T E R

Dvoretzky's Theorem

carried out at the

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1 Introduction

In this paper we are going to study convex, symmetric and compact subsetsets of an n dimensional real vector space X that have non-empty interior. We call such sets convex, symmetric bodies. Simple, yet important examples of symmetric, convex bodies are ellipsoids. Starting with a norm $\|\cdot\|$ on X, the closed unit ball $B := \{x \in X \mid \|x\| \leq 1\}$ also always is such a set. It turns out that there is a one-to-one correspondance between the norms on X and the convex, symmetric bodies of X. We can reconstruct the norm by using the equality $\|x\| = \inf\{t > 0 \mid (1/t)x \in B\}$. The precise statement is given in Lemma 2.3.1. The ellipsoids exactly correspond to the inner products, making X a Hilbert space. This means we can always translate questions about convex, symmetric bodies into questions about finite dimensional normed spaces and the other way around. We will often make use of this translation.

We will be concerned mostly with the question how close a given convex, symmetric body is to an ellipsoid. This corresponds to the question how close a given norm is to an inner product. Phrased differently once more we will treat the question how close a normed space is to euclidean space. As a distance measure we will use the so called multiplicative Banach-Mazur distance ρ_n, which defines the distance between two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ to be the minimum of all $\left\|T^{-1}\right\|_{op}$ over the set of all isomorphisms of vector spaces $T : X \to Y$ satisfying $\|T\|_{op} \leq 1$. The precise definition is given in Definition 3.1.1. Geometrically this means that we are looking for an ellipsoid that is included in the closed unit ball B of the norm $\|\cdot\|$ that we have to "blow up" the least in order for it to already contain B.

The final result of this paper will be Dvoretzky's Theorem 3.3.8. It states that if $\varepsilon > 0$ and we denote for each $n \in \mathbb{N}$ by $k(n)$ the maximum natural number such that for all normed real vector spaces $(Y, \|\cdot\|)$ of dimension n there exists a subspace V of Y of dimension $k(n)$ satisfying $\rho_n\left(\ell^2_{k(n)}, (V, \|\cdot\|)\right) < 1 + \varepsilon$ then we have $k(n) = \Omega(\log(n))$ in Bachmann-Landau notation.

In the course of the proof of Dvoretzky's theorem we will encounter the question how far two normed spaces can be at most apart from each other in the sense of the Banach-Mazur distance. In order to find an answer to this question, we take a closer look at the definition of the Banach-Mazur distance. Using Cramer's rule we have $\left\|T^{-1}\right\|_{op} = |\det T|^{-1} \|\operatorname{adj} T\|_{op}$, where $\operatorname{adj} T$ denotes the adjugate matrix. Since the adjugate matrix is a complicated object we will simply try to maximize $|\det T|$ and hope that this will give some estimate of the minimum of $\left\|T^{-1}\right\|_{op}$. Maximizing the absolute value of the determinant geometrically means, if we set $(X, \|\cdot\|_X) = \ell^2_n$, we want to find an ellipsoid E of maximum volume included in B. The Theorem 3.3.2 named after the mathematician Fritz John states that if one "blows up" E by a factor of $\sqrt{n}$ then it will already contain B. This gives a bound on the distance between two normed spaces, namely $\rho_n\left(\ell^2_n, (Y, \|\cdot\|)\right) \leq \sqrt{n}$.

Another very important tool we need for the proof of Dvoretzky's theorem is the so called concentration of measure phenomenon on the sphere. First we need to construct an appropriate measure on the sphere. Appropriate means in our case that we want it to be a probability measure on the Borel sets that is invariant under the action of $O(n)$, the orthogonal group. We will see in Theorem 2.1.7 that such a measure on the sphere is unique and we will see three different ways to construct such a measure, two of which we will make use of. For one construction we need to make use of a Haar measure on $O(n)$ that exists by Theorem 2.1.4.

The first result in the section about the concentration of measure phenomenon will be Theorem 3.2.2 that we will prove using the multiplicative Brunn-Minkowski inequality, Corollary 2.3.7. As a corollary we obtain Corollary 3.2.4 which states that for given $n \in \mathbb{N}^*$ and $\varepsilon > 0$ as well as a measurable function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ with median M and defining $A := \{f = M\}$ we have $\mathcal{H}^{n-1}(A_\varepsilon) \geq 1 - 4\exp\left(-n\varepsilon^2/4\right)$. In other words, f does not deviate far from its median too often. This allows us to prove the important Theorem 3.2.12 and as a consequence Theorem 3.3.5.

2 Preliminaries

2.1 Measure Theory

Definition 2.1.1. Let X be a Hausdorff space and $\mathfrak{B}(X)$ the Borel σ-Algebra of X. A measure $\mu : \mathfrak{B}(X) \to [0, +\infty]$ is called Radon measure if it is inner regular on the open sets, outer regular on the Borel sets and locally finite.

Definition 2.1.2. Let G be a group, $(X, \mathfrak{A}, \mu)$ a measure space, α a left group action of G on X and β a right group action of G on X. The measure μ is called *left invariant* under α if for all $A \in \mathfrak{A}$ and all $g \in G$ the equality $\mu(\alpha(g, A)) = \mu(A)$ is satisfied. Similarly, we call μ *right invariant* under β if for all $A \in \mathfrak{A}$ and all $g \in G$ the equality $\mu(\beta(A, g)) = \mu(A)$ is satisfied.

Definition 2.1.3. Let G be a Hausdorff space and a group. A Radon measure μ on G is called a *left Haar measure* if μ is left invariant under the group action $\alpha : G \times G \to G$ defined by $\alpha(g, h) := gh$. Similarly, μ is called a *right Haar measure* if it is right invariant under α.

The following theorem will be important for us. A proof can be found in [Lun21].

Theorem 2.1.4. If G is a locally compact Hausdorff topological group, then there exists a nontrivial left Haar measure μ on G. Furthermore, if ν is another Haar measure on G, then there exists a constant $c \geq 0$ such that $\nu = c\mu$. Respectively the theorem holds for right Haar measures.

Example 2.1.5. For $n \in \mathbb{N}$ the orthogonal group $O(n)$ is a compact Hausdorff topological group. By 2.1.4 there exists a unique Haar measure μ_n on $O(n)$ such that $\mu_n(O(n)) = 1$. We will keep this notation for the rest of the text.

Definition 2.1.6. Let G be a group, X a set and $\alpha : G \times X \to X$ a left group action of G on X. We say that α is *transitive* if for all $s, t \in X$ there exists $g \in G$ such that $\alpha(g, s) = t$. Let d be a metric on X. We say that α is a *left isometry* on (X, d) if for all $g \in G$ and for all $s, t \in X$ the equality $d(\alpha(g, s), \alpha(g, t)) = d(s, t)$. If α is transitive and a left isometry then we call (X, d) a *left homogeneous space* of α.

A bit more general form of the following theorem is proven in [MS86, p.2].

Theorem 2.1.7. Let (X, d) be a compact metric space and α a left group action of a compact Hausdorff topological group G on X. If (X, d) is a left homogeneous space of α and μ and ν are two Radon measures on X which are both left invariant under α then there exists $c \in [0, +\infty[$ such that $\nu = c\mu$ or $c\nu = \mu$.

Proof. By Theorem 2.1.4 there exists a unique right Haar measure ω on G that satisfies $\omega(G) = 1$. For $x, y \in X$ there exists $g \in G$ such that $x = \alpha(g, y)$. Hence, for every measurable function $f : X \to \mathbb{C}$ we have

$$\begin{aligned}\varphi(f) &:= \int_G f(\alpha(h,x))\omega(h) = \int_G f(\alpha(hg,y))\omega(h)\\ &= \int_G f(\alpha(u,y))\omega(ug^{-1}) = \int_G f(\alpha(u,y))\omega(u),\end{aligned}$$

because ω is right invariant under α. For every Borel set $A \subseteq X$ we obtain

$$\begin{aligned}\mu(A) = \omega(G)\mu(A) &= \int_G \mu(A)\omega(h) = \int_G \mu(h^{-1}A)\omega(h) = \int_G\int_X \chi_A(hx)\mu(x)\omega(h)\\ &= \int_X\int_G \chi_A(hx)\omega(h)\mu(x) = \varphi(\chi_A)\mu(X).\end{aligned}$$

Similarly, we obtain $\nu(A) = \varphi(\chi_A)\nu(X)$. If $\mu(X) = 0$ then we define $c := 0$ and have $\mu = c\nu$. If $\mu(X) > 0$ then we define $c := \nu(X)/\mu(X)$ and have

$$\nu(A) = \varphi(\chi_A)\nu(X) = \frac{\mu(A)}{\mu(X)}\nu(X) = c\mu(A).$$

□

Definition 2.1.8. Let $(X, \mathfrak{A}, \mu)$ be be a measure sapce and $(Y, \mathfrak{T})$ a measurable space. If $f : X \to Y$ is a measurable function then we call the measure $f_*(\mu) : \mathfrak{T} \to [0, +\infty]$ defined by $f_*(\mu)(B) := \mu\big(f^{-1}(B)\big)$ for all $B \in \mathfrak{T}$ the *pushforward measure* of μ by f.

Example 2.1.9. For $n \in \mathbb{N}^*$ let μ_n be the Haar measure on $O(n)$ introduced in Example 2.1.5 and define for all $x \in \mathbb{S}^{n-1}$ the function $f_x : O(n) \to \mathbb{S}^{n-1}$ by $f_x(R) := Rx$. The function f_x is continuous and consequently measurable. Thus, we may define $\mathcal{H}^{n-1}$ as the pushforward measure of μ_n by f_x. Since μ_n is a probability measure so is $\mathcal{H}^{n-1}$. For every Borel set A of $\mathbb{S}^{n-1}$ and every $Q \in O(n)$ we have

$$\begin{aligned}\mathcal{H}^{n-1}(QA) &= \mu_n\big(f_x^{-1}(QA)\big) = \mu_n(\{R \in O(n) \mid Rx \in QA\})\\ &= \mu_n\big(\{R \in O(n) \mid Q^{-1}Rx \in A\}\big) = \mu_n\big(Q^{-1}\{R \in O(n) \mid Q^{-1}Rx \in A\}\big)\\ &= \mu_n\big(\{Q^{-1}R \in O(n) \mid Q^{-1}Rx \in A\}\big) = \mu_n\big(f_x^{-1}(A)\big) = \mathcal{H}^{n-1}(A).\end{aligned}$$

By the uniqueness up to a constant stated in Theorem 2.1.7 we obtain that $\mathcal{H}^{n-1}$ does not depend on the choice of $x \in \mathbb{S}^{n-1}$.

Remark 2.1.10. The notation $\mathcal{H}^{n-1}$ for the measure defined in Example 2.1.9 stems from the fact that the measure we obtain is the normalised $(n-1)$-dimensional Hausdorff measure on the sphere $\mathbb{S}^{n-1}$ restriced to the Borel sets. In fact, the Hausdorff measure is invariant under isometries and $O(n)$ acts as an isometry on $\mathbb{S}^{n-1}$. By the uniqueness stated in Theorem 2.1.7 the measures are the same.

Definition 2.1.11. For every $n \in \mathbb{N}$ and every $p \in [1, +\infty]$ we denote by B_n^p the unit ball in ℓ_n^p, which is $\mathbb{R}^n$ endowed with the p-norm $\|\cdot\|_p$.

Example 2.1.12. For $n \in \mathbb{N}$ let ν_n be the unique Haar measure of the locally compact Hausdorff fopological group $(\mathbb{R}^n, +)$ satisfying $\nu_n\big(B_n^2\big) = 1$. This measure is a multiple of the Lebesgue measure λ_n. Since λ_n is invariant under the action of $O(n)$ on $\mathbb{R}^n$ the same holds true for ν_n. The set $U := B_n^2 \setminus \{0\}$ is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{S}^{n-1}$ defined by $f(x) := \|x\|_2^{-1} x$ is continuous and consequently measurable. We define the measure $\mathcal{H}^{n-1}$ to be the pushforward measure of ν_n restriced on U by f. For $R \in O(n)$ and every Borel set A of $\mathbb{S}^{n-1}$ we have

$$\begin{aligned}\mathcal{H}^{n-1}(RA) &= \nu_n\big(f^{-1}(RA)\big) = \nu_n\Big(\Big\{x \in U : \|x\|_2^{-1} x \in RA\Big\}\Big)\\ &= \nu_n\Big(R^{-1}\Big\{x \in U : \big\|R^{-1}x\big\|_2^{-1} R^{-1}x \in A\Big\}\Big) = \nu_n\big(f^{-1}(A)\big) = \mathcal{H}^{n-1}(A).\end{aligned}$$

Hence, $\mathcal{H}^{n-1}$ is invariant under the action of $O(n)$ and is consequently the same measure we already construced in Example 2.1.9 and in Remark 2.1.10. This also justifies that we are using the same notation.

2.2 Analysis

Lemma 2.2.1. The function $f :]1, +\infty[\to \mathbb{R}$ defined by

$$f(x) := \frac{x}{\sqrt{n}}\left(\left(1 - \frac{1}{n}\right) \Big/ \left(1 - \frac{1}{x^2}\right)\right)^{(n-1)/2} = \frac{x}{\sqrt{n}}\left(\frac{n-1}{n}\right)^{(n-1)/2}\left(\frac{x^2}{x^2-1}\right)^{(n-1)/2} \tag{2.1}$$

satisfies $f \geq 1$ and $f(x) = 1$ if and only if $x = \sqrt{n}$.

Proof. We have

$$\begin{aligned}f'(x) &= \left(\frac{n-1}{n}\right)^{(n-1)/2}\left(\frac{1}{\sqrt{n}}\left(\frac{x^2}{x^2-1}\right)^{(n-1)/2} - \frac{n-1}{2\sqrt{n}}\left(\frac{x^2}{x^2-1}\right)^{((n-1)/2)-1}\frac{2x^2}{(x^2-1)^2}\right)\\ &= \left(\frac{n-1}{n}\right)^{(n-1)/2}\left(\frac{x^2}{x^2-1}\right)^{(n-1)/2}\frac{1}{\sqrt{n}}\left(1 - \frac{n-1}{x^2-1}\right).\end{aligned}$$

From this we deduce that $f'(x) = 0$ if and only if $x = \sqrt{n}$. Since we have $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to 1+} f(x) = +\infty$ we obtain that f attains its minimum at $\sqrt{n}$. Finally, $f(\sqrt{n}) = 1$ which finishes the proof. □

Definition 2.2.2. A discrete probability distribution with probability density function $f : \mathbb{R} \to [0, 1]$ defined by

$$f(x) := \begin{cases} 2^{-1} & \text{if } x \in \{-1, 1\} \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{R}$ is called Rademacher distribution.

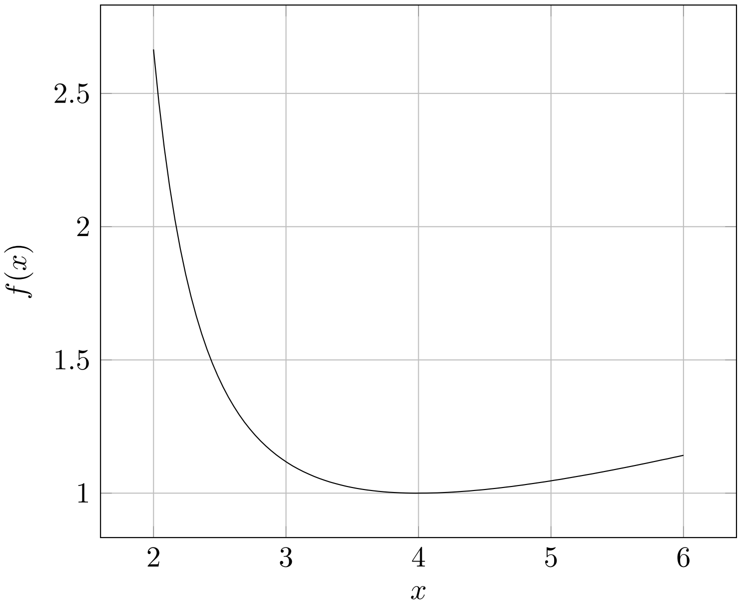

Figure 2.1: The function f from (2.1) for $n = 16$.

Proposition 2.2.3. For every $n \in \mathbb{N}^*$ the random variable $r_n : [0,1] \to \mathbb{R}$ defined by

$$r_n(t) := \operatorname{sgn}\left(\sin\left(2^n \pi t\right)\right) \tag{2.2}$$

for all $t \in [0,1]$ is Rademacher distributed. Furthermore, for every $p \in [1, +\infty[$ and all elements $x_1, \ldots, x_l$ of a normed space $(X, \|\cdot\|)$ the equality

$$\int_0^1 \left\| \sum_{i=1}^{l} r_i(t) x_i \right\|^p dt = \frac{1}{2^l} \sum_{\varepsilon \in \{-1,1\}^l} \left\| \sum_{i=1}^{l} \varepsilon_i x_i \right\|^p \tag{2.3}$$

holds true. Finally, the inequality

$$\int_0^1 \left\| \sum_{i=1}^{l} r_i(t) x_i \right\| dt \geq \max\left\{ \|x_i\| : 1 \leq i \leq l \right\} \tag{2.4}$$

is satisfied.

Proof. For every $n \in \mathbb{N}^*$ and every $k \in \{0, \ldots, 2^n - 1\}$ we define $B_{n,k} :=]k2^{-n}, (k+1)2^{-n}[$. For $t \in [0,1]$ we have

$$r(t) = 1 \Leftrightarrow t \in \bigsqcup_{k=0}^{2^{n-1}-1} B_{n,2k}$$

and

$$r(t) = -1 \Leftrightarrow t \in \bigsqcup_{k=0}^{2^{n-1}-1} B_{n,2k+1}.$$

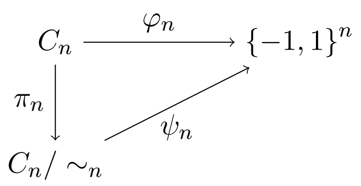

Figure 2.2: Factorization of φ_n.

It remains that $r(t)=0$ if and only if there exists $k\in\{0,\dots,2^n\}$ such that $t=k2^{-n}$. Denoting by λ the Lebesgue measure on $[0,1]$ we clearly have $\lambda(\{r=0\})=0$. Furthermore,

$$\lambda(\{r=1\})=\sum_{k=0}^{2^{n-1}-1}2^{-n}=\frac{1}{2}$$

Similarly, we obtain $\lambda(\{r=-1\})=2^{-1}$. This shows that r_n is Rademacher distributed.

We define the set

$$C_n:=\bigsqcup_{k=0}^{2^n-1}B_{n,k}$$

and on C_n an equivalence rlation by $t\sim_n s$ if and only if there exists $k\in\{0,\dots,2^n-1\}$ such that $t,s\in B_{n,k}$. Denote by $\pi_n: C_n\to C_n/\sim_n$ the natural function associated with the equivalence relation. For all $k\in\{0,\dots,2^n-1\}$ the function $r_k: C_n\to\mathbb{R}$ is continuous. Thus, $\varphi_n: C_n\to\mathbb{R}$ defined by $\varphi_n(t):=(r_1(t),\dots,r_n(t))$ is continuous as well. Since φ_n only has values in the set $\{-1,1\}^n$ it is an immediate consequence of continuity that $s\sim_n t$ implies $\varphi_n(s)=\varphi_n(t)$. Hence, we can factorize φ_n and obtain a new function $\psi_n: C_n/\sim_n\to\{-1,1\}^n$.

We prove by induction that for all $n\in\mathbb{N}^*$ the function ψ_n is injective. For the base case consider $t,u\in[0,1]$ such that $r_1(t)=\varphi_1(t)=\varphi_1(u)=r_1(u)$. This clearly implies $u\sim_1 t$. Thus, ψ_1 is injective. For the induction step we assume that ψ_n is injective for some $n\in\mathbb{N}^*$. If $t,u\in[0,1]$ such that $\psi_{n+1}\circ\pi_{n+1}(t)=\psi_{n+1}\circ\pi_{n+1}(u)$ then $\psi_n\circ\pi_n(t)=\psi_n\circ\pi_n(u)$ and $r_{n+1}(t)=r_{n+1}(u)$. By the induction hypothesis we have $\pi_n(t)=\pi_n(u)$ implying that there exists $k\in\{0,\dots,2^n-1\}$ such that $t,u\in B_{n+1,2k}\sqcup B_{n+1,2k+1}\subseteq B_{n,k}$. Since $r_{n+1}[B_{n+1,2k}]=\{1\}$ and $r_{n+1}[B_{n+1,2k+1}]=\{-1\}$ we have $\pi_{n+1}(t)=\pi_{n+1}(u)$ impyling that ψ_{n+1} is injective.

Since ψ_n is an injective function between two finite sets of the same caridnality it is also bijective. Hence, we have for every $p\in[1,+\infty[$

$$\int_0^1\left\|\sum_{i=1}^{l}r_i(t)x_i\right\|^p dt=\sum_{j=0}^{2^n-1}\int_{j2^{-n}}^{(j+1)2^{-n}}\|\langle\psi_l\circ\pi_l(t),x\rangle_2\|^p dt=\sum_{\varepsilon\in\{-1,1\}^l}\int_{\psi_l^{-1}(\varepsilon)}\left\|\sum_{i=1}^{l}\varepsilon_i x_i\right\|^p dt$$
$$=2^{-l}\sum_{\varepsilon\in\{-1,1\}^l}\left\|\sum_{i=1}^{l}\varepsilon_i x_i\right\|$$

We prove the last inequality by induction. In the base case $l = 1$ we obtain by (2.3)

$$\int_0^1 \|r_1(t)x_1\|dt = \frac{1}{2}(\|x_1\| + \|-x_1\|) = \|x_1\| \geq \max\{\|x_i\| : 1 \leq i \leq 1\}.$$

For the induction step we assume the statement is true for some $l \in \mathbb{N}^*$. Given $l+1$ vectors $x_1, \dots, x_{l+1}$ of X we may assume $\|x_i\| \geq \|x_{i+1}\|$ for all $i \in \{1, \dots, l\}$. Making use of (2.3) again we obtain

$$\begin{aligned}
\int_0^1 \left\| \sum_{i=1}^{l+1} r_i(t)x_i \right\| dt &= 2^{-(l+1)} \sum_{\varepsilon \in \{-1,1\}^{l+1}} \left\| \sum_{i=1}^{l+1} \varepsilon_i x_i \right\| \\
&= 2^{-(l+1)} \sum_{\varepsilon \in \{-1,1\}^{l}} \left(\left\| x_{l+1} + \sum_{i=1}^{l} \varepsilon_i x_i \right\| + \left\| -x_{l+1} + \sum_{i=1}^{l} \varepsilon_i x_i \right\| \right) \\
&\geq 2^{-(l+1)} \sum_{\varepsilon \in \{-1,1\}^{l}} \left\| 2 \sum_{i=1}^{l} \varepsilon_i x_i \right\| = 2^{-l} \sum_{\varepsilon \in \{-1,1\}^{l}} \left\| \sum_{i=1}^{l} \varepsilon_i x_i \right\|^p \\
&\geq \max\{\|x_i\| : 1 \leq i \leq l\} \geq \max\{\|x_i\| : 1 \leq i \leq l+1\}.
\end{aligned}$$

□

Definition 2.2.4. For every $z \in \mathbb{C}$ with $\Re(z) > 0$ we define the gamma function Γ by

$$\Gamma(z) := \int_0^{+\infty} x^{z-1} \exp(-x) dx.$$

The following inequality is called Gautschi's inequality. A proof can be found on the page [Wik20]

Lemma 2.2.5 (Gautschi)**.** If $x \in \mathbb{R}_+^*$ and $s \in]0, 1[$, then

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}.$$

Lemma 2.2.6. For all $n, m \in \mathbb{N}^*$ with $1 \leq m \leq n$ the inequality

$$\int_{\mathbb{S}^{n-1}} \max\{|x_i| : 1 \leq i \leq m\} \mathcal{H}^{n-1}(x) \geq \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\log(m)}{n}}$$

is satisfied.

Proof. For $m = 1$ the inequality is trivially satisfied, so we may assume $m, n \geq 2$ from now on. We define a function $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by $g(x) := \exp\left(-\pi\|x\|_2^2\right)\|x\|_2$. Denoting by $\mathfrak{B}(\mathbb{R}^n \setminus \{0\})$ the Borel σ-algebra of $\mathbb{R}^n \setminus \{0\}$ we define a measure $\mu : \mathfrak{B}(\mathbb{R}^n \setminus \{0\}) \to [0, 1]$ by

$$\mu(A) := \int_A g(x)dx = \int_A \|x\|_2 \exp\left(-\pi\|x\|_2^2\right) dx$$

Tranforming to spherical coordinates, substituting $u = \pi r^2$ and applying Lemma 2.2.5 we obtain

$$\begin{aligned}\mu(\mathbb{R}^n \setminus \{0\}) &= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} r^n \exp(-\pi r^2) dr d\mathcal{H}^{n-1} \\ &= \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \int_0^{+\infty} r^n \exp(-\pi r^2) dr \\ &= \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \pi^{-(n-1)/2} \int_0^{+\infty} u^{(n-1)/2} \exp(-u) du \\ &= \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \Gamma\Big(\frac{n+1}{2}\Big) \pi^{-(n-1)/2} = \frac{n}{2\sqrt{\pi}} \Gamma\Big(\frac{n}{2} + \frac{1}{2}\Big) \Big(\Gamma\Big(\frac{n}{2}+1\Big)\Big)^{-1} \le \frac{\sqrt{n}}{\sqrt{2\pi}}.\end{aligned} \tag{2.5}$$

Additionally, μ is invariant under the action of $O(n)$. In fact, if $R \in O(n)$ and $A \in \mathfrak{B}(\mathbb{R}^n \setminus \{0\})$ then

$$\begin{aligned}\mu(RA) &= \int_{RA} \|x\|_2 \exp\left(-\pi \|x\|_2^2\right) dx \\ &= \int_A \|Rx\|_2 \exp\left(-\pi \|Rx\|_2^2\right) |\det R| dx = \mu(A),\end{aligned}$$

because $|\det R| = 1$ and $\|Rx\|_2 = \|x\|_2$ for all $x \in \mathbb{R}^n$. The function $h : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ defined by $h(x) := x\|x\|_2^{-1}$ is continuous and consequently measurable. We define $\nu := h_*\mu$ as the pushforward measure of μ by h. For every $x \in R^n \setminus \{0\}$, every $y \in \mathbb{S}^{n-1}$ and every $R \in O(n)$ we have

$$h(x) = Ry \Leftrightarrow \left\|R^{-1}x\right\|_2^{-1} R^{-1}x = y \Leftrightarrow h \circ R^{-1}(x) = y.$$

Hence, for every Borel set A of $\mathbb{S}^{n-1}$ we obtain

$$\nu(RA) = \mu\big(h^{-1}(RA)\big) = \mu\big(R^{-1}h^{-1}(A)\big) = \mu\big(h^{-1}(A)\big) = \nu(A).$$

This shows that ν is invariant under the action of $O(n)$. By Theorem 2.1.7 there exists $a \in \mathbb{R}_+$ such that $a\mathcal{H}^{n-1} = \nu$. Using (2.5) we obtain

$$\begin{aligned}a &= a\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \nu(\mathbb{S}^{n-1}) = \mu\big(h^{-1}(\mathbb{S}^{n-1})\big) = \mu(\mathbb{R}^n \setminus \{0\}) \\ &= \frac{n}{2\sqrt{\pi}} \Gamma\Big(\frac{n}{2} + \frac{1}{2}\Big) \Big(\Gamma\Big(\frac{n}{2}+1\Big)\Big)^{-1} \le \frac{\sqrt{n}}{\sqrt{2\pi}}.\end{aligned}$$

For $\alpha > 0$ we have with the substitution $u = t - \alpha$

$$\begin{aligned}2\int_\alpha^{+\infty} \exp\left(-\pi(t^2 - \alpha^2)\right) dt &= 2\int_\alpha^{+\infty} \exp\left(-\pi\big((t-\alpha)^2 + 2t\alpha - 2\alpha^2\big)\right) dt \\ &= 2\exp\left(2\pi\alpha^2\right) \int_\alpha^{+\infty} \exp\left(-\pi(t-\alpha)^2\right) dt + 2\int_\alpha^{+\infty} \exp\left(-2\pi t\alpha\right) dt \\ = 2\exp\left(2\pi\alpha^2\right) \int_0^{+\infty} \exp\left(-\pi u^2\right) dt + \frac{\exp(-2\pi\alpha^2)}{\pi\alpha} &= \exp\left(2\pi\alpha^2\right) + \frac{\exp(-2\pi\alpha^2)}{\pi\alpha} \ge 1,\end{aligned} \tag{2.6}$$

because

$$\pi\alpha\big(\exp\left(2\pi\alpha^2\right)-1\big)+\exp\left(-2\pi\alpha^2\right)\geq 0 \Leftrightarrow \pi\alpha\exp\left(2\pi\alpha^2\right)+\exp\left(-2\pi\alpha^2\right)\geq\pi\alpha$$
$$\Leftrightarrow \exp\left(2\pi\alpha^2\right)+\frac{\exp(-2\pi\alpha^2)}{\pi\alpha}\geq 1.$$

From (2.6) we obtain

$$2\int_\alpha^{+\infty}\exp\left(-\pi t^2\right)dt\geq\exp\left(-\pi\alpha^2\right)$$
$$\Leftrightarrow\left(1-2\int_\alpha^{+\infty}\exp\left(-\pi t^2\right)dt\right)^m\leq\left(1-\exp\left(-\pi\alpha^2\right)\right)^m.$$

For $1\leq m\leq n$ define the function $\varphi_m:\mathbb{R}^n\to\mathbb{R}$ by $\varphi_m(x)=\max\{|x_i|:1\leq i\leq m\}$ and the set $E_m:=\left\{\varphi_m<(1/\sqrt{\pi})\sqrt{\log(m)}\right\}$. We obtain the inequality

$$\int_{\{\varphi_m<\alpha\}}\exp\left(-\pi\|x\|_2^2\right)dx=\left(\int_{-\alpha}^{\alpha}\exp\left(-\pi t^2\right)dt\right)^m\left(\int_{-\infty}^{\infty}\exp\left(-\pi t^2\right)dt\right)^{n-m}$$
$$=\left(1-2\int_\alpha^{+\infty}\exp\left(-\pi t^2\right)dt\right)^m\leq\left(1-\exp\left(-\pi\alpha^2\right)\right)^m$$

By [Kal14, p. 86] we have for all $m\geq 2$ and $k:=m-1$

$$\left(1-\frac{1}{m}\right)^{-m}=\left(\frac{m}{m-1}\right)^m=\left(\frac{k+1}{k}\right)^{k+1}=\left(1+\frac{1}{k}\right)^{k+1}>\left(1+\frac{1}{k}\right)^k\geq 2$$

Thus, for $\alpha=(1/\sqrt{\pi})\sqrt{\log(m)}$ we obtain

$$\int_{E_m}\exp\left(-\pi\|x\|_2^2\right)dx\leq\left(1-\exp\left(-\log(m)\right)\right)^m=\left(1-\frac{1}{m}\right)^m\leq\frac{1}{2}.$$

Combining everything we have

$$\int_{\mathbb{S}^{n-1}}\varphi_m\mathcal{H}^{n-1}=a^{-1}\int_{\mathbb{S}^{n-1}}\varphi_m\nu=a^{-1}\int_{\mathbb{R}^n\setminus\{0\}}\varphi_m\left(\|x\|_2^{-1}x\right)\mu(x)$$
$$=a^{-1}\int_{\mathbb{R}^n\setminus\{0\}}\|x\|_2^{-1}\varphi_m(x)\|x\|_2\exp\left(-\pi\|x\|_2^2\right)dx$$
$$=a^{-1}\int_{\mathbb{R}^n\setminus\{0\}}\varphi_m(x)\exp\left(-\pi\|x\|_2^2\right)dx$$
$$\geq a^{-1}\int_{\mathbb{R}^n\setminus E_m}\varphi_m(x)\exp\left(-\pi\|x\|_2^2\right)dx\geq\frac{\sqrt{\log(m)}}{\sqrt{\pi}a}\int_{\mathbb{R}^n\setminus E_m}\exp\left(-\pi\|x\|_2^2\right)dx$$
$$\geq\sqrt{\frac{\pi}{2}}\sqrt{\frac{\log(m)}{n}}\left(1-\int_{E_m}\exp\left(-\pi\|x\|_2^2\right)dx\right)\geq\frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\log(m)}{n}}.$$

□

2.3 Geomerty

The following Lemma 2.3.1 gives a geometric point of view on normed spaces. The main part of its proof can be found in [WKB20, p. 71] where the so called Minkowski-functional is studied.

Lemma 2.3.1. Let $n \in \mathbb{N}$ and $D := \{\|\cdot\| : \mathbb{R}^n \to \mathbb{R}; \|\cdot\| \text{ is a norm}\}$. Denoting by G the set of all convex, symmetric and compact subsetes of $\mathbb{R}^n$ with non-empty interior then the function $\varphi : D \to G$ defined by $\varphi(\|\cdot\|) := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is a bijection and its inverse is given by $\varphi^{-1}(L)(x) := \inf\{t > 0 \mid (1/t)x \in L\}$.

Definition 2.3.2. A vector space V over $\mathbb{R}$ together with a positive definite, symmetrical bilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is called a euclidean space. The function $\langle\cdot,\cdot\rangle$ is called an inner product.

The following is a result from linear algebra. A proof of it can be found in [Hav12, p. 376].

Theorem 2.3.3 (Singular value decomposition)**.** Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be two euclidean spaces and $f : V \to W$ a linear function with $r := \dim f(V)$. Then there exists an orthonormal basis $B = (b_1, b_2, \dots, b_k)$ of V and an orthonormal basis $C = (c_1, c_2, \dots, c_n)$ of W and $s_1 \geq s_2 \geq \dots \geq s_r > 0$ such that for all $i \in \{1, \dots, r\}$ the equality $f(b_i) = s_i c_i$ is satisfied and for all $i \in \{j \in \{1, \dots, k\} \mid j > r\}$ the equality $f(b_i) = 0$ holds.

Lemma 2.3.4. If $T : \mathbb{R}^k \to \mathbb{R}^n$ with $0 \leq k \leq n$ is an injective, linear function, then there exists a linear subspace V of $T(\mathbb{R}^k)$ and a constant $\alpha > 0$ such that $\dim V \geq k/2$ and $V \cap T(B_2^k) = \alpha(V \cap B_2^n)$.

Proof. Let $T : \mathbb{R}^k \to \mathbb{R}^n$ be an injective linear function, where $k \leq n$. We regard $\mathbb{R}^k$ and $\mathbb{R}^n$ as euclidean spaces with the usual scalar product. By Theorem 2.3.3, there exists an orthonormal basis $B = (b_1, \dots, b_k)$ of $\mathbb{R}^k$ and an orthonormal basis $C = (c_1, \dots, c_n)$ of $\mathbb{R}^n$ as well as $s_1 \geq s_2 \geq \dots \geq s_k > 0$ such that for all $i \in \{1, \dots, k\}$ the equality $T(b_i) = s_i c_i$ is satisfied.

Let us first consider the case that k is odd. For every $l \in \left\{1, \dots, \frac{k-1}{2}\right\}$ we define

$$\lambda_l := \sqrt{\frac{s_{k+1-l}^2 - s_{(k+1)/2}^2}{s_{k+1-l}^2 - s_l^2}} \in [0, 1]$$

and $\lambda_{k-l+1} := \sqrt{1 - \lambda_l^2}$. We set $u_l := \lambda_l b_l + \lambda_{k-l+1} b_{k-l+1}$ and $u_{(k+1)/2} := b_{(k+1)/2}$. Since B is an orthonormal basis we have

$$\|u_l\|_2^2 = \|\lambda_l b_l + \lambda_{k-l+1} b_{k-l+1}\|_2^2 = \lambda_l^2 + \lambda_{k-l+1}^2 = \lambda_l^2 + (1 - \lambda_l^2) = 1$$

as well as $\left\|u_{(k+1)/2}\right\|_2^2 = \left\|b_{(k+1)/2}\right\|_2^2 = 1$. Furthermore, we have for all $i, j \in \left\{1, \dots, \frac{k+1}{2}\right\}$ with $i \neq j$ that $\langle u_i, u_j\rangle = 0$, so $\left(u_1, \dots, u_{(k+1)/2}\right)$ forms an orthonormal system in $\mathbb{R}^k$. Since C is an orthonormal basis, we have

$$\begin{aligned}\|T(u_l)\|_2^2 &= \|\lambda_l T(b_l) + \lambda_{k-l+1} T(b_{k-l+1})\|_2^2 = \|\lambda_l s_l c_l + \lambda_{k-l+1} s_{k-l+1} c_{k-l+1}\|_2^2 \\ &= \lambda_l^2 s_l^2 + \lambda_{k-l+1}^2 s_{k-l+1}^2 = \lambda_l^2\left(s_l^2 - s_{k-l+1}^2\right) + s_{k-l+1}^2 = s_{(k+1)/2}^2\end{aligned}$$

as well as $\langle T(u_i), T(u_j)\rangle = 0$. So $\big(T(u_1), \dots, T\big(u_{(k+1)/2}\big)\big)$ forms an orthogonal system in $\mathbb{R}^n$, which consists of vectors that all have norm $s_{(k+1)/2}$. In fact, λ_l and λ_{k-l+1} were determined such that these properties are satisfied.

We define $V := \operatorname{span}\big(T(u_1), \dots, T\big(u_{(k+1)/2}\big)\big)$ and $\alpha := s_{(k+1)/2}$. Let us proof $V \cap T\big(B_2^k\big) = \alpha(V \cap B_2^n)$.

Consider $x \in B_2^k$ such that $T(x) \in V$. We can find $\alpha_1, \dots, \alpha_{(k+1)/2} \in \mathbb{R}$ such that

$$T(x) = \sum_{i=1}^{(k+1)/2} \alpha_i T(u_i)$$

and since T is injective we have

$$x = \sum_{i=1}^{(k+1)/2} \alpha_i u_i$$

Due to the fact that $\big(u_1, \dots, u_{(k+1)/2}\big)$ forms an orthonormal system in $\mathbb{R}^k$ we have

$$1 \geq \|x\|_2^2 = \sum_{i=1}^{(k+1)/2} \alpha_i^2 \tag{2.7}$$

and since $\big(T(u_1), \dots, T\big(u_{(k+1)/2}\big)\big)$ forms an orthogonal system in $\mathbb{R}^n$ we have

$$\|T(x)\|_2^2 = \sum_{i=1}^{(k+1)/2} \alpha_i^2 \|T(u_i)\|_2^2 = \sum_{i=1}^{(k+1)/2} \alpha_i^2 s_{(k+1)/2}^2 \leq s_{(k+1)/2}^2 = \alpha^2. \tag{2.8}$$

so we have $T(x) \in \alpha(V \cap B_2^n)$. In order to obtain the other conclusion we first use 2.8 and then obtain 2.7.

For the case that $k > 0$ is even, define a linear, injective function $\tilde{T} : \mathbb{R}^{k-1} \to \mathbb{R}^n$ by setting $\tilde{T}(e_i) := T(e_i)$ for all $1 \leq i \leq k-1$. Since $k-1$ is odd, we can apply the result from the previous case and obtain a linear subspace V of $\tilde{T}(\mathbb{R}^{k-1})$ such that $\dim V = \frac{(k-1)+1}{2} = \frac{k}{2}$ and $V \cap \tilde{T}\Big(B_2^{k-1}\Big) = \alpha(V \cap B_2^n)$. But it is easy to see that $V \subseteq T\big(\mathbb{R}^k\big)$ and $V \cap \tilde{T}\Big(B_2^{k-1}\Big) = V \cap T\big(B_2^k\big)$.

The case $k = 0$ is trivial. □

A proof of the following inequality can be found in [Klo22, p. 97].

Lemma 2.3.5 (Weighted AM-GM inequality)**.** For all $t \in [0,1]$ and for all $x, y \geq 0$ the inequality

$$tx + (1-t)y \geq x^t y^(1-t)$$

is satisfied.

At one point we will use the following results. The proofs given here follow the ones on [Wik22].

Theorem 2.3.6 (Prékopa-Leindler inequality)**.** If $n \in \mathbb{N}$, $t \in]0,1[$ and $f,g,h : \mathbb{R}^n \to \mathbb{R}_+$ are three measurable functions satisfying

$$h(tx+(1-t)y) \geq (f(x))^t (g(y))^{1-t} \tag{2.9}$$

for all $x, y \in \mathbb{R}^n$ then

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^t \left(\int_{\mathbb{R}^n} g(x)dx\right)^{1-t}.$$

Proof. We start by studying the measure of sets in one dimension. So let A, B two Borel-sets of $\mathbb{R}$. Let us first assume that there exists $k \in \mathbb{N}$ such that $A, B \subseteq [-k,k]$. In case $A \cap B = \{0\}$ we have $A \cup B \subseteq A + B$. Hence,

$$\lambda_n(A) + \lambda_n(B) = \lambda_n(A \cup B) \leq \lambda_n(A+B).$$

If $A \cap B \neq \{0\}$ then we define $\tilde{A} := (A - \sup A) \cup \{\sup A\}$ and $\tilde{B} := (B + \inf B) \cup \{\inf B\}$. Since $\tilde{A} \cap \tilde{B} = \{0\}$ we obtain

$$\lambda_n(A) + \lambda_n(B) = \lambda_n\left(\tilde{A}\right) + \lambda_n\left(\tilde{B}\right) \leq \lambda_n\left(\tilde{A}+\tilde{B}\right) = \lambda_n(A+B).$$

Now let A and B two Borel sets of $\mathbb{R}$ without any further assumptions. For all $k \in \mathbb{N}$ we define $A_k := A \cap [-k,k]$ and $B_k := B \cap [-k,k]$. Applying what we already showed for bounded subsets we obtain

$$\begin{aligned}\lambda_n(A) + \lambda_n(B) &= \lambda_n\left(\bigcup_{k\in\mathbb{N}} A_k\right) + \lambda_n\left(\bigcup_{k\in\mathbb{N}} B_k\right) = \lim_{k\to\infty} \left(\lambda_n(A_k) + \lambda_n(B_k)\right) \\ &\leq \lim_{k\to\infty} \left(\lambda_n(A_k + B_k)\right) = \lambda_n(A+B).\end{aligned}$$

Let $f,g,h : \mathbb{R}^n \to \mathbb{R}^+$ be three measurable functions that satisfy (2.9) for all $x, y \in \mathbb{R}^n$. We have for all $s \geq 0$

$$\lambda_n(\{h \geq s\}) \geq \lambda_n(t\{f \geq s\} + (1-t)\{g \geq s\}) \geq t\lambda_n(\{f \geq s\}) + (1-t)\lambda_n(\{g \geq s\}).$$

consequently, we have by Lemma 2.3.5

$$\begin{aligned}\int_{\mathbb{R}} h(x)dx &= \int_0^{+\infty} \lambda_n(\{h \geq s\})ds \geq t\int_0^{+\infty} \lambda_n(\{f \geq s\})ds + (1-t)\int_0^{+\infty} \lambda_n(\{g \geq s\})ds \\ &\geq \left(\int_{\mathbb{R}} f(x)dx\right)^t \left(\int_R g(x)dx\right)^{1-t}.\end{aligned}$$

For the induction step we assume the Prékopa-Leindlier inequality is satisfied in $\mathbb{R}^{n-1}$, where $n \in \mathbb{N}$ satisfies $n \geq 2$. Let $x, y \in \mathbb{R}^{n-1}$, $\alpha, \beta \in \mathbb{R}$, $t \in [0,1]$ and $\gamma := t\alpha + (1-t)\beta$. We have

$$\begin{aligned}h(tx+(1-t)y, \gamma) &= h(tx+(1-t)y, t\alpha+(1-t)\beta) = h(t(x,\alpha)+(1-t)(y,\beta)) \\ &\geq (f(x,\alpha))^t (g(y,\beta))^{1-t}.\end{aligned}$$

Defining

$$H(\gamma) := \int_{\mathbb{R}^{n-1}} h(z,\gamma)dz$$

and $F(\alpha)$ as well as $G(\beta)$ respectively we obtain by the induction hypothesis

$$H(\gamma) = \int_{\mathbb{R}^{n-1}} h(z,\gamma)dz \geq \left(\int_{\mathbb{R}^{n-1}} f(z,\alpha)dz\right)^t \left(\int_{\mathbb{R}^{n-1}} g(z,\beta)\right)^{1-t} = (F(\alpha))^t (G(\beta))^{1-t}.$$

Hence, we can apply the inequality in the one dimensional case and have

$$\begin{aligned}\int_{\mathbb{R}^n} h(x)dx = \int_{\mathbb{R}} H(\gamma)d\gamma &\geq \left(\int_{\mathbb{R}} F(\alpha)d\alpha\right)^t \left(\int_{\mathbb{R}} G(\beta)d\beta\right)^{1-t} \\ &= \left(\int_{\mathbb{R}^n} f(x)dx\right)^t \left(\int_{\mathbb{R}^n} g(x)dx\right)^{1-t}.\end{aligned}$$

This finishes the induction and consequently the proof. □

Corollary 2.3.7 (multiplicative Brunn-Minkowski inequality)**.** If $A, B \subseteq \mathbb{R}^n$ are two measurable sets and $t \in [0,1]$ then $\lambda_n(tA + (1-t)B) \geq (\lambda_n(A))^t (\lambda_n(B))^{1-t}$.

Proof. For $t \in \{0,1\}$ the inequality is trivially satisfied. From now on we assume $t \in]0,1[$. Let $A, B \subseteq \mathbb{R}^n$ be two measurable sets. Let $f := \chi_A$ and $g := \chi_B$ as well as h be the characteristc function of the set $tA + (1-t)B$. These functions satisfy for all $x, y \in \mathbb{R}^n$

$$h(tx + (1-t)y) \geq (f(x))^t (g(y))^{1-t}.$$

Thus, we can apply Theorem 2.3.6 and obtain

$$\begin{aligned}\lambda_n(tA + (1-t)B) = \int_{\mathbb{R}^n} h(x)dx &\geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^t \left(\int_{\mathbb{R}^n} g(x)dx\right)^{1-t} \\ &= (\lambda_n(A))^t (\lambda_n(B))^{1-t}.\end{aligned}$$

□

3 Dvoretzky's Theorem

3.1 Banach-Mazur distance

Definition 3.1.1. For all $n \in \mathbb{N}$ we denote by $Q(n)$ the set of all normed vector spaces over $\mathbb{R}$ of dimension n. If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two elements of $Q(n)$, then we define the *multiplicative Banach-Mazur distance* $\rho_n : Q(n) \times Q(n) \to \mathbb{R}_+^*$ by

$$\rho_n((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)) := \inf \Big\{ \|T\|_{op} \|T^{-1}\|_{op} : T \text{ is an isomorphism of vector spaces} \Big\},$$

where $\|T\|_{op}$ is the operator norm of T. If it is clear which norms the vector spaces are endowed with we sometimes write $\rho_n(X, Y) = \rho_n((X, \|\cdot\|_X), (Y, \|\cdot\|_Y))$. The fucntion $d_n : Q(n) \times Q(n) \to \mathbb{R}$ defined by $d_n(X, Y) := \log(\rho_n(X, Y))$ is called the *Banach-Mazur distance.*

Lemma 3.1.2. For all $n \in \mathbb{N}$ the multiplicative Banach-Mazur distance ρ_n is well defined and satisfies $\rho_n \geq 1$.

Proof. Let $n \in \mathbb{N}$ and $(X, \|\cdot\|_X)$ as well as $(Y, \|\cdot\|_Y)$ be two normed spaces that belong to $Q(n)$. Since X and Y have the same dimension, there exists at least one isomorphism of vector spaces $S : X \to Y$. Hence,

$$E_n := \Big\{ \|T\|_{op} \|T^{-1}\|_{op} \mid T : X \to Y \text{ is an isomorphism of vector spaces} \Big\}$$

is not the empty set.

Every $T \in E_n$ satisfies for every $y \in Y$ the inequality $\|y\|_Y \leq \|T\|_{op} \|T^{-1} y\|_X$ and consequently $\|T^{-1}\|_{op} \geq \|T\|_{op}^{-1}$. Thus, $\|T\|_{op} \|T^{-1}\|_{op} \geq \|T\|_{op} \|T\|_{op}^{-1} = 1$. From this we deduce $\rho_n(X, Y) \geq 1$. □

Lemma 3.1.3. If $n \in \mathbb{N}$ and $(X, \|\cdot\|_X)$ as well as $(Y, \|\cdot\|_Y)$ are two normed spaces which are elements of $Q(n)$ then the multiplicative Banach-Mazur distance ρ_n satisfies $\rho_n(X, Y) = 1$ if and only if there exists an isomorphism of vector spaces $T : X \to Y$ that is an isometry.

Proof. Let us first show the easy direction. If $T : X \to Y$ is an isomorphism of vector spaces that is an isometry, then for all $x \in X$ and all $y \in Y$ we have $\|Tx\|_Y = \|x\|_X$ and $\|T^{-1} y\|_X = \|y\|_Y$ implying $\|T\|_{op} = \|T^{-1}\|_{op} = 1$. Thus, $\rho_n(X, Y) \leq 1$. This gives togehter with Lemma 3.1.2 the equality $\rho_n(X, Y) = 1$.

Let us consider, on the other hand, $\rho_n(X, Y) = 1$. There exists a sequence of isomorphisms of vector spaces $T_n : X \to Y$, $n \in \mathbb{N}$, such that $\lim_{n \to +\infty} \|T_n\|_{op} \|T_n^{-1}\|_{op} = 1$. The sequence $S_n := \|T_n\|_{op}^{-1} T_n$ satisfies $\|S_n\|_{op} = 1$ for all $n \in \mathbb{N}$. Since the set of all homomorphisms of vector spaces from X to Y endowed with the operator norm is a finite dimensional normed space, the unit ball is compact in this space. Hence, there exists a subsequence $(S_{n_k})_{k\in\mathbb{N}}$ that converges to a homomorphism of vector spaces S. Clearly, $\|S\|_{op} = \lim_{k\to+\infty} \|S_{n_k}\|_{op} = 1$. The function S is even an isomorphism of vector spaces with the inverse $S^{-1} = \lim_{k\to\infty} S_{n_k}^{-1} = \lim_{k\to\infty} \|T_{n_k}\|_{op} T_{n_k}^{-1}$. Note that the inversion of an isomorphism of vector spaces is a continuous function. We also have have $\left\|S^{-1}\right\|_{op} = \lim_{k\to\infty} \|T_{n_k}\|_{op} \left\|T_{n_k}^{-1}\right\|_{op} = 1$. Thus, $\|S\|_{op}\left\|S^{-1}\right\|_{op} = 1$. □

Proposition 3.1.4. For all $n \in \mathbb{N}$ the Banach-Mazur distance d_n is a pseudometric on $Q(n)$.

Proof. Let $n \in \mathbb{N}$. By Lemma 3.1.2 the multiplicative Banach-Mazur distance satisfies $\rho_n \geq 1$. Hence, the Banach-Mazur distance $d_n = \log \circ \rho_n$ satisfies $d_n \geq 0$. Since for every normed vector space $(X, \|\cdot\|_X)$ of dimension n the identity function $\mathrm{id}_X : X \to X$ is an isomorphism of vector spaces that is an isometry, we have by Lemma 3.1.3 that $d_n(X, X) = 0$.

The symmetry of d_n follows easily from the definition of ρ_n. Given two normed vector spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ of dimension n and denoting by E_n the set of all isomorphisms of vector spaces $X \to Y$ and by F_n the set of all isomorphisms of vector spaces $Y \to X$ then the function $f_n : E_n \to F_n$ defined by $f_n(T) := T^{-1}$ is bijective and its own inverse. We obtain

$$\begin{aligned}\rho_n(Y, X) &= \inf\left\{\|f_n(T)\|_{op}\left\|f_n(T)^{-1}\right\|_{op} : T \in E_n\right\}\\ &= \inf\left\{\|T\|_{op}\left\|T^{-1}\right\|_{op} : T \in E_n\right\} = \rho_n(X, Y).\end{aligned}$$

It remains to show the subadditivity. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be three normed vector spaces of dimension n. If $T : X \to Y$ is an isomorphism of vector spaces and $S : Y \to Z$ is an isomorphism of vector spaces then $R := S \circ T$ is also an isomorphism of vector spaces satisfying

$$\rho_n(X, Z) \leq \|R\|_{op}\left\|R^{-1}\right\|_{op} = \|ST\|_{op}\left\|T^{-1}S^{-1}\right\|_{op} \leq \|S\|_{op}\left\|S^{-1}\right\|_{op}\|T\|_{op}\left\|T^{-1}\right\|_{op}.$$

Consequently, $\rho_n(X, Z) \leq \rho_n(X, Y)\rho_n(Y, Z)$ and from this we deduce

$$\begin{aligned}d_n(X, Z) &\leq \log\left(\rho_n(X, Y)\rho_n(Y, Z)\right) = \log\left(\rho_n(X, Y)\right) + \log\left(\rho_n(Y, Z)\right)\\ &= d_n(X, Y) + d_n(Y, Z).\end{aligned}$$

□

3.2 Concentration of measure

Definition 3.2.1. For every metric space X, every subset A of X and every $\varepsilon > 0$ we denote $A_\varepsilon := \{x \in X \mid d(A, x) < \varepsilon\}$.

Proposition 3.2.2. For every $n \in \mathbb{N}^*$, every Borel set A with $\mathcal{H}^{n-1}(A) > 0$ of $\mathbb{S}^{n-1}$ and every $\varepsilon \in]0,1]$ the inequality

$$\mathcal{H}^{n-1}(A_\varepsilon) \geq 1 - \frac{1}{\mathcal{H}^{n-1}(A)} \exp\left(-\frac{n\varepsilon^2}{4}\right)$$

is satisfied. Here, $\mathcal{H}^{n-1}$ is the measure defined in Example 2.1.9.

Proof. Given $\varepsilon \in]0,1]$, $n \in \mathbb{N}^*$ and a Borel set A of $\mathbb{S}^{n-1}$ satsifying $\mathcal{H}^{n-1}(A) > 0$ we define $\delta := \varepsilon^2/8$ and $B := \mathbb{S}^{n-1} \setminus A_\varepsilon$. For $a \in A$ and $b \in B$ we obtain

$$\begin{aligned}\left\|\frac{1}{2}(a+b)\right\|_2^2 &= \frac{1}{2}\left(\|a\|_2^2 + \|b\|_2^2 - \frac{1}{2}\|b-a\|_2^2\right) = \frac{1}{2}\left(2 - \frac{1}{2}\|b-a\|_2^2\right) = 1 - \frac{1}{4}\|b-a\|_2^2 \\ &< 1 - \frac{\varepsilon^2}{4} \leq \left(1 - \frac{\varepsilon^2}{8}\right)^2 = (1-\delta)^2.\end{aligned}$$

Consequently, $2^{-1}(a+b) \in (1-\delta)B_n^2$.

Define the function $f : B_n^2 \setminus \{0\} \to \mathbb{S}^{n-1}$ by $f(x) := \|x\|_2^{-1} x$ and $X := f^{-1}(A)$ as well as $Y := f^{-1}(B)$. Let $x \in X$ and $y \in Y$. We define $\gamma := \|x\|_2 / \|y\|_2$ and distinguish two cases.

The first case is $\gamma \leq 1$. In this case we have

$$\begin{aligned}\frac{1}{2}(x+y) &= \frac{1}{2}(\|x\|_2 f(x) + \|y\|_2 f(y)) = \frac{\|y\|_2}{2}(\gamma f(x) + f(y)) \\ &= \|y\|_2\left(\gamma\left(\frac{1}{2}(f(x)+f(y))\right) + (1-\gamma)\left(\frac{1}{2}f(y)\right)\right).\end{aligned}$$

Since $f(x) \in A$ and $f(y) \in B$ we have $2^{-1}(f(x)+f(y)) \in (1-\delta)B_n^2$. Furthermore, $2^{-1}f(y) \in (1-\delta)B_n^2$. Since $\|y\|_2(1-\delta)B_n^2$ is convex we have

$$\frac{1}{2}(x+y) \in \gamma\|y\|_2(1-\delta)B_n^2 + (1-\gamma)\|y\|_2(1-\delta)B_n^2 = \|y\|_2(1-\delta)B_n^2 \subseteq (1-\delta)B_n^2.$$

In the second case we have $\gamma > 1$ and hence, $\gamma^{-1} < 1$. Similarly to the first case we have

$$\begin{aligned}\frac{1}{2}(x+y) &= \frac{1}{2}(\|x\|_2 f(x) + \|y\|_2 f(y)) = \frac{\|x\|_2}{2}\left(f(x) + \gamma^{-1} f(y)\right) \\ &= \|x\|_2\left(\gamma^{-1}\left(\frac{1}{2}(f(x)+f(y))\right) + (1-\gamma^{-1})\left(\frac{1}{2}f(x)\right)\right).\end{aligned}$$

As in the first case we obtain $2^{-1}(x+y) \in (1-\delta)B_n^2$.

We just showed $2^{-1}(X+Y) \subseteq (1-\delta)B_n^2$. This implies together with the Brunn-Minkowski inequality from Corollary 2.3.7

$$(1-\delta)^n \lambda_n\left(B_n^2\right) \geq \lambda_n\left(\frac{1}{2}X + \frac{1}{2}Y\right) \geq \sqrt{\lambda_n(X)}\sqrt{\lambda_n(Y)}.$$

By Example 2.1.12 we have

$$\begin{aligned}1 - \mathcal{H}^{n-1}(A_\varepsilon) = \mathcal{H}^{n-1}(B) = \frac{\lambda_n(Y)}{\lambda_n(B_n^2)} &\leq (1-\delta)^{2n}\frac{\lambda_n\left(B_n^2\right)}{\lambda_n(X)} = \left(1 + \frac{-2n\delta}{2n}\right)^{2n}\frac{1}{\mathcal{H}^{n-1}(A)} \\ &\leq \exp\left(-2n\delta\right)\frac{1}{\mathcal{H}^{n-1}(A)} = \exp\left(-\frac{n\varepsilon^2}{4}\right)\frac{1}{\mathcal{H}^{n-1}(A)}.\end{aligned}$$

□

Definition 3.2.3. If $(X, \mathfrak{A}, \mathbb{P})$ is a probability space and $f : X \to \mathbb{R}$ is a measurable function then we call every real number M satisfying $\mathbb{P}\{f \leq M\} \geq 1/2$ and $\mathbb{P}\{f \geq M\} \geq 1/2$ a *median* of f.

Corollary 3.2.4. Let $n \in \mathbb{N}^*$ and $\varepsilon > 0$. If $f : \mathbb{S}^{n-1} \to \mathbb{R}$ is a measureable function and M is a median of f then the set $A := \{f = M\}$ satisfies $\mathcal{H}^{n-1}(A_\varepsilon) \geq 1 - 4\exp\big(-n\varepsilon^2/4\big)$.

Proof. Denoting $B := \{f \geq M\}$ and $C := \{f \leq M\}$ we $\mathcal{H}^{n-1}(B) \geq 1/2$ and $\mathcal{H}^{n-1}(C) \geq 1/2$ and consequently by Proposition 3.2.2

$$\mathcal{H}^{n-1}(B_\varepsilon) \geq 1 - \frac{1}{\mathcal{H}^{n-1}(B)} \exp\left(-\frac{n\varepsilon^2}{4}\right) \geq 1 - 2\exp\left(-\frac{n\varepsilon^2}{4}\right)$$

as well as

$$\mathcal{H}^{n-1}(C_\varepsilon) \geq 1 - \frac{1}{\mathcal{H}^{n-1}(C)} \exp\left(-\frac{n\varepsilon^2}{4}\right) \geq 1 - 2\exp\left(-\frac{n\varepsilon^2}{4}\right).$$

From this we deduce

$$\mathcal{H}^{n-1}(A_\varepsilon) = \mathcal{H}^{n-1}(B_\varepsilon \cap C_\varepsilon) \geq 1 - 4\exp\left(-\frac{n\varepsilon^2}{4}\right).$$

□

Proposition 3.2.5. If $f \in C\big(\mathbb{S}^{n-1}, \mathbb{R}\big)$ then there exists a unique median $M \in \mathbb{R}$ of f.

Proof. We start by defining a function $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(t) := \mathcal{H}^{n-1}(\{f \leq t\})$. If $\min f = \max f$ then f is a constant function and its value M is the unique median.

Else for $\min f \leq s < t \leq \max f$ there exists $x \in \mathbb{S}^{n-1}$ such that $s < f(x) < t$, because $\mathbb{S}^{n-1}$ is connected and f is continuous. By continuity of f, there exists $\delta > 0$ such that for all $y \in B(x, \delta)$ we have $s < f(y) < t$. Thus,

$$\begin{aligned}\varphi(t) - \varphi(s) &= \mathcal{H}^{n-1}(\{f \leq t\}) - \mathcal{H}^{n-1}(\{f \leq s\}) \\ &= \mathcal{H}^{n-1}(\{s < f \leq t\}) \geq \mathcal{H}^{n-1}(B(x,\delta)) > 0.\end{aligned}$$

From this the uniqueness of a median follows immediately. For the existence we define $M := \sup\{t \in \mathbb{R} \mid \varphi(t) \leq 1/2\}$. We have

$$\mathcal{H}^{n-1}(\{f \leq M\}) = \lim_{n\to+\infty} \mathcal{H}^{n-1}\left(\left\{f \leq M + \frac{1}{n}\right\}\right) \geq \lim_{n\to+\infty} \varphi\left(M + \frac{1}{n}\right) \geq \frac{1}{2}$$

and

$$\mathcal{H}^{n-1}(\{f < M\}) = \lim_{n\to\infty} \mathcal{H}^{n-1}\left(\left\{f \leq M - \frac{1}{n}\right\}\right) = \lim_{n\to+\infty} \varphi\left(M - \frac{1}{n}\right) \leq \frac{1}{2}.$$

Hence, M is the median of f. □

Definition 3.2.6. A *modulus of continuity* is an increasing function $\omega : [0, +\infty] \to [0, +\infty]$ such that $\omega(0) = 0$ and ω is continuous at 0.

Proposition 3.2.7. If (X, d_X) and (Y, d_Y) are two metric spaces and $f : X \to Y$ is a uniformly continuous function, then the function $\omega_f : [0, +\infty] \to [0, +\infty]$ defined by $\omega_f(t) := \sup\{d_Y(f(y), f(x)) \mid d_X(y, x) \le t\}$ is a modulus of continuity.

Proof. It is clear that ω_f is an increasing function and also that $\omega_f(0) = 0$. Given $\varepsilon > 0$ we find $\delta > 0$ such that for all $x, y \in X$ the inequality $d_X(y, x) < \delta$ implies $d_Y(f(y), f(x)) < \varepsilon$. Hence, for all $0 \le t < \delta$ we have $\omega_f(t) \le \varepsilon$, which means ω_f is continuous at 0. □

Definition 3.2.8. Let (X, d) be a metric space and Y a subset of X. For every $\varepsilon > 0$ we call a subset N of Y an ε-net of Y if

$$Y \subseteq \bigcup \{B(y, \varepsilon) \mid y \in N\}.$$

Definition 3.2.9. If $(X, \|\cdot\|)$ is a normed vector space then we denote

$$\begin{aligned} S(X) &:= \{x \in X \mid \|x\| = 1\} \\ B(X) &:= \{x \in X \mid \|x\| < 1\} \\ B[X] &:= \{x \in X \mid \|x\| \le 1\}. \end{aligned}$$

Lemma 3.2.10. Let $(X, \|\cdot\|)$ be a normed space of dimension $k \in \mathbb{N}$. For every $\theta \in]0, 1[$ there exists a θ-net N of $S(X)$ such that $\#N \le \left(1 + \frac{2}{\theta}\right)^k \le \exp\left(k \log\left(\frac{3}{\theta}\right)\right)$

Proof. Define the set

$$M := \{L \subseteq S(X) \mid \forall x, y \in L : (x \ne y \Rightarrow \|x - y\| \ge \theta)\}$$

Note that $M \ne \emptyset$ because for all $x \in S(X)$ we trivially have $\{x\} \in M$.

Since $S(X) \subseteq \bigcup \left\{y + \frac{\theta}{2} B(X), y \in S(X)\right\}$ and $S(X)$ is compact, there exists $l \in \mathbb{N}$ and $y_1, \dots, y_l \in S(X)$ such that $S(X) \subseteq \bigcup_{i=1}^{l} \left(y_i + \frac{\theta}{2} B(X)\right)$. We claim that for all $L \in M$ we have $\#L \le l$. Define $f : L \to 1, \dots, l$ by $f(x) := \min\left\{i \in \{1, \dots, l\} \mid x \in y_i + \frac{\theta}{2} B(X)\right\}$ for all $x \in L$. If $x, z \in L$ and $f(x) = f(z) = j$ then $x, z \in y_j + \frac{\theta}{2} B(X)$ which implies $\|x - z\| \le \|x - y_j\| + \|y_j - z\| < \theta$. By the definition of M this necessarily means $x = z$. Hence, f is injective and hence, by the pigeon hole principle $\#L \le l$. So $\{\#L \mid L \in M\} \subseteq \{1, \dots, l\}$, which means we can find $L \in M$ such that $\#L$ is a maximum.

We claim that L is a θ-net of $S(X)$ in X. Suppose this is not true, then there exists $z \in S(X) \setminus \bigcup \{x + \theta B(X) \mid x \in L\}$. This implies for all $x \in L$ that $\|x - z\| \ge \theta$ and in particular $z \notin L$. Hence, $K := L \cup \{z\} \in M$ and $\#K > \#L$, which infringes the maximality of $\#L$. Thus, L is a θ-net of $S(X)$. Finally, for all $x, y \in L$ with $x \ne y$ we have $\left(x + \frac{\theta}{2} B(X)\right) \cap \left(y + \frac{\theta}{2} B(X)\right) = \emptyset$, because if we found z in the intersection we had $\|x - y\| \le \|x - z\| + \|z - y\| < \theta$ which contradicts the definition of M. Denoting by λ_k the unique Haar measure on X satisfying $\lambda_k(B(X)) = 1$ we have

$$\left(1 + \frac{\theta}{2}\right)^k = \lambda_k\left(\left(1 + \frac{\theta}{2}\right) B(X)\right) \ge \lambda_k\left(\bigcup \left\{x + \frac{\theta}{2} B(X) \mid x \in L\right\}\right) = \#L \left(\frac{\theta}{2}\right)^k.$$

This implies

$$\#L \le \left(1 + \frac{2}{\theta}\right)^k \le \exp\left(k \log\left(\frac{3}{\theta}\right)\right).$$

□

Lemma 3.2.11. Let $n \in \mathbb{N}^*$ and $\varepsilon > 0$. If $f \in C(\mathbb{S}^{n-1}, \mathbb{R})$, the real number M is the median of f and $A := \{f = M\}$ then for every $N \subseteq \mathbb{S}^{n-1}$ with $\#N < (1/4)\exp\big((\varepsilon^2 n)/4\big)$ there exists $T \in O(n)$ such that $TN \subseteq A_\varepsilon$.

Proof. Let μ_n be the unique Haar measure on $O(n)$ satisfying $\mu_n(O(n)) = 1$. We choose some $y \in \mathbb{S}^{n-1}$ and denote by $\pi(y) : O(n) \to \mathbb{S}^{n-1}$ the function defined by $\pi(y)(T) := Ty$ for all $T \in O(n)$. Let $\mathcal{H}^{n-1} := \pi(y)_* T$ the pushforward measure as in Example 2.1.9. We then have by Corollary 3.2.4

$$\mu_n(\{T \in O(n) \mid Ty \in A_\varepsilon\}) = \mu_n\big(\pi(y)^{-1}(A_\varepsilon)\big) = \mathcal{H}^{n-1}(A_\varepsilon) \geq 1 - 4\exp\left(\frac{-\varepsilon^2 n}{4}\right).$$

Denoting $B := \{T \in O(n) \mid TN \subseteq A_\varepsilon\}$ and using the subadditivity of measures we obtain

$$\begin{aligned}
\mu_n(B) = \mu_n\left(\bigcap_{x \in N} \{T \in O(n) \mid Tx \in A_\varepsilon\}\right) &\geq 1 - \sum_{x \in \mathbb{N}} \mu_n\big(O(n) \setminus \pi(x)^{-1}(A_\varepsilon)\big) \\
&\geq 1 - \sum_{x \in N} 4\exp\left(\frac{-\varepsilon^2 n}{4}\right) = 1 - \#N 4\exp\left(\frac{-\varepsilon^2 n}{4}\right) \\
&> 1 - \frac{1}{4}\exp\left(\frac{\varepsilon^2 n}{4}\right) 4\exp\left(\frac{-\varepsilon^2 n}{4}\right) = 0.
\end{aligned}$$

So since the measure of B is greater than 0 we certainly have $B \neq \emptyset$. Thus, there exists $T \in B$ and by definition $TN \subseteq A_\varepsilon$. □

Theorem 3.2.12. Let $\varepsilon, \theta \in]0,1[$. Then there exists $c > 0$ and $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ satisfying $n \geq l$ and all $f \in C(\mathbb{S}^{n-1}, \mathbb{R})$ with median $M \in \mathbb{R}$ there exists $k \in \mathbb{N}$ that satisfies

$$k \geq c \frac{n\varepsilon^2}{4(\log(3) - \log(\theta))}$$

and a k-dimensional linear subspace E of $\mathbb{R}^n$ as well as a θ-net N of $E \cap \mathbb{S}^{n-1}$ such that for all $x \in N$ the inequality

$$|f(x) - M| \leq \omega_f(\varepsilon)$$

is satisfied and for all $x \in E \cap \mathbb{S}^{n-1}$ the inequality

$$|f(x) - M| \leq \omega_f(\varepsilon) + \omega_f(\theta)$$

is satisfied.

Proof. Given $\varepsilon, \theta \in]0,1[$ we define

$$l := \left\lceil \frac{4(\log(3) - \log(\theta) + \log(4))}{\varepsilon^2} \right\rceil + 1$$

and

$$c := \frac{1}{2}\left(1 - \left(4\frac{\log(4) + \log(3) - \log(\theta)}{l\varepsilon^2}\right)\right).$$

This assures that for all $n \in \mathbb{N}$ with $n \geq l$ we have

$$\begin{aligned} c\frac{n\varepsilon^2}{4(\log(3) - \log(\theta))} &\leq \frac{n\varepsilon^2}{4(\log(3) - \log(\theta))} - \frac{\log(4)}{\log(3) - \log(\theta)} - 1 \\ &\leq \left\lfloor \frac{n\varepsilon^2}{4(\log(3) - \log(\theta))} - \frac{\log(4)}{\log(3) - \log(\theta)} \right\rfloor =: k \end{aligned}$$

Let X be a linear subspace of $\mathbb{R}^n$ of dimension k and $A := \{f = M\}$. By Lemma 3.2.10 there exists a θ-net L of $S(X)$ such that

$$\#L < \exp\left(k(\log(3) - \log(\theta))\right) \leq \exp\left(\left(\frac{\varepsilon^2 n}{4} - \log(4)\right)\frac{\log(3) - \log(\theta)}{\log(3) - \log(\theta)}\right) = \frac{1}{4}\exp\left(\frac{\varepsilon^2 n}{4}\right).$$

By Lemma 3.2.11 there exists a $R \in O(n)$ such that $RL \subseteq A_\varepsilon$. Define $E := RX$ and $N := RL$. Clearly, N is a θ-net of $S(E)$ and by continuity of f and connectedness of $\mathbb{S}^{n-1}$ there exists $y \in \mathbb{S}^{n-1}$ such that $f(y) = M$. For every $x \in N$ we have $\|x - y\|_2 < \varepsilon$ and consequently

$$|f(x) - M| < |f(x) - f(y)| \leq \omega_f(\varepsilon).$$

For $z \in E \cap \mathbb{S}^{n-1}$ there exists $x \in N$ such that $\|z - x\|_2 < \theta$. Hence,

$$|f(z) - M| \leq |f(z) - f(x)| + |f(x) - f(y)| \leq \omega_f(\theta) + \omega_f(\varepsilon).$$

□

Lemma 3.2.13. If $(X, \|\cdot\|)$ is an n-dimensional normed space and $T : \ell_n^2 \to (X, \|\cdot\|)$ is an isomorphism of vector spaces such that for all homomorphisms of vector spaces $S : \ell_n^2 \to (X, \|\cdot\|)$ with $\|S\|_{op} \leq 1$ the inequality $|\det S| \leq |\det T|$ holds true, then $\|T\|_{op} = 1$.

Proof. Suppose $\|T\|_{op} < 1$. Then we define a new linear function $S : \mathbb{R}^n \to X$ by $S := \|T\|_{op}^{-1} T$. We remark that $\|T\|_{op} \neq 0$ because T is bijective. By absolute homogeneity of norms we have $\|S\|_{op} = \|T\|_{op}^{-1}\|T\|_{op} = 1$. By homogeneity of the determinant and the norm we have $|\det S| = \|T\|_{op}^{-n}|\det T| > |\det T|$. This contradicts the maximality condition we required for T. □

3.3 The distance between normed spaces and euclidean space

Theorem 3.3.1 (Dvoretzky-Rogers)**.** Let $(X, \|\cdot\|) \in Q(n)$ for some $n \in \mathbb{N}$ and $T : \ell_n^2 \to (X, \|\cdot\|)$ be an isomorphism of vector spaces such that for all homomorphisms of vector spaces $S : \ell_n^2 \to (X, \|\cdot\|)$ the inequality $|\det S| \leq |\det T|$ is satisfied. Then there exists an orthonormal basis $(x_1, \ldots, x_n)$ of ℓ_n^2 such that for all $i \in \{1, \ldots, n\}$ the inequalities

$$2^{-n/(n-(i-1))} \leq \|Tx_i\| \leq 1$$

hold true.

Proof. In case $n = 0$ we take $\emptyset$ as a basis of ℓ_0^2 and we are finished. So suppose from now on that $n \geq 1$. By Lemma 3.2.13 we have $\|T\|_{op} = 1$ and since the closed unit ball in ℓ_n^2 is compact there exists $x_1 \in B_n^2 \setminus \{0\}$ such that $\|Tx_1\| = 1$. This is the start of our recursive construction of the orthonormal basis of ℓ_n^2. Given i orthonormal vectors $x_1, \ldots, x_i \in \ell_n^2$ for $i \in \{1, \ldots, n-1\}$ we define x_{i+1} as a vector that maximizes $\|T\cdot\|$ on the compact set $\langle x_1, \ldots, x_i \rangle^\perp \cap B_n^2$. Clearly, we obtain an orthogonal base in this way. Suppose there was a $k \in \{1, \ldots, n\}$ such that $\|x_k\|_2 < 1$. Then we had

$$\left\| T\left(\frac{1}{\|x_k\|_2} x_k \right) \right\| = \frac{1}{\|x_k\|_2} \|Tx_k\| > \|Tx_k\|.$$

This contradicts the maximality condition for x_k. Hence, we showed that we have constructed an orthonormal base of ℓ_n^2.

For $k \in \{1, \ldots, n\}$ define a linear function $S_k : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S_k(x_i) := \begin{cases} \frac{1}{2} x_i & \text{if } i \in \{1, \ldots, k-1\} \\ \frac{1}{2\|Tx_k\|} x_i & \text{if } i \in \{k, \ldots, n\} \end{cases}$$

Let $a_1, \ldots, a_n$ be real numbers such that

$$\left\| \sum_{i=1}^{n} a_i x_i \right\|_2^2 = \sum_{i=1}^{n} a_i^2 = 1,$$

so this linear combination might represent any number on the sphere $\mathbb{S}^{n-1}$. We have

$$\left\| TS_k \sum_{i=1}^{k-1} a_i x_i \right\|^2 \leq \|T\|_{op}^2 \left\| \sum_{i=1}^{k-1} \frac{a_i}{2} x_i \right\|_2^2 = \frac{1}{4} \sum_{i=1}^{k-1} a_i^2 \|x_i\|_2^2 \leq \frac{1}{4}.$$

Since

$$\left\| 2\|Tx_k\| S_k \sum_{i=k}^{n} a_i x_i \right\|_2^2 = \sum_{i=k}^{n} a_i^2 \|x_i\|_2^2 \leq 1$$

we have

$$2\|Tx_k\| S_k \sum_{i=k}^{n} a_i x_i \in \langle x_1, \ldots, x_{k-1} \rangle^\perp \cap B_n^2.$$

Thus, by construction of the orthonormal base we obtain

$$2\|Tx_k\| \left\| TS_k \left(\sum_{i=k}^{n} a_i x_i \right) \right\| = \left\| T\left(2\|Tx_k\| S_k \left(\sum_{i=k}^{n} a_i x_i \right) \right) \right\| \leq \|Tx_k\|.$$

These two inequalities yield

$$\left\| TS_k \left(\sum_{i=1}^{n} a_i x_i \right) \right\| \leq \left\| TS_k \left(\sum_{i=1}^{k-1} a_i x_i \right) \right\| + \left\| TS_k \left(\sum_{i=k}^{n} a_i x_i \right) \right\| \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Hence, $\|TS_k\|_{op} \leq 1$ and

$$|\det T| \geq |\det TS_k| = |\det T||\det S_k|.$$

This yields

$$1 \geq |\det S_k| = \left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2\|Tx_k\|}\right)^{n-(k-1)} = 2^{-n}\|Tx_k\|^{-(n-(k-1))}$$

which is equivalent to

$$\|Tx_k\| \geq \left(\frac{1}{2}\right)^{\frac{n}{n-(k-1)}}.$$

□

Theorem 3.3.2 (F. John). **If** $(Y, \|\cdot\|)$ is a normed vector space of dimension $n \in \mathbb{N}$, then there exists an isomorphism of vector spaces $T : \ell_n^2 \to (Y, \|\cdot\|)$ that maximizes $|\det \cdot|$ on the set of all linear functions $S : \ell_n^2 \to (Y, \|\cdot\|)$ with $\|S\|_{op} \leq 1$. This function T satisfies $\|T\|_{op} = 1$ and $\rho_n(\ell_n^2, Y) \leq \|T^{-1}\|_{op} \leq \sqrt{n}$, where ρ_n denotes the multiplicative Banach-Mazur distance from Lemma 3.1.2.

Proof. One easily checks that for $n \in \{0, 1\}$ the theorem is true. Assume from now on $n \geq 2$. Since the set

$$B := \left\{S : \ell_n^2 \to (Y, \|\cdot\|) \mid S \text{ is an isomorphism of vector spaces and } \|S\|_{op} \leq 1\right\}$$

is a compact set and $\varphi : B \to \mathbb{R}$ defined by $\varphi(S) := |\det S|$ is a continuous function, there exists $T \in B$ where φ attains its maximum. By Lemma 3.2.13 we have $\|T\|_{op} = 1$.

Suppose $\|T^{-1}\|_{op} > \sqrt{n}$. There exists $z \in Y$ satisfying $\|z\| \leq 1$ such that $\sqrt{2n-1} > \|T^{-1}z\|_2 > \sqrt{n}$. There exists $R \in O(n)$ such that $R^{-1}T^{-1}z \in \mathbb{R}e_1$. Defining $S := TR$ we have $|\det S| = |\det T||\det R| = |\det T|$. Hence, $\|S\|_{op} = 1$ and there exists $\sqrt{2n-1} > \lambda > \sqrt{n}$ such that $S^{-1}z = \lambda e_1$.

We define $a := \lambda/\sqrt{n}$ and $b := \left(\sqrt{1 - 1/n}\right)/\left(\sqrt{1 - 1/\lambda^2}\right)$. Furthermore, let $U : \mathbb{R}^n \to X$ be the isomorphism of vector spaces defined by $Ue_1 := aSe_1$ and for all $k \in \{2, \ldots, n\}$ we set $Ue_k := bSe_k$. For $x \in B_n^2$ we have

$$\sum_{i=1}^{n} x_i^2 \leq 1 \Leftrightarrow 0 \leq \sum_{i=2}^{n} x_i^2 \leq 1 - x_1^2.$$

Furthermore, we obtain

$$\lambda^2 < 2n - 1 \Rightarrow x_1^2(\lambda^2 - n) \leq n - 1 \Leftrightarrow \lambda^2 x_1^2(\lambda^2 - n) \leq (n-1)\lambda^2$$
$$\Leftrightarrow \lambda^2 x_1^2 \frac{\lambda^2 - n}{n(\lambda^2 - 1)} + \frac{(n-1)\lambda^2}{n(\lambda^2 - 1)} \leq 1.$$

Thus, we have

$$\|Ux\|^2 \leq \|T\|_{op}^2 \left\| ax_1 e_1 + b\sum_{i=2}^{n} x_i e_i \right\|_2^2 = a^2 x_1^2 + b^2 \sum_{i=2}^{n} x_i^2 \leq a^2 x_1^2 + b^2(1 - x_1^2)$$
$$= \frac{\lambda^2}{n} x_1^2 + \frac{(n-1)\lambda^2}{n(\lambda^2 - 1)}(1 - x_1^2) = \lambda^2 x_1^2 \frac{\lambda^2 - n}{n(\lambda^2 - 1)} + \frac{(n-1)\lambda^2}{n(\lambda^2 - 1)} \leq 1.$$

This implies $\|U\|_{op} \leq 1$. By Lemma 2.2.1 we have $|\det U| = |\det T| ab^{n-1} > |\det T|$. This contradicts the maximality condition required for T. Hence, we have

$$\rho_n(\ell_2^n, Y) \leq \left\|T^{-1}\right\|_{op} \leq \sqrt{n}.$$

□

Lemma 3.3.3. Let $n \in \mathbb{N}$, the tuple $(X, \|\cdot\|)$ a normed vector space of dimension n and $\theta \in \mathbb{R}_+^*$. If E is a linear subspace of dimension $k \in \{0, \dots, n\}$ of X and N is a θ-net of $S(X) \cap E$ in $(E, \|\cdot\||_E)$, then for every $x \in S(X) \cap E$ there exists a sequence $(y_i)_{i\in\mathbb{N}}$ of elements of N and a sequence $(\delta_i)_{i\in\mathbb{N}}$ of real numbers such that for all $l \in \mathbb{N}$ the inequalities $|\delta_l| \leq \theta^l$ and

$$\left\| x - \sum_{i=0}^{l} \delta_i y_i \right\| < \theta^{l+1}$$

are satisfied.

Proof. For $n = 0$ or $k = 0$ the set $S(X) \cap E = \emptyset$ and the statement is trivially true. So assume $k, n > 0$ and let $x \in S(X) \cap E$ be given. We are going to construct the two sequences we are looking for recursively. First, we define $\delta_0 := 1$, which clearly satisfies $|\delta_0| \leq \theta^0 = 1$. Since N is a θ-net of $S(X) \cap E$ we find $y_0 \in \mathbb{N}$ such that $\theta^1 = \theta > \|x - y_0\| = \|x - \delta_0 y_0\|$. So the initial values of our recursion are set.

Let now for some $k \in \mathbb{N}$ the vectors $y_0, \dots, y_k \in N$ and the real numbers $\delta_0, \dots, \delta_k \in \mathbb{R}$ such that for all $l \in \{1, \dots, k\}$ the inequalities $|\delta_l| \leq \theta^l$ and

$$\left| x - \sum_{i=0}^{l} \delta_i y_i \right| < \theta^{l+1}.$$

are satisfied be given. We then define

$$\delta_{k+1} := \left| x - \sum_{i=0}^{k} \delta_i y_i \right| \quad \text{and} \quad z := \delta_{k+1}^{-1}\left(x - \sum_{i=0}^{k} \delta_i y_i \right).$$

We have $z \in E \cap S(X)$ and since N is a θ-net of this set, we can find $y_{k+1} \in N$ such that $|z - y_{k+1}| < \theta$. We obviously have $|\delta_{k+1}| \leq \theta^{k+1}$ and

$$\left| x - \sum_{i=0}^{k+1} \delta_i y_i \right| = |\delta_{k+1} z - \delta_{k+1} y_{k+1}| = |\delta_{k+1}||z - y_{k+1}| < \theta^{k+2}$$

□

Lemma 3.3.4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector space of dimension $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}_+^*$. Let $\theta \in]0,1[$ and E a linear subspace of dimension $k \in \{0,\dots,n\}$ of X. If $T : X \to Y$ is a homomorphism of vector spaces, N is a θ-net of $E \cap S(X)$ in $(E, \|\cdot\|_X|_E)$ and M is a real number such that for all $x \in N$ the inequality $|\|Tx\|_Y - M| \le \|T\|_{op}\varepsilon$ is staisfied then all $x \in E \cap S(X)$ satisfy

$$\frac{1-2\theta}{1-\theta}M - \frac{\|T\|_{op}\varepsilon}{1-\theta} \le \|Tx\|_Y \le \frac{M}{1-\theta} + \frac{\|T\|_{op}\varepsilon}{1-\theta}.$$

Proof. If $n = 0$ or $k = 0$ then $E \cap S(X) = \emptyset$ and the statement is trivially true. So assume $n, k > 0$ and let $x \in E \cap S(X)$ and $\theta \in]0,1[$ be given. By Lemma 3.3.3 we can find a sequence $(y_i)_{i\in\mathbb{N}}$ of vectors in N and a sequence $(\delta_i)_{i\in\mathbb{N}}$ of real numbers such that for all $l \in \mathbb{N}$ the inequalities $|\delta_l| \le \theta^l$ and

$$\left\| x - \sum_{i=0}^{l} \delta_i y_i \right\|_X < \theta^{l+1}$$

are satsified. Thus,

$$\begin{aligned}\|Tx\|_Y &\le \|T\|_{op}\left\| x - \sum_{i=0}^{l} \delta_i y_i \right\|_X + \left\| \sum_{i=0}^{l} \delta_i T y_i \right\|_Y \le \|T\|_{op}\theta^{l+1} + \sum_{i=0}^{l} |\delta_i| \|Ty_i\|_Y \\ &\le \|T\|_{op}\theta^{l+1} + \sum_{i=0}^{l} \theta^i \Big(M + \|T\|_{op}\varepsilon\Big) = \|T\|_{op}\theta^{l+1} + \frac{1-\theta^{l+1}}{1-\theta}\Big(M + \|T\|_{op}\varepsilon\Big).\end{aligned}$$

Since $\theta \in]0,1[$ we have

$$\|Tx\|_Y \le \lim_{l\to+\infty}\left(\|T\|_{op}\theta^{l+1} + \frac{1-\theta^{l+1}}{1-\theta}\Big(M + \|T\|_{op}\varepsilon\Big)\right) = \frac{1}{1-\theta}\Big(M + \|T\|_{op}\varepsilon\Big).$$

We find $z \in N$ such that $\|x - z\|_X < \theta$ and we define $u := \|x - z\|_X^{-1}(x - z) \in S(X) \cap E$. By the first part of the proof, we have $\|Tu\|_Y \le (1-\theta)^{-1}\Big(M + \|T\|_{op}\varepsilon\Big)$. Thus,

$$\|T(x - z)\|_Y = \|x - z\|_X \|Tu\|_Y \le \frac{\|x - z\|_X}{1-\theta}\Big(M + \|T\|_{op}\varepsilon\Big) \le \frac{\theta}{1-\theta}\Big(M + \|T\|_{op}\varepsilon\Big).$$

We conclude

$$\begin{aligned}\|Tx\|_Y &\ge \|Tz\|_Y - \|T(x - z)\|_Y \ge \Big(M - \|T\|_{op}\varepsilon\Big) - \frac{\theta}{1-\theta}\Big(M + \|T\|_{op}\varepsilon\Big) \\ &= \frac{(1-2\theta)M}{1-\theta} - \frac{\|T\|_{op}\varepsilon}{1-\theta}.\end{aligned}$$

□

Theorem 3.3.5. For every $\delta \in]0,1[$ there exists $c > 0$ and $l \in \mathbb{N}$ such that for every $n \in \mathbb{N}^*$ with $n \ge l$, every $(X, \|\cdot\|) \in Q(n)$ and every isomorphism of vector spaces $T : \ell_n^2 \to X$ there exists a linear subspace E of X with $\dim E \ge cnM^2\|T\|_{op}^{-2}$ such that $\rho_k\big(\ell_k^2, E\big) \le \frac{1+\delta}{1-\delta}$. Here M denotes the median of the function $r : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by $r(x) := \|Tx\|$.

Proof. Let $\delta \in]0,1[$ be given and define $\theta := 2^{-1}\delta(2+\delta)^{-1}$. This positive real number θ satisfies $\theta \in]0,1[$ as well as

$$\theta < \frac{\delta}{2+\delta} \Leftrightarrow \theta(2+\delta) < \delta \Leftrightarrow (\theta-1)(2+\delta) < \delta - (2+\delta)$$
$$\Leftrightarrow 2 < (1-\theta)(2+\delta) \Leftrightarrow \frac{1}{1-\theta} < 1 + \frac{\delta}{2}$$

and

$$\theta < \frac{\delta}{2+\delta} \Leftrightarrow \theta(2+\delta) < \delta \Leftrightarrow \theta(4-2+\delta) < 2-(2-\delta) \Leftrightarrow (2-\delta) - \theta(2-\delta) < 2 - 4\theta$$
$$\Leftrightarrow (2-\delta)(1-\theta) < 2(1-2\theta) \Leftrightarrow 1 - \frac{\delta}{2} < \frac{1-2\theta}{1-\theta}.$$

We define

$$\varepsilon := \frac{\delta}{4}(1-\theta).$$

By Theorem 3.2.12 there exists $c_1 > 0$ and $l \in \mathbb{N}$ such that for all $f \in C(\mathbb{S}^n, \mathbb{R})$ with median M there exists $k \in \mathbb{N}$, a k-dimensional linear subspace E of $\mathbb{R}^n$ and a θ-net N of $E \cap \mathbb{S}^{n-1}$ in $(E, \|\cdot\|_2|_E)$ such that

$$k \geq c_1 \frac{n\varepsilon^2}{4(\log(3) - \log(\theta))}$$

and for all $x \in N$ the inequality $|f(x) - M| \leq \omega_f(\varepsilon)$ holds true. We define

$$c := c_1 \frac{\varepsilon^2}{4(\log(3) - \log(\theta))}$$

Let $n \geq l$ and $(X, \|\cdot\|) \in Q(n)$. For $T : \ell_n^2 \to (X, \|\cdot\|)$ we have $M/\|T\|_{op} \leq 1$ and

$$k \geq c_1 \frac{n\varepsilon^2}{4(\log(3) - \log(\theta))} \geq c \frac{nM^2}{\|T\|_{op}^2}.$$

Furthermore, $|r(x) - M| \leq \omega_r(\varepsilon) \leq \|T\|_{op}\varepsilon$. By Lemma 3.3.4 all $x \in E \cap \mathbb{S}^{n-1}$ satisfy

$$\frac{1-2\theta}{1-\theta}M - \frac{\|T\|_{op}\varepsilon}{1-\theta} \leq \|Tx\| \leq \frac{M}{1-\theta} + \frac{\|T\|_{op}\varepsilon}{1-\theta}.$$

Thus, we have for $x \in E \cap \mathbb{S}^{n-1}$

$$\|Tx\| \leq \frac{M}{1-\theta} + \frac{\|T\|_{op}\varepsilon}{1-\theta} = M\left(1 + \frac{\delta}{2} + \frac{\varepsilon}{1-\theta}\right) \leq M(1+\delta).$$

This implies $\|T|_E\|_{op} \leq (1+\delta)M$. Furthermore, we have

$$(1-\delta)M \leq M\left(1 - \frac{\delta}{2} - \frac{\varepsilon}{1-\theta}\right) \leq \frac{1-2\theta}{1-\theta}M - \frac{\|T\|_{op}\varepsilon}{1-\theta} \leq \|Tx\|$$

For every $y \in S(Y)$ satisfying $T^{-1}y \in E$ we have

$$\left\|T^{-1}y\right\|_2 \le \left\|T^{-1}y\right\|_2 \frac{1}{M(1-\delta)} \left\|T\left(\left\|T^{-1}y\right\|_2^{-1} T^{-1}y\right)\right\| = \frac{1}{M(1-\delta)}\|y\|.$$

Hence, $\left\|T^{-1}|_{TE}\right\|_{op} \le (M(1-\delta))^{-1}$. Finally,

$$\rho_k\left(\ell_k^2, TE\right) \le \rho_k\left(\ell_k^2, E\right)\rho_k(E, TE) \le \|T|_E\|_{op}\left\|(T|_E)^{-1}\right\|_{op} \le \frac{1+\delta}{1-\delta}$$

□

Lemma 3.3.6. For all $n \in \mathbb{N}^*$, all $(X, \|\cdot\|) \in Q(n)$ and all isomorphisms of vector spaces $T : \mathbb{R}^n \to X$ satisfying $1 \le \|T\|_{op} \le \sqrt{n}$ the following holds true. If M is the median of the function $r : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by $r(x) := \|Tx\|$ and A is the expected value of r, then the inequality $|M - A| < 4\sqrt{\pi}$ is satisfied.

Proof. For $u \in \mathbb{R}_+^*$ we have

$$0 < \|T\|_{op} \le \sqrt{n} \Rightarrow \|T\|_{op}^2 \le n \Leftrightarrow -n\|T\|_{op}^{-2} \le -1 \Leftrightarrow -\frac{u^2 n}{4}\|T\|_{op}^{-2} \le -\frac{u^2}{4}$$

Applying Corollary 3.2.4 we obtain for $t = u\|T\|_{op}^{-1}$

$$\begin{aligned}\mathcal{H}^{n-1}\left(\left\{x \in S^{n-1} \mid |r(x) - M| > u\|T\|_{op}^{-1}\right\}\right) &\le 4\exp\left(-\frac{u^2 n}{4}\|T\|_{op}^{-2}\right) \\ &\le 4\exp\left(-\frac{u^2}{4}\right) = 4\exp\left(-\frac{t^2\|T\|_{op}^2}{4}\right) \le 4\exp\left(-\frac{t^2}{4}\right)\end{aligned}$$

From the definition of the expected value and since $\mathcal{H}^{n-1}\left(\mathbb{S}^{n-1}\right) = 1$ we have

$$\begin{aligned}|A - M| &\le \int_{\mathbb{S}^{n-1}} |r(x) - M|\mathcal{H}^{n-1}(x) = \int_0^{+\infty} \mathcal{H}^{n-1}\left(\left\{x \in \mathbb{S}^{n-1} \mid |r(x) - M| > t\right\}\right)dt \\ &\le 4\int_0^{+\infty} \exp\left(-\frac{t^2}{4}\right)dt = 4\sqrt{\pi}\end{aligned}$$

□

Lemma 3.3.7. For all $n \in \mathbb{N}^*$, all $(X, \|\cdot\|) \in Q(n)$ and all isomorphisms of vector spaces $T : \ell_n^2 \to (X, \|\cdot\|)$ satisfying $\|T\|_{op}\left\|T^{-1}\right\|_{op} \le \sqrt{n}$ the following holds true. If M is the median of the function $r : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by $r(x) := \|Tx\|$ and A is the expected value of r, then $2^{-1} \le AM^{-1} \le 1 + 4\sqrt{\pi}$ is satisfied.

Proof. Let us first treat the case $\|T\|_{op} \le \sqrt{n}$ and $\left\|T^{-1}\right\|_{op} \le 1$. For all $x \in \mathbb{S}^{n-1}$ the inequality $\left\|T^{-1}\right\|_{op}^{-1}\|x\|_2 \le r(x)$ is satisfied. Thus,

$$\mathcal{H}^{n-1}\left(\left\{x \in \mathbb{S}^{n-1} \mid \left\|T^{-1}\right\|_{op}^{-1} \le r(x)\right\}\right) = \mathcal{H}^{n-1}\left(\mathbb{S}^{n-1}\right) = 1 > \frac{1}{2}$$

and consequently $M \geq \left\|T^{-1}\right\|_{op}^{-1} \geq 1$. By Lemma 3.3.6 we have $|A - M| \leq 4\sqrt{\pi}$. Since $r \geq 0$ we obtain

$$|A| = A \leq M + 4\sqrt{\pi} \leq \left(1 + 4\sqrt{\pi}\right) M$$

By Markov's inequality we also have

$$\frac{1}{2} = \mathcal{H}^{n-1}(\{r \geq M\}) \leq \frac{A}{M}.$$

In case $\|T\|_{op} > \sqrt{n}$ or $\left\|T^{-1}\right\|_{op} > 1$ we define $S : \mathbb{R}^n \to X$ by $S := \sqrt{n}\|T\|_{op}^{-1} T$. We have $\|S\|_{op} = \sqrt{n}$ and

$$\left\|S^{-1}\right\|_{op} = \frac{\|T\|_{op}}{\sqrt{n}} \left\|T^{-1}\right\|_{op} \leq 1.$$

We define $\tilde{r} : \mathbb{S}^{n-1} \to \mathbb{R}$ by $r(x) := \|Sx\|$. If $\tilde{A}$ is the expected value of $\tilde{r}$ then $\tilde{A} = \sqrt{n}\|T\|_{op}^{-1}$. Furthermore, $\tilde{M} := \sqrt{n}\|T\|_{op}^{-1}$ is the median for S. We find ourselves in the situation of the first case and obtain

$$\frac{1}{2} = \frac{\tilde{A}}{\tilde{M}} = \frac{A}{M} \leq 1 + 4\sqrt{\pi}.$$

□

Theorem 3.3.8 (Dvoretzky)**.** For every $\varepsilon > 0$ there exists $c > 0$ and $l \in \mathbb{N}^*$ such that for all $n \in \mathbb{N}$ with $n \geq l$ and every normed space $(X, \|\cdot\|)$ of dimension n there exists $k \geq c \log(n)$ and a subspace E of X of dimension k such that $\rho_k\left(\ell_k^2, E\right) \leq 1 + \varepsilon$, where ρ_k is the multiplicative Banach-Mazur distance.

Proof. Let $\varepsilon > 0$ and define $\delta := 2^{-1}(\varepsilon/(2 + \varepsilon))$. We have $\delta \in]0, 1[$ and

$$\delta < \frac{\varepsilon}{2 + \varepsilon} \Leftrightarrow \delta(2 + \varepsilon) < \varepsilon \Leftrightarrow 1 + \delta < (1 + \varepsilon)(1 - \delta) \Leftrightarrow \frac{1 + \delta}{1 - \delta} < 1 + \varepsilon$$

By Theorem 3.3.5 there exists a constant $c_1 > 0$ and $l \in \mathbb{N}$ such that for every $n \geq l$, every normed space $(X, \|\cdot\|) \in Q(n)$ and every isomorphism $T : \ell_n^2 \to (X, \|\cdot\|)$ there exists a linear subspace E of X satisfying $\dim E \geq c_1 n (M/\|T\|_{op})^2$ such that $\rho_k(\ell_k^2, E) \leq \frac{1+\delta}{1-\delta}$. Here M denotes the median of the function $r : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by $r(x) := \|Tx\|$. Let $c_2 > 0$ be such that for all $n \geq l$ we have

$$\left(1 + 4\sqrt{\pi}\right) c_2 \sqrt{\frac{\log(n)}{n}} \leq \frac{1}{8} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\log\left(\lfloor n/2 \rfloor\right)}{n}}.$$

Finally, define $c := c_1 c_2^2$.

For a given normed space $(X, \|\cdot\|)$ we have by Theorem 3.3.2 that $\rho_n\left(\ell_n^2, X\right) \leq \sqrt{n}$. Thus, there exists an isomorphism of vector spaces $T : \mathbb{R}^n \to X$ maximizing $|\det \cdot|$ on the set of all homomorphisms of vector spaces $S : \ell_n^2 \to (X, \|\cdot\|)$ satisfying $\|S\|_{op} \leq 1$ and such that $\|T\|_{op} = 1$ and $\left\|T^{-1}\right\|_{op} \leq \sqrt{n}$. By Theorem 3.3.1 there exists an orthonormal basis $x_1, \dots, x_n$ of ℓ_n^2 such that for all $i \in \{1, \dots, n\}$ we have

$$2^{-n/(n-(i-1))} \leq \|Tx_i\| \leq 1.$$

In particular, we have for $1 \leq i \leq \lfloor n/2 \rfloor$

$$1 \geq \|Tx_i\| \geq 2^{-n/(n-\lfloor n/2 \rfloor+1)} \geq 2^{-n/(n/2)} = 2^{-2} = \frac{1}{4}$$

We define $r : \mathbb{S}^{n-1} \to \mathbb{R}$ by $r(x) := \|Tx\|$ and denote by A the expected value of r and by M the functions median. By Fubini's Theorem, with the definition (2.2) and the inequality (2.4) and by Lemma 2.2.6 we obtain

$$\begin{aligned}
A &= \int_0^1 A dt = \int_0^1 \int_{\mathbb{S}^{n-1}} r(x) \mathcal{H}^{n-1}(x) dt = \int_0^1 \int_{\mathbb{S}^{n-1}} \|Tx\| \mathcal{H}^{n-1}(x) dt \\
&= \int_0^1 \int_{\mathbb{S}^{n-1}} \left\| \sum_{i=1}^n a_i T x_i \right\| \mathcal{H}^{n-1}(a) dt = \int_0^1 \int_{\mathbb{S}^{n-1}} \left\| \sum_{i=1}^n r_i(t) a_i T x_i \right\| \mathcal{H}^{n-1}(a) dt \\
&= \int_{\mathbb{S}^{n-1}} \int_0^1 \left\| \sum_{i=1}^n r_i(t) a_i T x_i \right\| dt \mathcal{H}^{n-1}(a) \geq \int_{\mathbb{S}^{n-1}} \max\left\{\|a_i T x_i\| : 1 \leq i \leq n\right\} \mathcal{H}^{n-1}(a) \\
&\geq \int_{\mathbb{S}^{n-1}} \max\left\{|a_i| \|Tx_i\| : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\right\} \mathcal{H}^{n-1}(a) \\
&\geq \frac{1}{4} \int_{\mathbb{S}^{n-1}} \max\left\{|a_i| : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\right\} \mathcal{H}^{n-1}(a) \geq \frac{1}{4} \frac{1}{2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\log(\lfloor n/2 \rfloor)}{n}} \\
&\geq \left(1 + 4\sqrt{\pi}\right) c_2 \sqrt{\frac{\log(n)}{n}}
\end{aligned}$$

By Lemma 3.3.7 we have

$$M \geq \frac{A}{1 + 4\sqrt{\pi}} \geq \frac{1}{1 + 4\sqrt{\pi}} \left(1 + 4\sqrt{\pi}\right) c_2 \sqrt{\frac{\log(n)}{n}} = c_2 \sqrt{\frac{\log(n)}{n}}$$

By the choice of c_1 and l according to Lemma 3.3.5 we find for every $n \geq l$ a subspace E of X such that

$$\dim E \geq c_1 n \left(\frac{M}{\|T\|_{op}} \right)^2 \geq c_1 n c_2^2 \frac{\log(n)}{n} = c \log(n).$$

and $\rho_k\left(\ell_k^2, E\right) \leq \frac{1+\delta}{1-\delta} \leq 1 + \varepsilon$. □

Finally we can state a more geometric version of Dvoretzky's Theorem, which can also be found in [Sch13, p.2].

Corollary 3.3.9. For every $\varepsilon > 0$ there exists $c > 0$ and $l \in \mathbb{N}^*$ such that for all $n \in \mathbb{N}$ satisfying $n \geq l$ and every convex, symmetric body P of $\mathbb{R}^n$ there exists $k \geq c \log(n)$, a subsapce V of $\mathbb{R}^n$ of dimension k and $\alpha > 0$ such that

$$\alpha\left(V \cap B_n^2\right) \subseteq V \cap P \subseteq (1 + \varepsilon)\alpha\left(V \cap B_n^2\right).$$

Proof. Let $\varepsilon > 0$. By Theorem 3.3.8 there exists a constant $\tilde{c} > 0$ and $l \in \mathbb{N}^*$ such that for all $n \in \mathbb{N}$ with $n \geq l$ and every normed space $(X, \|\cdot\|)$ of dimension n there exists $k \geq \tilde{c}\log(n)$ and a subspace E of X of dimension k such that $\rho_k\big(E, \ell_k^2\big) \leq 1+\varepsilon$, where ρ_k denotes the multiplicative Banach-Mazur distance. We define $c := \tilde{c}/2$.

Let $n \geq l$ and P a convex, symmetric and compact subset of $\mathbb{R}^n$ with non-empty interior. By Lemma 2.3.1 there exists a unique norm $\|\cdot\|$ on $\mathbb{R}^n$. such that $P = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. By the choice of l and $\tilde{c}$ according to Theorem 3.3.8 there exists $k \geq \tilde{c}\log(n)$ and a subspace E of X of dimension k such that $\rho\big(\ell_k^2, (E, \|\cdot\||_E)\big) \leq 1+\varepsilon$. Hence, there exists an isomorphism of vector spaces $T : \ell_k^2 \to (E, \|\cdot\||_E)$ such that $\|T\|_{op} = 1$ and $\big\|T^{-1}\big\|_{op} = 1+\varepsilon$. By Lemma 2.3.4 there exists a subspace V of E and a constant $\alpha > 0$ such that $\dim V \geq k/2 \geq c\log(n)$ and $V \cap T\big(B_k^2\big) = \alpha\big(V \cap B_n^2\big)$. For every $y \in P$ we have $\big\|T^{-1}y\big\|_2 \leq \big\|T^{-1}\big\|_{op}\|y\| \leq (1+\varepsilon)\|y\| = 1+\varepsilon$. Hence, $y \in V \cap T\big((1+\varepsilon)B_k^2\big) = (1+\varepsilon)\alpha\big(V \cap B_n^2\big)$. Thus,

$$\alpha\big(V \cap B_n^2\big) = V \cap T\big(B_k^2\big) \subseteq V \cap P \subseteq (1+\varepsilon)\alpha\big(V \cap B_n^2\big).$$

□

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