Fixed Point Theorems For Set-Valued Maps

Bachelor's thesis in functional analysis

Institute for Analysis and Scientific Computing
Vienna University of Technology

Andreas Widder, July 2009.

Preface

In this thesis we provide an introduction to fixed point theory for set-valued maps. It is not our goal in the present work to give an outline of the status quo of fixed point theory with all its newest achievements, but rather to give a thorough overview of the basic results in this discipline. It then should be possible for the reader to better and easier understand the newest developments as generalizations and continuations of the results we present here.

After a short recollection about classical fixed point theorems for single-valued maps, we will first give an introduction to the theory of set-valued maps. We will generalize the notion of continuity for such maps and also give an example on how one can reduce problems involving set-valued maps to classical maps.
We then show how to achieve fixed point theorems by using two principles:
First we will generalize the well known concept of contractivity of a map. Thus we will achieve results similar to the famous Banach Fixed Point Theorem. We will also adress the question of convergence of fixed points for sequences of set-valued contraction.
Then we will derive fixed point theorems for maps from geometrical properties. There, we will see generalizations of the important fixed point theorems by Schauder and Tychonoff. We will also give a small excursus to the concept of the KKM-principle and give a few examples of results, that can be achieved that way.

Contents

Chapter 1

Classical Fixed Point Theorems

Definition 1.1 *A point x of a space X is called a fixed point of a function $f : X \to X$, if*

$$f(x) = x.$$

Many mathematical problems, originating from various branches of mathematics, can be equivalently formulated as fixed point problem, meaning that one has to find a fixed point of some function F. Fixed point theorems provide sufficient conditions under which there exists a fixed point for a given function, and thus allow us to guarantee the existence of a solution of the original problem. Because of the wide variety of uses, fixed point theorems are of great interest in many mathematical disciplines.
There are four general principles by which results are achieved:

1. Contractivity: the function under consideration is Lipschitz-continuous with a Lipschitz-Constant < 1.

2. Geometry: the domain and/or the range of the function has certain geometrical properties (e.g. compactness or convexity).

3. Homotopy: there exists a function homotopic to a given function and a homotopy with certain properties

4. Set-Theory: the viewed space is ordered (and may have more set-theoretical properties) and the function satisfies relations between a point and its image regarding this order.

Of course these are not all means by which fixed point theorems can be achieved. In some cases one can generalize these concepts, e.g. the conctractivity principle can be extended to non-expansive cases, meaning the Lipschitz-Constant is allowed to be ≤ 1. Fixed point theorems may also arise as byproducts of, at first sight, unrelated concepts. An important example for this would be the fixed point index which arises in topological degree theory.

In the following, we will give (without proofs) basic examples for a fixed point theorem belonging to 1-4.

1.1 The Banach Fixed Point Theorem

This is probably the most well-known fixed-point theorem. This theorem is oustanding among fixed point thoerems, because it not only guarantees existence of a fixed point, but also its uniqueness, an approximative method to actually find the fixed point, and a priori and a posteriori estimates for the rate of convergence.

Definition 1.2 *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces. A function* $f : X \to Y$ *is called Lipschitz, if there exists a* $l > 0$ *such that for all* $x_1, x_2 \in X$ *the inequality*

$$d_Y(f(x_1), f(x_2)) \leq l d_X(x_1, x_2)$$

holds. The number $k = \inf\{l \in \mathbb{R} | \forall x_1, x_2 \in X,\, d_Y(f(x_1), f(x_2)) \leq l d_X(x_1, x_2)\}$ *is called the Lipschitz constant of* f*.*
If $k < 1$*, then* f *is called a contraction.*

Theorem 1.3 (Banach Fixed Point Theorem) *Let* (X, d) *be a complete metric space and let* f *be a contraction with Lipschitz constant* k*. Then* f *has an unique fixed point.*
In more deatil: If $x_0 \in X$*, then the sequence* $\{x_n\}_{n=0}^{\infty}$ *with*

$$x_{n+1} = f(x_n), \qquad n \geq 0,$$

converges to a uniquely defined fixed point ζ *of* f*, and we have*

$$d(x_n, \zeta) \leq k^n (1-k)^{-1} d(x_1, x_0) \quad \text{and} \quad d(x_{n+1}, \zeta) \leq k(1-k)^{-1} d(x_n, x_{n+1}).$$

1.2 The Schauder Fixed Point Theorem

The fixed point theorem by Schauder is one of the most basic ones, when it comes to dealing with geometrical properties. In fact, many other fixed point theorems in this category are proven by reducing it to the Schauder Theorem.
To state it, one needs the following definition:

Definition 1.4 *Let* X *and* Y *be Banach spaces. A function* $f : X \to Y$ *is called compact, if* f *maps bounded sets to relatively compact sets.*

Theorem 1.5 (Schauder Fixed Point Theorem) *Let* X *be a Banach space,* M *be a nonempty convex subset of* X*, and* $f : M \to M$ *be continuous. If furthermore*

- M *is closed and bounded and* f *compact*

or

- M *is compact,*

then f *has a fixed point.*

`Remark:` This theorem stays true, if we interchange 'Banach space' and 'locally convex topological vector space' in the definition and the theorem. The result is then known as the Tychonoff Fixed Point Theorem.

1.3 The Schaefer Fixed Point Theorem

Definition 1.6 *Two continuous functions* f *and* g *from a topological space* X *to a topological space* Y *are called homotopic, if there exists a continuous function* $H : X \times [0,1] \to Y$ *with* $H(x,0) = f(x)$ *and* $H(x,1) = g(x)$, $x \in X$*.*
The function H *is called a homotopy.*

There are two kinds of theorems in this category: Some deal with the question, under which conditions the existence of a fixed point is invariant under a homotopy, while others look for conditions under which a fixed point emerges. The Schaefer Theorem is of the first kind.

Theorem 1.7 (Schaefer Fixed Point Theorem) *Let X be a normed space and let $f : X \to X$ be continuous and compact. Define $H : X \times [0,1] \to X$ as $H(x, \alpha) = \alpha f(x)$. If, for each $\alpha \in (0,1)$, the set $\{x \in X : x = H(x, \alpha)\}$ is bounded, then f has a fixed point.*

Homotopic fixed point theorems and geometrical ones are often connected in some way. For example, the Schaefer Theorem can be derived from the Schauder Theorem using only geometrical means. On the other hand, the Schauder Theorem can also be derived from the Leray-Schauder fixed point index by homotopical means.

1.4 The Bourbaki-Kneser Fixed Point Theorem

Fixed point theorems of this, fourth, category are applied to progressive or regressive functions (meaning $x \leq f(x)$ or $f(x) \leq x$, respectively) or monotone functions. An interesting feature of these theorems is, that there are generally no further restrictions to the function. The Bourbaki-Kneser theorem is the most basic one of this type. It follows immediately from Zorn's Lemma.

Theorem 1.8 (Bourbaki-Kneser Fixed Point Theorem) *Let M be an ordered set and let $f : M \to M$ satisfy $x \leq f(x)$ for all $x \in M$. If every totally ordered subset of M has a supremum, then f has a fixed point.*

Apparently, this theorem remains true, if '$\leq$' is replaced by '$\geq$' and 'supremum' by 'infimum'. It is interesting to note that the Bourbaki-Kneser Fixed Point Theorem can be used to show the equivalence of Zorn's Lemma and the Axiom of Choice.

Chapter 2

Set-valued Maps

For a set Y, denote by $\mathscr{P}(Y)$ the power set of Y.
By a set-valued map, we mean a map

$$T : X \to \mathscr{P}(Y)$$

which thus assings to each point $x \in X$ a subset $T(x) \subseteq Y$. Note that a map $S : X \to Y$ can be identified with a set-valued map $S' : X \to \mathscr{P}(Y)$ by setting $S'(x) = \{S(x)\}$. We will refer to a map $S : X \to Y$ as a single-valued map.
For $T : X \to \mathscr{P}(Y)$ and $M \subseteq X$ we define

$$T(M) = \bigcup_{x \in M} T(x),$$

and the graph $G(T)$ of T will be the set

$$G(T) = \{(x, y) : x \in X, y \in T(x)\}.$$

Set-valued maps are important objects for many applications, for example in game theory or mathematical economics. They also often arise when studying certain optimization problems or when dealing with variational inequalities (for examples see Zeidler [Z1]).

2.1 Continuity

We generalize the concept of continuity to set-valued maps.

Definition 2.1 *Let X and Y be topological spaces and $T : X \to \mathscr{P}(Y)$ a set-valued map.*

1. *T is called upper semi-continuous, if for every $x \in X$ and every open set V in Y with $T(x) \subseteq V$, there exists a neighbourhood $U(x)$ such that $T(U(x)) \subseteq V$.*

2. *T is called lower semi-continuous, if for every $x \in X$, $y \in T(x)$ and every neighbourhood $V(y)$ of y, there exists a neighbourhood $U(x)$of (x) such that*

$$T(u) \cap V(y) \neq \emptyset, \quad u \in U(x).$$

3. *T is called continuous if it is both upper semi-continuous and lower semi-continuous.*

Using simple topological arguments, these definitions can be equivalently stated in a simpler formulation. The preimage $T^{-1}(A)$ of a set $A \subseteq Y$ under a set-valued map T is defined as

$$T^{-1}(A) = \{x \in X : T(x) \cap A \neq \emptyset\}$$

Note that, unlike for single valued maps, the inclusion $T(T^{-1}(A)) \subseteq A$ need not hold. However, unless $T^{-1}(A) = \emptyset$, we have $T(T^{-1}(A)) \cap A \neq \emptyset$.

Proposition 2.2 *Let X and Y be topological spaces and $T : X \to \mathscr{P}(Y)$ a set-valued map.*

1. *T is upper semi-continuous if and only if $T^{-1}(A)$ is closed for all closed sets $A \subseteq Y$.*
2. *T is lower semi-continuous if and only if $T^{-1}(A)$ is open for all open sets $A \subseteq Y$.*

Proof:

1. Suppose T is upper semi-continuous and $A \subseteq Y$ closed. Choose $x \in (T^{-1}(A))^C$, then $T(x) \subseteq A^C$. Since A^C is open, there exists a neighbourhood $U(x)$ of x, such that $T(U(x)) \subseteq A^C$. Therefore $U(x) \subseteq (T^{-1}(A))^C$ and it follows that $T^{-1}(A)$ is closed.
   Conversely suppose $T^{-1}(A)$ is closed for all closed $A \subseteq Y$. Let $x \in X$ $V \subseteq Y$ be open with $T(x) \subseteq V$. Then V^C is closed and by assumption so is $T^{-1}(V^C)$. Moreover $x \notin T^{-1}(V^C)$. Hence there exists a neighbourhood of x with $U(x) \subseteq (T^{-1}(V^C))^C$. This neighbourhood apparently satisfies $T(U(x)) \subseteq V$.

2. Suppose T is lower semi-continuous and $A \subseteq Y$ open. Assume that $x \in T^{-1}(A)$, and choose $y \in T(x) \cap A$. Since A is open, there exists a neighbourhood of y with $V(y) \subseteq A$. Because of the lower semi-continuity of T, there exists a neighbourhood $U(x)$ of x with $T(u) \cap V(y) \neq \emptyset$ for all $u \in U(x)$. This means $U(x) \subseteq T^{-1}(V(y)) \subseteq T^{-1}(A)$. It follows that $T^{-1}(A)$ is open.
   Conversely suppose $T^{-1}(A)$ is open for every open set $A \subseteq Y$. Assume that $x \in X$, $y \in T(x)$ and $V(y)$ is an open neighbourhood of y. Then $T^{-1}(V(y))$ is open and $x \in T^{-1}(V(y))$. Therefore there exists a neighbourhood $U(x)$ of x with $U(x) \subseteq T^{-1}(V(y))$. It follows that $T(u) \cap V(y) \neq \emptyset$ for all $u \in U(x)$.

□

Proposition (2.2) shows that for single-valued maps lower semi-continuity is identical with continuity in the classical sense
When only dealing with subsets of the spaces X and Y the terms open and closed naturally mean open and closed in the induced topologies.

In some special cases, the graph of a map can be used to characterize semi-continuity. We will give an example:

Theorem 2.3 *Let X and Y be compact spaces and $T : X \to \mathscr{P}(Y)$ a set-valued mapping. Assume that $T(x)$ is closed for all $x \in X$. Then T is upper semi-continuous if and only if $G(T)$ is closed in $X \times Y$.*

Proof: Suppose T is upper semi-continuous and choose $(x, y) \in X \times Y$ with $(x, y) \notin G(T)$. Then y does not belong to the closed set $T(x)$. Since Y is compact there exist two neighbourhoods $V_1$of y and V_2 of $T(x)$ with $V_1 \cap V_2 = \emptyset$. Since T is upper semi-continuous, we can find a neighbourhood U of x with $T(U) \subseteq V_2$. Thus $U \times V_1$ is a neighbourhood of (x, y) with $U(\times V_1) \cap G(T) = \emptyset$, and therefore $G(T)$ is closed.

Conversely assume that $G(T)$ is closed and that T is not upper semi-continuous at a point $x \in X$. Choose an open set $V \subsetneq Y$ containing $T(x)$. Note that because T is not uppersemi-continuous we can find such a set V. Let $\{U_i\}_{i\in I}$ be the family of all neighbourhoods of x. Because T is not upper semi-continuous we have for every $U_j \in \{U_i\}_{i\in I}$

$$G(T) \cap (\overline{U_j} \times (Y \backslash V)) \neq \emptyset$$

Since $G(T)$, $\overline{U_j}$ and $Y\backslash V$ are all closed the intersection is also closed and when looking at it for different $U_j \in \{U_i\}_{i\in I}$ it is obvious, that this intersections have the finite intersection property. Because $X \times Y$ with the product topology is compact it follows, that

$$\bigcap_{i\in I} G(T) \cap (\overline{U_i} \times (Y \backslash V)) \neq \emptyset.$$

The only point in the intersection of all U_i is x, therefore there exists $y \in Y$, such that (x, y) is in this intersection. Then $y \in T(x) \cap Y\backslash V$. But this contradicts our choice of V as a neighbourhood of $T(x)$. Therefore T must be upper semi-continuous. □

2.2 Selection Theorems

If $T : X \to \mathscr{P}(Y)$ is a set-valued mapping, we call a single-valued mapping $t : X \to Y$ a selection of T, if

$$t(x) \in T(x) \quad \text{for all } x \in X.$$

The existence of a selection is obviously equivalent to the fact, that $T(x) \neq \emptyset$ for all $x \in X$. Often it is practical to reduce a problem involving set-valued maps to a question about single-valued maps. Hence, selections are an important tool, and it is of interest to show existence of selections with additional properties. We will give one example of a statement of this kind.

Michael's Selection Theorem

This selection theorem guarantees the existence of a continuous selection.
Recall, that a topological space is called paracompact if every open cover has a locally finite refinement, where a collection of sets is called locally finite, if every point of a set has a neighbourhood which intersects at most finitely many of these sets. Important examples for paracompact spaces are compact spaces, metric spaces and locally compact spaces with a countable basis.
Paracompactness is an important property since it guarantees existence of partitions of unity. Recall that the family $\{f_i\}_{i\in I}$ of continuous mappings is called a partition of unity subordinate to the open covering $\{O_j\}_{j\in J}$, if for every i, there is a j such that $\text{supp}(f_i) \subseteq O_j$. Furthermore, for all $x \in X$, $0 \leq f_i(x) \leq 1$ and $\sum_i f_i(x) = 1$ hold and for fixed x there is a open set O_x with $x \in O_x$ such that at most finitely many f_i are not identically zero on O_x.

Theorem 2.4 *Let $T : X \to \mathscr{P}(Y)$ be a lower semi-continuous set-valued map. If*

- *X is paracompact,*
- *Y is a Banach-Space,*
- *$T(x)$ is nonempty, closed and convex for all $x \in X$,*

then there exists a continuous selection $t : X \to Y$ *of* T.

Proof: As a first step we show, that for each $\epsilon > 0$ there exists a continuous map $f : X \to Y$ such that

$$d(f(x), T(x)) < \epsilon, \quad x \in X. \tag{2.1}$$

where d is the distance on Y induced by its norm.

Fix $\epsilon > 0$ and choose a selection $m : X \to Y$. Let $B_\epsilon(m(x))$ denote the open ball with radius ϵ and center $m(x)$. Since T is lower semi-continuous, for each $x \in X$ there exists an open neighbourhood $U(x)$ of x such that

$$T(u) \cap B_\epsilon(m(x)) \neq \emptyset, \quad u \in U(x). \tag{2.2}$$

Since X is paracompact there exists a partition of unity $\{f_\alpha\}$ subordinate to the open covering $\{U(x)\}_{x \in X}$. Set

$$f(x) = \sum_\alpha f_\alpha(x) m(x_\alpha),$$

then f is a continuous function of X into Y. If $f_\alpha(x) > 0$ for some α, then $x \in U(x_\alpha)$ and therefore by (2.2)

$$m(x_\alpha) \in T(x) + B_\epsilon(0).$$

Since $T(x) + B_\epsilon(0)$ is convex, and $f(x)$ a finite convex combination of elements from $f_\alpha(x)m(x_\alpha)$ with $f_\alpha(x) > 0$, we have $f(x) \in T(x) + B_\epsilon(0)$. Hence $d(f(x), T(x)) \leq \epsilon$, i.e. f satisfies (2.1).

In the second step, we construct the requested selection. Set $\epsilon_n = 2^{-n}$. We will define inductively a sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous mappings $f_n : X \to Y$ with

$$d(f_n(x), T(x)) < \epsilon_n, \quad x \in X,\ n = 1, 2, \ldots \tag{2.3}$$

$$d(f_n(x), f_{n-1}(x)) < \epsilon_{n-1}, \quad x \in X,\ n = 2, 3, \ldots \tag{2.4}$$

As we showed in the first step, there exists f_1 with $d(f_1(x), T(x)) < 1/2$, $x \in X$.

Assume that $n \geq 2$ and we already have constructed $f_1, \ldots, f_{n-1}$. For each $x \in X$ we define

$$G(x) = (f_{n-1}(x) + B_{\epsilon_{n-1}}(0)) \cap T(x)$$

By the induction hypothesis, $G(x)$ is not empty. Since $T(x)$ convex, so is $G(x)$. Further $G : X \to \mathscr{P}(Y)$ is lower semi-continuous, since f_{n-1} is continuous and T lower semi-continuous. So we can apply the first part of our proof also to G, since the only additional property of T is, that $T(x)$ is closed, which was not used in that argument. Therefore there exists a continuous map $f_n : X \to Y$ such that (2.3) holds. By construction f_n also satisfies (2.4).

Since $\sum \epsilon_n$ converges, $\{f_n\}_{n \in \mathbb{N}}$ is a uniform Cauchy sequence and hence converges to a continuous map $t : X \to Y$. Since $T(x)$ is closed, t is a selection of T. $\square$

Application to Differential Inclusions

We now show an easy application of Michael's theorem and consider the initial value problem

$$x'(t) \in F(x(t), t), \quad x(t_0) = x_0 \tag{2.5}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. $F(x,t)$ gives the set of all possible velocities of the system at a time t. Such differential inclusions arise when modeling systems for which we have no complete description. We are looking for a solution $x : \mathbb{R} \to \mathbb{R}^n$.

Theorem 2.5 (Generalized Peano Theorem) *Suppose we have $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, where $(x_0, t_0) \in U$. Let*

$$F : U \subseteq \mathbb{R}^{n+1} \to \mathscr{P}(\mathbb{R}^n)$$

be a lower semi-continuous map such that $F(x,t)$ is a nonempty closed and convex set in $\mathbb{R}^n$ for all $(x,t) \in U$. Then the initial value problem (2.5) has a C^1-solution $x = x(t)$ in a neighbourhood of t_0.

Proof: By Michael's Selection Theorem there exists a continuous selection $f : U \to \mathbb{R}^n$ of F. By the classical Peano Theorem, the initial value problem

$$x'(t) = f(x(t), t), \quad x(t_0) = x_0$$

has (locally at t_0) a solution $x(t)$. This function solves (2.5). □

Chapter 3

Fixed Point Theorems for set-valued Maps. Contraction Principle

Definition 3.1 *A point x of a space X is called a fixed point of a set-valued map $T : X \to \mathscr{P}(X)$, if*

$$x \in T(x)$$

For single-valued maps, we presented four principles, by which fixed point theorems can be achieved. For set-valued maps we will only consider fixed point theorems achieved due to contractivity or by geometrical means.
Although there exist a few results by homotopic means, they are applicable only in very special settings. For more details, see for example Sim, Xu, Yuan [SXY].
Achieving fixed point theorems for set-valued maps by set theoretic means is also more complicated. The approach shown in the introduction certainly works only for a map from a space X into itself. Still, some results were achieved when looking for common fixed points of two set-valued maps, but only with additional assumptions about compatibility with metric values. For an example, see Beg, Butt [BB].

3.1 Hausdorff-Metric

The key to the classical Banach fixed point theorem is that one is working in a complete metric space. To get an analogous result for set-valued mappings, we have to equip the powerset of a metric space with a metric.
Let (X, d) be a metric space. For each two nonempty elements M and N of $\mathscr{P}(X)$ we define

$$\begin{aligned} D(x, N) &= \inf \{d(x, y) : y \in N\} \in [0, \infty), \quad x \in M, \\ D(M, N) &= \sup \{D(x, N) : x \in M\} \in [0, \infty]. \end{aligned}$$

Let us remark, that $D(M, N)$ being small thus means, that each point of M is close to some point of N. The value of $D(M, N)$ will certainly be finite, if M and N are bounded.

Lemma 3.2 *Let (X, d) be a metric space and let M, N, and Q be nonempty, bounded elements of $\mathscr{P}(X)$. Then*

1. $D(M, N) = 0$ *if and only if* $M \subseteq \overline{N}$,
2. $D(M, N) \leq D(M, Q) + D(Q, N)$.

Proof: The first assertion follows directly from the definition of $D(M, N)$. We come to the proof of 2.
We have

$$d(m, n) \leq d(m, q) + d(q, n), \quad m \in M, n \in N, q \in Q$$

By taking the infimum over $n \in \mathbb{N}$, the inequality

$$D(m, N) \leq d(m, q) + D(q, N), \quad m \in M, q \in Q$$

follows. By taking the supremum over all $q \in Q$ we obtain

$$D(m, N) \leq D(m, Q) + D(Q, N), \quad m \in M$$

and by taking the supremum over all $m \in M$ the assertion follows. □

In general the equality $D(M, N) = D(N, M)$ need not hold, thus we need to symmetrize.

Definition 3.3 *The Hausdorff semi-metric on the family of all nonempty bounded subsets of a metric space is defined as*

$$\delta(M, N) = \max\{D(M, N), D(N, M)\}.$$

Theorem 3.4 *On the family of all nonempty bounded and closed sets of a metric space the Hausdorff semi-metric is a metric.*

Proof: Symmetry is built into the definition of δ, the equivalence

$$\delta(M, N) = 0 \Leftrightarrow M = N$$

holds since we restrict ourselves to closed subsets of X, and the triangle inequality since

$$D(M, N) \leq D(M, Q) + D(Q, M) \leq \delta(M, Q) + \delta(Q, N),$$
$$D(N, M) \leq D(N, Q) + D(Q, M) \leq \delta(N, Q) + \delta(Q, M).$$

□

In the following, we will denote by $\mathrm{CB}(X)$ the family of all nonempty closed and bounded subsets of the metric space (X, d). Note, that different metrics on X result in different Hausdorff metrics on $\mathrm{CB}(X)$. The following theorem shows, that the completeness of the metric space (X, d) is preserved by going to the space $(\mathrm{CB}(X), \delta)$.

Theorem 3.5 *If (X, d) is a complete metric space, then $(CB(X), \delta)$ is a complete metric space, where δ is the Hausdorff metric induced by d.*

Proof: Let $\{M_n\}_{n \in \mathbb{N}}$ be a sequence of bounded and closed subsets of X and suppose it is a Cauchy sequence with respect to δ. We define

$$M = \bigcap_{n=1}^{\infty} (\overline{\bigcup_{m=n}^{\infty} (M_m)})$$

and show that

$$\lim \delta(M_n, M) = 0.$$

Boundedness of M follows easily from the definition and the fact, that $\{M_n\}$ is Cauchy sequence. Hence, $M \in \mathrm{CB}(X)$.
For a given $\epsilon > 0$, choose a number N_ϵ such that

$$\delta(M_m, M_n) < \epsilon, \quad n, m \geq N_\epsilon$$

We claim that

$$\delta(M, M_{N_\epsilon}) \leq 2\epsilon. \tag{3.1}$$

Once this claim has been established, we obtain

$$\delta(M, M_n) \leq \delta(M, M_{N_\epsilon}) + \delta(M_{N_\epsilon}, M_n) < 2\epsilon + \epsilon = 3\epsilon, \quad n \geq N_\epsilon$$

and the assertion of the theorem will follow.

To prove (3.1) we need to show that:

1. $D(x, M_{N_\epsilon}) \leq 2\epsilon$ for all $x \in M$
2. $D(y, M) \leq 2\epsilon$ for all $y \in M_{N_\epsilon}$.

We first consider the set

$$A_i = \{x \in X : D(x, M_i) \leq \epsilon\}.$$

Note that A_i is closed. Since $\{M_n\}$ is a Cauchy sequence, we have $M_n \subseteq A_{N_\epsilon}$ for $n \geq N_\epsilon$. By the definition of M

$$M \subseteq \overline{\bigcup_{i \geq N_\epsilon} A_i},$$

and therefore $M \subseteq A_{N_\epsilon}$ which implies 1.

Next consider the sequence

$$n_i = N_{\epsilon/2^i}, \quad i \geq 0.$$

Since $\{M_n\}$ is a Cauchy sequence, we can inductively construct a sequence m_{n_i} with

- $m_{n_i} \in M_{n_i}, \quad i \geq 0,$
- $m_{n_0} = y,$
- $d(m_{n_i}, m_{n_{i-1}}) \leq \epsilon/2^{i-1}, \quad i \geq 1.$

The sequence $\{m_{n_i}\}$ is a Cauchy sequence, because

$$d(m_{n_p}, m_{n_q}) \leq \epsilon \sum_{i=q+1}^{p} 1/2^{i-1}\epsilon/2^{q-1}, \quad p > q.$$

Since (X, d) is complete there exists the limit $m = \lim m_{n_i}$, and from the definition of M we have $m \in M$. Since

$$d(m_{n_p}, m_{n_0}) = d(m_{n_p}, y) \leq 2\epsilon$$

by taking the limit we get

$$d(m, y) \leq 2\epsilon$$

From this 2. follows. Together, we have established our claim (3.1). □

3.2 Contractive Maps

We can extend the existence part of the Banach Fixed Point Theorem to set-valued contraction mappings.

Theorem 3.6 *Let (X, d) be a complete metric space, and let $T : X \to CB(X)$ be a set-valued contraction where $CB(X)$ is endowed with the Hausdorff metric induced by d. Then T has a fixed point.*

Proof: Denote by k the Lipschitz constant of T.
We are going to construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ with

$$d(x_n, x_{n+1}) \leq \delta(T(x_{n-1}), T(x_n)) + k^n, \quad x_n \in T(x_{n-1}),\ n \geq 1. \tag{3.2}$$

For x_0 we choose an arbitrary point of X, and for x_1 an arbitrary point of $T(x_0)$. Then we can find a point $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq \delta(T(x_0), T(x_1)) + k$$

holds. By repating this appropriately, we can construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ with the desired properties.
From (3.2) we obtain the estimate

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \delta(T(x_{n-1}), T(x_n)) + k^n \\ &\leq kd(x_{n-1}, x_n) + k^n \\ &\leq k(\delta(T(x_{n-2}), T(x_{n-1})) + k^{n-1}) + k^n \\ &\leq k^2 d(x_{n-2}, x_{n-1}) + 2k^n \leq \cdots \\ &\leq k^n d(x_0, x_1) + nk^n \end{aligned}$$

This implies that, for any $n, m \geq 0$,

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq \sum_{l=n}^{n+m-1} (k^l d(x_0, x_1) + lk^l). \end{aligned}$$

Therefore $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, it converges to some point $x \in X$. Because T is a Lipschitz mapping, this implies that $T(x_n)$ converges to $T(x)$. This means that for each $\epsilon > 0$ there is an N_ϵ such that for all $n \geq N_\epsilon$

$$\delta(T(x_n), T(x)) < \epsilon.$$

Assume $x \notin T(x)$. Then

$$D(x, T(x)) = \lambda$$

for some $\lambda > 0$. Choose $\epsilon = \lambda/2$. Since $x_{n+1} \in T(x_n)$, for all $n \geq N_\epsilon$ we have

$$d(x_n, x) > \lambda/2.$$

This is a contradiction to $x_n \to x$, and thus we have $x \in T(x)$. □

We now give a second theorem of this kind, in which the assumptions about the contractivity of T are loosened some. In order to formulate this result, we need the two following definitions:

Definition 3.7 *A complete metric space (X, d) is called ϵ-chainable, if, for any points $a, b \in X$ and fixed $\epsilon > 0$, there exists a finite set of points*

$$a = x_0, x_1, \cdots, x_{n-1}, x_n = b$$

such that

$$d(x_{i-1}, x_i) \leq \epsilon \quad i = 1, 2, 3, \cdots, n.$$

Each such set is called an ϵ-chain.

Definition 3.8 *Let (X, d) be a complete metric space, and let $\epsilon > 0$ and $\lambda \in [0, 1)$. A set-valued mapping $T : X \to CB(X)$ is called an (ϵ, λ)-uniformly local contraction if*

$$\delta(T(x), T(y)) \leq \lambda d(x, y)$$

whenever $d(x, y) < \epsilon$.

Theorem 3.9 *Let (X, d) be an ϵ-chainable space and $T : X \to CB(X)$ a set-valued (ϵ, λ)-uniformly local contraction. Then T has a fixed point.*

Proof: We define a new metric on X by

$$\widetilde{d}(x, y) = \inf \left\{ \sum_{i=1}^{n} d(x_{i-1}, x_i) \right\}$$

where $x_0 = x$, $x_n = y$ and the infimum is taken over all ϵ-chains $(x_0, \cdots, x_n)$.
It is easy to see, that $\widetilde{d}$ is indeed a metric on X. Since $d(x, y) \leq \widetilde{d}(x, y)$ holds trivially and $d(x, y) = \widetilde{d}(x, y)$ if $d(x, y) < \epsilon$, $(X, \widetilde{d})$ is also a complete metric space. We denote by δ_ϵ the Hausdorff metric induced by $\widetilde{d}$.
Let $x, y \in X$ and consider an ϵ-chain

$$x = x_0, x_1, \cdots, x_{n-1}, x_n = y.$$

We have $d(x_{i-1}, x_i) < \epsilon$ for $i = 1, 2, \cdots, n$, and since T is an (ϵ, λ)-uniformly locally contraction we have $\delta(T(x_{i-1}), T(x_i)) \leq \lambda d(x_{i-1}, x_i)$. It follows that

$$\begin{aligned} \delta_\epsilon(T(x), T(y)) &\leq \sum_{i=1}^{n} \delta_\epsilon(T(x_{i-1}), T(x_i)) \\ &= \sum_{i=1}^{n} \delta(T(x_{i-1}), T(x_i)) \\ &\leq \lambda \sum_{i=1}^{n} d(x_{i-1}, x_i) \end{aligned}$$

Taking the infimum over all possible ϵ-chains we obtain

$$\delta_\epsilon(T(x), T(y)) \leq \lambda \widetilde{d}(x, y)$$

Therefore T is a contraction mapping with respect to the metrices $\widetilde{d}$ and δ_ϵ. Hence, we can apply Theorem (3.6). □

3.3 Sequences of Contractive Maps

We now want to consider the following problem: Assume we have a sequence of contractive mappings T_i and a sequence x_i, where for every i the point x_i is a fixed point of T_i. If the sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to T_0, does $\{x_i\}_{i\in\mathbb{N}}$ (or at least a subsequence) converge to a fixed point of T_0?
Without further assumptions, this is not the case: Suppose $(X, d) = (\mathbb{R}, d)$ with a metric d so $\mathbb{R}$ is bounded, $T_n(x) = \mathbb{R}$ and $x_n = n$. Then every x_n is a fixed point, but there is no converging subsequence.
For this reason we restrict ourselves to set-valued mappings with compact values. We will denote by $\mathrm{K}(X)$ the family of all nonempty compact subsets of X. Naturally $\mathrm{K}(X) \subseteq \mathrm{CB}(X)$, and therefore $(\mathrm{K}(X), \delta)$ is a metric space. $\mathrm{K}(X)$ preserves many important properties of X:

Proposition 3.10 *Let δ be the Hausdorff metric on $K(X)$ induced by the metric d of (X, d).*

1. *If (X, d) is a complete metric space, then $(K(X), \delta)$ is a complete metric space.*
2. *If (X, d) is a compact metric space, then $(K(X), \delta)$ is a compact metric space.*
3. *If (X, d) is a locally compact metric space, then $(K(X), \delta)$ is a locally compact metric space.*

Proof: 1) Let $\{M_n\}$ be a Cauchy sequence in $\mathrm{K}(X)$. Since this is naturally also a Cauchy sequence in $\mathrm{CB}(X)$, it follows from Theorem (3.5), that the limit $\lim M_n$ exists in $\mathrm{CB}(X)$. We show, that $M = \lim_{n\to\infty} M_n$ is compact. To do that it is sufficient to show that M is totally bounded.
Let $\epsilon > 0$ and choose N_ϵ such that for all $n \geq N_\epsilon$ we have

$$\delta(M, M_n) \leq \epsilon/2.$$

Since the sets M_n are compact, we find a finite number of spheres of radius $\epsilon/2$ which cover M_{N_ϵ}. Obviously the spheres with the same centers and radius ϵ cover M.

2) We have to show, that $\mathrm{K}(X)$ is totally bounded. Therefore we fix $\epsilon > 0$. The set of open balls $\{B_\epsilon(x)|x \in X\}$ is an open cover of X. Since X is compact there is a finite set J such that $X \subseteq \{B_\epsilon(x_j)|j \in J\}$. Let $Y = \{x_j|j \in J\}$. We look at the collection of nonempty subsets of Y, $\mathscr{P}_0(Y) = \mathscr{P}(Y)\backslash\{\emptyset\}$. Since Y is finite, so is $\mathscr{P}_0(Y)$. Also every $Z \in \mathscr{P}_0(Y)$ is compact and thus $\mathscr{P}_0(Y) \subseteq \mathrm{K}(X)$. We show that the balls $\{B_\epsilon(Z)|Z \in \mathscr{P}_0(Y)\}$ cover $\mathrm{K}(X)$.
For any $A \in \mathrm{K}(X)$ let $Z \in \mathscr{P}_0(Y)$ be defined by

$$Z = \{y \in Y|D(y, A) < \epsilon\}.$$

We also define

$$N_\epsilon(A) = \{x \in X|D(x, A) < \epsilon\}.$$

Then $Z \subseteq N_\epsilon(A)$ by construction. We also claim that $A \subseteq N_\epsilon(Z)$: If not then there exists a point $a \in A$ such that $D(a, Z) \geq \epsilon$. By our choice of Y there exists a $y \in Y$ such that $d(y, a) < \epsilon$. By definition of Z, $y \in Z$. Thus $D(a, Z) < \epsilon$. A contradiction.
Now since $Z \subseteq N_\epsilon(A)$ and $A \subseteq N_\epsilon(Z)$, we have $\delta(Z, A) < \epsilon$.

3) We show that every $A \in \mathrm{K}(X)$ has a compact neighbourhood. Since A is compact in X and X is locally compact, there exist finitely many compact sets $A_1, \cdots A_N$ such that

$\mathfrak{A} = \bigcup_{n=1}^{N} A_n \supseteq A$. $\mathfrak{A}$ is as a finite union of compact sets compact and thus $(\mathfrak{A}, d|_{\mathfrak{A}\times\mathfrak{A}})$ is a compact metric space. Therefore $(\mathrm{K}(\mathfrak{A}), \delta_{\mathfrak{A}})$ is a compact metric space and in particular locally compact. So there exists a compact neighbourhood V of A in $\mathrm{K}(\mathfrak{A})$. Since $(\mathrm{K}(\mathfrak{A}), \delta_{\mathfrak{A}}) = (\mathrm{K}(X)|_{K(\mathfrak{A})}, \delta|_{K(\mathfrak{A})\times K(\mathfrak{A})})$, V is also a compact neighbourhood of A in $(\mathrm{K}(X), \delta)$.

□

The following lemma shows us, how we can construct a Lipschitz mapping in $\mathrm{K}(X)$ from a set-valued Lipschitz mapping.

Lemma 3.11 *Let $T : X \to K(Y)$ be a Lipschitz mapping with Lipschitz constant k. Then for any compact $K \subseteq X$,*

$$A_T(K) = \bigcup_{x \in K} T(x)$$

is a compact subset of Y, i.e. $A_T(K) \in K(Y)$.
The thusly defined map A_T :$K(X) \to$$K(Y)$ is a Lipschitz mapping with Lipschitz constant k.

Proof: We show that every sequence of $A_T(K)$ has a convergent subsequence.
Let $\{a_i\}$ be a sequence in $A_T(K)$. Then $a_i \in T(k_i)$ for some $k_i \in K$. So we have a sequence $\{k_i\}$ in the compact set K, and therefore there exists a subsequence $\{k_{i_j}\}$ which converges to some $k \in K$. Since T is a Lipschitz mapping, $\{T(k_{i_j})\}_{j=1}^{\infty}$ converges to $T(k)$. Therefore all limit points of $\{a_{i_j}\}$ lie in $T(k)$. So to an arbitrary limit point x we can choose a subsequence of $\{a_{i_j}\}$ that converges to $x \in T(k) \subseteq A_T(K)$. Thus we have found a convergent subsequence of $\{a_i\}$.
So A_T is well defined map $\mathrm{K}(X) \to \mathrm{K}(Y)$. We now show, that A_T is a Lipschitz map.
By using the definition of the Hausdorff metric we gain

$$\begin{aligned}
D_Y(A_T(F_1), A_T(F_2)) &= \sup_{u \in A_T(F_1)} \inf_{v \in A_T(F_2)} d_Y(u, v) \\
&= \sup_{u \in A_T(F_1)} \inf_{y \in F_2} D_Y(u, T(y)) \\
&= \sup_{x \in F_1} \inf_{y \in F_2} D_Y(T(x), T(y)) \\
&\leq \sup_{x \in F_1} \inf_{y \in F_2} \delta_Y(T(x), T(y)) \\
&\leq k \sup_{x \in F_1} \inf_{y \in F_2} d_X(x, y) = k D_X(F_1, F_2).
\end{aligned}$$

Analogue we get that $D_Y(A_T(F_2), A_T(F_1)) \leq k D_X(F_2, F_1)$ and thus

$$\delta_Y(A_T(F_1), A_T(F_2)) \leq k \delta_X(F_1, F_2).$$

□

To prove the main result of this section, we first need a few results about sequences of single-valued contraction mappings. For the following three propositions, f_i will denote a single-valued contraction mapping of a metric space (X, d) into itself with a fixed point a_i for $i = 0, 1, 2, \cdots$.

Proposition 3.12 *If all the mappings $\{f_i\}_{i=1}^{\infty}$ have the same Lipschitz constant $k < 1$, and the sequence $\{f_i\}_{i=1}^{\infty}$ converges pointwise to f_0, then the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a_0.*

Proof: By the pointwise convergence, we can choose to given $\epsilon > 0$ an N_ϵ, such that

$$d(f_n(a_0), f_0(a_0)) \leq \epsilon(1-k)$$

for all $n \geq N_\epsilon$. Then

$$d(a_n, a_0) = d(f_n(a_n), f_0(a_0)) \leq d(f_n(a_n), f_n(a_0)) + d(f_n(a_0), f_0(a_0)) \leq kd(a_n, a_0) + \epsilon(1-k)$$

which yields

$$d(a_n, a_0) \leq \epsilon, \quad n \geq N_\epsilon.$$

□

Proposition 3.13 *If the sequence* $\{f_i\}_{i=1}^\infty$ *converges uniformly to* f_0*, then the sequence* $\{a_i\}_{i=1}^\infty$ *converges to* a_0*.*

Proof: To given $\epsilon > 0$ choose N_ϵ, such that for all $x \in X$ and $n \geq N_\epsilon$

$$d(f_n(x), f_0(x)) \leq \epsilon(1-k_0)$$

where $k_0 < 1$ is the Lipschitz constant of f_0. Then

$$d(a_i, a_0) = d(f_i(a_i), f_0(a_0)) \leq d(f_i(a_i), f_0(a_i)) + d(f_0(a_i), f_0(a_0)) \leq \epsilon(1-k_0) + k_0 d(a_i, a_0).$$

This yields

$$d(a_i, a_0) \leq \epsilon, \quad i \geq N_\epsilon.$$

□

Proposition 3.14 *If the space* (X, d) *is locally compact and the sequence* $\{f_i\}_{i=1}^\infty$ *converges pointwise to* f_0*, then the sequence* $\{a_i\}_{i=1}^\infty$ *converges to* a_0*.*

Proof: Let $\epsilon > 0$ and assume ϵ is sufficiently small, so that

$$K(a_0, \epsilon) = \{x \in X : d(a_0, x) \leq \epsilon\}$$

is a compact subset of X. Then, since $\{f_i\}_{i=1}^\infty$ is as a sequence of Lipschitz mappings equicontinuous and converges pointwise to f_0 and since $K(a_0, \epsilon)$ is compact, the sequence $\{f_i\}_{i=1}^\infty$ converges uniformly on $K(a_0, \epsilon)$ to f_0. Choose N_ϵ such that if $i \geq N_\epsilon$, then

$$d(f_i(x), f_0(x)) \leq (1-k_0)\epsilon$$

for all $x \in K(a_0, \epsilon)$, where $k_0 < 1$ is the Lipschitz constant for f_0. Then, if $i \geq N_\epsilon$ and $x \in K(a_0, \epsilon)$,

$$d(f_i(x), a_0) \leq d(f_i(x), f_0(x)) + d(f_0(x), f_0(a_0)) \leq (1-k_0)\epsilon + k_0 d(x, a_0) \leq (1-k_0)\epsilon + k_0\epsilon = \epsilon.$$

This proves that if $i \geq N_\epsilon$, then f_i maps $K(a_0, \epsilon)$ into itself. Letting g_i be the restriction of f_i to $K(a_0, \epsilon)$ for each $i \geq N_\epsilon$, we see that each g_i is a contraction mapping of $K(a_0, \epsilon)$ into itself. Since $K(a_0, \epsilon)$ is a complete metric space, g_i has a fixed point for each $i \geq N_\epsilon$ which must, from the definition of g_i and the fact that f_i has only one fixed point, be a_i. Hence, $a_i \in K(a_0, \epsilon)$ for each $i \geq N_\epsilon$. It follows that the sequence $\{a_i\}_{i=1}^\infty$ of fixed points converges to a_0. □

Lemma 3.15 *Let (X, d) be a metric space, let $T_i : X \to CB(X)$ be a set-valued contraction mapping with fixed point x_i for each $i = 1, 2, \cdots$, and let $T_0 : X \to CB(X)$ be a set-valued contraction mapping. If the sequence $\{T_i\}_{i=1}^{\infty}$ converges pointwise to T_0 and if $\{x_{i_j}\}_{j=1}^{\infty}$ is a convergent subsequence of $\{x_i\}_{i=1}^{\infty}$ then $\{x_{i_j}\}_{j=1}^{\infty}$ converges to a fixed point of T_0.*

Proof: Set $x_0 = \lim_{j\to\infty} x_{i_j}$ and fix $\epsilon > 0$. Choose N_ϵ such that

$$\delta(T_{i_j}(x_0), T_0(x_0)) \leq \frac{\epsilon}{2} \quad \text{and} \quad d(x_{i_j}, x_0) \leq \frac{\epsilon}{2}$$

for all $j \geq N_\epsilon$. Then for such j, we have

$$\delta(T_{i_j}(x_{i_j}), T_0(x_0)) \leq \delta(T_{i_j}(x_{i_j}), T_{i_j}(x_0)) + \delta(T_{i_j}(x_0), T_0, (x_0)) \leq d(x_{i_j}, x_0) + \delta(T_{i_j}(x_0), T_0, (x_0)) \leq \epsilon.$$

This shows, that $\lim_{j\to\infty} T_{i_j}(x_{i_j}) = T_0(x_0)$. And since $x_{i_j} \in T_{i_j}(x_{i_j})$ for each j, it follows, that $x_0 \in T_0(x_0)$. □

Theorem 3.16 *Let (X, d) be a complete metric space, let $T_i : X \to K(X)$ be a set-valued contraction mapping with fixed point x_i for each $i = 1, 2, \cdots$, and let $T_0 : X \to K(X)$ be a set-valued contraction mapping. Suppose one of the following holds:*

1. *Each of the mappings T_i, $i \geq 1$ has the same Lipschitz constant $k < 1$ and the sequence $\{T_i\}_{i=1}^{\infty}$ converges pointwise to T_0.*
2. *The sequence $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T_0.*
3. *(X, d) is a locally compact space and the sequence $\{T_i\}_{i=1}^{\infty}$ converges pointwise to T_0.*

Then there exists a subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that $\{x_{i_j}\}_{j=1}^{\infty}$ converges to a fixed point of T_0.

Proof: For each i define the map $A_{T_i} : \mathrm{K}(X) \to \mathrm{K}(X)$ as in Lemma (3.11). Then, A_{T_i} is a contraction mapping and therefore has a unique fixed point $F_i \in \mathrm{K}(X)$. If the sequence $\{T_i\}_{i=1}^{\infty}$ converges pointwise to T_0, then $\{T_i\}_{i=1}^{\infty}$ converges uniformly on compact subsets of X to T_0 and therefore $\{A_{T_i}\}_{i=1}^{\infty}$ converges pointwise on $\mathrm{K}(X)$ to A_{T_0}. If $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T_0, then $\{A_{T_i}\}_{i=1}^{\infty}$ converges uniformly on $\mathrm{K}(X)$ to A_{T_0}. So we can use Proposition (3.12) with assumption 1, Proposition (3.13) with assumption 2 and the Propositions (3.14) and (3.10) with assumption 3 to conclude, that the sequence $\{F_i\}_{i=1}^{\infty}$ converges to F_0. We now show, that $K = \bigcup_{i=0}^{\infty}(F_i)$ is compact.
Let $\{y_k\}_{k=1}^{\infty}$ be a sequence in K. Then either one of the F_i contains infinitely many y_k or each contains only finitely many. In the first case, we can choose convergent subsequence in the compact set F_i. In the second case, every limit point of $\{y_k\}_{k=1}^{\infty}$ must lie in F_0, because of the convergence of the sequence $\{F_i\}_{i=1}^{\infty}$. So we can choose a subsequence, that converges to one of these limit point.
By the Banach Fixed Point Theorem (1.3), the sequence $\{A_{T_i}^n(x_i)\}_{n=1}^{\infty}$ converges to F_i. And since $x_i \in A_{T_i}^n(x_i)$ for all $n > 0$, it follows that $x_i \in F_i$. Hence, $\{x_i\}_{i=1}^{\infty}$ is a sequence in the compact set K. Thus there exists a convergent subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ which, by Lemma (3.15), converges to x_0. □

Chapter 4

Fixed Point Theorems for set-valued Maps. Geometric Principle

4.1 The KKM-Principle

The KKM-Principle borrows its name from a lemma by Knaster, Kuratowski, and Mazurkiewicz, see the below Lemma (4.2). By the KKM-principle one means deriving results from this, or similar, results. Such techniques are used not only in fixed point theory, but also in mathematical economics, the study of variational inequalities, best approximation theory, and other disciplines. We will use it to prove the classical fixed point theorem of Brouwer for single-valued maps. Then we will show examples for fixed point theorems for set-valued maps derived from the KKM-Lemma.
To prove it, we will need the Sperner Lemma.

Lemma 4.1 (Sperner) *Let S be an n-dimensional Simplex with vertices $\{p_i | i = 0, \cdots, n\}$. For $0 \leq k \leq n$ the face with vertices $p_{i_0}, \cdots, p_{i_k}$ will be denoted by $p_{i_0} p_{i_1} \cdots p_{i_k}$. Divide S simplicialy, i.e. into finitely many simplices $T_1, \cdots, T_J$ of the same dimension such that the intersection of two of them is either empty, a common k-dimensional face or a common vertex and that $\bigcup_{j=1}^{J} T_j = S$.*
To every vertex e of a subsimiplex we assign a number $\nu(e) \in \{0, \cdots, n\}$ such that

$$\begin{array}{c} \text{if } e \text{ lies on a } k\text{-dimensional face } p_{i_0} p_{i_1} \cdots p_{i_k} \; (0 \leq k \leq n) \text{ of } S \\ \text{then } \nu(e) \in \{i_0, i_1, \cdots, i_k\}. \end{array} \tag{4.1}$$

Then there exist an odd number of subsimplices T_j such that each $i \in \{0, \cdots, n\}$ is assigned to a vertex of T_j by $\nu(e)$.

Proof: For $n = 0$ the assertion is trivial. So we want to prove it for $n > 0$ under the assumption, that it is true for $n - 1$.
We call a subsimplex T_j of S representative if it fulfills our assumption. Analogue we call a face representative, in short r-face, is $\nu(e)$ assigns to the vertices of this face all numbers $0, 1, \cdots, n-1$. We then set

$$\begin{aligned} \rho &= \text{number of representative subsimplices} \\ \sigma &= \text{number of r-faces lying on the boundary of } S \\ \alpha(T_j) &= \text{number of r-faces of a subsimplex } T_j \end{aligned}$$

and first show, that

$$\rho \equiv \sigma \pmod 2. \tag{4.2}$$

It is obvious, that if T_j is representative, $\alpha(T_j) = 1$ need hold. If T_j is not representative, then either $\alpha(T_j) = 0$ or $\alpha(T_j) = 2$, depending if one of the numbers $0, \cdots n-1$ is missing on the vertices of T_j or not. It follows that

$$\rho \equiv \sum \alpha(T_j) \pmod 2, \tag{4.3}$$

where the sum is taken over all subsimplices T_j of the division. Because of our condition, that this division is simplicial, every r-face is counted once or twice, depending if it is on the boundary of S or not. Therefore we have $\sigma \equiv \sum \alpha(T_j)$ (mod 2) and because of (4.3) this yields (4.2).

If we now look at one of the from $p_0 p_1 \cdots p_{n-1}$ different (n-1)-dimensional faces Z of S, then under the vertices of Z one of the points $p_i, 0 \leq i \leq n-1$ is missing. By (4.1) Z contains no point e for which $\nu(e) = i$ and in conclusion no r-face. That means that all r-faces, that lie on the boundary of S, are contained in the face $p_0 p_1 \cdots p_{n-1}$. So σ denotes the number of representative subsimplices of the simplicial division of the (n-1)-dimensioanl simplex $p_0 p_1 \cdots p_{n-1}$. It follows, since we know that the assertion is true for n-1, that σ is odd. The assertion follows by (4.2). □

For us, the important property of the Sperner Lemma will be, that, since 0 is an even number, it guarantees the existence of at least one such subsimplex.

Lemma 4.2 (KKM-Lemma) *Let S be a simplex in $\mathbb{R}^n$ with vertices $p_0, \cdots, p_n$. Let $\{A_0, \cdots, A_n\}$ be a family of closed sets such that for each subset $J \subseteq \{0, \cdots, n\}$ with $J = \{j_0, \cdots, j_k\}$ we have*

$$p_{j_0} p_{j_1} \cdots p_{j_k} \subseteq \bigcup_{j \in J} A_j,$$

then

$$\bigcap_{i=0}^{n} A_i \neq \emptyset.$$

Proof: Fix $m > 0$ and divide S simplicialy such that each subsimplex has a diameter $< 2^{-m}$. Let e be an arbitrary point of a subsimplex and let $p_{i_0} p_{i_1} \cdots p_{i_k}$ be the lowest-dimensional simplex containing e.
By our assumption $p_{i_0} p_{i_1} \cdots p_{i_k} \subseteq A_{i_0} \cup \cdots \cup A_{i_k}$, and therefore there exists at least one index $i_j \, (0 \leq j \leq k)$, such that $e \in A_{i_j}$. If we set $\nu(e) = i_j$ the condition of Lemma (4.1) is fulfilled. So we have a representative subsimplex which we can denote with $e_0^m e_1^m \cdots e_n^m$, by setting $\nu(e_i^m) = i$. Because $e \in A_{\nu(e)}$ holds, we have $e_i^m \in A_i$.
By letting m grow towards infinity, we can assume without loss of generality, that we gain a convergent sequence $\{e_0^m\}_{m=1}^{\infty}$. Set $a = \lim_{m \to \infty} e_0^m$. Because the diameter of the subsimplices converges to 0 with growing m, we have that $a = \lim_{m \to \infty} e_i^m$ for each $i = 0, \cdots, n$. Because each A_i is closed, this yields $a \in \bigcap_{i=0}^{n} A_i$. □

As one can see from the nature of the lemma, the KKM-Lemma can be used for existence theorems. Since fixed point theorems fall into that category, it naturally has its applications in fixed point theory. We want to note however, that the main applications of the KKM-Principle lie not there, but in the fields already noted at the beginning of this section.
However, it allows us to give a short proof of the Brouwer Fixed Point Theorem. This is surely one of the most important fixed point theorems for single-valued maps. Most fixed point theorems that deal with geometrical properties are in some way derived from this theorem.

Theorem 4.3 (Brouwer Fixed Point Theorem) *Let $C \subseteq \mathbb{R}^n$ be nonempty, compact, and convex. Then every continuous map $f : C \to C$ has a fixed point.*

Proof: Let S be a simplex with vertices $p_0, \cdots, p_n$. We denote each $x \in S$ by its unique representation as convex combination of the vertices of S, $x = c_0 p_0 + \cdots + c_n p_n$, where $c_i \geq 0$, and $c_0 + \cdots c_n = 1$. Let $G : S \to S$ be continuous. Define $n + 1$ maps $g_i : \mathbb{R} \to \mathbb{R}$ by $G(x) = g_0(c_0)p_0 + \cdots + g_n(c_n)p_n$.
Set $A_i = \{x \in S | g_i(c_i) \leq c_i\}$. We show that the sets $A_0, \cdots, A_n$ satisfy the conditions of the KKM-Lemma.
Because of the continuity of g, each A_i is closed. Suppose that a point x of a k-dimensional simplex $p_{i_0} p_{i_1} \cdots p_{i_k}$ does not lie in any of the sets $A_{i_0}, \cdots, A_{i_k}$. Then we would have $g_{i_l}(c_{i_l}) > c_{i_l}$ for $0 \leq l \leq k$ and thus $\sum_{l=0}^{k} g_{i_l}(c_{i_l}) > \sum_{l=0}^{k} c_{i_l}$. A contradiction, since the left sum can not be greater than 1 by the definition of convex combinations and the right sum is equal to one because $x \in p_{i_0} p_{i_1} \cdots p_{i_k}$.
Thus we have a point $a \in \bigcap_{i=0}^{n} A_i$ by Lemma (4.2). Because of our definition of the A_i we have for $x = a$: $g_i(c_i) \leq c_i$ for each $i = 0, \cdots, n$. Thus

$$1 = g_0(c_0) + \cdots g_n(c_n) \leq c_0 + \cdots + c_n = 1$$

and hence $g_i(c_i) = c_i$. This yields $G(a) = a$.
Since C and S are both nonempty compact convex subsets of $\mathbb{R}^n$, one can show, that there exists a homeomorphism $h : S \to C$. Then $h^{-1} \circ f \circ h : S \to S$ is a continuous map and therefore has a fixed point a: $(h^{-1} \circ f \circ h)(a) = a$. We therefore get, that $h(a)$ is a fixed point of f. □

We now want to give two examples of fixed point theorems for set-valued maps derived by the KKM-Principle. First, we will give an example for a fixed point theorem that is derived from a generalized version of the KKM-Lemma. Then we will show results from a similar lemma.

Lemma 4.4 (Generalized KKM-Lemma) *Let X be a topological vector space, M be a nonempty subset of X, and $F : M \to \mathscr{P}(X)$. Assume that*

- *$F(x)$ is nonempty and closed for all $x \in M$,*
- *$F(x_0)$ is compact for at least one $x_0 \in M$,*
- *for every finite subset $\{x_1, \cdots x_n\} \subseteq M$ we have*

$$co\{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i).$$

Then

$$\bigcap_{x \in M} F(x) \neq \emptyset.$$

Proof: 1) If M is finite, co(M) is homeomorphic to a simplex in $\mathbb{R}^n$. Therefore we can apply Lemma (4.2).
2) Let M be infinite and suppose $\bigcap_{x \in M} F(x) = \emptyset$. Because $F(x_0)$ is compact for some x_0, there exists by the finite intersection property a tuple $\{x_1, \cdots, x_m\}$ with $\bigcap_{i=1}^{m} F(x_i) = \emptyset$. This contradicts 1). □

Theorem 4.5 *Let C be a nonempty compact convex subset of a topological vector space X. Let $T : C \to \mathscr{P}(C)$ be a set valued map such that*

- *$T(x)$ is closed for each $x \in C$,*
- *$T^{-1}(y)$ is open for each $y \in C$.*

Then T has a fixed point.

Proof: Define a map $S : C \to \mathscr{P}(C)$ by $S(y) = C \setminus T^{-1}(y)$. Then $S(y)$ is nonempty and closed in C, and therefore compact. Note that $C = \bigcup\{T^{-1}(y) | y \in C\}$. Given any x_0 in C choose a $y_0 \in T(x_0)$. Then $x_0 \in T^{-1}(y_0)$. Thus

$$\bigcap_{y \in C} S(y) = \bigcap_{y \in C} (C \setminus T^{-1}(y)) = [\bigcup_{y \in C} T^{-1}(y)]^C = \emptyset,$$

where by $[A]^C$ we denote the complement of A. By Lemma (4.4) there exist $y_i, i = 1, \cdots n$ such that the convex combination $w = \sum_{i=1}^n \lambda_i y_i \notin \bigcup_{i=1}^n S(y_i)$. Hence, we have $w \in C \setminus \bigcup_{i=1}^n S(y_i) = \bigcap_{i=1}^n T^{-1}(y_i)$. Thus $w \in T^{-1}(y_i)$ for each $i = 1, \cdots n$. and therefore $y_i \in T(w)$. Since $T(w)$ is convex, we have $w = \sum_{i=1}^n \lambda_i y_i \in T(w)$. □

For the next example we use a KKM-like lemma to prove two fixed point theorems. That this lemma is closely related to the KKM-Lemma (or more exactly its generalization) can be seen in its proof.

Lemma 4.6 *Let M be a nonempty convex subset of a topological vector space X. Let $F : M \to \mathscr{P}(X)$ be a set valued map that satisfies*

(a) $x \in F(x)$ for each $x \in M$

(b) $F(x_0)$ is compact for some $x_0 \in M$,

(c) the set $A(x) = \{y \in M | x \notin F(y)\}$ is convex for each $x \in M$,

(d) the intersection of $F(x)$ with any finite dimensional subspace of X is closed for each $x \in M$,

(e) $F(x_0) \cap F(x)$ is closed for each $x \in M$.

Then $\bigcap_{x \in M} F(x) \neq \emptyset$.

Proof: First we show, that for any finite set $\{x_1, \cdots, x_m\} \subseteq M$, $\text{co}\{x_1, \cdots, x_m\} \subseteq \bigcup_{i=1}^m F(x_i)$. Suppose there is an $x \in \text{co}\{x_1, \cdots, x_m\}$ with $x \notin \bigcup_{i=1}^m F(x_i)$. Then each $x_i \in A(x)$. Since $A(x)$ is convex and $x \in \text{co}\{x_1, \cdots, x_m\}$, we have $x \in A(x)$, which means $x \notin F(x)$. This contradicts *(a)*.

Next we show, that the $F(x)$ have the finite intersection property. Assume therefore that $\bigcap_{i=1}^m F(x_i) = \emptyset$. Denote by L the finite dimensional subspace spanned by $\{x_1, \cdots, x_m\}$ and let $C = \text{co}\{x_1, \cdots, x_m\} \subseteq L$. Because $L \cap F(x_i)$ is closed, we have $D(x, L \cap F(x_i)) = 0$ if and only if $x \in L \cap F(x_i)$. Since $\bigcap_{i=1}^m L \cap F(x_i) = \emptyset$ by assumption the function $f : C \to \mathbb{R}, f(x) = \sum_{i=1}^m D(x, L \cap F(x_i))$ is never zero for any $x \in C$. So by setting

$$g(x) = \frac{1}{f(x)} \sum_{i=1}^m D(x, L \cap F(x_i))\, x_i$$

we have a continuous function $g: C \to C$. By Theorem (4.3) g has a fixed point $x_0 \in C$. Denote with $I = \{i | D(x_0, L \cap F(x_i)) \neq 0\}$. Then $x_0 \notin \bigcup\{F(x_i)|i \in I\}$. But since

$$x_0 = g(x_0) \in \mathrm{co}\{x_i | i \in I\} \subseteq \bigcup\{F(x_i)|i \in I\}$$

we have a contradiction and thus the $F(x)$ have the finite intersection property.
So now we have that for each finite set $\{x_1, \cdots, x_m\}$ the $(\bigcap_{i=1}^m F(x_i)) \cap F(x_0)$ is a nonempty intersection of closed sets in the compact set $F(x_0)$. Therefore we have $\bigcap_{x \in M} F(x) \neq \emptyset$. □

Theorem 4.7 *Let K be a nonempty convex subset of a topological vector space X. Let $T: K \to \mathscr{P}(K)$ be a set-valued map such that*

(i) $T(x)$ is nonempty and convex for each $x \in K$,

(ii) for some $x_0 \in K$, $[T^{-1}(x_0)]^C$ is compact,

(iii) the intersection of $[T^{-1}(x)]^C$ with any finite dimensional subspace of X is closed for each $x \in K$,

(iv) $[T^{-1}(x)]^C \cap [T^{-1}(x_0)]^C$ is closed for each $x \in K$.

Then T has a fixed point.

Proof: Assume there is no point $x \in K$ with $x \in T(x)$. This implies that there is no $x \in K$ such that $x \in T^{-1}(x)$. Set $F(x) = [T^{-1}(x)]^C$. Then we have *(a)* $x \in F(x)$ for each $x \in K$. We also have, that

$$A(x) = \{y \in K | x \notin F(y)\} = \{y \in K | x \notin [T^{-1}(y)]^C\} = \{y \in K | y \in T^{-1}(y)\} = T(x)$$

is convex by *(i)*, which is condition *(c)* of Lemma (4.6). Conditions *(b)*, *(d)*, and *(e)* follow from assumptions *(ii)*, *(iii)* and *(iv)*, respectively.
Hence, there is a point x_0 such that $x_0 \in \bigcap_{x \in K} F(x)$. So $x_0 \in [T^{-1}(x)]^C$ for each $x \in K$, which means $x_0 \notin T^{-1}(x)$ for any $x \in K$. Since $x_0 \in K = \bigcup_{x \in K} T^{-1}(x)$, we have a contradiction. □

Theorem 4.8 *Let K be a nonempty convex subset of a topological vector space X. Let $T: K \to \mathscr{P}(K)$ be a set-valued map such that*

(i) $T(x)$ is nonempty for each $x \in K$,

(ii) for some $x_0 \in K$, $[T(x_0)]^C$ is compact,

(iii) $T^{-1}(x)$ is convex for each $x \in K$ (but may be empty),

(iv) the intersection of $[T(x)]^C$ with any finite dimensional subspace of X is closed,

(v) $[T(x)]^C \cap [T(x_0)]^C$ is closed for each $x \in K$,

(vi) $\bigcup_{x \in K} T(x) = K$.

Then T has a fixed point.

Proof: Assume, that T has no fixed point and set $F(x) = [T(x)]^C$. Then we have condition *(a)* of Lemma (4.6). Since

$$A(x) = \{y \in K | x \notin F(y)\} = \{y \in K | x \in T(y)\} = T^{-1}(x)$$

is convex by *(iii)*, we also have *(c)*. The conditions *(ii)*, *(iv)*, and *(v)* imply, respectively, conditions *(b)*, *(d)*, and *(e)* of Lemma (4.6). Therefore there is a point $x_0 \in K$ such that $x_0 \in \bigcap_{x \in K} F(x) = \bigcap_{x \in K} [T^{-1}(x)]^C$. This implies $x_0 \notin \bigcup_{x \in K} T(x)$ which is a contradiction to *(vi)*. □

We now give to each of this theorems a corollary, in which the assumptions are easier to verify.

Corollary 4.9 *Let K be a nonempty convex subset of a topological vector space X. Let $T : K \to \mathscr{P}(K)$ be a set-valued map such that*

1. *$T(x)$ is nonempty and convex for each $x \in K$,*
2. *for some $x_0 \in K$, $[T^{-1}(x)]^C$ is compact,*
3. *$T^{-1}(x)$ is open for each $x \in K$.*

Then T has a fixed point.

Proof: Since $T^{-1}(x)$ is open, each $[T^{-1}(x)]^C$ is closed. We can apply Theorem (4.7). □

Corollary 4.10 *Let K be a nonempty convex subset of a topological vector space X. Let $T : K \to \mathscr{P}(K)$ be a set-valued map such that*

1. *$T(x)$ is nonempty and open for each $x \in K$,*
2. *for some $x_0 \in K$, $[T(x_0)]^C$ is compact,*
3. *$T^{-1}(x)$ is convex for each $x \in K$ (but may be empty),*
4. *$\bigcup_{x \in K} T(x) = K$.*

Then T has a fixed point.

Proof: Since each $T(x)$ is open the $[T(x)]^C$ are closed and we can apply Theorem (4.8). □

`Remark:` We want to bring attention to the fact, that in this fixed point theorems we derived by the KKM-principle, there is no mention of continuity. This is remarkable, since in the following sections, this will be a basic assumption on the viewed map.
Also note, that these theorems and their corollaries belong to the rare kind of fixed point theorems for set-valued maps, that can't be applied to single-valued maps, since the condition, that $T(x)$ or $T^{-1}(x)$ is open, can not be met in that case (except for some trivial examples).
We therefore see, that the strength of the KKM-principle lies therin, that with it, one can derive theorems with rather unusual assumptions.

4.2 The Kakutani Fixed Point Theorem

The Kakutani-Fixed Point Theorem was the first fixed point result about set-valued mappings. It is a generalization of the fixed point theorem by Brouwer. The statement is as follows:

Theorem 4.11 (Kakutani Fixed Point Theorem) *Let K be a nonempty compact convex subset of $\mathbb{R}^n$. Let $T : K \to \mathscr{P}(K)$ satisfy*

- *for each $x \in K$, $T(x)$ is nonempty closed and convex,*
- *T is upper semi-continuous.*

Then T has a fixed point.

Note, that by Theorem (2.3) the upper semi-continuity of T is in this case equivalent to the closedness of the graph $G(T)$ in $K \times K$.
Before we come to the proof of this theorem, we will need a few preliminary results, that will allow us to state it in this general form. To show them, we will need the concept of retraction mappings:

Definition 4.12 *We say that X is a retract of a topological space Y, if $X \subseteq Y$ and there exists a continuous mapping $r : Y \to X$ such that r restricted to X acts like the identity mapping on X.*
The map r is called a retraction.

The following lemma is a well known result in functional analysis which we will give without proof. We will denote by $\mathring{M}$ the interior of M, and by ∂M the boundary of M.

Lemma 4.13 *Let $M \subseteq \mathbb{R}^n$ be closed and convex, and let $0 \in \mathring{M}$. Then the Minkowksi functional*

$$g_M(x) = \inf\{c > 0 | x \in cM\}$$

is a continuous real function on $\mathbb{R}^n$ such that

1. *$g_M(ax) = a g_M(x)$ for $a \geq 0$,*
2. *$g_M(x + y) \leq g_M(x) + g_M(y)$,*
3. *$0 \leq g_M(x) < 1$ if $x \in \mathring{M}$,*
4. *$g_M(x) > 1$ if $x \notin M$,*
5. *$g_m(x) = 1$ if $x \in \partial M$.*

In the following lemma $M - x$ will denote the set $\{y - x | y \in M\}$.

Lemma 4.14 *Let M be a nonempty closed and convex subset of $\mathbb{R}^n$ with nonempty interior and $x_0 \in \mathring{M}$. Then the radial retraction defined by*

$$r(x) = x / \max\{1, g_{M-x_0}(x - x_0)\}$$

is a retraction of $\mathbb{R}^n$ onto M.

Proof: Trivial. □

Theorem 4.15 *A nonempty closed convex subset M of $\mathbb{R}^n$ is a retract of any larger subset N with $M \subseteq N \subseteq Y$.*

Proof: Equip N with the trace topology of $\mathbb{R}^n$.
If $\mathring{M} \neq \emptyset$ this follows immediately from Lemma (4.14). If $\mathring{M} = \emptyset$ then $M \subseteq \mathbb{R}^m$ for some $m < n$. Let r be the radial retraction of $\mathbb{R}^m$ onto M and P_m the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$. Then $r \circ P_m$ is a retraction of $\mathbb{R}^n$ onto M. The assertion follows again by Lemma (4.14). □

The following lemma will allow us to restrict ourselves to very simple subsets of $\mathbb{R}^n$ when proving the Kakutani Theorem.

Lemma 4.16 *If Theorem (4.11) holds for $U \subseteq \mathbb{R}^n$ then it also holds for any retract of U.*

Proof: Let r be a retraction of U onto V. Let $T : V \to \mathscr{P}(V)$ have the same properties as in Theorem (4.11). Define a map $S : U \to \mathscr{P}(U), S(x) = T(r(x))$. Due to the continuity of r and Proposition (2.2), S also fulfills all conditions of Theorem (4.11). Therefore S has a fixed point x so that $x \in S(x) = T(r(x)) \subseteq V$. Thus $r(x) = x$ and $x \in T(x)$. □

Now, to prove Theorem (4.11), we can restrict ourselves to proving it for simplices.
Proof of Theorem (4.11): Let S be a simplex with $K \subseteq S$ and let $U : S \to \mathscr{P}(S)$ satisfy the properties required in the theorem. Denote by S^i be a simplical division of S where each simplex in S^i has a diameter $< 2^i$. To each vertex x^i of S^i choose an arbitrary $y^i \in U(x^i)$. We construct a map $u_i : S \to S$ by setting $u(x^i) = y^i$ and extending it linearly in each simplex of S^i. Thus u_i is a continuous map of the compact convex set S into itself and therefore there exists a fixed point x_i of u_i by the Brouwer Fixed Point Theorem. Since S is compact, there exists a subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$, that converges to some $x_0 \in S$.
Let Δ_i be a simplex of S^i which contains x_i. Let $x_0^i, x_1^i, \cdots, x_n^i$ be the vertices of Δ_i. Then it is obvious, that $\{x_k^{i_j}\}_{j=1}^{\infty}$ converges to x_0 for each $k \in \{0, \cdots, n\}$. Further, $x_i = \sum_{k=0}^{n} \lambda_k^i x_k^i$ for some λ_k^i with $\lambda_k^i \geq 0$ and $\sum_{k=0}^{n} \lambda_k^i = 1$. We then set $y_k^i = u_i(x_k^i)$. W.l.o.g. we can assume, that the $y_k^{i_j}$ converge to a y_k and the $\lambda_k^{i_j}$ to a λ_k with $\lambda_k \geq 0$ and $\sum_{k=0}^{n} \lambda_k = 1$.
By Theorem (2.3) the graph $G(U)$ of U is closed. Since $(x_k^{i_j}, y_k^{i_j})$ is in $G(U)$ and converges to (x_0, y_k), it follows that $y_k \in U(x_0)$ for each k. Thus

$$x_0 = \lim_{j\to\infty} x_{i_j} = \lim_{j\to\infty} u_{i_j}(x_{i_j}) = \lim_{j\to\infty} \sum_{k=0}^{n} \lambda_k^{i_j} u(x_k^{i_j}) = \lim_{j\to\infty} \sum_{k=0}^{n} \lambda_k^{i_j} y_k^{i_j} = \sum_{k=0}^{n} \lambda_k y_k \in U(x_0)$$

since $U(x_0)$ is convex, and therefore the theorem holds for the simplex S.
Since $K \subseteq S$, K is a retract of S by Theorem (4.15). Hence, we can apply Lemma (4.16), which completes the proof. □

It is easy to see, that the Kakutani Theorem implies the Brouwer Fixed Point Theorem. Since we used only the Brouwer Theorem to prove it, we see, that these two theorems are in fact equivalent.

4.3 Generalizations

We now want to generalize the Kakutani Theorem, which only holds in $\mathbb{R}^n$, to more general spaces. Namely, to seperable Banach spaces, where we will get fixed point theorems similar to the Schauder Theorem. Later, we will also give a generalization of Tychonoffs Fixed Point Theorem in topological vector spaces.

Although the results for Banach spaces could be derived as corollaries from the results in topological vector spaces, we will give the proofs here, because the methods differ profoundly.

4.3.1 Banach spaces

Again, we will first need a few preliminary results.

Definition 4.17 *We define the Hilbert cube $\mathscr{H}_0$ as the subset of the Hilbert space l^2 consisting of all points $a = (a_1, a_2, \cdots)$ with $|a_r| \leq r^{-1}$ for all r.*

We will denote by P_n the projection of l^2 onto an n-dimensial subspace given by

$$P_n(a_1, a_2, \cdots) = (a_1, a_2, \cdots, a_n, 0, 0, \cdots).$$

We then have, that $P_n(\mathscr{H}_0)$ is compact for each $n \in \mathbb{N}$, and that

$$\|P_n a - a\| \leq \left(\sum_{n+1}^{\infty} r^{-2}\right)^{\frac{1}{2}} < \epsilon,$$

for each $\epsilon > 0$ and n sufficiently large.
From that we can easily derive, that $\mathscr{H}_0$ is a compact convex set.

Proposition 4.18 *Theorem (4.11) holds for the Hilbert cube $\mathscr{H}_0$.*

Proof: Let $U : \mathscr{H}_0 \to \mathscr{P}(\mathscr{H}_0)$ satisfy the conditions of Theorem (4.11). Then the map $P_n \circ U : P_n\mathscr{H}_0 \to \mathscr{P}(P_n\mathscr{H}_0)$ also satisfies them: that the set $(P_n \circ U)(x)$ is closed and convex for each $x \in P_n\mathscr{H}_0$ is trivial and if we choose n sufficiently large, it is also nonempty. To see, that $P_n \circ U$ is also upper semi-continuous, we observe, that the graph $G(P_n \circ U)$ of $P_n \circ U$ is the projection of $G(U)$ onto $P_n\mathscr{H}_0 \times P_n\mathscr{H}_0$. By Theorem (2.3) $G(U)$, and therefore also $G(P_n \circ U)$, is closed and thus $P_n \circ U$ upper semi-continuous.
Since we can identify $P_n\mathscr{H}_0$ with a nonempty compact convex subset of $\mathbb{R}^n$, by the Kakutani Theorem $P_n \circ U$ has a fixed point $y_n \in P_nU(y_n)$. Thus $y_n = P_n z_n$ with $z_n \in U(y_n)$.
W.l.o.g we may assume, that the sequence $\{y_n\}_{n=0}^{\infty}$ converges to some $y \in \mathscr{H}_0$. Since we have

$$\|y_n - z_n\| = \|P_n z_n - z_n\| \to 0 \text{ for } n \to \infty$$

we see, that $\{z_n\}_{n=0}^{\infty}$ also converges to y. Thus the points $(y_n, z_n) \in G(U)$ converge to $(y, y) \in G(U)$ since $G(U)$ is closed. This means, that y is a fixed point for U. □

We are now able to reduce the Banach space case to a problem in a Hilbert space.

Lemma 4.19 *Every compact convex subset K of a Banach space X is homeomorphic, under a linear mapping, to a compact convex subset of $\mathscr{H}_0$.*

Proof: W.l.o.g assume, that K is a subset of the unit ball in X. Since K and the linear hull span(K) of K are separable, we can choose a sequence $\{x_n\}_{n=0}^{\infty}$ that lies dense in span(K). For each $n \in \mathbb{N}$ choose an f_n in the dual space X' such that

$$f_n(x_n) = \frac{\|x_n\|}{n}, \ \|f_n\| = 1/n.$$

Then the mapping $F : x \mapsto (f_1(x), f_2(x), \cdots)$ maps K into $\mathscr{H}_0$. We can see that F is a bounded linear operator on X to l^2. Further we have for $x, y \in$ span(K), $x \neq y$ and x_n sufficiently close to $x - y$

$$\|f_n(x) - f_n(y)\| \geq \|f_n(x_n)\| - \|f_n(x - y - x_n)\| \geq \frac{\|x_n\|}{n} - \frac{\|(x - y) - x_n\|}{n} > 0.$$

So F is injective on span(K) and thus bijective on K to $F(K)$. Since F is also continuous, it is a homeomorphism.
Since compactnes and convexity are preserved under linear homeomorphisms, $F(K)$ is compact and convex. □

Theorem 4.20 *Let K be a nonempty compact convex subset of a Banach space X. Let $T : K \to \mathscr{P}(K)$ satisfy*

- *for each $x \in K$, $T(x)$ is nonempty closed and convex,*
- *T is upper semi-continuous.*

Then T has a fixed point.

Proof: By Lemma (4.19), K is homeomorphic, under a linear mapping, to a compact convex subset M of the Hilbert cube $\mathscr{H}_0$. By Theorem (4.15), M is a retract of $\mathscr{H}_0$. Since the theorem holds for $\mathscr{H}_0$, it also does for M by Lemma (4.16).
Let f be the linear homeomorphism of K onto M. Then $f \circ T \circ f^{-1} : M \to \mathscr{P}(M)$ has a fixed point $y \in (f \circ T \circ f^{-1})(y)$. Thus $f^{-1}(y) \in (T \circ f^{-1})(y)$, and $f^{-1}(y)$ is a fixed point for T. □

`Remark:` Theorem (4.15) and Lemma (4.16) were stated only for the spaces $\mathbb{R}^n$, but the proofs hold, as they are, also for Hilbert spaces.

This fixed point theorem, together with the next one, is the analogon of the Schauder Fixed Point Theorem we saw in the first chapter. To prove the second part, we need the following result by Mazur.

Lemma 4.21 (Mazur) *Let X be a Banach space and $K \subseteq X$ be relatively compact. Then the convex hull co(K) of K is relatively compact.*

Proof: For each $\epsilon > 0$ choose finitely many $(y_i)_{i=1}^m$ such that the ϵ-balls $B_\epsilon(y_i)$ cover K. We define $\widetilde{K} = \{y_1, \cdots, y_m\}$ and $\widetilde{X} = \text{span}\{y_1, \cdots, y_m\}$. Set $R = \max_{1 \le i \le m} \|y_i\|$. Then co$(\widetilde{K}) \subseteq B_R(0)$ in $\widetilde{X}$ and since $\widetilde{X}$ is finite dimensional, co$(\widetilde{K})$ is relatively compact. Let $x = \sum_{k=1}^N \lambda_k x_k \in$co$(K)$ be a convex combination of $x_1, \cdots, x_N \in K$. To each $x_k, k = 1, \cdots N$, there exists a y_{i_k} with $\|x_k - y_{i_k}\| < \epsilon$. Thus, we have for $\widetilde{y} = \sum_{k=1}^N \lambda_k y_{i_k} \in$co$(\widetilde{K})$ the estimate

$$\|x - \widetilde{y}\| \le \sum_{k=1}^N \lambda_k \|x_k - y_{i_k}\| < \epsilon \sum_{k=1}^N \lambda_k = \epsilon.$$

The relatively compact set co$(\widetilde{K})$ is covered by finitely many $B_\epsilon(\widetilde{y}_i), i = 1, \cdots, p$. So for every $x \in$co(K), there exists an i such that

$$\|x - \widetilde{y}_i\| \le \|x - \widetilde{y}\| + \|\widetilde{y} - \widetilde{y}_i\| < 2\epsilon.$$

Thus the balls $(B_{2\epsilon}(\widetilde{y}_i))_{i=1}^p$ provide a open cover of co(K) and therefore co(K) is relatively compact. □

Corollary 4.22 (Bohnenblust-Karlin Fixed Point Theorem) *Let M be a nonempty closed convex subset of a Banach space X. Let $T : M \to \mathscr{P}(M)$ satisfy*

- *for each $x \in M$, $T(x)$ is nonempty closed and convex,*
- *T is upper semi-continuous,*
- *$T(M)$ is relatively compact.*

Then T has a fixed point.

Proof: Since $T(M)$ is relatively compact the closure of the convex hull $\overline{co}(T(M))$ is compact by Lemma (4.21). If we restrict T to $\overline{co}(T(M))$, we can apply Theorem (4.20). □

4.3.2 Topological vector spaces

Next, we want to consider fixed point theorems in topological vector spaces, where we will prove analogous results to the Tychonoff Fixed Point Theorem and even a generalization of it.

Theorem 4.23 (Fan-Glicksberg Fixed Point Theorem) *Let X be a locally convex topological vector space and let $M \subseteq X$ be nonempty compact and convex. Let $T : M \to \mathscr{P}(M)$ satisfy*

- *for each $x \in M$, $T(x)$ is nonempty closed and convex,*
- *T is upper semi-continuous.*

Then T has a fixed point.

In order to prove this, we need the following lemma.

Lemma 4.24 *If C is closed set in X, then the sets*

$$\begin{aligned} Q &= \{x \in M | x \in T(x) + C\}, \\ P &= \{(x, y) | x \in M, y \in T(x) + C\} \end{aligned}$$

are also closed.

Proof: We show that $(M \times X) \backslash P$ is open. Let $(x_0, y_0) \notin P$, i.e., $x_0 \in M$ and $y_0 \notin T(x) + C$. Then there exists a neighbourhood U of zero with

$$(y_0 + U) \cap (T(x_0) + C + U) = \emptyset.$$

Since T is upper semi-continuous, there exists a neighbourhood $V(x_0)$ of x_0 in M with

$$v \in V(x_0) \Rightarrow T(x) \subseteq T(x_0) + U,$$

and hence

$$x \in V(x_0) \Rightarrow (y_0 + U) \cap (T(x) + C) = \emptyset$$

Therefore a neighbourhood of (x_0, y_0) in $M \times X$ does not belong to P.
Analogous arguments show, that Q is also closed. □

Proof of Theorem (4.23): Let B be a neighborhood basis of zero in X which consists of open, balanced, and convex sets. For every $U \in B$ we define

$$S_U = \{x \in M | x \in T(x) + \overline{U}\}.$$

By Lemma (4.24), each of this sets is closed. Below we will show, that S_U is nonempty for each $U \in B$. Then it follows from the choice of B, that the intersection of finitely many S_U is nonempty. By the finite intersection property, there exists an x with $x \in \bigcap_{U \in B} S_U$. This means that $x \in T(x)$.
We now have to show that $S_U \neq \emptyset$. Since M is compact, we can choose finitely many points $x_1, \cdots, x_m \in M$ such that $(x_k + U)_{k=1}^m$ form a covering of M. We set $K = \text{co}\{x_1, \cdots, x_m\}$ and define

$$T_U(x) = (T(x) + \overline{U}) \cap K.$$

Because of $T(x) \subseteq M$ and $U = -U$ it follows that $T_U(x)$ is nonempty, convex, and closed. The set K lies in a finite-dimensional subspace of X which can be identified with $\mathbb{R}^n$. According to Lemma (4.24) the map $T_U : K \to \mathscr{P}(K)$ has a closed graph. The Kakutani Theorem implies the existence of a point x with $x \in T_U(x)$, i.e., $S_U \neq \emptyset$. □

To get an analogous result to the Bohnenblust-Karlin Theorem, we would need an analogous result of Lemma (4.21) in topological vector spaces. Although it is possible to generalize Mazurs result that way, by taking a different path we can achieve a more general result without great difficulties.

Lemma 4.25 *Let X and Y be topological vector spaces. Let $M \subseteq X$ be convex and suppose $F, G : M \to \mathscr{P}(Y)$ satisfy*

- *F is upper semi-continuous,*
- *$F(x)$ is nonempty closed and convex for each $x \in M$,*
- *$F(M)$ is relatively compact*
- *for each $y \in F(M)$, $G^{-1}(y)$ is convex,*
- *$\bigcup_{x \in M} (\mathring{G(x)})$ covers $\overline{F(M)}$.*

Then there exists a point $x_0 \in M$ such that $F(x_0) \cap G(x_0) \neq \emptyset$.

Proof: Since $\overline{F(M)}$ is compact and covered by the open sets $(\mathring{G(x)})$, there is a finite set $N = x_1, \cdots, x_n$ in M such that $F(M) \subseteq \bigcup_{x \in N} (\mathring{G(x)})$. Let $\{f_1, \cdots, f_n\}$ be the partition of unity subordinated to this cover and set $P = \text{co}(N)$. We define $f : \overline{F(M)} \to P$ by

$$f(y) = \sum_{i=1}^{n} f_i(y) x_i = \sum_{i \in N_y} f_i(y) x_i$$

for $y \in \overline{F(M)} \subseteq Y$, where $i \in N_y$ if and only if $f_i(y) \neq 0$, which implies that $y \in (\mathring{G(x_i)})$. Then, for $i \in N_y$, $x_i \in G^{-1}(y)$. Obviously f is continuous and by our assumptions we have $f(y) \in \text{co}\{x_i | i \in N_y\} \subseteq G^{-1}(y)$ for each $y \in F(M)$.
Since $P \subseteq M$ is finite dimensional compact and convex, we can apply the Kakutani Theorem to $f \circ F : P \to \mathscr{P}(P)$. Thus we have a fixed point $x_0 \in P \subseteq M$. Since $x_0 \in (f \circ F)(x_0)$ and $f^{-1}(x_0) \subseteq G(x_0)$, we have $F(x_0) \cap G(x_0) \neq \emptyset$. □

Lemma 4.26 *Let M be a nonempty convex subset of a topological vector space X. Let $T : M \to \mathscr{P}(M)$ be upper semi-continuous and $T(x)$ nonempty, closed, and convex for each $x \in M$, and $T(M)$ relatively compact.*
If $\mathfrak{U} = \{V_i | i \in I\}$ is a finite family covering $\overline{T(M)}$ and V_i is an open convex subset of M for all $i \in I$, then there exists a $V \in \mathfrak{U}$ such that $V \cap T(V) \neq \emptyset$.

Proof: Choose an $x_i \in V_i$ for each $i \in I$ and set $Z = \text{co}\{x_i | i \in I\}$. We define a map $\widetilde{T} : Z \to \mathscr{P}(M)$ as the restriction of T on Z. We further define a map $G : Z \to \mathscr{P}(M)$ by $G^{-1}(x) = Z \cap \bigcap\{V_i | x \in V_i\}$ for each $x \in M$. Then each $G^{-1}(x)$ is open and convex. We also have for any $z \in Z$ and $x \in M$

$$z \in G^{-1}(x) \Leftrightarrow (\text{for each } i, x \in V_i \Rightarrow z \in V_i).$$

We show that $G(z) \subseteq G^{-1}(z)$ and therefore also $\text{co}(G(z)) \subseteq G^{-1}(z)$.
For any $x \in G(z)$, we have $z \in G^{-1}(x)$. Suppose $x \notin G^{-1}(z) \subseteq \bigcap\{V_i | z \in V_i\}$. Then $x \in \bigcup\{V_j | z \notin V_j\}$ and hence, there exists V_j such that $x \in V_j$ and $z \notin V_j$. This contradicts $z \in G^{-1}(x)$.
Since $\overline{\widetilde{T}(Z)}$ is compact and $\overline{\widetilde{T}(M)} \subseteq \bigcup\{V_i | i \in I\}$, $\overline{\widetilde{T}(Z)}$ is covered by a finite number of $G^{-1}(z)$'s.
By Lemma (4.25) there exist $x_0 \in Z$ and $y_0 \in \widetilde{T}(x_0) \cap \text{co}(G(x_0)) = T(x_0) \cap \text{co}(G(x_0)) \neq \emptyset$. Since

$$y_0 \in \text{co}(G(x_0)) \subseteq G^{-1}(x_0) = \bigcap\{V_i | x_0 \in V_i\},$$

we can choose a V_j such that $x_0 \in V_j$ and $y_0 \in V_j$. Note that $y_0 \in V_j \cap T(x_0)$. Hence $y_0 \in V_j \cap T(V_j) \neq \emptyset$. □

Theorem 4.27 (Himmelberg Fixed Point Theorem) *Let M be a nonempty convex subset of a topological vector space X. Let $T : M \to \mathscr{P}(M)$ satisfy*

- *for each $x \in M$, $T(x)$ is nonempty closed and convex,*
- *T is upper semi-continuous,*
- *$T(M)$ is relatively compact.*

Then T has a fixed point.

Proof: Let V be a symmetric neighbourhood of 0 in X. Since $\overline{T(M)}$ is compact we can choose finitely many points $x_i \in M$ and open convex sets $C_i \subseteq V$ such that

$$\overline{T(M)} \subseteq \bigcup\{(x_i + C_i) \cap M | i \in I\}.$$

By Lemma (4.26) there exists a $j \in I$ such that $W \cap T(W) \neq \emptyset$ where $W = (x_j + C_j) \cap M$. Thus there exist a $x_V \in W$ and a $y_V \in W \cap T(x_V)$. Then

$$x_V - y_V \in (x_j + C_j) - (x_j + C_j) = C_j - C_j \subseteq V + V. \tag{4.4}$$

Since the balanced 0-neighbourhoods with the set-theoretical inclusion are directed, the $\{y_V\}$ form a net. And since $y_V \in T(x_V) \subseteq \overline{T(M)}$ and $\overline{T(M)}$ is compact the net $\{y_V\}$ has a subnet converging to a point $x_0 \in M$. The corresponding subnet of $\{x_V\}$ also converges to $x_0 \in M$ by (4.4).
Since T can be viewed as an upper semi-continuous map from M into $\mathscr{P}(\overline{T(M)})$, the graph

$G(T)$ of T is closed by Theorem (2.3) (review the proof to see, that this implication remains true, even if M is not compact).
Now, since $(x_V, y_V) \in G(T)$ converges to $(x_0, x_0) \in G(T)$, we have that $x_0 \in T(x_0)$. □

`Remark:` Had we taken the different route by generalizing Lemma (4.21) to locally convex spaces and arguing like in the proof of Corollary (4.22), we would have gotten the same result but with the additional restriction, that M would have had to be closed too.

Set-valued fixed point theorems naturally always deal with mappings from a set into its own powerset. With our last theorem, we want to give an example, how the concept of fixed point theorems can be generalized.
This leads us to the theory of best approximation, for which the following result by Ky Fan is one of the basic tools. There the question then is: if a mapping does not go from a set into itself but can have values anywhere in the viewed space, can one choose a point that is at least not mapped far away from the set in some sense?

Theorem 4.28 (Ky Fan's Best Approximation Theorem) *Let C be a nonempty compact convex subset of a topological vector space X. Let p be a continuous seminorm on X and denote by $d_p(A, B) = \inf\{p(a-b) | a \in A, b \in B\}$. If $F : C \to \mathscr{P}(X)$ is a continuous set-valued map whith compact convex values, then there exists $y \in C$ such that*

$$d_p(y, F(y)) = d_p(F(y), C).$$

If p is a norm and $d_p(F(y), C) > 0$, then $y \in \partial C$.

Proof: We define a map $Q : C \to \mathscr{P}(C)$ by $Q(x) = \{y \in C | d_p(y, F(x)) = d_p(F(x), C)\}$ and want to apply Theorem (4.23). We first show that Q is upper semi-continuous.
Let A be a closed subset of C. By Proposition (2.2) we have to show that

$$Q^{-1}(A) = \{x \in C | A \cap Q(x) \neq \emptyset\} = \{x \in C | A \cap \{y \in C | \mathrm{dist}(y, F(x)) = \mathrm{dist}(F(x), C)\} \neq \emptyset\}$$

is closed. Let $\{x_i\}_{i \in I}$ be a convergent net in $Q^{-1}(A)$. To every x_i we can choose a $y_i \in A \cap Q(x_i)$. Since C is compact, we can assume without loss of generality, that the y_i converge to a $y_0 \in A \subseteq C$. Denote by x_0 the limit of $\{x_i\}_{i \in I}$, then we have because of the continuity of F

$$d_p(y_0, F(x_0)) = \lim_{i \in I} d_p(y_i, F(x_i)) = \lim_{i \in I} d_p(F(x_i), C) = d_p(F(x_0), C).$$

Thus $y_0 \in A \cap Q(x_0)$ and therefore $Q^{-1}(A)$ is closed.
Since C is compact, $Q(x) \neq \emptyset$ for each $x \in C$. It is also easy to see, that $Q(x)$ is closed: Let $\{y_j\}_{j \in J} \subseteq Q(x)$ converge to y, then

$$d_p(y, F(x)) = \lim_{j \in J} d_p(y_j, F(x)) = d_p(F(x), C).$$

and thus $y \in Q(x)$. Now choose $x, y \in Q(x)$. Since $F(x)$ is compact we can choose $a, b \in F(x)$ with $p(x-a) = p(y-b) = d_p(F(x), C)$. Set $z = \lambda x + (1-\lambda)y$ with $\lambda \in (0, 1)$. Since $F(x)$ is convex, $c = \lambda a + (1-\lambda)b \in F(x)$. We therefore have

$$p(z-c) = p(\lambda x + (1-\lambda)y - \lambda a - (1-\lambda)b) \leq \lambda p(x-a) + (1-\lambda)p(y-b) = d_p(F(x), C).$$

Since $p(z-c) < d_p(F(x), C)$ can not be true, we have equality and hence $z \in Q(x)$. Thus $Q(x)$ is convex.

We are now able to apply the Fan-Glicksberg Theorem. That is, there is a point $x \in C$ such that $d_p(x, F(x)) = d_p(F(X), C)$.
For the additional assertion, assume that p is a norm, $d_p(F(x), C) = d_p(x, F(x)) > 0$ and $x \in \mathring{C}$. We then may choose $y \in F(x)$ such that $p(y - x) = d_p(x, F(x))$. Since C is convex, there exists $z \in C \cap \mathrm{co}\{x, y\} \backslash \{x\}$. Since p is a norm $p(z - x) > 0$. Then

$$d_p(F(x), C) \leq p(z - y) < p(x - y) = d_p(x, F(x)) = d_p(F(x), C),$$

a contradiction. Hence $x \in \partial C$. □

Under special conditions, these theorems can turn into fixed point theorems. For example we can derive a fixed point theorem from the Ky Fan Theorem without any additional work.

Corollary 4.29 *Let C be a nonempty compact convex subset of a Banach space X. If $F : C \to \mathscr{P}(X)$ is a continuous set-valued map with compact convex values, and if $F(x) \cap C \neq \emptyset$ for each $x \in C$, then F has a fixed point.*

Overview of achieved Results

We now give a overview of our main fixed point theorems with their necessary conditions on the space X, the set M, and the set-valued map T and their counterparts for single-valued maps.

Theorem	X	M	$T : M \to \mathscr{P}(M)$	s.v. analogon
Theorem (3.6)	complete metric space	—	T is a contraction with values in $\mathrm{CB}(X)$	Banach Fixed Point Theorem
Theorem (3.9)	ϵ-chainable complete metric space	—	T is a (ϵ, λ)-uniformly local contraction with values in $\mathrm{CB}(X)$	—
Theorem (4.5)	locally convex topological vector space	nonempty, compact, convex	$\forall x \in M : T(x)$ is closed, $\forall y \in M : T^{-1}(y)$ is open	—
Corollary (4.9)	locally convex topological vector space	nonempty, convex	$\forall x \in M : T(x) \neq \emptyset$, $\forall x \in M : T(x)$ is convex, $\exists x_0 \in M : [T^{-1}(x_0)]^C$ is compact, $\forall y \in M : T^{-1}(y)$ is open	—
Corollary (4.10)	locally convex topological vector space	nonempty, convex	$\forall x \in M : T(x) \neq \emptyset$, $\forall x \in M : T(x)$ is open, $\exists x_0 \in M : [T(x_0)]^C$ is compact, $\forall y \in M : T^{-1}(y)$ is convex, $\bigcup_{x \in M} T(x) = M$	—
Theorem (4.11) (Kakutani Fixed Point Theorem)	$\mathbb{R}^n$	nonempty, compact, convex	$\forall x \in M : T(x) \neq \emptyset$, $\forall x \in M : T(x)$ is closed and convex, T is upper semi-continuous	Brouwer Fixed Point Theorem
Theorem (4.20)	Banach space	nonempty, compact, convex	$\forall x \in M : T(x) \neq \emptyset$, $\forall x \in M : T(x)$ is closed and convex, T is upper semi-continuous	Schauder Fixed Point Theorem
Corollary (4.22) (Bohnenblust-Karlin Fixed Point Theorem)	Banach space	nonempty, closed, convex	$\forall x \in M : T(x) \neq \emptyset$, $\forall x \in M : T(x)$ is closed and convex, T is upper semi-continuous, $T(M)$ is relatively compact	Schauder Fixed Point Theorem
Theorem (4.23) (Fan-Glicksberg Fixed Point Theorem)	locally convex topological vector space	nonempty, compact, convex	$\forall x \in M : T(x) \neq \emptyset$, $\forall x \in M : T(x)$ is closed and convex, T is upper semi-continuous	Tychonoff Fixed Point Theorem
Theorem (4.27) (Himmelberg Fixed Point Theorem)	locally convex topological vector space	nonempty, convex	$\forall x \in M : T(x) \neq \emptyset$, $\forall x \in M : T(x)$ is closed and convex, T is upper semi-continuous, $T(M)$ is relatively compact	—

Bibliography

[BB] I.BEG, A.R.BUTT: *Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces,* Nonlinear Analysis (2009), doi:10.1016/j.na.2009.02.027.

[DG] J.DUGUNDJI, A.GRANAS: *KKM maps and variational inequalities,* Annali della Scuola Normale Superiore di Pisa, Classe di Science 4^{e} série, tome 5, n°4 (1978), p.679-682

[I] V.I.ISTRATESCU: *Fixed Point Theory*, D. Reidel Publishing Company, 1981.

[KKM] B.KNASTER, K.KURATOWSKI, AND S.MAZURKIEWICZ: *Ein Beweis des Fixpunktsatzes für n-dimensionale simplexe*, Fundamenta Mathematicae 14 (1929) 132-137.

[N1] S.B.NADLER,JR: *Sequences of contractions and fixed points*, Pacific J. Math. 27 (1968), 579-585.

[N2] S.B.NADLER,JR: *Multi-Valued contraction mappings*, Pacific J. Math. 30 (1969) 475-488.

[P] S.PARK: *Generalizations of the Idzik fixed point theorem*, available online at http://www.math.uncc.edu/ adow/Park.pdf

[R] R.C.ROBINSON: *The Hausdorff metric and Hemicontinuity*, available online at http://www.math.northwestern.edu/ clark/320/2001/metric.pdf

[SXY] B.SIM, H.K.XU, G.X.Z.YUAN: *The homotopic invariance for fixed point of set-valued nonexpansive mappings,* in Y.J.Cho (Ed), Fixed Point Theory an Applications, Nova Science, Huntington, NY,(2000),pp,113-125.

[SWS] S.SINGH, B.WATSON, P.SRIVASTAVA: *Fixed Point Theory and Best Approximation: The KKM-map Principle*, Kluwer Academic Publishers, 1997.

[S] D.R.SMART: *Fixed Point Theorems*, Cambridge University Press, 1974.

[Z1] E.ZEIDLER: *Nonlinear Functional Analysis and its Applications I (Fixed-Point Theorems)*, Springer, 1986.

[Z2] E.ZEIDLER: *Nonlinear Functional Analysis and its Applications IV (Applications to Mathematical Physics)*, Springer, 1988.