$$

Via Proposition 2.13 follows that the integrand $e \mapsto \left(e + (H(z) - e)^{-1}\right) H'(z) (H(z) - e)^{-1}$ is a holomorphic families of bounded operators. Therefore, the integral exists, even in the Riemann-sense.

It remains to justify that the operator $H(z)$ can be exchanged with the integration in (62). This can be done using a standard argument: The integral $P'_\Gamma(z)$ can be approximated by a convergent family of Riemann-sums $r_k \in \mathcal{H}$. Similarly, term (63) is approximated by numbers h_k computed from Riemann-sums of the integral. If in both sequences of Riemann-sums the same points on Γ are chosen where the integrands are to be evaluated, it holds $h_k = H(z)r_k$ because $H(z)$ as a linear operator commutes with finite summation. But both, r_k and h_k converge, and $H(z)$ is closed by the definition of families of type (A). Hence

$$H(z)P'_\Gamma(z) = H(z) \lim_{k \to \infty} r_k = \lim_{k \to \infty} H(z) r_k = \lim_{k \to \infty} h_k$$

[18] In fact, $P_\Gamma(z)$ is a projection with $\mathcal{R}P_\Gamma(z) \subseteq \mathcal{D}H(z)$, and $H(z)P_\Gamma(z)$ is a bounded operator with $P_\Gamma(z)H(z) \subseteq H(z)P_\Gamma(z)$. Moreover, $\sigma H(z)_{|\mathcal{R}P_\Gamma(z)} = \sigma_0$ is the part σ_0 of $\sigma H(z)$ surrounded by Γ and $\sigma H(z)_{|\mathcal{R}(1-P_\Gamma(z))} = \sigma H(z) \setminus \sigma_0$. For these and further properties of such integrals see the thorough treatise in [GGK, I] for bounded operators and its extension to unbounded ones in [GGK, XV]. The eigenprojections are introduced in [K, I§5], and used in [RS, XII], too.

If $H(z)$ is a selfadjoint (or, more generally, normal) operator, $P_\Gamma(z)$ is an orthogonal projection. This is clear because then it equals the orthogonal eigenprojection $E(\sigma_0)$, where E denotes the spectral family of $H(z)$. For an extensive introduction to spectral families see [BN]. In [K, VI§5] the case of spectral families of selfadjoint operators is examined.

and this is just the integral (63). $\heartsuit$

Proposition 2.16 *Let H be a selfadjoint operator pencil which is an analytic family of type (A) around some $z \in I\!R$, and let $H(z')$ have compact resolvent there.*

If H is monotonously increasing on some real interval around z, then $\frac{de_\iota(z)}{dz}(z) \geq 0$ for all the families e_ι of eigenvalues defined in Proposition 2.14.

Proof: Remember that $x_\iota(z)$ is the eigenvector corresponding to the eigenvalue $e_\iota(z)$. Secondly, remember $\|x_\iota(z')\| = 1$ for $z' \in \mathbb{R} \cap \mathcal{D}H$ by Proposition 2.14. So, we have

$$e_\iota(z') = \langle H(z')x_\iota(z'), x_\iota(z')\rangle \tag{64}$$

for real [19] z' near z. Now we compute

$$\begin{aligned} e_\iota'(z) &= \langle H'(z)x_\iota(z), x_\iota(z)\rangle + \langle H(z)x_\iota'(z), x_\iota(z)\rangle + \langle H(z)x_\iota(z), x_\iota'(z)\rangle && (65)\\ &= \langle H'(z)x_\iota(z), x_\iota(z)\rangle + \langle x_\iota'(z), H(z)x_\iota(z)\rangle + \langle H(z)x_\iota(z), x_\iota'(z)\rangle \\ &= \langle H'(z)x_\iota(z), x_\iota(z)\rangle + e_\iota\left(\langle x_\iota'(z), x_\iota(z)\rangle + \langle x_\iota(z), x_\iota'(z)\rangle\right) \\ &= \langle H'(z)x_\iota(z), x_\iota(z)\rangle + e_\iota\left(\frac{d\,\|x_\iota(z)\|^2}{dz}(z)\right) = \langle H'(z)x_\iota(z), x_\iota(z)\rangle && (66) \end{aligned}$$

From this we see $e_\iota'(z) \geq 0$ because $H'(z) \geq 0$ holds in consequence of the monotony of H.

To remain honest, it is necessary to justify the expression $H(z)x_\iota'(z)$. Let Γ be a positively oriented closed Jordan-curve in $\rho H(z)$ around $e_\iota(z)$, and let $P_\Gamma(z)$ be the projection defined in Proposition 2.15. Then $x_\iota(z') = P_\Gamma(z')x_\iota(z')$ is valid for z' near z. As x_ι and P_Γ are analytic families of vectors and bounded operators, respectively, this implies $x_\iota'(z) = \frac{d(P_\Gamma(z)x_\iota(z))}{dz}(z) = P_\Gamma'(z)x_\iota + P_\Gamma(z)x_\iota'(z)$. The right hand term is contained in $\mathcal{D}H(z)$, because $\mathcal{R}P_\Gamma(z) \subseteq \mathcal{D}H(z)$ (see footnote 18) and $\mathcal{R}P_\Gamma(z) \subseteq \mathcal{D}H(z)$ because of Proposition 2.15. $\heartsuit$

Remark 2.10 It should be noted here that there is a short and elementary proof for $e_\iota'(z) \geq 0$ which does not need $x_\iota'(z) \in \mathcal{D}H(z)$. The cumbersome way including the Propositions 2.12, 2.13, and 2.15 has been chosen because these propositions appear to be valuable themselves. You may forget about all of them, however, when you compute (again, for simplicity admitting real numbers z' only):

$$\begin{aligned} e_\iota'(z) &= \lim_{z'\to z} \frac{\langle H(z')x_\iota(z'), x_\iota(z')\rangle - \langle H(z)x_\iota(z), x_\iota(z)\rangle}{z'-z} \\ &= \lim_{z'\to z}\left(\left\langle H(z')x_\iota(z'), \frac{x_\iota(z') - x_\iota(z)}{z'-z}\right\rangle\right. \end{aligned}$$

[19] To admit only real z' is for the sake of simplicity. The result, however is true since e_ι is holomorphic. Note that the complete expressions $e_\iota(z) = \frac{\langle H(z)P_\Gamma(z)x_\iota, P_\Gamma(z)x_\iota\rangle}{\langle P_\Gamma(z)x_\iota, P_\Gamma(z)x_\iota\rangle}$ would have to be used when H is not selfadjoint or $z \notin I\!R$. In [RS] nondegenerate eigenvalues of $H(z)$ are calculated in this way.

$$
\begin{aligned}
& \quad + \left\langle \frac{H(z') - H(z)}{z' - z} x_\iota(z'), x_\iota(z) \right\rangle + \left\langle H(z) \frac{x_\iota(z') - x_\iota(z)}{z' - z}, x_\iota(z) \right\rangle \Bigg) \\
= & \lim_{z' \to z} \Bigg(\left\langle H(z') x_\iota(z'), \frac{x_\iota(z') - x_\iota(z)}{z' - z}, \right\rangle \\
& \quad + \left\langle x_\iota(z'), \frac{H(z') - H(z)}{z' - z} x_\iota(z) \right\rangle + \left\langle \frac{x_\iota(z') - x_\iota(z)}{z' - z}, H(z) x_\iota(z) \right\rangle \Bigg) \\
= & \lim_{z' \to z} \Bigg(\left\langle e_\iota(z') x_\iota(z'), \frac{x_\iota(z') - x_\iota(z)}{z' - z}, \right\rangle \\
& \quad + \left\langle x_\iota(z'), \frac{H(z') - H(z)}{z' - z} x_\iota(z) \right\rangle + \left\langle \frac{x_\iota(z') - x_\iota(z)}{z' - z}, e_\iota(z) x_\iota(z) \right\rangle \Bigg) \\
= & \langle e_\iota(z) x_\iota(z), x_\iota'(z) \rangle + \langle x_\iota(z), H'(z) x_\iota(z) \rangle + \langle x_\iota'(z), e_\iota(z) x_\iota(z) \rangle \\
= & \langle x_\iota(z), H'(z) x_\iota(z) \rangle + e_\iota \left(\langle x_\iota(z), x_\iota'(z) \rangle + \langle x_\iota'(z), x_\iota(z) \rangle \right) \\
= & \langle x_\iota(z), H'(z) x_\iota(z) \rangle + e_\iota \left(\frac{d \left\| x_\iota(z) \right\|^2}{dz}(z) \right) = \langle x_\iota(z), H'(z) x_\iota(z) \rangle + 0 \geq 0.
\end{aligned}
$$

Remark 2.11 The assertion of Proposition 2.16 holds true in the slightly more general context when $H(z)$ does not have a compact resolvent but e is an isolated eigenvalue of finite multiplicity. This can be seen as soon as it has been proved (and this is actually done in [K, VII§3.1] and [RS, XII.2]) that the eigenvalues e and the associated eigenvectors x are values of analytic functions similar to those in Proposition 2.14. Only the set of eigenvectors $x(z)$ does not span $\mathcal{H}$ any more, in general. But Proposition 2.16 does not use this fact.

Remark 2.12 In Section 1.1 we have studied the monotonous selfadjoint pencil $a \mapsto A_1(a)$. Remember Propositions 1.5 and 1.6, which stated monotony of the eigenvalues for this particular pencil.

2.5 General results for compact T

Let us return to the specific pencil M. From now on, we assume that T is compact. An important case is the case of a perturbation of finite rank ν. Then we may set $\mathcal{H}' = \mathbb{C}^\nu$ and S becomes a regular, hermitian $\nu \times \nu$-matrix. The pencil M becomes a $\nu \times \nu$-matrix pencil holomorphic in ρA.

If T is compact, so is S. Consequently, S^{-1} has a compact resolvent. Therefore, we shall use

Proposition 2.17 *If T is compact and selfadjoint, and $z \in \rho A$, then $M(z)$ has compact resolvent, and the number of eigenvalues of $M(z)$ is the number of eigenvalues of S^{-1}.*

Proof: The operator S^{-1} has compact resolvent since T and hence S are compact. For $z \in \rho A$ the term $Q^*(A-z)^{-1}Q$ is a bounded operator, and [K, IV§3.6,Theorem 3.17] becomes applicable, which states the first assertion.

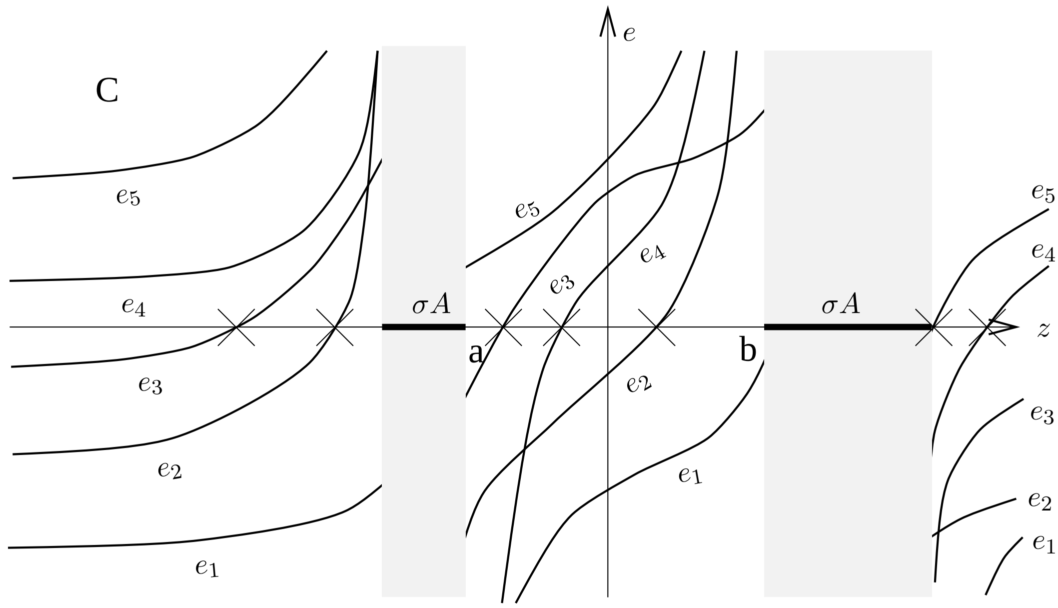

Figure 7: The functions e_ι, sketched. Wherever one of them crosses the z-axis in ρA, $A+T$ has an eigenvalue.

The second one is implied by the first, since $M(z)$ then has pure point spectrum. Hence the number of eigenvalues of S^{-1} and $M(z)$ is determined by the Hilbert-dimension of the space $\mathcal{H}'$, which both of S^{-1} and $M(z)$ act on. ♡

Let us note the following convention: If we speak of a number of eigenvalues, we count multiplicity. This means that we take the dimension of the span of the eigenspaces of the eigenvalues as the number of eigenvalues under consideration: For instance, if an operator or pencil has one nondegenerate eigenvalue and two degenerate eigenvalues, one of, say, degree two, the other of degree five of degeneracy, we count that the operator or pencil has a total of eight eigenvalues.

Now, let us summarize the important properties of M defined by (60) which are of interest in this context:

Proposition 2.18 *The pencil M defined by (60) is a monotonous, selfadjoint analytic family of type (A) on ρA, and $M(z)$ has compact resolvent there.*

For all families e_ι which have been defined in Proposition 2.14 it holds

$$\forall z \in \rho A \cap I\!R: \; {e_\iota}'(z) > 0 \tag{67}$$

or

$$\forall z \in \rho A: \; {e_\iota}'(z) = 0 \;\wedge\; e_\iota(z) \neq 0. \tag{68}$$

Line (68) holds for an index ι, if and only if $e_\iota(z) = \underline{e}_\iota$ and $x_\iota(z) = \underline{x}_\iota$ are an eigenvalue and an associated eigenvector of S^{-1} such that $\underline{x}_\iota \in \ker(A-z)^{-1}Q$.

Hence, the case that (68) holds for a function e_ι is only possible if A is a proper relation. which equals an eigenvector $\underline{x}_\iota$ of S^{-1} then, Proof: From Proposition 2.17 we know that $M(z)$ has compact resolvent for $z \in \rho A$.

The pencil M is monotonous on $\mathbb{R} \cap \rho A$. This follows from

$$\frac{d \langle M(z)x, x\rangle}{dz} = \left\langle Q^*(A-z)^{-2}Qx, x \right\rangle = \left\|(A-z)^{-1}Qx\right\|^2 \geq 0 \tag{69}$$

for $z \in \rho A$ and $x \neq 0$.

The only thing left to be proved before Proposition 2.16 can be applied is that M, considered on ρA, is an analytic family of type (A). But this is obvious, since S^{-1} is selfadjoint and hence closed, and $Q^*(A-z)^{-1}Q$ is bounded and selfadjoint for $z \in \rho A$. Therefore, the sum $M(z)$ is closed. Further, from [K, V§4.1, Theorem 4.3] follows that $M(z)$ is as well selfadjoint. Moreover $\mathcal{D}M(z) = \mathcal{D}S^{-1}$ is independent of z.

The inequality (67) is strict for all functions e_ι with the possible exception of up to $\dim\ker(A-z)^{-1}Q$ of them. This inequality to be strict follows from the result in (66) in the proof of Proposition 2.16, and the fact, that the eigenvectors x_ι of $M(z_0)$ are [20] pairwise linearly independent. Hence, equality in (69) can only hold for $\dim\ker(A-z)^{-1}Q$ of them.

In these cases, when $e_\iota'(z) = 0$ for an arbitrary $z \in \rho A$, (69) implies $(A-z)^{-1}Qx_\iota(z) = 0$. That is why

$$e_\iota(z)x_\iota(z) = M(z)x_\iota(z) = S^{-1}x_\iota(z). \tag{70}$$

This implies that $e_\iota(z)$ is an eigenvalue $\underline{e}_\iota$ of S^{-1}, hence $e_\iota(z) \neq 0$. Moreover, $x_\iota(z)$ obviously is an eigenvector $\underline{x}_\iota$ of S^{-1} which lies in $\ker(A-z)^{-1}Q$. However, $\ker(A-z)^{-1}Q$ does not change for $z \in \rho A$. Therefore $\underline{x}_\iota$ is eigenvector of $M(z')$ for all $z' \in \rho A$, and the associated eigenvalue is constantly $\underline{e}_\iota$. That is the reason why $\forall z' \in \rho A:\ e_\iota(z') = \underline{e}_\iota$. ♡

Remark 2.13 In fact, the pencil M is not only monotonous, but it is an operator-valued Nevanlinna-function, because

$$\begin{aligned}
\Im \langle M(z)x, x\rangle &= \underbrace{\Im \left\langle S^{-1}x, x\right\rangle}_{0} + \Im \left\langle Q^*(A-z)^{-1}Qx, x\right\rangle = \Im\left\langle (A-z)^{-1}Qx, Qx\right\rangle \\
&= \Im \langle x', y' - zx'\rangle = \Im \langle x', -zx'\rangle = \Im z \langle x', x'\rangle \geq 0
\end{aligned}$$

for $\Im z > 0$, and $x \in \mathcal{D}M(z)$. It has been substituted $(A-z)^{-1}Qx = x'$, and thus for $y' = Qx + zx'$ holds $(x', y') \in A$.

2.6 $\sigma(A+T)$ below and above σA

In this section, the effect is studied which the eigenvalues of the perturbing operator T have on $\sigma(A+T)$ below (or above) σA.

Definition 2.8 Let A be a symmetric relation. Then we define

$$A \geq 0 \Leftrightarrow \forall (x,y) \in A:\ \langle y, x\rangle \geq 0.$$

We say $A \geq m_-$ if $A - m_- \geq 0$.

[20] In fact, they are pairwise orthogonal, see Proposition 2.14.

Remark 2.14 It holds $A \geq 0$ if and only if $\sigma A \geq 0$. Moreover, $(A-z)^{-1} \geq 0$ is true then, for all $z < 0$.

These are wellknown properties of semibounded selfadjoint operators See e.g. [K, V§3.10] for the former one. The latter one follows then by

$$\left\langle (A-z)^{-1}x, x \right\rangle = \left\langle (A-z)^{-1}x, (A-z)(A-z)^{-1}x \right\rangle = \left\langle (A-z)(A-z)^{-1}x, (A-z)^{-1}x \right\rangle \geq 0.$$

In order to prove these properties for selfadjoint proper relations, decompose A into the orthogonal sum of an operator $A_{\mathbb{R}}$ and a proper relation A_∞ with $\mathcal{D}A_\infty = \{0\}$.

Proposition 2.19 *Let T be a compact selfadjoint operator, $\sigma T \cap (-\infty, 0)$ consisting of $\kappa_- \in \mathbb{N} \cup \{\infty\}$ eigenvalues, and A be semibounded from below by $m_- = \min \sigma A$, $m_- \in \mathbb{R}$.*

Then $\sigma(A+T)$ below m_- consists of isolated eigenvalues of finite multiplicity. These are no more than κ_-.

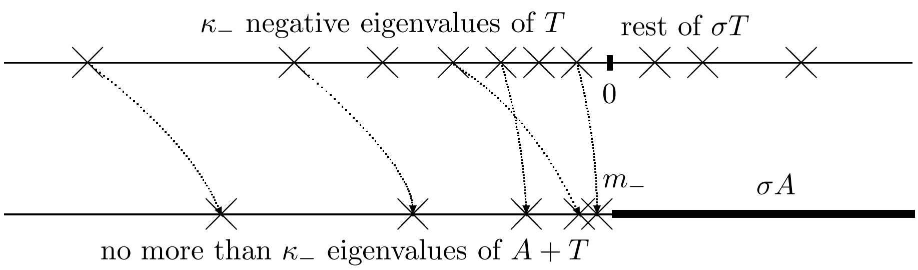

Figure 8: Illustration of Proposition 2.19.

<u>Proof</u>: We have $M(-\infty) = S^{-1}$, increasing monotonously. The pencil M has an eigenvalue z whenever $0 \in \sigma_p M(z)$, or equivalently $0 = e_\iota(z)$ for an index ι. We see from Proposition 2.18 that only κ_- times an eigenvalue of $M(z)$ on its way upward can cross 0. This is so because for $z = -\infty$ there are no more than κ_- eigenvalues of $M(z)$ below 0, and eigenvalues cannot appear somewhere or suddenly increase their multiplicity when z is changed a bit, because from Proposition 2.14 is clear that the eigenvectors $x_\iota(z)$ span the whole of $\mathcal{H}'$. That is why the related eigenvalues $e_\iota(z)$ are the only points in $\sigma M(z)$ and the multiplicity of a value e as an eigenvalue of $M(z)$ is just the number of indices ι such that $e_\iota(z)$ equals this value e.

Proposition 2.18 assures that no function e_ι has more than one zero below m_-. Note that if A is a proper relation, those among the functions e_ι with $e'_\iota(z) = 0$ are nonzero by Proposition 2.18.
$\heartsuit$

Of course, the same reasoning as above can be used to prove the symmetric proposition

Proposition 2.20 *Let T be a compact selfadjoint operator, $\sigma T \cap (0, +\infty)$ consisting of $\kappa_+ \in \mathbb{N} \cup \{\infty\}$ eigenvalues, and let A be semibounded from above by $m_+ = \max \sigma A$, $m_+ \in \mathbb{R}$.*

Then $\sigma(A+T)$ above m_+ consists of isolated eigenvalues of finite multiplicity. These are no more than κ_+.

Remark 2.15 Instead of using $\lim_{z\to-\infty} M(z) = S^{-1}$ in the proof of Proposition 2.19 to get some kind of "initial number" of eigenvalues, Proposition 2.17 could be used, as well.

However, without further consideration (see Section 2.8), Proposition 2.17 does not allow to distinguish between the effects of positive and negative eigenvalues of T.

Remark 2.16 A different approach to Propositions 2.19 and 2.20 is the use of the min-max-principle (in the form of [RS, Theorem XIII.1]). See Proposition 2.26 for an application.

To a certain degree, the min-max-principle can even be used as a surrogate for Proposition 2.16 in the case studied above, thus allowing a limited possibility to study operators T which have both eigenvalues and continuous spectrum. After decomposition of T into a compact part T_0 and the part T_1 which has continuous spectrum, such research can be done with the help of Proposition 2.25 and the use of M, when T is viewed as the sum of two subsequent perturbations T_0 and T_1. However, the fact that T_0 and T_1 correspond to a representation of T according to *orthogonal* subspaces of $\mathcal{H}$ is not exploited then. That is the reason why the min-max-principle will yield stronger results.

Nevertheless, the straightforward application of the min-max-principle is only possible as long as we want to know something about the structure of $\sigma(A+T)$ at its maximum or minimum. To get results on eigenvalues of $A+T$ inside gaps of σA (see section 2.7 below), the approach presented in the proof of Proposition 2.19 above seems more promising.

2.7 $\sigma(A+T)$ in gaps of σA

In this section, we formulate results on the change which takes place inside gaps of the spectrum of the selfadjoint operator or relation A after the selfadjoint compact perturbation T has been switched on. The methods applied are so similar to those presented in section 2.6 that some parts of the proofs will rather be sketched than repeated in full detail.

Proposition 2.21 *Let T be a compact selfadjoint operator, and have $\kappa \in \mathbb{N}\cup\{\infty\}$ nonzero eigenvalues, and let $(a,b) \subset \rho A$, $a<b$, be a real gap in σA.*

Then $\sigma(A+T)\cap(a,b)$ consists of isolated eigenvalues of finite multiplicity. These are no more than κ.

Proof: Each of the κ functions e_ι has maximally one zero in (a,b). ♡

2.8 Corresponding eigenvalues of T and $A+T$

The results of the previous section are of the type that the number of eigenvalues of T determines the maximal possible number of eigenvalues of $A+T$ in the gaps of A. In the case of the unbounded gaps $(-\infty, m_-)$ and (m_+, ∞), we have even been able to distinguish between the effect of positive and negative eigenvalues of T, a distinction that inside a narrow gap is messed up a bit, of course. The above facts suggest that there is some correspondence not only between the sets of eigenvalues as such, but between the individual eigenvalues of T and those of $A+T$,

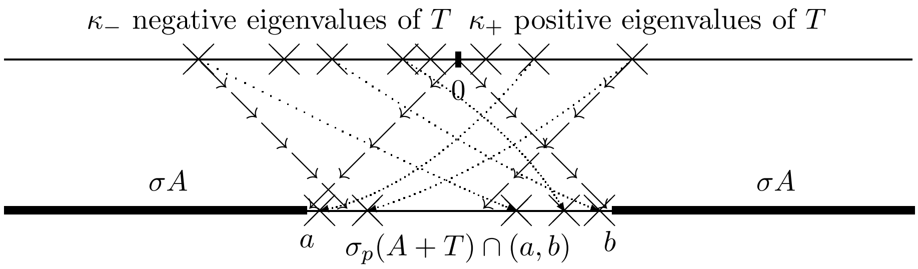

Figure 9: Illustration of Proposition 2.21. The arrows between the illustrations of σT and of $\sigma(A+T)\cap(a,b)$ point to the boundaries of the areas of (a,b) where the eigenvalues of $A+T$ correponding to positive respectively negative eigenvalues of T can lie. See Proposition 2.23 for an explanation.

as well. Such a correspondence will be established and studied in the present section. To speak precisely: Each eigenvalue $z_\iota \in (a,b) \subset \rho A$ of $A+T$ will be mapped to a corresponding nonzero eigenvalue of T. This mapping will be injective, but in general not bijective.

We shall study the eigenvalues in a nonvoid gap $(a,b) \subseteq \rho A \cap \mathbb{R}$, $a \in \{-\infty\} \cup \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$.

The correspondence is the following (see Figure 10): Fix a decomposition of T such that (49–50) and (55) hold.

Then each eigenvalue $\underline{z}_\iota$ of T is itself an eigenvalue $\underline{z}_\iota$ of S if $z_\iota \neq 0$ (see Proposition 2.5). The reciprocal $\frac{1}{\underline{z}_\iota}$ is an eigenvalue $\underline{e}_\iota$ of S^{-1}. After a bounded perturbation by $Q^*(A-z)^{-1}Q$ the numbers $\underline{e}_\iota$ correspond to the eigenvalues $e_\iota(z)$ of $M(z)$ and hence to the analytic functions e_ι. The values $z_\iota \in (a,b)$ where $e_\iota(z_\iota) = 0$ are the eigenvalues of M and hence $A+T$ in (a,b) via Proposition 2.9.

Note that the functions e_ι need not vanish in (a,b). Hence there may be eigenvalues of T that do not correspond to eigenvalues of $A+T$ in (a,b). However the converse correspondence must hold true in any case: All eigenvalues of $A+T$ in (a,b) must correspond to eigenvalues of T.

$$\begin{array}{ccccccccccccc} T & \longleftarrow & S & \longleftrightarrow & S^{-1} & & \longleftarrow & & M(z) & \longleftrightarrow & M & \longleftrightarrow & A+T \\ \underline{z}_\iota & = & \underline{z}_\iota & \cdot^{-1} & \underline{e}_\iota & = e_\iota(0,z) & \sim & e_\iota(1,z) = & e_\iota(z) & e_\iota(z_\iota)=0 & z_\iota & = & z_\iota \end{array}$$

Figure 10: The correspondence between the eigenvalues of T and those of $A+T$ inside $(a,b) \subset \rho A$, sketched.

The only step vague so far in this correspondence is the step from $\underline{e}_\iota$ to $e_\iota(z)$. To make it precise, we define the r-dependent analytic family of operator-pencils by

$$M(r,z) = S^{-1} + rQ^*(A-z)^{-1}Q. \tag{71}$$

Let $e_\iota(r,z)$ denote the eigenvalues of $M(r,z)$, and enumerate them by ι such that $\underline{e}_\iota = e_\iota(0,z)$.

It is clear that for $z \in (a,b)$ we get that $\sigma M(0,z) = \sigma S^{-1}$ is changed analytically into $\sigma M(1,z) = \sigma M(z)$ if r moves from 0 up to 1. Hence, every eigenvalue $\underline{e}_\iota = e_\iota(0,z)$ of S^{-1} is moved analytically into an eigenvalue $e_\iota(1,z) = e_\iota(z)$ of $M(z)$, and vice versa if r falls from 1 to 0.

Just a word about degenerate eigenvalues: From Proposition 2.2 we know that the multiplicity of equal eigenvalues of $A+T$ and M is the same. And from Proposition 2.14, we know that a degenerate eigenvalue of $M(r,z)$ may split up if r changes, but the total multiplicity stays constant. Hence, if $\underline{z}$ is an eigenvalue of T there are exactly as many indices ι such that $\underline{z} = \underline{z}_\iota$ as the multiplicity of $\underline{z}$ is. Thus we get just as many functions $(r,z) \mapsto e_\iota(r,z)$ which attain the same value $\frac{1}{\underline{z}}$ for $r=0$, provided $\underline{z} \neq 0$. Hence, the correspondence of eigenvalues of $A+T$ to eigenvalues of T is injective in this direction.

Let us conclude the notes on the correspondence of eigenvalues of T and $A+T$ established: All that justifies the so far tacit convention to use the same indices ι for the eigenvalues of S^{-1} and the corresponding families e_ι of eigenvalues.

Proposition 2.22 *If $(a,b) \subset \rho A$, with $-\infty \leq a < b \leq +\infty$, then*

$$a < z < b \quad \Rightarrow \quad e_\iota(z) \in \left[\underline{e}_\iota + \frac{1}{a-z}, \underline{e}_\iota + \frac{1}{b-z}\right]$$

holds.

Proof: Consider the family $r \mapsto e_\iota(r,z)$. Let x_ι be an eigenvector of the eigenvalue $e_\iota(r,z)$. Then the computation which has been done to obtain the result in (66) can be continued. We only replace the derivative $\frac{d}{dz}$ by $\frac{\partial}{\partial r}$ now:

$$\begin{aligned}
\frac{\partial e_\iota(r,z)}{\partial r}(r) &= \left\langle \frac{\partial M(r,z)}{\partial r}(r) x_\iota(r,z), x_\iota(r,z) \right\rangle = \left\langle Q^*(A-z)^{-1} Q x_\iota(r,z), x_\iota(r,z) \right\rangle \\
&\in \Theta Q^*(A-z)^{-1}Q \subseteq \text{convex hull of } \sigma Q^*(A-z)^{-1}Q \\
&\quad \subseteq \text{convex hull of } \sigma(A-z)^{-1} \subseteq \left[\frac{1}{a-z}, \frac{1}{b-z}\right]. \qquad (72)
\end{aligned}$$

We have used $\|x_\iota(r,z)\| = 1$ for real r and z from Proposition 2.14. Then the fact has been exploited that the numerical range[21] $\Theta(A-z)^{-1} = \{\langle (A-z)^{-1}x, x\rangle : x \in \mathcal{H} \wedge \|x\| = 1\}$ of the selfadjoint operator $(A-z)^{-1}$ is contained in the convex hull of its spectrum.[22]

From (72) we get the assertion by integration:

$$e_\iota(z) - \underline{e}_\iota = \int_0^1 \frac{de_\iota(r,z)}{dr}(r)dr \in \left[\frac{1}{a-z}, \frac{1}{b-z}\right].$$

♡

Following the construction of the correspondence between the eigenvalues of T and $A+T$ we obtain the main result of this section. It tells us that the eigenvalues emerging in a gap of σA are closer to the borders of the gap than the eigenvalues of T are close to 0. Remember here Figures 8 and 9, where this fact already has been taken into account.

[21] See [K, V§3.2].

[22] This can easily be proved from $A_+ 0 \geq 0 \Rightarrow \sigma A_+ \geq 0$, confer Remark 2.14.

Proposition 2.23 *Let T be compact and selfadjoint, and $(a,b) \subset \rho A$, with $-\infty \leq a < b \leq +\infty$. Let $z_\iota \in (a,b)$ be an eigenvalue of $A+T$. Let $\underline{z}_\iota$ be the corresponding, hence nonzero, eigenvalue of T. Then it holds:*

$$\begin{aligned} \underline{z}_\iota > 0 \quad &\Rightarrow \quad z_\iota \leq \underline{z}_\iota + a, \\ \underline{z}_\iota < 0 \quad &\Rightarrow \quad z_\iota \geq \underline{z}_\iota + b. \end{aligned}$$

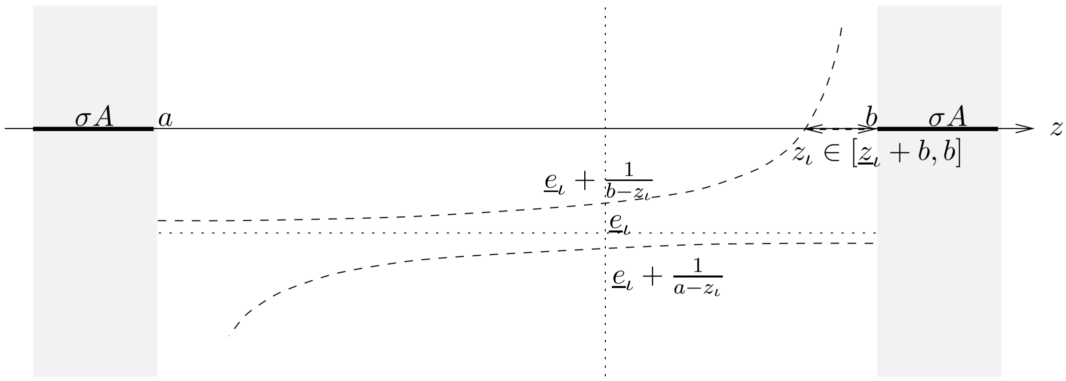

Figure 11: Bounding the position of z_ι.

Proof: Combining the definition of z_ι and Proposition 2.22, we know

$$0 = e_\iota(z_\iota) \in \left[\underline{e}_\iota + \frac{1}{a - z_\iota}, \underline{e}_\iota + \frac{1}{b - z_\iota} \right].$$

This is equivalent to the following lines:

$$\frac{1}{a - z_\iota} \leq -\underline{e}_\iota \leq \frac{1}{b - z_\iota},$$

$$\frac{1}{z_\iota - a} \geq \underline{e}_\iota \geq \frac{1}{b - z_\iota},$$

$$\underline{e}_\iota = 0 \vee \left(\underline{e}_\iota > 0 \wedge z_\iota - a \leq \frac{1}{\underline{e}_\iota} \right) \vee \left(\underline{e}_\iota < 0 \wedge z_\iota - b \geq \frac{1}{\underline{e}_\iota} \right),$$

$$\left(\underline{z}_\iota > 0 \wedge z_\iota \leq \underline{z}_\iota + a \right) \vee \left(\underline{z}_\iota < 0 \wedge z_\iota \geq \underline{z}_\iota + b \right),$$

where $\underline{e}_\iota = \frac{1}{\underline{z}_\iota}$ has been applied. ♡

Remark 2.17 In the case $a = -\infty$, Proposition 2.23 states that there are no eigenvalues of the operator $A+T$ in (a,b) which correspond to positive eigenvalues of T at all. This is so because then all functions e_ι corresponding to positive $\underline{z}_\iota$ are bounded from zero by $e_\iota(z) \geq \underline{e}_\iota > 0$ via Proposition 2.22.

Analogously, negative eigenvalues of T do not correspond to eigenvalues in $\sigma(A+T) \cap (a, b)$ if $b = +\infty$.

An easy consequence of Proposition 2.23 is

Proposition 2.24 *Under the assumptions of Proposition 2.23, eigenvalues $z_\iota \in (a, b)$ of $A+T$ can only accumulate at a and b.*

More precise, the eigenvalues z_ι corresponding to positive eigenvalues of T may accumulate only at a, those corresponding to negative ones only at b.

Finally let us note the following generalisation of the wellknown[23] theorem which states the upper semicontinuity of spectra. Perhaps, it makes Proposition 2.23 seem even more plausible.

Proposition 2.25 *Let A and T be selfadjoint, and let T be bounded. If $(a, b) \subseteq \rho A$ and $\sigma T \subseteq [a', b']$ then $(a+b', b+a') \subseteq \rho(A+T)$. Note that this interval is empty if $b-a \leq b'-a'$.*

Proof: We have $\left\|T - \frac{a'+b'}{2}\right\| = \frac{b'-a'}{2}$. Hence, by the wellknown theorem which we remembered above $\left(a + \frac{b'-a'}{2}, b - \frac{b'-a'}{2}\right) \subseteq \rho(A+T-\frac{a'+b'}{2})$. This, however, is equivalent to $(a+b', b-a') \subseteq \rho(A+T)$ what can be seen by addition of $\frac{a'+b'}{2}$. ♡

2.9 Perturbation by T with compact resolvent

Application and mathematical taste want to get to know something about T having a compact resolvent. However, there cannot be gained much without further assumptions.

Remark 2.18 If T has compact resolvent, we can choose an $a \in \rho T$ and write $A+T$ as $(A+a)+(T-a)$, so removing the kernel of the perturbing operator, and hence, set $\mathcal{H}' = \mathcal{H}$, $Q = Q^* = 1$, and $S = T - a$ instead of (56–59).

Second, let us note that T has compact resolvent if and only if S^{-1} is compact.

The pencil M defined by (60), which has been used excessively in the previous sections, cannot be applied as easily as before, since $M(z)$ in general does not have compact resolvent if S is not compact. That is why we remember Remark 2.16: We will resort to the min-max-principle.

Proposition 2.26 *Let $A \geq m_-$ and T be semibounded from below, and T have a compact resolvent. Let $\underline{z}_\iota$ be the eigenvalues of T, numbered in ascending order: $\underline{z}_1 \leq \underline{z}_2 \leq \underline{z}_3 \leq \ldots$*

Then $\sigma_p(A+T)$ consists [24] *of isolated eigenvalues z_ι of finite multiplicity, numbered in ascending order, and it holds $z_\iota \geq \underline{z}_\iota + m_-$.*

[23]See e.g. [K, IV§3.1, Remark 3.2] and note there that $\left\|(A-z)^{-1}\right\| = \frac{1}{\operatorname{dist}(z, \sigma A)}$ for selfadjoint A).

[24]Note that $A+T$ does not have to be selfadjoint.

Proof: By $\operatorname{codim}_{\mathcal{D}(A+T)}$ we denote the algebraic codimension with respect to the space $\mathcal{D}(A+T)$. The min-max-principle tells us

$$\begin{aligned} z_\iota &\geq \sup_{\operatorname{codim}_{\mathcal{D}(A+T)}\mathcal{H}_\iota=\iota-1}\left(\inf_{x\in\mathcal{H}_\iota\wedge\|x\|=1}\langle(A+T)x,x\rangle\right) \\ &\geq \sup_{\operatorname{codim}_{\mathcal{D}(A+T)}\mathcal{H}_\iota=\iota-1}\left(\inf_{x\in\mathcal{H}_\iota\wedge\|x\|=1}\langle Tx,x\rangle+m_-\right) \\ &\geq \inf_{x\in\mathcal{D}T\cap\underline{\mathcal{H}}_\iota\wedge\|x\|=1}\langle Tx,x\rangle+m_-=\underline{z}_\iota+m_-. \end{aligned}$$

when $\underline{\mathcal{H}}_\iota$ denotes the orthogonal complement of the span of the eigenspaces of the eigenvalues $\underline{z}_1, \ldots, \underline{z}_{\iota-1}$ of T. In other words, the span of the eigenvectors of T to the eigenvalues $\underline{z}_\iota$, $\underline{z}_{\iota+1}$, and so on. Hence $\langle Tx,x\rangle \geq z_\iota$ holds if $x\in\mathcal{D}T\cap\underline{\mathcal{H}}_\iota$. ♡

Remark 2.19 Of course, a proposition symmetric to Proposition 2.26 can be formulated and proved for the case of $A\leq m_+$ and T bounded from above, and for $\sigma(A+T)\cap(m_+,+\infty)$. This is done just as Proposition 2.20 has been formulated symmetric to Proposition 2.19.

2.10 One-dimensional perturbation as an example

Example 2.1 Set $I=[-1,+1]$ and $\mathcal{H}=\mathcal{L}^2(I)$. Let u and v with $\|v\|=1$ be continuous functions on I. The function u is assumed to be real-valued. Consider the operator $A: y\mapsto (t\mapsto u(t)t\cdot y(t))$ and the one-dimensional perturbation $T(a)=a\cdot v\otimes v^*$. By $v\otimes v^*$ we denote the tensor-product $y\mapsto v\cdot(v^*y)$, confer Section 3.1.

Obviously $\overline{\mathcal{R}T(a)}=\mathbb{C}\,v$ holds. Then we may define[25]

$$\begin{aligned} \mathcal{H}' &= \mathbb{C}, \\ Q &= v, \\ P &= v^*, \\ S &= a. \end{aligned}$$

With these definitions, (49–50) and (55) are satisfied.

The pencil M, which characterizes the eigenvalues of $A+T$ outside $u(I)$ is then given by

$$M(z)=a^{-1}+v^*(u-z)^{-1}v=a^{-1}+\int_{-1}^{+1}\frac{|v|^2}{u-z}dt=a^{-1}+f(z).$$

Hence, z is an eigenvalue of $A+T$ if and only if (2) holds, and $f(z)=-a^{-1}$.

The similarity to the one-dimensional extension problem discussed in Section 1 is striking (confer (7)). Hence, the tools developed there to study embedded eigenvalues, for instance, become applicable here, again, and similar facts hold. That is why we shall rather give an impression by Figure 12 than generate a list of idle propositions.

[25]The definition $Q=v$ is to be understood in the sense of Section 1, that is: $Q:\mathbb{C}\to\mathcal{H}:c\mapsto cv$.

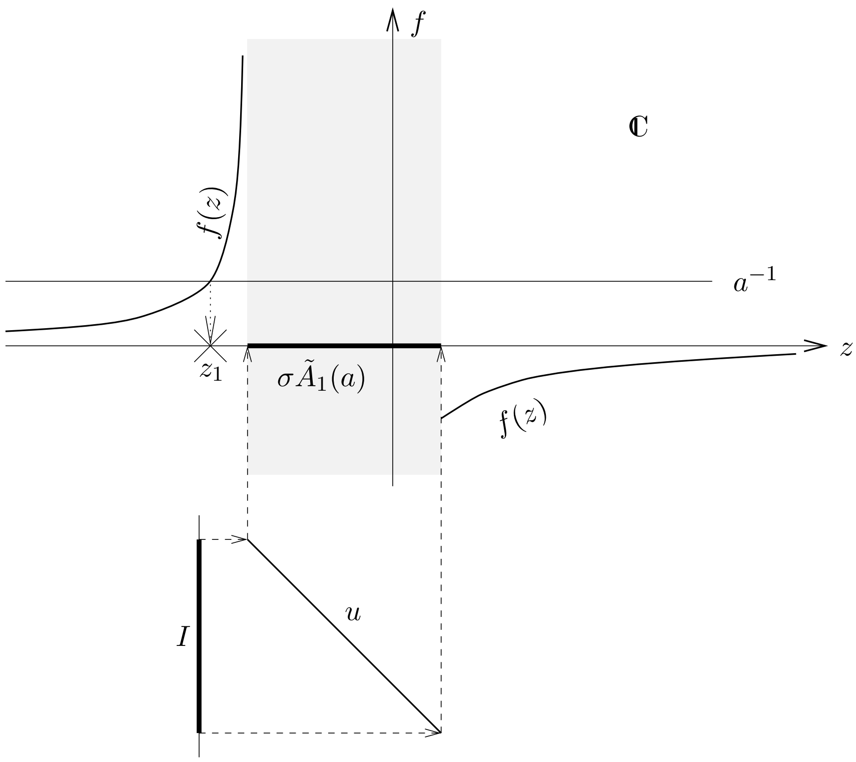

Figure 12: Illustration of the characteristic equation $f(z) = -a^{-1}$.

3 Coupled systems

As an application of the theory of section 2 we take a look at a special kind of perturbation, which we call coupling. A selfadjoint operator of the form

$$\tilde{A}_0 = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},$$

which represents two independent systems, is changed to an operator

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

We shall call B the coupling operator. The selfadjoint operator

$$\tilde{B} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \tag{73}$$

is the perturbing one. The dimension of B will be called the dimension of the coupling. The rank of the perturbation is the double rank of the coupling.

Note that $\tilde{A}$ can be viewed as the result of an operator-extension instead of an operator perturbation, too. But we will impose compactness assumptions on B only. In this case it is more adequate to view $\tilde{A}$ as a perturbed system.

3.1 A one-dimensional coupling problem as motivation

Let u_1, u_2, v, and w be continuous functions on $I = [-1, 1]$, where u_1 and u_2 are real-valued. Further assume $u_2 < 0 < u_1$ to hold on I.

Consider the operator on $\mathcal{L}^2(I)^2$

$$\tilde{A} = \begin{pmatrix} u_1 & v \otimes w^* \\ w \otimes v^* & u_2 \end{pmatrix},$$

where, as in section 1, u_1 and u_2 are to be understood as multiplication operators on $\mathcal{L}^2(I)$ and the tensor product $v \otimes w^*$ is the operator

$$v \otimes w^* : \mathcal{L}^2(I) \to \mathcal{L}^2(I) : f \mapsto \langle f, w \rangle\, v,$$

and similar for $w \otimes v^*$. Then, $\tilde{A}$ is selfadjoint in the natural inner product on $\mathcal{L}^2(I)^2$.

We know $\sigma\tilde{A}_0 = \mathcal{R}u_1 \cup \mathcal{R}u_2$ for the operator

$$\tilde{A}_0 = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

representing the uncoupled systems. Hence, the spectrum of $\tilde{A}_0$ looks as shown in Figure 13.

Note, that as a consequence of u_1 and u_2 being continuous functions,

$$\sigma\tilde{A}_0 = \sigma_{ess}\tilde{A}_0 = u_1(I) \cup u_2(I)$$

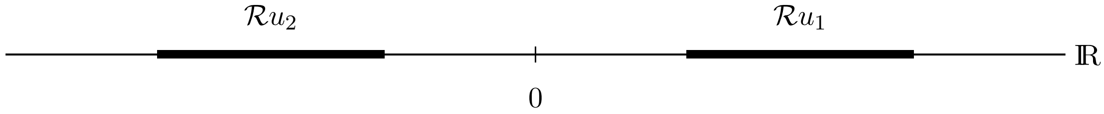

Figure 13: $\sigma\tilde{A}_0$

holds.

Obviously, the coupling operator $B = v \otimes w^*$ is one-dimensional. Hence, $\tilde{B}$ is two-dimensional, and Weyl's Theorem yields $\sigma_{ess}\tilde{A} = \sigma_{ess}\tilde{A}_0$.

We now are interested in the spectrum of $\tilde{A}$ outside $\sigma\tilde{A}_0$. It turns out that $\sigma\tilde{A} \setminus \sigma\tilde{A}_0$ is discrete. Using the above notation, the effect of this coupling is the following:

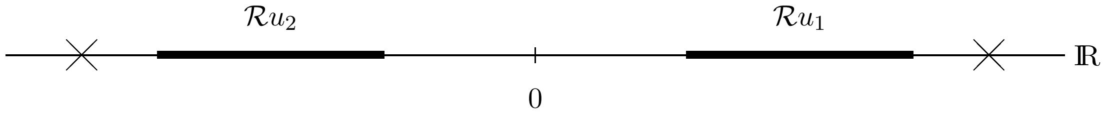

Figure 14: $\sigma\tilde{A}$

Proposition 3.1 *The spectrum $\sigma\tilde{A}$ outside $\sigma\tilde{A}_0$ consists of isolated eigenvalues. There are no more than one in $(-\infty, \min u_2)$, no more than one in $(\max u_1, +\infty)$, but none in $(\max u_2, \min u_1)$ (see Figure 14).*

Proof: The spectrum of $\tilde{A}$ in $\rho\tilde{A}_0$ is completely determined by the spectrum of the pencil $L_1(z) = z - u_1 + v \otimes w^*(u_2 - z)^{-1} w \otimes v^*$ or, equivalently, of $L_2(z) = z - u_2 + w \otimes v^*(u_1 - z)^{-1} v \otimes w^*$ (see e.g. [1, Behauptung 1], or Proposition 2.3).

We now take into account that the coupling is one-dimensional, and so the terms $g(z) = v^*(u_1 - z)^{-1} v$ and $h(z) = w^*(u_2 - z)^{-1} w$ are scalars: $L_1(z) = z - u_1 + h(z) v \otimes v^*$. But this pencil, in return, also characterizes the spectrum of the operator-pencil $z \mapsto \begin{pmatrix} u_1 - z & h(z)v \\ v^* & 1 \end{pmatrix}$, which, in return again, is the spectrum of the pencil $H : z \mapsto -1 + h(z) v^*(u_1 - z)^{-1} v = -1 + g(z)h(z)$.

Now note that g and h are real-valued and increase monotonously on those real intervals where they are holomorphic. Moreover, for $z \in \mathbb{R}$ the following implications hold:

- $z < \mathcal{R}u_1 \Rightarrow g(z) \geq 0$,
- $z > \mathcal{R}u_1 \Rightarrow g(z) \leq 0$.

Finally, we know $\lim_{|z| \to \infty} g(z) = 0$. Analogous facts are valid for h. Hence, we have for the product $g(z)h(z)$:

$$\begin{array}{rcll} z < \min u_2 & \Rightarrow & g(z)h(z) \geq 0, & \text{monotonically increasing from } 0 \\ \max u_2 < z < \min u_1 & \Rightarrow & g(z)h(z) \leq 0 & \\ \max u_1 < z & \Rightarrow & g(z)h(z) \geq 0, & \text{monotonically decreasing to } 0 \end{array}$$

From this behaviour, we immediately see that H has no more than two eigenvalues in $\mathbb{R} \setminus (u_1(I) \cup u_2(I))$, one may be above $\max u_1$, and one below $\min u_2$. ♡

3.2 General compact coupling operators

The treatise of Section 2 suggests that even the cases of more general coupling operators behave quite like the one-dimensional case.

The only questions which we have not covered by the theory yet are evident from the example in Section 3.1:

- Is it generally true that there are no more eigenvalues above and below $\sigma C \cup \sigma A$ than is the rank of the coupling?
- Is it generally true that there are no eigenvalues in a real interval separating σC and σA?

The answer to both questions is "yes". To show this, the special structure of the perturbation $\tilde{B}$ has to be exploited.

The first "yes" for compact B follows from Propositions 2.19 and 2.20 as soon as we have shown:

Proposition 3.2 *The spectrum $\sigma\tilde{B}$ is symmetric with respect to 0, and the multiplicities of eigenvalues z and $-z$ of $\sigma\tilde{B}$ are the same.*

Proof: We show a bit more, namely, that $\tilde{B}$ and $-\tilde{B}$ are unitarily equivalent:

$$\begin{pmatrix} 0 & -B \\ -B^* & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

♡

The second "yes" follows from

Proposition 3.3 *If there exist $a, b \in \mathbb{R}$ such that $C \le a < b \le A$, then $(a, b) \subseteq \rho\tilde{A}$.*

Proof: (See [2, Theorem 2.1].) For $C \le a < z < b \le A$ it holds

$$L(z) = \underbrace{z - A}_{\le z-b} + \underbrace{B(C-z)^{-1}B^*}_{\le 0} \le z - b.$$

Hence, $L(z)$ is boundedly invertible. Then Proposition 2.8 implies that $\tilde{A} - z$ is boundedly invertible, too. (Instead of Proposition 2.8, [1, Folgerung 1] can be applied.) ♡

4 Numerical Examples

The following numerical examples have been computed on an RS6000 computer. To compute the eigenvalues of the symmetric matrices $M(z)$ the NAG-library-routine `f02faf` has been used. Its definition is the same as that of the BLAS `dsyev`.

To compute the integrals in the examples where $A = y \mapsto (t \mapsto t \cdot y(t))$, the NAG-library-routine `d01ahf` has been applied. When the integral has had to be evaluated as Cauchy principal value, the NAG-library-routine `d01aqf` has been tried. It is supposed to compute the Hilbert-transform $\int_I \frac{g(t)}{t-z} dt$ over an interval $I = [t_-, t_+]$. However, these expectations turned out to be too high. The accuracy achieved could not even satisfy the minimal requirement "to give a nice plot". That is why a separate routine had to be written that computes Cauchy principal values as

$$(CP) \int_{t_-}^{t_+} \frac{g(t)}{t-z} dt = \begin{cases} \int_{t_-}^{z} \frac{g(t)+g(2z-t)}{t-z} dt + \int_{2z-t_-}^{t_+} \frac{g(t)}{t-z} dt & \ldots\ a < z \leq \frac{a+b}{2} \\ \int_{z}^{t_+} \frac{g(t)+g(2z-t)}{t-z} dt + \int_{t_-}^{2z-t_+} \frac{g(t)}{t-z} dt & \ldots\ \frac{a+b}{2} < z < b \\ \int_{t_-}^{t_+} \frac{g(t)}{t-z} dt & \ldots\ z \notin (a,b) \end{cases} .$$

This routine calls `d01ahf`.

In ardent opposition to the humble FORTRAN programming language, C has been used. Consequently, the sourcecode is too ugly to be published.

4.1 Examples concerning the one-dimensional extension problem

In the Examples 4.1–4.2, we have set $u(t) = t$ on $I = [-1, +1]$.

Example 4.1 Figure 15 shows a plot of f_{CP} for $v(t) = \cos 2\pi t$.

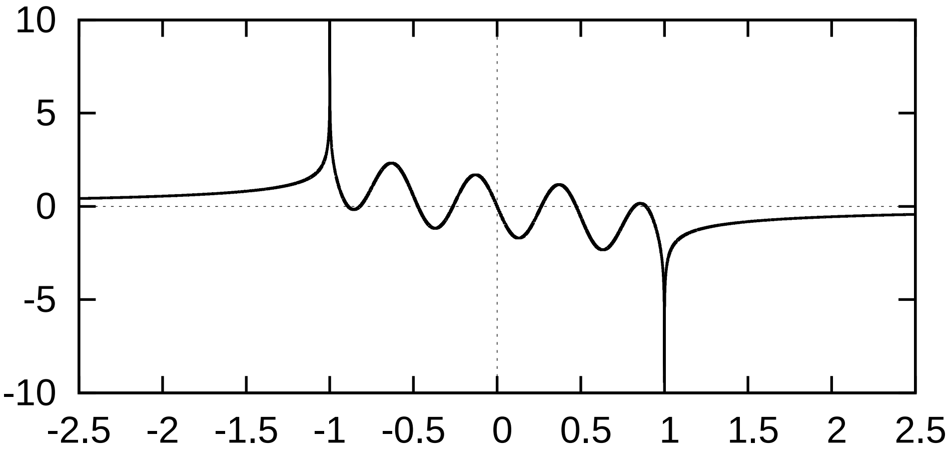

Figure 15: $f_{CP}(z) = (CP) \int_{-1}^{+1} \frac{\cos^2 2\pi t}{t-z} dt$

Example 4.2 Figure 16 shows a plot of f_{CP} for $v(t) = \sin 2\pi t$. Note here that f is continuous now at -1 and 1. The reason is that $\sin(\pm 2\pi) = 0$.

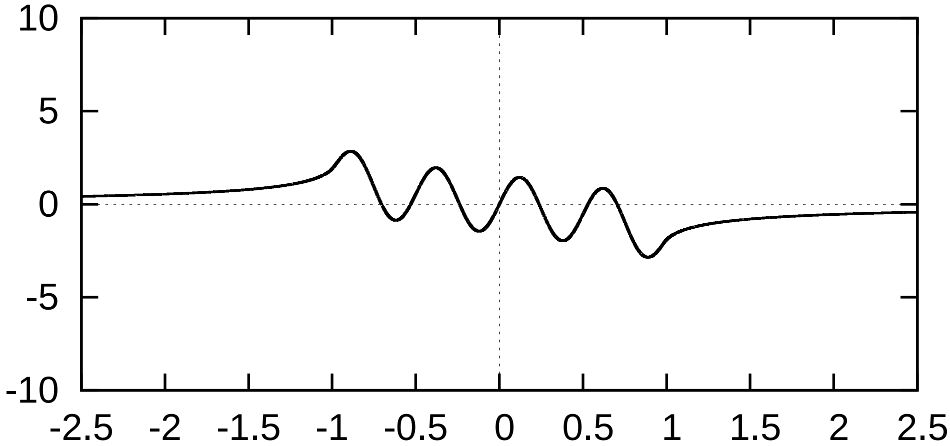

Figure 16: $f_{CP}(z) = (CP)\int_{-1}^{+1} \frac{\sin^2 2\pi t}{t-z} dt$

4.2 Examples concerning the perturbation of a finite dimensional operator

In the Examples 4.3–4.4, we have set[26]

$$A = \begin{pmatrix} 1 &&&&&&& \\ & \frac{5}{6} &&&&&& \\ && \frac{2}{3} &&&&& \\ &&& \frac{1}{2} &&&& \\ &&&& -\frac{1}{2} &&& \\ &&&&& -\frac{2}{3} && \\ &&&&&& -\frac{5}{6} & \\ &&&&&&& -1 \end{pmatrix}, \tag{74}$$

$$S^{-1} = \begin{pmatrix} 2 &&& \\ & 1 && \\ && -1 & \\ &&& -2 \end{pmatrix}, \tag{75}$$

and

$$Q_{\text{Exp 4.3}} = \begin{pmatrix} \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \end{pmatrix}, \qquad Q_{\text{Exp 4.4}} = \begin{pmatrix} \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \end{pmatrix}.$$

[26] We often do not write all zero entries in the matrices.

Example 4.3 Figure 17 shows a plot of the eigenvalues of $M(z)$ when z varies and $Q = Q_{\text{Exp 4.3}}$.

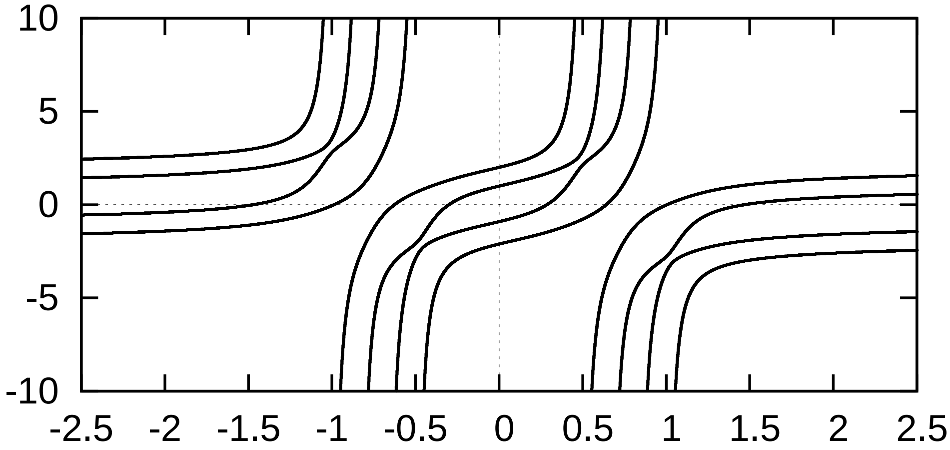

Figure 17: $z \mapsto \sigma_p \left(S^{-1} + Q^*_{\text{Exp 4.3}}(A - z)^{-1} Q_{\text{Exp 4.3}}\right)$

Example 4.4 Figure 18 shows a plot of the eigenvalues of $M(z)$ when z varies and $Q = Q_{\text{Exp 4.4}}$.

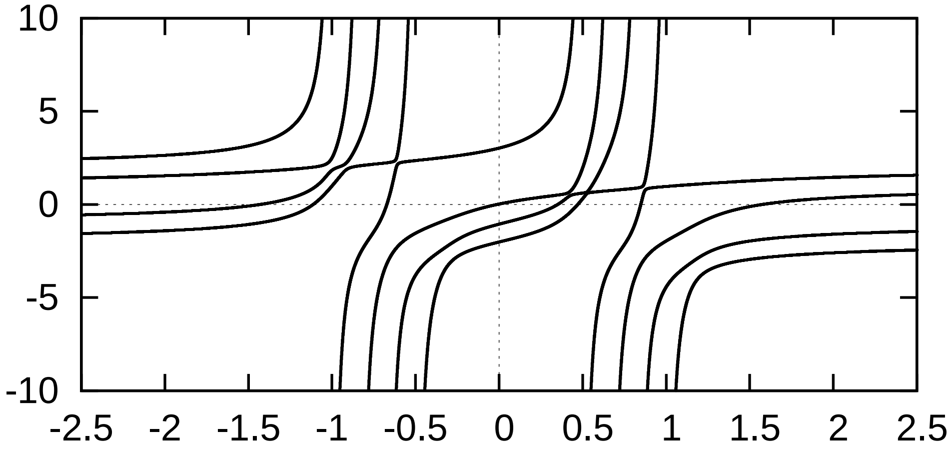

Figure 18: $z \mapsto \sigma_p \left(S^{-1} + Q^*_{\text{Exp 4.4}}(A - z)^{-1} Q_{\text{Exp 4.4}}\right)$

4.3 Examples concerning the finite dimensional perturbation of $y \mapsto (t \mapsto t \cdot y(t))$

The operator A in this section is the same as in Section 4.1: $(Af)(t) = t \cdot f(t)$ on $\mathcal{L}^2([-1, +1])$. The matrix S^{-1} has been chosen to be defined by (75) and hence to be the same as in Section 4.2. Now, we set

$$Q_{\text{Exp 4.5}} = \begin{pmatrix} t \mapsto \cos \pi t & t \mapsto \cos 2\pi t & t \mapsto \cos 3\pi t & t \mapsto \cos 4\pi t \end{pmatrix}$$

and

$$Q_{\text{Exp 4.6}} = \begin{pmatrix} t \mapsto \sin \pi t & t \mapsto \sin 2\pi t & t \mapsto \sin 3\pi t & t \mapsto \sin 4\pi t \end{pmatrix}.$$

Example 4.5 Figure 19 shows the movement of the eigenvalues of $M(z)$ for $Q = Q_{\text{Exp 4.5}}$. If $z \in (-1, 1)$, the integrals to compute $Q^*(A - z)^{-1}Q$ have been evaluated as Cauchy principal values to show potential embedded eigenvalues of M.

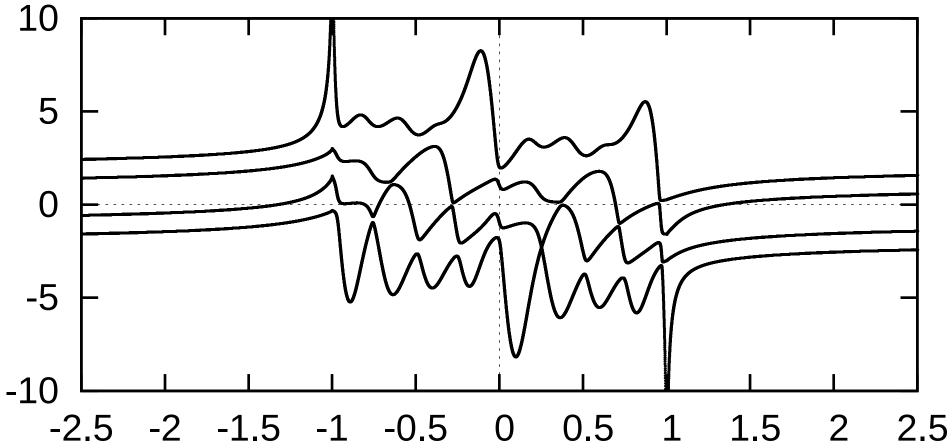

Figure 19: $z \mapsto \sigma_p\left(M(z) = S^{-1} + Q^*_{\text{Exp 4.5}}(A - z)^{-1}Q_{\text{Exp 4.5}}\right)$.

Example 4.6 Figure 20 shows the movement of the eigenvalues of $M(z)$ for $Q = Q_{\text{Exp 4.6}}$. If $z \in (-1, 1)$, Cauchy principal values have been used, confer Example 4.5.

4.4 Examples concerning coupling

In the Examples 4.7–4.8, we have set

$$A = \begin{pmatrix} 1 & & & \\ & \frac{5}{6} & & \\ & & \frac{2}{3} & \\ & & & \frac{1}{2} \end{pmatrix} \quad , \quad C = \begin{pmatrix} -\frac{1}{2} & & & \\ & -\frac{2}{3} & & \\ & & -\frac{5}{6} & \\ & & & -1 \end{pmatrix}.$$

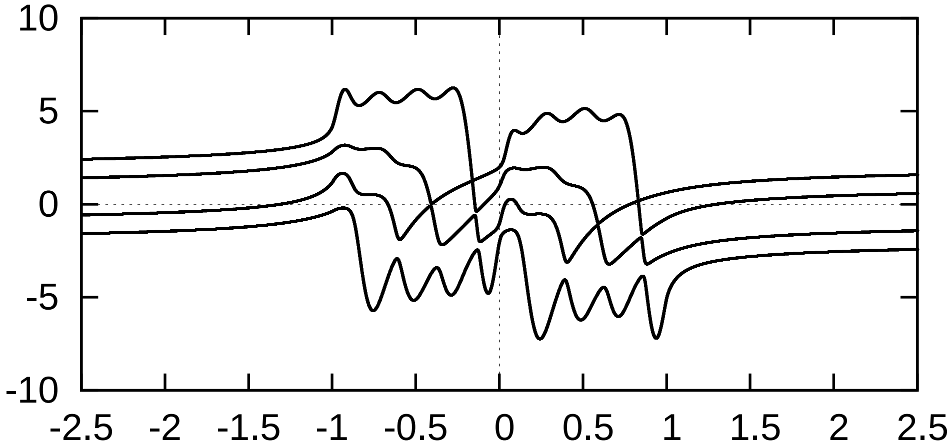

Figure 20: $z \mapsto \sigma_p\Big(M(z) = S^{-1} + Q^*_{\text{Exp 4.6}}(A - z)^{-1}Q_{\text{Exp 4.6}}\Big)$.

Hence, the matrix $\tilde{A}_0 = \begin{pmatrix} A & \\ & C \end{pmatrix}$ is just the matrix defined by (74), which has been studied in Section 4.2, too. The coupling operators have been chosen to be

$$B_{\dots} = T^{-1}J_{\dots}T,$$

where

$$J_{\text{Exp 4.7}} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \qquad J_{\text{Exp 4.8}} = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

and $T = \begin{pmatrix} 4 & -1 & 1 & 1 \\ 1 & 4 & 1 & 0 \\ -1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}$. In order to ease the notation, we understand $\tilde{B}_{\text{Exp 4.7}}$ as the matrix $\tilde{B}$ defined via (73) from $B_{\text{Exp 4.7}}$, and similarly for $B_{\text{Exp 4.8}}$.

Example 4.7 Figure 21 shows a plot of the eigenvalues of $\tilde{A}_0 + b\tilde{B}_{\text{Exp 4.7}}$ when b varies. For $b \to \pm\infty$ only two eigenvalues may leave the interval $\left[\frac{1}{2}, 1\right]$, and this may only occur towards the right. Similarly, two eigenvalues may leave $\left[-1, -\frac{1}{2}\right]$ to the left.

This example impressively shows the consequences of the theoretical considerations done in Section 3. That only two may leave on each side follows from the fact that $B_{\text{Exp 4.7}}$ has rank 2. That they cannot leave the interval into the gap $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is implied by Proposition 3.3.

Example 4.8 Figure 22 shows a plot of the eigenvalues of $\tilde{A}_0 + b\tilde{B}_{\text{Exp 4.8}}$ when b varies.

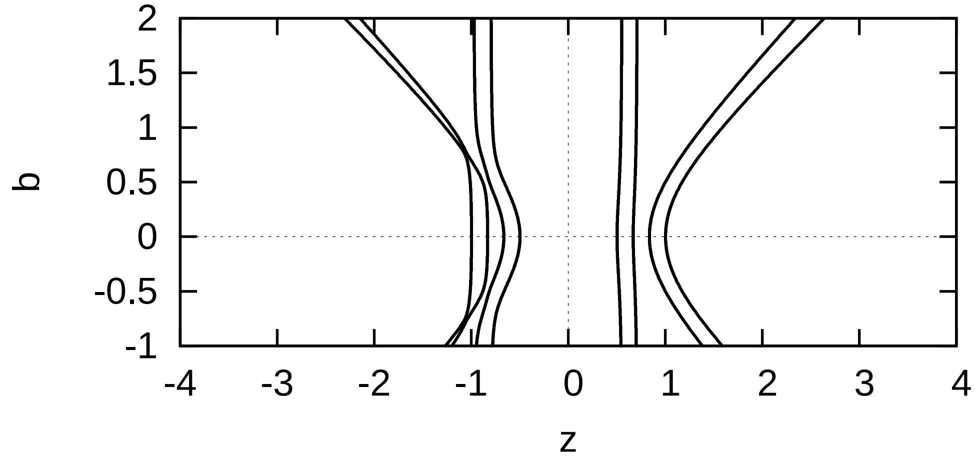

Figure 21: $b \mapsto \sigma_p\left(\tilde{A}_0 + b\tilde{B}_{\text{Exp 4.7}}\right)$

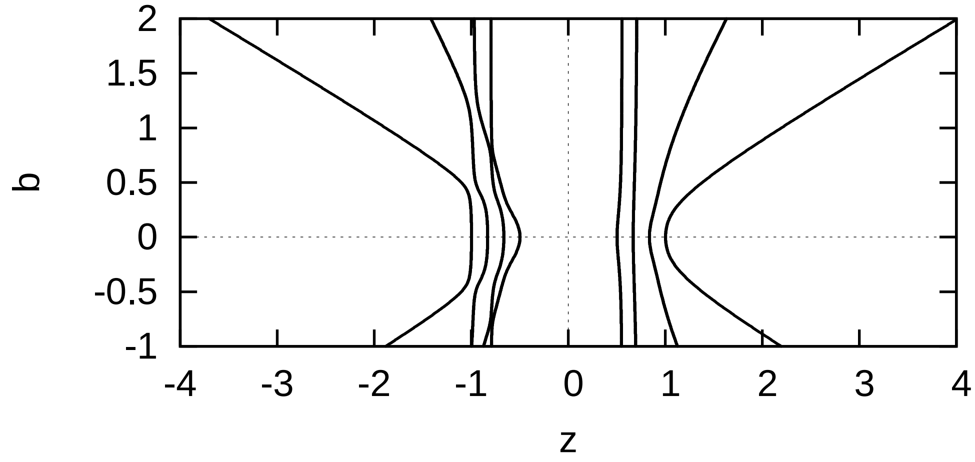

Figure 22: $b \mapsto \sigma_p\left(\tilde{A}_0 + b\tilde{B}_{\text{Exp 4.8}}\right)$

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