Semi-discretization of a scalar conservation law with stochastic forcing

Fabian Kleon Zehetgruber

under the supervision of M. Julien Reygner

1. Introduction

1.1. **What is this paper about?** This document is a presentation of the authors work during his internship at CERMICS. The central object studied during the internship was the stochastic partial differential equation (SPDE)

$$\mathrm{d}u_t + \operatorname{div}\left(A(u_t)\right)\mathrm{d}t = \operatorname{div}\left(B(u_t)\nabla u_t\right)\mathrm{d}t + \Phi(u_t)\mathrm{d}W_t \tag{1}$$

on the d-dimensional torus $\mathbb{T}^d$ for some $d \in \mathbb{N}$, where we define the **natural numbers** as $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We assume that W is an r-dimensional **Brownian motion** on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ for some $r \in \mathbb{N}$ and that $\Phi : L^2\left(\mathbb{T}^d\right) \to \mathfrak{S}\left(\mathbb{R}^r, L^2\left(\mathbb{T}^d\right)\right)$, where $\mathfrak{S}(\cdot)$ denotes the set of all **Hilbert-Schmidt operators**. It is unnecessary to work with the Hilbert-Schmidt operators $\mathfrak{S}$ when working on the finite-dimensional space $\mathbb{R}^r$, but for extensions to infinite dimensions it becomes necessary. We refer the reader to [3] for an introduction to the infinite dimensional setting. If we do not specify any filtration $(\mathfrak{F}_t)_{t\in[0,\infty[}$ together with the Brownian motion W we always consider the **filtration generated by the process** W denoted by $\left(\mathfrak{F}_t^W\right)_{t\in[0,\infty[}$. For a definition of this filtration see for example [10, p. 3].

Assumption 1.1. *We denote by $(e_k)_{k=1}^r$ the **canonical basis** of $\mathbb{R}^r$ and assume that there exists a family of functions $(\sigma_k)_{k=1}^r$ in $C\left(\mathbb{T}^d \times \mathbb{R}\right)$ such that for each $k \in \{1, \dots, r\}$, each $u \in L^2\left(\mathbb{T}^d\right)$ and each $x \in \mathbb{T}^d$ the equality $\left(\Phi\left(u\right)e_k\right)(x) = \sigma_k\left(x, u(x)\right)$ holds true. Furthermore, we suppose that there exists a constant $C_1 \in \mathbb{R}$ such that for all $x, y \in \mathbb{T}^d$ and all $u, v \in \mathbb{R}$ the inequalities*

$$\sum_{k=1}^{r} \left|\sigma_k\left(x, u\right)\right|^2 \leq C_1 \left(1 + |u|^2\right)$$

and

$$\sum_{k=1}^{r} \left|\sigma_k\left(y, v\right) - \sigma_k\left(x, u\right)\right|^2 \leq C_1 \left(|y - x|^2 + |v - u|^2\right)$$

are satisfied.

For the next assumption we first need a definition.

Definition 1.1. *Given two metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called **locally Lipschitz continuous** if for every $x \in X$ there exists a neighborhood U of X and a constant $C_x \in \mathbb{R}$ such that for every $z_1, z_2 \in U$ the inequality $d_Y\left(f\left(z_2\right), f\left(z_1\right)\right) \leq C_x d_X\left(z_2, z_1\right)$ is satisfied.*

Throughout the paper we denote $a := A'$.

Assumption 1.2. *We assume $A \in C^2\left(\mathbb{R}, \mathbb{R}^d\right)$ and the existence of a constant $C_2 \in \mathbb{R}$ and a natural number $p \in \mathbb{N}$ such that for all $u \in \mathbb{R}$ the inequality*

$$\left\|a\left(u\right)\right\|_{\mathbb{R}^d} \leq C_2\left(1 + |u|^p\right) \tag{2}$$

holds. Furthermore, we assume that A'' is locally Lipschitz continuous and $A(0) = 0$.

Assumption 1.3. *We assume that $B : \mathbb{R} \to \mathbb{R}^{d\times d}$ is a measurable function and that for every $u \in \mathbb{R}$ the matrix $B(u)$ is symmetric and positive definite. Moreover, we suppose that there exists $\gamma \in]1/2, +\infty[$ and a constant $C_3 \in \mathbb{R}$ such that for all $u, v \in \mathbb{R}$ that satisfy $|v-u| < 1$ the inequality $\left|\sqrt{B}(v) - \sqrt{B}(u)\right| \le C_3 |v-u|^{\gamma}$ holds.*

Under Assumptions 1.1, 1.2 and 1.3 existence and uniqueness of a so called kinetic solution for each initial condition u_0 to (1) was shown in [4].

Our paper comprises a derivation of a possible semi-discrete version of (1), where the d-dimensional torus $\mathbb{T}^d$ is discretized with an appropriate mesh $\mathcal{T}$, but the time t is still continuous. Let us think of the mesh $\mathcal{T}$ as a finite partition of $[0, 1[^d$ consisting of polytopes. We refer to [6] for more details on such meshes. This **semi-discrete problem** associated with (1) we derive is a stochastic differential equation (sde) of the form

$$\mathrm{d}U_t = b(U_t)\mathrm{d}t + \sigma(U_t)\,\mathrm{d}W_t, \tag{3}$$

where the drift vector b and the dispersion matrix σ do not explicitly depend on time. Given $X \in \mathbb{R}^{\mathcal{T}}$ and an r-dimensional Brownian motion W it had already been shown in [11] that under certain conditions (3) admits a unique **strong solution** U^X with initial condition X.

The existence of a unique strong solution allows us to define a **family of Markovian transition functions** $(P_t)_{t\in[0,+\infty[}$ on $\mathbb{R}^{\mathcal{T}}_0$. What is the space $\mathbb{R}^{\mathcal{T}}_0$? To give its definition we start by endowing the space $\mathbb{R}^{\mathcal{T}}$ with the scalar product

$$\langle U, V\rangle_{\mathbb{R}^{\mathcal{T}}} := \sum_{K\in\mathcal{T}} \lambda^d(K)\, U^K V^K.$$

We denote by λ^d the d-dimensional **Lebesgue measure** and by $\mathbf{1} \in \mathbb{R}^{\mathcal{T}}$ the vector that has all of its entries equal to one. We define

$$\mathbb{R}^{\mathcal{T}}_0 := \left\{U \in \mathbb{R}^{\mathcal{T}} \mid \langle U, \mathbf{1}\rangle_{\mathbb{R}^{\mathcal{T}}} = 0\right\}. \tag{4}$$

We immediately recognize $\mathbb{R}^{\mathcal{T}}_0$ as a $(|\mathcal{T}|-1)$-dimensional subvectorspace of $\mathbb{R}^{\mathcal{T}}$. It is the hyperplane orthogonal to $\mathbf{1}$. We will see in Proposition 3.1 sufficient conditions for which a solution to (3) with initial condition in $\mathbb{R}^{\mathcal{T}}_0$ stays in this space for all times t. Under these conditions it was shown in [11] that this family $(P_t)_{t\in[0,+\infty[}$ admits an invariant measure $\mu_{\mathcal{T}}$ on $\mathbb{R}^{\mathcal{T}}_0$.

In addition to $\mathbb{R}^{\mathcal{T}}_0$ we define for each $1 \le p \le +\infty$ the space

$$L^p_0\left(\mathbb{T}^d\right) := \left\{v \in L^p\left(\mathbb{T}^d\right) \mid \int_{\mathbb{T}^d} v(x)\lambda^d(\mathrm{d}x) = 0\right\}.$$

The main question during the authors internship was whether there exists a probability measure μ on $L^1_0\left(\mathbb{T}^d\right)$ that is the limit of the measures $\mu_{\mathcal{T}}$ when the mesh $\mathcal{T}$ is getting fine. If such a probability measure μ does exist one could ask the question whether this is an invariant measure associated with the spde (1).

To tackle these questions we tried to use techniques similar to the ones used in [5]. The internship is ending and we have not been able to find solutions to the questions we just presented. This paper contains a presentation of the work we have done during the internship. It also shows the difficulties we encountered and presents some specific questions that remain open.

In section 2 we present preliminary results that we will use later in the text. In section 3 we start from the initial spde (1) and derive the semi-discrete problem. Furthermore, we discuss the existence and uniqueness of a solution to the semi-discrete problem as well as existence and uniqueness of an invariant measure associated with the semi-discrete problem. In section 4 we introduce a so called kinetic formulation of the semi-discrete problem. Starting from this kinetic formulation we discuss the core questions of the internship in section 5.

1.2. **What is the purpose of this internship report?** The author does not dare to name a purpose of this work. We could say that there is no purpose. We do not know whether the statements made in this internship report are true or false. This is true irrespective of the question whether logical errors have or have not been made in this internship report. Logic namely is but a set of tools to deduce one statement from another. The deduced statement is only true if the statement one started from in the first place was true. This report is based on the axioms commonly used in modern mathematics. If they are true then what is written here is true as well (except logical errors have happened in the deduction of these statements). Yet who can tell whether the axioms we started from are true? Let us not mistake a logically correct argumentation for discovery of truth.

2. Preliminaries

In this section we will state some preliminary results that we will need later on. These results mainly concern stochastic differential equations, Markov processes, the discretization of the torus and harmonic analysis.

2.1. **Discretization of the d-dimensional torus.** In order to discretize the d-dimensional torus we think of it as the hypercube $[0,1[^d$, but we always have to keep in mind that opposite **facets** are identified with each other. We will not give a precise definition of a general mesh $\mathcal{T}$ on $\mathbb{T}^d$, but the reader should think of it as an appropriate finite partition of $[0,1[^d$ consisting of polytopes as already mentioned in section 1. The polytopes touching a facet of the hypercube have to match appropriately the ones on the opposite facet. We also do not define precisely what it means for a general mesh to become finer. For a sufficiently nice mesh $\mathcal{T}$ we can think of it as the shrinking of the maximal diameter among all diameters of the polytopes in the mesh. For more details on the mesh we refer to [6].

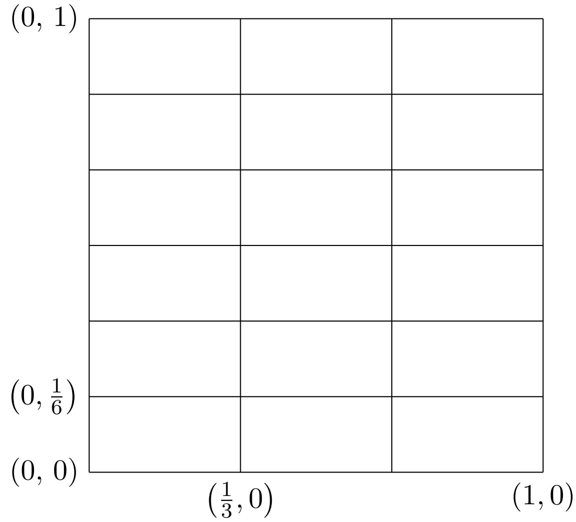

FIGURE 1. The regular mesh of rectangular cuboids in $\mathbb{T}^2$ corresponding to $M = \mathrm{diag}\,(3, 6)$.

As already mentioned in the introduction we will obtain an invariant measures $\mu_{\mathcal{T}}$ on $\mathbb{R}_0^{\mathcal{T}}$ for every mesh $\mathcal{T}$ and we want to know whether these measures converge to a probability measure μ on $L_0^1\left(\mathbb{T}^d\right)$. What do we mean by convergence of the measures $\mu_{\mathcal{T}}$ that are all defined on a different space $\mathbb{R}_0^{\mathcal{T}}$? In fact, we can associate with $U \in \mathbb{R}^{\mathcal{T}}$ the function $u \in L^1\left(\mathbb{T}^d\right)$ given by

$$u\left(x\right) := \sum_{K \in \mathcal{T}} U^K \chi_K\left(x\right),$$

where χ_K denotes the **indicator function** of K. If $U \in \mathbb{R}_0^{\mathcal{T}}$ then it is clear that the associated u is an element of $L_0^1\left(\mathbb{T}^d\right)$. We can regard $\mu_{\mathcal{T}}$, by abuse of notation, as the push-forward of this embedding which makes it a measure on $L_0^1\left(\mathbb{T}^d\right)$. Thus, when we talk about convergence of the measures $\mu_{\mathcal{T}}$ we may look at them as measures on $L_0^1\left(\mathbb{T}^d\right)$.

While we leave the imagination of a general mesh to the reader we want to give a precise definition of a particular regular mesh consisting of rectangular cuboids. We characterize such a mesh by a diagonal matrix $M \in \mathbb{N}_0^{d\times d}$ with strictly positive entries on the diagonal. The vertices of this particular mesh are given by $M^{-1}\mathbb{Z}^d \cap [0, 1[^d$ which can be identified with $\mathbb{T}_M^d := M^{-1}\mathbb{Z}^d/\mathbb{Z}^d$. The facets of the mesh are simply given by all hyper-planes that are orthogonal to one of the base vectors $(e_j)_{j=1}^d$ and that intersect a vertex. This way we obtain, as stated above, a regular mesh of rectangular cuboids. An example of such a regular mesh is shown in Figure 1. We will often abuse our notation by writing $x \in \mathbb{T}_M^d$ when we actually think of the corresponding $x \in M^{-1}\mathbb{Z}^d \cap [0, 1[^d$.

Remark 2.1. *Some readers might wonder why we are working with diagonal matrices when we could just as well work with vectors containing the entries on the diagonal of the matrix. One advantage of a matrix M is that we can write its inverse M^{-1}, while this is not possible for a vector without an additional definition. Furthermore, the author of this document believes that there might be some generalizations to other matrices, for example symmetric positive ones. A notation in matrix form might facilitate such a generalization.*

It remains to define what it means for the mesh to become finer. We endow the set of all diagonal $\mathbb{R}^{d\times d}$-matrices with a relation $B \preceq C$ if for all $j \in \{1,\dots,d\}$ the equation $B_{jj} \le C_{jj}$ is satisfied, making it a directed set. The mesh $\mathbb{T}^d_M$ is finer than the mesh $\mathbb{T}^d_L$ if $M^{-1} \preceq L^{-1}$, or equivalently, $L \preceq M$. The mesh becomes "infinitely fine" if $M \to +\infty$ or $M^{-1} \to 0$, where at least the first "limit" should not be understood in a topological sense but must be understood in the framework of the directed set. Given a net (a_M) in a topological space we can define what convergence $\lim_{M\to+\infty} a_M$ means. The notation $M \to +\infty$ must be understood in this sense.

2.2. **Stochastic differential equations.** The main reference we use for this subsection is [10]. Let $d, r \in \mathbb{N}$ and for each $1 \le j \le d$ and $1 \le k \le r$ let $b_j, \sigma_{Jk} : [0,+\infty[\times\mathbb{R}^{\mathcal{T}} \to \mathbb{R}$ be Borel-measurable functions. We call $b = (b_J)_{J\in\mathcal{T}}$ the **drift vector** and $\sigma = (\sigma_{Jk})^r_{J\in\mathcal{T},k=1}$ the **dispersion matrix**. We are going to state some results concerning the stochastic differential equation

$$\mathrm{d}U_t = b(t,U_t)\mathrm{d}t + \sigma(t,U_t)\,\mathrm{d}W_t, \tag{5}$$

where W is an r-dimensional Brownian motion. For this endeavor we introduce a probability measure μ on $\mathbb{R}^d$ and follow [10, p. 285] to develop the concept of a strong solution of (5). We denote by $\mathfrak{s}(\mathfrak{D})$ the σ-algebra generated by a set of sets $\mathfrak{D}$.

Definition 2.1. *A **filtration** on a measurable space $(\Omega,\mathfrak{F})$ is a family of sub-σ-fields $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ of $\mathfrak{F}$ such that for each $0 \le s \le t < +\infty$ the inclusions $\mathfrak{F}_s \subseteq \mathfrak{F}_t \subseteq \mathfrak{F}$ are satisfied. For such a filtration we define*

$$\mathfrak{F}_\infty := \mathfrak{s}\left(\bigcup_{t\in[0,+\infty[} \mathfrak{F}_t\right).$$

We choose a probability space $(\Omega,\mathfrak{F},\mathbb{P})$ as well as an r-dimensional Brownian motion $W = \left(W_t, \mathfrak{F}^W_t; 0 \le t < +\infty\right)$ on it. We assume also that this space is rich enough to accommodate a random vector X taking values in $\mathbb{R}^{\mathcal{T}}$, independent of $\mathfrak{F}^W_\infty$, and with given distribution μ. For every $0 \le t < +\infty$ we consider the left-continuous filtration $\mathfrak{G}_t := \mathfrak{s}(X, W_s; 0 \le s \le t)$ as well as the collection of null-sets

$$\mathcal{N} := \{N \subseteq \Omega \mid \exists G \in \mathfrak{G}_\infty \text{ with } N \subseteq G \text{ and } \mathbb{P}(G) = 0\},$$

and create the **augmented filtration** given for every $0 \leq t < +\infty$ by

$$\mathfrak{F}_t := \mathfrak{s}\left(\mathfrak{G}_t \cup \mathcal{N}\right). \tag{6}$$

Definition 2.2. *A strong solution of the stochastic differential equation* (5) *on a given probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$ *and with respect to the fixed Brownian motion* W *and initial condition* X*, is a process* U *with continuous sample paths and with the following properties:*

1. U *is adapted to the augmented filtration* $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ *given by* (6)*,*
2. $\mathbb{P}(U_0 = X) = 1$*,*
3. *for every* $J \in \mathcal{T}$*,* $1 \leq k \leq r$ *and* $0 \leq t < +\infty$ *the equation*
$$\mathbb{P}\left(\int_0^t \left(|b_J(s, U_s)| + |\sigma_{Jk}(s, U_s)|^2\right) \mathrm{d}s < +\infty\right) = 1$$
is satisfied and
4. *the equation*
$$U_t = U_0 + \int_0^t b(s, U_s)\,\mathrm{d}s + \int_0^t \sigma(s, U_s)\,\mathrm{d}W_s; \quad 0 \leq t < +\infty$$
holds almost surely.

Remark 2.2. *The important case for us will be a deterministic initial condition* $X \in \mathbb{R}^{\mathcal{T}}$*. By this we mean* $\mu(\{X\}) = 1$*. This is a very special case and we do not need to demand explicitly from the probability space that it is rich enough to accommodate the random vector* X*, since* $\tilde{X} : \Omega \to \mathbb{R}^{\mathcal{T}}$ *defined by* $\tilde{X}(\omega) := X$ *for all* $\omega \in \Omega$ *can be defined for every* $\Omega \neq \emptyset$*. Thus, given a Brownian motion on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$ *we can talk about a strong solution with initial condition* $X \in \mathbb{R}^{\mathcal{T}}$ *without any additional condition.*

Furthermore, $\sigma(X) = \{\emptyset, \Omega\}$ *is a particularly simple* σ*-algebra and we have* $\mathfrak{G}_t = \mathfrak{F}_t^W$*. Consequently, the augmented filtration* $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ *is no more than* $\left(\mathfrak{F}_t^W\right)_{t\in[0,+\infty[}$ *"extended by its null-sets".*

Definition 2.3. *A filtration* $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ *satisfies the* ***usual condition*** *if it is right-continuous and* $\mathfrak{F}_0$ *contains all the* $\mathbb{P}$*-negligible events in* $\mathfrak{F}$*.*

By $s \wedge t$ we denote the minimum of s and t.

Definition 2.4. *A* ***weak solution*** *up to an* ***explosion time*** *of* (5) *is a triple* (U, W)*,* $(\Omega, \mathfrak{F}, \mathbb{P})$*,* $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ *such that*

1. $(\Omega, \mathfrak{F}, \mathbb{P})$ *is a probability space and* $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ *is a filtration of sub-*σ*-fields of* $\mathfrak{F}$ *satisfying the usual condition,*
2. $U = (U_t)_{t\in[0,+\infty[}$ *is a continuous, adapted* $\mathbb{R}^{\mathcal{T}} \cup \{\infty\}$ *valued process with* $\|X_0\| < +\infty$ *almost surely, and* $W = (W_t, \mathfrak{F}_t)_{t\in[0,+\infty[}$ *is an* r*-dimensional Brownian motion,*
3. *for all* $\Gamma \in \mathcal{B}\left(\mathbb{R}^{\mathcal{T}}\right)$ *the equality* $\mu(\Gamma) = \mathbb{P}(U_0 \in \Gamma)$ *is satisfied,*

(4) *for every $n \in \mathbb{N}$ and*

(7)
$$S_n := \inf\left\{t \in [0, +\infty[: \|U_t\|^2_{\mathbb{R}^{\mathcal{T}}} \geq n\right\}$$

the equality

$$\mathbb{P}\left(\int_0^{t \wedge S_n} \left(|b_J(s, U_s)| + |\sigma_{Jk}(s, U_s)|^2\right) \mathrm{d}s < +\infty\right) = 1$$

is satisfied for all $J \in \mathcal{T}$ and $1 \leq k \leq r$, and the equality

(8)
$$U_{t \wedge S_n} = U_0 + \int_0^{t \wedge S_n} b(s, U_s) \,\mathrm{d}s + \int_0^{t \wedge S_n} \sigma(s, U_s) \,\mathrm{d}W_s, \text{ for all } t \in [0, +\infty[$$

is $\mathbb{P}$-almost surely satisfied.

We refer to $S := \lim_{n\to\infty} S_n$ as the explosion time of U. The triple is called a weak solution if $\mathbb{P}(S = +\infty) = 1$.

The following Theorem, which is [9, Theorem 2.3], assures the existence of a weak solution up to an explosion time of an sde of the form (3) under very weak conditions.

Theorem 2.1. *If $b : \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ and $\sigma : \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}\times r}$ are both continuous then for every probability measure μ on $\mathbb{R}^{\mathcal{T}}$ with compact support there exists a weak solution up to an explosion time of (3) with initial distribution μ.*

In the following we introduce notions of uniqueness for a solution of an sde.

Definition 2.5. *Let the drift vector $b(t, x)$ and dispersion matrix $\sigma(t, x)$ be given. Suppose that, whenever W is an r-dimensional Brownian motion on some $(\Omega, \mathfrak{F}, \mathbb{P})$, X is an independent, $\mathbb{R}^{\mathcal{T}}$-valued random vector, $(\mathfrak{F}_t)$ is given as in the definition of the strong solution, and U, $\tilde{U}$ are two strong solutions of (5) relative to W both with initial condition X, then $\mathbb{P}\left(U_t = \tilde{U}_t; 0 \leq t < \infty\right) = 1$. Under these conditions, we say that* ***strong uniqueness*** *holds for the pair (b, σ).*

For weak solutions we also introduce the notion of **pathwise uniqueness**.

Definition 2.6. *Suppose that whenever (U, W), $(\Omega, \mathfrak{F}, \mathbb{P})$, $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ and the triple $\left(\tilde{U}, W\right)$, $(\Omega, \mathfrak{F}, \mathbb{P})$, $\left(\tilde{\mathfrak{F}}_t\right)_{t\in[0,+\infty[}$ are weak solutions to (5) with common Brownian motion W (relative to possibly different filtrations) on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and with common initial value, i.e. $\mathbb{P}\left(U_0 = \tilde{U}_0\right) = 1$, the two processes U and $\tilde{U}$ are indistinguishable:*

$$\mathbb{P}\left(U_t = \tilde{U}_t; \forall 0 \leq t < +\infty\right) = 1.$$

We say then that pathwise uniqueness holds for equation (5).

The following theorem combines [10, Theorem 5.2.5] and [10, Remark 5.3.3].

Theorem 2.2. *Suppose that for every $n \in \mathbb{N}$ there exists a constant $K_n > 0$ such that for every $t \geq 0$, $\|X\|_{\mathbb{R}^{\mathcal{T}}} \leq n$ and $\|Y\|_{\mathbb{R}^{\mathcal{T}}} \leq n$:*

$$\|b(t,x) - b(t,y)\|_{\mathbb{R}^{\mathcal{T}}} + \|\sigma(t,x) - \sigma(t,y)\|_{\mathbb{R}^{\mathcal{T}\times r}} \leq K_n \|Y - X\|_{\mathbb{R}^{\mathcal{T}}}.$$

Then strong uniqueness and pathwise uniqueness hold for the stochastic differential equation (5).

In the sequel we will need the following definition.

Definition 2.7. *For every $t \in [0, +\infty[$ we introduce $\varphi_t : (C[0,+\infty[)^{\mathcal{T}} \to (C[0,+\infty[)^{\mathcal{T}}$ which we define by $\varphi_t(f)(s) := f(s \wedge t)$. We define for each $t \in [0,+\infty[$ the set*

$$\mathcal{B}_t\left((C[0,+\infty[)^{\mathcal{T}}\right) := \varphi_t^{-1}\left(\mathcal{B}\left((C[0,+\infty[)^{\mathcal{T}}\right)\right).$$

The following Theorem, which can be found as [10, Corollary 5.3.23] tells us that weak existence and pathwise uniqueness assure the existence of a strong solution. For the definition of $\widehat{\mathcal{B}}_t$ see [10, Problem 5.3.21].

Theorem 2.3. *Suppose that the stochastic differential equation* (5) *has a weak solution (U, W), $(\Omega, \mathfrak{F}, \mathbb{P})$, $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ with initial distribution μ, and suppose that pathwise uniqueness holds for* (5)*. Then there exists a $\mathcal{B}\left(\mathbb{R}^{\mathcal{T}}\right) \times \mathcal{B}\left((C[0,+\infty[)^r\right)/\mathcal{B}\left((C[0,+\infty[)^{\mathcal{T}}\right)$-measurable function*

$$h : \mathbb{R}^d \times (C[0,+\infty[)^r \to (C[0,+\infty[)^{\mathcal{T}}$$

which is also $\widehat{\mathcal{B}}_t/\mathcal{B}_t\left((C[0,+\infty[)^{\mathcal{T}}\right)$-measurable for every fixed $0 \leq t < \infty$, such that

$$U = h(U_0, W), \quad \mathbb{P}\text{-almost surely.}$$

Moreover, given any probability space $\left(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}}\right)$ rich enough to support an $\mathbb{R}^{\mathcal{T}}$-valued random variable X with distribution μ and an independent Brownian motion $\tilde{W}$, the process $\tilde{U} := h\left(X, \tilde{W}\right)$ is a strong solution of (5) *with initial condition X.*

Remark 2.3. *We recall Remark 2.2 and observe that for a given Brownian motion $\tilde{W}$ and a deterministic initial condition $X \in \mathbb{R}^{\mathcal{T}}$ we can omit the condition that $\left(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}}\right)$ is rich enough to support an $\mathbb{R}^{\mathcal{T}}$-valued random variable X with distribution μ and an independent Brownian motion $\tilde{W}$ in Theorem 2.3.*

Definition 2.8. *We say that* ***uniqueness in the sense of probability law*** *holds for equation* (5) *if for any two weak solutions (U, W), $(\Omega, \mathfrak{F}, \mathbb{P})$,*

$(\mathfrak{F}_t)$ *and* $\left(\tilde{U},\tilde{W}\right)$, $\left(\tilde{\Omega},\tilde{\mathfrak{F}},\mathbb{P}\right)$, $\left(\tilde{\mathfrak{F}}_t\right)$ *with the same initial distribution the two processes* U *and* $\tilde{U}$ *have the same law.*

Definition 2.9. *The stochastic differential equation* (3) *is said to be* ***well posed*** *if for every initial condition* $X \in \mathbb{R}^{\mathcal{T}}$ *it admits a weak solution which is unique in the sense of probability law.*

If the sde (3) is well posed then for an initial value $X \in \mathbb{R}^d$ there exists a solution $\left(U^X, W\right)$, $(\Omega,\mathfrak{F},\mathbb{P})$, $(\mathfrak{F}_t)$ of 3 which is unique in the sense of probability law. This solution induces a measure $\mathbb{P}^X$ on the measurable space $\left((C[0,+\infty[)^{\mathcal{T}}, \mathcal{B}\left((C[0,+\infty[)^{\mathcal{T}}\right)\right)$ defined by

$$\mathbb{P}^X(F) := \mathbb{P}\left(\left\{\omega \in \Omega \mid U^X(\omega) \in F\right\}\right).$$

Definition 2.10. *For every* $s \geq 0$ *we introduce a* ***shift operator*** $\theta_s : (C[0,+\infty[)^{\mathcal{T}} \to (C[0,+\infty[)^{\mathcal{T}}$ *that is defined by* $\theta_s f(t) := f(t+s)$ *for all* $t \geq 0$.

The next Theorem is a special case of [10, Theorem 5.4.20]. It assures under certain conditions that the solution of (3) satisfies the **Markov property**.

Theorem 2.4. *If* b *and* σ *are bounded on compact subsets of* $\mathbb{R}^{\mathcal{T}}$ *and* (3) *is well posed then for every* $s \geq 0$, $F \in \mathcal{B}\left((C[0,+\infty[)^{\mathcal{T}}\right)$ *and* $X \in \mathbb{R}^{\mathcal{T}}$ *the equality*

$$\mathbb{P}^X\left(\theta_s^{-1}F \mid \mathcal{B}_s\right)(f) = \mathbb{P}^{f(s)}(F)$$

is $\mathbb{P}^X$*-almost surely satisfied.*

2.3. **Markov processes.** In this subsection we will present some results about Markov processes. Our main reference for this subsection is [2].

Definition 2.11. *Let* $(\Omega_1,\mathfrak{F}_1)$ *and* $(\Omega_2,\mathfrak{F}_2)$ *be two measurable spaces. A* ***Markov kernel*** *with source* $(\Omega_1,\mathfrak{F}_1)$ *and target* $(\Omega_2,\mathfrak{F}_2)$ *is a function* $P : \Omega_1 \times \mathfrak{F}_2 \to [0,1]$ *such that*

1. *for every* $A \in \mathfrak{F}_2$ *the function* $\omega \mapsto P(\omega, A)$ *is* $\mathfrak{F}_1$*-measurable and*
2. *for every* $\omega \in \Omega_1$ *the function* $A \mapsto P(\omega, A)$ *is a probability measure on* $(\Omega_2,\mathfrak{F}_2)$.

Definition 2.12. *Let* E *be a* ***Polish space*** *with metric* ρ *and endowed with the Borel* σ*-algebra* $\mathcal{B}(E)$*. A family of Markov kernels* $(P_t)_{t\in[0,+\infty[}$ *with source* E *and target* E *is called a family of Markovian transition functions if*

1. *for every* $s,t \geq 0$, $x \in E$ *and* $\Gamma \in \mathcal{B}(E)$ *the equality*

$$P_{t+s}(x,\Gamma) = \int_E P_s(y,\Gamma)\, P_t(x,\mathrm{d}y)$$

is satisfied and

(2) $P_0(x,\Gamma) = \chi_\Gamma(x)$ *holds for each* $x \in E$ *and* $\Gamma \in \mathcal{B}(E)$.

Definition 2.13. *For a measurable space* $(\Omega, \mathfrak{F})$ *we denote by* $B_b(\Omega)$ *the set of all functions* $f : \Omega \to \mathbb{R}$ *that are measurable and bounded. We denote by* $\mathcal{M}(\Omega)$ *the set of all real valued measures on* $(\Omega, \mathfrak{F})$ *and by* $\mathcal{M}_1(\Omega)$ *the set of all probability measures on* $(\Omega, \mathfrak{F})$. *If* E *is a topological space then by* $C_b(E)$ *we denote the set of all functions* $f : E \to \mathbb{R}$ *that are continuous and bounded.*

Definition 2.14. *The* ***Markovian transition semi-group*** *associated with a family of Markovian transition functions* P_t *is a semi-group of linear operators* $(Q_t)_{t\in[0,+\infty[}$ *on* $B_b(E)$ *defined by the formula*

$$Q_t\varphi(x) := \int_E \varphi(y)\, P_t(x, \mathrm{d}y).$$

Similarly, we define for $\mu \in \mathcal{M}_1(E)$ *the function* Q_t^* *by*

$$Q_t^*\mu(\Gamma) := \int_E P_t(x,\Gamma)\, \mu(\mathrm{d}x).$$

Definition 2.15. *A family of Markovian transition functions* $(P_t)_{t\in[0,+\infty[}$ *is said to be* ***stochastically continuous*** *if for all* $x \in E$ *and all* $\delta > 0$ *the equation*

$$\lim_{t\to 0} P_t(x, B(x,\delta)) = 1$$

holds true. Let $(Q_t)_{t\in[0,+\infty[}$ *be the associated transition semi-group. It is called* ***Feller*** *if for every* $t \in [0,+\infty[$ *and every* $\varphi \in C_b(E)$ *we have* $Q_t\varphi \in C_b(E)$.

Lemma 2.1. *Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space and for every* $X \in \mathbb{R}^{\mathcal{T}}$ *the stochastic process* U^X *be* $\mathbb{P}$*-almost surely continuous and* $\mathbb{P}\left(U_0^X = X\right) = 1$. *If we define for every* $t \in [0,+\infty[$ *a function* $P_t : \mathbb{R}^{\mathcal{T}} \times \mathcal{B}\left(\mathbb{R}^{\mathcal{T}}\right) \to \mathbb{R}$ *by* $P_t(X,\Gamma) := \mathbb{P}\left(U_t^X \in \Gamma\right)$ *and assume that* $(P_t)_{t\in[0,+\infty[}$ *is a family of Markovian transition functions then this family is always stochastically continuous.*

Proof. Let $X \in \mathbb{R}^{\mathcal{T}}$ and $\rho > 0$. Since U^X is almost surely continuous, the net of indicator functions $\left(\chi_{\{U_t^X \in B(X,\rho)\}}\right)_{t\in[0,+\infty[}$ tends $\mathbb{P}$-almost surely to 1 when $t \to 0$. This net can clearly be dominated by 1 and we therefore obtain due to the dominated convergence theorem

$$\lim_{t\to 0} \mathbb{P}\left(U_t^X \in B(X,\rho)\right) = \lim_{t\to 0} \int_\Omega \chi_{\{U_t^X \in B(X,\rho)\}}(\omega)\, \mathbb{P}(\mathrm{d}\omega) = 1.$$

□

Definition 2.16. *Let* $(P_t)_{t\in[0,+\infty[}$ *be a family of Markovian transition functions and* $(Q_t)_{t\in[0,+\infty[}$ *the associated transition semi-group. A probability measure* $\mu \in \mathcal{M}_1(E)$ *is said to be invariant or stationary with respect to* $(P_t)_{t\in[0,+\infty[}$ *if for every* $0 \le t < +\infty$ *we have* $Q_t^*\mu = \mu$.

Definition 2.17. *Let E be a* ***Hausdorff*** *space and $\mathfrak{F}$ a σ-algebra containing the topology. A collection of probability measures M defined on $\mathfrak{F}$ is called* ***tight*** *if for any $\varepsilon > 0$ there is a compact subset K_ε of E such that for all $\mu \in M$ the inequality $\mu(K_\varepsilon) > 1-\varepsilon$ is satisfied.*

Given a Polish space E and a stochastically continuous family of Markovian transition functions $(P_t)_{t\in[0,+\infty[}$ we introduce for every $T > 0$ the functions $R_T : E \times \mathcal{B}(E) \to \mathbb{R}$ defined by

$$R_T(x, \Gamma) := \frac{1}{T} \int_0^T P_t(x, \Gamma)\,\mathrm{d}t$$

and $R_T^* : \mathcal{M}_1(E) \to \mathcal{M}_1(E)$ defined by

$$R_T^* \nu(\Gamma) := \int_E R_T(x, \Gamma)\,\nu(\mathrm{d}x)$$

for every $\Gamma \in \mathcal{B}(E)$. The following Theorem, [2, Corollary 3.1.2], gives some conditions under which an invariant measure for $(P_t)_{t\in[0,+\infty[}$ exists.

Theorem 2.5. *Let $(P_t)_{t\in[0,+\infty[}$ be a family of Markovian transition functions and $(Q_t)_{t\in[0,+\infty[}$ the associated transition semi-group that we assume to be stochastically continuous and Feller. If for some $\mu \in \mathcal{M}_1(E)$ and some increasing sequence $T_n \to \infty$ the sequence $\left(R_{T_n}^* \mu\right)$ is tight, then there exists an invariant measure for $(P_t)_{t\in[0,+\infty[}$.*

2.4. **Harmonic analysis.** Since we work on the d-dimensional torus we can hope to apply results from harmonic analysis. Indeed, this theory will be useful to us and we will remind the reader of parts of it in this subsection. We start by introducing the Fourier transform on some groups that are relevant to us. The first one is the Fourier transform on $\mathbb{R}^d$ endowed with the Lebesgue measure λ^d. We denote it by $\mathcal{F} : L^2\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$ and define it for every **Schwartz function** $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ by the formula

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} f(x) \exp\left(-2\pi \mathrm{i} \langle x, \xi\rangle_{\mathbb{R}^d}\right) \lambda^d(\mathrm{d}x) = \langle \mathrm{c}_\xi, f\rangle_{L^2\left(\mathbb{R}^d\right)},$$

where $\mathrm{c}_\xi(x) := \exp\left(2\pi \mathrm{i} \langle \xi, x\rangle_{\mathbb{R}^d}\right)$. The inverse is given by $\mathcal{F}^{-1}f(x) := \langle \mathrm{c}_{-x}, f\rangle_{L^2\left(\mathbb{R}^d\right)}$. We omit in the notation the dependence on the dimension $d \in \mathbb{N}$.

The second Fourier transform we introduce is the one on the d-dimensional torus $\mathbb{T}^d$ which we denote by $\mathcal{G} : L^2\left(\mathbb{T}^d\right) \to L^2\left(\mathbb{Z}^d\right)$ and which we define by

$$\mathcal{G}f(\xi) = \int_{\mathbb{T}^d} f(x) \mathrm{e}^{-2\pi \mathrm{i} \langle \xi, x\rangle} \lambda^d(\mathrm{d}x) = \langle \mathrm{c}_\xi, f\rangle_{L^2\left(\mathbb{T}^d\right)}.$$

We make an abuse of notation by writing λ^d for the **Haar measure** on $\mathbb{T}^d$ that satisfies $\lambda^d\left(\mathbb{T}^d\right) = 1$. This is justified since we may think of the

d-dimensional torus $\mathbb{T}^d$ as $[0,1[^d$ which can be endowed with the Lebesgue measure λ^d. The inverse $\mathcal{G}^{-1}$ is given by

$$\mathcal{G}^{-1}g(x) = \sum_{k\in\mathbb{Z}^d} g(k)\mathrm{e}^{2\pi\mathrm{i}\langle k,x\rangle_d} = \langle \mathrm{c}_{-x}, g\rangle_{L^2\left(\mathbb{Z}^d\right)},$$

where $\mathbb{Z}^d$ is endowed with the counting measure $|\cdot|$.

The Fourier transform of yet another family of groups will be useful in the sequel, namely on the discretization of the d-dimensional torus $\mathbb{T}^d_M$ that we introduced in the subsection 2.1. We endow $\mathbb{T}^d_M$ with the probability measure $\lambda^d_M := \det\left(M^{-1}\right)|\cdot|$. In order to define the Fourier transform we have to identify the dual group of $\mathbb{T}^d_M$. It turns out that this is the group $\mathbb{Z}^d_M := \mathbb{Z}^d/M\mathbb{Z}^d$. Some readers might notice that $\mathbb{Z}^d_M$ is, regarded as a group, isomorphic to $\mathbb{T}^d_M$, which raises the question why we introduce a new notation. One reason is that we endow $\mathbb{Z}^d_M$ with the counting measure $|\cdot|$ which is in general different from λ^d_M. Therefore, while they are isomorphic as groups, the two spaces differ as measure spaces. Another reason is that we want to think of $\mathbb{T}^d_M$ as an approximation of the d-dimensional torus $\mathbb{T}^d$, while we think of $\mathbb{Z}^d_M$ as an approximation of $\mathbb{Z}^d$.

This being said, the Fourier transform $\mathcal{G}_M : L^2\left(\mathbb{T}^d_M\right) \to L^2\left(\mathbb{Z}^d_M\right)$ is defined by

$$\mathcal{G}_M f(\xi) := \det\left(M^{-1}\right)\sum_{x\in\mathbb{T}^d_M} f(x)\mathrm{e}^{-2\pi\mathrm{i}\langle\xi,x\rangle} = \langle c_\xi, f\rangle_{L^2\left(\mathbb{T}^d_M\right)}.$$

Its inverse is given by

$$\mathcal{G}_M^{-1} g\left(x\right) = \sum_{\xi\in\mathbb{Z}^d_M} g(\xi)\mathrm{e}^{2\pi\mathrm{i}\langle\xi,x\rangle} = \langle c_{-x}, g\rangle_{L^2\left(\mathbb{Z}^d_M\right)}.$$

Finally, we also define $\mathcal{H}_M : L^2\left(M\mathbb{T}^d\right) \to L^2\left(M^{-1}\mathbb{Z}^d\right)$ that is given by

$$\mathcal{H}_M f\left(\xi\right) := \int_{M\mathbb{T}^d} f\left(x\right)\exp\left(-2\pi\mathrm{i}\left\langle\xi,x\right\rangle_{\mathbb{R}^d}\right)\lambda^d\left(\mathrm{d}x\right).$$

The inverse is defined by

$$\mathcal{H}_M^{-1} g\left(x\right) := \sum_{\xi\in M^{-1}\mathbb{Z}^d} g\left(\xi\right)\exp\left(2\pi\mathrm{i}\left\langle x,\xi\right\rangle_{\mathbb{R}^d}\right)\det\left(M^{-1}\right).$$

We remark that for $M = I$ we have $\mathcal{H}_M = \mathcal{H}_I = \mathcal{G}$, where I denotes the identity function. With these definitions at hand we can perform some computations that will be useful in the sequel.

Lemma 2.2. *For every $a > 0$ and $g_a, h_a : \mathbb{R} \to \mathbb{R}$ defined by $g\left(t\right) := \chi_{[0,+\infty[}\left(t\right)\mathrm{e}^{-at}$ and $h\left(t\right) := \mathrm{e}^{-a|t|}$ we have $\mathcal{F}g_a\left(\tau\right) = \frac{1}{a+\mathrm{i}2\pi\tau}$ and $\mathcal{F}h_a\left(\tau\right) = \frac{2a}{a^2+4\pi^2\tau^2}$.*

To finish this subsection we want to establish a common frame for all Fourier transforms introduced above. We could call this general point of view "abstract harmonic analysis", which is the title of the main reference [8] used for this section.

Definition 2.18. *A* ***locally compact abelian group*** *is a topological group G with a topology that is locally compact and Hausdorff.*

Definition 2.19. *If G is a locally compact abelian group then the* ***dual group*** *$\widehat{G}$ is the set of all continuous group homomorphisms from G to the circle group $\mathbb{T}$. The dual group is endowed with the topology given by uniform convergence on compact sets.*

In the following lemma we study the dual groups of the groups we encountered in the definition of the Fourier transforms.

Lemma 2.3. *The function $\Psi : \mathbb{R}^d \to \widehat{\mathbb{R}^d}$ defined by $\xi \mapsto \mathrm{c}_\xi$ is an isomorphism of groups. Similarly, the restrictions $\mathbb{Z}^d \to \widehat{\mathbb{T}^d}$ and $\mathbb{Z}^d_M \to \widehat{\mathbb{T}^d_M}$ are isomorphisms of groups.*

The identifications stated in Lemma 2.3 were already silently used in the definitions of the Fourier transforms above. With them the Fourier transforms defined above enter into the general framework we establish in the sequel.

Definition 2.20. *Let G be a locally compact abelian group with a Haar measure μ and $\widehat{G}$ its dual group. If $f \in L^1(G)$, then the Fourier transform is the function $\widehat{f} : \widehat{G} \to \mathbb{C}$ given by*

$$\widehat{f}(\xi) := \int_G \overline{\xi(x)} f(x) \mu(\mathrm{d}x).$$

Given a Haar measure μ on the group G the following theorem lets us identify a unique dual measure $\widehat{\mu}$ on the dual group.

Theorem 2.6. *Let G be a locally compact abelian group. For each Haar measure μ on G there exists a unique Haar measure $\widehat{\mu}$ on $\widehat{G}$ such that whenever $f \in L^1(G)$ and $\widehat{f} \in L^1\left(\widehat{G}\right)$, we have for μ-almost every $x \in G$ the equality*

$$f(x) = \int_{\widehat{G}} \widehat{f}(\xi) \xi(x) \nu(\mathrm{d}\xi).$$

The measure $\widehat{\mu}$ is called the dual measure of μ.

In the following lemma we identify the dual measures of the measures we defined above when introducing the Fourier transforms.

Lemma 2.4. *The dual measure of the Lebesgue measure on $\mathbb{R}^d$ is itself. The dual measure of the Haar measure λ^d on $\mathbb{T}^d$ such that $\lambda^d\left(\mathbb{T}^d\right) = 1$ is the counting measure $|\cdot|$ on $\mathbb{Z}^d$. The dual measure of λ^d_M on $\mathbb{T}^d_M$ is the counting measure $|\cdot|$ on $\mathbb{Z}^d_M$.*

Our last goal in this section is a general version of the Poisson summation formula. We start with a locally compact abelian group G endowed with a Haar measure μ. Furthermore, we assume that H is a closed subgroup of G that is also endowed with a Haar measure ν. For $f \in L^1(G)$ and $x \in G$ we define $g_{f,x} : H \to \mathbb{C}$ by $g_{f,x}(z) := f(xz)$. The first part of [8, Theorem 28.54] is stated in the following lemma.

Lemma 2.5. *The function $g_{f,x}$ is ν-measurable.*

This allows us to define $h_f : G \to \mathbb{C}$ by

$$h_f(y) := \int_H g_{f,y}(z)\, \nu(\mathrm{d}z) = \int_H f(yz)\, \nu(\mathrm{d}z).$$

Since we have for all $y \in G$ and $w \in H$ the equality $h_f(yw) = h_f(y)$ the function $u_f : G/H \to \mathbb{C}$ defined by $u_f(xH) := h_f(x)$ is well defined. The second part of [8, Theorem 28.54] is stated in the following theorem.

Theorem 2.7. *There exists a unique Haar measure ρ on G/H such that for all $f \in L^1(G)$ the equality*

$$\int_{G/H} u_f \rho = \int_G f(x)\, \mu(\mathrm{d}x)$$

holds true.

In order to state the Poisson summation formula we need in addition to the measure ρ on G/H also the notion of an annihilator that is introduced in the next definition.

Definition 2.21. *Let G be a locally compact abelian group and let $\widehat{G}$ be its dual group. For any subset H of G we call the of all $\xi \in \widehat{G}$ such that $\xi(H) = 1$ the **annihilator** of H in G.*

The following theorem, [7, Theorem 23.25], gives us a relation between the annihilator of H in G and the dual group $\widehat{G/H}$ of G/H.

Theorem 2.8. *The group $\widehat{G/H}$ is topologically isomorphic with the annihilator of H in G.*

In the following statement of [8, p. 31.46] we will sometimes write $\widehat{G/H}$ when actually meaning the annihilator of H in G. Theorem 2.8 justifies this abuse of notation.

Theorem 2.9 (Poisson summation formula)**.** *If $u_f : G/H \to \mathbb{C}$ is a continuous function and $\widehat{u_f} \in L^1\left(\widehat{G/H}, \widehat{\rho}\right)$ then for every $x \in G$ the equality*

$$\int_H f(xy)\, \nu(\mathrm{d}y) = \int_H g_{f,x}(y)\, \nu(\mathrm{d}y) = \int_{\widehat{G/H}} \widehat{f}(\xi)\, \xi(x)\, \widehat{\mu}(\mathrm{d}\xi).$$

We want to study the Poisson summation formula in two specific cases involving the Fourier transforms we defined at the beginning of this subsection.

Example 2.1. *The first example we give is when $G = \mathbb{R}^d$ is endowed with the Lebesgue measure and $H = \mathbb{Z}^d$ is the closed subgroup endowed with the counting measure. The quotient group $\mathbb{T}^d = G/H$ can be identified with $[0,1[^d$ which can by abuse of notation also be endowed with the Lebesgue measure λ^d. We already saw in Lemma 2.3 that the dual group of $\mathbb{T}^d$ can be identified with $\mathbb{Z}^d$ which is the annihilator of $\mathbb{Z}^d$ in $\mathbb{R}^d$. If $\widehat{u}_f \in L^1\left(\mathbb{Z}^d\right)$ we have for every $x \in \mathbb{R}^d$ the equality*

$$\sum_{k\in\mathbb{Z}^d} f\left(k+x\right) = \sum_{l\in\mathbb{Z}^d} \mathcal{F}f\left(l\right)\exp\left(2\pi\mathrm{i}\left\langle l,x\right\rangle_{\mathbb{R}^d}\right).$$

Example 2.2. *For the second example we introduce $M \in \mathbb{N}_0^{d\times d}$ as we already did above and define $G = M^{-1}\mathbb{Z}^d$. As a closed subgroup we take $H = \mathbb{Z}^d$ and obtain $G/H = \mathbb{T}^d_M$. We already identified the dual space of $\mathbb{T}^d_M$ with $\mathbb{Z}^d_M$. With these definitions and $\widehat{u}_f \in L^1\left(\mathbb{Z}^d_M\right)$ we obtain for every $x \in M^{-1}\mathbb{Z}^d$ by the Poisson summation formula*

$$\sum_{k\in\mathbb{Z}^d} f\left(k+x\right) = \sum_{l\in\mathbb{Z}^d_M} \mathcal{H}_M^{-1}f\left(l\right)\exp\left(2\pi\mathrm{i}\left\langle l,x\right\rangle_{\mathbb{R}^d}\right).$$

3. The semi-discrete problem

In this section our starting point is the discretization of the d-dimensional torus $\mathbb{T}^d$ we talked about in the subsection 2.1. We derive a semi-discrete version of (1), where the time is still continuous. We based our derivation on ideas from [11] and [12]. The semi-discrete version will turn out to be an sde of the form (3) and we will study this sde in one of the subsections as it has been done in [11].

3.1. Derivation of the semi-discrete problem for a general mesh.

In order to obtain a semi-discrete version of (1) we pick $J \in \mathcal{T}$, we multiply (1) by the indicator function χ_J and integrate over $\mathbb{T}^d$. This way we obtain

$$\begin{aligned}
&\mathrm{d}\int_J u_t(x)\lambda^d\left(\mathrm{d}x\right) + \int_J \operatorname{div} A\left(u_t(x)\right)\lambda^d\left(\mathrm{d}x\right)\mathrm{d}t\\
&= \int_J \operatorname{div}\left(B\left(u_t(x)\right)\nabla u_t(x)\right)\lambda^d\left(\mathrm{d}x\right)\mathrm{d}t + \int_J \sigma\left(x,u_t(x)\right)\lambda^d\left(\mathrm{d}x\right)\mathrm{d}W_t.
\end{aligned}$$

The definition

$$U_t^J = \frac{1}{\lambda^d\left(J\right)}\int_J u_t\left(x\right)\lambda^d\left(\mathrm{d}x\right)$$

and an application of the generalized Stokes theorem yield

$$
\begin{aligned}
\mathrm{d}U_t^J &= \frac{1}{\lambda^d(J)} \sum_{K \in \mathcal{N}(J)} \int_{\overline{J} \cap \overline{K}} \langle \overline{n}_{J\to K}, B(u_t(y))\nabla u_t(y)\rangle_{\mathbb{R}^d} \, \mathcal{H}^{d-1}(\mathrm{d}y) \\
&- \frac{1}{\lambda^d(J)} \sum_{K \in \mathcal{N}(J)} \int_{\overline{J} \cap \overline{K}} \langle \overline{n}_{J\to K}, A(u_t(y))\rangle_{\mathbb{R}^d} \, \mathcal{H}^{d-1}(\mathrm{d}y) \\
&+ \frac{1}{\lambda^d(J)} \int_J \sigma(x, u_t(x)) \, \lambda^d(\mathrm{d}x) \, \mathrm{d}W_t,
\end{aligned}
$$

where $\overline{n}_{J\to K}$ is normalized and normal to the facet between J and K pointing outward from J towards K and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional **Hausdorff measure** on $\mathbb{R}^d$. For every $J \in \mathcal{T}$ we denote by $\mathcal{N}(J)$ the set of all polytopes that are neighbors of J. Let $(\Lambda_{J\to K})$ be a family of strictly positive numbers, where the index set of the family are all $(J,K) \in \mathcal{T}\times\mathcal{T}$ such that $K \in \mathcal{N}(J)$.

Assumption 3.1. *Let $B = 0$ and $\Lambda_{J\to K} > 0$ for each $J \in \mathcal{T}$ and $K \in \mathcal{N}(J)$. Furthermore, we assume $\Lambda_{J\to K} = \Lambda_{K\to J}$.*

We suppose that Assumption 3.1 holds since we do not know how to discretize the term containing B in the present setting of a general mesh $\mathcal{T}$. We define for every $J \in \mathcal{T}$, $1 \le k \le r$ and $U \in \mathbb{R}^{\mathcal{T}}$ the matrix entry

$$
\sigma_{J,k}(U) := \frac{1}{\lambda^d(J)} \int_J \sigma_k\left(x, U^J\right) \lambda^d(\mathrm{d}x). \tag{9}
$$

For further discretization we want to use the Engquist-Osher scheme. We introduce a function $\theta : \mathbb{R}\times\mathbb{R} \to \{0, \pm 1\}$ that is defined by

$$
\theta(\xi, v) = \begin{cases} 1 & \text{if } 0 < \xi \le v, \\ -1 & \text{if } v \le \xi < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}
$$

Recall from the introduction that $a = A'$. We define $a : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by

$$
a(\overline{n}, u) := \max\left(0, \langle a(u), \overline{n}\rangle_{\mathbb{R}^d}\right)
$$

and $A : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by

$$
A(\overline{n}, v) := \int_{\mathbb{R}} a(\overline{n}, u)\, \theta(u, v)\, \lambda(\mathrm{d}u).
$$

For every $\kappa > 0$ we define the numerical fluxes $A, \overline{A}_\kappa : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$
A(\overline{n}, u, v) := A(\overline{n}, u) - A(-\overline{n}, v)
$$

and

$$
\overline{A}_\kappa(\overline{n}, u, v) := A(n, u, v) + \kappa(u - v).
$$

We also define for each $\overline{n} \in \mathbb{R}^d$ the functions $A_{\overline{n}}, \overline{A}_{\kappa,\overline{n}} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by $A_{\overline{n}}(u,v) := A(\overline{n}, u, v)$ and $\overline{A}_{\kappa,\overline{n}}(u,v) := \overline{A}_\kappa(\overline{n}, u, v)$.

Lemma 3.1. *Let us make Assumption 1.2. For every $\overline{n} \in \mathbb{R}^d$ we have $A_{\overline{n}}, \overline{A}_{\overline{n}} \in C^1\left(\mathbb{R}^2, \mathbb{R}\right)$. Furthermore, for all $u, v \in \mathbb{R}$ the equalities*

$$D_1 A_{\overline{n}}(u,v) = a(\overline{n}, u) \quad \text{and} \quad D_2 A_{\overline{n}}(u,v) = -a(-\overline{n}, v) \tag{11}$$

as well as

$$D_1 \overline{A}_{\kappa,\overline{n}}(u,v) = a(\overline{n}, u) + \kappa \quad \text{and} \quad D_2 \overline{A}_{\kappa,\overline{n}}(u,v) = -a(-\overline{n}, v) - \kappa. \tag{12}$$

Proof. Since $A \in C^2\left(\mathbb{R}, \mathbb{R}^d\right)$ by Assumption 1.2 it follows immediately that for each $\overline{n} \in \mathbb{R}^d$ the function $a_{\overline{n}} : \mathbb{R} \to \mathbb{R}$ given by $a_{\overline{n}}(u) := a(n, u)$ is continuous. It follows that $A_{\overline{n}}, \overline{A}_{\kappa,\overline{n}} \in C^1(\mathbb{R}, \mathbb{R})$. Computing the derivatives of these functions we obtain (11) and (12). □

Definition 3.1. *We say that a numerical flux $A : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is **monotone** if it is non-decreasing in its second argument and non-increasing in its third argument.*

We denote by $\mathcal{S}^{d-1}$ the $(d-1)$-dimensional sphere in $\mathbb{R}^d$.

Lemma 3.2. *Let us again make Assumption 1.2. The numerical fluxes A and $\overline{A}_\kappa$ are both monotone numerical fluxes. Furthermore, A and $\overline{A}_\kappa$ are **symmetric** in a sense that for each $\overline{n} \in \mathbb{R}^d$ and every $u, v \in \mathbb{R}$ the equality*

$$A(-\overline{n}, v, u) = -A(\overline{n}, u, v)$$

*is satisfied. The numerical fluxes A and $\overline{A}_\kappa$ are also **consistent**, meaning for every $\overline{n} \in \mathbb{R}^d$ and every $u \in \mathbb{R}$ the equality*

$$\overline{A}_\kappa(\overline{n}, u, u) = \langle A(u), \overline{n} \rangle_{\mathbb{R}^d}$$

is satisfied. Moreover, there exists $\tilde{C}_2 \in \mathbb{R}$ such that for every $\overline{n} \in \mathcal{S}^{d-1}$ and $u, v \in \mathbb{R}$ the inequalities

$$\left| D_1 \overline{A}_{\kappa,\overline{n}}(u,v) \right| \leq \tilde{C}_2 \left(1 + |u|^p\right) \quad \text{and} \quad \left| D_2 \overline{A}_{\kappa,\overline{n}}(u,v) \right| \leq \tilde{C}_2 \left(1 + |v|^p\right)$$

hold.

Proof. The monotonicity of A and $\overline{A}_\kappa$ follows immediately from (11) and (12).

In order to show symmetry we compute

$$A(-\overline{n}, v, u) = A(-\overline{n}, v) - A(\overline{n}, u) = -A(\overline{n}, u, v)$$

and

$$\overline{A}_\kappa(-\overline{n}, v, u) = -A(\overline{n}, u, v) + \kappa(v - u) = -\overline{A}_\kappa(\overline{n}, u, v).$$

We observe that

$$A(n, u, u) = \overline{A}_\kappa(n, u, u) = \int_{\mathbb{R}} \langle a(v), \overline{n} \rangle_{\mathbb{R}^d} \theta(v, u) \lambda(\mathrm{d}v) = \langle A(u), \overline{n} \rangle_{\mathbb{R}^d},$$

and deduce consistency of A and $\overline{A}_\kappa$.

By Lemma 3.1 and (2) we have for every $\overline{n} \in \mathcal{S}^{d-1}$ and $u, v \in \mathbb{R}$ the inequalities

$$\left|D_1\overline{A}_{\kappa,\overline{n}}\left(u,v\right)\right| \leq \left\|a\left(u\right)\right\|_{\mathbb{R}^d} + \kappa \leq C_2\left(1+\left|u\right|^p\right) + \kappa \leq \tilde{C}_2\left(1+\left|u\right|^p\right),$$

and

$$\left|D_2\overline{A}_{\kappa,\overline{n}}\left(u,v\right)\right| \leq \left\|a\left(v\right)\right\|_{\mathbb{R}^d} + \kappa \leq C_2\left(1+\left|v\right|^p\right) + \kappa \leq \tilde{C}_2\left(1+\left|v\right|^p\right).$$

□

We define $\tilde{A} : \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ by

$$\left(\tilde{A}\left(U\right)\right)_J := \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d\left(J\right)} A\left(\overline{n}_{J\to K}, U^J, U^K\right) \tag{13}$$

and $L_\Lambda : \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ by

$$\left(L_\Lambda U\right)_J := \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d\left(J\right)} \Lambda_{J\to K}\left(U^K - U^J\right). \tag{14}$$

We define $b : \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ by $b := L_\Lambda - \tilde{A}$. For $U \in \mathbb{R}^{\mathcal{T}}$ and $J \in \mathcal{T}$ we have by Lemma 3.3 the equality

$$\left(b\left(U\right)\right)_J = -\sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d\left(J\right)} \overline{A}_{\Lambda_{J\to K}}\left(\overline{n}_{J\to K}, U^J, U^K\right). \tag{15}$$

The semi-discrete problem is an sde of the from (3) with drift vector b defined by (15) and dispersion matrix σ given by (9). With the following lemma we may pass to an assumption that is a bit weaker.

Lemma 3.3. *For all $u \in \mathbb{R}$ the equality*

$$\sum_{K\in\mathcal{N}(J)} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left\langle A\left(u\right), \overline{n}_{J\to K}\right\rangle_{\mathbb{R}^d} = 0$$

holds.

Proof. An application of the generalized Stokes theorem shows

$$\sum_{K\in\mathcal{N}(J)} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left\langle A\left(u\right), \overline{n}_{J\to K}\right\rangle_{\mathbb{R}^d}$$
$$= \sum_{K\in\mathcal{N}(J)} \int_{\overline{J}\cap\overline{K}} \left\langle A\left(u\right), \overline{n}_{J\to K}\right\rangle_{\mathbb{R}^d} \mathcal{H}^{d-1}\left(\mathrm{d}y\right) = \int_J \operatorname{div} A\left(u\right)\lambda^d\left(\mathrm{d}x\right) = 0.$$

□

Remark 3.1. *If we do not assume $A(0) = 0$ we can define*

$$\overline{A}_\kappa\left(\overline{n}, u, v\right) := A\left(n, u, v\right) + \kappa\left(u - v\right) + \left\langle A\left(0\right), \overline{n}\right\rangle_{\mathbb{R}^d}.$$

By Lemma 3.3 the function b is still the same, which means we still obtain the same semi-discrete problem. The numerical flux A loses in this case the consistency and $\overline{A}_\kappa$ loses the symmetry, but this does not pose any problem.

3.2. **The semi-discrete problem for regular mesh of rectangular cuboids.** In this subsection we want to have a closer look at the semi-discrete problem in the case of a regular mesh of rectangular cuboids $\mathbb{T}_M^d$ as it was introduced in subsection 2.1. In this case we make assumptions that are slightly different from Assumption 3.1, but still lead to a semi-discrete problem of the same type as in the last section. The assumption we make here is the following.

Assumption 3.2. *We assume that* $\Lambda, B \in \left(\mathbb{R}_0^+\right)^{d\times d}$ *are constant, diagonal matrices. For* Λ *we even demand strict positivity. This means for all* $j \in \{1, \ldots, d\}$ *we assume* $\Lambda_{jj} > 0$.

While in Assumption 3.1 we supposed $B = 0$ we allow for certain $B \neq 0$ in Assumption 3.2. When it comes to Λ, on the other hand, we become more restrictive. For $J \in \mathcal{T}$ and $K \in \mathcal{N}(J)$ we now have $\Lambda_{J\to K} = \Lambda_{jj}$ if $\overline{J} \cap \overline{K}$ is perpendicular to the canonical basis vector e_j.

Instead of an indexing over $\mathcal{T}$ we now index over $\mathbb{T}_M^d$. Each $x \in \mathbb{T}_M^d$ has the polytope $x + M^{-1}[0,1[^d$ associated with it. This being said we obtain for every $x \in \mathbb{T}_M^d$ and $0 \leq k \leq r$ an entry of the dispersion matrix σ given by

$$\sigma_{x,k}(U) := \det(M) \int_{x+M^{-1}\mathbb{T}^d} g_k(y, U^x)\, \lambda^d(\mathrm{d}y)$$

For every $j \in \{1, \ldots, d\}$ we define $a_j, a_j^{\pm} : \mathbb{R} \to \mathbb{R}$ by

$$a_j(u) := \langle a(u), e_j\rangle_{\mathbb{R}^d}, \quad a_j^+(u) := \max(0, a_j(u))$$

and $a_j^-(u) := a_j^+(u) - a_j(u)$. Furthermore, we introduce $A_j^{\pm} : \mathbb{R} \to \mathbb{R}$ and $A_j : \mathbb{R}^2 \to \mathbb{R}$, which are given

$$A_j^{\pm}(u) := \int_{\mathbb{R}} a_j^{\pm}(v)\, \theta(v, u)\, \lambda(\mathrm{d}v) \quad \text{and} \quad A_j(u, v) := A_j^+(u) - A_j^-(v)$$

The diffusion vector b for the semi-discrete problem is thus given by

$$\begin{aligned} (b(U))_x &= -\sum_{j=1}^{d} M_{jj}\left(A_j\left(U^x, U^{x+e_j}\right) - A_j\left(U^{x-e_j}, U^x\right)\right) \\ &\quad + \sum_{j=1}^{d} M_{jj}\left(\Lambda_{jj} + M_{jj}B_{jj}\right)\left(U^{x+e_j} - 2U^x + U^{x-e_j}\right). \end{aligned} \tag{16}$$

This formula shows that despite the nonzero B the semi-discrete problem has the same form as the one for the general mesh. In the case of our current mesh we can rewrite (16) by introducing a discretization of the differential operator $D_{M,j} : L^2\left(\mathbb{T}_M^d; \mathbb{R}\right) \to L^2\left(\mathbb{T}_M^d; \mathbb{R}\right)$, where we note that

$L^2\left(\mathbb{T}_M^d;\mathbb{R}\right)=\mathbb{R}^{\mathbb{T}_M^d}$. For each $U\in L^2\left(\mathbb{T}_M^d;\mathbb{R}\right)=\mathbb{R}^{\mathbb{T}_M^d}$ it is given by

$$D_{M,j}U\left(x\right):=\frac{U\left(x\right)-U\left(x-M^{-1}e_j\right)}{M_{jj}^{-1}}$$

We also introduce a discretization of $\operatorname{div}\left(B\nabla u\right)$ which we denote by $\Delta_{M,B}:L^2\left(\mathbb{T}_M^d;\mathbb{R}\right)\to L^2\left(\mathbb{T}_M^d;\mathbb{R}\right)$ and define by

$$\Delta_{M,B}:=\sum_{j=1}^{d}M_{jj}B_{jj}\left(\left(D_{M,j}\right)^*+D_{M,j}\right).$$

For $B=I$ the operator

$$\Delta_M:=\Delta_{M,I}=\sum_{j=1}^{d}\left(D_{M,j}\right)^*D_{M,j}$$

is a discretization of the Laplace operator. We remark that we omit the dependence on d of the operators in our notation. With these notions we can write

$$b\left(U\right)=-\sum_{j=1}^{d}\Big(\left(D_{M,j}\right)^*A_j^-\left(U\right)+D_{M,j}A_j^+\left(U\right)\Big)+\Delta_{M,B+M^{-1}\Lambda}U.$$

3.3. **Study of the semi-discrete problem.** In this subsection we study the semi-discrete problem. The results we obtain hold both for the semi-discrete problem in case of a general mesh $\mathcal{T}$ with Assumption 3.1 as well as for the semi-discrete problem in case of a regular mesh of rectangular cuboids with Assumption 3.2.

A sum over $J\sim K$ signifies a sum over all facets. Our main reference for this subsection is [11]. We start by stating [11, Proposition A.4].

Lemma 3.4. *For all $U\in\mathbb{R}^{\mathcal{T}}$ we have*

$$\sum_{J\sim K}\mathcal{H}\left(\overline{J}\cap\overline{K}\right)\int_{U^J}^{U^K}A\left(\overline{n}_{J\to K},z,z\right)\mathrm{d}z=0.$$

Proof. Let $U\in\mathbb{R}^{\mathcal{T}}$. We choose some arbitrary $m\in\mathbb{R}^d\setminus\{0\}$ and define for every $J\in\mathcal{T}$ the two sets

$$\mathcal{N}^+\left(J\right):=\left\{K\in\mathcal{N}\left(J\right)\mid\left\langle\overline{n}_{J\to K},m\right\rangle_{\mathbb{R}^d}\geq 0\right\}\tag{17}$$

and $\mathcal{N}^-\left(J\right):=\mathcal{N}\left(J\right)\setminus\mathcal{N}^+\left(J\right)$. By Lemma 3.3 we have for every $\xi\in\mathbb{R}$ the equality

$$\begin{aligned}&\sum_{K\in\mathcal{N}^+(J)}\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left\langle A\left(\xi\right),\overline{n}_{J\to K}\right\rangle_{\mathbb{R}^d}\\&=\sum_{K\in\mathcal{N}^-(J)}\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left\langle A\left(\xi\right),\overline{n}_{K\to J}\right\rangle_{\mathbb{R}^d}\end{aligned}\tag{18}$$

Integrating (18) in ξ from 0 to U^J and summing over all $J \in \mathcal{T}$ we obtain

$$\begin{aligned}
&\sum_{J\in\mathcal{T}} \sum_{K\in\mathcal{N}^+(J)} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right) \int_0^{U^J} \left\langle A\left(\xi\right), \overline{n}_{J\to K}\right\rangle_{\mathbb{R}^d} \mathrm{d}\xi \\
&= \sum_{J\sim K, K\in\mathcal{N}^+(J)} \mathcal{H}\left(\overline{J}\cap\overline{K}\right) \int_0^{U^J} \left\langle A\left(\xi\right), \overline{n}_{J\to K}\right\rangle_{\mathbb{R}^d} \mathrm{d}\xi \\
&= \sum_{K\sim J, J\in\mathcal{N}^+(K)} \mathcal{H}\left(\overline{K}\cap\overline{J}\right) \int_0^{U^K} \left\langle A\left(\xi\right), \overline{n}_{K\to J}\right\rangle_{\mathbb{R}^d} \mathrm{d}\xi \\
&= \sum_{J\sim K, K\in\mathcal{N}^-(J)} \mathcal{H}\left(\overline{J}\cap\overline{K}\right) \int_0^{U^K} \left\langle A\left(\xi\right), \overline{n}_{J\to K}\right\rangle_{\mathbb{R}^d} \mathrm{d}\xi \\
&= \sum_{J\in\mathcal{T}} \sum_{K\in\mathcal{N}^+(J)} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right) \int_0^{U^K} \left\langle A\left(\xi\right), \overline{n}_{J\to K}\right\rangle_{\mathbb{R}^d} \mathrm{d}\xi.
\end{aligned}$$

Subtraction one side of this equality from the other side immediately yields (18). □

With this definition we state the following Lemma.

Lemma 3.5. *We make Assumption 1.2 and Assumption 3.1 or 3.2. For all* $U \in \mathbb{R}^{\mathcal{T}}$ *the equality*

$$\left\langle U, L_\Lambda U\right\rangle_{\mathbb{R}^{\mathcal{T}}} = -\sum_{J\sim K} \Lambda_{J\to K}\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left(U^K - U^J\right)^2$$

holds and the inequality

$$\left\langle U, \tilde{A}\left(U\right)\right\rangle_{\mathbb{R}^{\mathcal{T}}} \geq 0$$

is satisfied.

Proof. Let $U \in \mathbb{R}^{\mathcal{T}}$. Since for all $J \in \mathcal{T}$ and all $K \in \mathcal{N}(J)$ we have $\Lambda_{J\to K} = \Lambda_{K\to J}$ we obtain

$$\begin{aligned}
&\left\langle U, L_\Lambda U\right\rangle_{\mathbb{R}^{\mathcal{T}}} = \sum_{J\in\mathcal{T}} \sum_{K\in\mathcal{N}(J)} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right) \Lambda_{J\to K}\left(U^K - U^J\right) U^J \\
&= \sum_{J\sim K} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left(\Lambda_{J\to K}\left(U^K - U^J\right)U^J - \Lambda_{K\to J}\left(U^J - U^K\right)U^K\right) \\
&= -\sum_{J\sim K} \Lambda_{J\to K}\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left(U^K - U^J\right)^2.
\end{aligned}$$

Due to the Lemmata 3.2 and 3.4 we obtain

$$\left\langle U, \tilde{A}(U) \right\rangle_{\mathbb{R}^{\mathcal{T}}} = \sum_{J \in \mathcal{T}} \sum_{K \in \mathcal{N}(J)} \mathcal{H}^{d-1}\left(\overline{J} \cap \overline{K}\right) U^J A\left(\overline{n}_{J \to K}, U^J, U^K\right)$$

$$= \sum_{J \sim K} \mathcal{H}^{d-1}\left(\overline{J} \cap \overline{K}\right) \left(U^J - U^K\right) A\left(\overline{n}_{J \to K}, U^J, U^K\right)$$

$$= \sum_{J \sim K} \mathcal{H}^{d-1}\left(\overline{J} \cap \overline{K}\right) \int_{U^J}^{U^K} \left(A\left(\overline{n}_{J \to K}, \xi, \xi\right) - A\left(\overline{n}_{J \to K}, U^J, U^K\right)\right) \mathrm{d}\xi \geq 0.$$

□

Let (U, W), $(\Omega, \mathfrak{F}, \mathbb{P})$, $(\mathfrak{F}_t)_{t \in [0,+\infty[}$ be a weak solution up to an explosion time of the semi-discrete problem. For every $n \in \mathbb{N}$ and $\Psi : [0, +\infty[\times \mathbb{R}^d \to \mathbb{R}$ we obtain by Itô's rule [10, Theorem 5.3.6]

(19)

$$\Psi\left(t, U_{t \wedge S_n}\right) = \Psi\left(0, U_0\right) + \int_0^{t \wedge S_n} \frac{\partial \Psi}{\partial t}\left(s, U_s\right) \mathrm{d}s$$

$$+ \sum_{J \in \mathcal{T}} \int_0^{t \wedge S_n} \frac{\partial \Psi}{\partial X^J}\left(s, U_s\right) \left(L_\Lambda U_s\right)_J - \sum_{J \in \mathcal{T}} \int_0^{t \wedge S_n} \frac{\partial \Psi}{\partial X^J}\left(s, U_s\right) \left(\tilde{A}\left(U_s\right)\right)_J \mathrm{d}s$$

$$+ \sum_{k=1}^{r} \sum_{J \in \mathcal{T}} \int_0^{t \wedge S_n} \frac{\partial \Psi}{\partial X^J}\left(s, U_s\right) \sigma_{J,k}\left(U_s\right) \mathrm{d}W_s^k$$

$$+ \frac{1}{2} \sum_{k=1}^{r} \sum_{J,K \in \mathcal{T}} \int_0^{t \wedge S_n} \frac{\partial^2 \Psi}{\partial X^J \partial X^K}\left(s, U_s\right) \sigma_{J,k}\left(U_s\right) \sigma_{K,k}\left(U_s\right) \mathrm{d}s.$$

In order to obtain the existence of a strong solution to the semi-discrete problem we need to make Assumption 1.1 stricter.

Assumption 3.3. *The function σ does not depend on u and there exists a constant $\tilde{C}_1 \in \mathbb{R}$ such that for all $x \in \mathbb{T}^d$ and all $u \in \mathbb{R}$ the inequality*

$$\sum_{k=1}^{r} \left|\sigma_k\left(x, u\right)\right|^2 = \sum_{k=1}^{r} \left|\sigma_k\left(x\right)\right|^2 \leq \tilde{C}_1$$

holds true. Furthermore, we assume that for all $u \in \mathbb{R}$ and $k \in \{1, \ldots, r\}$ the equality

$$\int_{\mathbb{T}^d} \sigma_k\left(x, u\right) \lambda^d\left(\mathrm{d}x\right) = 0.$$

is satisfied.

Theorem 3.1. *Let us make Assumptions 1.2 and 3.3 as well as Assumption 3.1 or 3.2. If $X \in \mathbb{R}^{\mathcal{T}}$ then the semi-discrete problem admits a weak solution.*

Proof. Let $X \in \mathbb{R}^{\mathcal{T}}$. By Theorem 2.1 there exists a weak solution (U, W), $(\Omega, \mathfrak{F}, \mathbb{P})$, $(\mathfrak{F}_t)_{t \in [0,+\infty[}$ up to an explosion time S of the semi-discrete problem with initial condition $X \in \mathbb{R}^{\mathcal{T}}$. An application of Itô's rule as in (19) for $\Psi(t,U) := \|U\|^2_{\mathbb{R}^{\mathcal{T}}}$ gives

$$\mathbb{E}\left[\|U_{t\wedge S_n}\|^2_{\mathbb{R}^{\mathcal{T}}}\right] = \mathbb{E}\left[\|U_0\|^2_{\mathbb{R}^{\mathcal{T}}}\right] + 2\mathbb{E}\left[\int_0^{t\wedge S_n} \langle U_s, L_\Lambda(U_s)\rangle_{\mathbb{R}^{\mathcal{T}}}\,\mathrm{d}s\right]$$

$$-2\mathbb{E}\left[\int_0^{t\wedge S_n}\left\langle U_s, \tilde{A}(U_s)\right\rangle_{\mathbb{R}^{\mathcal{T}}}\mathrm{d}s\right] + \sum_{k=1}^{r}\sum_{J\in\mathcal{T}}\lambda^d(J)\,\mathbb{E}\left[\int_0^{t\wedge S_n}|\sigma_{J,k}(U_s)|^2\,\mathrm{d}s\right].$$

By Lemma 3.5 we have

$$\mathbb{E}\left[\int_0^{t\wedge S_n}\langle U_s, L_\Lambda(U_s)\rangle_{\mathbb{R}^{\mathcal{T}}}\,\mathrm{d}s\right]$$

$$= -\sum_{J\sim K}\Lambda_{J\to K}\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\mathbb{E}\left[\int_0^{t\wedge S_n}\left(U_s^K - U_s^J\right)^2\mathrm{d}s\right] \leq 0$$

and

$$-\mathbb{E}\left[\int_0^{t\wedge S_n}\left\langle U_s, \tilde{A}(U_s)\right\rangle_{\mathbb{R}^{\mathcal{T}}}\mathrm{d}s\right] \leq 0.$$

Due to Assumption 3.3 we have for every $J \in \mathcal{T}$ the inequality

$$\lambda^d(J)\sum_{k=1}^{r}|\sigma_{J,k}(U_s)|^2 \leq \sum_{k=1}^{r}\int_J\left|\sigma_k\left(y, U_s^J\right)\right|^2\mathrm{d}y \leq \tilde{C}_1\lambda^d(J).$$

Combining all of this we obtain

$$\mathbb{E}\left[\|U_{t\wedge S_n}\|^2_{\mathbb{R}^{\mathcal{T}}}\right] \leq \|U_0\|^2_{\mathbb{R}^{\mathcal{T}}} + t\tilde{C}_1.$$

By Markov's inequality we have

$$\mathbb{P}(S_n \leq t) = \mathbb{P}\left(\|U_{t\wedge S_n}\|^2_{\mathbb{R}^{\mathcal{T}}} \geq n\right) \leq \frac{\mathbb{E}\left[\|U_{t\wedge S_n}\|^2_{\mathbb{R}^{\mathcal{T}}}\right]}{n} \leq \frac{\|U_0\|^2_{\mathbb{R}^{\mathcal{T}}} + \tilde{C}_1 t}{n}.$$

Consequently,

$$1 - \mathbb{P}(S = +\infty) = \mathbb{P}(S < +\infty) = \lim_{l\to\infty}\mathbb{P}(S \leq l) = \lim_{l\to\infty}\lim_{n\to\infty}\mathbb{P}(S_n \leq l) = 0.$$

□

Theorem 3.2. *Let us make Assumptions 1.2 and 3.3 as well as Assumption 3.1 or 3.2. For every r-dimensional Brownian motion W and every $X \in \mathbb{R}^{\mathcal{T}}$ there exists a unique strong solution with initial condition X to the semi-discrete problem.*

Proof. Let $X \in \mathbb{R}^{\mathcal{T}}$ and W an r-dimensional Brownian motion. By Theorem 2.2 strong uniqueness as well as pathwise uniqueness hold for the semi-discrete problem that has the form 3. By Theorem 3.1 there exists a weak solution $\left(U, \tilde{W}\right)$, $(\Omega, \mathfrak{F}, \mathbb{P})$, $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ with initial condition X of the semi-discrete problem. The existence of a strong solution is a consequence of Theorem 2.3. □

Proposition 3.1. *Let* $X \in \mathbb{R}_0^{\mathcal{T}}$ *and* $\left(U^X, W\right)$, $(\Omega, \mathfrak{F}, \mathbb{P})$, $(\mathfrak{F}_t)_{t\in[0,+\infty[}$ *be a weak solution of the semi-discrete problem with initial condition* X. *If Assumptions 1.2 and 3.3 as well as Assumption 3.1 or 3.2 are satisfied then for* $\mathbb{P}$*-almost all* $\omega \in \Omega$ *and all* $t \in [0, +\infty[$ *we have* $U_t^X(\omega) \in \mathbb{R}_0^{\mathcal{T}}$.

Proof. By the Definition 2.4 equation (8) is $\mathbb{P}$-almost surely satisfied. Therefore, we obtain $\mathbb{P}$-almost surely and for every $t \in [0, +\infty[$ the equation

$$\begin{aligned}
&\sum_{J\in\mathcal{T}} \lambda^d(J)\, U_t^{X,J} = \sum_{J\in\mathcal{T}} \lambda^d(J)\, X^J + \int_0^t \sum_{J\in\mathcal{T}} \lambda(J) \left(\tilde{A}(U_s)\right)_J \mathrm{d}s \\
&+ \int_0^t \sum_{J\in\mathcal{T}} \lambda^d(J)\,(L_\Lambda U_s)_J\, \mathrm{d}s + \int_0^t \sum_{J\in\mathcal{T}} \lambda^d(J) \sum_{k=1}^r \sigma_{J,k}(U_s)\, \mathrm{d}W_s^k \\
&= \int_0^t \sum_{J\in\mathcal{T}} \lambda(J) \left(\tilde{A}(U_s)\right)_J \mathrm{d}s + \int_0^t \sum_{J\in\mathcal{T}} \lambda^d(J)\,(L_\Lambda U_s)_J\, \mathrm{d}s.
\end{aligned}$$

The observations

$$\begin{aligned}
&\sum_{J\in\mathcal{T}} \lambda^d(J) \left(\tilde{A}(U_s)\right)_J = \sum_{J\in\mathcal{T}} \sum_{K\in\mathcal{N}(J)} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right) A\left(\overline{n}_{J\to K}, U_s^J, U_s^K\right) \\
&= \sum_{J\sim K} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left(A\left(\overline{n}_{J\to K}, U_s^J, U_s^K\right) - A\left(\overline{n}_{K\to J}, U_s^K, U_s^J\right)\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{J\in\mathcal{T}} \lambda^d(J)\,(L_\Lambda U_s)_J = \sum_{J\in\mathcal{T}} \sum_{K\in\mathcal{N}(J)} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right) \Lambda_{J\to K}\left(U_s^K - U_s^J\right) \\
&= \sum_{J\sim K} \mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)\left(\Lambda_{J\to K}\left(U_s^K - U_s^J\right) + \Lambda_{K\to J}\left(Us^J - U_s^K\right)\right) = 0
\end{aligned}$$

conclude the proof. □

If $U_0 \in \mathbb{R}_0^{\mathcal{T}}$ then by Proposition 3.1 the solution U of the semi-discrete problem remains under certain assumptions in $\mathbb{R}_0^{\mathcal{T}}$. This is crucial in order to show the following [11, Proposition 3.17].

Proposition 3.2 (L^1-contraction)**.** *Let us make Assumptions 1.2 and 3.3 as well as Assumption 3.1 or 3.2. If* W *is an* r*-dimensional Wiener process,* $X, Y \in \mathbb{R}_0^{\mathcal{T}}$ *and* U^X *respectively* U^Y *are strong solutions of the semi-discrete problem with initial conditions* X *respectively* Y *then for all* $0 \le s \le t$ *the inequality*

$$\left\|U_t^Y - U_t^X\right\|_{L^1(\mathbb{R}^{\mathcal{T}})} \le \left\|U_s^Y - U_s^X\right\|_{L^1(\mathbb{R}^{\mathcal{T}})}$$

is satisfied.

Let W be an r-dimensional Brownian motion. In view of Theorem 3.2 we denote for every $X \in \mathbb{R}^{\mathcal{T}}$ by U^X the unique strong solution with initial condition X of the semi-discrete problem. For every $t \in [0,+\infty[$ and every $\varphi \in B_b\left(\mathbb{R}^d\right) \to \mathbb{R}$ we define $h_{t,\varphi} : \mathbb{R}_0^{\mathcal{T}} \to \mathbb{R}$ by

$$h_{t,\varphi}(X) := \mathbb{E}\left[\varphi\left(U_t^X\right)\right] = \int_\Omega \varphi\left(U_t^X(\omega)\right)\mathbb{P}(\omega)$$

Proposition 3.1 justifies the statement of the following Proposition 3.3, which is [11, Corollary 3.18]. The proof relies on Proposition 3.2.

Proposition 3.3. *Let us make Assumptions 1.2 and 3.3 as well as Assumption 3.1 or 3.2. For every $\varphi \in C_b\left(\mathbb{R}_0^{\mathcal{T}};\mathbb{R}\right)$ the function $h_{t,\varphi}$ is an element of $C_b\left(\mathbb{R}_0^{\mathcal{T}};\mathbb{R}\right)$.*

In order to see in detail why we restrict ourselves to the space $\mathbb{R}_0^{\mathcal{T}}$ we refer to [11]. For every $t \in [0,+\infty[$ we introduce $P_t : \mathbb{R}_0^{\mathcal{T}} \times \mathcal{B}\left(\mathbb{R}_0^{\mathcal{T}}\right) \to \mathbb{R}$ which we define by

$$P_t(X,\Gamma) := \mathbb{P}\left(U_t^X \in \Gamma\right) = h_{t,\chi_\Gamma}(X). \tag{20}$$

Due to Proposition 3.1 the following statement makes sense.

Proposition 3.4. *For every $\Gamma \in \mathcal{B}\left(\mathbb{R}_0^{\mathcal{T}}\right)$ the function h_{t,χ_Γ} is a measurable function.*

Proof. Let us define the set

$$\mathfrak{D} := \left\{\Gamma \in \mathcal{B}\left(\mathbb{R}_0^{\mathcal{T}}\right) \mid h_{t,\chi_\Gamma} \in B_b\left(\mathbb{R}^d;\mathbb{R}\right)\right\}.$$

We clearly have $\Omega \in \mathfrak{D}$, because $h_{t,\chi_\Omega} = 1$ is clearly Borel measurable and bounded. Let $C, D \in \mathfrak{D}$ and $C \subseteq D$. We then have

$$h_{t,\chi_{D\setminus C}}(X) = \int_\Omega \left(\chi_D\left(U_t^X\right) - \chi_C\left(U_t^X\right)\right)\mathbb{P} = h_{t,\chi_D}(X) - h_{t,\chi_C}(X),$$

which shows that $D \setminus C \in \mathfrak{D}$. Let finally $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathfrak{D}$ such that for all $m, n \in \mathbb{N}$ the inequality $m \le n$ implies $\Gamma_m \subseteq \Gamma_n$. We define $\Gamma := \bigcup_{n\in\mathbb{N}} \Gamma_n$. We obtain

$$h_{t,\chi_\Gamma}(X) = \lim_{n\to\infty} \int_\Omega \chi_{\Gamma_n}\left(U_t^X\right)\mathbb{P} = \lim_{n\to\infty} h_{t,\chi_{\Gamma_n}}(X).$$

As the point-wise limit of measurable functions $h_{t,\Gamma}$ is measurable and it is bounded by the constant function 1. All of this together shows that $\mathfrak{D}$ is a Dynkin system.

Realling the discussion of (4) there exists an isomorphism of vector-spaces $\Psi : \mathbb{R}^{|\mathcal{T}|-1} \to \mathbb{R}_0^{\mathcal{T}}$. We define

$$\mathfrak{P} := \left\{\Gamma \in \mathcal{B}\left(\mathbb{R}_0^{\mathcal{T}}\right) \mid \forall 1 \le j \le (|\mathcal{T}|-1)\, \exists a_j \le b_j : \Gamma = \Psi\left(\prod_{j=1}^{|\mathcal{T}|-1} [a_j, b_j[\right)\right\}.$$

Let $\Gamma \in \mathfrak{P}$ with

$$\Gamma = \Psi \left(\prod_{j=1}^{|\mathcal{T}|-1} [a_j, b_j[\right).$$

There exists a sequence $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$ of continuous functions that are uniformly bounded by 1 and that converge to $\chi_{\Psi^{-1}(\Gamma)} = \chi_\Gamma \circ \Psi^{-1}$. Defining for every $n \in \mathbb{N}$ the function $\varphi_n := \tilde{\varphi}_n \circ \Psi^{-1}$ we observe that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is still unifromly bounded and we have

$$h_{t,\chi_\Gamma}(X) = \int_\Omega \lim_{n\to\infty} \varphi_n \left(U_t^X\right) \mathbb{P} = h_{t,\varphi_n}(X).$$

Since all h_{t,φ_n} are measurable functions by Proposition 3.3, it follows that h_{t,χ_Γ} is also measurable and therefore $\Gamma \in \mathfrak{D}$. The set $\mathfrak{P}$ is stable by intersection and we can apply the Sierpiński-Dynkin π-λ theorem to obtain $\mathcal{B}\left(\mathbb{R}_0^{\mathcal{T}}\right) = \mathfrak{s}(\mathfrak{P}) \subseteq \mathfrak{D}$. Consequently, $\mathfrak{D} = \mathcal{B}\left(\mathbb{R}_0^{\mathcal{T}}\right)$. □

Theorem 3.3. *Let us make Assumptions 1.2 and 3.3 as well as Assumption 3.1 or 3.2. The family of functions $(P_t)_{t\in[0,+\infty[}$ given by (20) is a stochastically continuous family of Markovian transition functions that is even Feller.*

Proof. Proposition 3.4 ensures that for every $t \in [0, +\infty[$ is a Markov kernel. For every $X \in \mathbb{R}_0^{\mathcal{T}}$ and every $\Gamma \in \mathcal{B}\left(\mathbb{R}_0^{\mathcal{T}}\right)$ we have

$$P_0(X, \Gamma) = \mathbb{P}\left(U_0^X \in \Gamma\right) = \mathbb{P}(X \in \Gamma) = \chi_\Gamma(X).$$

In order to show the Markov property we define for every $t \geq 0$ the function $\varphi_t : (C[0, +\infty[)^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ by $\varphi_t(f) := f(t)$. We observe that for every $t \in [0, +\infty[$, every $\Gamma \in \mathcal{B}\left(\mathbb{R}^{\mathcal{T}}\right)$ and each $X \in \mathbb{R}^{\mathcal{T}}$ the equalities

$$(\varphi_t)_* \mathbb{P}^X(\Gamma) = \mathbb{P}\left(U_t^X \in \Gamma\right) = P_t(X, \Gamma).$$

This shows that the the measure $P_t(X, \cdot)$ is the push forward measure of $\mathbb{P}^X$ by φ_t. For $s, t \geq 0$, $X \in \mathbb{R}^{\mathcal{T}}$ and $\Gamma \in \mathcal{B}\left(\mathbb{R}^{\mathcal{T}}\right)$ we obtain due to Theorem 2.4 and the transformation formula the inequalities

$$\begin{aligned}
P_{s+t}(X, \Gamma) &= \int_{(C[0,+\infty[)^{\mathcal{T}}} \chi_{\theta_t^{-1}\varphi_s^{-1}(\Gamma)}(f)\, \mathbb{P}^X(\mathrm{d}f) \\
&= \int_{(C[0,+\infty[)^{\mathcal{T}}} \mathbb{P}^X\left(\theta_t^{-1}\varphi_s^{-1}(\Gamma) \mid \mathcal{B}_s\right) \mathbb{P}^X(\mathrm{d}f) \\
&= \int_{(C[0,+\infty[)^{\mathcal{T}}} \mathbb{P}^{f(t)}\left(\varphi_s^{-1}(\Gamma)\right) \mathbb{P}^X(\mathrm{d}f) \\
&= \int_{(C[0,+\infty[)^{\mathcal{T}}} P_s(\varphi_t(f), \Gamma)\, \mathbb{P}^X(\mathrm{d}f) \\
&= \int_{\mathbb{R}^{\mathcal{T}}} P_s(Y, \Gamma)\, P_t(X, \mathrm{d}Y).
\end{aligned}$$

At this point we have shown that $(P_t)_{t\in[0,+\infty[}$ is a family of Markov transition functions.This family is by Lemma 2.1 stochastically continuous. The Feller property is a direct consequence of Proposition 3.3. □

The following theorem was shown as [11, Theorem 3.5] and [11, Theorem A.7].

Theorem 3.4. *Under Assumptions 1.2 and 3.3 as well as Assumption 3.1 or 3.2 the family of Markovian transition kernels $(P_t)_{t\in[0,+\infty[}$ admits a unique invariant measure $\mu_{\mathcal{T}} \in \mathcal{M}_1\left(\mathbb{R}_0^{\mathcal{T}}\right)$.*

4. Kinetic formulation

In the previous section we reminded the reader of some resulsts from [11], namely that the semi-discrete problem admits under certain conditions a unique strong solution and under even stricter conditions there exists a corresponding invariant measure $\mu_{\mathcal{T}}$ on $\mathbb{R}_0^{\mathcal{T}}$. Recall from the introduction that we asked whether the probability measures $\mu_{\mathcal{T}}$ converge towards a probability measure μ on $L_0^1\left(\mathbb{T}^d\right)$ when the mesh $\mathcal{T}$ is getting fine. We want to try using techniques similar to the ones used in [5] in order to find an answer to this question. For this endeavor it is necessary to introduce a kinetic formulation of the semi-discrete problem.

4.1. **Kinetic formulation for a general mesh.** Let $\mathcal{T}$ be a mesh of the d-dimensional torus $\mathbb{T}^d$ and Λ as in subsection 3.1. We introduce $a : \mathbb{R} \to \left(\mathbb{R}^{\mathcal{T}}\right)^{\left(\mathbb{R}^{\mathcal{T}}\right)}$ and $l_\Lambda \in \left(\mathbb{R}^{\mathcal{T}}\right)^{\left(\mathbb{R}^{\mathcal{T}}\right)}$ that we define by

$$a(\xi) F(J) := \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d(J)} \left(a\left(\overline{n}_{J\to K},\xi\right) F_J - a\left(\overline{n}_{K\to J},\xi\right) F_K\right).$$

and

$$l_\Lambda F(J) := \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d(J)} \Lambda_{J\to K}\left(F_K - F_J\right)$$

Furthermore, for every $J \in \mathcal{T}$ and every $K \in \mathcal{N}(J)$ we define $\mathfrak{m}_{J\to K} : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned} \mathfrak{m}_{J\to K}(\xi,u,v) &:= A\left(\overline{n}_{J\to K},\xi,v\right)\theta(\xi,u) \\ - \left(A\left(\overline{n}_{K\to J},\xi\right) - A\left(\overline{n}_{K\to J},v\right)\right)\theta(\xi,v) &+ \kappa_{J\to K}(\xi - v)\left(\theta(\xi,u) - \theta(\xi,v)\right) \end{aligned}$$

Given $X \in \mathbb{R}^{\mathcal{T}}$, a probability space $(\Omega,\mathfrak{F},\mathbb{P})$ and an r-dimensional Brownian motion there exists a unique strong solution U of the semi-discrete problem with initial condition X. This is ensured by Theorem 3.2. For each $J \in \mathcal{T}$ and $t \in [0,+\infty[$ we define the function $F_J : \mathbb{R}^2 \to \mathbb{R}$ by $F_J(t,\xi) := \theta\left(\xi, U_t^J\right)$ as well as $\mathfrak{m}_J : [0,+\infty[\times\mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{m}_J(t,\xi) := \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d(J)} \mathfrak{m}_{J\to K}\left(\xi, U_t^J, U_t^K\right).$$

Furthermore, we introduce for every $k \in \{1, \dots, r\}$ the two distributions $P_{k,t}^J, Q_t^J$ on $\mathbb{R}$ that are given by

$$\left\langle P_{k,t}^J, \varphi \right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} := \sigma_{J,k}\left(U_t\right) \varphi\left(U_t^J\right)$$

and

$$\left\langle Q_t^J, \varphi \right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} := \int_{\mathbb{R}} \varphi\left(\xi\right) \mathfrak{m}_J\left(t,\xi\right) \lambda\left(\mathrm{d}\xi\right) - \varphi\left(U_t^J\right) \sum_{k=1}^{r} \left|\sigma_{J,k}\left(U_t\right)\right|^2$$

for every $\varphi \in \mathcal{D}\left(\mathbb{R}\right)$. We use the notation $\mathcal{D}\left(\mathbb{R}\right) = C_c^\infty\left(\mathbb{R}\right)$ and denote by $\mathcal{D}'\left(\mathbb{R}\right)$ its dual space that is the space of distributions.

Proposition 4.1. *Let $T \in [0, \infty[$. For every $\psi \in \left(C_c^1\left([0,T] \times \mathbb{R}\right)\right)^{\mathcal{T}}$ and $\tilde{\psi}_{J,t} : \mathbb{R} \to \mathbb{R}$ defined as $\tilde{\psi}_{J,t}\left(\xi\right) := \psi_J\left(t,\xi\right)$ for every $J \in \mathcal{T}$ and $t \in [0,T]$ the equation*

$$\begin{aligned}
&\sum_{J\in\mathcal{T}} \lambda^d\left(J\right) \int_{\mathbb{R}} \left(\psi_J\left(T,\xi\right) F_J\left(T,\xi\right) - \psi_J\left(0,\xi\right) F_J\left(0,\xi\right)\right) \mathrm{d}\xi \\
&= \sum_{J\in\mathcal{T}} \lambda^d\left(J\right) \int_0^T \int_{\mathbb{R}} \frac{\partial \psi_J}{\partial t}\left(t,\xi\right) F_J\left(t,\xi\right) \lambda\left(\mathrm{d}\xi\right) \mathrm{d}t \\
&+ \sum_{J\in\mathcal{T}} \lambda^d\left(J\right) \int_0^T \int_{\mathbb{R}} \psi_J\left(t,\xi\right) \left(\left(l_\Lambda - a\left(\xi\right)\right) F\left(t,\xi\right)\right)_J \mathrm{d}\xi \mathrm{d}t \\
&+ \sum_{J\in\mathcal{T}} \lambda^d\left(J\right) \int_0^T \left\langle \left(Q_t^J\right)', \tilde{\psi}_{J,t} \right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} \mathrm{d}t \\
&+ \sum_{k=1}^{r} \sum_{J\in\mathcal{T}} \lambda^d\left(J\right) \int_0^T \left\langle P_{k,t}^J, \tilde{\psi}_{J,t} \right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} \mathrm{d}W_t^k
\end{aligned} \tag{21}$$

is $\mathbb{P}$-almost surely satisfied. This is what we call the ***kinetic formulation*** *of the semi-discrete scheme.*

Proof. Let $T > 0$ and $\eta \in C^1\left([0,T]\right)$. To derive the kinetic formulation we fix $\varphi \in C_c^\infty\left(\mathbb{R}\right)$ and define $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi\left(u\right) := \int_{-\infty}^{u} \varphi\left(\xi\right) \mathrm{d}\xi. \tag{22}$$

An application of Itô's rule like in (19) for $\Psi_J(t,U) := \eta(t)\Phi\left(U^J\right)$ for some $J \in \mathcal{T}$ yields

(23)

$$\eta(t)\Phi\left(U_t^J\right) = \eta(0)\Phi\left(U_0^J\right) + \int_0^t \eta'(s)\Phi\left(U_s^J\right)\mathrm{d}s$$
$$+ \int_0^t \eta(s)\varphi\left(U_s^J\right)\left(\left(L_\Lambda - A\right)(U_s)\right)_J \mathrm{d}s$$
$$+ \sum_{k=1}^r \left(\int_0^t \eta(s)\varphi\left(U_s^J\right)\sigma_{J,k}(U_s)\mathrm{d}W_s^k + \frac{1}{2}\int_0^t \eta(s)\varphi'\left(U_s^J\right)\left|\sigma_{J,k}(U_s)\right|^2 \mathrm{d}s\right).$$

Since for each $J \in \mathcal{T}$ and $K \in \mathcal{N}(J)$ the equality $A(\overline{n}_{J\to K}, 0) = 0$ holds, we obtain

$$\varphi\left(U_s^J\right)\left(\left(L_\Lambda - A\right)(U_s)\right)_J$$
$$= \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d(J)}\left(\varphi\left(U_s^K\right)A\left(\overline{n}_{K\to J}, U_s^K\right) - \varphi\left(U_s^J\right)A\left(\overline{n}_{J\to K}, U_s^J\right)\right)$$
$$- \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d(J)}\left(\varphi\left(U_s^K\right) - \varphi\left(U_s^J\right)\right)A\left(\overline{n}_{K\to J}, U_s^K\right)$$
$$+ \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d(J)}\Lambda_{J\to K}\left(\varphi\left(U_s^K\right)U_s^K - \varphi\left(U_s^J\right)U_s^J\right)$$
$$- \sum_{K\in\mathcal{N}(J)} \frac{\mathcal{H}^{d-1}\left(\overline{J}\cap\overline{K}\right)}{\lambda^d(J)}\Lambda_{J\to K}\left(\varphi\left(U_s^K\right) - \varphi\left(U_s^J\right)\right)U_s^K$$
$$= \int_{\mathbb{R}} \varphi(\xi)\left(\left(l_\Lambda - a(\xi)\right)F(t,\xi)\right)_J \lambda(\mathrm{d}\xi) + \int_{\mathbb{R}} \varphi(\xi)\,\mathfrak{m}_J(t,\xi)\,\lambda(\mathrm{d}\xi).$$

Inserting this into (23) together with a density argument gives (21). □

4.2. **Kinetic formulation for a regular mesh of rectangular cuboids.** We want to specify the kinetic formulation (21) in the case of a regular mesh of rectangular cuboids. For this endeavor we define for every $j \in \{1,\dots,d\}$ the function $a_j : \mathbb{R} \to \mathbb{R}$ by $a_j(\xi) := \langle a(\xi), e_j\rangle_{\mathbb{R}^d}$. Furthermore, we introduce $a_j^+, a_j^- : \mathbb{R} \to \mathbb{R}$ given by

$$a_j^+(\xi) := \max\left(0, a_j(\xi)\right) \quad \text{and} \quad a_j^-(\xi) := a_j^+(\xi) - a_j(\xi).$$

With these functions we have

$$\tilde{a}(\xi)(F) = \sum_{j=1}^d \left(a_j^-(\xi)\left(D_{d,j}^M\right)^* F + a_j^+(\xi)\, D_{d,j}^M F\right).$$

Furthermore, we observe that l_Λ corresponds to $\Delta_{M,B+M^{-1}\Lambda}$, with the additional $B \neq 0$ as the only difference. For every $p \in \{1,\dots,d\}$ we introduce

$\mathfrak{m}_p^-, \mathfrak{m}_p^+ : [0,+\infty[\times\mathbb{R}\to\mathbb{R}$ that we define by

$$\begin{aligned}\mathfrak{m}_p^-(\xi,u,v) &:= \left(A_p^-(\xi)-A_p^-(v)\right)(\theta(\xi,u)-\theta(\xi,v))\\ &+M_{pp}^2\left(B_{pp}+M_{pp}^{-1}\Lambda_{pp}\right)(\xi-v)(\theta(\xi,u)-\theta(\xi,v))\end{aligned}$$

and

$$\begin{aligned}\mathfrak{m}_p^+(\xi,u,v) &:= \left(A_p^+(\xi)-A_p^+(u)\right)(\theta(\xi,v)-\theta(\xi,u))\\ &+M_{pp}^2\left(B_{pp}+M_{pp}^{-1}\Lambda_{pp}\right)(\xi-u)(\theta(\xi,v)-\theta(\xi,u))\end{aligned}$$

and define for each $x\in\mathbb{T}_M^d$ the so called **discrete kinetic entropy defect measure** $\mathfrak{m}_x : [0,+\infty[\times\mathbb{R}\to\mathbb{R}$ by

$$\mathfrak{m}_x(t,\xi)=\sum_{p=1}^{d}\left(\mathfrak{m}_p^-\left(\xi,U^x,U^{x+e_j}\right)+\mathfrak{m}_p^+\left(\xi,U^{x-e_j},U^x\right)\right).$$

5. Do the invariant measures converge?

In this section we tackle the core question of the internship directly. We want to remind ourselves of the fact that for each mesh $\mathcal{T}$ of $\mathbb{T}^d$ there exists by Theorem 3.4 a unique invariant measure $\mu_{\mathcal{T}}$ on $\mathbb{R}_0^{\mathcal{T}}$ associated with the semi-discrete problem. We want to know whether or not these measures converge to a measure μ on $L_0^1\left(\mathbb{T}^d\right)$. In order to answer this question we want to use techniques that are similar to the ones used in [5]. We will restrict ourselves to the regular meshes of rectangular cuboids $\mathbb{T}_M^d$ defined above. Starting from the kinetic formulation we will obtain a new representation of U_t, the solution of the semi-discrete problem. Using this representation we will try to estimate the expected value of a certain norm of U_t. We hope that these estimations are uniform in M and allow us to deduce $\lim_{M\to\infty}\mu_M=\mu$ for a probability measure μ on $L_0^1\left(\mathbb{T}^d\right)$. We remark that these estimations would also allow us to apply the Krylov-Bogolyubov theorem 2.5 which would again ensure the existence of an invariant measure μ_M.

5.1. Decomposition of the solution and other preparations.

We fix $\alpha\in]0,1]$, and $\beta,\gamma>0$ and define a new function $\tilde{a}_{\alpha,\beta,\gamma}(\xi):=\tilde{a}(\xi)+$

$\gamma(-\Delta_M)^\alpha+\beta$ and $l_{\alpha,\beta,\gamma}:=\Delta_{M,B+M^{-1}\Lambda}+\gamma(\Delta_M)^\alpha+\beta$. With these definitions and for $x\in\mathbb{T}_M^d$ we rewrite the kinetic formulation (21) in the form

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(\psi_x(T,\xi)(F(T,\xi))_x-\psi_x(0,\xi)(F(0,\xi))_x\right)\mathrm{d}\xi\\
&=\int_0^T\int_{\mathbb{R}}\frac{\partial\psi_x}{\partial t}(t,\xi)(F(t,\xi))_x\,\mathrm{d}\xi\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{R}}\psi_x(t,\xi)\left((l_{\alpha,\beta,\gamma}-\tilde{a}_{\alpha,\beta,\gamma}(\xi))F(t,\xi)\right)_x\lambda(\mathrm{d}\xi)\,\mathrm{d}t\\
&+\int_0^T\left\langle(Q_t^x)',\tilde{\psi}_{x,t}\right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}\mathrm{d}t+\sum_{k=1}^r\int_0^T\left\langle P_{k,t}^x,\tilde{\psi}_{x,t}\right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}\mathrm{d}W_t^k
\end{aligned}
\tag{24}
$$

Choosing in particular the function $\psi_x(t,\xi):=\left((\exp((s-t)A_{\alpha,\beta,\gamma}))^*e_x\right)_x$ we obtain $U_t^x=U_t^{0,x}+U_t^{1,x}+U_t^{2,x}+U_t^{3,x}$, where

$$U_t^{0,x}:=\int_{\mathbb{R}}\left(\exp(-t\tilde{a}_{\alpha,\beta,\gamma})F(0,\xi)\right)_x\mathrm{d}\xi$$

$$U_t^{1,x}:=\int_0^t\int_{\mathbb{R}}\left(\exp(-(t-s)\tilde{a}_{\alpha,\beta,\gamma})\,l_{\alpha,\beta,\gamma}F(s,\xi)\right)_x\lambda(\mathrm{d}\xi)\,\mathrm{d}s$$

$$U_t^{2,x}:=\sum_{k=1}^r\int_0^t\left\langle P_{k,s}^x,\tilde{\psi}_{x,t}\right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}\mathrm{d}W_s^k$$

$$U_t^{3,x}:=\int_0^t\left\langle(Q_s^x)',\tilde{\psi}_{x,s}\right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}\mathrm{d}s.$$

What we just have accomplished is the decomposition of U. We continue this subsection with some preparations for the next subsections.

Lemma 5.1. *The operator $\mathcal{G}_{d,M}D_{d,j}^M(\mathcal{G}_{d,M}):L^2(\mathbb{Z}_M^d)\to L^2(\mathbb{Z}_M^d)$ is a multiplication operator corresponding to $\left(\rho_{d,j}^M(k)\right)_{k\in\mathbb{Z}_M^d}$, where*

$$\rho_{d,j}^M(k):=M_{jj}\left(1-\exp\left(-2\pi\mathrm{i}\langle Me_j,k\rangle_d\right)\right).$$

The operator $\mathcal{G}_{d,M}\Delta_M^d(\mathcal{G}_{d,M}):L^2(\mathbb{Z}_M^d)\to L^2(\mathbb{Z}_M^d)$ is a multiplication operator corresponding to $\left(\ell_d^M(k)\right)_{k\in\mathbb{Z}_M^d}$, where

$$\ell_d^M(k)=2\sum_{j=1}^d M_{jj}^2\left(1-\cos\left(2\pi\langle Me_j,k\rangle_d\right)\right).$$

Let us define $\Gamma:\mathbb{R}^d\to\mathbb{R}^d$ by $(\Gamma(z))_j:=1-\cos(2\pi z_j)$ and $\omega:\mathbb{R}^d\to\mathbb{R}^d$ by $(\omega(z))_j:=\sin(2\pi z_j)$.

Proposition 5.1. *The operator* $\mathcal{G}_M \tilde{a}_{\alpha,\beta,\gamma} (\mathcal{G}_M)^{-1} : L^2\left(\mathbb{Z}_M^d\right) \to L^2\left(\mathbb{Z}_M^d\right)$ *is a multiplication operator corresponding to*

$$\left(\left\langle |a|(\xi), M\Gamma\left(M^{-1}k\right)\right\rangle_{\mathbb{R}^d} + \mathrm{i}\left\langle a(\xi), M\omega\left(M^{-1}k\right)\right\rangle_{\mathbb{R}^d} + \gamma \ell_M^{\alpha}(k) + \beta\right)_{k\in\mathbb{Z}_M^d}.$$

We introduce $\zeta_M : \mathbb{Z}_M^d \to \mathbb{R}$ defined by

$$\zeta_M(k) := \left\|M\omega\left(M^{-1}k\right)\right\|_{\mathbb{R}^d}.$$

For every $s \geq 0$ we define $\|\cdot\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)} : L^2\left(\mathbb{T}_M^d\right) \to \mathbb{R}$ by

$$\|U\|^2_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)} := \sum_{k\in\mathbb{Z}_M^d} |\zeta_M(k)|^{2s} \left|(\mathcal{G}_M U)_k\right|^2.$$

Moreover, we define $\alpha_M, \delta_M, \eta_M : \mathbb{Z}_M^d \to \mathbb{R}$ by

$$\begin{aligned}
\alpha_M(k) &:= \left\|M\omega\left(M^{-1}k\right)\right\|^{-1}_{\mathbb{R}^d} M\Gamma\left(M^{-1}k\right),\\
\delta_M(k) &:= \frac{1}{2}\left\|M\omega\left(M^{-1}k\right)\right\|^{-1}_{\mathbb{R}^d}\left(\gamma\ell_M^{\alpha}(k) + \beta\right) \quad \text{and}\\
\eta_M(k) &:= \left\|M\omega\left(M^{-1}k\right)\right\|^{-1}_{\mathbb{R}^d} M\omega\left(M^{-1}k\right).
\end{aligned}$$

Defining $\Psi_{M,k} : \mathbb{R} \to \mathbb{R}$ by

$$\Psi_{M,k}(\xi) := \langle |a|(\xi), \alpha_M(k)\rangle_{\mathbb{R}^d} + \delta_M(k)$$

we can make the following assumption.

Assumption 5.1. *We suppose that all entries of M are odd numbers. In addition to Assumption 1.2 we want to assume that there exists a constant $C_4 \in \mathbb{R}$ and $b \in]0,1]$ such that for all $k \in \mathbb{Z}_M^d$, for all $t \in \mathbb{R}$ and for all $\varepsilon > 0$ the inequality*

$$\int_{\{\xi\in\mathbb{R} | t < \langle a(\xi), \eta_M(k)\rangle < t+\varepsilon\}} \left|\frac{\Psi_{M,k}(\xi)}{\delta_M(k)}\right|^2 \mathrm{d}\xi \leq C_4 \varepsilon^b$$

is satisfied.

On this assumption we can make the following remark

Remark 5.1. *We have not yet understood in detail to what extent Assumption 5.1 restricts the choice of A. We would like to have a constant C_4 that does not depend on M. In case $|a|$ is a bounded function there exists a constant $C \in \mathbb{R}$ such that*

$$\left|\frac{\Psi_{M,k}(\xi)}{\delta_M(k)}\right|^2 \leq C.$$

It then would be sufficient to demand

$$\iota(\varepsilon) := \sup_{\alpha\in\mathbb{R}, \beta\in\mathcal{S}^{d-1}} \lambda\left(\left\{\xi \in \mathbb{R} : \left|\alpha + \langle\beta, a(\xi)\rangle_{\mathbb{R}^d}\right| < \varepsilon\right\}\right) \leq C_4\varepsilon^b,$$

which is exactly the assumption made in [5]. Furthermore, we have

$$\lim_{M\to+\infty}\left|\frac{\Psi_{M,k}(\xi)}{\delta_M(k)}\right|^2=1,$$

which leads again to the assumption in [5].

Lemma 5.2. *Under Assumption 5.1 the function* $\|\cdot\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}$ *is a norm on the space* $L_0^2\left(\mathbb{T}_M^d\right)=\mathbb{R}_0^{\mathbb{T}_M^d}$.

Proof. It is not difficult to show that for all $U,V\in L_0^2\left(\mathbb{T}_M^d\right)$ we have $\|U\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}\geq 0$ and

$$\|U+V\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}\leq\|U\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}+\|V\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}.$$

Furthermore, we have $\|0\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}=0$ since $\mathcal{G}_M 0=0$. It remains to show that from $\|U\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}=0$ it follows that $U=0$. Thus, let us suppose that $\|U\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}=0$ for some $U\in L_0^2\left(\mathbb{T}_M^d\right)$. It is clear that for all $k\in\mathbb{Z}_M^d$ with $\zeta_M(k)\neq 0$ we have $\left(\mathcal{G}_M(U)\right)_k=0$. Since all M are odd there is only one $k\in\mathbb{Z}_M^d$ such that $\zeta_M(k)=0$, namely $k=0$. In this case we obtain

$$\left(\mathcal{G}_M U\right)_0=\sum_{x\in\mathbb{T}_M^d}U^x=0$$

since $U\in L_0^2\left(\mathbb{T}_M^d\right)$. □

Under the Assumptions 3.1 or 3.2 as well as 3.3 and 5.1 our final goal will be to estimate

$$\int_0^T\mathbb{E}\left(\|U_t\|_{\mathcal{H}^s\left(\mathbb{T}_M^d\right)}\right)\mathrm{d}t$$

for an appropriate s. This should allow us to verify the hypothesis of Theorem 2.5 which immediately assures the existence of an invariant measure μ_M for a fixed M. If we get an estimate that is uniform in M we hope that we are able to show convergence of μ_M to a probability measure μ for $M\to\infty$. We continue to follow [5] and estimate each term of the decomposition of U separately.

5.2. **Estimation of the first term.** In this subsection we discuss the estimation of U^0. For each $k\in\mathbb{Z}_M^d$ we define $b_{M,k}:\mathbb{R}\times[0,+\infty[\to\mathbb{R}$ by

$$b_{M,k}(\xi,t):=\frac{\langle a(\xi),\eta_M(k)\rangle_{\mathbb{R}^d}-t}{2\delta_M(k)}.$$

For every $k\in\mathbb{T}_M^d$ and every $\xi\in\mathbb{R}$ we define

$$g_{M,k,\xi}(t)=\frac{1}{2\delta_M(k)}\chi_{[0,+\infty[}\left(b_{M,k}(\xi,t)\right)\exp\left(b_{M,k}(\xi,t)\right)$$

and

$$h_{M,k,\xi}(t) = 2\frac{\langle |a|(\xi), \alpha_M(k)\rangle_{\mathbb{R}^d} + \delta_M(k)}{\left(\langle |a|(\xi), \alpha_M(k)\rangle_{\mathbb{R}^d} + \delta_M(k)\right)^2 + t^2}.$$

By Proposition 2.2 we obtain

$$\mathcal{F}(g_{M,k,\xi})(\tau) := \frac{1}{1 - 2\mathrm{i}\tau\delta_M(k)} \mathrm{e}^{-\mathrm{i}\tau\langle a(\xi), \eta_M(k)\rangle_{\mathbb{R}^d}}$$

and

$$\mathcal{F}(h_{M,k,\xi})(\tau) := \mathrm{e}^{-|\tau|\left(\langle |a|(\xi), \alpha_M(k)\rangle_{\mathbb{R}^d} + \delta_M(k)\right)}.$$

Furthermore, we define for every $k \in \mathbb{T}_M^d$ the functions $H_{M,k}, \tilde{H}_{M,k} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$H_{M,k}(t, \xi) := \mathrm{e}^{-t\left(\left\langle |a|(\xi), M\Gamma\left(M^{-1}k\right)\right\rangle_{\mathbb{R}^d} + \mathrm{i}\left\langle a(\xi), M\omega\left(M^{-1}k\right)\right\rangle_{\mathbb{R}^d} + \gamma\ell_M^{\alpha}(k) + \beta\right)}$$

and

$$\tilde{H}_{M,k}(\tau, \xi) := \mathrm{e}^{-\tau\left(\langle |a|(\xi), \alpha_M(k)\rangle_{\mathbb{R}^d} + \mathrm{i}\langle a(\xi), \eta_M(k)\rangle_{\mathbb{R}^d} + \delta_M(k)\right)}.$$

For every $k \in \mathbb{Z}_M^d \setminus \{0\}$ we substitute $\tau = \left\|M\omega\left(M^{-1}k\right)\right\|_{\mathbb{R}^d} t$ and apply the inequality $\mathrm{e}^{-a} \le \left(1 + a^2\right)^{-1}$ in order to obtain

$$\begin{aligned}
&\int_0^T \left| \int_{\mathbb{R}} H_{M,k}(t, \xi) \left(\mathcal{G}_M(F(0, \xi))\right)_k \mathrm{d}\xi \right|^2 \lambda(\mathrm{d}t) \\
&= \zeta_M^{-1}(k) \int_0^{\zeta_M(k)T} \mathrm{e}^{-2\tau\delta_M(k)} \left| \int_{\mathbb{R}} \tilde{H}_{M,k}(\tau, \xi) \left(\mathcal{G}_M(F(0, \xi))\right)_k \mathrm{d}\xi \right|^2 \mathrm{d}\tau \\
&\le \zeta_M^{-1}(k) \int_0^{+\infty} \mathrm{e}^{-2\tau\delta_M(k)} \left| \int_{\mathbb{R}} \tilde{H}_{M,k}(\tau, \xi) \left(\mathcal{G}_M(F(0, \xi))\right)_k \mathrm{d}\xi \right|^2 \mathrm{d}\tau \\
&\le \zeta_M^{-1}(k) \int_0^{+\infty} \left| \int_{\mathbb{R}} \frac{1}{1 - \mathrm{i}2\tau\delta_M(k)} \tilde{H}_{M,k}(\tau, \xi) \left(\mathcal{G}_M(F(0, \xi))\right)_k \mathrm{d}\xi \right|^2 \mathrm{d}\tau
\end{aligned}$$

Introducing for every $k \in \mathbb{T}_M^d$ and $\xi \in \mathbb{R}$ the functions $r_{M,k}, u_{M,k} : \mathbb{R} \to \mathbb{R}$ given by

$$r_{M,k,\xi}(s) := \int_{\mathbb{R}} g_{M,k,\xi}(s) \Psi_{M,k}(\xi) \left|\left(\mathcal{G}_M(F(0, \xi))\right)_k\right| \mathrm{d}\xi$$

and

$$u_{M,k}(s) := \frac{1}{|\delta_M(k)|^2 + s^2}$$

and applying the Plancherel theorem and Young's convolution inequality we obtain

$$\begin{aligned}
&\int_0^{+\infty}\left|\int_{\mathbb{R}}\frac{1}{1-\mathrm{i}2\tau\delta_M(k)}\tilde{H}_{M,k}(\tau,\xi)\left(\mathcal{G}_M F(0,\xi)\right)_k \mathrm{d}\xi\right|^2\mathrm{d}\tau\\
&\leq\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\mathcal{F}(g_{M,k,\xi})(\tau)\,\mathcal{F}(h_{M,k,\xi})(\tau)\left(\mathcal{G}_M F(0,\xi)\right)_k \mathrm{d}\xi\right|^2\mathrm{d}\tau\\
&=\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\mathcal{F}(g_{M,k,\xi}*h_{M,k,\xi})(\tau)\left(\mathcal{G}_M F(0,\xi)\right)_k \mathrm{d}\xi\right|^2\mathrm{d}\tau\\
&\leq\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(g_{M,k,\xi}*h_{M,k,\xi})(t)\left|\left(\mathcal{G}_M F(0,\xi)\right)_k\right|\mathrm{d}\xi\right)^2\mathrm{d}t\\
&\leq\left\|r_{M,k,\xi}*u_{M,k}\right\|_{L^2(\mathbb{R})}^2\leq\left\|r_{M,k,\xi}\right\|_{L^2(\mathbb{R})}^2\left\|u_{M,k}\right\|_{L^1(\mathbb{R})}^2.
\end{aligned}$$

We have got

$$\begin{aligned}
&\left\|u_{M,k}\right\|_{L^1(\mathbb{R})}^2\left\|r_{M,k,\xi}\right\|_{L^2(\mathbb{R})}^2\\
&=\left(\frac{\pi}{\delta_M(k)}\right)^2\int_{\mathbb{R}}\left|\int_{\mathbb{R}}g_{M,k,\xi}(s)\,\Psi_{M,k}(\xi)\left(\mathcal{G}_M\left(F(0,\xi)\right)\right)_k\mathrm{d}\xi\right|^2\mathrm{d}s\\
&=\pi^2\int_{\mathbb{R}}\left|\int_{\mathbb{R}}g_{M,k,\xi}(s)\,\frac{\Psi_{M,k}(\xi)}{\delta_M(k)}\left(\mathcal{G}_M\left(F(0,\xi)\right)\right)_k\mathrm{d}\xi\right|^2\mathrm{d}s
\end{aligned}$$

Now we can proceed as in the proof of the averaging Lemma [1, Lemma 2.4] and obtain

$$\begin{aligned}
&\int_0^T\left|\int_{\mathbb{R}}G_{M,k}(t,\xi)\left(\mathcal{G}_M\left(F(0,\xi)\right)\right)_k\mathrm{d}\xi\right|^2\lambda(\mathrm{d}t)\\
&\leq C\frac{1}{\zeta_M(k)}\left(\delta_M(k)\right)^{b-1}\int_{\mathbb{R}}\left|\left(\mathcal{G}_M\left(F(0,\xi)\right)\right)_k\right|^2\mathrm{d}\xi\\
&\leq C\gamma^{b-1}\left(\zeta_M(k)\right)^{-(2\alpha(1-b)+b)}\int_{\mathbb{R}}\left|\left(\mathcal{G}_M\left(F(0,\xi)\right)\right)_k\right|^2\mathrm{d}\xi.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
&\int_0^T\left\|U_{0,t}\right\|_{\mathcal{H}^{\alpha(1-b)+b}\left(\mathbb{T}_M^d\right)}\leq C\gamma^{b-1}\sum_{k\in\mathbb{Z}_M^d}\left|\int_{\mathbb{R}}\left(\mathcal{G}_M\left(F(0,\xi)\right)\right)_k\right|^2\mathrm{d}\xi\\
&=C\gamma^{b-1}\sum_{x\in\mathbb{Z}_M^d}\int_{\mathbb{R}}\left|\left(F(0,\xi)\right)_x\right|^2\mathrm{d}\xi=C\gamma^{b-1}\det M^{-1}\sum_{x\in\mathbb{T}_M^d}\left|U_0^x\right|\\
&\leq C\gamma^{b-1}\left\|U_0\right\|_{L^1\left(\mathbb{T}_M^d\right)},
\end{aligned}$$

where $C\in\mathbb{R}$ is a constant.

5.3. **Encountered difficulties and outlook on possible future work.** In this subsection we talk about difficulties we encountered during our work and we try to give an outlook on possible future work that can be done in continuation of this work.

It is most natural to continue following [5] and see if everything works out well in the discrete case. While there is hope that we can estimate U^1 using similar techniques as in the previous subsection, a result similar to [5, Lemma 10] is probably necessary in order to estimate U^2 and U^3. Before we discuss this lemma we introduce for every $t \in [0,+\infty[$ the functions $h_t^{\alpha,\beta} : \mathbb{R}^d \to \mathbb{R}$ and $K_t^{\alpha,\beta} : \mathbb{T}^d \to \mathbb{R}$ that we define by

$$h_t^{\alpha,\beta}(x) = \|x\|_{\mathbb{R}^d}^{\beta} \exp\left(-\|x\|_{\mathbb{R}^d}^{2\alpha}\right)$$

and

$$K_t^{\alpha,\beta}(x) := \sum_{k\in\mathbb{Z}^d} h_t^{\alpha,\beta}(k) \exp\left(2\pi \mathrm{i} \langle k, x\rangle_{\mathbb{R}^d}\right).$$

We denote $h^{\alpha,\beta} := h_1^{\alpha,\beta}$. With these notions we proceed by stating a lemma whichs proof can be found in the new, improved version of [5, Lemma 10]. See the discussion after Lemma 5.5 for more details on the new and the old version of [5, Lemma 10].

Lemma 5.3. *For every $\alpha \in]0,1]$, every $\beta \geq 0$ and every $d \in \mathbb{N}$ we have*

$$\left\|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\right\|_{L^1\left(\mathbb{R}^d\right)} < +\infty.$$

Keeping this lemma in mind we prove the following.

Lemma 5.4. *For every $\alpha \in]0,1]$, every $\beta \geq 0$, every $d \in \mathbb{N}$ every $1 \leq p \leq +\infty$ and all $t > 0$ the inequality*

$$\left\|(-\Delta)^{\beta/2} \exp\left(-t\left(-\Delta\right)^{\alpha}\right)\right\|_{L^p\left(\mathbb{T}^d\right)\to L^p\left(\mathbb{T}^d\right)} \leq \left\|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\right\|_{L^1\left(\mathbb{R}^d\right)} t^{-\beta/(2\alpha)}$$

holds.

Proof. Defining

$$K_t^{\alpha,\beta}(x) := \sum_{k\in\mathbb{Z}^d} h_t^{\alpha,\beta}(k) \mathrm{e}^{2\pi\mathrm{i}\langle k,x\rangle}$$

we obtain for $\varphi \in C^2\left(\mathbb{T}^d\right)$

$$\begin{aligned}\left(K_t^{\alpha,\beta} * \varphi\right)(x) &= \sum_{k\in\mathbb{Z}^d} h_t^{\alpha,\beta}(k)\mathcal{G}\varphi(k)\mathrm{e}^{2\pi\mathrm{i}\langle k,x\rangle} \\ &= t^{\beta/(2\alpha)}\left(-\Delta\right)^{\beta/2} \exp\left(-t\left(-\Delta\right)^{\alpha}\right)\varphi(x).\end{aligned}$$

The Poisson summation formula gives

$$K_t^{\alpha,\beta}(x) = \sum_{l\in\mathbb{Z}^d} \mathcal{F}^{-1}\left(h_t^{\alpha,\beta}\right)(l+x) = t^{-1/(2\alpha)} \sum_{l\in\mathbb{Z}^d} \mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\left(\frac{l+x}{t^{1/(2\alpha)}}\right).$$

Consequently, we have

$$\left\|K_t^{\alpha,\beta}\right\|_{L^1\left(\mathbb{T}^d\right)} \leq t^{-1/(2\alpha)} \int_{\mathbb{T}^d} \sum_{l\in\mathbb{Z}^d} \left|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\left(\frac{l+x}{t^{1/(2\alpha)}}\right)\right| \mathrm{d}x$$
$$= t^{-1/(2\alpha)} \int_{\mathbb{R}^d} \left|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\left(\frac{y}{t^{1/(2\alpha)}}\right)\right| \mathrm{d}y = \int_{\mathbb{R}^d} \left|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)(z)\right| \mathrm{d}z.$$

Finally, by Young's convolution inequality we have

$$\left\|(-\Delta)^{\beta/2} \exp\left(-t\left(-\Delta\right)^{\alpha}\right)\varphi\right\|_{L^p\left(\mathbb{T}^d\right)} \leq t^{-\beta/(2\alpha)} \left\|K_t^{\alpha,\beta}\right\|_{L^1\left(\mathbb{T}^d\right)} \|\varphi\|_{L^p\left(\mathbb{T}^d\right)}$$
$$\leq \left\|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\right\|_{L^1\left(\mathbb{R}^d\right)} t^{-\beta/(2\alpha)} \|\varphi\|_{L^p\left(\mathbb{T}^d\right)}.$$

By Lemma 5.3 we have $\left\|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\right\|_{L^1\left(\mathbb{R}^d\right)} < +\infty$. □

The old Lemma [5, Lemma 10] is more general than Lemma 5.4 and in the proof the function $H_t^{\alpha,\beta,per} : \mathbb{T}^d \to \mathbb{R}$ given by

$$H_t^{\alpha,\beta,per}(x) := \sum_{l\in\mathbb{Z}^d} \mathcal{F}^{-1}\left(h_t^{\alpha,\beta}\right)(l+x)$$

plays a role. In particular, the proof relies on the fact that for all $1 \leq q \leq +\infty$ the inequality $\left\|H_t^{\alpha,0,per}\right\|_{L^q\left(\mathbb{T}^d\right)} \leq \left\|\mathcal{F}^{-1}\left(h_t^{\alpha,0}\right)\right\|_{L^q\left(\mathbb{R}^d\right)}$ holds true. The following lemma shows an instance where this inequality is wrong.

Lemma 5.5. *There exists* $\alpha \in]0,1]$, $\beta \geq 0$, $d \in \mathbb{N}$, $1 \leq q \leq +\infty$ *and* $t \in [0,+\infty[$ *such that*

$$\left\|H_t^{\alpha,\beta,per}\right\|_{L^q\left(\mathbb{T}^d\right)} > \left\|\mathcal{F}^{-1}\left(h_t^{\alpha,\beta}\right)\right\|_{L^q\left(\mathbb{R}^d\right)}.$$

Proof. We prove this statement by contradiction, meaning we assume the opposite of the statement of the lemma is true. We have for $d = 1$ the equalities

$$\begin{aligned}
K_t^{1/2,0}(x) &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{-t|k|}\mathrm{e}^{2\pi\mathrm{i}kx} = \sum_{k=0}^{+\infty} \mathrm{e}^{-tk}\mathrm{e}^{2\pi\mathrm{i}kx} + \sum_{l=1}^{+\infty} \mathrm{e}^{-tl}\mathrm{e}^{-2\pi\mathrm{i}lx} \\
&= \frac{1}{1-\mathrm{e}^{-t}\mathrm{e}^{2\pi\mathrm{i}x}} + \frac{\mathrm{e}^{-t}\mathrm{e}^{-2\pi\mathrm{i}x}}{1-\mathrm{e}^{-t}\mathrm{e}^{-2\pi\mathrm{i}x}} \\
&= \frac{1-\mathrm{e}^{-t}\mathrm{e}^{-2\pi\mathrm{i}x} + \mathrm{e}^{-t}\mathrm{e}^{-2\pi\mathrm{i}x}\left(1-\mathrm{e}^{-t}\mathrm{e}^{2\pi\mathrm{i}x}\right)}{\left(1-\mathrm{e}^{-t}\mathrm{e}^{2\pi\mathrm{i}x}\right)\left(1-\mathrm{e}^{-t}\mathrm{e}^{-2\pi\mathrm{i}x}\right)} \\
&= \frac{1-\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}-\mathrm{e}^{-t}\left(\mathrm{e}^{2\pi\mathrm{i}x}+\mathrm{e}^{-2\pi\mathrm{i}x}\right)} \\
&= \left(\frac{\mathrm{e}^{t}-\mathrm{e}^{-t}}{2}\right)\left(\frac{\mathrm{e}^{t}+\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{2\pi\mathrm{i}x}+\mathrm{e}^{-2\pi\mathrm{i}x}}{2}\right)^{-1} \\
&= \frac{\sinh(t)}{\cosh(t)-\cos(2\pi x)}
\end{aligned}$$

Therefore $K_t^{1/2,0}(x) \to 1$ for $t \to \infty$ for every $x \in \mathbb{T}$. We have for $1 \le q \le +\infty$ that

$$\begin{aligned}
1 = \lim_{t\to\infty} K_t^{1/2,0}(1/2)\, |\mathbb{T}|^{1/q} &\le \lim_{t\to+\infty} \left\| K_t^{1/2,0} \right\|_{L^q(\mathbb{T})} \\
&\le \lim_{t\to+\infty} K_t^{1/2,0}(0)\, |\mathbb{T}|^{1/q} = 1.
\end{aligned}$$

By the Poisson summation formula we thus have for $1 < q \le +\infty$.

$$\begin{aligned}
1 &= \lim_{t\to+\infty} \left\| K_t^{1/2,0} \right\|_{L^q(\mathbb{T})} = \lim_{t\to+\infty} \left\| H_t^{1/2,0,per} \right\|_{L^q(\mathbb{T})} \\
&\le \lim_{t\to+\infty} \left\| \mathcal{F}^{-1}\left(h_t^{\alpha,0}\right) \right\|_{L^q(\mathbb{R})} = \lim_{t\to+\infty} t^{-1/q'} \left\| \mathcal{F}^{-1}\left(h^{\alpha,0}\right) \right\|_{L^q(\mathbb{R})} = 0.
\end{aligned}$$

This is clearly a contradiction. □

We contacted Mister Julien Vovelle, one of the authors of [5], to inform him about this error in the proof of [5, Lemma 10]. He quickly came up with a slight variation of the statement of [5, Lemma 10] and a proof for the new statement that still allows for usage in the sequel of [5]. We state the new lemma with its proof here.

Lemma 5.6. *For every $\alpha \in]0,1]$, every $\beta \ge 0$, every $d \in \mathbb{N}$ every $1 \le p \le q \le +\infty$ there exists a constant $C \in \mathbb{R}$ such that for all $t > 0$ and all $L_0^p\left(\mathbb{T}^d\right)$ the inequality*

$$\left\| (-\Delta)^{\beta/2} \exp\left(-t\,(-\Delta)^{\alpha}\right) v \right\|_{L^q\left(\mathbb{T}^d\right)} \le C t^{-d/(2\alpha)(1/p-1/q)-\beta/(2\alpha)} \left\| v \right\|_{L^p\left(\mathbb{T}^d\right)}$$

holds.

Proof. Let $\alpha \in]0,1]$, $\beta \geq 0$, $d \in \mathbb{N}$ and $1 \leq p \leq q \leq +\infty$. We define

$$\tilde{\beta} := d\left(\frac{1}{p} - \frac{1}{q}\right).$$

By the Sobolev inequality an Lemma 5.4 there exists a constant $\tilde{C} \in \mathbb{R}$ such that for all $t > 0$ and all $v \in L_0^p\left(\mathbb{T}^d\right)$ we have

$$\begin{aligned}
&\left\|(-\Delta)^{\beta/2} \exp\left(-t\left(-\Delta\right)^{\alpha}\right) v\right\|_{L^q\left(\mathbb{T}^d\right)} \\
&\leq \tilde{C}\left\|(-\Delta)^{\left(\tilde{\beta}+\beta\right)/2} \exp\left(-t\left(-\Delta\right)^{\alpha}\right) v\right\|_{L^P\left(\mathbb{T}^d\right)} \\
&\leq \tilde{C}\left\|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\right\|_{L^1\left(\mathbb{R}^d\right)} t^{-\left(\tilde{\beta}+\beta\right)/(2\alpha)} \|v\|_{L^p\left(\mathbb{T}^d\right)}.
\end{aligned}$$

□

We would like to find a discrete version of Lemma 5.4. For this purpose we define $\sqrt{\Gamma} : \mathbb{R}^d \to \mathbb{R}^d$ given by $\left(\sqrt{\Gamma}\left(z\right)\right)_j := \sqrt{2}\sqrt{\left(\Gamma\left(z\right)\right)_j}$. Furthermore, we introduce $g_{M,t}^{\alpha,\beta} : \mathbb{Z}^d \to \mathbb{C}$ given by

$$g_{M,t}^{\alpha,\beta}\left(z\right) := h^{\alpha,\beta}\left(t^{1/(2\alpha)} M \sqrt{\Gamma}\left(M^{-1}z\right)\right)$$

and $K_{M,t}^{\alpha,\beta} : \mathbb{T}_M^d \to \mathbb{C}$ given by

$$K_{M,t}^{\alpha,\beta}\left(x\right) = \sum_{k \in \mathbb{Z}_M^d} g_{M,t}^{\alpha,\beta}\left(k\right) \exp\left(2\pi \mathrm{i}\left\langle k, x\right\rangle_{\mathbb{R}^d}\right).$$

We define $g_M^{\alpha,\beta} := g_{M,1}^{\alpha,\beta}$ and

$$\sup_{S \succeq 0}\left(\sum_{x \in S^{-1}\mathbb{Z}^d}\left|\int_{S\mathbb{T}^d} g_S^{\alpha,\beta}(\xi) \mathrm{e}^{2\pi\mathrm{i}\langle x,\xi\rangle} \mathrm{d}\xi\right| \det S^{-1}\right) =: C_5 \in [0, +\infty].$$

It is an important question whether C_5 is or is not finite. A better understanding of the functions involved might help answering this question, which is why we added Figure 2. We also believe that we can show

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\right\|_{L^1\left(\mathbb{R}^d\right)} &= \lim_{S\to+\infty} \lim_{R\to+\infty} \sum_{x \in S^{-1}\mathbb{Z}^d}\left|\int_{R\mathbb{T}^d} g_R^{\alpha,\beta}(\xi) \mathrm{e}^{2\pi\mathrm{i}\langle x,\xi\rangle} \mathrm{d}\xi\right| \det S^{-1} \\
&= \lim_{R\to+\infty} \lim_{S\to+\infty} \sum_{x \in S^{-1}\mathbb{Z}^d}\left|\int_{R\mathbb{T}^d} g_R^{\alpha,\beta}(\xi) \mathrm{e}^{2\pi\mathrm{i}\langle x,\xi\rangle} \mathrm{d}\xi\right| \det S^{-1},
\end{aligned}$$

which indicates

$$\left\|\mathcal{F}^{-1}\left(h^{\alpha,\beta}\right)\right\|_{L^1\left(\mathbb{R}^d\right)} = \lim_{S\to+\infty} \sum_{x \in S^{-1}\mathbb{Z}^d}\left|\int_{S\mathbb{T}^d} g_S^{\alpha,\beta}(\xi) \mathrm{e}^{2\pi\mathrm{i}\langle x,\xi\rangle} \mathrm{d}\xi\right| \det S^{-1}.$$

We have not yet managed to show that $C_5 < +\infty$, but in case this is true we obtain the following lemma.

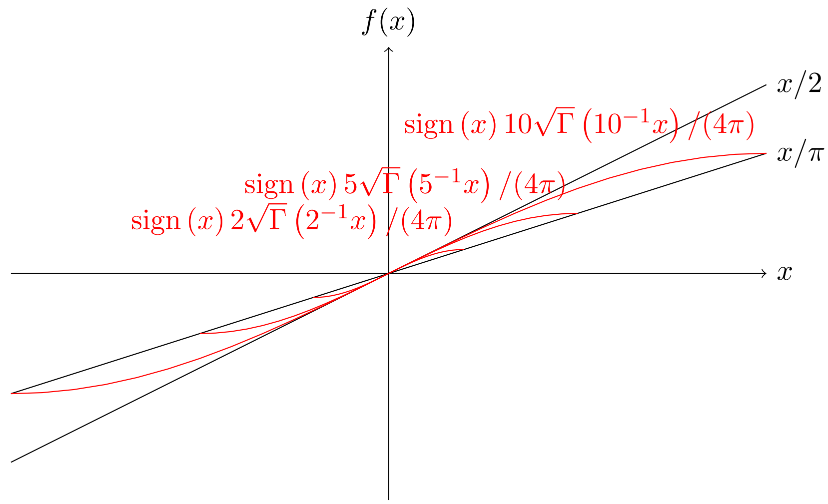

FIGURE 2. The function $z \mapsto \operatorname{sign}(z)\, M\sqrt{\Gamma}\left(M^{-1}z\right)$ for different M and scaled by $(4\pi)^{-1}$.

Lemma 5.7. *For every $\alpha \in]0,1]$, every $\beta \geq 0$, every $d \in \mathbb{N}$, every $1 \leq p \leq +\infty$, all $t > 0$ and all M the inequality*

$$\left\| (-\Delta_M)^{\beta/2} \exp\left(-t\left(-\Delta_M\right)^{\alpha}\right) \right\|_{L^p\left(\mathbb{T}_M^d\right) \to L^p\left(\mathbb{T}_M^d\right)} \leq C_5 t^{-\beta/(2\alpha)}$$

holds true.

Proof. For $\varphi \in C^2\left(\mathbb{T}_M^d\right)$ we have

$$\begin{aligned}\left(K_{M,t}^{\alpha,\beta} * \varphi\right)(x) &= \sum_{k \in \mathbb{Z}_M^d} g_{M,t}^{\alpha,\beta}(k) \mathcal{G}_M \varphi(k) \mathrm{e}^{2\pi \mathrm{i}\langle k,x\rangle} \\ &= t^{\beta/(2\alpha)} \left(-\Delta_M\right)^{\beta/2} \exp\left(-t\left(-\Delta_M\right)^{\alpha}\right) \varphi(x)\end{aligned}$$

The Poisson summation formula gives

$$K_{M,t}^{\alpha,\beta}(x) = \sum_{l \in \mathbb{Z}^d} \mathcal{H}_M\left(g_{M,t}^{\alpha,\beta}\right)(l+x) = t^{-1/(2\alpha)} \sum_{l \in \mathbb{Z}^d} \mathcal{H}_M\left(g_{M,t}^{\alpha,\beta}\right)\left(\frac{l+x}{t^{1/(2\alpha)}}\right)$$

Consequently, we obtain

$$\left\| K_{M,t}^{\alpha,\beta} \right\|_{L^1\left(\mathbb{T}_M^d\right)} \leq C_5.$$

Finally, by Young's convolution inequality we have

$$\begin{aligned}&\left\| (-\Delta_M)^{\beta/2} \exp\left(-t\left(-\Delta_M\right)^{\alpha}\right) \varphi \right\|_{L^p\left(\mathbb{T}^d\right)} \\ &\leq t^{-\beta/(2\alpha)} \left\| K_{M,t}^{\alpha,\beta} \right\|_{L^1\left(\mathbb{T}_M^d\right)} \|\varphi\|_{L^p\left(\mathbb{T}_M^d\right)} \leq t^{-\beta/(2\alpha)} C_5 \|\varphi\|_{L^p\left(\mathbb{T}_M^d\right)}.\end{aligned}$$

□

The following is an outlook on possible future work.

(1) Is it possible to weaken Assumption 3.3? What happens if we allow for dependence of σ on u?.
(2) A rigorous proof of 3.4 should in the authors opinion be added. In particular, we deal with a family of positive numbers Λ, while in [11] the family Λ is just one value. Furthermore, in our case the function σ might depend on U, while this is not the case in [11].
(3) The density argument used to obtain (21) needs to be rigorously executed.
(4) Assumption 5.1 should be studied. What happens when M has entries that are not odd? How restrictive is this assumption for a? What happens if we allow $b > 1$?
(5) Is an estimation of U^1 possible with the techniques used for the estimation of U^0.
(6) Is the constant C_5 finite?
(7) Does a discrete version of the new version of [5, Lemma 10] hold true? Is it even necessary in our discrete case?

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