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THE BANACH SPACE-VALUED INTEGRALS OF RIEMANN, MCSHANE, HENSTOCK-KURZWEIL AND BOCHNER

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Preface

The transparent definition of the Riemann integral admits an easy extension of the integral to Banach space-valued functions, which was first studied by Graves [11] in 1927. On the other hand, in 1933 Bochner [4] gave a natural extension of the Lebesgue integral to functions which take values in Banach spaces. The Bochner integral inherits most of the *good* properties of Lebesgue integral (a.e. dominated convergence theorem), whereas the definition is not as simple as the one of the Banach space-valued Riemann integral due to the requirement of measure theory.

In an attempt to remedy the technical deficiencies of the Riemann integral while being devoted to its simple definition at the same time, a real-valued integration theory based on the concept of generalized Riemann integral sums was initiated by Kurzweil [15] and independently by Henstock [12] around 1960. The generalized Riemann integral obtained in this way—the so-called Henstock-Kurzweil integral (which is equivalent to the narrow Denjoy integral, the Luzin integral and the Perron integral)— has the properties that it

- ⇝ integrates all functions that have primitives,
- ⇝ integrates all improper Riemann integrable functions,
- ⇝ integrates all Lebesgue integrable functions,
- ⇝ generalizes the monotone convergence theorem and the dominated convergence theorem,
- ⇝ requires no knowledge about measure theory or topology,

and hence is more general than the Lebesgue integral.

In 1969 McShane [18] proposed a slight modification in the definition of the Henstock-Kurzweil integral, which leads to an integral that is equivalent to the Lebesgue integral for real-valued functions—the McShane integral.

When investigations of the Banach space-valued versions of Henstock-Kurzweil and McShane integrals started around 1990 by the work of Gordon [9], it was somewhat surprising that the equivalence of Bochner (as natural extension of the Lebesgue integral) and McShane integrability does not hold for general Banach spaces. In fact, the class of Bochner integrable functions is very restrictive.

This text presents a short survey on the differences in the concepts of Riemann, McShane, Henstock-Kurzweil and Bochner integrability for functions $f : I \to X$, where I is a compact interval in $\mathbb{R}$ and X denotes a Banach space.

In Chapter 1 an elementary introduction to the Riemann sum type integrals of Riemann, McShane and Henstock-Kurzweil is given.

Chapter 2 is devoted to a short presentation of the Bochner integral.

Finally, in Chapter 3 the interrelations of the those integrals is investigated: It is shown that every Bochner integrable function is McShane and hence also Henstock-Kurzweil integrable and that the converse assumption is not true for infinite-dimensional Banach spaces. Moreover an example of a function, which is Henstock-Kurzweil integrable but neither McShane nor Bochner integrable, is presented.

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1 Banach Space-Valued Riemann Sum Type Integrals

1.1 Systems, partitions and gauges

1.1.1 Definition Let a compact interval $I \subset \mathbb{R}$ be given[1].

⇝ We call a pair (t, J) of a *tag* $t \in \mathbb{R}$ and a compact interval $J \subset \mathbb{R}$ a *tagged interval*.

⇝ Two compact intervals $J, L \subset \mathbb{R}$ are called *non-overlapping*, if $J^\circ \cap L^\circ = \emptyset$.

⇝ Let $\mathfrak{J}$ be a finite index set. Then a finite collection

$$\{(t_j, I_j) : j \in \mathfrak{J}\} \tag{1.1}$$

of pairwise non-overlapping tagged intervals is called an *M-system in I* if $I_j \subseteq I$ for all $j \in \mathfrak{J}$.

Notation

For notational simplicity we will omit the always finite index set $\mathfrak{J}, \mathfrak{L}$ etc. in expressions like (1.1) and simply write $\{(t_j, I_j)\}, \{(s_l, L_l)\}$ etc. insofar as it is clear that $j \in \mathfrak{J}$, $l \in \mathfrak{L}$ in expression like sums or integrals.

⇝ We call an M-system $\{(t_j, I_j)\}$ in I, for which $t_j \in I_j$ for all $j \in \mathfrak{J}$ holds, a *K-system in I*.

⇝ We denote an M-system, respectively K-system $\{(t_j, I_j)\}$ in I as *M-partition*, respectively *K-partition* if

$$\bigcup_{j \in \mathfrak{J}} I_j = I.$$

⇝ We call a function $\delta : I \to \mathbb{R}^+$ *gauge on I*, and say a tagged interval (t, J) is *δ-fine* if $J \subseteq B_{\delta(t)}(t)$, where $B_{\delta(t)}(t)$ denotes the open ball in $(\mathbb{R}, |\cdot|)$ centered at t with the radius $\delta(t)$. Moreover, M-systems or K-systems are called δ-fine if all tagged intervals (t_j, I_j) are δ-fine with respect to the same gauge δ.

We observe that every K-system is also an M-system in I; analogously, every K-Partition is also an M-Partition of I.

The following lemma will give rise to the definitions of integrals in the next section. We follow the proof of Kurtz, Kurzweil and Swartz in [14].

1.1.2 Lemma (Cousin) *For every gauge $\delta : I \to \mathbb{R}^+$ there exists a δ-fine K-partition of I.*

Proof Assume that a compact interval $I = [a, b]$ is given and define

$$E := \{t \in (a, b] : [a, t] \text{ has a } \delta\text{-fine } K\text{-partition}\}.$$

Hence, it suffices to show $b \in E$.
At first, we notice that $B_{\delta(a)}(a) \cap (a, b)$ is not empty. Thus $\{(a, [a, x])\}$ with any $x \in B_{\delta(a)}(a) \cap (a, b) \subset E$ is a δ-fine K-partition of $[a, x]$ and E cannot be empty.
Next we show that $t_{\max} := \sup E$ is an element of E: As δ is defined at $t_{\max} \in [a, b]$, we can choose $x \in B_{\delta(t_{\max})}(t_{\max})$ on the assumption $t_{\max} \geq x \in E$. Now let $\{(t_j, I_j)\}$ be a δ-fine K-partition of $[a, x]$. Then $\{(t_j, I_j)\} \cup \{(t_{\max}, [x, t_{\max}])\}$ is a δ-fine K-partition of $[a, t_{\max}]$; we infer $t_{\max} \in E$.

[1] I may consist of only one point, i.e. $I = [a, a]$ for $a \in \mathbb{R}$.

Finally suppose $t_{\max} < b$. Then we can choose $w \in B_{\delta(t_{\max})}(t_{\max}) \cap (t_{\max}, b)$. For any δ-fine K-partition $\{(t_j, I_j)\}$ of $[a, t_{\max}]$ we know that $\{(t_j, I_j)\} \cup \{(t_{\max}, [t_{\max}, w])\}$ is a δ-fine K-partition of $[a, w]$. As $t_{\max} < w$ is a contradiction to the definition of E, we infer $t_{\max} = b$. ■

1.2 Definitions of Banach Space-Valued Riemann Sum Type Integrals

1.2.1 Definition Let a function $f : I \to X$ be given where X is a Banach space with the norm $\|\cdot\|_X$ and the compact interval I is endowed with the Lebesgue measure λ.

⇝ f is said to be Riemann integrable and $x \in X$ its *Riemann integral* if for every $\varepsilon > 0$ there exists a constant gauge $\delta \in \mathbb{R}^+$ such that for every δ-fine K-partition $\{(t_i, I_i)\}$ of I the inequality

$$\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) - x \right\|_X < \varepsilon \tag{1.2}$$

holds.

⇝ f is said to be *McShane integrable* and $y \in X$ its *McShane integral* if for every $\varepsilon > 0$ there exists a gauge $\delta : I \to \mathbb{R}^+$ such that for every δ-fine M-partition $\{(t_i, I_i)\}$ of I the inequality

$$\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) - y \right\|_X < \varepsilon \tag{1.3}$$

holds.

⇝ f is said to be *Henstock-Kurzweil integrable* and $z \in X$ its *Henstock-Kurzweil integral* if for every $\varepsilon > 0$ there exists a gauge $\delta : I \to \mathbb{R}^+$ such that for every δ-fine K-partition $\{(t_i, I_i)\}$ of I the inequality

$$\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) - z \right\|_X < \varepsilon \tag{1.4}$$

holds.

⇝ If a subset $E \subseteq I$ is given, a function $f : I \to X$ is called *integrable over* E if the function $\mathbb{1}_E \cdot f$ is integrable.

Notation

We denote the set of all Riemann integrable functions with $\mathcal{R}$, the set of all McShane integrable functions with $\mathcal{M}$ and the set of all Henstock-Kurzweil integrable functions with $\mathcal{HK}$.

The Riemann integral, the McShane integral and the Henstock-Kurzweil integral of a function f are denoted by

$$\overset{\mathcal{R}}{\int_I} f\,, \quad \overset{\mathcal{M}}{\int_I} f \quad \text{and} \quad \overset{\mathcal{HK}}{\int_I} f \quad \text{respectively.}$$

The following result immediately follows from the definition above:

1.2.2 Proposition *The inclusions $\mathcal{R} \subseteq \mathcal{HK}$ and $\mathfrak{M} \subseteq \mathcal{HK}$ hold and if the corresponding integrals of $f : I \to X$ exist, they do coincide.*

Proof To show $\mathcal{R} \subseteq \mathcal{HK}$ we notice that the Riemann integral is a restriction of the Henstock-Kurzweil integral by assuming the gauge to be constant in $\mathbb{R}^+$.

Next we show $\mathcal{M} \subseteq \mathcal{HK}$: When compared to an M-partition of I, a K-partition of I imposes a greater restriction by assuming the tag to be in the corresponding interval of the partition. Thus the number of δ-fine partitions in Definition 1.2.1 of the Henstock-Kurzweil integral is decreased which results in the Henstock-Kurzweil integral being more general than the McShane integral, i.e. $\mathcal{M} \subseteq \mathcal{HK}$.

Clearly, if the Riemann or the McShane integrals exist, they do coincide with the Henstock-Kurzweil integral. ∎

In Chapter 3 we will even observe that the inclusions $\mathcal{R} \subset \mathcal{HK}$ and $\mathcal{M} \subset \mathcal{HK}$ are proper.

1.3 Elementary Properties of Banach Space-Valued Riemann Sum Type Integrals

In this section we will start with a Cauchy criterion for the existence of the McShane integral:

1.3.1 Proposition *A function $f : I \to X$ is McShane integrable iff for every $\varepsilon > 0$ there exists a gauge $\delta : I \to \mathbb{R}^+$ such that for every δ-fine M-partitions $\{(t_i, I_i)\}$ and $\{(s_j, J_j)\}$ of I the inequality*

$$\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) - \sum_{j \in \mathfrak{J}} f(s_j)\lambda(J_j) \right\|_X < \varepsilon \tag{1.5}$$

holds.

Proof Let be $f \in \mathcal{M}$. According to Definition 1.2.1 for every $\varepsilon > 0$ there exists a gauge $\delta : I \to \mathbb{R}^+$ such that for every δ-fine M-partition $\{(t_i, I_i)\}$ of I the inequality

$$\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) - \mathcal{M}\!\!\int_I f \right\|_X < \varepsilon$$

holds. Thus we obtain

$$\begin{aligned} &\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) - \sum_{j \in \mathfrak{J}} f(s_j)\lambda(J_j) \right\|_X \leq \\ \leq &\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) - \mathcal{M}\!\!\int_I f \right\|_X + \left\| \sum_{j \in \mathfrak{J}} f(s_j)\lambda(J_j) - \mathcal{M}\!\!\int_I f \right\|_X < 2\varepsilon \end{aligned}$$

and (1.5) holds for any δ-fine M-partitions $\{(t_i, I_i)\}$ and $\{(s_j, J_j)\}$ of I.

On the other hand, to prove the converse statement, denote

$$S(\delta) := \left\{ \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) : \{(t_i, I_i) : i \in \mathfrak{T}\} \text{ is a } \delta\text{-fine } M\text{-partition of } I \right\} \subseteq X$$

and by $\mathcal{F}$ the set of all $S(\delta)$, $\delta : I \to \mathbb{R}^+$. Every set $S(\delta) \in \mathcal{F}$ is nonempty because for every gauge δ there exists a δ-fine M-partition $\{(t_i, I_i)\}$ of I due to Lemma 1.1.2. Moreover,

for any two gauges δ_1, δ_2 on I the function $\delta_{\min} \coloneqq \min\{\delta_1, \delta_2\} : I \to \mathbb{R}^+$ defines again a gauge on I and clearly each $\delta_{\min}$-fine M-partition $\{(t_i, I_i)\}$ of I is also δ_1-fine and δ_2-fine. Thus the intersection of any two sets $S(\delta_1), S(\delta_2) \in \mathcal{F}$ contains again a set $S(\delta_{\min}) \in \mathcal{F}$, i.e. $S(\delta_{\min}) \subseteq S(\delta_1) \cap S(\delta_2)$. We infer that $\mathcal{F}$ is a Filter base on $(X, \|\cdot\|_X)$.
According to our premise, for every $\varepsilon > 0$ there is an $S(\delta) \in \mathcal{F}$ such that $\operatorname{diam} S(\delta) \le \varepsilon$ since (1.5) holds for any two δ-fine M-partitions $\{(t_i, I_i)\}$, $\{(s_j, J_j)\}$ of I; hence the filter base $\mathcal{F}$ is Cauchy.
Due to the completeness of the Banach space X, $\mathcal{F}$ converges to a single point $s \in X$, i.e.

$$\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) - s \right\|_X < \varepsilon$$

and $\sum_{i \in \mathfrak{T}} f(t_i)\lambda(I_i) \in S(\delta)$ whenever $\{(t_i, I_i)\}$ is an arbitrary δ-fine M-partition of I. We conclude $f \in \mathcal{M}$ by Definition 1.2.1. ∎

1.3.2 Corollary *Assume that $f : I \to X$ is McShane integrable and let $J \subseteq I$ be a compact interval. Then f is McShane integrable over J.*

Proof By Proposition 1.3.1 for any given $\varepsilon > 0$ there exists a gauge $\delta : I \to \mathbb{R}^+$ such that for every δ-fine M-partitions $\{(t_i, I_i)\}$ and $\{(s_j, J_j)\}$ of I inequality (1.5) is satisfied.
Let $\{(\tau_k, K_k)\}$ and $\{(\sigma_l, L_l)\}$ be arbitrary δ-fine M-partitions of the interval J. The compact complement $I \setminus J^\circ$ can be expressed as a finite union of compact intervals contained in I. By taking an arbitrary δ-fine M-partition of each of those intervals we obtain a finite collection $\{(\rho_m, M_m)\}$ of tagged intervals which together with $\{(\tau_k, K_k)\}$ or $\{(\sigma_l, L_l)\}$ form two δ-fine M-partitions of the interval I.
Taking the difference of the integral sums corresponding to these two δ-fine M-partitions of I, we can see that its value is

$$\sum_{k \in \mathfrak{K}} f(\tau_k)\lambda(K_k) - \sum_{l \in \mathfrak{L}} f(\sigma_l)\lambda(L_l)$$

because the remaining $\sum_{m \in \mathfrak{M}} f(\rho_m)\lambda(M_m)$ is the same for both of them. Whence by (1.5) we obtain

$$\left\| \sum_{k \in \mathfrak{K}} f(\tau_k)\lambda(K_k) - \sum_{l \in \mathfrak{L}} f(\sigma_l)\lambda(L_l) \right\|_X < \varepsilon .$$

By Proposition 1.3.1 this implies the McShane integrability of f on J. ∎

1.3.3 Proposition *Assume that $J, K \subset \mathbb{R}$ are non-overlapping compact intervals such that $J \cup K$ is again an interval in $\mathbb{R}$. If a function $f : J \cup K \to X$ is McShane integrable on each of the intervals J and K, then f is McShane integrable on the interval $J \cup K$ and*

$$\mathcal{M}\!\!\int_{J \cup K} f = \mathcal{M}\!\!\int_J f + \mathcal{M}\!\!\int_K f . \tag{1.6}$$

Proof Denote by $\{w\} = J \cap K$ the common face of both intervals J and K in $\mathbb{R}$.
By assumption, there is a gauge δ_1 on J and a gauge δ_2 on K such that for every δ_1-fine M-partition $\{(t_i, J_i)\}$ of J we have

$$\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(J_i) - \mathcal{M}\!\!\int_J f \right\|_X < \varepsilon$$

and for every δ_2-fine M-partition $\{(s_j, K_j)\}$ of K we have

$$\left\| \sum_{j\in\mathfrak{J}} f(s_j)\lambda(K_j) - \mathcal{M}\!\!\int_K f \right\|_X < \varepsilon.$$

Define $\delta : J \cup K \to \mathbb{R}^+$ by

$$\delta(t) := \begin{cases} \min\{\delta_1(t), \operatorname{dist}(t,w)\} & \text{if } t \in J \setminus \{w\} \\ \min\{\delta_1(t), \delta_2(t)\} & \text{if } t = w \\ \min\{\delta_2(t), \operatorname{dist}(t,w)\} & \text{if } t \in K \setminus \{w\}. \end{cases}$$

Let $\{(t_i, I_i)\}$ be a δ-fine M-partition of $J \cup K$. Note that by definition of δ, we have $I_i \subseteq B_{\delta(t_i)}(t_i) \cap J \subseteq J \setminus \{w\}$ whenever $t_i \in J \setminus \{w\}$, and similarly $I_i \subseteq B_{\delta(t_i)}(t_i) \cap K \subseteq K \setminus \{w\}$ whenever $t_i \in K \setminus \{w\}$. Hence there exists at least one tagged interval $(t_i, I_i) \in \{(t_i, I_i)\}$ with $t_i = w$.

Consider the tagged intervals (t_i, I_i) for which $t_i = w$. Then $(t_i, I_i \cap J)$ is δ_1-fine and $(t_i, I_i \cap K)$ is δ_2-fine while for the corresponding term in the integral sum we have

$$f(t_i)\lambda(I_i) = f(t_i)\lambda(I_i \cap J) + f(t_i)\lambda(I_i \cap K).$$

The system of tagged intervals $\{(t_i, I_i) : t_i \in J\} \cup \{(t_i, I_i \cap J) : t_i = w\}$ is a δ_1-fine M-partition of J and the system of tagged intervals $\{(t_i, I_i) : t_i \in K\} \cup \{(t_i, I_i \cap K) : t_i = w\}$ is a δ_2-fine M-partition of K.

Now we have

$$\left\| \sum_{i\in\mathfrak{T}} f(t_i)\lambda(I_i) - \mathcal{M}\!\!\int_J f - \mathcal{M}\!\!\int_K f \right\|_X =$$

$$= \left\| \sum_{\substack{i\in\mathfrak{T} \\ t_i\in J\setminus\{w\}}} f(t_i)\lambda(I_i) + \sum_{\substack{i\in\mathfrak{T} \\ t_i=w}} f(t_i)\lambda(I_i) + \sum_{\substack{i\in\mathfrak{T} \\ t_i\in K\setminus\{w\}}} f(t_i)\lambda(I_i) - \mathcal{M}\!\!\int_J f - \mathcal{M}\!\!\int_K f \right\|_X =$$

$$= \left\| \sum_{\substack{i\in\mathfrak{T} \\ t_i\in J\setminus\{w\}}} f(t_i)\lambda(I_i) + \sum_{\substack{i\in\mathfrak{T} \\ t_i=w}} f(t_i)\big(\lambda(I_i \cap J) + \lambda(I_i \cap K)\big) + \sum_{\substack{i\in\mathfrak{T} \\ t_i\in K\setminus\{w\}}} f(t_i)\lambda(I_i) - \mathcal{M}\!\!\int_J f - \mathcal{M}\!\!\int_K f \right\|_X \le$$

$$\le \left\| \sum_{\substack{i\in\mathfrak{T} \\ t_i\in J\setminus\{w\}}} f(t_i)\lambda(I_i) + \sum_{\substack{i\in\mathfrak{T} \\ t_i=w}} f(t_i)\lambda(I_i \cap J) - \mathcal{M}\!\!\int_J f \right\|_X + \left\| \sum_{\substack{i\in\mathfrak{T} \\ t_i=w}} f(t_i)\lambda(I_i \cap K) + \sum_{\substack{i\in\mathfrak{T} \\ t_i\in K\setminus\{w\}}} f(t_i)\lambda(I_i) - \mathcal{M}\!\!\int_K f \right\|_X <$$

$$< 2\varepsilon.$$

Hence f is McShane integrable on $J \cup K$ and (1.6) holds. ■

1.3.4 Facts Let functions $f, g : I \to X$ and $c \in \mathbb{R}$ be given.

⇝ Let $f, g \in \mathcal{HK}$; then the integral sums for cf equal c times the integral sums for f and the integral sums for $f + g$ are the sum of integral sums for f and g. Thus $cf + g$ is Henstock-Kurzweil integrable and

$$\mathcal{HK}\!\!\int_I (cf + g) = c\,\mathcal{HK}\!\!\int_I f + \mathcal{HK}\!\!\int_I g.$$

Analogous statements also hold for the Riemann and the McShane integral.

⇝ Following the proof of Proposition 1.3.1 with the replacement of arbitrary gauges by constant gauges $\delta \in \mathbb{R}^+$ and M-partitions by K-partitions yields the analogous result for the Riemann integral:

f is Riemann integrable iff for every $\varepsilon > 0$ there exists a constant gauge $\delta \in \mathbb{R}^+$ such that for every δ-fine K-partitions $\{(t_i, I_i)\}$ and $\{(s_j, J_j)\}$ of I inequality (1.5) holds.

⇝ If we replace M-partitions by K-partitions in the proof of Proposition 1.3.1, then *f is Henstock-Kurzweil integrable iff for every $\varepsilon > 0$ there exists a gauge $\delta : I \to \mathbb{R}^+$ such that for every δ-fine K-partitions $\{(t_i, I_i)\}$ and $\{(s_j, J_j)\}$ of I inequality (1.5) holds.*

⇝ Similarly Corollary 1.3.2 and Proposition 1.3.3 can be adapted:
If $f : I \to X$ is Henstock-Kurzweil (Riemann) integrable, then f is Henstock-Kurzweil (Riemann) integrable over every compact subset $J \subseteq I$.

If $J, K \subset \mathbb{R}$ are non-overlapping compact intervals such that $J \cup K$ is again an interval and f is Henstock-Kurzweil (Riemann) integrable on each of the intervals J and K, then f is Henstock-Kurzweil (Riemann) integrable on $J \cup K$ and

$$\mathcal{HK}\!\!\int_{J\cup K} f = \mathcal{HK}\!\!\int_J f + \mathcal{HK}\!\!\int_K f \qquad \text{or} \qquad \mathcal{R}\!\!\int_{J\cup K} f = \mathcal{R}\!\!\int_J f + \mathcal{R}\!\!\int_K f.$$

⇝ Let a gauge $\delta : I \to \mathbb{R}^+$ be given. Then for every δ-fine K-partition $\{(t_i, I_i)\}$ of I the value of the integral sum $\sum_{i\in\mathfrak{T}} f(t_i)\lambda(I_i)$ remains unchanged if we assume that either all of the tags of $\{(t_i, I_i)\}$ occur as endpoints or each tag of $\{(t_i, I_i)\}$ occurs only once since every tagged interval $(t_i, I_i) = (t_i, [c_i, d_i])$ is δ-fine iff the K-system $\{(t_i, [c_i, t_i]), (t_i, [t_i, d_i])\}$ of I is δ-fine and $f(t_i)\lambda([c_i, d_i]) = f(t_i)\lambda([c_i, t_i]) + f(t_i)\lambda([t_i, d_i])$ for each $i \in \mathfrak{T}$.
Note that this statement does not hold for arbitrary δ-fine M-partitions of I.

2 The Bochner Integral

2.1 Simple Functions and Measurability

2.1.1 Definition Let X again be a Banach space endowed with the norm $\|\cdot\|_X$.

⇝ A function $f : I \to X$ is called *simple* if it is of the form

$$f = \sum_{m \in \mathfrak{M}} \mathbb{1}_{E_m} x_m \tag{2.1}$$

where $(E_m)_{m \in \mathfrak{M}}$ is a finite set of Lebesgue measurable subsets of I such that $x_m \in X$, $E_m \cap E_l = \emptyset$ for $m \neq l$, $m, n \in \mathfrak{M}$ and $I = \bigcup_{m \in \mathfrak{M}} E_m$.

⇝ The *integral of a simple function* f is defined by

$$\int_I f = \sum_{m \in \mathfrak{M}} \lambda(E_m) x_m \tag{2.2}$$

⇝ A function $f : I \to X$ is called *measurable* if there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions with

$$\lim_{n \to \infty} \|f_n(t) - f(t)\|_X = 0$$

for almost all $t \in I$.

Notation

Denote by $\mathcal{J}$ the set of all simple functions defined on I.

2.1.2 Facts

⇝ Clearly $\mathcal{J}$ is a linear space and if $f : I \to X$ is a simple function then also $\|f\|_X : I \to \mathbb{R}_0^+$ is a simple function.

⇝ The integral of simple functions is a linear mapping $\int_I : \mathcal{J} \to X$.

⇝ Let f be a simple function as in (2.1). By virtue of (2.2) and

$$\left\| \int_A f \right\|_X = \left\| \sum_{m \in \mathfrak{M}} \lambda(A \cap E_m) x_m \right\|_X \leq \sum_{m \in \mathfrak{M}} \lambda(A \cap E_m) \|x_m\|_X = \int_A \|f\|_X$$

we gain the inequality

$$\left\| \int_A f \right\|_X \leq \int_A \|f\|_X \tag{2.3}$$

for every measurable $A \subseteq I$.

⇝ Obviously every $f \in \mathcal{J}$ is measurable.

⇝ Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of simple functions corresponding to a measurable $f : I \to X$. Since

$$\Big| \|f_n(t)\|_X - \|f(t)\|_X \Big| \leq \|f_n(t) - f(t)\|_X$$

for all $t \in I$, we infer

$$\lim_{n \to \infty} \|f_n(t)\|_X = \|f(t)\|_X$$

almost everywhere in I; therefore $\|f\|_X$ is measurable.

The space $\mathcal{J}$ can be equipped with a seminorm:

2.1.3 Definition The mapping

$$\begin{aligned} \|\cdot\|_1 : \mathcal{J} &\longrightarrow \mathbb{R}_0^+ \\ f &\longmapsto \int_I \|f\|_X \end{aligned}$$

is called the *L-seminorm* on the space of simple functions $\mathcal{J}$.

Using the fact that, given any two measurable finite partitions $(E_m)_{m\in\mathfrak{M}}$, $(F_n)_{n\in\mathfrak{N}}$ of I, $(E_m \cap F_n)_{(m,n)\in\mathfrak{M}\times\mathfrak{N}}$ constitutes again such a finite partition, it is easy to see that $\|\cdot\|_1$ has indeed the properties of a seminorm on $\mathcal{J}$:

(JN1) $\|cf\|_1 = |c| \cdot \|f\|_1$ for every $f \in \mathcal{J}$ and $c \in \mathbb{R}$,

(JN2) $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$ for every $f, g \in \mathcal{J}$.

For $f := \mathbb{1}_{\{t\}}x$ with $t \in I$, $x \in X$ such that $\|x\|_X \neq 0$, the implication $\|f\|_1 = 0 \Longrightarrow f = 0$ does not hold. Henceforth $\|\cdot\|_1$ is not a norm on $\mathcal{J}$.

2.2 Sequences of Simple Functions

Let us now consider sequences of simple functions equipped with the L-seminorm $\|\cdot\|_1$ given in the previous section.

2.2.1 Definition Let sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ with $f_n, g_n \in \mathcal{J}$ be given.

⇝ $(f_n)_{n\in\mathbb{N}}$ is called *L-zero* if

$$\lim_{n\to\infty} \|f_n\|_1 = 0.$$

⇝ $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are called *equivalent* if their difference $f_n - g_n$ is L-zero.

⇝ $(f_n)_{n\in\mathbb{N}}$ is called *L-Cauchy* if for every $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that $\|f_q - f_r\|_1 < \varepsilon$ for any $q, r \geq N_\varepsilon$.

2.2.2 Facts

⇝ Using (JN1) and (JN2) it is easy to show that the set of L-Cauchy sequences of simple functions has the structure of a linear space.

⇝ For a given L-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions we have

$$\Big| \|f_q(t)\|_X - \|f_r(t)\|_X \Big| \leq \|f_q(t) - f_r(t)\|_X \qquad \text{for } q, r \in \mathbb{N},\ t \in I$$

and thus

$$\begin{aligned} \Big\| \|f_q\|_X - \|f_r\|_X \Big\|_1 &= \int_I \Big| \|f_q(t)\|_X - \|f_r(t)\|_X \Big| \leq \\ &\leq \int_I \|f_q(t) - f_r(t)\|_X = \|f_q - f_r\|_1 . \end{aligned}$$

This means that the sequence $(\|f_n\|_X)_{n\in\mathbb{N}}$ of real-valued simple functions is L-Cauchy.

2.2.3 Lemma *Let $(f_n)_{n\in\mathbb{N}}$ be an L-Cauchy sequence of simple functions defined on I. Then there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, which converges pointwise almost everywhere to some function $f : I \to X$ and for every $\varepsilon > 0$ there is a measurable $E \subseteq I$ with $\lambda(E) < \varepsilon$ such that this subsequence converges uniformly on $I \setminus E$.*

Proof As the sequence $(f_n)_{n\in\mathbb{N}}$ is L-Cauchy, for every $k \in \mathbb{N}$ there exists an $N_k \in \mathbb{N}$ such that if $n, r \geq N_k$, then

$$\|f_n - f_r\|_1 < \frac{1}{2^{2k}}.$$

Without loss of generality we can assume $N_k < N_{k+1}$ for every $k \in \mathbb{N}$. Then for $m \geq n$ we have

$$\|f_{N_m} - f_{N_n}\|_1 < \frac{1}{2^{2n}}.$$

Next we define the series

$$f_{N_1}(t) + \sum_{n\in\mathbb{N}} \Big(f_{N_{n+1}}(t) - f_{N_n}(t)\Big) = \lim_{n\to\infty} f_{N_n}(t) \tag{2.4}$$

for $t \in I$ and show that it converges absolutely almost everywhere in I to an element in X and that this convergence is uniform except for a set with arbitrary small measure.

For $n \in \mathbb{N}$ set

$$M_n := \left\{ t \in I : \left\|f_{N_{n+1}}(t) - f_{N_n}(t)\right\|_X \geq \frac{1}{2^n} \right\}. \tag{2.5}$$

Then

$$\frac{\lambda(M_n)}{2^n} = \int\limits_{M_n} \frac{1}{2^n} \leq \int\limits_{M_n} \left\|f_{N_{n+1}}(t) - f_{N_n}(t)\right\|_X \leq$$
$$\leq \int\limits_I \left\|f_{N_{n+1}}(t) - f_{N_n}(t)\right\|_X = \left\|f_{N_{n+1}} - f_{N_n}\right\|_1 < \frac{1}{2^{2n}}$$

and we infer

$$\lambda(M_n) < \frac{1}{2^n}.$$

Define

$$Z_n := \bigcup_{i\geq n} M_i.$$

Then $Z_{n+1} \subseteq Z_n$ for all $n \in \mathbb{N}$ and

$$\lambda(Z_n) \leq \sum_{j\geq n} \lambda(M_j) < \sum_{j\geq n} \frac{1}{2^j} = \frac{1}{2^{n-1}}.$$

By (2.5) we obtain for $t \notin Z_n$ and $k \geq n$ the estimate

$$\left\|f_{N_{k+1}}(t) - f_{N_k}(t)\right\|_X < \frac{1}{2^k}.$$

Therefore, the series

$$\sum_{k\geq n} \Big(f_{N_{k+1}}(t) - f_{N_k}(t)\Big)$$

and hence also the series in (2.4) converges absolutely and uniformly on $I \setminus Z_n$.

Let $\varepsilon > 0$ be be given. Setting $N := Z_n$, we have for sufficiently large $n \in \mathbb{N}$

$$\lambda(N) = \lambda(Z_n) < \frac{1}{2^{n-1}} < \varepsilon.$$

Henceforth the series (2.4) converges absolutely and uniformly on $I \setminus N$. Clearly, also $(f_{N_n})_{n\in\mathbb{N}}$ converges absolutely and uniformly on $I \setminus N$.
If we take

$$M := \bigcap_{n\in\mathbb{N}} Z_n,$$

then $\lambda(M) = 0$. Furthermore, if $t \notin M$, then $t \notin Z_n$ for at least one $n \in \mathbb{N}$. Thus, the series in (2.4) converges for $t \notin M$, and therefore $\lim\limits_{n\to\infty} f_{N_n}(t)$ exists for almost all $t \in I$. ■

2.2.4 Lemma *Consider the L-Cauchy sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ of X-valued simple functions. Then the following statements hold:*

(CS1) *The limits $\lim\limits_{n\to\infty} \int_I f_n$ and $\lim\limits_{n\to\infty} \int_I g_n$ exist in X.*

(CS2) *If $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are equivalent then $\lim\limits_{n\to\infty} \int_I f_n = \lim\limits_{n\to\infty} \int_I g_n$.*

(CS3) *If $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ converge almost everywhere to a function $f : I \to X$, then $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are equivalent.*

Proof First we show (CS1):
For simple functions $f_n \in \mathcal{J}$, $n \in \mathbb{N}$ using (2.3) we have

$$\left\| \int_I f_q - \int_I f_r \right\|_X = \left\| \int_I (f_q - f_r) \right\|_X \le \int_I \|f_q - f_r\|_X = \|f_q - f_r\|_1$$

for $q, r \in \mathbb{N}$. This means that the sequence of integrals $\left(\int_I f_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence, convergent, i.e. the limit $\lim\limits_{n\to\infty} \int_I f_n$ exists.

Next we prove (CS2):
Given $\varepsilon > 0$, by (CS1) and the equivalence of the L-Cauchy sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ there exists an $N \in \mathbb{N}$ such that for every $r \in \mathbb{N}$, $r \ge N$

$$\left\| \int_I f_r - \lim_{n\to\infty} \int_I f_n \right\|_X < \varepsilon, \qquad \left\| \int_I g_r - \lim_{n\to\infty} \int_I g_n \right\|_X < \varepsilon,$$

$$\|f_r - g_r\|_1 = \int_I \|f_r - g_r\|_X < \varepsilon.$$

This gives

$$\left\| \lim_{n\to\infty} \int_I f_n - \lim_{n\to\infty} \int_I g_n \right\|_X \le \left\| \lim_{n\to\infty} \int_I f_n - \int_I f_r \right\|_X + \left\| \int_I f_r - \int_I g_r \right\|_X + \\ + \left\| \int_I g_r - \lim_{n\to\infty} \int_I g_n \right\|_X < 2\varepsilon + \int_I \|f_r - g_r\|_X < 3\varepsilon.$$

Finally we show (CS3):
For $h_n := f_n - g_n$, we have $\lim\limits_{n\to\infty} h_n(t) = 0$ for almost all $t \in I$ and the sequence $(h_n)_{n\in\mathbb{N}}$ is L-Cauchy, i.e. for given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for $r, q \ge N$ we have

$\|h_q - h_r\|_1 < \varepsilon$. Thus the sequences of integrals $\big(\int_I h_n\big)_{n\in\mathbb{N}}$ is convergent. Due to Facts 2.2.2, $(\|h_n\|)_{n\in\mathbb{N}}$ is also L-Cauchy and therefore $\big(\int_I \|h_n\|_X\big)_{n\in\mathbb{N}}$ convergent.

It remains to prove $\lim\limits_{n\to\infty} \int_I \|h_n\|_X = 0$.

We define $M := \{t \in I : h_N(t) \neq 0\} \subseteq I$. For $n \geq N$ we have

$$\int\limits_{I\setminus M} \|h_n\|_X = \int\limits_{I\setminus M} \|h_n - h_N\|_X \leq \int\limits_I \|h_n - h_N\|_X = \|h_n - h_N\|_1 < \varepsilon$$

since $h_N(t) = 0$ for $t \in I \setminus M$.

As a consequence of Lemma 2.2.3 there exists a measurable subset $Z \subseteq M$ with

$$\lambda(Z) < \frac{\varepsilon}{\sup\limits_{t\in I} \|h_N(t)\|_X + 1}$$

and a subsequence $(h_{n_s})_{s\in\mathbb{N}}$ which converges to 0 uniformly on the set $M \setminus Z$. Hence, there exists an $s_0 \in \mathbb{N}$ with $s_0 \geq N$ such that for $s \geq s_0$ and for $t \in M \setminus Z$ we have

$$\|h_{n_s}(t)\|_X < \frac{\varepsilon}{\lambda(I)}.$$

Therefore, for all $s \geq s_0$ we have

$$\int\limits_{M\setminus Z} \|h_{n_s}(t)\|_X < \frac{\varepsilon\lambda(M \setminus Z)}{\lambda(I)} \leq \varepsilon.$$

Moreover, for all $s \geq s_0$ we obtain

$$\begin{aligned}\int\limits_Z \|h_{n_s}(t)\|_X &\leq \int\limits_Z \|h_{n_s}(t) - h_N(t)\|_X + \int\limits_Z \|h_N(t)\|_X \leq \\ &\leq \|h_{n_s} - h_N\|_1 + \sup_{t\in I} \|h_N(t)\|_X \,\lambda(Z) < \\ &< \varepsilon + \frac{\varepsilon}{\sup\limits_{t\in I} \|h_N(t)\|_X + 1} \sup_{t\in I} \|h_N(t)\|_X < 2\varepsilon.\end{aligned}$$

Henceforth

$$\begin{aligned}\|h_{n_s}\|_1 = \int\limits_I \|h_{n_s}(t)\|_X &= \int\limits_{I\setminus M} \|h_{n_s}(t)\|_X + \int\limits_{M\setminus Z} \|h_{n_s}(t)\| + \int\limits_Z \|h_{n_s}(t)\|_X < \\ &< \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon,\end{aligned}$$

and we infer $\lim\limits_{s\to\infty} \int_I \|h_{n_s}\|_X = 0$. As $\big(\int_I \|h_n\|_X\big)_{n\in\mathbb{N}}$ converges, we conclude $\lim\limits_{n\to\infty} \int_I \|h_n\|_X = 0$. ∎

2.3 Definition of the Bochner Integral

By virtue of (CS1) on page 13 we can assign a value $x_{(f_n)_{n\in\mathbb{N}}} \in X$ to every L-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions by the relation

$$x_{(f_n)_{n\in\mathbb{N}}} := \lim_{n\to\infty} \int\limits_I f_n.$$

Moreover, by using (CS2) we can assure that the same value $x_{(f_n)_{n\in\mathbb{N}}} \in X$ belongs to all L-Cauchy sequences which are equivalent to the sequence $(f_n)_{n\in\mathbb{N}}$.
Henceforth the following definition makes sense.

2.3.1 Definition Consider a function $f : I \to X$.

⇝ We say that the L-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions *determines* f if it converges to f almost everywhere in I, i.e.

$$\lim_{n\to\infty} \|f_n(t) - f(t)\|_X = 0$$

for almost all $t \in I$.

⇝ f is said to be *Bochner integrable* and $x \in X$ its *Bochner integral* if there is an L-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions which determines f and

$$x = \lim_{n\to\infty} \int_I f_n .$$

Notation

We denote by $\mathcal{B}$ the set of all Bochner integrable functions. The Bochner integral of a function f is denoted by

$$\mathcal{B}\!\!\!\!\int_I f .$$

2.4 Elementary Properties of the Bochner Integral

2.4.1 Facts

⇝ The set $\mathcal{B}$ forms a linear vector space of measurable functions because

$$c\,\mathcal{B}\!\!\!\!\int_I f + \mathcal{B}\!\!\!\!\int_I g = \lim_{n\to\infty}\Big(c\int_I f_n + \int_I g_n\Big) = \lim_{n\to\infty}\int_I (cf_n + g_n) = \mathcal{B}\!\!\!\!\int_I cf + g \qquad c \in \mathbb{R}$$

for every Bochner integrable functions f and g determined by $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ respectively.

⇝ Let $f \in \mathcal{B}$ be determined by a L-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions. In Facts 2.2.2 we observed that then $(\|f_n\|_X)_{n\in\mathbb{N}}$ is an L-Cauchy sequence of real-valued simple functions. Furthermore $\lim_{n\to\infty} \|f_n(t)\|_X = \|f(t)\|_X$ holds for almost all $t \in I$. We infer that $\|f\|_X : I \to \mathbb{R}_0^+$ is integrable by Definition 2.3.1.

⇝ Applying Definition 2.3.1 to our considerations above we obtain

$$\int_I \|f\|_X = \lim_{n\to\infty}\int_I \|f_n\|_X = \lim_{n\to\infty} \|f_n\|_1 \tag{2.6}$$

for $f \in \mathcal{B}$ determined by $(f_n)_{n\in\mathbb{N}}$.

$\rightsquigarrow$ Using Definition 2.3.1, (2.3) and (2.6) on $f \in \mathcal{B}$ determined by $(f_n)_{n\in\mathbb{N}}$ we obtain

$$\left\| \int_I f \right\|_X = \left\| \lim_{n\to\infty} \int_I f_n \right\|_X \leq \lim_{n\to\infty} \int_I \|f_n\|_X = \int_I \|f\|_X .$$

Thus we have the inequality

$$\left\| \int_I f \right\|_X \leq \int_I \|f\|_X . \tag{2.7}$$

$\rightsquigarrow$ If $f : I \to X$ is 0 almost everywhere in I, then the L-Cauchy sequence of simple functions from Definition 2.3.1 can be chosen as functions which are identically 0. Thus $f \in \mathcal{B}$ with $\int_I f = 0$.

$\rightsquigarrow$ Let functions $f \in \mathcal{B}$ and $g : I \to X$ be given such that $f(t) = g(t)$ for almost all $t \in I$. Then $g - f = 0$ almost everywhere in I and hence $g = g - f + f$ is Bochner integrable.

By Lemma 2.2.4 and Facts 2.4.1 we know that $\lim_{n\to\infty} \|f_n\|_1$ does not depend on the choice of the L-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ which determines the same f; therefore the L-seminorm $\|\cdot\|_1$ defined for simple functions $f \in \mathcal{J}$ can be extended to functions $f \in \mathcal{B}$:

2.4.2 Definition The mapping

$$\|\cdot\|_1 : \mathcal{B} \longrightarrow \mathbb{R}_0^+$$
$$f \longmapsto \int_I \|f\|_X = \lim_{n\to\infty} \|f_n\|_1$$

where f is determined by the L-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions is called the *L-seminorm* on the space of Bochner integrable functions $\mathcal{B}$.

The properties (JN1) and (JN2) for $(\mathcal{J}, \|\cdot\|_1)$ on page 13 are directly carried over to $(\mathcal{B}, \|\cdot\|_1)$:

(BN1) $\|cf\|_1 = |c| \cdot \|f\|_1$ for every $f \in \mathcal{B}$ and $a \in \mathbb{R}$,

(BN2) $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$ for every $f, g \in \mathcal{B}$.

2.4.3 Lemma *If $f \in \mathcal{B}$ and $(f_n)_{n\in\mathbb{N}}$ is an L-Cauchy sequence in $\mathcal{J}$ determining f, then*

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0.$$

Thus the space $\mathcal{J}$ of simple functions is dense in $\mathcal{B}$ with respect to the L-seminorm $\|\cdot\|_1$.

Proof Let $(f_n)_{n\in\mathbb{N}}$ be an L-Cauchy sequence in $\mathcal{J}$ which determines f. According to Definition 2.2.1 for every $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that $\|f_r - f_q\|_1 < \varepsilon$ for all $r, q \geq N_\varepsilon$. Let us fix $r \geq N_\varepsilon$ and put $g_q := f_r - f_q \in \mathcal{J}$. Then $\lim_{q\to\infty} g_q(t) = f_r(t) - f(t)$ for almost all $t \in I$. Since $\|g_l - g_k\|_1 = \|f_l - f_k\|_1$ for $l, k \in \mathbb{N}$, the sequence $(g_q)_{q\in\mathbb{N}}$ is L-Cauchy and determines $f_r - f \in \mathcal{B}$. Hence

$$\|f - f_r\|_1 = \lim_{q\to\infty} \|g_q\|_1 = \lim_{q\to\infty} \|f_q - f_r\|_1 \leq \varepsilon.$$

Thus we have $\lim_{r\to\infty} \|f_r - f\|_1 = 0$. ■

2.4.4 Proposition *The space $(\mathcal{B}, \|\cdot\|_1)$ is complete.*

Proof Assume that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{B}, \|\cdot\|_1)$. By Lemma 2.4.3 for every $n \in \mathbb{N}$ there exists a simple function $f_n \in \mathcal{J}$ such that

$$\|g_n - f_n\|_1 < \frac{1}{n}.$$

Hence we gain a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions for which

$$\|f_q - f_r\|_1 \leq \|f_q - g_q\|_1 + \|g_q - g_r\|_1 \, \|g_r - f_r\|_1 < \frac{1}{q} + \frac{1}{r} + \|g_q - g_r\|_1$$

for all $r, q \geq N_\varepsilon$ holds. Therefore $(f_n)_{n\in\mathbb{N}}$ is L-Cauchy. By Lemma 2.2.3 the sequence $(f_n)_{n\in\mathbb{N}}$ contains an L-Cauchy subsequence $(f_{n_s})_{s\in\mathbb{N}}$ which converges almost everywhere in I to a certain function $f : I \to X$. For this subsequence we have

$$\|g_{n_s} - f\|_1 \leq \|g_{n_s} - f_{n_s}\|_1 + \|f_{n_s} - f\|_1 \qquad \text{for } s \in \mathbb{N}$$

and thus the subsequence $(g_{n_s})_{s\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ converges in the seminorm $\|\cdot\|_1$ to f. As $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, $(g_n)_{n\in\mathbb{N}}$ converges with respect to $\|\cdot\|_1$. This implies that $(\mathcal{B}, \|\cdot\|_1)$ is complete. ∎

The following statement allows us to give another definition of the Bochner integral which is equivalent to Definition 2.3.1:

2.4.5 Corollary *A function $f : I \to X$ belongs to $\mathcal{B}$ iff there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{J}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for almost all $t \in I$ and $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.*

Proof Assume $f \in \mathcal{B}$; then we have $\lim_{n\to\infty} \|f_n - f\|_1 = 0$ for an arbitrary L-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{J}$ which determines f, due to Lemma 2.4.3. By Definition 2.3.1 such a sequence does exist.

Conversely, every sequence of simple functions which converges with respect to $\|\cdot\|_1$, is also L-Cauchy. ∎

2.4.6 Remark Due to (2.7) in Facts 2.4.1, we have

$$\left\| \mathcal{B}\!\!\!\int_I (f - g) \right\|_X \leq \mathcal{B}\!\!\!\int_I \|f - g\|_X = \|f - g\|_1$$

for functions $f, g \in \mathcal{B}$. This estimate makes it possible to define an equivalence relation $\sim$ on $\mathcal{B}$ (or $\mathcal{J}$) that identifies functions f, g which $\|f - g\|_1 = 0$, as it is usual in the Lebesgue theory. Then $\|\cdot\|_1$ defines a norm on the factor space $\mathcal{B}/\!\sim$ and $(\mathcal{B}/\!\sim, \|\cdot\|_1)$ is a Banach space due to Proposition 2.4.4.

As a consequence we obtain the result that the Bochner integral is equivalent to the Lebesgue integral for $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, i.e. for functions $f : I \to \mathbb{R}$. To comprehend this, we use the fact that the factor space of real-valued simple functions $\mathcal{J}/\!\sim$ is dense in the Banach space L^1 of Lebesgue integrable functions with respect to the norm $\|\cdot\|_1$. Hence both L^1 and $\mathcal{B}/\!\sim$ are Banach space completions of $\mathcal{J}/\!\sim$ and therefore isometrically isomorphic.

Now consider f is Lebesgue integrable with $\int_I f \, \mathrm{d}\lambda \in \mathbb{R}$. Then we can write $f = f^+ - f^-$, where $f^+ := \max\{f, 0\} \in L^1$ and $f^- := \max\{-f, 0\} \in L^1$. Thus is suffices to consider Lebesgue integrable functions $f : I \to \mathbb{R}_0^+$.

By definition (cf. i.e. [8], Definition 122K) we have a sequence $(\acute{f}_n)_{n\in\mathbb{N}}$ of non-negative simple functions which $\acute{f}_n \nearrow f$. Hence for every $\varepsilon > 0$ there exits an $N \in \mathbb{N}$ such that

$$\int_I f \, \mathrm{d}\lambda - \int_I \acute{f}_n < \varepsilon \qquad \text{for } n \geq N.$$

Therefore, we also have

$$\int_I |\acute{f}_q - \acute{f}_r| \, \mathrm{d}\lambda = \int_I (\acute{f}_q - \acute{f}_r) \, \mathrm{d}\lambda \leq$$
$$\leq \int_I (f - \acute{f}_r) \, \mathrm{d}\lambda = \int_I f \, \mathrm{d}\lambda - \int_I \acute{f}_r \, \mathrm{d}\lambda < \varepsilon$$

for $q \geq r \geq N$ and the sequence $(\acute{f}_n)_{n\in\mathbb{N}}$ is L-Cauchy. By Definition 2.3.1 we conclude

$$\int_I f \, \mathrm{d}\lambda = \lim_{n\to\infty} \int_I \acute{f}_n = \mathcal{B}\!\!\!\int_I f \in \mathbb{R}$$

and the isometric isomorphism between L^1 and $\mathcal{B}/\sim$ is canonical, i.e. $L^1 = \overline{\mathcal{J}/\sim}^{\|\cdot\|_1} = \mathcal{B}/\sim$.

The considerations given above allow us to give an alternate approach to the Lebesgue integral via L-Cauchy sequences of simple functions. For instance the textbooks [2], [5] or [17] follow this approach.

2.5 Absolute Integrability of the Bochner Integral

Remark 2.4.6 provides a convenient way to transfer several results from the Lebesgue theory to the Bochner integral. We will show this exemplarily for two major results.

2.5.1 Proposition *A function $f : I \to X$ is Bochner integrable iff f is measurable and $\|f\|_X : I \to \mathbb{R}_0^+$ is Bochner integrable.*

Proof Let $f \in \mathcal{B}$; then f is measurable and $\|f\|_X : I \to \mathbb{R}_0^+$ is Bochner integrable by Facts 2.4.1.
Conversely, let f be a measurable function satisfying

$$\mathcal{B}\!\!\!\int_I \|f\|_X < \infty.$$

By Definition 2.1.1 there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{J}$ such that $\lim_{n\to\infty} \|f(t) - f_n(t)\| = 0$ almost everywhere in I. Define $A_N := \{t \in I : \|f_n(t)\|_X \leq 2\,\|f(t)\|_X\}$ for $n \in \mathbb{N}$. Clearly, every A_n is Lebesgue measurable. Hence every $\acute{f}_n(t) := \mathbb{1}_{A_n}(t) f_n(t)$ is a simple function of the form (2.1) and $\lim_{n\to\infty} \|f(t) - \acute{f}_n(t)\|_X = 0$ almost everywhere in I.
As both f and $\acute{f}_n$, $n \in \mathbb{N}$ are measurable, we observe that every real-valued function $\|\acute{f}_n(t) - f(t)\|_X : I \to \mathbb{R}_0^+$ is measurable by Facts 2.1.2. Therefore $\|\acute{f}_n(t) - f(t)\|_X$ is also measurable in the sense of Lebesgue theory for every $n \in \mathbb{N}$.
Since $\|f(t)\|_X$ is Lebesgue integrable by Remark 2.4.6 and $\|\acute{f}_n(t)\|_X \leq 2\,\|f(t)\|_X$ point-wise in I, the dominated convergence theorem from Lebesgue theory can be applied by the estimate $\|\acute{f}_n(t) - f(t)\|_X \leq 3\,\|f(t)\|_X$ point-wise in I for every $n \in \mathbb{N}$. Thus

$$\lim_{n\to\infty} \mathcal{B}\!\!\!\int_I \|\acute{f}_n(t) - f(t)\|_X = \lim_{n\to\infty} \int_I \|\acute{f}_n(t) - f(t)\|_X \, \mathrm{d}\lambda = 0$$

and we conclude that f is Bochner integrable by Corollary 2.4.5. ■

Results from the theory of Lebesgue integration do not carry over to the Bochner integral if there are non-negativity assumptions involved. For instance, there are no analogues of

Fatou's lemma or the monotone convergence theorem. Nevertheless we do have the following analogue of the dominated convergence theorem:

2.5.2 **Proposition** (Dominated convergence theorem) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Bochner integrable functions $f_n : I \to X$ which converges to $f : I \to X$ almost everywhere in I.*

If there exists a Bochner integrable functions $g : I \to \mathbb{R}$ such that $\|f_n(t)\| \leq |g(t)|$ almost everywhere in I for every $n \in \mathbb{N}$, then f is Bochner integrable and we have

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \int_I f_n = \int_I f.$$

Proof As every f_n is measurable, by Definition 2.1.1 there exists a simple function $\acute{f}_n$ for every $n \in \mathbb{N}$ such that $\|\acute{f}_n(t) - f_n(t)\| < \frac{1}{2^n}$ almost everywhere in I. Since

$$\|\acute{f}_n(t) - f(t)\| \leq \|\acute{f}_n(t) - f_n(t)\|_X + \|f_n(t) - f(t)\|_X,$$

we infer that $\lim\limits_{n\to\infty} \|\acute{f}_n(t) - f(t)\| = 0$ almost everywhere in I for the sequence $(\acute{f}_n)_{n\in\mathbb{N}}$ of simple functions. Hence f is measurable.

We observe that every real-valued function $\|f_n(t) - f(t)\|_X : I \to \mathbb{R}_0^+$ is measurable by Facts 2.1.2. Therefore $\|f_n(t) - f(t)\|_X$ is also measurable in the sense of Lebesgue theory for every $n \in \mathbb{N}$.

Since $g(t)$ is Lebesgue integrable by Remark 2.4.6, the dominated convergence theorem from Lebesgue theory can be applied by the estimate $\|f_n(t) - f(t)\|_X \leq 2\,|g(t)|$ almost everywhere in I for every $n \in \mathbb{N}$. We conclude

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X = \lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X \,\mathrm{d}\lambda = 0$$

and

$$\lim_{n\to\infty} \left\| \int_I \big(f_n - f\big) \right\|_X \leq \lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X = 0.$$

■

3 Comparison of Banach Space-Valued Integrals

In this chapter we compare the concepts of Bochner and Riemann type sum integrals introduced in the previous chapters.

3.1 McShane Integrability and the Bochner Integral

3.1.1 Lemma *Assume that $f : I \to X$ is Bochner integrable and let $\varepsilon > 0$ be given. Then there is a gauge $\omega : I \to \mathbb{R}^+$ and $\eta \in (0, \varepsilon)$ such that the following statement holds:*

If $\{(t_m, H_m)\}$ is an ω-fine M-system for which $\sum_{m\in\mathfrak{M}} \lambda(H_m) < \eta$, then

$$\sum_{m\in\mathfrak{M}} \|f(t_m)\|_X \,\lambda(H_m) < \varepsilon.$$

Proof Let us set

$$E_j := \{t \in I : j-1 \le \|f(t)\|_X < j\} \qquad \text{for } j \in \mathbb{N}.$$

Since $\|f\|_X$ is integrable by Facts 2.4.1, the sets E_j, $j \in \mathbb{N}$ are measurable and $E_i \cap E_j = \emptyset$ for $i \neq j$, while $\bigcup_{j\in\mathbb{N}} E_j = I$. Moreover we have

$$\sum_{j\in\mathbb{N}} (j-1)\lambda(E_j) \le \int_I \|f(t)\|_X$$

and therefore

$$\sum_{j\in\mathbb{N}} j\,\lambda(E_j) \le \int_I \|f(t)\|_X + \sum_{j\in\mathbb{N}} \lambda(E_j) = \int_I \|f(t)\|_X + \lambda(I) < \infty. \tag{3.1}$$

Since λ is a regular measure on I, for every $j \in \mathbb{N}$, there exists a relatively open set $G_j \subseteq I$ for which $E_j \subseteq G_j$ and

$$\lambda(G_j) < \lambda(E_j) + \frac{1}{2^j}.$$

Together with (3.1), this yields

$$\sum_{j\in\mathbb{N}} j\,\lambda(G_j) < \sum_{j\in\mathbb{N}} j\,\lambda(E_j) + \sum_{j\in\mathbb{N}} \frac{j}{2^j} < \infty.$$

Assume that $\varepsilon_0 > 0$ is given. Then there is an $r \in \mathbb{N}$ such that

$$\sum_{j>r} j\,\lambda(G_j) < \varepsilon_0.$$

For $t \in I$ there is exactly one $j(t) \in \mathbb{N}$ such that $t \in E_{j(t)}$. Let us choose a gauge ω on I such that $I \cap B_{\omega(t)}(t) \subseteq G_{j(t)}$.

If $\{(t_m, H_m)\}$ is an ω-fine M-system with $\sum_{m\in\mathfrak{M}} \lambda(H_m) < \eta$, then we have $t_m \in E_{j(t_m)}$, $H_m \subseteq B_{\omega(t_m)}(t_m) \subseteq G_{j(t_m)}$ and $\|f(t_m)\|_X < j(t_m)$ for every tagged interval (t_m, H_m) of $\{(t_m, H_m)\}$. Henceforth we infer

$$\begin{aligned}\sum_{m\in\mathfrak{M}} \|f(t_m)\|_X \,\lambda(H_m) &\le \sum_{\substack{m\in\mathfrak{M}\\ j(t_m)\le r}} j(t_m)\lambda(H_m) + \sum_{\substack{m\in\mathfrak{M}\\ j(t_m)> r}} j(t_m)\lambda(H_m) \le \\ &\le r \sum_{\substack{m\in\mathfrak{M}\\ j(t_m)\le r}} \lambda(H_m) \qquad + \sum_{\substack{m\in\mathfrak{M}\\ j(t_m)> r}} j(t_m)\lambda(G_{j(t_m)}) < r\eta + \varepsilon_0.\end{aligned}$$

Finally, by taking $\varepsilon_0 < \frac{\varepsilon}{2}$ and $\eta < \frac{\varepsilon}{2r+1}$ we obtain the desired result. ■

3.1.2 Theorem *If $f : I \to X$ is Bochner integrable, then f is McShane integrable and*

$$(\mathcal{B})\int_I f = (\mathcal{M})\int_I f. \tag{3.2}$$

Proof Suppose that $\varepsilon > 0$ is given. Moreover, assume that $E \subseteq I$ is an arbitrary measurable set. For $E^{\mathsf{C}} := I \setminus E$, we apparently have $I = E \cup E^{\mathsf{C}}$, and both E and E^{C} are λ-regular sets. In this situation there exist open sets $G \subset \mathbb{R}$ and $H \subset \mathbb{R}$ such that $E \subseteq G$, $E^{\mathsf{C}} \subseteq H$ and

$$\lambda(G) < \lambda(E) + \varepsilon \qquad \lambda(H) < \lambda(E^{\mathsf{C}}) + \varepsilon.$$

Let us define a gauge $\delta : I \to \mathbb{R}^+$ such that the implications

$$t \in E \Longrightarrow B_{\delta(t)}(t) \cap I \subseteq G \qquad t \in E^{\mathsf{C}} \Longrightarrow B_{\delta(t)}(t) \cap I \subseteq H$$

hold true.
Let $\{(t_i, J_i)\}$ be an arbitrary δ-fine M-partition of I; then

$$\sum_{i \in \mathfrak{I}} \mathbb{1}_E(t_i)\lambda(J_i) = \sum_{\substack{i \in \mathfrak{I} \\ t_i \in E}} \lambda(J_i) \leq \lambda(G) < \lambda(E) + \varepsilon \tag{3.3}$$

and similarly

$$\sum_{i \in \mathfrak{I}} \mathbb{1}_{E^{\mathsf{C}}}(t_i)\lambda(J_i) = \sum_{\substack{i \in \mathfrak{I} \\ t_i \in E^{\mathsf{C}}}} \lambda(J_i) \leq \lambda(H) < \lambda(E^{\mathsf{C}}) + \varepsilon.$$

Further we have

$$\sum_{i \in \mathfrak{I}} \mathbb{1}_I(t_i)\lambda(J_i) = \sum_{i \in \mathfrak{I}} \lambda(J_i) = \lambda(I)$$

and also $\mathbb{1}_I = \mathbb{1}_E + \mathbb{1}_{E^{\mathsf{C}}}$. This yields

$$\begin{aligned}\sum_{i \in \mathfrak{I}} \mathbb{1}_E(t_i)\lambda(J_i) &= \sum_{i \in \mathfrak{I}} \mathbb{1}_I(t_i)\lambda(J_i) - \sum_{i \in \mathfrak{I}} \mathbb{1}_{E^{\mathsf{C}}}(t_i)\lambda(J_i) > \\ &> \lambda(I) - \big(\lambda(E^{\mathsf{C}}) + \varepsilon\big) = \lambda(E) - \varepsilon.\end{aligned}$$

This inequality, together with (3.3) and the fact that $(\mathcal{B})\int_I \mathbb{1}_E = \lambda(E)$, implies

$$\left| \sum_{i \in \mathfrak{I}} \mathbb{1}_E(t_i)\lambda(J_i) - (\mathcal{B})\int_I \mathbb{1}_E \right| = \left| \sum_{i \in \mathfrak{I}} \mathbb{1}_E(t_i)\lambda(J_i) - \lambda(E) \right| < \varepsilon. \tag{3.4}$$

As $\varepsilon > 0$ was arbitrary, we get $(\mathcal{M})\int_I \mathbb{1}_E = (\mathcal{B})\int_I \mathbb{1}_E$.
Next assume $y \in X$; then the function $\mathbb{1}_E\, y : I \to X$ belongs to $\mathcal{B}$ and

$$(\mathcal{B})\int_I \mathbb{1}_E\, y = \lambda(E)y = y\,(\mathcal{B})\int_I \mathbb{1}_E.$$

Starting again with $\varepsilon > 0$ and choosing the gauge δ as above, we obtain

$$\left\| \sum_{i\in\mathfrak{I}} \mathbb{1}_E(t_i) y \lambda(J_i) - (\mathcal{B})\!\int_I \mathbb{1}_E\, y \right\|_X = \left\| \Big(\sum_{i\in\mathfrak{I}} \mathbb{1}_E(t_i)\lambda(J_i) - (\mathcal{B})\!\int_I \mathbb{1}_E \Big) y \right\|_X =$$
$$= \left| \sum_{i\in\mathfrak{I}} \mathbb{1}_E(t_i)\lambda(J_i) - (\mathcal{B})\!\int_I \mathbb{1}_E \right| \|y\|_X < \varepsilon \|y\|_X$$

for every δ-fine M-partition $\{(t_i, J_i)\}$ of I, due to (3.4). Hence $(\mathcal{B})\!\int_I \mathbb{1}_E\, y = (\mathcal{M})\!\int_I \mathbb{1}_E\, y$. This immediately yields $(\mathcal{B})\!\int_I g = (\mathcal{M})\!\int_I g$ for an arbitrary simple function $g : I \to X$, by the linearity of both integrals.

Now let $(\acute{f}_q)_{q\in\mathbb{N}}$ be a sequence of simple functions which determines $f \in \mathcal{B}$. Set $A_q \coloneqq \{t \in I : \|\acute{f}_q(t)\|_X \le \|f(t)\|_X + 1\}$ for $q \in \mathbb{N}$ and observe that each A_q is Lebesgue measurable. Hence $(f_q)_{q\in\mathbb{N}}$ is a sequence of simple functions $f_q(t) \coloneqq \mathbb{1}_{A_q}(t)\acute{f}_q(t)$ which also determines f and satisfies the inequality $\|f_q(t)\|_X \le \|f(t)\|_X + 1$.

Since

$$\lim_{r\to\infty} \left\| (\mathcal{B})\!\int_I f_r - (\mathcal{B})\!\int_I f \right\|_X = 0,$$

there exists an $N_\varepsilon \in \mathbb{N}$ such that

$$\left\| (\mathcal{B})\!\int_I f_q - (\mathcal{B})\!\int_I f \right\|_X < \varepsilon \qquad \text{for } q \ge N_\varepsilon.$$

Due to Lemma 3.1.1, there is a gauge $\omega_1 : I \to \mathbb{R}^+$ and $\eta \in (0, \varepsilon)$ such that

$$\sum_{m\in\mathfrak{M}} \|f(t_m)\|_X \lambda(H_m) < \varepsilon$$

whenever $\{(t_m, H_m)\}$ is an ω_1-fine M-system of I, for which $\sum_{m\in\mathfrak{M}} \lambda(H_m) < \eta$.

Moreover, fix an integer $q \ge N_\varepsilon$ and assume a gauge ω_2 on I such that

$$\left\| \sum_{i\in\mathfrak{I}} f_q(t_i)\lambda(J_i) - (\mathcal{B})\!\int_I f_q \right\|_X = \left\| \sum_{i\in\mathfrak{I}} f_q(t_i)\lambda(J_i) - (\mathcal{M})\!\int_I f_q \right\|_X < \varepsilon$$

for the simple function f_q and every ω_2-fine M-partition $\{(t_i, J_i)\}$ of I. The function $\delta(t) \coloneqq \min\{\omega_1(t), \omega_2(t)\} : I \to X$ defines again a gauge on I. Clearly, every δ-fine M-partition of I is also ω_1-fine and ω_2-fine. For such a δ-fine M-partition $\{(t_i, J_i)\}$ we infer

$$\left\| \sum_{i\in\mathfrak{I}} f(t_i)\lambda(J_i) - (\mathcal{B})\!\int_I f \right\|_X \le$$
$$\le \left\| \sum_{i\in\mathfrak{I}} f(t_i)\lambda(J_i) - \sum_{i\in\mathfrak{I}} f_q(t_i)\lambda(J_i) \right\|_X + \left\| \sum_{i\in\mathfrak{I}} f_q(t_i)\lambda(J_i) - (\mathcal{M})\!\int_I f_q \right\|_X + \left\| (\mathcal{B})\!\int_I f_q - (\mathcal{B})\!\int_I f \right\|_X \le$$
$$\le \left\| \sum_{i\in\mathfrak{I}} f(t_i)\lambda(J_i) - \sum_{i\in\mathfrak{I}} f_q(t_i)\lambda(J_i) \right\|_X + 2\varepsilon. \tag{3.5}$$

We need an estimate for the sum in (3.5). To this aim take $\alpha \in \big(0, \min\big\{\frac{\eta}{2\lambda(I)}, \frac{\eta}{2}\big\}\big)$. Then, due to Lemma 2.2.3, the sequence $(f_q)_{q\in\mathbb{N}}$ can be chosen in such a way that there exists a measurable set $Z_\alpha \subseteq I$ with $\lambda(Z_\alpha) < \frac{\alpha}{2}$, for which the sequence $(f_q)_{q\in\mathbb{N}}$ converges to the function f uniformly on $I \setminus Z_\alpha$.

For the measurable and λ-regular set Z_α there is a relatively open set $G_\alpha \subseteq I$ such that $Z_\alpha \subseteq G_\alpha$ and $\lambda(G_\alpha) < \alpha$.

Define the closed set $G_\alpha^{\complement} := I \setminus G_\alpha \subseteq I \setminus Z_\alpha$. Henceforth $\lambda(I \setminus G_\alpha^{\complement}) = \lambda(G_\alpha) < \alpha$ and there is an $N_\alpha \in \mathbb{N}$ such that $\|f_q(t) - f(t)\|_X < \alpha$ for $q \geq q_\alpha$ and $t \in G_\alpha^{\complement}$.

As f_q is a simple function for fixed $q \geq N_\varepsilon$, it is of the form

$$f_q = \sum_{m\in\mathfrak{M}} \mathbb{1}_{E_{q_m}} y_{q_m},$$

where $(E_{q_m})_{m\in\mathfrak{M}}$ is a finite set of measurable sets $E_{q_m} \subseteq I$ such that $y_{q_m} \in X$, $E_{q_m} \cap E_{q_l} = \emptyset$ for all $m \neq l$, $m, l \in \mathfrak{M}$ and $I = \bigcup\limits_{m\in\mathfrak{M}} E_{q_m}$.

By λ-regularity of the Lebesgue measurable sets E_{q_m}, $m \in \mathfrak{M}$, there exist compact sets F_{q_m} with $F_{q_m} \subseteq E_{q_m}$ and

$$\lambda(E_{q_m} \setminus F_{q_m}) < \frac{\eta}{2\,|\mathfrak{M}|} \qquad \text{for } m \in \mathfrak{M}.$$

Therefore we have

$$\lambda\Big(\bigcup_{m\in\mathfrak{M}} (E_{q_m} \setminus F_{q_m})\Big) < \sum_{m\in\mathfrak{M}} \frac{\eta}{2\,|\mathfrak{M}|} = \frac{\eta}{2}.$$

Moreover, we set $A_{q_m} := G_\alpha^{\complement} \cap F_{q_m}$ for every $m \in \mathfrak{M}$. These sets A_{q_m} are compact and $A_{q_m} \cap A_{q_l} = \emptyset$ for $m \neq l$, $m, l \in \mathfrak{M}$. Therefore the distance of any two different sets A_{q_m} is positive, i.e. there is a $\rho > 0$ such that if $t \in A_{q_m}$, $s \in A_{q_l}$ and $m \neq l$ for $m, l \in \mathfrak{M}$, then $\operatorname{dist}(t, s) > \rho$.

From

$$\begin{aligned} I \setminus \bigcup_{m\in\mathfrak{M}} A_{q_m} &= \bigcup_{m\in\mathfrak{M}} E_{q_m} \setminus \bigcup_{m\in\mathfrak{M}} (G_\alpha^{\complement} \cap F_{q_m}) \subseteq \bigcup_{m\in\mathfrak{M}} (E_{q_m} \setminus F_{q_m}) \cup \bigcup_{m\in\mathfrak{M}} (E_{q_m} \setminus G_\alpha^{\complement}) = \\ &= \bigcup_{m\in\mathfrak{M}} (E_{q_m} \setminus F_{q_m}) \cup (I \setminus G_\alpha^{\complement}), \end{aligned}$$

we derive

$$\lambda\big(I \setminus \bigcup_{m\in\mathfrak{M}} A_{q_m}\big) \leq \sum_{m\in\mathfrak{M}} \lambda(E_{q_m} \setminus F_{q_m}) + \lambda(I \setminus G_\alpha^{\complement}) < \frac{\eta}{2} + \alpha < \eta.$$

Assume now the gauge ϑ on I, defined by

$$\vartheta(x) := \begin{cases} \min\big\{\operatorname{dist}\big(t, \bigcup\limits_{m\in\mathfrak{M}} A_{q_m}\big),\, \delta(t)\big\} & \text{if } t \in I \setminus \bigcup\limits_{m\in\mathfrak{M}} A_{q_m} \\ \delta(t) & \text{otherwise} \end{cases}$$

and note that $B_{\vartheta(t)}(t) \cap I \subseteq I \setminus \bigcup\limits_{m\in\mathfrak{M}} A_{q_m}$ as long as $t \in I \setminus \bigcup\limits_{m\in\mathfrak{M}} A_{q_m}$, because the set $I \setminus \bigcup\limits_{m\in\mathfrak{M}} A_{q_m}$ is relatively open in I. Hence, for each tagged interval (t_i, J_i), $t_i \notin \bigcup\limits_{m\in\mathfrak{M}} A_{q_m}$ of an arbitrary ϑ-fine M-partition $\{(t_i, J_i)\}$ of I, that

$$J_i \subseteq B_{\vartheta(t_i)}(t_i) \cap I \subseteq I \setminus \bigcup_{m\in\mathfrak{M}} A_{q_m},$$

i.e $J_i \cap A_{q_m} = \emptyset$ for all $m \in \mathfrak{M}$.

From

$$\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}} J_i \subseteq I\setminus\bigcup_{m\in\mathfrak{M}}A_{q_m},$$

we conclude

$$\lambda\Big(\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}} J_i\Big) \le \lambda\Big(I\setminus\bigcup_{m\in\mathfrak{M}}A_{q_m}\Big) < \eta. \tag{3.6}$$

Now we can split the sum in (3.5) into two parts, one with $t_i \in \bigcup\limits_{m\in\mathfrak{M}} A_{q_m}$ and the other with $t_i \notin \bigcup\limits_{m\in\mathfrak{M}} A_{q_m}$, i.e.

$$\sum_{i\in\mathfrak{T}}\big(f(t_i)-f_q(t_i)\big)\lambda(J_i) = \sum_{\substack{i\in\mathfrak{T}\\ t_i\in\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}}\big(f(t_i)-f_q(t_i)\big)\lambda(J_i) + \sum_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}}\big(f(t_i)-f_q(t_i)\big)\lambda(J_i).$$

If $t_i \notin \bigcup\limits_{m\in\mathfrak{M}} A_{q_m}$, then $\lambda\Big(\bigcup\limits_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup_{m\in\mathfrak{M}}A_{q_m}}} J_i\Big) < \eta$ due to (3.6) and

$$\begin{aligned}\Bigg\|\sum_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}}\big(f(t_i)-f_q(t_i)\big)\lambda(J_i)\Bigg\|_X &\le \sum_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}}\|f(t_i)\|_X\,\lambda(J_i) + \sum_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}}\|f_q(t_i)\|_X\,\lambda(J_i) \le \\ &\le \sum_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}}\big(2\,\|f(t_i)\|_X+1\big)\lambda(J_i) \le 2\varepsilon+\eta < 3\varepsilon \end{aligned} \tag{3.7}$$

by Lemma 3.1.1, since $\lambda\Big(\bigcup\limits_{\substack{i\in\mathfrak{T}\\ t_i\notin\bigcup_{m\in\mathfrak{M}}A_{q_m}}} J_i\Big) < \eta$.

On the other hand, if $t_i \in \bigcup\limits_{m\in\mathfrak{M}} A_{q_m} \subseteq G_\alpha^{\complement} \subseteq I\setminus Z_\alpha$, then $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on $I\setminus Z_\alpha$ according to our assumptions above. As $N_\alpha\in\mathbb{N}$ was chosen such that for every $q\ge N_\alpha$ we have $\|f(t_i)-f_q(t_i)\|_X<\alpha$, we get

$$\Bigg\|\sum_{\substack{i\in\mathfrak{T}\\ t_i\in\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}}\big(f(t_i)-f_q(t_i)\big)\lambda(J_i)\Bigg\|_X < \alpha\sum_{\substack{i\in\mathfrak{T}\\ t_i\in\bigcup\limits_{m\in\mathfrak{M}}A_{q_m}}}\lambda(J_i) \le \alpha\sum_{i\in\mathfrak{T}}\lambda(J_i) = \alpha\lambda(I) < \varepsilon. \tag{3.8}$$

Putting together all these estimates (3.5), (3.7) and (3.8), we ultimately obtain

$$\Bigg\|\sum_{i\in\mathfrak{T}} f(t_i)\lambda(J_i) - \mathcal{B}\!\!\int_I f\Bigg\|_X < 2\varepsilon+3\varepsilon+\varepsilon = 6\varepsilon$$

for every ϑ-fine M-partition $\{(t_i,J_i)\}$. This finally implies the existence of $\mathcal{M}\!\!\int_I f \in X$ and our desired equality of both integrals in (3.2). ■

By Theorem 3.1.2 the inclusion $\mathcal{B}\subseteq\mathcal{M}$ prevails. We will show that the converse inclusion $\mathcal{M}\subseteq\mathcal{B}$ is not true for an infinite-dimensional Banach space. For this purpose we will need the concept of *unconditional convergence*:

3.1.3 Definition A series $\sum\limits_{k\in\mathbb{N}} x_k$ of elements $x_k \in X$, $k \in \mathbb{N}$ of a Banach space X is said to be *unconditionally convergent* to an $s \in X$ if for every $\varepsilon > 0$ there is a finite subset $\mathfrak{E}_\varepsilon$ of $\mathbb{N}$ such that

$$\Big\| s - \sum_{k\in\mathfrak{L}\cup\mathfrak{E}_\varepsilon} x_k \Big\|_X < \varepsilon$$

for all finite subsets $\mathfrak{L}$ of $\mathbb{N}$.

From basic analysis we know the *Riemann series theorem* (cf. i.e. [1], Theorem 8.9) which states that a series $\sum\limits_{k\in\mathbb{N}} x_k$ of elements $x_k \in \mathbb{R}$, $k \in \mathbb{N}$ is unconditionally convergent iff $\sum\limits_{k\in\mathbb{N}} x_k$ is absolutely convergent.
The following example shows that this equivalence does not need to be true in infinite-dimensional Banach spaces:

3.1.4 Example For $k \in \mathbb{N}$ set $x_k := \big(0, \ldots 0, \frac{1}{k}, 0, \ldots\big) \in \ell_2$. We have $\|x_k\|_{\ell_2} = \frac{1}{k}$ and $\sum\limits_{k\in\mathbb{N}} \|x_k\|_{\ell_2} = \sum\limits_{k\in\mathbb{N}} \frac{1}{k} = \infty$; thus the series is not absolutely convergent.
On the other hand, the series $\sum\limits_{k\in\mathbb{N}} x_k$ converges unconditionally to $s = \big(1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\big) \in \ell_2$ with $\|s\|^2_{\ell_2} = \sum\limits_{k\in\mathbb{N}} \frac{1}{n^2} < \infty$. ◀

Moreover, the well-known *Dvoretzky-Rogers theorem* asserts that in every infinite-dimensional Banach space there exists an unconditionally but non-absolutely convergent series (cf. [7], Theorem 13.38).

3.1.5 Lemma *Given $\varepsilon > 0$, suppose that $(c_i, z_i)_{i\in\mathfrak{T}}$ is a finite set of elements $c_i \in [0,1]$, $z_i \in X$. Assume*

$$\Big\| \sum_{q\in\mathfrak{Q}} z_q \Big\|_X < \varepsilon$$

for each subset $\mathfrak{Q}$ of $\mathfrak{T}$. Then

$$\Big\| \sum_{i\in\mathfrak{T}} c_i z_i \Big\|_X < \varepsilon.$$

Proof With no loss of generality we can assume that $0 \le c_1 \le \ldots \le c_k \le 1$, where $\mathfrak{T} = \{1, 2 \ldots k\}$. Then

$$\begin{aligned}
\sum_{i\in\mathfrak{T}} c_i z_i &= c_1(z_1 + \ldots + z_k) - c_1(z_2 + \ldots + z_k) + \sum_{i\in\mathfrak{T}\setminus\{1\}} c_i z_i = \\
&= c_1(z_1 + \ldots + z_k) + c_2(z_2 + \ldots + z_k) - \\
&\quad - c_1(z_2 + \ldots + z_k) - c_2(z_3 + \ldots + z_k) + \sum_{i\in\mathfrak{T}\setminus\{1,2\}} c_i z_i = \\
&= c_1(z_1 + \ldots + z_k) + (c_2 - c_1)(z_2 + \ldots + z_k) + \\
&\quad + (c_3 - c_2)(z_3 + \ldots + z_k) + \ldots + (c_k - c_{k-1}) z_k.
\end{aligned}$$

This gives

$$\begin{aligned}
\Big\| \sum_{i\in\mathfrak{T}} c_i z_i \Big\|_X &\le c_1 \Big\| \sum_{i\in\mathfrak{T}} z_i \Big\|_X + (c_2 - c_1) \Big\| \sum_{i\in\mathfrak{T}\setminus\{1\}} z_i \Big\|_X + \ldots + (c_k - c_{k-1}) \|z_k\|_X < \\
&< \varepsilon\big(c_1 + (c_2 - c_1) + (c_3 - c_2) + \ldots + (c_k - c_{k-1})\big) = \varepsilon \max_{i\in\mathfrak{T}} c_i \le \varepsilon.
\end{aligned}$$

■

3.1.6 Proposition *If X is an infinite-dimensional Banach space then there exists a function $f : I \to X$ which is McShane integrable but not Bochner integrable.*

Proof Assume that $\sum_{j\in\mathbb{N}} z_j$ is an unconditionally convergent series for which $\sum_{j\in\mathbb{N}} \|z_j\|_X = \infty$. Such a series exists in every Banach space with $\dim X = \infty$ due to the Dvoretzky-Rogers theorem (cf. [7], Theorem 13.38).

Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of open intervals $K_j \subset I$ such that $K_j \cap K_i = \emptyset$ for $i \neq j$; $i, j \in \mathbb{N}$. Then we have

$$\sum_{j\in\mathbb{N}} \lambda(K_j) \leq \lambda(I) < \infty.$$

Additionally, denote $K := \bigcup_{j\in\mathbb{N}} K_j$ and $K^{\mathsf{C}} := I \setminus K$. Apparently the set $K \subset I$ is open. Moreover, set

$$y_j := \frac{z_j}{\lambda(K_j)} \qquad \text{for } j \in \mathbb{N}$$

and note that the series $s := \sum_{j\in\mathfrak{I}} y_j \lambda(K_j) = \sum_{j\in\mathfrak{I}} z_j \in X$ is unconditionally convergent according to our premise, while $\sum_{j\in\mathfrak{I}} \|y_j\|_X \lambda(K_j) = \infty$.

Thus for given $\varepsilon > 0$ there is a finite subset $\mathfrak{E}_\varepsilon$ of $\mathbb{N}$ such that

$$\Big\| s - \sum_{j\in\mathfrak{L}\cup\mathfrak{E}_\varepsilon} y_j \lambda(K_j) \Big\|_X < \frac{\varepsilon}{6} \tag{3.9}$$

for every finite subset $\mathfrak{L}$ of $\mathbb{N}$.

Put $N_\varepsilon := \max \mathfrak{E}_\varepsilon$; for every finite subset $\mathfrak{Q} \subset \mathbb{N} \setminus \{1, \dots N_\varepsilon\} \subseteq \mathbb{N} \setminus \mathfrak{E}_\varepsilon$ we infer

$$\begin{aligned} \Big\| \sum_{j\in\mathfrak{Q}} y_j \lambda(K_j) \Big\|_X &= \Big\| \sum_{j\in\mathfrak{Q}\cup\mathfrak{E}_\varepsilon} y_j \lambda(K_j) - \sum_{j\in\mathfrak{E}_\varepsilon} y_j \lambda(K_j) \Big\|_X \leq \\ &\leq \Big\| \sum_{j\in\mathfrak{Q}\cup\mathfrak{E}_\varepsilon} y_j \lambda(K_j) - s \Big\|_X + \Big\| s - \sum_{j\in\mathfrak{E}_\varepsilon} y_j \lambda(K_j) \Big\|_X < \frac{\varepsilon}{3}. \end{aligned} \tag{3.10}$$

Moreover, by (3.9) we have

$$\Big\| s - \sum_{j\leq N_\varepsilon} y_j \lambda(K_j) \Big\|_X < \frac{\varepsilon}{6}. \tag{3.11}$$

Define

$$f(t) := \begin{cases} y_j & \text{if } t \in K_j,\ j \in \mathbb{N} \\ 0 & \text{if } t \in K^{\mathsf{C}}. \end{cases}$$

Let $\eta > 0$ be such that

$$0 < \eta < \frac{\varepsilon}{3\big(1 + \sum_{j\leq N_\varepsilon} \|y_j\|_X\big)}$$

and by regularity of the Lebesgue measure on I let $G \subset I$ be a relatively open set for which $K^{\mathsf{C}} \subset G$ and $\lambda(G) < \lambda(K^{\mathsf{C}}) + \eta$.

Define a gauge $\delta : I \to \mathbb{R}^+$ by

$$\delta(t) := \begin{cases} \operatorname{dist}(t, I \setminus K_j) & \text{if } t \in K_j \setminus G,\ j \in \mathbb{N} \\ \min\left(\operatorname{dist}(t, I \setminus K_j),\ \operatorname{dist}(t, I \setminus G)\right) & \text{if } t \in K_j \cap G,\ j \in \mathbb{N} \\ \operatorname{dist}(t, I \setminus G) & \text{if } t \in G \setminus K \\ 1 & \text{otherwise} \end{cases}$$

and note that $B_{\delta(t)}(t) \cap I \subseteq K_j$ whenever $t \in K_j$, $j \in \mathbb{N}$, and $B_{\delta(t)}(t) \cap I \subseteq G$ whenever $t \in G$, in particular for $t \in K^{\mathsf{C}}$.
Let $\{(t_i, J_i)\}$ be a δ-fine M-partition of I; then by (3.11) we have

$$\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(J_i) - s \right\|_X < \frac{\varepsilon}{6} + \left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(J_i) - \sum_{j \le N_\varepsilon} y_j \lambda(K_j) \right\|_X. \tag{3.12}$$

Moreover, denote $K_{\underline{N}} := \bigcup_{j \le N_\varepsilon} K_j$ and $K_{\overline{N}} := \bigcup_{j > N_\varepsilon} K_j$, and split the sum

$$\sum_{i \in \mathfrak{T}} f(t_i)\lambda(J_i) = \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K}} f(t_i)\lambda(J_i)$$

into two parts

$$\begin{aligned} \sum_{i \in \mathfrak{T}} f(t_i)\lambda(J_i) &= \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_{\underline{N}}}} f(t_i)\lambda(J_i) + \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_{\overline{N}}}} f(t_i)\lambda(J_i) = \\ &= \sum_{j \le N_\varepsilon} \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} f(t_i)\lambda(J_i) + \sum_{j > N_\varepsilon} \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} f(t_i)\lambda(J_i) = \\ &= \sum_{j \le N_\varepsilon} y_j \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} \lambda(J_i) + \sum_{j > N_\varepsilon} y_j \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} \lambda(J_i). \end{aligned}$$

Then we obtain

$$\begin{aligned} &\left\| \sum_{i \in \mathfrak{T}} f(t_i)\lambda(J_i) - \sum_{j \le N_\varepsilon} y_j \lambda(K_j) \right\|_X \le \\ \le &\left\| \sum_{j \le N_\varepsilon} y_j \left(\sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} \lambda(J_i) - \lambda(K_j) \right) \right\|_X + \left\| \sum_{j > N_\varepsilon} y_j \left(\sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} \lambda(J_i) \right) \right\|_X. \end{aligned} \tag{3.13}$$

The right term

$$\left\| \sum_{j > N_\varepsilon} y_j \left(\sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} \lambda(J_i) \right) \right\|_X$$

consists of a finite number of nonzero terms only since $t_i \in K_j$ just for $i \in \mathfrak{T}$ and $|\mathfrak{T}| < \infty$. Hence, we have

$$0 < \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} \lambda(J_i) \le \lambda(K_j)$$

for $j \in \mathfrak{Q} := \{j \in \mathbb{N} : j > N_\varepsilon,\ \exists i \in \mathfrak{T} : t_i \in K_j\}$. As $t_i \in K_j$ implies $I_i \subseteq K_j$, the index set $\mathfrak{Q}$ is finite. Define $c_j \in [0, 1]$ for $j \in \mathfrak{Q}$ such that

$$0 < \sum_{\substack{i \in \mathfrak{T} \\ t_i \in K_j}} \lambda(J_i) = c_j \lambda(K_j).$$

Now we can apply Lemma 3.1.5 and get from (3.10) that

$$\Big\|\sum_{j>m} y_j\Big(\sum_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}\lambda(J_i)\Big)\Big\|_X = \Big\|\sum_{j\in\mathfrak{Q}} y_j\Big(\sum_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}\lambda(J_i)\Big)\Big\|_X = \Big\|\sum_{j\in\mathfrak{Q}} c_j\big(y_j\lambda(K_j)\big)\Big\|_X < \frac{\varepsilon}{3}. \tag{3.14}$$

It remains to give an estimate for the remaining term

$$\Big\|\sum_{j\le N_\varepsilon} y_j\Big(\sum_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}\lambda(J_i)-\lambda(K_j)\Big)\Big\|_X$$

in (3.13). Hence, by definition of the relatively open set $G\supset K^{\mathsf{C}}$ and the gauge δ, we obtain

$$\begin{aligned}\lambda\big(K\setminus\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}J_i\big) &= \lambda(K)-\lambda\big(\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}J_i\big)=\lambda(I)-\lambda(K^{\mathsf{C}})-\lambda\big(\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}J_i\big)=\\ &=\lambda\big(\bigcup_{i\in\mathfrak{T}}J_i\big)-\lambda(K^{\mathsf{C}})-\lambda\big(\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}J_i\big)=\lambda\big(\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\in K^{\mathsf{C}}}}J_i\big)-\lambda(K^{\mathsf{C}})\le\lambda(G)-\lambda(K^{\mathsf{C}})<\eta.\end{aligned}$$

We infer

$$0\le\lambda(K_j)-\sum_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}\lambda(J_i)=\lambda\big(K_j\setminus\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}J_i\big)\le\lambda\big(K\setminus\bigcup_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}J_i\big)<\eta \qquad \text{for } j\le N_\varepsilon.$$

Henceforth

$$\Big\|\sum_{j\le N_\varepsilon} y_j\Big(\sum_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}\lambda(J_i)-\lambda(K_j)\Big)\Big\|_X \le \sum_{j\le N_\varepsilon}\|y_j\|_X\Big(\lambda(K_j)-\sum_{\substack{i\in\mathfrak{T}\\ t_i\in K_j}}\lambda(J_i)\Big)\le\eta\sum_{j\le N_\varepsilon}\|y_j\|_X<\frac{\varepsilon}{3}. \tag{3.15}$$

Finally, by combining the estimates (3.12), (3.13), as well as (3.14) and (3.15) we conclude

$$\Big\|\sum_{i\in\mathfrak{T}} f(t_i)\lambda(J_i)-s\Big\|_X<\frac{\varepsilon}{6}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}<\varepsilon.$$

Thus the McShane integral $\mathcal{M}\!\!\int_I f$ exists and

$$\mathcal{M}\!\!\int_I f = s = \sum_{j\in\mathbb{N}} y_j\lambda(K_j).$$

On the other hand, since the series $\sum_{j\in\mathbb{N}} y_j\lambda(K_j)$ does not converge absolutely according to our premises, we have

$$\mathcal{B}\!\!\int_I \|f\|_X = \mathcal{M}\!\!\int_I \|f\|_X = \sum_{j\in\mathbb{N}}\|y_j\|_X\,\lambda(K_j)=\infty$$

and the Bochner integral $\mathcal{B}\!\!\int_I f$ does not exist due to Facts 2.4.1. ■

3.1.7 Remark By Proposition 3.1.6 we infer that the inclusion $\mathcal{B}\subset\mathcal{M}$ is proper for infinite-dimensional Banach spaces X. The situation is different for the Euclidean space $(\mathbb{R},|\cdot|)$. In fact, the real-valued McShane integral is equivalent to the Lebesgue integral (cf. [10], Theorem 10.13) and therefore, by Remark 2.4.6, also to the Bochner integral.

Note that the approach via the McShane integral—as opposed to the Lebesgue or Bochner theory—does not require any knowledge of measure theory at all, as one could write $\lambda([a,b])$ for $(b-a)$ in Definition 1.2.1.

3.2 The Integrals of McShane and Henstock-Kurzweil

It is not clear from Definition 1.2.1, how the two types of tagged partitions yield the different integrals of McShane and Henstock-Kurzweil. Obviously, these two integrals share several common properties (cf. Facts 1.3.4). The following example, which is due to Gordon in [10], consists of a Henstock-Kurzweil integrable function $f : [0,1] \to \mathbb{R}$, which is not Lebesgue/Bochner/McShane integrable.

3.2.1 **Example** Let $s := \sum_{n\in\mathbb{N}} a_n$ be a conditionally convergent series and define the sequence $(K_n)_{n\in\mathbb{N}}$ of non-overlapping compact intervals $K_n := [2^{-n}, 2^{-n+1}] \subset [0,1]$. Note that $\bigcup_{n\in\mathbb{N}} K_n = (0,1]$ and for $\alpha > 0$ there is an $N_\alpha \in \mathbb{N}$ such that $\lambda(K_n) < \alpha$ for every $n \geq N_\alpha$.
Furthermore define

$$f : [0,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto \sum_{n\in\mathbb{N}} \mathbb{1}_{K_n^\circ}(x) 2^n a_n.$$

To show that $f \in \mathcal{HK}$, assume that $M > 1$ is a bound for the sequence $(a_n)_{n\in\mathbb{N}}$ and let a positive $\varepsilon < 1$ be given. Choose $N_\varepsilon \in \mathbb{N}$ such that $|a_n| < \varepsilon$ and $\left|\sum_{k\geq n} a_k\right| < \varepsilon$ for all $n \geq N_\varepsilon$.
Set $K_n^\complement := [0,1] \setminus K_n$, $n \in \mathbb{N}$ and define a gauge $\delta : [0,1] \to \mathbb{R}^+$ by

$$\delta(x) := \begin{cases} \operatorname{dist}(x, K_n^\complement) & \text{if } x \in K_n^\circ \\ \dfrac{\varepsilon}{4^n M} & \text{if } x = 2^{-n+1} \\ 2^{-N_\varepsilon} & \text{if } x = 0. \end{cases}$$

Clearly, $B_{\delta(x)}(x) \cap [0,1] \subseteq K_n$ whenever $x \in K_n^\circ$, $n \in \mathbb{N}$.
Now suppose that $\{(t_j, J_j)\}$ is a δ-fine K-partition of $[0,1]$. Due to the last statement in Facts 1.3.4, we can assume that each tag t_j occurs as an endpoint of the corresponding interval J_j with no loss of generalization. Therefore, 0 must be a tag by choice of the gauge δ. Moreover, if $t_j \in K_n^\circ$ then $J_j \subset K_n^\circ$, and if $t_j = 2^{-n+1}$ then $J_j \subset K_n$ or $J_j \subset K_{n+1}$. As a result each tagged interval (t_j, J_j) with $t_j \neq 0$ is contained in a certain K_n.
Apparently, there is a tagged interval $(0,[0,\beta]) \in \{(t_j, J_j)\}$ with $0 < \beta < 2^{-N_\varepsilon}$, by choice of the gauge δ. Hence there is exactly one integer $q \geq N_\varepsilon$ such that $\beta \in [2^{-q}, 2^{-q+1}] = K_q$.
For each $n \in \mathbb{N}$, $1 \leq n \leq q$, let $\{(t_j^n, J_j^n) : j \in \mathfrak{J}_n\}$ be the subset of $\{(t_j, J_j)\}$ that has intervals in K_n.
Supposed that $1 \leq n < q$, both 2^{-n} and 2^{-n+1} are tags of the K-system $\{(t_j^n, J_j^n) : j \in \mathfrak{J}_n\}$ as each tag t_j^n occurs as an endpoint of the corresponding interval J_j^n, according to our assumption above. Moreover $f(2^{-n}) = f(2^{-n+1}) = 0$ by choice of f.
Now we have

$$\sum_{\substack{j\in\mathfrak{J}_n \\ t_j^n \neq 2^{-n} \\ t_j^n \neq 2^{-n+1}}} 2^n \lambda(J_j^n) = 2^n \Big(\sum_{j\in\mathfrak{J}_n} \lambda(J_j^n) - \sum_{t_j^n \in \{2^{-n}, 2^{-n+1}\}} \lambda(J_j^n) \Big) \geq$$

$$\geq 2^n \big(2^{-n} - \delta(2^{-n}) - \delta(2^{-n+1})\big) \geq 2^n \big(2^{-n} - 2\delta(2^{-n+1})\big) = 1 - \frac{2\varepsilon}{2^n M}$$

for every K-system $\{(t_j^n, J_j^n) : j \in \mathfrak{J}_n\}$.

According to this estimate, we can choose k_n, $1 - \frac{2\varepsilon}{2^n M} \leq k_n < 1$ such that

$$\sum_{j \in \mathfrak{J}_n} f(t_j^n)\lambda(J_j^n) = 2^n a_n \Big(\sum_{j \in \mathfrak{J}_n} \lambda(J_j^n) - \sum_{t_j^n \in \{2^{-n}, 2^{-n+1}\}} \lambda(J_j^n) \Big) = k_n a_n \qquad \text{for every } n < q.$$

It follows that

$$\left| \sum_{j \in \mathfrak{J}_n} f(t_j^n)\lambda(J_j^n) - a_n \right| = (1 - k_n)\,|a_n| < \frac{2\varepsilon\,|a_n|}{2^n M} \leq \frac{2\varepsilon}{2^n}$$

for $n < q$. Thus

$$\begin{aligned} \left| \sum_{j \in \mathfrak{J}} f(t_j)\lambda(J_j) - s \right| &= \left| \sum_{n \leq q} \sum_{j \in \mathfrak{J}_n} f(t_j^n)\lambda(J_j^n) - \sum_{n \leq q} a_n - \sum_{n > q} a_n \right| \leq \\ &\leq \sum_{n \leq q} \left| \sum_{j \in \mathfrak{J}_n} f(t_j^n)\lambda(J_j^n) - a_n \right| + \left| \sum_{n > q} a_n \right| \leq \\ &\leq \sum_{n < q} \frac{2\varepsilon}{2^n} + |a_q| + \varepsilon < 4\varepsilon. \end{aligned}$$

Hence f is Henstock-Kurzweil integrable on $[0,1]$ and $\mathcal{HK}\!\!\int_{[0,1]} f = s$.

On the other hand, note that f is not Lebesgue Integrable on $[0,1]$ because

$$\int_{[0,1]} |f| \, \mathrm{d}\lambda = \sum_{n \in \mathbb{N}} \int_{K_n} |f| \, \mathrm{d}\lambda = \sum_{n \in \mathbb{N}} |a_n| = \infty.$$

By Remark 2.4.6 and Remark 3.1.7 we conclude that f is neither Bochner nor McShane integrable. ◀

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