February 28, 2024 – University of Hong Kong

Optimal complexity of adaptive FEM for nonlinear PDEs

Dirk Praetorius

joint work with

Gregor Gantner, Michael Innerberger, Ani Miraçi, Julian Streitberger

TU Wien
Institute of Analysis and Scientific Computing

Outline

What is all about?

A nonlinear model problem

AFEM with exact solver

AFEM with Kačanov linearization

Numerical experiment

AFEM with guaranteed optimal complexity

Conclusions

What is all about?

Well-known AFEM algorithm

NumPDEs
ASC ▸ TU WIEN

Input: $\mathcal{T}_0, \quad 0 < \theta \leq 1$

For each $\ell = 0, 1, 2, \ldots$ do

- **SOLVE:** compute discrete solution $u_\ell^\star$
- **ESTIMATE:** compute error indicators $\eta_\ell(T, u_\ell^\star)$ for all $T \in \mathcal{T}_\ell$
- **MARK:** determine $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^\star)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^\star)^2$
- **REFINE:** employ newest vertex bisection $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

Output: Discrete solutions $u_\ell^\star$ and corresponding estimators $\eta_\ell(u_\ell^\star)$

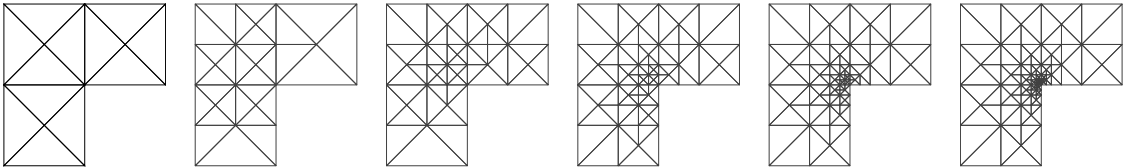
- in practice: $\mathcal{M}_\ell$ is always chosen with quasi-minimal cardinality

Dörfler: SIAM J. Numer. Anal., 33 (1996)

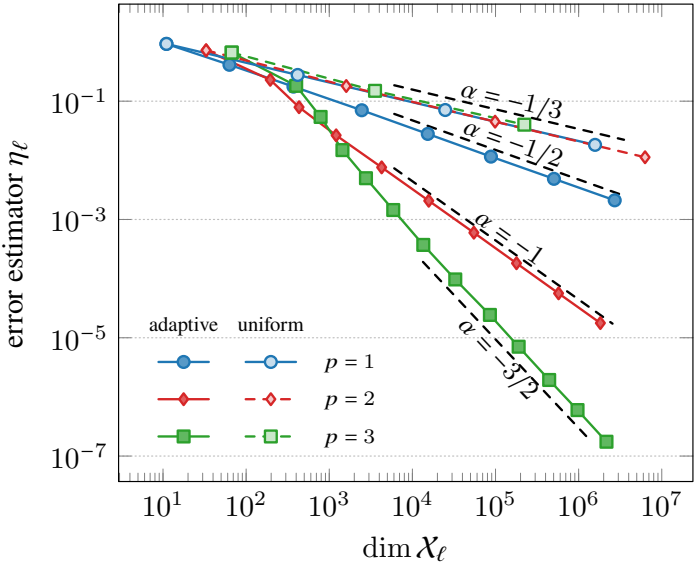
Stevenson: Found. Comput. Math., 7 (2007)

Example: AFEM for Laplace model problem

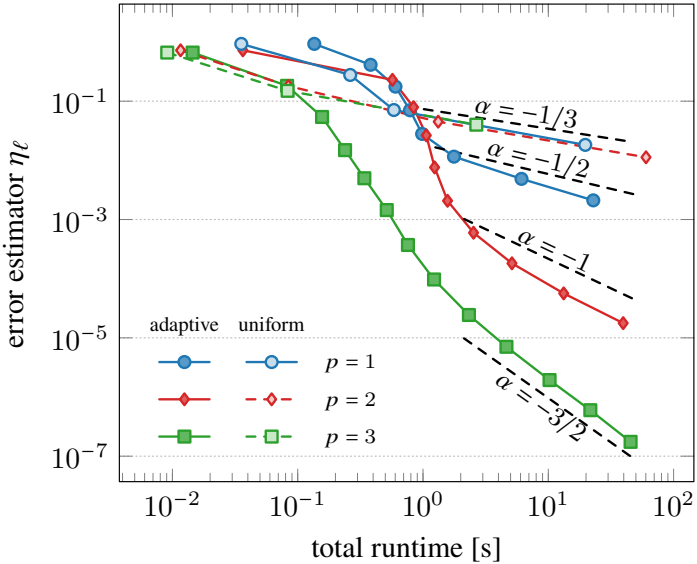
- solve $-\Delta u = 0$ in Ω subject to appropriate boundary conditions
- prescribed exact solution $u(x) = r^{2/3}\sin(2\varphi/3)$
- SOLVE – ESTIMATE – MARK – REFINE driven by standard error estimator $\eta_\ell(u_\ell)$



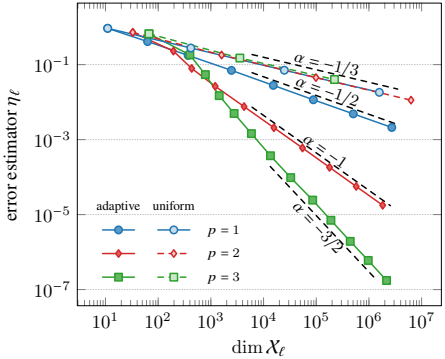
Convergence wrt. dofs

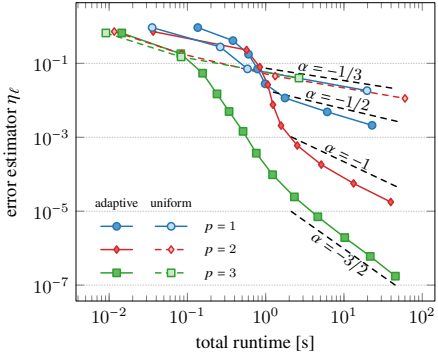


Convergence wrt. time



Describing convergence rates





- clear: $\dim \mathcal{X}_\ell \simeq \#\mathcal{T}_\ell$ for fixed p
- $\mathfrak{R}(\alpha) := \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell < \infty$

- assume: $\mathtt{work}(\mathcal{T}_{\ell'}) \simeq \#\mathcal{T}_{\ell'}$ for SEMR
- $\widehat{\mathfrak{R}}(\alpha) := \sup_{\ell \in \mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \Big)^\alpha \eta_\ell < \infty$

Linear convergence yields rates = complexity

- $\mathfrak{R}(\alpha) := \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell < \infty$ describes rate w.r.t. $\dim \mathcal{X}_\ell$
- $\widehat{\mathfrak{R}}(\alpha) := \sup_{\ell\in\mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'}\Big)^\alpha \eta_\ell < \infty$ describes rate w.r.t. (idealized) total cost/runtime

linear convergence $\eta_\ell \le C q^{\ell-\ell'} \eta_{\ell'} \quad \forall \ell' = 0,\dots,\ell$

$$\Longrightarrow \quad \mathfrak{R}(\alpha) \le \widehat{\mathfrak{R}}(\alpha) \le \frac{C}{(1-q^{1/\alpha})^\alpha}\mathfrak{R}(\alpha) \quad \text{by geometric series}$$

▸ Proof: $\#\mathcal{T}_{\ell'} \le \mathfrak{R}(\alpha)^{1/\alpha}\eta_{\ell'}^{-1/\alpha} \quad \forall \ell' = 0,\dots,\ell$

$$\Longrightarrow \quad \sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \le \mathfrak{R}(\alpha)^{1/\alpha} \sum_{\ell'=0}^{\ell} \eta_{\ell'}^{-1/\alpha} \le \mathfrak{R}(\alpha)^{1/\alpha} C^{1/\alpha} \Big(\sum_{\ell'=0}^{\ell} q^{(\ell-\ell')/\alpha}\Big)\eta_\ell^{-1/\alpha}$$

$$\Longrightarrow \quad \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'}\Big)\eta_\ell^{1/\alpha} \le \mathfrak{R}(\alpha)^{1/\alpha} \frac{C^{1/\alpha}}{1-q^{1/\alpha}}$$

Some(!) literature...

- optimal complexity for adaptive wavelet FEM: Cohen, Dahmen, DeVore (2001, 2003), ...
- plain convergence: Dörfler (1996), Morin, Nochetto, Siebert (2000), Veeser (2005), Mekchay, Nochetto (2005), Morin, Siebert, Veeser (2008), Siebert (2011), Garau, Morin, Zuppa (2011), Aurada, Ferraz-Leite, Praetorius (2012), Gantner, Praetorius (2021), ...
- linear convergence: Diening, Kreuzer (2008), Cáscon, Kreuzer, Nochetto, Siebert (2008), Becker, Mao (2009), Kreuzer, Siebert (2011), Belenki, Diening, Kreuzer (2012), Garau, Morin, Zuppa (2012), Cáscon, Nochetto (2012), Feischl, Führer, Praetorius (2014), ...
- optimal convergence rates: Binev, Dahmen, DeVore (2004), Stevenson (2007), Cáscon, Kreuzer, Nochetto, Siebert (2008), Belenki, Diening, Kreuzer (2012), Garau, Morin, Zuppa (2012), Cáscon, Nochetto (2012), Feischl, Führer, Praetorius (2014), Carstensen, Feischl, Page, Praetorius (2014), ...
- optimal complexity via perturbation analysis: Stevenson (2007), Carstensen, Gedicke (2012), Carstensen, Feischl, Page, Praetorius (2014), Haberl, Praetorius, Schimanko, Vohralík (2021), ...
- parameter-robust full linear convergence: Führer, Haberl, Praetorius, Schimanko (2018), Gantner, Haberl, Praetorius, Schimanko (2021), Heid, Praetorius, Wihler (2021), Bringmann, Feischl, Miraçi, Praetorius, Streitberger (2023+), Miraçi, Praetorius, Streitberger (2024+), ...

Take home messages, part I

1. analysis of AFEM should focus rather on rates w.r.t. complexity/time than dofs
2. linear convergence is key to relate dofs and complexity/time
3. provided that all modules SOLVE – ESTIMATE – MARK – REFINE have linear cost
   - SOLVE is critical (beyond 1D)
   - ESTIMATE is clear (with idealized quadrature)
   - MARK is known (Stevenson 2007 with binning, Pfeiler-Praetorius 2020 for minimal $\mathcal{M}_\ell$)
   - REFINE is known (Binev-Dahmen-DeVore 2004, Stevenson 2008)
4. our current developments focus on parameter-robust convergence
   - older works rely on a perturbation argument (and thus are not parameter-robust)

Binev, Dahmen, DeVore: Numer. Math., 97 (2004)
Stevenson: Found. Comput. Math., 7 (2007)
Stevenson: Math. Comp., 77 (2008)
Pfeiler, Praetorius: Math. Comp., 89 (2020)

A nonlinear model problem

Model problem

Strong formulation with scalar nonlinearity

$$\begin{aligned} -\operatorname{div}\left(\mu(|\nabla u^\star|^2)\nabla u^\star\right) &= f \quad \text{in } \Omega \\ u^\star &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Example for strongly monotone nonlinearities

- nonlinear material laws $\boldsymbol{M} = \chi(|\boldsymbol{H}|)\,\boldsymbol{H}$ in magnetostatics
- together with $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$
- e.g., $\mu(|\nabla u^\star|^2)\nabla u^\star = \left(1 + \dfrac{1}{\sqrt{1 + |\nabla u^\star|^2}}\right)\nabla u^\star$

Zeidler: Nonlinear functional analysis and its applications, Part II/B

Strongly monotone nonlinearities

Weak formulation

- find $u^\star \in H^1_0(\Omega)$ s.t. $\quad \langle \mu(|\nabla u^\star|^2)\nabla u^\star \,,\, \nabla v\rangle_{L^2(\Omega)} = \langle f \,,\, v\rangle_{L^2(\Omega)} \quad \forall\, v \in H^1_0(\Omega)$

- LHS defines operator $\mathcal{A}\colon H^1_0(\Omega) \to H^{-1}(\Omega)$ that is
  - strongly monotone $\qquad M\,\|u-v\|^2 \le \langle \mathcal{A}u - \mathcal{A}v \,,\, u-v\rangle$
  - Lipschitz continuous $\qquad \langle \mathcal{A}u - \mathcal{A}v \,,\, w\rangle \le L\,\|u-v\|\,\|w\|$

$\Longrightarrow$ existence and uniqueness of weak solution $u^\star \in H^1_0(\Omega)$

- e.g., $\quad M\,(t-s) \le \mu(t^2)t - \mu(s^2)s \le L\,(t-s) \quad \forall\, 0 \le s \le t$

Zeidler: Nonlinear functional analysis and its applications, Part II/B

PDE inherent energy structure

- energy functional $\mathcal{J}(v) := \frac{1}{2}\int_\Omega \int_0^{|\nabla v(x)|^2} \mu(t)\,\mathrm{d}t\,\mathrm{d}x - \langle f\,,\,v\rangle_{L^2(\Omega)}$

Lemma

- $\frac{M}{2}\,|\!|\!|u^\star - v|\!|\!|^2 \le \mathcal{J}(v) - \mathcal{J}(u^\star) \le \frac{L}{2}\,|\!|\!|u^\star - v|\!|\!|^2 \qquad \forall\, v \in H^1_0(\Omega)$
- i.e., $u^\star$ is equivalent to minimization of $\mathcal{J}$ over $H^1_0(\Omega)$

- energy distance $\mathbb{D}^2(u,v) := \mathcal{J}(v) - \mathcal{J}(u)$
  - note: $\mathbb{D}^2(u^\star, v) \simeq |\!|\!|u^\star - v|\!|\!|^2 \ge 0$

Zeidler: Nonlinear functional analysis and its applications, Part II/B

Galerkin discretization

NumPDEs
ASC ▸ TUWIEN

Weak formulation

- find $u^\star \in H^1_0(\Omega)$ s.t. $\langle \mu(|\nabla u^\star|^2)\nabla u^\star \,,\, \nabla v\rangle_{L^2(\Omega)} = \langle f \,,\, v\rangle_{L^2(\Omega)} \quad \forall\, v \in H^1_0(\Omega)$

- consider conforming discrete subspace $\mathcal{X}_\ell := \mathcal{P}^p(\mathcal{T}_\ell) \cap H^1_0(\Omega) \subset H^1_0(\Omega)$

Discrete formulation

- find $u^\star_\ell \in \mathcal{X}_\ell$ s.t. $\langle \mu(|\nabla u^\star_\ell|^2)\nabla u^\star_\ell \,,\, \nabla v_\ell\rangle_{L^2(\Omega)} = \langle f \,,\, v_\ell\rangle_{L^2(\Omega)} \quad \forall\, v_\ell \in \mathcal{X}_\ell$

- all analytical properties transfer from $H^1_0(\Omega)$ to $\mathcal{X}_\ell$
- clear: existence and uniqueness of $u^\star_\ell$ and equivalence to energy minimization over $\mathcal{X}_\ell$
- Pythagorean identity $\mathbb{D}^2(u^\star, u^\star_\ell) + \mathbb{D}^2(u^\star_\ell, v_\ell) = \mathbb{D}^2(u^\star, v_\ell)$ for energy difference

- but: corresponds to nonlinear discrete system so that $u^\star_\ell$ can hardly be computed exactly

Take home messages, part II

1. second-order elliptic PDE in divergence form with strongly monotone scalar nonlinearity
2. existence and uniqueness of $u^\star \in H^1_0(\Omega)$
3. equivalence with energy minimization problem
4. conforming Galerkin discretization inherits all properties
5. but $u^\star_\ell$ can hardly be computed exactly (and at linear cost)

AFEM with exact solver

Well-known AFEM algorithm

NumPDEs
ASC ▸ TUWIEN

Input: $\mathcal{T}_0, \quad 0 < \theta \leq 1$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE:** compute discrete solution $u_\ell^\star$
- **ESTIMATE:** compute error indicators $\eta_\ell(T, u_\ell^\star)$ for all $T \in \mathcal{T}_\ell$
- **MARK:** determine $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^\star)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^\star)^2$
- **REFINE:** employ newest vertex bisection $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

Output: Discrete solutions $u_\ell^\star$ and corresponding estimators $\eta_\ell(u_\ell^\star)$

- assumption: $u_\ell^\star$ is computed exactly

Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl., 5 (2012)

Linear convergence

NumPDEs
ASC ▸ TUWIEN

Theorem (Garau, Morin, Zuppa '12)

- arbitrary $0 < \theta \le 1$

$\Longrightarrow$ exists $0 < q, \gamma < 1$ s.t.

- contraction $\mathrm{H}_{\ell+1}^{1/2} \le q\,\mathrm{H}_\ell^{1/2}$ with $0 \le \mathrm{H}_\ell := \mathbb{D}^2(u^\star, u_\ell^\star) + \gamma\eta_\ell(u_\ell^\star)^2$

$\Longrightarrow$ linear convergence $\eta_\ell(u_\ell^\star) \lesssim q^{\ell-\ell'}\eta_{\ell'}(u_{\ell'}^\star)$ for all $\ell' \le \ell$

1. estimator reduction $\eta_{\ell+1}(u_{\ell+1}^\star) \le q\,\eta_\ell(u_\ell^\star) + C\,\|u_{\ell+1}^\star - u_\ell^\star\|$
2. energy equivalence $\mathbb{D}^2(u^\star, v) \simeq \|u^\star - v\|^2$
3. reliability $\|u^\star - u_\ell^\star\| \lesssim \eta_\ell(u_\ell^\star)$
4. Pythagorean identity $\mathbb{D}^2(u^\star, u_{\ell+1}^\star) + \mathbb{D}^2(u_{\ell+1}^\star, u_\ell^\star) = \mathbb{D}^2(u^\star, u_\ell^\star)$

Cáscon, Kreuzer, Nochetto, Siebert: SIAM J. Numer. Anal. 46 (2008)

Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl., 5 (2012)

Optimal convergence rates

NumPDEs
ASC ▸ TU WIEN

Theorem (Garau, Morin, Zuppa '12)

- arbitrary $\alpha > 0$
- sufficiently small $0 < \theta \leq 1$
- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with quasi-minimal cardinality

$$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}} (\#\mathcal{T}_\ell)^\alpha \, \eta_\ell(u_\ell^\star) \simeq \|u^\star\|_{\mathbb{A}_\alpha} := \sup_{N \geq \#\mathcal{T}_0} N^\alpha \Big[\min_{\#\mathcal{T}_+ \leq N} \eta_+(u_+^\star) \Big]$$

1. linear convergence of $\eta_\ell(u_\ell^\star)$
2. discrete reliability
3. fine properties of mesh-refinement

Stevenson: Found. Comput. Math., 7 (2007)

Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl., 5 (2012)

Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

Idealized optimal complexity

- suppose: $u_\ell^\star$ can be computed at linear cost

- then: linear convergence $+$ optimal rates $\implies$ optimal complexity

- $$\sup_{\ell\in\mathbb{N}} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'}\Big)^\alpha \eta_\ell(u_\ell^\star) \simeq \sup_{\ell\in\mathbb{N}} (\#\mathcal{T}_\ell)^\alpha \eta_\ell(u_\ell^\star) \simeq \|u^\star\|_{\mathbb{A}_\alpha}$$

- if $u^\star$ can be approximated at rate α over dofs,
  then AFEM approximates $u^\star$ at rate α over complexity/time

Take home messages, part III

1. second-order elliptic PDE in divergence form with strongly monotone scalar nonlinearity
2. same AFEM results (linear convergence + optimal rates) as for linear PDEs
3. optimal complexity would follow if $u_\ell^\star$ could be computed at linear cost

AFEM with Kačanov linearization

Linearized discretization

NumPDEs
ASC ▸ TUWIEN

Discrete formulation

- find $u_\ell^\star \in \mathcal{X}_\ell$ s.t. $\langle \mu(|\nabla u_\ell^\star|^2)\nabla u_\ell^\star \,,\, \nabla v_\ell\rangle_{L^2(\Omega)} = \langle f \,,\, v_\ell\rangle_{L^2(\Omega)} \quad \forall\, v_\ell \in \mathcal{X}_\ell$

Kačanov linearization: Linearized discrete formulation

- given $u_\ell^k \in \mathcal{X}_\ell$, find $u_\ell^{k+1} \in \mathcal{X}_\ell$ s.t. $\langle \mu(|\nabla u_\ell^k|^2)\nabla u_\ell^{k+1} \,,\, \nabla v_\ell\rangle_{L^2(\Omega)} = \langle f \,,\, v_\ell\rangle_{L^2(\Omega)} \,\forall\, v_\ell \in \mathcal{X}_\ell$

- corresponds to linear discrete SPD system (i.e., Laplace-type system)

- recall: $M\,(t-s) \le \mu(t^2)t - \mu(s^2)s \le L\,(t-s) \quad \forall\, 0 \le s \le t$

- still idealized: linear SPD system will be solved exactly at linear cost

Convergence of Kačanov linearization

- given $u_\ell^k \in \mathcal{X}_\ell$, find $u_\ell^{k+1} \in \mathcal{X}_\ell$ s.t. $\langle \mu(|\nabla u_\ell^k|^2)\nabla u_\ell^{k+1}, \nabla v_\ell\rangle_{L^2(\Omega)} = \langle f, v_\ell\rangle_{L^2(\Omega)} \; \forall v_\ell \in \mathcal{X}_\ell$

Proposition (Heid, P., Wihler '21)

- exists $0 < q < 1$ s.t. $\quad 0 \le \mathbb{D}(u_\ell^\star, u_\ell^{k+1}) \le q\,\mathbb{D}(u_\ell^\star, u_\ell^k)$

1. $\|u_\ell^{k+1} - u_\ell^k\|^2 \le \frac{2}{M}\,\mathbb{D}^2(u_\ell^{k+1}, u_\ell^k)$ from energy calculations
2. $\|u_\ell^\star - u_\ell^k\| \le \frac{L}{M}\,\|u_\ell^{k+1} - u_\ell^k\|$ from problem and Kačanov formulation
3. $\mathbb{D}^2(u_\ell^\star, u_\ell^k) \le \frac{L}{2}\,\|u_\ell^\star - u_\ell^k\|^2$ from energy equivalence

$$\Longrightarrow \quad 0 \le \mathbb{D}^2(u_\ell^\star, u_\ell^{k+1}) = \mathbb{D}^2(u_\ell^\star, u_\ell^k) - \mathbb{D}^2(u_\ell^{k+1}, u_\ell^k)$$
$$\le \mathbb{D}^2(u_\ell^\star, u_\ell^k) - \frac{M}{2}\|u_\ell^{k+1} - u_\ell^k\|^2 \le \mathbb{D}^2(u_\ell^\star, u_\ell^k) - \frac{M^3}{2L^2}\,\|u_\ell^\star - u_\ell^k\|^2 \le \Big[1 - \frac{M^3}{L^3}\Big]\,\mathbb{D}^2(u_\ell^\star, u_\ell^k)$$

Heid, Praetorius, Wihler: Comput. Methods Appl. Math., 21 (2021)

A-posteriori error control on linearization

- exists $0 < q < 1$ s.t. $\quad 0 \leq \mathbb{D}(u_\ell^\star, u_\ell^{k+1}) \leq q\,\mathbb{D}(u_\ell^\star, u_\ell^k)$

$$\Longrightarrow \quad \|u_\ell^\star - u_\ell^k\| \simeq \mathbb{D}(u_\ell^\star, u_\ell^k) \leq \frac{q}{1-q}\,\mathbb{D}(u_\ell^k, u_\ell^{k-1}) \leq \frac{q}{1-q}\,\mathbb{D}(u_\ell^\star, u_\ell^{k-1}) \simeq \|u_\ell^\star - u_\ell^{k-1}\|$$

- reliability & stability of residual error estimator

$$\begin{aligned} 0 \leq \mathbb{D}(u^\star, u_\ell^k) \simeq \|u^\star - u_\ell^k\| \leq \|u^\star - u_\ell^\star\| + \|u_\ell^\star - u_\ell^k\| &\overset{\text{reliability}}{\lesssim} \eta_\ell(u_\ell^\star) + \|u_\ell^\star - u_\ell^k\| \\ &\overset{\text{stability}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^\star - u_\ell^k\| \\ &\overset{\text{linearization}}{\lesssim} \eta_\ell(u_\ell^k) + \mathbb{D}(u_\ell^k, u_\ell^{k-1}) \end{aligned}$$

Stopping criterion for linearization

- a-posteriori estimate: $0 \le \mathbb{D}(u^\star, u_\ell^k) \simeq \|u^\star - u_\ell^k\| \lesssim \eta_\ell(u_\ell^k) + \mathbb{D}(u_\ell^k, u_\ell^{k-1})$

- idea: equilibrate $\eta_\ell(u_\ell^k)$ and $\mathbb{D}(u_\ell^k, u_\ell^{k-1})$

$\Longrightarrow$ stop linearization for $K = k$ as soon as $\mathbb{D}(u_\ell^K, u_\ell^{K-1}) \le \lambda\, \eta_\ell(u_\ell^K)$

- nested iteration: $u_\ell^0 := u_{\ell-1}^K$

$\Longrightarrow$ reliable a-posteriori error control for all u_ℓ^k but u_0^0

AFEM with contractive linearization

NumPDEs
ASC ▸ TU WIEN

Input: $\mathcal{T}_0$, u_0^0, $0 < \theta \le 1$, $\lambda > 0$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE & ESTIMATE:** For $k = 1, 2, 3, \dots, K$, **repeat**
  - compute u_ℓ^k
  - compute $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

  until $\mathbb{D}(u_\ell^K, u_\ell^{K-1}) \le \lambda\, \eta_\ell(u_\ell^K)$

- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^K)^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^K)^2$

- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, $u_{\ell+1}^0 := u_\ell^K$

Output: Discrete solutions u_ℓ^k and corresponding estimator $\eta_\ell(u_\ell^k) + |\!|\!| u_\ell^k - u_\ell^{k-1} |\!|\!|$

- note: number of solver steps $K = K(\ell)$ might vary with ℓ

Full linear convergence

Theorem (Gantner, Haberl, P., Schimanko '21 + Heid, P., Wihler '21)

- $\mathcal{Q} := \big\{(\ell,k) \in \mathbb{N}_0^2 \,:\, u_\ell^k \text{ computed by algorithm}\big\}$
- $|\ell,k| \;:=\; \#\big\{(\ell',k') \in \mathcal{Q} \,:\, u_{\ell'}^{k'} \text{ computed earlier than } u_\ell^k\big\}$
- quasi-error $\quad \mathrm{H}_\ell^k := \big[\!\big|\!\big| u^\star - u_\ell^k \big|\!\big|\!\big| + \eta_\ell(u_\ell^k)\big]$
- arbitrary $\quad u_0^0, \quad$ arbitrary $0 < \theta \le 1, \quad$ arbitrary $\lambda > 0$

$\Longrightarrow$ exists $\quad 0 < q < 1 \quad$ such that $\quad \big\|\!\big| u^\star - u_\ell^k \big|\!\big\| \lesssim \mathrm{H}_\ell^k \lesssim q^{|\ell,k|-|\ell',k'|}\mathrm{H}_{\ell'}^{k'}$

- proof: linear convergence $\quad \Longleftrightarrow \quad \displaystyle\sum_{\substack{(\ell,k)\in\mathcal{Q}\\ |\ell,k|>|\ell',k'|}} \mathrm{H}_\ell^k \lesssim \mathrm{H}_{\ell'}^{k'} \qquad$ (as observed in [CFPP'14])

Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)
Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)
Heid, Praetorius, Wihler: Comput. Methods Appl. Math., 21 (2021)
Bringmann, Feischl, Miraçi, Praetorius, Streitberger: arXiv:2311.15738 (2023+)

Still idealized optimal complexity

- suppose: Kačanov iterate $u_\ell^k \in \mathcal{X}_\ell$ can be computed at linear cost

Theorem (Gantner, Haberl, P., Schimanko '21)

- arbitrary $\alpha > 0$
- $\|u^\star\|_{\mathbb{A}_\alpha} := \sup_{N \geq \#\mathcal{T}_0} N^\alpha \big[\min_{\#\mathcal{T}_+ \leq N} \eta_+(u_+^\star) \big] < \infty$
- sufficiently small $0 < \theta \leq 1$ and $\lambda > 0$

$$\Longrightarrow \quad \|u^\star\|_{\mathbb{A}_\alpha} \lesssim \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'| \leq |\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \mathrm{H}_\ell^k \lesssim \max\big\{ \|u^\star\|_{\mathbb{A}_\alpha}, \mathrm{H}_0^0 \big\}$$

- recall: $\mathrm{H}_\ell^k = \big[|||u - u_\ell^k||| + \eta_\ell(u_\ell^k) \big]$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

Linear convergence yields rates = (idealized) complexity

- full linear convergence

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^{\alpha} \mathrm{H}_\ell^k \simeq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^\alpha \mathrm{H}_\ell^k$$

and $\displaystyle \sup_{(0,k)\in\mathcal{Q}} (\#\mathcal{T}_0)^\alpha \mathrm{H}_0^k \lesssim \mathrm{H}_0^0$

- sufficiently small $\lambda > 0$

$$\Longrightarrow \quad \sup_{\substack{(\ell,k)\in\mathcal{Q}\\ \ell>0}} (\#\mathcal{T}_\ell)^\alpha \mathrm{H}_\ell^k \lesssim \sup_{(\ell,0)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^\alpha \mathrm{H}_\ell^K \simeq \sup_{(\ell,0)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^\alpha \mathrm{H}_\ell^\star \simeq \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell(u_\ell^\star)$$

- sufficiently small $0 < \theta \le 1$

$$\Longrightarrow \quad \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell(u_\ell^\star) \simeq \|u^\star\|_{\mathbb{A}_\alpha}$$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

Take home messages, part IV

1. can include contractive Kačanov linearization into AFEM algorithm / analysis
2. solver termination criterion by equibalance of error contributions
3. full linear convergence proves contraction of quasi-error independently of
   - either mesh-refinement
   - or another solver (linearization) step
4. full linear convergence relates rates and (idealized) complexity
   - remains to include iterative solver for linearized SPD system
5. optimal rates/complexity follow from perturbation analysis for AFEM with exact solver

Numerical experiment

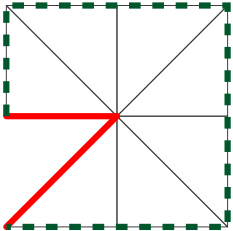
Setting

mixed BVP

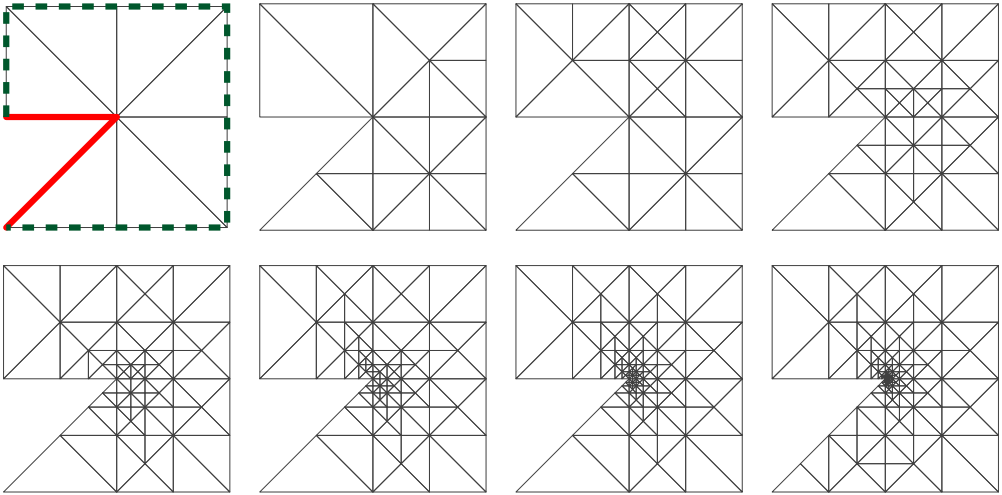
$$
\begin{aligned}
-\operatorname{div}\big(\,\mu(|\nabla u^\star|)\,\nabla u^\star\,\big) &= f \quad \text{in } \Omega \\
\mu(|\nabla u^\star|)\,\nabla u^\star \cdot \boldsymbol{n} &= g \quad \text{on } \Gamma_N \\
u^\star &= 0 \quad \text{on } \Gamma_D
\end{aligned}
$$

- $\mu(t) := 2 + \dfrac{1}{\sqrt{1+t^2}} \implies \alpha = 2, \quad L = 3$ w.r.t. $|||\cdot||| = \|\nabla(\cdot)\|_{L^2(\Omega)}$

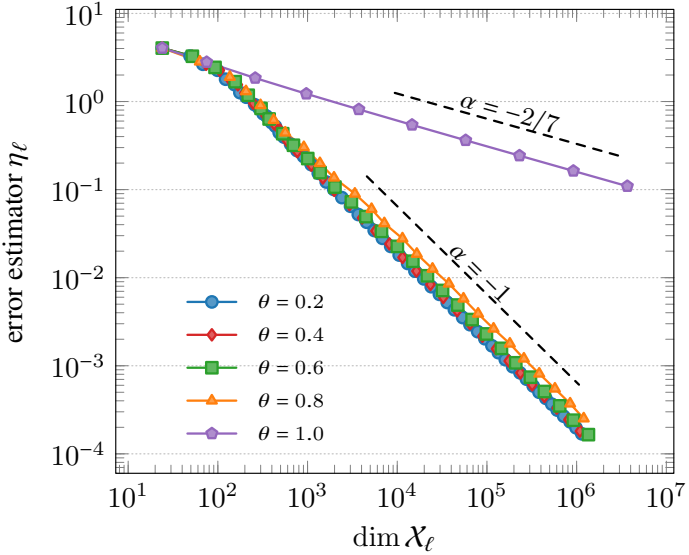
- $$
\begin{aligned}
\eta_\ell(T, u_\ell) = h_T^2 \, &\|f + \operatorname{div}\big(\,\mu(|\nabla u_\ell|)\,\nabla u_\ell\,\big)\|^2_{L^2(T)} \\
&+ h_T \, \|g - \mu(|\nabla u_\ell|)\,\nabla u_\ell \cdot \boldsymbol{n}\|^2_{L^2(\partial T \cap \Gamma_N)} \\
&+ h_T \, \|\, [\![\, \mu(|\nabla u_\ell|)\,\nabla u_\ell \cdot \boldsymbol{n} \,]\!] \,\|^2_{L^2(\partial T \cap \Omega)}
\end{aligned}
$$



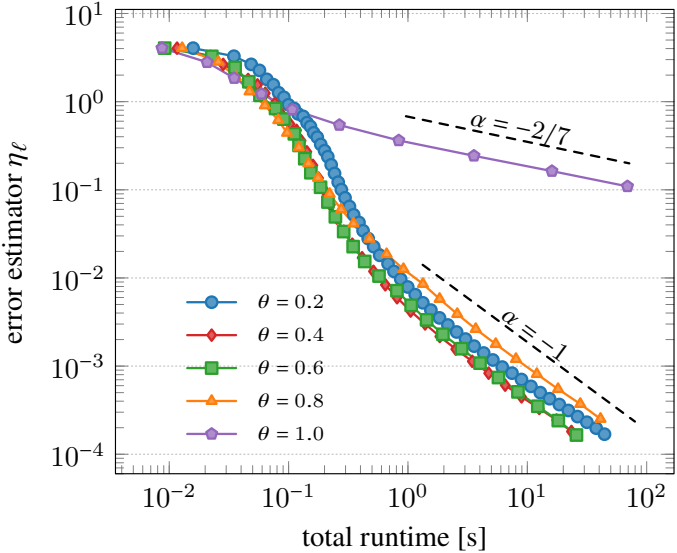
Adaptive meshes



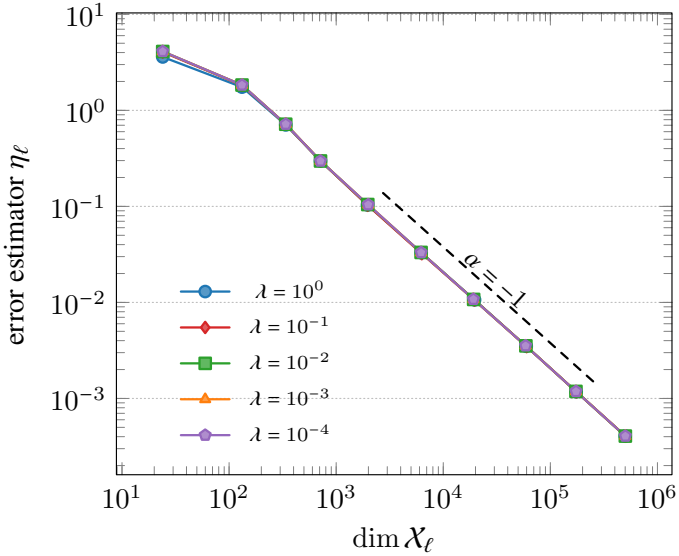
Dependence on θ for $\lambda = 10^{-2}$ $(p = 2)$



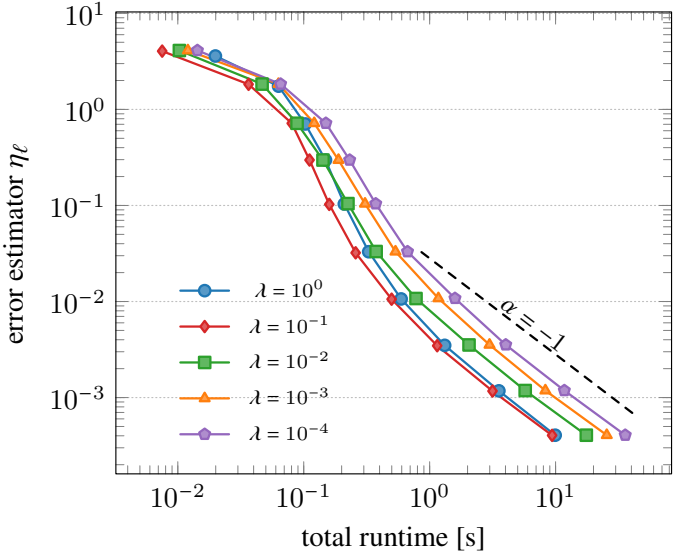
Dependence on θ for $\lambda = 10^{-2}$ wrt. time $(p = 2)$



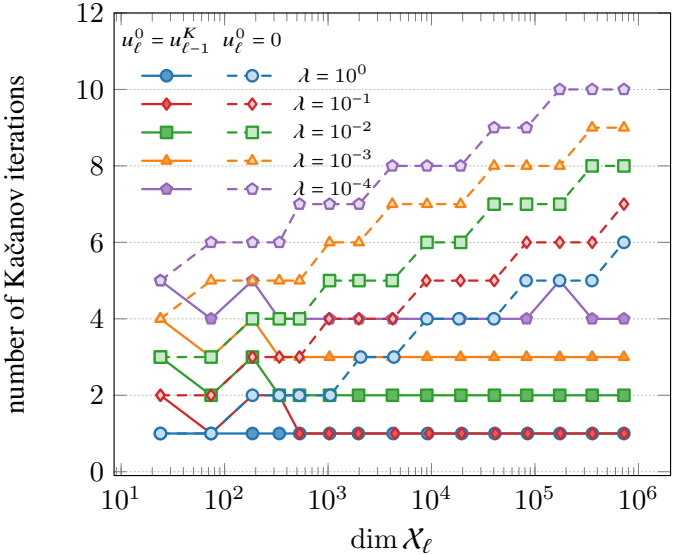
Dependence on λ for $\theta = 0.2$ $(p = 2)$



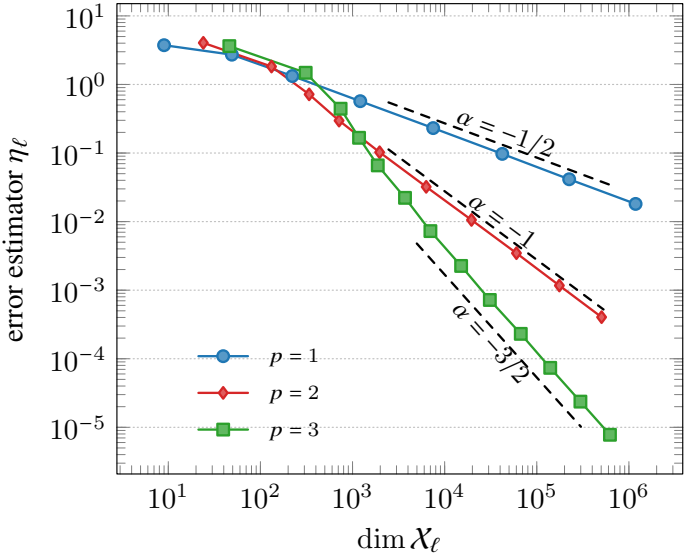
Dependence on λ for $\theta = 0.2$ wrt. time $(p = 2)$



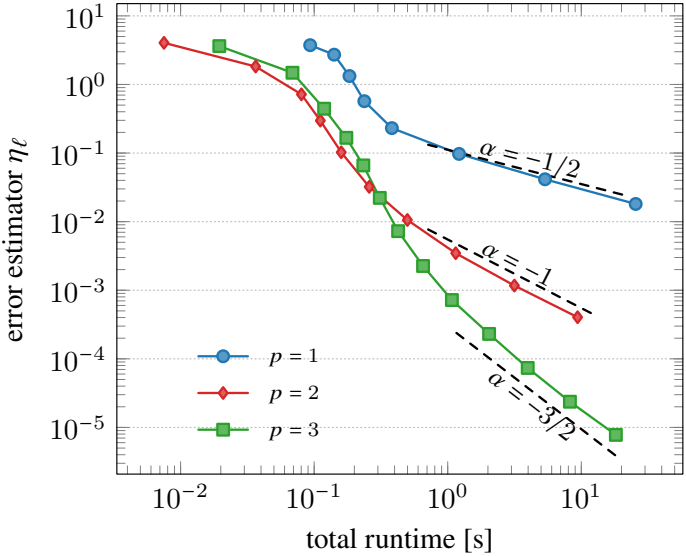
Nested iteration vs. $u_\ell^0 := 0$ $(p = 2,\ \theta = 0.2)$



Optimal convergence w.r.t. dofs $(\theta = 0.2,\ \lambda = 0.1)$



Optimal convergence w.r.t. time $(\theta = 0.2,\ \lambda = 0.1)$



AFEM with guaranteed optimal complexity

Iterative algebraic solvers

NumPDEs
ASC ▸ TUWIEN

Linearized discrete formulation

- given $u_\ell^{k,J} \in \mathcal{X}_\ell$, find $u_\ell^{k+1,\star} \in \mathcal{X}_\ell$ s.t. $\langle \mu(|\nabla u_\ell^{k,J}|^2)\nabla u_\ell^{k+1,\star}, \nabla v_\ell\rangle_{L^2(\Omega)} = \langle f, v_\ell\rangle_{L^2(\Omega)} \; \forall v_\ell \in \mathcal{X}_\ell$

- Compute algebraic iterates $u_\ell^{k+1,j} \approx u_\ell^{k+1,\star}$ via iterative solver

$\Longrightarrow$ three nested loops: mesh-refinement $(\ell) \mapsto$ linearization $(k) \mapsto$ algebraic solver (j)

$\Longrightarrow$ triple index set $(\ell, k, j) \in \mathcal{Q} \subset \mathbb{N}_0^3$ with iterates $u_\ell^{k,j}$

- $\|u_\ell^{k+1,\star} - u_\ell^{k+1,j+1}\| \leq q\,\|u_\ell^{k+1,\star} - u_\ell^{k+1,j}\| \quad \forall j \in \mathbb{N}_0$
- computing $u_\ell^{k+1,j+1}$ from $u_\ell^{k+1,j}$ has linear cost

- applies to contractive PCG [CNX12] or multigrid [WZ17], [IMPS24] solvers

Chen, Nochetto, Xu: Numer. Math., 120 (2012)

Wu, Zheng: Appl. Numer. Math., 113 (2017)

Innerberger, Miraçi, Praetorius, Streitberger: M2AN, 58 (2024)

Termination of algebraic solver

NumPDEs
ASC ▸ TUWIEN

Equilibration criterion [HPSV21]

- stop algebraic solver if $\quad \|\!| u_\ell^{k,J} - u_\ell^{k,J-1} |\!\| \le \mu \, [\, \eta_\ell(u_\ell^{k,J}) + \|\!| u_\ell^{k,J} - u_\ell^{k-1,J} |\!\| \,]$
- stop linearization if $\quad \|\!| u_\ell^{K,J} - u_\ell^{K-1,J} |\!\| \le \lambda \, \eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ full linear convergence for arbitrary λ but sufficiently small μ

Energy-based criterion [MPS24]

- stop algebraic solver if $\quad \|\!| u_\ell^{k,J} - u_\ell^{k,J-1} |\!\| \lesssim \mathbb{D}(u_\ell^{k,J}, u_\ell^{k,J-1})$
- stop linearization if $\quad \|\!| u_\ell^{K,J} - u_\ell^{K-1,J} |\!\| \le \lambda \, \eta_\ell(u_\ell^{K,J})$

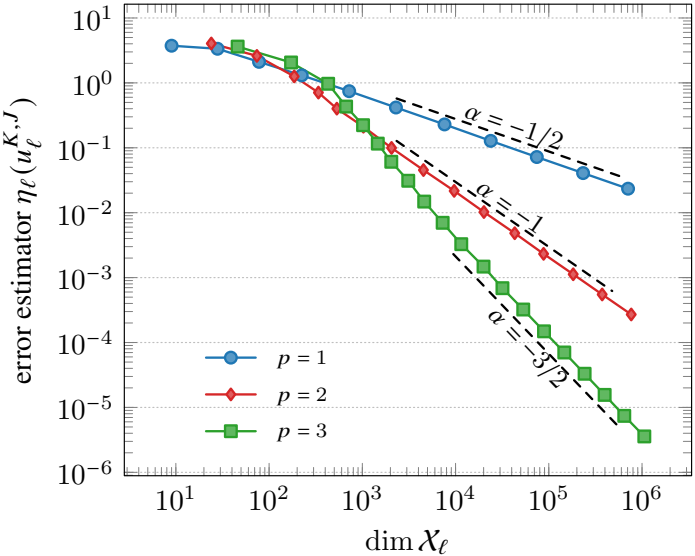
$\Longrightarrow$ full linear convergence for arbitrary λ

- optimal rates/complexity follow from perturbation analysis for AFEM with exact solver

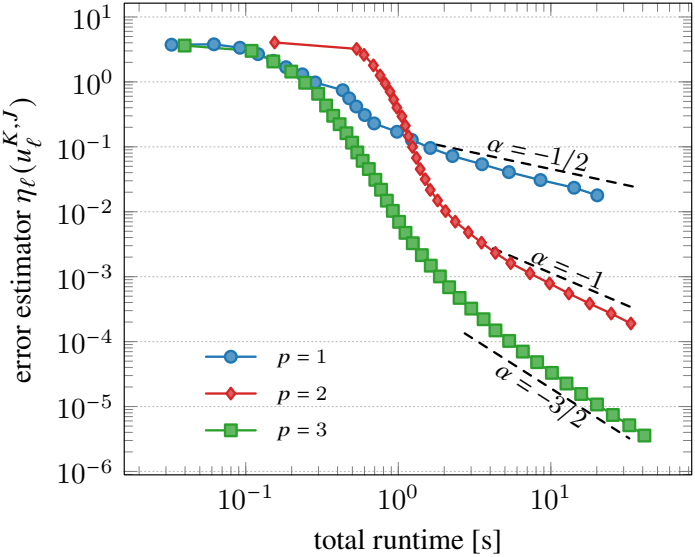
Haberl, Praetorius, Schimanko, Vohralík: Numer. Math., 147 (2021)

Miraçi, Praetorius, Streitberger: arXiv:2401.17778 (2024+)

Optimal convergence w.r.t. dofs $(\theta = 0.2,\ \lambda = 0.1)$



Optimal convergence w.r.t. time $(\theta = 0.2,\ \lambda = 0.1)$



Conclusions

Conclusion

- new: abstract framework for full linear convergence
  - i.e., contraction independently of mesh-refinement, linearization step, or algebraic solver step
  - applies to (strongly monotone) energy minimization problems
  - exploits only usual properties of residual error estimators
  - relies only on contractive linearization / algebraic solver
- implication: AFEM with (optimal) rates = complexity
  - guaranteed convergence for any choice of adaptivity parameters
  - optimal complexity for sufficiently small parameters
- extensions: analysis extends to
  - other linearization strategies (Zarantonello iteration, adaptively damped Newton method)
  - semilinear and nonsymmetric PDEs (yet strongly monotone)
  - goal-oriented adaptivity
- current restrictions:
  - only $p = 1$ rigorously proved for quasilinear PDEs
  - only simple scalar nonlinearities for quasilinear PDEs

Thank you for your attention!

Michael Innerberger, Dirk Praetorius:
MooAFEM: An object oriented Matlab code for higher-order adaptive FEM for (nonlinear) elliptic PDEs
Appl. Math. Comput., 442 (2023), #127731 or: homepage → software → mooafem

Philipp Bringmann, Michael Feischl, Ani Miraçi, Dirk Praetorius, Julian Streitberger:
On full linear convergence and optimal complexity of adaptive FEM with inexact solver
Preprint arXiv:2311.15738 (2023)

Ani Miraçi, Dirk Praetorius, Julian Streitberger:
Parameter-robust full linear convergence and optimal complexity of adaptive iteratively linearized FEM for nonlinear PDEs
Preprint arXiv:2401.17778 (2024)

Dirk Praetorius

TU Wien

Institute of Analysis and Scientific Computing

dirk.praetorius@asc.tuwien.ac.at

https://www.asc.tuwien.ac.at/praetorius/

NumPDEs
Workgroup on Numerics of PDEs

Axioms of adaptivity

NumPDEs
ASC ▸ TUWIEN

$\forall \mathcal{T}_H \quad \forall \mathcal{T}_h \in \mathrm{refine}(\mathcal{T}_H) \quad \forall v_H \in \mathcal{X}_H \quad \forall v_h \in \mathcal{X}_h$

(A1) $$\Big| \Big(\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \eta_h(T, v_h)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \eta_H(T, v_H)^2 \Big)^{1/2} \Big| \le C_{\mathrm{stab}} \, |\!|\!| v_h - v_H |\!|\!|$$

(A2) $$\sum_{T \in \mathcal{T}_h \setminus \mathcal{T}_H} \eta_h(T, v_H)^2 \le q_{\mathrm{red}} \sum_{T \in \mathcal{T}_H \setminus \mathcal{T}_h} \eta_H(T, v_H)^2$$

(A3) $$|\!|\!| u_h^\star - u_H^\star |\!|\!|^2 \le C_{\mathrm{rel}}^2 \sum_{T \in \mathcal{T}_H \setminus \mathcal{T}_h} \eta_H(T, u_H^\star)^2$$

$\forall \ell, N \ge 0 \quad \forall \varepsilon > 0$

(A4) $$\sum_{k=\ell}^{N} \big(|\!|\!| u_{k+1}^\star - u_k^\star |\!|\!|^2 - \varepsilon \eta_k(u_k^\star)^2 \big) \le C_{\mathrm{orth}}(\varepsilon) \, \eta_\ell(u_\ell^\star)^2$$