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Habilitation Thesis:

Multiscale problems in mechanics of materials

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Contents

Introduction

A great amount of materials exhibit interesting behaviors, being the result of complex microstructures, the outcome of involved in-time evolutions, the effect of the action of internal variables, or the macroscopic counterpart of atomistic interactions. Understanding the interplay of such different material scales is thus a key problem in materials science. Indeed, it is crucial for the description of the physics behind phenomena, which are not yet fully understood, for the development of innovative metamaterials, as well as for the exploration of their industrial applications.

This habilitation thesis focuses on the analysis of phenomena arising in materials science and characterized by the presence of multiple scales, with techniques borrowed from the theory of partial differential equations and from the calculus of variations. The thesis is subdivided into three chapters, corresponding to three different research directions.

Chapter 1 is concerned with the mathematical description of composite materials, and with the identification of limiting effective models capturing the macroscopic behavior associated with the presence of different kinds of microstructures. I present here a selection of my papers in this setting.

The first part of **Chapter 1** is devoted to a result obtained in [21] in collaboration with Irene Fonseca (Carnegie Mellon University). Our analysis departs from the observation that in many applications, in order to establish the macroscopic behavior of a system presenting a periodic microstructure, we are led to the problem of finding integral representations for limits as ε goes to zero of oscillating integral energies

$$u_\varepsilon \mapsto \int_\Omega f\Big(x, \frac{x}{\varepsilon^\alpha}, u_\varepsilon(x)\Big)\,dx,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, and the fields u_ε are subjected to x-dependent differential constraints as

$$\sum_{i=1}^N A^i\Big(\frac{x}{\varepsilon^\beta}\Big)\frac{\partial u_\varepsilon(x)}{\partial x_i} \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l),\ 1<p<+\infty, \tag{0.1}$$

or in divergence form

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(A^i\Big(\frac{x}{\varepsilon^\beta}\Big)u_\varepsilon(x)\Big) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l),\ 1<p<+\infty, \tag{0.2}$$

with $A^i(x) \in \mathbb{M}^{l\times d}$ for every $x \in \mathbb{R}^N$, $i = 1,\cdots,N$, $d, l \geq 1$, and where α, β are two nonnegative parameters.

Oscillating divergence-type constraints as in (0.2) appear in the homogenization theory of systems of second order elliptic partial differential equations. Indeed, if $u_\varepsilon = \nabla v_\varepsilon$, with $v_\varepsilon \in W^{1,p}(\Omega)$ for every $\varepsilon > 0$, and $A^i(x) = A(x) \in \mathbb{M}^{N\times N}$ for $i = 1,\cdots,N$, then considering (0.2) reduces to the homogenization problem of finding the effective behavior of (weak) limits of v_ε, where

$$\operatorname{div}\Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla v_\varepsilon\Big) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega),\ 1<p<+\infty.$$

These problems have been extensively studied in the literature (see, e.g., [2], [9, Chapter 1, Section 6], [15], and the references therein).

In the first part of **Chapter 1** we present an analysis of the limit case in which $\alpha = 0$ and $\beta > 0$, namely the energy density is independent of the first two variables, and the fields $\{u_\varepsilon\}$ are subject to (0.2). The opposite limit scenario $\alpha > 0, \beta = 0$ and (0.1) (i.e., the energy density is oscillating but the differential constraint is fixed and in "non-divergence" form) is the subject of [22] (see also [23]). In particular, we analize the setting in which the coefficients A^i are nonconstant L^∞-maps, $A^i \in L^\infty(\mathbb{R}^N;\mathbb{M}^{l\times d})$ for every $i = 1,\ldots,N$, the energies under consideration are of the type

$$u_\varepsilon \mapsto \int_\Omega f(u_\varepsilon(x))\,dx,$$

where the energy density f satisfies standard p-growth assumptions, $u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^d)$, and

$$\mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(A^i\Big(\frac{x}{\varepsilon}\Big)u_\varepsilon(x)\Big) \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l)$$

for all $1 \leq q < p$. Our analysis includes the case when $q = p$ if the coefficients A^i are smooth. However, in the general situation when the maps A^i are only bounded, the assumption $1 \leq q < p$ is required, in order to satisfy some truncation and p-equiintegrability arguments. Our main results are a characterization of the limiting homogenized energy and the observation that, as opposed to the case in which the operators A^i are constant, the homogenized energy $\mathcal{F}_{\mathscr{A}}$ might not, in principle, be local, i.e., we can not expect that there exists $f_{hom} : \Omega \times \mathbb{R}^d \to [0,+\infty)$ such that

$$\mathcal{F}_{\mathscr{A}}(u) = \int_\Omega f_{hom}(x, u(x))\,dx. \tag{0.3}$$

We provide, in fact, an explicit example showing that locality in the sense of (0.3) may fail even when the function f is convex in its second variable.

The second part of **Chapter 1** focuses on the problem of identifying lower dimensional models describing thin three-dimensional structures. This is a classical question in mechanics of materials, which, since the early '90s, has been studied successfully by means of variational techniques. In particular, starting from the seminal papers [1, 29, 30, 32] hierarchies of limiting models have been deduced by Γ-convergence, depending on the scaling of the elastic energy with respect to the thickness parameter.

The first homogenization results in nonlinear elasticity have been proved in [10] and [37]. In these two papers, A. Braides and S. Müller assume p-growth of a stored energy density W that oscillates periodically in the in-plane direction. They show that, as the periodicity scale goes to zero, the elastic energy converges to a homogenized integral functional whose density is obtained by means of an infinite-cell homogenization formula.

In [7, 11] the authors treat simultaneously homogenization and dimension reduction for thin plates, in the membrane regime and under p-growth assumptions of the stored energy density. More recently, in [31], [38], and [44] models for homogenized plates have been derived under physical growth conditions for the energy density.

In the second part of **Chapter 1** we present a multiscale version of the results in [31] and [44]. Let

$$\Omega_h := \omega \times \left(-\tfrac{h}{2}, \tfrac{h}{2}\right)$$

be the reference configuration of a nonlinearly elastic thin plate, where ω is a bounded domain in $\mathbb{R}^2$, and $h > 0$ is the thickness parameter. We assume that the plate undergoes the action of two in-plane homogeneity scales: a coarser one, henceforth denoted by $\varepsilon(h)$, and a finer one, $\varepsilon^2(h)$, where $\{h\}$ and $\{\varepsilon(h)\}$ are monotone decreasing sequences of positive numbers, $h \to 0$ and $\varepsilon(h) \to 0$ as $h \to 0$. The rescaled nonlinear elastic energy is given by

$$\mathcal{J}^h(v) := \frac{1}{h} \int_{\Omega_h} W\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla v(x)\right) dx$$

for every deformation $v \in W^{1,2}(\Omega_h; \mathbb{R}^3)$, where the stored energy density W is periodic in its first two arguments and satisfies classical assumptions in nonlinear elasticity, as well as a nondegeneracy condition in a neighborhood of the set of proper rotations (see [31, 38, 44]). We focus on the scaling of the energy which corresponds to Kirchhoff's plate theory, and we consider sequences of deformations $\{v^h\} \subset W^{1,2}(\Omega_h; \mathbb{R}^3)$ verifying

$$\limsup_{h \to 0} \frac{\mathcal{J}^h(v^h)}{h^2} < +\infty. \tag{0.4}$$

The main result of this section, proved in [12] jointly with Laura Bufford (former PhD-student at Carnegie Mellon University) and Irene Fonseca, is an identification of the effective energies arising as limiting descriptions of the rescaled elastic energies $\left\{\frac{\mathcal{J}^h(v^h)}{h^2}\right\}$, and depending on the interaction of the two homogeneity scales with the thickness parameter.

The main difference with respect to [31] and [44] is in the structure of the homogenized energy densities, which are obtained by means of a double pointwise minimization, first with respect to the faster periodicity scale, and then with respect to the slower one, and to the x_3 variable.

The third and last part of **Chapter 1** is devoted to the mathematical modeling of metamaterials. These are artificially engineered composites whose heterogeneities are optimized in order to improve structural performances. Due to their special mechanical properties, arising as a result of complex microstructures, metamaterials play a key role in industrial applications and are an increasingly active field of research. Two natural questions when dealing with composite materials are how the effective material response is influenced by the geometric distribution of its components, and how the mechanical properties of the components impact the overall macroscopic behavior of the metamaterial.

In the result presented in this last part of **Chapter 1** (and obtained jointly with Carolin Kreisbeck (University of Utrecht) and Rita Ferreira (KAUST) in [20]), we investigate these questions for a special class of metamaterials with two characteristic features that are of relevance in a number of applications: (i) the material consists of two components arranged in a highly anisotropic way into periodically alternating layers, and (ii) the (elasto)plastic properties of the two components exhibit strong differences, in the sense that one is rigid, while the other one is considerably softer, thus allowing for large (elasto)plastic deformations.

The analysis of variational models for such layered high-contrast materials was initiated in [13]. There, the authors derive a macroscopic description for a two-dimensional model in the context of geometrically nonlinear but rigid elasticity, assuming that the softer component can be deformed along a single active slip system with linear self-hardening. These results have been extended to general dimensions, to energy densities with p-growth for $1 < p < +\infty$, and to the case with non-trivial elastic energies, which allows treating very stiff (but not necessarily rigid) layers, see [14].

In the third part of **Chapter 1** we carry the ideas of [13] forward to a model for plastic composites without linear hardening, in the spirit of [18], and we study the effective behavior of a two-dimensional variational model within finite crystal plasticity for high-contrast bilayered composites. Precisely, we consider materials arranged into periodically alternating thin horizontal strips of an elastically rigid component and a softer one with one active slip system. The energies arising from these modeling assumptions are of integral form, featuring linear growth and non-convex differential constraints. This change turns the variational problem in [13], having quadratic growth (cf. also [16, 17]), into one with energy densities that grow merely linearly.

The main novelty lies in the fact that the homogenization analysis must be performed in the class BV of functions of bounded variation (see [4]) to account for concentration phenomena. This gives rise to conceptual mathematical difficulties: on the one hand, the standard convolution techniques commonly used for density arguments in BV or SBV cannot be directly applied because they do not preserve the intrinsic constraints of the problem; on the other hand, constraint-preserving approximations in this weaker setting of BV are rather challenging, as one needs to simultaneously regularize the absolutely continuous part of the distributional derivative of the functions and accommodate their jump sets. A crucial first step in the asymptotic analysis is the characterization of rigidity properties of limits of admissible deformations in the space BV of functions of bounded variation. In particular, we prove that, under suitable assumptions, the two-dimensional body may split horizontally into finitely many pieces, each of which undergoes shear deformation and global rotation. This allows us to identify a potential candidate for the homogenized limit energy, which we show to be a lower bound on the Γ-limit. Our main result is to show, in the framework of non-simple materials, a complete Γ-convergence analysis, including an explicit homogenization formula, for a regularized model with an anisotropic penalization in the in-layer direction.

Chapter 2 of the thesis focuses on the emergence of Wulff shapes in crystallization problems. The content of **Chapter 2** is the subject of [24] and [25], and is based on a collaboration with Paolo Piovano (University of Vienna) and Ulisse Stefanelli (University of Vienna).

In the last decades an increasing interest has arisen for carbon-based materials, such as carbon nanotubes, fullerenes and ultra-thin graphite films, due to their unexpected electromagnetic properties, e.g., superconductivity and anomalous quantum Hall effects. One of the most promising materials (investigated among others by the Nobel prizes Geim and Novoselov) is graphene, which can be seen as the basic constituent of more complex carbon-based structures. This material ideally corresponds to a regular, two-dimensional layer of carbon atoms. Each atom is covalently bonded to three neighbors. These covalent bonds are of sp^2-hybridized type and ideally form $2\pi/3$ angles in a plane, so that graphene patches can be identified as subsets of an infinite hexagonal lattice.

In order to describe these bonds, some phenomenological interaction energies (including two- and three-body interaction terms) have been presented and partially validated. The arrangement of carbon atoms in the two-dimensional crystal emerges then as the global effect of the combination of local atomic interactions, and can be seen as the result of a geometric optimization process: by identifying the configuration of n carbon atoms with their positions $\{x_1, \ldots, x_n\} \subset \mathbb{R}^2$, one minimizes a given configurational energy $E : \mathbb{R}^{2n} \to \mathbb{R} \cup \{\infty\}$ and proves that the minimizers are indeed subsets of a regular hexagonal lattice. The configurational energies for carbon feature a decomposition $E = E_2 + E_3$ where E_2 corresponds to an attractive-repulsive two-body interaction, favouring some preferred spacing of the atoms, and E_3 encodes three-body interactions, expressing the specific geometry of sp^2 covalent bonding in carbon.

The above variational viewpoint brings the study of graphene geometries into the realm of the so-called crystallization problems. In the hexagonal setting, the crystallization problem for a finite number of carbon atoms is studied in [35] where the periodicity of ground states as well as the exact quantification of the ground-state energy is obtained.

In the first part of **Chapter 2** we present an equivalent characterization of graphene flakes as particle configurations maximizing a discrete "area" and minimizing a discrete "perimeter". Our analysis moves from the consideration that, as the configurational energy favors bonding, ground states are expected to have minimal perimeter, since boundary atoms have necessarily less neighbors. This heuristics is here made precise by providing a new identification of ground states based on a crystalline isoperimetric inequality. Indeed, we prove that ground states correspond to isoperimetric extremizers and we determine the exact isoperimetric constant. Analogous results had been obtained in [33, 34] for the square lattice. As a byproduct of our isoperimetric characterization we are able to investigate the edge geometry of graphene patches. Graphene atoms tend to naturally arrange themselves into hexagonal samples whose edges can have, roughly speaking, two shapes: they can either form zigzag or armchair structures. We prove here that hexagonal configurations having armchair edges do not satisfy the isoperimetric equality, whereas those with zigzag edges do.

The minimality of the ground-state perimeter gives rise to the emergence of large polygonal clusters as the number of atoms n increases. Indeed, one is interested in identifying a so-called Wulff shape to which all properly rescaled ground states converge. This had been successfully obtained for both the triangular [6, 39] and the square lattice [33, 34] beforehand, showing that ground states approach a hexagon and a square, respectively, as $n \to \infty$. Quite remarkably, in both the triangular and the square case it has been proved that ground states differ from the Wulff shape by at most $\mathrm{O}(n^{3/4})$ atoms, this bound being sharp. This is what is usually referred to as the $n^{3/4}$-law.

Relying on our novel discrete isoperimetric inequality, our main result is an analysis of the asymptotic behavior of graphene patches as the number of particles grows, proving their convergence to a limit macroscopic hexagonal Wulff shape. In particular, ground states with n atoms in two dimensional graphene sheets are shown to deviate from suitable hexagonal configurations with zigzag edges and from a limit hexagonal Wulff shape by at most $K_h n^{3/4} + \mathrm{o}(n^{3/4})$ particles. The constant K_h is explicitly computed and proved to be sharp.

A parallel analysis in the triangular lattice is presented in [25] and in the second part of **Chapter 2**, allowing to provide a characterization of minimizers of the so-called "edge-isoperimetric" problem, which plays a key role in the variational description of many classifications and clustering tasks. Extremizers of the edge-isoperimetric problem are shown to deviate from suitable hexagonal configurations in the triangular lattice and from the Wulff shape by at most $K_t n^{3/4} + \mathrm{o}(n^{3/4})$ particles. Our result provides a new, alternative proof of the $n^{3/4}$-law in the triangular lattice. As a by-product of our analysis an explicit sharp value for K_t is also identified.

Our estimates in the triangular and hexagonal lattice provide a measure in different topologies of the fluctuation of the isoperimetric configurations with respect to suitable hexagonal configurations.

The mathematical modeling of inelastic phenomena is a very active research area, at the triple point between mathematics, physics, and materials science. **Chapter 3** is devoted to two results related to the modeling of inelastic phenomena in a dynamic setting.

In the first part of **Chapter 3** we discuss a new approximation result for solutions to the problem of dynamic perfect plasticity for the classical Prandtl-Reuss model

$$\rho\ddot{u} - \nabla\cdot\sigma = 0, \tag{0.5}$$

$$\sigma = \mathbb{C}(Eu - p), \tag{0.6}$$

$$\partial H(\dot{p}) \ni \sigma_D \tag{0.7}$$

describing the plastic behavior of metals. In the expression above, $u(t) : \Omega \to \mathbb{R}^3$ is the (time-dependent) displacement of a body with reference configuration $\Omega \subset \mathbb{R}^3$ and density $\rho > 0$, and $\sigma(t) : \Omega \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ is its stress. Equation (0.5) describes conservation of momenta. The constitutive relation (0.6) relates the stress $\sigma(t)$ to the linearized strain $Eu(t) := (\nabla u(t) + \nabla u(t)^\top)/2 : \Omega \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ and the (deviatoric) plastic strain $p(t) : \Omega \to \mathbb{M}^{3\times 3}_D$ (deviatoric tensors) via the fourth-order elasticity tensor $\mathbb{C}$. Finally, the differential inclusion (0.7) expresses the plastic-flow rule: $H : \mathbb{M}^{3\times 3}_D \to [0, +\infty)$ is a positively 1-homogeneous, convex dissipation function, σ_D stands for the deviatoric part of the stress, and the symbol ∂ is the subdifferential in the sense of Convex Analysis. The system is driven by imposing a nonhomogeneous time-dependent boundary displacement.

Our main result, obtained in [26] jointly with Ulisse Stefanelli, consists in recovering weak solutions to the dynamic perfect plasticity system (0.5)-(0.7) by minimizing a sequence of parameter-dependent convex functionals over entire trajectories, and by passing to the limit as the parameter tends to zero. In particular, we consider the Weighted-Inertia-Dissipation-Energy (WIDE) functional of the form

$$I_\varepsilon(u,p) = \int_0^T \int_\Omega \exp\left(-\frac{t}{\varepsilon}\right) \left(\frac{\rho\varepsilon^2}{2}|\ddot{u}|^2 + \varepsilon H(\dot{p}) + \frac{1}{2}(Eu - p) : \mathbb{C}(Eu - p)\right) dx\, dt, \tag{0.8}$$

defined on suitable admissible classes of entire trajectories $t \in [0,T] \mapsto (u(t), p(t)) : \Omega \to \mathbb{R}^3 \times \mathbb{M}^{3\times 3}_D$ fulfilling given boundary-displacement and initial conditions (on u and p, respectively). The name of the functional reflects the fact that it is given by the sum of the inertial term $\rho|\ddot{u}|^2/2$, the dissipative term $H(\dot{p})$, and the energy term $(Eu - p) : \mathbb{C}(Eu - p)/2$, weighted by different powers of ε, as well as by the function $\exp(-t/\varepsilon)$.

For all $\varepsilon > 0$ one can prove that (a suitable relaxation of) the convex functional I_ε admits minimizers $(u^\varepsilon, p^\varepsilon)$ which indeed approximate solutions to the dynamic perfect plasticity system (0.5)-(0.7). In particular, by computing the corresponding Euler-Lagrange equations one finds that the minimizers $(u^\varepsilon, p^\varepsilon)$ weakly solve the elliptic-in-time approximating relations

$$\varepsilon^2\rho\ddot{\ddot{u}}^\varepsilon - 2\varepsilon^2\rho\dddot{u}^\varepsilon + \rho\ddot{u}^\varepsilon - \nabla\cdot\sigma^\varepsilon = 0, \tag{0.9}$$

$$\sigma^\varepsilon = \mathbb{C}(Eu^\varepsilon - p^\varepsilon), \tag{0.10}$$

$$-\varepsilon(\partial H(\dot{p}^\varepsilon))^\bullet + \partial H(\dot{p}^\varepsilon) \ni \sigma^\varepsilon_D, \tag{0.11}$$

complemented by Neumann conditions at the final time T.

The dynamic perfect plasticity system (0.5)-(0.7) is formally recovered by taking $\varepsilon \to 0$ in system (0.9)-(0.11). The main result presented in the first part of **Chapter 3** consists in making this intuition rigorous, resulting in a new approximation theory for dynamic perfect plasticity.

Existence results for (0.5)-(0.7) are indeed quite classical. In the dynamic case $\rho > 0$ both the first existence results due to Anzellotti and Luckhaus [5] and their recent revisiting by Babadjian and Mora [8] are based on viscosity techniques. With respect to the available existence theories our approach is new, for it does not rely on viscous approximation but rather on a global variational method.

We briefly outline the main steps of the proof. First, by time discretization we prove a uniform energy estimate for minimizers of the WIDE functionals selected via time-discrete to continuum Γ-convergence. This uniform upper bound allows to deduce compactness and convergence of the sequence of ε-dependent weak solutions to (0.9)-(0.11) to weak solutions to (0.5)-(0.7). A key point in our argument is to show that the limit stress and plastic strain satisfy (0.7). This indeed does not follow directly by the uniform energy estimate but is rather obtained by proving a delicate ε-dependent energy equality. The proof of this last result follows closely the strategy of [41, Theorem 2.5 (c)]. The main additional difficulties in our setting are due to the linear growth of the dissipation function.

The WIDE approach in the dynamic case $\rho > 0$ has been the object of a long-standing conjecture by De Giorgi on semilinear waves [28]. The conjecture was proved in [42] for finite-time intervals and then by Serra and Tilli in [40] for the whole time semiline, that is in its original formulation. De Giorgi himself pointed out in [28] the interest of extending the method to other dynamic problems. The result presented in the first part of **Chapter 3** delivers the first realization of De Giorgi's suggestion in the context of Continuum Mechanics.

The second part of **Chapter 3** concerns a system of PDEs and differential inclusions describing the combination of linearized perfect plasticity and damage effects in a dynamic setting for viscoelastic media. This analysis has been performed jointly with Ulisse Stefanelli and Tomáš Roubíček (Czech Academy of Sciences and Charles University) in [27].

Plasticity and damage are inelastic phenomena providing the macroscopical evidence of defect formation and evolution at the atomistic level. Plasticity results from the accumulation of slip defects (dislocations), which determine the behavior of a body to change from elastic and reversible to plastic and irreversible, once the magnitude of the stress reaches a certain threshold and a plastic flow develops. Damage evolution originates from the formation of cracks and voids in the microstructure of the material.

A vast literature concerning damage in viscoelastic materials, both in the quasistatic and the dynamical setting is currently available. We refer, e.g., to [36] and the references therein for an overview of the main results.

The focus of the second part of **Chapter 3** is on providing a rigorous analysis of an isothermal and isotropic model for viscoelastic media combining both small-strain perfect plasticity and damage effects in a dynamic setting.

A motivation for tackling the simultaneous occurrence of dynamical perfect plasticity and damaging is the mathematical modeling of cataclasite zones in geophysics. During fast slips, lithospheric faults in elastic rocks tend to emit elastic (seismic) waves, which in turn determine the occurrence of (tectonic) earthquakes, and the local arising of cataclasis. This latter phenomenon consists in a gradual fracturing of mineral grains into core zones of lithospheric faults, which tend to arrange themselves into slip bands, sliding plastically on each other without further fracturing of the material. On the one hand, cataclasite core zone are often very narrow (sometimes centimeters wide) in comparison with the surrounding compact rocks (which typically extend for many kilometers), and can be hence modeled for rather small time scales (minutes of ongoing earthquakes or years between them, rather than millions of years) via small-strain perfect (no-gradient) plasticity. On the other hand the partially damaged area surrounding the thin cataclasite core can be relatively wide, and thus calls for a modeling via gradient-damage theories).

The novelty of the contribution presented in the second part of **Chapter 3** is threefold. First, we extend the mathematical modeling of damage-evolution effects to an inelastic setting. Second, we characterize the interaction between damage onset and plastic slips formation in the framework of perfect plasticity, with no gradient regularization and in the absence of hardening. Third, we complement the study of dynamic perfect plasticity, by keeping track of the effects of damage both on the plastic yield surface, and on the viscoelastic behavior of the material.

The analysis of the model considered in the second part of **Chapter 3** presents several technical challenges. Perfect plasticity allows for plastic strain concentrations along the (possibly infinitesimally thin) slip-bands and calls for weak formulations in the spaces of bounded Radon measures for plastic strains and bounded-deformation (BD) for displacements (see, e.g., [43]). This requires a delicate notion of stress-strain duality. Considering inertia and the related kinetic energy renders the analysis quite delicate because of the interaction of possible elastic waves with nonlinearly responding slip bands.

The proof strategy relies on a staggered discretization scheme, in which at each time-step we first identify the damage variable as a solution to the damage evolution equation, and we then determine the plastic strain and elastic displacements as minimizer of a damage-dependent energy inequality. The strong convergence of the time-discrete elastic strains, needed for the limit passage in the damage flow rule, relies on a non-standard higher order test. The convergence of the elastic strains is then achieved by means of a delicate limsup estimate. The flow rule is recovered, in the limit, in the form of an energy balance.

The thesis is organized as follows: **Chapter** 1 is based on the papers [21, 12, 20]. The content of **Chapter** 2 are the two publications [24, 25]. **Chapter** 3 involves the two works [26, 27].

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Chapter 1

Effective theories for composite materials

This chapter consists of the following papers:

1) E. Davoli, I. Fonseca.
Homogenization of integral energies under periodically oscillating differential constraints.
Calc. Var. Partial Differential Equations **55** (2016), 1–60.

2) L. Bufford, E. Davoli, I. Fonseca.
Multiscale homogenization in Kirchhoff's nonlinear plate theory.
Math. Models Methods Appl. Sci. **25** (2015), 1765–1812.

3) E. Davoli, R.A. Ferreira, C.C. Kreisbeck.
Homogenization in BV of a model for layered composites in finite crystal plasticity.
Submitted, 2019. *Preprint arXiv:1901.11517.*

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Calculus of Variations

Homogenization of integral energies under periodically oscillating differential constraints

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Abstract A homogenization result for a family of integral energies

$$u_\varepsilon \mapsto \int_\Omega f(u_\varepsilon(x))\,dx, \quad \varepsilon \to 0^+,$$

is presented, where the fields u_ε are subjected to periodic first order oscillating differential constraints in divergence form. The work is based on the theory of $\mathscr{A}$-quasiconvexity with variable coefficients and on two-scale convergence techniques.

Mathematics Subject Classification 49J45 · 35D99 · 49K20

1 Introduction

This paper is the first step toward a complete understanding of homogenization problems for oscillating energies subjected to oscillating linear differential constraints, in the framework of $\mathscr{A}$-quasiconvexity with variable coefficients. To be precise, we initiate the study of integral representations for limits of oscillating integral energies

$$u_\varepsilon \mapsto \int_\Omega f\left(x, \frac{x}{\varepsilon^\alpha}, u_\varepsilon(x)\right) dx,$$

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where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $\varepsilon \to 0^+$, and the fields $u_\varepsilon \in L^p(\Omega; \mathbb{R}^d)$ are subjected to periodically oscillating differential constraints such as

$$\mathscr{A}_\varepsilon u_\varepsilon := \sum_{i=1}^N A^i\left(\frac{x}{\varepsilon^\beta}\right)\frac{\partial u_\varepsilon(x)}{\partial x_i} \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l), \tag{1.1}$$

or in divergence form

$$\mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon := \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(A^i\left(\frac{x}{\varepsilon^\beta}\right)u_\varepsilon(x)\right) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l), \tag{1.2}$$

with $1 < p < +\infty$, $A^i(x) \in \mathrm{Lin}\,(\mathbb{R}^d; \mathbb{R}^l) \equiv \mathbb{M}^{l\times d}$ for every $x \in \mathbb{R}^N$, $i = 1, \dots, N$, $d, l \geq 1$, and where α, β are two nonnegative parameters. Here, and in what follows, $\mathbb{M}^{l\times d}$ stands for the linear space of matrices with l rows and d columns.

Oscillating divergence-type constraints as in (1.2) appear in the homogenization theory of systems of second order elliptic partial differential equations. Indeed, if $u_\varepsilon = \nabla v_\varepsilon$, with $v_\varepsilon \in W^{1,p}(\Omega)$ for every $\varepsilon > 0$, and $A^i(x) = A(x) \in \mathbb{M}^{N\times N}$ for $i = 1, \dots, N$, then considering (1.2) reduces to the homogenization problem of finding the effective behavior of (weak) limits of v_ε, where

$$\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla v_\varepsilon\right) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega),\ 1 < p < +\infty.$$

These problems have been extensively studied in the literature (see e.g. [2], [6, Chapter 1, Section 6], [10], and the references therein). Similar differential constraints play a key role also in optimal design and minimum compliance analysis. In fact if $l = N = 3$, $d = 9$, if $u^\varepsilon = e^\varepsilon \in L^2(\Omega; \mathbb{M}^{3\times 3})$ represent linearized elastic strains associated to Ω, and

$$[A^i(x)\xi]_j := [\mathbb{C}(x)\xi]_{ij} \quad \text{for } i, j = 1, \dots, 3,$$

where $\mathbb{C}$ is a positive definite, linearized elasticity tensor associated to Ω, then (1.2) leads to the effective behavior of elastic quasi-equilibria e^ε satisfying

$$\mathrm{div}\left(\mathbb{C}\left(\frac{x}{\varepsilon^\beta}\right)e^\varepsilon(x)\right) \to 0 \quad \text{strongly in } W^{-1,2}(\Omega; \mathbb{R}^3).$$

We refer to, e.g., [5] for an overview on this kind of problems.

Different regimes are expected to arise depending on the relation between α and β. Here we will consider $\beta > 0$, and we will assume that the energy density f is constant in the first two variables but the differential constraint in divergence form (1.2) oscillates periodically. The limit scenario $\alpha > 0$, $\beta = 0$ and (1.1) (treated in [14] for constant coefficients), i.e., the energy density is oscillating but the differential constraint is fixed is analyzed in [13]. The situation in which $\alpha > 0$ and $\beta > 0$, will be the subject of forthcoming papers.

The key tool for our analysis is the notion of $\mathscr{A}$-quasiconvexity. For $i = 1 \dots, N$, consider matrix-valued maps $A^i \in C^\infty(\Omega; \mathbb{M}^{l\times d})$, and define $\mathscr{A}$ as the differential operator such that

$$\mathscr{A} v(x) := \sum_{i=1}^N A^i(x)\frac{\partial v(x)}{\partial x_i}, \quad x \in \Omega,$$

for $v \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$, where $\frac{\partial v}{\partial x_i}$ is to be interpreted in the sense of distributions. We require that the operator $\mathscr{A}$ satisfies a uniform constant-rank assumption (see [20]) i.e., there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank}\left(\sum_{i=1}^{N} A^i(x)w_i\right) = r \quad \text{for every } w \in \mathbb{S}^{N-1}, \tag{1.3}$$

uniformly with respect to x, where $\mathbb{S}^{N-1}$ is the unit sphere in $\mathbb{R}^N$. The properties of $\mathscr{A}$-quasiconvexity in the case of constant coefficients were first investigated by Dacorogna in [11], and then studied by Fonseca and Müller in [16] (see also [12]). In [23] Santos extended the analysis of [16] to the case in which the coefficients of the differential operator $\mathscr{A}$ depend on the space variable.

Definition 1.1 Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function, let Q be the unit cube in $\mathbb{R}^N$ centered at the origin,

$$Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^N,$$

and denote by $C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ the set of smooth maps which are Q-periodic in $\mathbb{R}^N$. Consider the set

$$\mathcal{C}_x := \left\{ w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) : \int_Q w(y)\,dy = 0, \ \sum_{i=1}^{N} A^i(x) \frac{\partial w(y)}{\partial y_i} = 0 \right\}.$$

For a.e. $x \in \Omega$, the *$\mathscr{A}$-quasiconvex envelope* of f in $x \in \Omega$ is defined as

$$\xi \mapsto Q_{\mathscr{A}} f(x, \xi) := \inf \left\{ \int_Q f(\xi + w(y))\,dy : w \in \mathcal{C}_x \right\}.$$

f is said to be *$\mathscr{A}$-quasiconvex* if $f(\xi) = Q_{\mathscr{A}} f(x, \xi)$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$.

We remark that when $\mathscr{A} :=$ curl, i.e., when $v = \nabla\phi$ for some $\phi \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^m)$, then $d = m \times N$, then $\mathscr{A}$-quasiconvexity reduces to Morrey's notion of quasiconvexity (see [1,4,18,19]).

The following theorem was proved in [23] in the more general case when f is a Carathéodory function, generalizing the corresponding results [16, Theorems 3.6 and 3.7] in the case of constant coefficients (i.e. $A^i(x) \equiv A^i \in \mathbb{M}^{l\times d}$ for every $i = 1, \ldots, N$).

Theorem 1.2 *Let Ω be an open bounded domain in $\mathbb{R}^N$, let $A^i \in C^\infty(\Omega; \mathbb{M}^{l\times d}) \cap W^{1,\infty}(\Omega; \mathbb{M}^{l\times d})$, $i = 1, \ldots, N$, $d \geq 1$, $1 < p < +\infty$, and assume that the operator $\mathscr{A}$ satisfies the constant rank condition (1.3). Let $f : \mathbb{R}^d \to [0, +\infty)$ be a continuous function satisfying*

(i) $0 \leq f(v) \leq C(1 + |v|^p)$,
(ii) $|f(v_1) - f(v_2)| \leq C(1 + |v_1|^{p-1} + |v_2|^{p-1})|v_1 - v_2|$

for all $v, v_1, v_2 \in \mathbb{R}^d$, and for some $C > 0$. Then $\mathscr{A}$-quasiconvexity is a necessary and sufficient condition for lower semicontinuity of the functional

$$v \mapsto \int_\Omega f(v(x))\,dx$$

for sequences $v_\varepsilon \rightharpoonup v$ weakly in $L^p(\Omega; \mathbb{R}^d)$ and such that $\mathscr{A}v_\varepsilon \to 0$ strongly in $W^{-1,p}(\Omega; \mathbb{R}^l)$.

In the case of constant coefficients, Braides et al. [7] provided an integral representation formula for relaxation problems in the context of $\mathscr{A}$-quasiconvexity and presented (via Γ-convergence) homogenization results for periodic integrands evaluated along $\mathscr{A}$-free fields. Their homogenization results were later generalized in [14], where Fonseca and Krömer worked still in the framework of constant coefficients but under weaker assumptions on the energy density f.

This paper is devoted to extending the previous homogenization results to the case in which $\mathscr{A}$ is a differential operator with nonconstant L^∞-coefficients, the energies under consideration are of the type

$$u_\varepsilon \mapsto \int_\Omega f(u_\varepsilon(x))\,dx,$$

where $u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^d)$, and

$$\mathscr{A}_\varepsilon^{\text{div}} u_\varepsilon := \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(A^i\left(\frac{x}{\varepsilon}\right)u_\varepsilon(x)\right) \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l)$$

for all $1 \le q < p$. We point out that the result in Theorem 1.2 [23] covers the case $q = p$. Our analysis includes the case when $q = p$ if the operator $\mathscr{A}$ has smooth coefficients. However, in the general situation when $\mathscr{A}$ has bounded coefficients, the assumption $1 \le q < p$ is required, in order to satisfy some truncation and p-equiintegrability arguments (see the proofs of Theorems 4.2, 5.1).

Our starting point is a characterization of the set $\mathcal{C}^{\mathscr{A}}$ of limits of $\mathscr{A}_\varepsilon^{\text{div}}$-vanishing fields u_ε. We show in Proposition 3.5 that a function $u \in L^p(\Omega;\mathbb{R}^d)$ belongs to $\mathcal{C}^{\mathscr{A}}$ if and only if there exists a map $w \in L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^d))$ such that $\int_Q w(x,y)\,dy = 0$ for a.e. $x \in \Omega$,

$$u_\varepsilon \overset{2-s}{\to} u + w$$

strongly two-scale in $L^p(\Omega \times Q;\mathbb{R}^d)$ (see Definition 2.1), and $u+w$ satisfies the differential constraints

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)(u(x)+w(x,y))\,dy\right) = 0 \tag{1.4}$$

in $W^{-1,p}(\Omega;\mathbb{R}^l)$, and

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}(A^i(y)(u(x)+w(x,y))) = 0 \tag{1.5}$$

in $W^{-1,p}(Q;\mathbb{R}^l)$ for a.e. $x \in \Omega$. This generalizes the classical characterization of 2-scale limits of solutions to linear elliptic partial differential equations in divergence form in [2, Theorem 2.3] to the case of first order linear systems.

For every $u \in \mathcal{C}^{\mathscr{A}}$, we denote by $\mathcal{C}_u^{\mathscr{A}}$ the class of maps w as above. We then prove that the homogenized energy is given by the functional

$$\mathscr{F}_{\mathscr{A}}(u) := \begin{cases} \inf_{r>0} \inf\left\{ \liminf_{n\to+\infty} \overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(u_n) : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right\} \\ \text{if } u \in \mathcal{C}^{\mathscr{A}}, \\ +\infty \quad \text{otherwise in } L^p(\Omega;\mathbb{R}^d), \end{cases}$$

where

$$\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(v) := \begin{cases} \inf\left\{ \int_\Omega f(v(x)+w(x,y))\,dy\,dx : w \in \mathcal{C}_v^{\mathscr{A}(n\cdot)},\ \|w\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \leq r \right\} \\ \text{if } v \in \mathcal{C}_r^{\mathscr{A}(n\cdot)}, \\ +\infty \quad \text{otherwise in } L^p(\Omega;\mathbb{R}^d), \end{cases}$$

the classes $\mathcal{C}_v^{\mathscr{A}(n\cdot)}$ are defined analogously to $\mathcal{C}_v^{\mathscr{A}}$ by replacing the operators $A^i(\cdot)$ with $A^i(n\cdot)$ in (1.4) and (1.5), and

$$\mathcal{C}_r^{\mathscr{A}(n\cdot)} := \{v \in L^p(\Omega;\mathbb{R}^d) : \exists w \in \mathcal{C}_v^{\mathscr{A}(n\cdot)} \ \text{ with } \|w\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \leq r\}, \quad r > 0.$$

Our main result is the following.

Theorem 1.3 *Let* $1 < p < +\infty$*. Let* $A^i \in L^\infty(Q;\mathbb{M}^{l\times d})$*,* $i = 1,\dots,N$*, and let* $f : \mathbb{R}^d \to [0,+\infty)$ *be a continuous map satisfying the growth condition*

$$0 \leq f(v) \leq C(1+|v|^p) \ \textit{ for every } v \in \mathbb{R}^d, \quad \textit{and some } C > 0.$$

Then, for every $u \in \mathcal{C}^{\mathscr{A}}$ *there holds*

$$\begin{aligned} &\inf\left\{ \liminf_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \ \ \textit{weakly in } L^p(\Omega;\mathbb{R}^d) \right. \\ &\qquad \left. \textit{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \ \ \textit{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \textit{ for every } 1 \leq q < p \right\} \\ &= \inf\left\{ \limsup_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \ \ \textit{weakly in } L^p(\Omega;\mathbb{R}^d) \right. \\ &\qquad \left. \textit{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \ \ \textit{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \textit{ for every } 1 \leq q < p \right\} = \mathscr{F}_{\mathscr{A}}(u). \end{aligned}$$

Remark 1.4 (i) As a consequence of Theorem 1.2, we expected the homogenized energy to be related to the effective energy for an "$\mathscr{A}$-quasiconvex" envelope of the function f, with the role of the differential constraint $\mathscr{A}$ to be replaced by the limit constraints (1.4) and (1.5). We stress the fact that here the oscillatory behavior of the differential constraint as $\varepsilon \to 0$ forces the relaxation with respect to (1.4) and (1.5) and the homogenization in the differential constraint to happen somewhat simultaneously. Indeed, for every n the functional $\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}$ is obtained as a truncated version of a relaxation with respect to the limit differential constraints dilated by a factor n, and is evaluated on a fixed element of a sequence of maps approaching u, whereas the limit functional $\mathscr{F}_{\mathscr{A}}(u)$ is deduced by a "diagonal" procedure, as n tends to $+\infty$.

(ii) The truncation in the definition of the functionals $\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}$ plays a key role in the proof of the limsup inequality

$$\begin{aligned} &\inf\left\{ \limsup_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \ \ \text{weakly in } L^p(\Omega;\mathbb{R}^d) \right. \\ &\qquad \left. \text{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \ \ \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \text{ for every } 1 \leq q < p \right\} \leq \mathscr{F}_{\mathscr{A}}(u), \end{aligned}$$

because it provides boundedness of the "recovery sequences" and thus allows us to apply a diagonalization argument (see Step 3 in the proof of Proposition 4.12).

(iii) The functional $\mathscr{F}_{\mathscr{A}}$ is identified, in general, by means of an asymptotic characterization (see Theorem 1.3). In Theorem 5.6 we prove that in the case in which f is convex this reduces to a non-asymptotic cell formula.

(iv) We remark that, as opposed to the case in which the operators A^i are constant, we cannot expect the homogenized energy to be local, i.e., that there exists $f_{hom} : \Omega \times \mathbb{R}^d \to [0, +\infty)$ such that

$$\mathcal{F}_{\mathscr{A}}(u) = \int_{\Omega} f_{hom}(x, u(x))\, dx. \tag{1.6}$$

We show in Example 5.7 that locality in the sense of (1.6) may fail even when the function f is convex.

As in [14], the proof of this result is based on the so-called *unfolding operator*, introduced in [8,9] (see also [24,25] and Sect. 2.2). A first difference with [14, Theorem 1.1] (i.e., with the case in which the operators A^i are constant) is the fact that we are unable to work with exact solutions of the system $\mathscr{A}_{\varepsilon}^{\text{div}} u_{\varepsilon} = 0$, but instead we consider sequences of asymptotically $\mathscr{A}_{\varepsilon}^{\text{div}}$-vanishing fields. As pointed out in [23], in the case of variable coefficients the natural framework is pseudo-differential operators. In this setting, we do not project directly onto the kernel of a differential constraint $\mathscr{A}$, but rather we construct an "approximate" projection operator P such that for every field $v \in L^p$, the $W^{-1,p}$ norm of $\mathscr{A} P v$ is controlled by the $W^{-1,p}$ norm of v itself (we refer to [23, Subsection 2.1] for a detailed explanation of this issue, and to the references therein for a treatment of the main properties of pseudo-differential operators).

The crucial difference with respect to the case of constant coefficients is the structure of the set $\mathcal{C}^{\mathscr{A}}$. In the case in which the condition $\mathscr{A}_{\varepsilon}^{\text{div}} u_{\varepsilon} \to 0$ is replaced by $\mathscr{A} u_{\varepsilon} = 0$, with $\mathscr{A}$ being independent of the space variable, (1.4) and (1.5) decouple (see [14, Theorem 1.2]) becoming separate requirements on w and u. However, in our situation they can not be dealt with separately, and this forces the structure of the homogenized energy to be much more involved.

The oscillatory behavior of the differential constraint and its ε-dependent structure render this problem quite technical due to the difficulty in obtaining a suitable projection operator on the limit differential constraint. Moreover, due to the coupling between (1.4) and (1.5) and the dependence of the operators on ε, the pseudo-differential operators method cannot be applied directly here. In order to solve this problem, in Lemma 3.3 we are led to impose a uniform invertibility requirement on the differential operator. To be precise, we require $l \times N = d$ and we assume that there exists a positive constant γ such that the operator $\mathcal{A}(y) \in \text{Lin}\,(\mathbb{R}^d; \mathbb{R}^d)$, defined as

$$\mathcal{A}(y)\xi := \begin{pmatrix} (A^1(y)\xi)^T \\ \vdots \\ (A^N(y)\xi)^T \end{pmatrix} \in \mathbb{M}^{N\times l} \cong \mathbb{R}^d \quad \text{for every } \xi \in \mathbb{R}^d,$$

satisfies

$$(H) \quad \mathcal{A}(y)\lambda \cdot \lambda \geq \gamma |\lambda|^2 \quad \text{for every } \lambda \in \mathbb{R}^d \text{ and } y \in \mathbb{R}^N.$$

We remark that assumption (H) is quite natural, as it represents a higher-dimensional version of the classical uniform ellipticity assumption (see e.g. [2, (2.2)]). We refer to Remark 3.1 for a discussion on the relationship between (H) and the constant rank assumption (1.3).

The strategy of our argument consists in first proving Theorem 1.3 in the case in which the operators A^i are smooth. The general case is then deduced by means of an approximation argument of bounded operators by smooth ones, and by an application of Severini–Egoroff's theorem and p-equiintegrability (see Sect. 5).

Our main theorem is consistent with the relaxation results obtained in [7] in the case of constant coefficients. When the linear operators A^i are constant, we prove in Sect. 5.1 that the homogenized energy $\mathscr{F}_{\mathscr{A}}$ and Theorem 1.3 reduce to the $\mathscr{A}$-quasiconvex envelope of f and [7, Theorem 1.1], respectively.

This article is organized as follows. In Sect. 2 we introduce notation and recall some preliminary results on two-scale convergence and on the unfolding operator. In Sect. 3 we provide a characterization of the limits of $\mathscr{A}_\varepsilon^{\mathrm{div}}$-vanishing fields (see Proposition 3.5). Section 4 is devoted to the proof of our main result, Theorem 1.3, for smooth operators $\mathscr{A}_\varepsilon^{\mathrm{div}}$. The argument is extended to the case in which $\mathscr{A}_\varepsilon^{\mathrm{div}}$ are only bounded in Sect. 5.

2 Preliminary results

Throughout this paper $\Omega \subset \mathbb{R}^N$ is an open bounded domain and $\mathcal{O}(\Omega)$ is the set of open subsets of Ω. Q is the unit cube in $\mathbb{R}^N$ centered at the origin and with normals to its faces parallel to the vectors in the standard orthonormal basis of $\mathbb{R}^N$, $\{e_1, \dots, e_N\}$, i.e.,

$$Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^N.$$

Given $1 < p < +\infty$, we denote by p' its conjugate exponent, that is

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Whenever a map $v \in L^p, C^\infty, \dots$, is Q-periodic, that is

$$v(x + e_i) = v(x) \quad i = 1, \dots, N$$

for a.e. $x \in \mathbb{R}^N$, we write $v \in L^p_{\mathrm{per}}, C^\infty_{\mathrm{per}}, \dots$, respectively. We will implicitly identify the spaces $L^p(Q)$ and $L^p_{\mathrm{per}}(\mathbb{R}^N)$. We designate the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$ by $|A|$. We adopt the convention that C will stand for a generic positive constant, whose value may change from expression to expression in the same formula.

2.1 Two-scale convergence

We recall the definition and some properties of two-scale convergence which apply to our framework. For a detailed treatment of the topic we refer to, e.g., [2,17,22]. Throughout this subsection $1 < p < +\infty$.

Definition 2.1 If $v \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d))$ and $\{u_\varepsilon\} \in L^p(\Omega; \mathbb{R}^d)$, we say that $\{u_\varepsilon\}$ *weakly two-scale converge to* v in $L^p(\Omega \times Q; \mathbb{R}^d)$, $u_\varepsilon \overset{2-s}{\rightharpoonup} v$, if

$$\int_\Omega u_\varepsilon(x) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Q v(x, y) \cdot \varphi(x, y)\, dy\, dx$$

for every $\varphi \in L^{p'}(\Omega; C_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d))$.

We say that $\{u_\varepsilon\}$ *strongly two-scale converge to* v in $L^p(\Omega \times Q; \mathbb{R}^d)$, $u_\varepsilon \overset{2-s}{\to} v$, if $u_\varepsilon \overset{2-s}{\rightharpoonup} v$ and

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega; \mathbb{R}^d)} = \|v\|_{L^p(\Omega \times Q; \mathbb{R}^d)}.$$

Bounded sequences in $L^p(\Omega;\mathbb{R}^d)$ are pre-compact with respect to weak two-scale convergence. To be precise (see [2, Theorem 1.2]),

Proposition 2.2 *Let $\{u_\varepsilon\}\subset L^p(\Omega;\mathbb{R}^d)$ be bounded. Then there exists $v\in L^p(\Omega;L^p_{\rm per}(\mathbb{R}^N;\mathbb{R}^d))$ such that, up to a subsequence, $u_\varepsilon \overset{2-s}{\rightharpoonup} v$ weakly two-scale, and, in particular,*

$$u_\varepsilon \rightharpoonup u := \int_Q v(x,y)\,dy \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d).$$

The following result will play a key role throughout the paper in the proofs of limsup inequalities (see [2, Lemma1.3], [25, Lemma2.1], and [14, Proposition 2.4, Lemma 2.5 and Remark 2.6]).

Proposition 2.3 *Let $v\in L^p(\Omega;C_{\rm per}(\mathbb{R}^N;\mathbb{R}^d))$ or $v\in L^p_{\rm per}(\mathbb{R}^N;C(\overline{\Omega};\mathbb{R}^d))$. Then the sequence $\{u_\varepsilon\}$, defined as*

$$u_\varepsilon(x) := v\left(x,\frac{x}{\varepsilon}\right)$$

is p-equiintegrable, and

$$u_\varepsilon \overset{2-s}{\to} v \quad \text{strongly two-scale in } L^p(\Omega;\mathbb{R}^d).$$

2.2 The unfolding operator

We collect here the definition and some properties of the *unfolding operator* (see e.g. [8,9, 24,25]).

Definition 2.4 Let $u\in L^p(\Omega;\mathbb{R}^d)$. For every $\varepsilon>0$ the unfolding operator $T_\varepsilon : L^p(\Omega;\mathbb{R}^d)\to L^p(\mathbb{R}^N;L^p_{\rm per}(\mathbb{R}^N;\mathbb{R}^d))$ is defined componentwise as

$$T_\varepsilon(u)(x,y) := u\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor + \varepsilon(y-\lfloor y\rfloor)\right) \quad \text{for a.e. } x\in\Omega \text{ and } y\in\mathbb{R}^N, \tag{2.1}$$

where u is extended by zero outside Ω and $\lfloor\cdot\rfloor$ denotes the least integer part.

Proposition 2.5 *T_ε is a nonsurjective linear isometry from $L^p(\Omega;\mathbb{R}^d)$ to $L^p(\mathbb{R}^N\times Q;\mathbb{R}^d)$.*

The next theorem relates the notion of two-scale convergence to L^p convergence of the unfolding operator (see [25, Proposition 2.5 and Proposition 2.7], [17, Theorem 10]).

Theorem 2.6 *Let Ω be an open bounded domain and let $v\in L^p(\Omega;L^p_{\rm per}(\mathbb{R}^N;\mathbb{R}^d))$. Assume that v is extended to be 0 outside Ω. Then the following conditions are equivalent:*

(i) $u^\varepsilon \overset{2-s}{\rightharpoonup} v$ *weakly two-scale in* $L^p(\Omega\times Q;\mathbb{R}^d)$,
(ii) $T_\varepsilon u_\varepsilon \rightharpoonup v$ *weakly in* $L^p(\mathbb{R}^N\times Q;\mathbb{R}^d)$.

Moreover,

$$u^\varepsilon \overset{2-s}{\to} v \quad \text{strongly two-scale in } L^p(\Omega\times Q;\mathbb{R}^d)$$

if and only if

$$T_\varepsilon u_\varepsilon \to v \quad \text{strongly in } L^p(\mathbb{R}^N\times Q;\mathbb{R}^d).$$

The following proposition is proved in [14, Proposition A.1].

Proposition 2.7 *If $u\in L^p(\Omega;\mathbb{R}^d)$ (extended by 0 outside Ω) then*

$$\|u-T_\varepsilon u\|_{L^p(\mathbb{R}^N\times Q;\mathbb{R}^d)} \to 0$$

as $\varepsilon\to 0$.

3 Characterization of limits of $\mathscr{A}_\varepsilon^{\mathrm{div}}$-vanishing fields

Let $1 < p < +\infty$, and for every $\varepsilon > 0$ denote by $\mathscr{A}_\varepsilon^{\mathrm{div}} : L^p(\Omega; \mathbb{R}^d) \to W^{-1,p}(\Omega; \mathbb{R}^l)$ the first order differential operator

$$\mathscr{A}_\varepsilon^{\mathrm{div}} u := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(A^i\left(\frac{x}{\varepsilon}\right)u(x)\right) \tag{3.1}$$

for $u \in L^p(\Omega; \mathbb{R}^d)$. In this section we focus on the case in which the operators A^i are smooth and Q-periodic, $A^i \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, for all $i = 1, \dots, N$. We will also require that $N \times l = d$, and for every $y \in \mathbb{R}^N$ the operator $\mathcal{A}(y) \in \mathrm{Lin}\,(\mathbb{R}^d; \mathbb{R}^d)$, defined as

$$\mathcal{A}(y)\xi := \begin{pmatrix} (A^1(y)\xi)^T \\ \vdots \\ (A^N(y)\xi)^T \end{pmatrix} \in \mathbb{M}^{N\times l} \cong \mathbb{R}^d \quad \text{for every } \xi \in \mathbb{R}^d, \tag{3.2}$$

satisfies the uniform ellipticity condition

$$\mathcal{A}(y)\lambda \cdot \lambda \geq \gamma|\lambda|^2 \quad \text{for every } \lambda \in \mathbb{R}^d \text{ and } y \in \mathbb{R}^N \tag{3.3}$$

where $\gamma > 0$ is a positive constant.

Remark 3.1 We observe that if $\mathscr{A}$ satisfies the uniform constant rank assumption (1.3) with $r = d$, then the linear operator $\mathcal{A}(y)$ defined in (3.2) is injective (and hence invertible, in view of the Rank Theorem).

Indeed, for $r = d$ property (1.3) yields

$$\sum_{i=1}^{N} A^i(y)w_i v = 0 \quad \text{if and only if } v = 0,$$

for every $y \in Q$, and $w \in \mathbb{S}^{N-1}$. In particular, choosing $w = e_i$, $i = 1, \dots, N$, we deduce that

$$A^i(y)\xi = 0 \quad \text{if and only if} \quad \xi = 0,$$

for every $i = 1, \dots, N$, and for all $y \in Q$. Thus $\mathcal{A}(y)\xi = 0$ if and only if $\xi = 0$. However, the constant rank assumption (1.3) with $r = d$ is not enough to guarantee that the uniform ellipticity condition (3.3) holds true.

We also notice that the converse implication is false, namely there exist first order operators satisfying (3.3) and with constant rank strictly less than d. The operator $\mathscr{A}$ defined in Sect. 5.3 provides an explicit example.

We first state a corollary of [16, Lemma 2.14].

Lemma 3.2 *Let $1 < p < +\infty$ and consider the differential operator*

$$\mathrm{div} : L^p(Q; \mathbb{R}^N) \to W^{-1,p}(Q)$$

defined as

$$\mathrm{div}\, R := \sum_{i=1}^{N} \frac{\partial R_i(y)}{\partial y_i} \quad \textit{for every } R \in L^p(Q; \mathbb{R}^N).$$

Then, there exists an operator

$$\mathbb{T} : L^p(Q;\mathbb{R}^N) \to L^p(Q;\mathbb{R}^N)$$

such that

(P1) $\mathbb{T}$ *is linear and bounded, and vanishes on constant maps,*
(P2) $\mathbb{T}\circ\mathbb{T}R = \mathbb{T}R$ *and* $\operatorname{div}(\mathbb{T}R) = 0$ *for every* $R \in L^p(Q;\mathbb{R}^N)$,
(P3) *there exists a constant* $C = C(p) > 0$ *such that*

$$\|R - \mathbb{T}R\|_{L^p(Q;\mathbb{R}^N)} \le C\|\operatorname{div}R\|_{W^{-1,p}(Q)},$$

for all $R \in L^p(Q;\mathbb{R}^N)$ *with* $\int_Q R(y)\,dy = 0$,
(P4) *if* $\{v^n\}$ *is bounded in* $L^p(\Omega\times Q;\mathbb{R}^N)$ *and p-equiintegrable in* $\Omega\times Q$, *then setting* $w^n(x,\cdot) := \mathbb{T}v^n(x,\cdot)$ *for a.e.* $x\in\Omega$, *the sequence* $\{w^n\}$ *is p-equiintegrable in* $\Omega\times Q$,
(P5) *if* $\psi \in C^1(\Omega; C^1_{per}(\mathbb{R}^N;\mathbb{R}^N)) \cap W^{2,2}(\Omega; W^{2,2}_{per}(\mathbb{R}^N;\mathbb{R}^N))$ *then setting* $\varphi(x,\cdot) := \mathbb{T}\psi(x,\cdot)$ *for every* $x \in \mathbb{R}^N$, *there holds* $\varphi \in C^1(\Omega; C^1_{per}(\mathbb{R}^N;\mathbb{R}^N))$.

Using the previous result we can prove the following projection lemma.

Lemma 3.3 *Let* $1 < p < +\infty$, *let* $A^i \in L^\infty(Q;\mathbb{M}^{l\times d})$, $i = 1,\dots,N$, *with* $\mathcal{A}$ *satisfying the invertibility requirement in* (3.3). *Let* $\{v^n\}, v \subset L^p(\Omega\times Q;\mathbb{R}^d)$ *be such that*

$$v^n \rightharpoonup v \quad \text{weakly in } L^p(\Omega\times Q;\mathbb{R}^d), \tag{3.4}$$

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)v^n(x,y)\,dy\right) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l), \tag{3.5}$$

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(A^i(y)v^n(x,y)\right) \to 0 \quad \text{strongly in } L^p(\Omega; W^{-1,p}(Q;\mathbb{R}^l)). \tag{3.6}$$

Then there exists a subsequence $\{v^{n_k}\}$ *and a sequence* $\{w^k\} \subset L^p(\Omega\times Q;\mathbb{R}^d)$ *such that*

$$v^{n_k} - w^k \to 0 \quad \text{strongly in } L^p(\Omega\times Q;\mathbb{R}^d), \tag{3.7}$$

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)w^k(x,y)\,dy\right) = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^l), \tag{3.8}$$

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(A^i(y)w^k(x,y)\right) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x\in\Omega. \tag{3.9}$$

Proof We first notice that (3.4)–(3.6) imply that

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)v(x,y)\,dy\right) = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^l),$$

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(A^i(y)v(x,y)\right) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x\in\Omega.$$

By linearity, it is enough to consider the case in which $v = 0$. Moreover, up to a translation and a dilation, we can assume that Ω is compactly contained in Q.

By the compact embedding of $L^p(\Omega;\mathbb{R}^d)$ into $W^{-1,p}(\Omega;\mathbb{R}^d)$, and by (3.4) and (3.5), for every $\varphi\in C_c^\infty(\Omega;[0,1])$ there holds

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)\varphi(x)v^n(x,y)\,dy\right)$$
$$=\sum_{i=1}^N \varphi(x)\frac{\partial}{\partial x_i}\left(\int_Q A^i(y)v^n(x,y)\,dy\right)+\sum_{i=1}^N\frac{\partial\varphi(x)}{\partial x_i}\left(\int_Q A^i(y)v^n(x,y)\,dy\right)\to 0$$

strongly in $W^{-1,p}(\Omega;\mathbb{R}^d)$. On the other hand, by (3.6),

$$\left\|\sum_{i=1}^N\frac{\partial}{\partial y_i}\Big(A^i(y)v^n(x,y)\Big)\right\|_{W^{-1,p}(Q;\mathbb{R}^l)}\to 0\quad\text{strongly in } L^p(\Omega).$$

Therefore, we may consider a sequence $\{\varphi_k\}\subset C_c^\infty(\Omega;[0,1])$ with $\varphi_k\nearrow 1$ and such that, setting $\tilde{v}_k^n:=\varphi_k v^n$ and extending $\tilde{v}_k^n$ by zero to $Q\backslash\Omega$ and then periodically, there holds

$$\tilde{v}_k^n\rightharpoonup 0\quad\text{weakly in } L^p(Q\times Q;\mathbb{R}^d),$$
$$\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(\int_Q A^i(y)\tilde{v}_k^n(x,y)\,dy\right)\to 0\quad\text{strongly in } W^{-1,p}(Q;\mathbb{R}^l),$$
$$\left\|\sum_{i=1}^N\frac{\partial}{\partial y_i}\Big(A^i(y)\tilde{v}_k^n(x,y)\Big)\right\|_{W^{-1,p}(Q;\mathbb{R}^l)}\to 0\quad\text{strongly in } L^p(Q),$$

as $n\to+\infty$, $k\to+\infty$.

By a diagonal argument we extract a subsequence $\hat{v}^k:=\tilde{v}_k^{n(k)}$ such that

$$\hat{v}^k\rightharpoonup 0\quad\text{weakly in } L^p(Q\times Q;\mathbb{R}^d),\tag{3.10}$$
$$\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(\int_Q A^i(y)\hat{v}^k(x,y)\,dy\right)\to 0\quad\text{strongly in } W^{-1,p}(Q;\mathbb{R}^l),\tag{3.11}$$
$$\left\|\sum_{i=1}^N\frac{\partial}{\partial y_i}\Big(A^i(y)\hat{v}^k(x,y)\Big)\right\|_{W^{-1,p}(Q;\mathbb{R}^l)}\to 0\quad\text{strongly in } L^p(Q).\tag{3.12}$$

Define the maps

$$R_i^k(x,y):=A^i(y)\hat{v}^k(x,y)\quad\text{for a.e. } x\in Q \text{ and } y\in Q,\quad i=1,\dots,N,$$

and let $R^k\in L^p(Q\times Q;\mathbb{R}^d)$ be given by

$$R_{ij}^k:=(R_i^k)_j,\quad\text{for all } i=1,\dots,N,\ j=1\dots,l.$$

By (3.10)–(3.12),

$$R^k\rightharpoonup 0\quad\text{weakly in } L^p(Q\times Q;\mathbb{R}^d),\tag{3.13}$$
$$\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(\int_Q R_i^k(x,y)\,dy\right)\to 0\quad\text{strongly in } W^{-1,p}(Q;\mathbb{R}^l),\tag{3.14}$$
$$\left\|\sum_{i=1}^N\frac{\partial}{\partial y_i}(R_i^k(x,y))\right\|_{W^{-1,p}(Q;\mathbb{R}^l)}\to 0\quad\text{strongly in } L^p(Q).\tag{3.15}$$

Using Lemma 3.2, we consider the projection operators $\mathbb{T}_x$ and $\mathbb{T}_y$ onto the kernel of the divergence operator with respect to x and the divergence operator with respect to y in the set Q. We have

$$\begin{aligned}&\Big\|\mathbb{T}_x\Big(\int_Q R^k(x,y)\,dy-\int_Q\int_Q R^k(w,y)\,dy\,dw\Big)\\&\quad-\Big(\int_Q R^k(x,y)\,dy-\int_Q\int_Q R^k(w,y)\,dy\,dw\Big)\Big\|_{L^p(Q;\mathbb{R}^d)}\\&\quad\le C\Big\|\sum_{i=1}^N\frac{\partial}{\partial x_i}\Big(\int_Q R^k_i(x,y)\,dy\Big)\Big\|_{W^{-1,p}(Q;\mathbb{R}^l)},\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}&\Big\|\mathbb{T}_y\Big(R^k(x,y)-\int_Q R^k(x,z)\,dz\Big)-\Big(R^k(x,y)-\int_Q R^k(x,z)\,dz\Big)\Big\|_{L^p(Q\times Q;\mathbb{R}^d)}\\&\quad\le C\Big\|\Big\|\sum_{i=1}^N\frac{\partial}{\partial y_i}(R^k_i(x,y))\Big\|_{W^{-1,p}(Q;\mathbb{R}^l)}\Big\|_{L^p(Q)},\end{aligned}\tag{3.17}$$

which in turn yields

$$\begin{aligned}&\Big\|\int_Q\mathbb{T}_y\Big(R^k(x,y)-\int_Q R^k(x,z)\,dz\Big)\,dy\Big\|_{L^p(Q;\mathbb{R}^d)}\\&\quad=\Big\|\int_Q\Big[\mathbb{T}_y\Big(R^k(x,y)-\int_Q R^k(x,z)\,dz\Big)-\Big(R^k(x,y)-\int_Q R^k(x,z)\,dz\Big)\Big]\,dy\Big\|_{L^p(Q;\mathbb{R}^d)}\\&\quad\le C\Big\|\Big\|\sum_{i=1}^N\frac{\partial}{\partial y_i}(R^k_i(x,y))\Big\|_{W^{-1,p}(Q;\mathbb{R}^l)}\Big\|_{L^p(Q)}.\end{aligned}\tag{3.18}$$

Set

$$\begin{aligned}S^k(x,y):=&\,\mathbb{T}_y\Big(R^k(x,y)-\int_Q R^k(x,z)\,dz\Big)-\int_Q\Big(\mathbb{T}_y\Big(R^k(x,z)-\int_Q R^k(x,\xi)\,d\xi\Big)\Big)\,dz\\&+\mathbb{T}_x\Big(\int_Q R^k(x,z)\,dz-\int_Q\int_Q R^k(w,z)\,dz\,dw\Big)+\int_Q\int_Q R^k(w,z)\,dz\,dw\end{aligned}$$

for a.e. $(x,y)\in Q\times Q$. Combining (3.13)–(3.16), we deduce the inequality

$$\begin{aligned}&\|R^k-S^k\|_{L^p(Q\times Q;\mathbb{R}^d)}\\&\quad\le\Big\|\mathbb{T}_y\Big(R^k(x,y)-\int_Q R^k(x,z)\,dz\Big)-\Big(R^k(x,y)-\int_Q R^k(x,z)\,dz\Big)\Big\|_{L^p(Q\times Q;\mathbb{R}^d)}\\&\quad+\Big\|\mathbb{T}_x\Big(\int_Q R^k(x,z)\,dz-\int_Q\int_Q R^k(w,z)\,dz\,dw\Big)\\&\quad-\Big(\int_Q R^k(x,z)\,dz-\int_Q\int_Q R^k(w,z)\,dz\,dw\Big)\Big\|_{L^p(Q;\mathbb{R}^d)}\\&\quad+\Big\|\int_Q\mathbb{T}_y\Big(R^k(x,z)-\int_Q R^k(x,\xi)\,d\xi\Big)\,dz\Big\|_{L^p(Q;\mathbb{R}^d))}+\Big|\int_Q\int_Q R^k(x,y)\,dy\,dx\Big|,\end{aligned}\tag{3.19}$$

whose right-hand-side converges to zero as $k \to +\infty$. On the other hand, by Lemma 3.2 there holds

$$\sum_{i=1}^{N} \frac{\partial S^k_{ir}(x,y)}{\partial y_i} = 0 \quad \text{in } W^{-1,p}(Q) \quad \text{for a.e. } x \in Q, \tag{3.20}$$

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\int_Q S^k_{ir}(x,y)\,dy\right) = 0 \quad \text{in } W^{-1,p}(Q), \tag{3.21}$$

for every k, for all $r = 1, \dots, l$.

Finally, define

$$w^k(x,y) := \mathcal{A}(y)^{-1}\begin{pmatrix} S^k_1(x,y) \\ \vdots \\ S^k_N(x,y) \end{pmatrix} \quad \text{for a.e. } x \in \Omega \text{ and } y \in \mathbb{R}^N$$

(where the components S^k_i are defined analogously to the maps R^k_i). Properties (3.8) and (3.9) follow directly from (3.20) and (3.21). Condition (3.7) is a consequence of the boundedness of $\mathcal{A}^{-1}$ (due to (3.3)) and (3.19).

Remark 3.4 By property (P4) in Lemma 3.2, the boundedness of the operators A^i, $i = 1, \dots, N$, and the uniform invertibility condition (3.3), it follows that if $\{v^n\}$ is p-equiintegrable, then $\{w^k\}$ is p-equiintegrable as well.

In view of property (P5) in Lemma 3.2 if $A^i \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1 \dots, N$, and $\{v^n\} \subset C^\infty_c(\Omega; C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d))$, then the sequence $\{w^k\}$ constructed in the proof of Lemma 3.2 inherits the same regularity.

In order to characterize the limit differential constraint, for $u \in L^p(\Omega; \mathbb{R}^d)$ and $n \in \mathbb{N}$ we introduce the classes

$$\begin{aligned} \mathcal{C}^{\mathscr{A}(n\cdot)}_u := \Bigg\{ & w \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d)) : \int_Q w(x,y)\,dy = 0 \quad \text{for a.e. } x \in \Omega, \\ & \times \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\int_Q A^i(ny)(u(x) + w(x,y))\,dy\right) = 0 \quad \text{in } W^{-1,p}(\Omega; \mathbb{R}^l), \\ & \times \sum_{i=1}^{N} \frac{\partial}{\partial y_i}\Big(A^i(ny)(u(x) + w(x,y))\Big) = 0 \quad \text{in } W^{-1,p}(Q; \mathbb{R}^l) \text{ for a.e. } x \in \Omega \Bigg\}, \end{aligned} \tag{3.22}$$

and

$$\mathcal{C}^{\mathscr{A}(n\cdot)} := \{u \in L^p(\Omega; \mathbb{R}^d) : \mathcal{C}^{\mathscr{A}(n\cdot)}_u \neq \emptyset\}. \tag{3.23}$$

For simplicity we will also adopt the notation $\mathcal{C}^{\mathscr{A}}_u := \mathcal{C}^{\mathscr{A}(1\cdot)}_u$ and $\mathcal{C}^{\mathscr{A}} := \mathcal{C}^{\mathscr{A}(1\cdot)}$. Lemma 3.3 allows us to provide a first characterization of the set $\mathcal{C}^{\mathscr{A}}$ in the case in which $A^i \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1 \dots, N$.

Proposition 3.5 *Let $1 < p < +\infty$. Let $A^i \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1, \dots, N$, with $\mathcal{A}$ satisfying the invertibility requirement in (3.3). Let $\mathcal{C}^{\mathscr{A}}$ be the class introduced in (3.23) and let $\mathscr{A}^{\mathrm{div}}_\varepsilon$ be the operator defined in (3.1). Then*

$$\mathcal{C}^{\mathscr{A}} = \Big\{ u \in L^p(\Omega;\mathbb{R}^d) : \textit{ there exists a sequence } \{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d) \textit{ such that}$$
$$u_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega;\mathbb{R}^d) \textit{ and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l) \Big\}. \tag{3.24}$$

Moreover, for every $u \in \mathcal{C}^{\mathscr{A}}$ *and* $w \in \mathcal{C}_u^{\mathscr{A}}$ *there exists a sequence* $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d)$ *such that*

$$u_\varepsilon \xrightarrow{2-s} u + w \quad \textit{strongly two-scale in } L^p(\Omega \times Q;\mathbb{R}^d),$$

and

$$\mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l).$$

Proof Denote by $\mathcal{D}$ the set in the right-hand side of (3.24). We divide the proof into two steps.

Step 1 We first show the inclusion

$$\mathcal{D} \subset \mathcal{C}^{\mathscr{A}}.$$

Let $u \in \mathcal{D}$, and let $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d)$ be such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d) \tag{3.25}$$

and

$$\mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l). \tag{3.26}$$

Consider a test function $\psi \in C_c^1(\Omega;\mathbb{R}^l)$. We have

$$\left\langle \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon,\ \psi \right\rangle \to 0, \tag{3.27}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $W^{-1,p}(\Omega;\mathbb{R}^l)$ and $W_0^{1,p'}(\Omega;\mathbb{R}^l)$. By definition of the operators $\mathscr{A}_\varepsilon^{\mathrm{div}}$,

$$\left\langle \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon,\ \psi \right\rangle = -\int_\Omega \sum_{i=1}^N A^i\left(\frac{x}{\varepsilon}\right) u_\varepsilon(x) \cdot \frac{\partial \psi(x)}{\partial x_i}\, dx \quad \text{for every } \varepsilon > 0.$$

By Proposition 2.2 there exists a map $w \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d))$ with $\int_Q w(x,y)\,dy = 0$ such that, up to the extraction of a (not relabeled) subsequence

$$u_\varepsilon \overset{2-s}{\rightharpoonup} v \quad \text{weakly two-scale} \tag{3.28}$$

where

$$v(x,y) := u(x) + w(x,y), \tag{3.29}$$

for a.e. $x \in \Omega$, $y \in \mathbb{R}^N$. Hence, by the definition of two-scale convergence,

$$\left\langle \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon,\ \psi \right\rangle \to -\int_\Omega \int_Q \sum_{i=1}^N A^i(y) v(x,y) \cdot \frac{\partial \psi(x)}{\partial x_i}\, dy\, dx,$$

and by (3.27) we have that

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)(u(x)+w(x,y))\,dy\right)=0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^l). \tag{3.30}$$

Let now $\varphi \in C^1_{\text{per}}(\mathbb{R}^N;\mathbb{R}^l)$, $\psi \in C^1_c(\Omega)$, and consider the sequence of test functions

$$\varphi_\varepsilon(x) := \varepsilon\varphi\left(\frac{x}{\varepsilon}\right)\psi(x) \quad \text{for } x \in \mathbb{R}^N.$$

The sequence $\{\varphi_\varepsilon\}$ is uniformly bounded in $W_0^{1,p'}(\Omega;\mathbb{R}^l)$, therefore by (3.26)

$$\left\langle \mathscr{A}_\varepsilon^{\text{div}} u_\varepsilon,\, \varphi_\varepsilon \right\rangle \to 0, \tag{3.31}$$

with

$$\left\langle \mathscr{A}_\varepsilon^{\text{div}} u_\varepsilon,\, \varphi_\varepsilon \right\rangle = -\int_\Omega \sum_{i=1}^{N} A^i\left(\frac{x}{\varepsilon}\right)u_\varepsilon(x)\cdot\left(\frac{\partial\varphi}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\psi(x)+\varepsilon\varphi\left(\frac{x}{\varepsilon}\right)\frac{\partial\psi(x)}{\partial x_i}\right)dx$$

for every ε. Passing to the subsequence of $\{u_\varepsilon\}$ extracted in (3.28), and applying the definition of two-scale convergence, we obtain

$$\int_\Omega\int_Q \sum_{i=1}^{N} A^i(y)v(x,y)\cdot\frac{\partial\varphi(y)}{\partial y_i}\psi(x)\,dy\,dx=0$$

for every $\varphi \in C^1_{\text{per}}(\mathbb{R}^N;\mathbb{R}^l)$ and $\psi \in C^1_c(\Omega)$. By density, this equality still holds for an arbitrary $\varphi \in W_0^{1,p'}(Q;\mathbb{R}^l)$, and so

$$\sum_{i=1}^{N} \frac{\partial}{\partial y_i}\left(A^i(y)(u(x)+w(x,y))\right)=0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega. \tag{3.32}$$

Combining (3.30) and (3.32), we deduce that $u \in \mathcal{C}^{\mathscr{A}}$.

Step 2 We claim that $\mathcal{C}^{\mathscr{A}} \subset \mathcal{D}$. Let $u \in \mathcal{C}^{\mathscr{A}}$, let $w \in \mathcal{C}_u^{\mathscr{A}}$, and set

$$v(x,y) := u(x)+w(x,y) \quad \text{for a.e } x \in \Omega \text{ and } y \in \mathbb{R}^N.$$

Let $\{v_\delta\} \subset C_c^\infty(\Omega; C^\infty_{\text{per}}(\mathbb{R}^N;\mathbb{R}^d))$ be such that

$$v_\delta \to v \quad \text{strongly in } L^p(\Omega\times Q;\mathbb{R}^d). \tag{3.33}$$

The sequence $\{v_\delta\}$ satisfies both (3.5) and (3.6), hence by Lemma 3.3 and Remark 3.4 we can construct a sequence $\{\hat{v}_\delta\} \subset C^\infty(\Omega; C^\infty_{\text{per}}(\mathbb{R}^N;\mathbb{R}^d))$ such that

$$\hat{v}_\delta \to v \quad \text{strongly in } L^p(\Omega\times Q;\mathbb{R}^d), \tag{3.34}$$

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)\hat{v}_\delta(x,y)\,dy\right)=0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^l), \tag{3.35}$$

and

$$\sum_{i=1}^{N} \frac{\partial}{\partial y_i}(A^i(y)\hat{v}_\delta(x,y))=0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega. \tag{3.36}$$

Consider now the maps

$$u^\varepsilon_\delta(x) := \hat{v}_\delta\left(x, \frac{x}{\varepsilon}\right)$$

for every $x \in \Omega$. By Proposition 2.3 we have

$$u^\varepsilon_\delta \xrightarrow{2-s} \hat{v}_\delta \quad \text{strongly two-scale in } L^p(\Omega \times Q; \mathbb{R}^d)$$

as $\varepsilon \to 0$, and hence, by Theorem 2.6

$$T_\varepsilon u^\varepsilon_\delta \to \hat{v}_\delta \quad \text{strongly in } L^p(\mathbb{R}^N \times Q; \mathbb{R}^d) \tag{3.37}$$

(where T_ε is the unfolding operator defined in (2.1)). We observe that by (3.36),

$$\sum_{i=1}^N \frac{\partial A^i}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\hat{v}_\delta\left(x, \frac{x}{\varepsilon}\right) + A^i\left(\frac{x}{\varepsilon}\right)\frac{\partial \hat{v}_\delta}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right) = 0 \tag{3.38}$$

for all $x \in \Omega$, for every ε and δ. Moreover, by Propositions 2.2 and 2.3,

$$\begin{aligned}\sum_{i=1}^N A^i\left(\frac{x}{\varepsilon}\right)\frac{\partial \hat{v}_\delta}{\partial x_i}\left(x, \frac{x}{\varepsilon}\right) &\rightharpoonup \sum_{i=1}^N \int_Q A^i(y)\frac{\partial \hat{v}_\delta}{\partial x_i}(x, y)\, dy \\ &= \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)\hat{v}_\delta(x, y)\, dy\right) = 0 \end{aligned}\tag{3.39}$$

as $\varepsilon \to 0$, weakly in $L^p(\Omega; \mathbb{R}^d)$, where the last equality follows by (3.35). Finally, since

$$\begin{aligned}\mathscr{A}^{\mathrm{div}}_\varepsilon u^\varepsilon_\delta(x) &= \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(A^i\left(\frac{x}{\varepsilon}\right)u^\varepsilon_\delta(x)\right) \\ &= \frac{1}{\varepsilon}\sum_{i=1}^N \left[\frac{\partial A^i}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\hat{v}_\delta\left(x, \frac{x}{\varepsilon}\right) + A^i\left(\frac{x}{\varepsilon}\right)\frac{\partial \hat{v}_\delta}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right)\right] \\ &\quad + \sum_{i=1}^N A^i\left(\frac{x}{\varepsilon}\right)\frac{\partial \hat{v}_\delta}{\partial x_i}\left(x, \frac{x}{\varepsilon}\right),\end{aligned}$$

by (3.38), (3.39), and the compact embedding of L^p into $W^{-1,p}$, we conclude that

$$\mathscr{A}^{\mathrm{div}}_\varepsilon u^\varepsilon_\delta \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l), \tag{3.40}$$

as $\varepsilon \to 0$. Collecting (3.34), (3.37), and (3.40), we deduce that

$$\lim_{\delta \to 0}\lim_{\varepsilon \to 0}\left(\|T_\varepsilon u^\varepsilon_\delta - (u + w)\|_{L^p(\Omega \times Q)} + \|\mathscr{A}^{\mathrm{div}}_\varepsilon u^\varepsilon_\delta\|_{W^{-1,p}(\Omega; \mathbb{R}^l)}\right) = 0.$$

By Attouch's diagonalization lemma [3, Lemma 1.15 and Corollary 1.16], there exists a subsequence $\{\delta(\varepsilon)\}$ such that

$$\lim_{\varepsilon \to 0}\left(\|T_\varepsilon u^\varepsilon_{\delta(\varepsilon)} - (u + w)\|_{L^p(\Omega \times Q)} + \|\mathscr{A}^{\mathrm{div}}_\varepsilon u^\varepsilon_{\delta(\varepsilon)}\|_{W^{-1,p}(\Omega; \mathbb{R}^l)}\right) = 0.$$

Setting $u^\varepsilon := u^\varepsilon_{\delta_\varepsilon}$, we finally obtain

$$\mathscr{A}^{\mathrm{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l)$$

and

$$u^\varepsilon \overset{2-s}{\to} u + w \quad \text{strongly two-scale in } L^p(\Omega \times Q; \mathbb{R}^d),$$

and hence, by Proposition 2.2,

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d).$$

This yields that $u \in \mathcal{D}$ and completes the proof of the proposition. □

Remark 3.6 The regularity of the operators A^i played a key role in Step 2. In the case in which $A^i \in L^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1, \dots, N$, but we have no further smoothness assumptions on the operators, the argument in Step 1 still guarantees that

$$\Big\{u \in L^p(\Omega; \mathbb{R}^d) : \text{there exists a sequence } \{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^d) \text{ such that}$$
$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d)$$
$$\text{and } \mathscr{A}^{\text{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \quad \text{for every } 1 \le q < p\Big\} \subset \mathcal{C}^{\mathscr{A}}. \tag{3.41}$$

Indeed, arguing as in Step 1 we obtain that there exists $w \in L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d))$ with $\int_Q w(x, y)\,dy = 0$, such that

$$u_\varepsilon \overset{2-s}{\rightharpoonup} u + w \quad \text{weakly two-scale in } L^p(\Omega \times Q; \mathbb{R}^d),$$

and

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)(u(x) + w(x, y))\,dy\right) = 0 \quad \text{in } W^{-1,q}(\Omega; \mathbb{R}^l)$$
$$\sum_{i=1}^N \frac{\partial}{\partial y_i}(A^i(y)(u(x) + w(x, y))) = 0 \quad \text{in } W^{-1,q}(Q; \mathbb{R}^l) \text{ for a.e. } x \in \Omega, \tag{3.42}$$

for all $1 \le q < p$. Since $u + w \in L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d))$, it follows that (3.42) holds also for $q = p$. Therefore we deduce the inclusion (3.41).

The proof of the opposite inclusion, on the other hand, is not a straightforward consequence of Proposition 3.5. In fact, in the case in which the operators A^i are only bounded, the second conclusion in Remark 3.4 does not hold anymore, and we are not able to guarantee that the projection operator provided by Lemma 3.3 preserves the regularity of smooth functions. Therefore, the measurability of the maps u^ε_δ is questionable (see [2, discussion below Definition 1.4]). This difficulty will be overcome in Lemma 5.3 by means of an approximation of the operators A^i with C^∞ operators.

4 Homogenization for smooth operators

We recall that

$$\mathcal{F}^r_{\mathscr{A}(n\cdot)}(u) := \inf\left\{\int_\Omega \int_Q f(u(x) + w(x, y))\,dy\,dx : w \in \mathcal{C}^{\mathscr{A}(n\cdot)}_u, \|w\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \le r\right\} \tag{4.1}$$

for every $u \in \mathcal{C}_r^{\mathscr{A}(n\cdot)}$ and $r > 0$, where

$$\mathcal{C}_r^{\mathscr{A}(n\cdot)} := \{v \in L^p(\Omega;\mathbb{R}^d) : \exists w \in \mathcal{C}_v^{\mathscr{A}(n\cdot)} \quad \text{with } \|w\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \le r\},$$

$$\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(u) := \begin{cases} \mathcal{F}^r_{\mathscr{A}(n\cdot)}(u) & \text{if } u \in \mathcal{C}_r^{\mathscr{A}(n\cdot)}, \\ +\infty & \text{otherwise in } L^p(\Omega;\mathbb{R}^d), \end{cases} \tag{4.2}$$

for every $r > 0$, and

$$\mathscr{F}_{\mathscr{A}}(u) := \begin{cases} \inf_{r>0} \inf \left\{ \liminf_{n\to+\infty} \overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(u_n) : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d) \right\} \\ \text{if } u \in \mathcal{C}^{\mathscr{A}}, \\ +\infty \quad \text{otherwise in } L^p(\Omega;\mathbb{R}^d). \end{cases} \tag{4.3}$$

Remark 4.1 We observe that for every $u \in \mathcal{C}^{\mathscr{A}}$ there holds

$$\mathcal{S} := \left\{ \{u_n\} : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\ u_n \in \mathcal{C}^{\mathscr{A}(n\cdot)} \quad \text{for every } n \in \mathbb{N} \right\} \neq \emptyset$$

and $\mathscr{F}_{\mathscr{A}}(u) < +\infty$. Indeed, let $u \in \mathcal{C}^{\mathscr{A}}$ and $w \in \mathcal{C}_u^{\mathscr{A}}$. Then a change of variables and the periodicity of w yield immediately that

$$\int_Q w(x, ny)\, dy = 0 \quad \text{for a.e. } x \in \Omega$$

and

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_Q (A^i(ny)(u(x) + w(x, ny)))\, dy \right) = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^l).$$

Proving that

$$\sum_{i=1}^n \frac{\partial}{\partial y_i}(A^i(ny)(u(x) + w(x, ny))) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega$$

is equivalent to showing that

$$\sum_{i=1}^n \frac{\partial}{\partial y_i}(A^i(y)(u(x) + w(x, y))) = 0 \quad \text{in } W^{-1,p}(nQ;\mathbb{R}^l) \text{ for a.e. } x \in \Omega. \tag{4.4}$$

To this purpose, arguing as in Step 2 of the proof of Proposition 3.5, construct $\{\hat{v}^\delta\} \subset C^\infty(\Omega;\ C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d))$ such that

$$\hat{v}^\delta \to u + w \quad \text{strongly in } L^p(\Omega \times Q;\mathbb{R}^d), \tag{4.5}$$

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_Q (A^i(ny)\hat{v}^\delta(x, y))\, dy \right) = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^l), \tag{4.6}$$

$$\sum_{i=1}^n \frac{\partial}{\partial y_i}(A^i(y)\hat{v}^\delta(x, y)) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega. \tag{4.7}$$

By the smoothness and the periodicity of $\{\hat{v}^\delta\}$ there holds

$$\sum_{i=1}^{n} \frac{\partial}{\partial y_i}(A^i(y)\hat{v}^\delta(x,y)) = 0 \quad \text{in } W^{-1,p}(nQ;\mathbb{R}^l) \text{ for a.e. } x \in \Omega$$

and (4.4) follows in view of (4.5). By the previous argument, $w(x,ny) \in \mathcal{C}_u^{\mathscr{A}(n\cdot)}$, therefore the set $\mathcal{S}$ contains always the sequence $u_n := u$ for every n.

Theorem 4.2 *Let $1 < p < +\infty$. Let $A^i \in C^\infty_{\text{per}}(\mathbb{R}^N;\mathbb{M}^{l\times d})$, $i = 1,\dots,N$, assume that the operator $\mathcal{A}$ satisfies the invertibility requirement in (3.3), and let $\mathscr{A}_\varepsilon^{\text{div}}$ be the operator defined in (3.1). Let $f : \mathbb{R}^d \to [0,+\infty)$ be a continuous function satisfying the growth condition*

$$0 \le f(v) \le C(1+|v|^p) \quad \textit{for every } v \in \mathbb{R}^d, \tag{4.8}$$

where $C > 0$. Then, for every $u \in L^p(\Omega;\mathbb{R}^d)$ there holds

$$\begin{aligned} &\inf\Big\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega;\mathbb{R}^d) \\ &\qquad \textit{and } \mathscr{A}_\varepsilon^{\text{div}} u_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)\Big\} \\ &= \inf\Big\{\limsup_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega;\mathbb{R}^d) \\ &\qquad \textit{and } \mathscr{A}_\varepsilon^{\text{div}} u_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)\Big\} = \mathscr{F}_{\mathscr{A}}(u). \end{aligned}$$

Before starting the proof of Theorem 4.2, we first state without proving a corollary of [16, Lemma 2.15] and one of [14, Lemma 2.8], and we prove an adaptation of [16, Lemma 2.15] to our framework.

Lemma 4.3 *Let $1 < p < +\infty$. Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega;\mathbb{R}^N)$ such that*

$$\operatorname{div} u_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,p}(\Omega) \tag{4.9}$$

and

$$u_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega;\mathbb{R}^N).$$

Then there exists a p-equiintegrable sequence $\{\tilde{u}_\varepsilon\}$ such that

$$\begin{aligned} &\operatorname{div}\tilde{u}_\varepsilon = 0 \quad \textit{in } W^{-1,p}(\Omega) \quad \textit{for every } \varepsilon, \\ &\tilde{u}_\varepsilon - u_\varepsilon \to 0 \quad \textit{strongly in } L^q(\Omega;\mathbb{R}^N) \quad \textit{for every } 1 \le q < p, \\ &\tilde{u}_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega;\mathbb{R}^N). \end{aligned}$$

Remark 4.4 A direct adaptation of the proof of [16, Lemma 2.15] yields also that the thesis of Lemma 4.3 still holds if we replace (4.9) with the condition

$$\operatorname{div} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega)$$

for every $1 \le q < p$.

Lemma 4.5 *Let $1 < p < +\infty$, and let $D \subset Q$. Let $\{u_\varepsilon\} \subset L^p(D;\mathbb{R}^N)$ be p-equiintegrable, with*

$$u_\varepsilon \rightharpoonup 0 \quad \textit{weakly in } L^p(D;\mathbb{R}^N),$$

and

$$\operatorname{div} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(D).$$

Then there exists a p-equiintegrable sequence $\{\tilde{u}_\varepsilon\} \subset L^p(Q;\mathbb{R}^N)$ *such that*

$$\begin{aligned}
&\tilde{u}_\varepsilon - u_\varepsilon \to 0 \quad \text{strongly in } L^p(D;\mathbb{R}^N),\\
&\tilde{u}_\varepsilon \to 0 \quad \text{strongly in } L^p(Q\setminus D;\mathbb{R}^N),\\
&\operatorname{div} \tilde{u}_\varepsilon = 0 \quad \text{in } W^{-1,p}(Q),\\
&\|\tilde{u}_\varepsilon\|_{L^p(Q;\mathbb{R}^N)} \le C\|u_\varepsilon\|_{L^p(D;\mathbb{R}^N)},\\
&\int_Q \tilde{u}_\varepsilon(x)\,dx = 0 \quad \text{for every } \varepsilon.
\end{aligned}$$

More generally, we have

Lemma 4.6 *Let* $1 < p < +\infty$, $u \in L^p(\Omega;\mathbb{R}^d)$ *and let* $\{u^\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d)$ *be such that*

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^p(\Omega;\mathbb{R}^d) \tag{4.10}$$

$$\mathscr{A}^{\operatorname{div}}_\varepsilon u_\varepsilon \to 0 \text{ strongly in } W^{-1,p}(\Omega;\mathbb{R}^l). \tag{4.11}$$

Then there exists a p-equiintegrable sequence $\{\tilde{u}_\varepsilon\}$ *such that*

$$\begin{aligned}
&\tilde{u}_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\\
&\mathscr{A}^{\operatorname{div}}_\varepsilon \tilde{u}_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \quad \text{for every } 1 \le q < p,\\
&\tilde{u}_\varepsilon - u_\varepsilon \to 0 \quad \text{strongly in } L^q(\Omega;\mathbb{R}^d) \quad \text{for every } 1 \le q < p.
\end{aligned}$$

The following propositions is a corollary of [14, Proposition 3.5 (ii)].

Proposition 4.7 *Let* $f : \mathbb{R}^d \to [0,+\infty)$ *be a continuous map satisfying* (4.8) *for some* $1 < p < +\infty$. *Let* $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d)$ *be a bounded sequence and let* $\{\tilde{u}_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d)$ *be p-equiintegrable and such that*

$$u_\varepsilon - \tilde{u}_\varepsilon \to 0 \quad \text{in measure}.$$

Then

$$\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx \ge \liminf_{\varepsilon\to 0}\int_\Omega f(\tilde{u}_\varepsilon(x))\,dx.$$

Moreover, if $g : \mathbb{R}^N \times \mathbb{R}^d \to [0,+\infty)$ *satisfies*

(i) $g(\cdot,\xi)$ *is measurable and Q-periodic for every* $\xi \in \mathbb{R}^d$,
(ii) $g(y,\cdot)$ *is continuous for a.e.* $y \in \mathbb{R}^N$,
(iii) *there exists a constant C such that*

$$0 \le g(y,\xi) \le C(1+|\xi|^p) \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and } \xi \in \mathbb{R}^d,$$

then

$$\liminf_{\varepsilon\to 0}\int_\Omega g\Big(\frac{x}{\varepsilon}, u_\varepsilon(x)\Big)\,dx \ge \liminf_{\varepsilon\to 0}\int_\Omega g\Big(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon(x)\Big)\,dx.$$

The next proposition is another corollary of [14, Proposition 3.5 (ii)].

Proposition 4.8 *Let* $1 \le p < +\infty$ *and* $\lambda \in (0,1]$. *Let* $g : \mathbb{R}^N \times \mathbb{R}^d \to [0,+\infty)$ *be such that*

(i) $g(\cdot, \xi)$ *is measurable and* Q*-periodic for every* $\xi \in \mathbb{R}^d$,
(ii) $g(y, \cdot)$ *is continuous for a.e.* $y \in \mathbb{R}^N$,
(iii) *there exists a constant* C *such that*

$$0 \leq g(y, \xi) \leq C(1 + |\xi|^p) \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and } \xi \in \mathbb{R}^d,$$

and let V *be a* p*-equiintegrable subset of* $L^p(\Omega \times Q; \mathbb{R}^d)$. *Then there exists a constant* C *such that*

$$\left\| g\left(\frac{y}{\lambda}, v_1(x, y)\right) - g\left(\frac{y}{\lambda}, v_2(x, y)\right) \right\|_{L^1(\Omega \times Q)} \leq C \|v_1 - v_2\|_{L^p(\Omega \times Q; \mathbb{R}^d)}$$

for every $v_1, v_2 \in V$.

We now start the proof of Theorem 4.2. First we prove the liminf inequality. The argument relies on the use of the *unfolding operator* (see Sect. 2.2 and [14, Appendix A] and the references therein).

Proposition 4.9 *Under the assumptions of Theorem* 4.2 *for every* $u \in L^p(\Omega; \mathbb{R}^d)$ *there holds*

$$\inf \left\{ \liminf_{\varepsilon \to 0} \int_\Omega f(u_\varepsilon(x))\, dx : u_\varepsilon \rightharpoonup u \text{ weakly in } L^p(\Omega; \mathbb{R}^d) \right.$$
$$\left. \text{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \text{ strongly in } W^{-1,p}(\Omega; \mathbb{R}^l) \right\} \geq \mathscr{F}_{\mathscr{A}}(u). \tag{4.12}$$

Proof Let $\mathcal{C}^{\mathscr{A}}$ be the class introduced in (3.23). We first notice that, by Proposition 3.5, if $u \in L^p(\Omega; \mathbb{R}^d) \setminus \mathcal{C}^{\mathscr{A}}$ then

$$\left\{ \{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^d) : u_\varepsilon \rightharpoonup u \text{ weakly in } L^p(\Omega; \mathbb{R}^d) \right.$$
$$\left. \text{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \text{ strongly in } W^{-1,p}(\Omega; \mathbb{R}^l)\} \right\} = \emptyset,$$

hence (4.12) follows trivially.

Define $g : \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty)$ as

$$g(y, \xi) = f(\mathcal{A}(y)^{-1}\xi) \quad \text{for every } y \in \mathbb{R}^N, \text{ and } \xi \in \mathbb{R}^d.$$

By the continuity of f, g is measurable with respect to the first variable (it is the composition of a continuous function with a measurable one), and continuous with respect to the second variable. By (4.8), there holds

$$0 \leq g(y, \xi) \leq C(1 + |\mathcal{A}(y)^{-1}\xi|^p) \leq C(1 + |\xi|^p), \tag{4.13}$$

where the last inequality follows by the uniform invertibility assumption (3.3). We divide the proof of the proposition into five steps.

Step 1 We first show that for every $u \in \mathcal{C}^{\mathscr{A}}$ there holds

$$\begin{aligned}
&\inf\Big\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\\
&\qquad \text{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)\Big\}\\
&\geq \inf_{w\in\mathcal{C}_u^{\mathscr{A}}} \inf\Big\{\liminf_{n\to+\infty}\int_\Omega g\Big(\frac{x}{\varepsilon_n}, v_{\varepsilon_n}(x)\Big)\,dx :\\
&\qquad v_{\varepsilon_n} \rightharpoonup \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\\
&\qquad \operatorname{div} v_{\varepsilon_n} \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l),\\
&\qquad \text{and } \mathcal{A}\Big(\frac{x}{\varepsilon_n}\Big)^{-1} v_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\Big\}.
\end{aligned}\tag{4.14}$$

Indeed let $u \in \mathcal{C}^{\mathscr{A}}$ and let $\{u_\varepsilon\}$ be as in (4.14). Up to the extraction of a subsequence $\{\varepsilon_n\}$,

$$\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx = \lim_{n\to+\infty}\int_\Omega f(u_{\varepsilon_n}(x))\,dx,$$

and by Proposition 2.2 there exists $w \in \mathcal{C}_u^{\mathscr{A}}$ such that

$$u_{\varepsilon_n} \overset{2-s}{\rightharpoonup} u + w \quad \text{weakly two-scale in } L^p(\Omega\times Q;\mathbb{R}^d).\tag{4.15}$$

By the definition of g it is straightforward to see that

$$\lim_{n\to+\infty}\int_\Omega f(u_{\varepsilon_n}(x))\,dx = \lim_{n\to+\infty}\int_\Omega g\Big(\frac{x}{\varepsilon_n}, \mathcal{A}\Big(\frac{x}{\varepsilon_n}\Big)u_{\varepsilon_n}(x)\Big)\,dx.$$

Setting $v_{\varepsilon_n} := \mathcal{A}\Big(\frac{x}{\varepsilon_n}\Big)u_{\varepsilon_n}$, in view of the assumptions on $\{u_\varepsilon\}$ in (4.14), it follows that

$$\operatorname{div} v_{\varepsilon_n} = \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(A^i\Big(\frac{x}{\varepsilon_n}\Big)u_{\varepsilon_n}(x)\Big) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l),$$

and

$$\mathcal{A}\Big(\frac{x}{\varepsilon_n}\Big)^{-1} v_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d).\tag{4.16}$$

Finally, by (4.15) for every $\varphi \in L^{p'}(\Omega;\mathbb{R}^d)$ there holds

$$\begin{aligned}
\lim_{n\to+\infty}\int_\Omega v_{\varepsilon_n}(x)\cdot\varphi(x)\,dx &= \lim_{n\to+\infty}\int_\Omega u_{\varepsilon_n}(x)\cdot\mathcal{A}\Big(\frac{x}{\varepsilon_n}\Big)^T\varphi(x)\,dx\\
&= \int_\Omega\int_Q (u(x)+w(x,y))\cdot\mathcal{A}(y)^T\varphi(x)\,dy\,dx.
\end{aligned}$$

This completes the proof of (4.14).

Step 2 We claim that

$$\begin{aligned}
&\inf_{w\in\mathcal{C}_u^{\mathscr{A}}}\ \inf\Bigg\{\liminf_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, v_{\varepsilon_n}(x)\right)dx:\\
&\quad v_{\varepsilon_n}\rightharpoonup\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy\quad\text{weakly in } L^p(\Omega;\mathbb{R}^d),\\
&\quad \operatorname{div} v_{\varepsilon_n}\to 0\quad\text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l),\\
&\quad \text{and } \mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} v_{\varepsilon_n}\rightharpoonup u\quad\text{weakly in } L^p(\Omega;\mathbb{R}^d)\Bigg\}\\
&\geq \inf_{w\in\mathcal{C}_u^{\mathscr{A}}}\ \inf\Bigg\{\liminf_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy+\tilde{v}_{\varepsilon_n}(x)\right)dx:\\
&\quad \{\tilde{v}_{\varepsilon_n}\}\text{ is p-equiintegrable},\\
&\quad \tilde{v}_{\varepsilon_n}\rightharpoonup 0\quad\text{weakly in } L^p(\Omega;\mathbb{R}^d),\\
&\quad \operatorname{div}\tilde{v}_{\varepsilon_n}=0\quad\text{in } W^{-1,p}(\Omega;\mathbb{R}^l),\ \text{and}\\
&\quad \mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1}\tilde{v}_{\varepsilon_n}\rightharpoonup u(x)-\left(\int_Q \mathcal{A}(z)^{-1}\,dz\right)\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy\\
&\quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\Bigg\}.
\end{aligned}\tag{4.17}$$

Let $\{v_{\varepsilon_n}\}$ be as in (4.17). By Lemma 4.3, we construct a p-equiintegrable sequence $\{\bar{v}_{\varepsilon_n}\}$ such that

$$\operatorname{div}\bar{v}_{\varepsilon_n}=0\quad\text{in } W^{-1,p}(\Omega;\mathbb{R}^l),\tag{4.18}$$

$$v_{\varepsilon_n}-\bar{v}_{\varepsilon_n}\to 0\quad\text{strongly in } L^q(\Omega;\mathbb{R}^d)\quad\text{for every } 1\leq q<p,\tag{4.19}$$

$$\bar{v}_{\varepsilon_n}\rightharpoonup\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy\quad\text{weakly in } L^p(\Omega;\mathbb{R}^d).\tag{4.20}$$

Moreover, by Proposition 4.7,

$$\lim_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, v_{\varepsilon_n}(x)\right)dx\geq\liminf_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, \bar{v}_{\varepsilon_n}(x)\right)dx.$$

By (4.20) and by the uniform invertibility assumption (3.3), there exists a constant C such that

$$\left\|\mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1}\bar{v}_{\varepsilon_n}\right\|_{L^p(\Omega;\mathbb{R}^d)}\leq C\quad\text{for every } n.$$

Therefore, there exists a map $\phi\in L^p(\Omega;\mathbb{R}^d)$ such that, up to the extraction of a (not relabeled) subsequence,

$$\mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1}\bar{v}_{\varepsilon_n}\rightharpoonup\phi\quad\text{weakly in } L^p(\Omega;\mathbb{R}^d).\tag{4.21}$$

By the properties of $\{v_{\varepsilon_n}\}$ in (4.17) and (4.19), the convergence in (4.21) holds for the entire sequence $\{\bar{v}_{\varepsilon_n}\}$ and

$$\phi=u.$$

Claim (4.17) follows by setting

$$\tilde{v}_{\varepsilon_n}(x) := \bar{v}_{\varepsilon_n} - \int_Q \mathcal{A}(y)(u(x) + w(x, y))\,dy \quad \text{for a.e. } x \in \Omega.$$

Indeed, by (4.20) we have

$$\tilde{v}_{\varepsilon_n} \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d),$$

and since $w \in \mathcal{C}_u^{\mathscr{A}}$, by (4.18)

$$\operatorname{div} \tilde{v}_{\varepsilon_n} = 0 \quad \text{in } W^{-1,p}(\Omega; \mathbb{R}^l).$$

Finally,

$$\mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} \tilde{v}_{\varepsilon_n}(x) = \mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} \bar{v}_{\varepsilon_n}(x) - \mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} \int_Q \mathcal{A}(y)(u(x) + w(x, y))\,dy$$

for a.e. $x \in \Omega$, therefore by (4.21) and Riemann–Lebesgue lemma (see e.g. [15]),

$$\mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} \tilde{v}_{\varepsilon_n} \rightharpoonup u(x) - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right) \int_Q \mathcal{A}(y)(u(x) + w(x, y))\,dy$$

weakly in $L^p(\Omega; \mathbb{R}^d)$.

Step 3 We show that for every $w \in \mathcal{C}_u^{\mathscr{A}}$ and every p-equiintegrable sequence $\{\tilde{v}_{\varepsilon_n}\}$ satisfying

$$\tilde{v}_{\varepsilon_n} \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d), \tag{4.22}$$

$$\operatorname{div} \tilde{v}_{\varepsilon_n} = 0 \quad \text{in } W^{-1,p}(\Omega; \mathbb{R}^l), \tag{4.23}$$

$$\mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} \tilde{v}_{\varepsilon_n} \rightharpoonup u(x) - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right) \int_Q \mathcal{A}(y)(u(x) + w(x, y))\,dy \tag{4.24}$$

weakly in $L^p(\Omega; \mathbb{R}^d)$, there exists a p-equiintegrable family $\{v_{\nu,n} : \nu \in \mathbb{N},\ n \in \mathbb{N}\}$ such that

$$\operatorname{div} v_{\nu,n} = 0 \quad \text{in } W^{-1,p}(\Omega; \mathbb{R}^l), \tag{4.25}$$

$$v_{\nu,n} \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \quad \text{as } n \to +\infty, \tag{4.26}$$

$$\mathcal{A}\left(\nu \left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor x\right)^{-1} v_{\nu,n}(x) \rightharpoonup u - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right) \int_Q \mathcal{A}(y)(u(x) + w(x, y))\,dy \tag{4.27}$$

weakly in $L^p(\Omega; \mathbb{R}^d)$, as $n \to +\infty$ and

$$\begin{aligned} &\liminf_{n\to+\infty} \int_\Omega g\left(\frac{x}{\varepsilon_n}, \int_Q \mathcal{A}(y)(u(x) + w(x, y))\,dy + \tilde{v}_{\varepsilon_n}(x)\right) dx \\ &\geq \sup_{\nu\in\mathbb{N}} \liminf_{n\to+\infty} \int_\Omega g\left(\nu \left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor x, \int_Q \mathcal{A}(y)(u(x) + w(x, y))\,dy + v_{\nu,n}(x)\right) dx. \end{aligned} \tag{4.28}$$

To prove the claim we argue as in [14, Proposition 3.8]. Fix $\nu \in \mathbb{N}$, let

$$\theta_{\nu,n} := \nu\varepsilon_n \left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor \in [0, 1]$$

and set

$$k_{\nu,n} := \frac{\theta_{\nu,n}}{\nu\varepsilon_n} \in \mathbb{N}_0.$$

We notice that

$$\theta_{\nu,n} \to 1 \quad \text{as } n \to +\infty. \tag{4.29}$$

Without loss of generality, we can assume that $\Omega \subset\subset Q$. By Lemma 4.5 we extend every map $\tilde{v}_{\varepsilon_n}$ to a map $\tilde{\tilde{v}}_{\varepsilon_n} \in L^p(Q;\mathbb{R}^d)$ such that $\{\tilde{\tilde{v}}_{\varepsilon_n}\}$ is p-equiintegrable, and satisfies the following properties

$$\begin{aligned}
&\tilde{\tilde{v}}_{\varepsilon_n} - \tilde{v}_{\varepsilon_n} \to 0 \quad \text{strongly in } L^p(\Omega;\mathbb{R}^d),\\
&\tilde{\tilde{v}}_{\varepsilon_n} \to 0 \quad \text{strongly in } L^p(Q\backslash\Omega;\mathbb{R}^d),\\
&\operatorname{div}\tilde{\tilde{v}}_{\varepsilon_n} = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l).
\end{aligned} \tag{4.30}$$

In particular, by (4.24) and (4.30) it follows that

$$\mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1}\tilde{\tilde{v}}_{\varepsilon_n}(x) \rightharpoonup u(x) - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right)\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy \tag{4.31}$$

weakly in $L^p(\Omega;\mathbb{R}^d)$. By Proposition 4.7 and the definition of $\theta_{\nu,n}$ and $k_{\nu,n}$,

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + \tilde{v}_{\varepsilon_n}(x)\right)dx\\
&\quad\geq \liminf_{n\to+\infty}\int_\Omega g\left(\nu\frac{k_{\nu,n}}{\theta_{\nu,n}}x, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + \tilde{\tilde{v}}_{\varepsilon_n}(x)\right)dx.
\end{aligned} \tag{4.32}$$

For $\Omega' \subset\subset \Omega$ fixed, there holds $\theta_{\nu,n}\Omega' \subset \Omega$ for n large enough. Since g is nonnegative (see (4.13)), by (4.32) we have

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + \tilde{v}_{\varepsilon_n}(x)\right)dx\\
&\quad\geq \liminf_{n\to+\infty}\int_{\theta_{\nu,n}\Omega'} g\left(\nu\frac{k_{\nu,n}}{\theta_{\nu,n}}x, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + \tilde{\tilde{v}}_{\varepsilon_n}(x)\right)dx\\
&\quad= \liminf_{n\to+\infty}(\theta_{\nu,n})^N\int_{\Omega'} g\left(\nu k_{\nu,n}x, \int_Q \mathcal{A}(y)(u(\theta_{\nu,n}x)+w(\theta_{\nu,n}x,y))\,dy + \tilde{\tilde{v}}_{\varepsilon_n}(\theta_{\nu,n}x)\right)dx\\
&\quad\geq \liminf_{n\to+\infty}\int_{\Omega'} g\left(\nu k_{\nu,n}x, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + \tilde{\tilde{v}}_{\varepsilon_n}(\theta_{\nu,n}x)\right)dx
\end{aligned} \tag{4.33}$$

where the last inequality follows by (4.29), and since

$$u(\theta_{\nu,n}x) + w(\theta_{\nu,n}x,y) - u(x) - w(x,y) \to 0 \quad \text{strongly in } L^p(\Omega\times Q;\mathbb{R}^d).$$

Letting Ω' tend to Ω, by the p-equiintegrability of $\{\tilde{\tilde{v}}_{\varepsilon_n}\}$, there holds

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + \tilde{v}_{\varepsilon_n}(x)\right)dx\\
&\quad\geq \liminf_{n\to+\infty}\int_\Omega g\left(\nu k_{\nu,n}x, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + \tilde{\tilde{v}}_{\varepsilon_n}(\theta_{\nu,n}x)\right)dx.
\end{aligned} \tag{4.34}$$

Set

$$v_{\nu,n}(x) := \tilde{\tilde{v}}_{\varepsilon_n}(\theta_{\nu,n}x) \quad \text{for a.e. } x\in\Omega,\ \nu\in\mathbb{N} \text{ and } n \geq n_0(\nu),$$

where $n_0(\nu)$ is big enough so that $\theta_{\nu,n}\Omega \subset Q$ for $n \geq n_0(\nu)$. Inequality (4.28) is a direct consequence of (4.34). The p-equiintegrability of $\{v_{\nu,n}\}$, (4.25), and (4.26) follow in view of (4.29) and (4.30).

To conclude the proof of the claim it remains to establish (4.27). We first remark that, by (3.3) and (4.26), the sequence $\left\{\mathcal{A}\left(\nu\left\lfloor \frac{1}{\nu\varepsilon_n}\right\rfloor x\right)^{-1} v_{\nu,n}(x)\right\}$ is uniformly bounded in $L^p(\Omega;\mathbb{R}^d)$. Therefore, there exists a map $L \in L^p(\Omega;\mathbb{R}^d)$ such that, up to the extraction of a (not relabeled) subsequence, there holds

$$\mathcal{A}\left(\nu\left\lfloor \frac{1}{\nu\varepsilon_n}\right\rfloor x\right)^{-1} v_{\nu,n}(x) \rightharpoonup L(x) \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d). \tag{4.35}$$

Let $\varphi \in C_c^\infty(\Omega;\mathbb{R}^d)$. Then,

$$\int_\Omega \mathcal{A}\left(\nu\left\lfloor \frac{1}{\nu\varepsilon_n}\right\rfloor x\right)^{-1} v_{\nu,n}(x)\cdot\varphi(x)\,dx \to \int_\Omega L(x)\cdot\varphi(x)\,dx. \tag{4.36}$$

For n big enough, $\theta_{\nu,n}\operatorname{supp}\varphi \subset \Omega$. Hence,

$$\begin{aligned}
&\int_\Omega \mathcal{A}\left(\nu\left\lfloor \frac{1}{\nu\varepsilon_n}\right\rfloor x\right)^{-1} v_{\nu,n}(x)\cdot\varphi(x)\,dx\\
&\quad= \frac{1}{(\theta_{\nu,n})^N}\int_{\theta_{\nu,n}\operatorname{supp}\varphi} \mathcal{A}\left(\frac{y}{\varepsilon_n}\right)^{-1}\tilde{\tilde{v}}_{\varepsilon_n}(y)\cdot\varphi\left(\frac{y}{\theta_{\nu,n}}\right)dy\\
&\quad= \frac{1}{(\theta_{\nu,n})^N}\int_\Omega \mathcal{A}\left(\frac{y}{\varepsilon_n}\right)^{-1}\tilde{\tilde{v}}_{\varepsilon_n}(y)\cdot\left(\varphi\left(\frac{y}{\theta_{\nu,n}}\right)-\varphi(y)\right)dy\\
&\qquad+ \frac{1}{(\theta_{\nu,n})^N}\int_\Omega \mathcal{A}\left(\frac{y}{\varepsilon_n}\right)^{-1}\tilde{\tilde{v}}_{\varepsilon_n}(y)\cdot\varphi(y)\,dy.
\end{aligned} \tag{4.37}$$

By (3.3) the first term in the right-hand side of (4.37) is bounded by

$$\begin{aligned}
&\left|\frac{1}{(\theta_{\nu,n})^N}\int_\Omega \mathcal{A}\left(\frac{y}{\varepsilon_n}\right)^{-1}\tilde{\tilde{v}}_{\varepsilon_n}(y)\cdot\left(\varphi\left(\frac{y}{\theta_{\nu,n}}\right)-\varphi(y)\right)dy\right|\\
&\quad\leq C\|\tilde{\tilde{v}}_{\varepsilon_n}\|_{L^p(\Omega;\mathbb{R}^d)}\|\nabla\varphi\|_{L^\infty(\Omega;\mathbb{M}^{d\times N})}\sup_{y\in\Omega}\left|\frac{y}{\theta_{\nu,n}}-y\right|,
\end{aligned}$$

which is infinitesimal as $n \to +\infty$ due to (4.29). By (4.29) and (4.31), the second term in the right-hand side of (4.37) satisfies

$$\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{(\theta_{\nu,n})^N}\int_\Omega \mathcal{A}\left(\frac{y}{\varepsilon_n}\right)^{-1}\tilde{\tilde{v}}_{\varepsilon_n}(y)\cdot\varphi(y)\,dy\\
&\quad= \int_\Omega \left[u(x) - \int_Q \mathcal{A}(z)^{-1}\,dz\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy\right]\cdot\varphi(x)\,dx.
\end{aligned}$$

Arguing by density, we conclude that

$$L(x) = u(x) - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right)\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy \quad \text{for a.e. } x\in\Omega$$

and (4.35) holds for the entire sequence. This completes the proof of (4.27).

Step 4 We claim that for every $w \in \mathcal{C}_u^{\mathscr{A}}$, and every p-equiintegrable family $\{v_{\nu,n} : \nu \in \mathbb{N},\ n \in \mathbb{N}\}$ satisfying (4.25)–(4.27), there exists a p-equiintegrable family $\{w_{\nu,n} : \nu \in \mathbb{N},\ n \in \mathbb{N}\}$ such that

$$\operatorname{div}_y w_{\nu,n}(x,y) := \sum_{i=1}^{N} \frac{\partial w_{\nu,n}}{\partial y_i}(x,y) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega, \tag{4.38}$$

$$w_{\nu,n} \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega \times Q;\mathbb{R}^d), \quad \text{as } n \to +\infty, \tag{4.39}$$

$$\int_Q w_{\nu,n}(x,y)\,dy = 0 \quad \text{for a.e. } x \in \Omega, \tag{4.40}$$

$$\mathcal{A}\left(\left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor y\right)^{-1} w_{\nu,n}(x,y) \rightharpoonup u(x) - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right) \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy \tag{4.41}$$

weakly in $L^p(\Omega;\mathbb{R}^d)$ as $n \to +\infty$ and $\nu \to +\infty$, in this order, and

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_\Omega g\left(\nu \left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor x, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + v_{\nu,n}(x)\right) dx \\
&\geq \liminf_{n\to+\infty} \int_\Omega \int_Q g\left(\left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor y, \int_Q \mathcal{A}(z)(u(x)+w(x,z))\,dz + w_{\nu,n}(x,y)\right) dy\,dx + \sigma_\nu,
\end{aligned} \tag{4.42}$$

where $\sigma_\nu \to 0$ as $\nu \to +\infty$. Let $\{v_{\nu,n}\}$ be as above. We argue similarly to [14, Proof of Proposition 3.9]. We extend u, w, and $\{v_{\nu,n}\}$ to 0 outside Ω, and define

$$\begin{aligned}
Q_{\nu,z} &:= \frac{1}{\nu} z + \frac{1}{\nu} Q, \quad z \in \mathbb{Z}^N, \\
\mathbb{Z}_\nu &:= \{z \in \mathbb{Z}^N : Q_{\nu,z} \cap \Omega \neq \emptyset\}, \\
I_{\nu,n} &:= \int_\Omega g\left(\nu \left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor x, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + v_{\nu,n}(x)\right) dx.
\end{aligned}$$

By a change of variables, since $g(\cdot,0) = 0$ by (4.13), and by the periodicity of g in its first variable, we obtain the following chain of equalities

$$\begin{aligned}
I_{\nu,n} &= \sum_{z\in\mathbb{Z}_\nu} \frac{1}{\nu^N} \int_Q g\left(\nu \left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor \left(\frac{z}{\nu} + \frac{\eta}{\nu}\right), \int_Q \mathcal{A}(y)\left(u\left(\frac{z}{\nu}+\frac{\eta}{\nu}\right) + w\left(\frac{z}{\nu}+\frac{\eta}{\nu}, y\right)\right) dy \right.\\
&\qquad \left. + v_{\nu,n}\left(\frac{z}{\nu}+\frac{\eta}{\nu}\right)\right) d\eta \\
&= \sum_{z\in\mathbb{Z}_\nu} \int_{Q_{\nu,z}} \int_Q g\left(\left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor \eta, \int_Q \mathcal{A}(y)\left(u\left(\frac{\lfloor \nu x \rfloor}{\nu}+\frac{\eta}{\nu}\right) + w\left(\frac{\lfloor \nu x \rfloor}{\nu}+\frac{\eta}{\nu}, y\right)\right) dy \right.\\
&\qquad \left. + v_{\nu,n}\left(\frac{\lfloor \nu x \rfloor}{\nu}+\frac{\eta}{\nu}\right)\right) d\eta\,dx \\
&= \sum_{z\in\mathbb{Z}_\nu} \int_{Q_{\nu,z}} \int_Q g\left(\left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor \eta, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + T_{\frac{1}{\nu}} v_{\nu,n}(x,\eta)\right) d\eta\,dx + \sigma_\nu
\end{aligned}$$

where $T_{\frac{1}{\nu}}$ is the unfolding operator defined in (2.1), and

$$\sigma_\nu := \sum_{z\in\mathbb{Z}_\nu} \int_{Q_{\nu,z}} \int_Q \Bigg\{ g\left(\left\lfloor \frac{1}{\nu\varepsilon_n}\right\rfloor \eta, \int_Q \mathcal{A}(y)\left(u\left(\frac{\lfloor \nu x \rfloor}{\nu}+\frac{\eta}{\nu}\right)+w\left(\frac{\lfloor \nu x \rfloor}{\nu}+\frac{\eta}{\nu}, y\right)\right) dy \right.$$
$$\left. + T_{\frac{1}{\nu}} v_{\nu,n}(x,\eta)\right) - g\left(\left\lfloor \frac{1}{\nu\varepsilon_n}\right\rfloor \eta, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + T_{\frac{1}{\nu}} v_{\nu,n}(x,\eta)\right)\Bigg\}\, d\eta\, dx.$$

By Proposition 2.7,

$$\left\| \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy - \int_Q \mathcal{A}(y)\left(T_{\frac{1}{\nu}} u(x,\eta)+T_{\frac{1}{\nu}} w((x,\eta),y)\right) dy \right\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \to 0$$

as $\nu \to +\infty$. Moreover, by Proposition 2.7, the sequence

$$\left\{ \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy - \int_Q \mathcal{A}(y)\left(T_{\frac{1}{\nu}} u(x,\eta)+T_{\frac{1}{\nu}} w((x,\eta),y)\right) dy \right\}$$

is p-equiintegrable and

$$\Bigg\| \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy$$
$$- \int_Q \mathcal{A}(y)\left(T_{\frac{1}{\nu}} u(x,\eta)+T_{\frac{1}{\nu}} w((x,\eta),y)\right) dy \Bigg\|_{L^p\left(\left(\left(\cup_{z\in\mathbb{Z}_\nu} Q_{\nu,z}\setminus\Omega\right)\times Q;\mathbb{R}^d\right)\right)} \to 0$$

as $\nu \to +\infty$. Hence, by Proposition 4.8, $\sigma_\nu = \sigma_\nu(\Omega) \to 0$ as $\nu \to +\infty$.

We set

$$\hat{v}_{\nu,z,n}(y) := T_{\frac{1}{\nu}} v_{\nu,n}\left(\frac{z}{\nu}, y\right) \quad \text{for a.e. } y \in Q, \text{ for every } z \in \mathbb{Z}_\nu.$$

For fixed ν and $z \in \mathbb{Z}_\nu$, the sequence $\{\hat{v}_{\nu,z,n}\}$ is p-equiintegrable. Moreover,

$$\operatorname{div}_y \hat{v}_{\nu,z,n} = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l)$$

and

$$\hat{v}_{\nu,z,n} \rightharpoonup 0 \quad \text{weakly in } L^p(Q;\mathbb{R}^d), \text{ as } n \to +\infty.$$

Setting $w_{\nu,z,n} := \hat{v}_{\nu,z,n} - \int_Q \hat{v}_{\nu,z,n}(y)\,dy$, the sequence $\{w_{\nu,z,n}\} \subset L^p(Q;\mathbb{R}^d)$ is p-equiintegrable and such that

$$\begin{aligned} &\operatorname{div}_y w_{\nu,z,n} = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l),\\ &w_{\nu,z,n} \rightharpoonup 0 \quad \text{weakly in } L^p(Q;\mathbb{R}^d), \text{ as } n \to +\infty\\ &\int_Q w_{\nu,z,n}(y)\,dy = 0, \end{aligned}$$

and

$$w_{\nu,z,n} - \hat{v}_{\nu,z,n} \to 0 \quad \text{strongly in } L^p(Q;\mathbb{R}^d), \text{ as } n \to +\infty. \tag{4.43}$$

By applying again Proposition 4.8 we obtain

$$\int_{Q_{\nu,z}}\int_Q g\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y, \int_Q \mathcal{A}(\xi)(u(x)+w(x,\xi))\,d\xi + T_{\frac{1}{\nu}}v_{\nu,n}\left(\frac{z}{\nu},y\right)\right)dy\,dx$$
$$\geq \int_{Q_{\nu,z}}\int_Q g\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y, \int_Q \mathcal{A}(\xi)(u(x)+w(x,\xi))\,d\xi + w_{\nu,z,n}(y)\right)dy\,dx + \tau_{z,\nu,n}$$

with

$$\tau_{z,\nu,n} \to 0 \quad \text{as } n \to +\infty.$$

Therefore, by (4.13) and since

$$\Omega \subset \cup_{z\in\mathbb{Z}^N} Q_{\nu,z},$$

we deduce (4.42) with

$$w_{\nu,n}(x,y) := \sum_{z\in\mathbb{Z}_\nu}\chi_{Q_{\nu,z}\cap\Omega}(x)w_{\nu,z,n}(y) \quad \text{for a.e. } x\in\Omega,\ y\in Q.$$

We observe that for ν fixed only a finite number of terms in the sum above are different from zero, hence properties (4.38), (4.39) and (4.40) follow immediately.

To prove (4.41), we notice that the sequence $\left\{\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}w_{\nu,n}(x,y)\right\}$ is uniformly bounded in $L^p(\Omega\times Q;\mathbb{R}^d)$ by (3.3) and (4.39), therefore it is enough to work with a convergent subsequence and check that the limit is uniquely determined. Fix $\varphi \in C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Q;\mathbb{R}^d))$ and set

$$\psi(x) := u(x) - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right)\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy$$

for a.e. $x\in\Omega$, $\psi := 0$ outside Ω, and

$$\mathbb{Z}^\nu_\varphi := \{z\in\mathbb{Z}^N : (Q_{\nu,z}\times Q)\cap \operatorname{supp}\varphi \neq \emptyset\}.$$

Then

$$\int_\Omega\int_Q \mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1} w_{\nu,n}(x,y)\cdot\varphi(x,y)\,dy\,dx$$
$$= \sum_{z\in\mathbb{Z}^\nu_\varphi}\int_{(Q_{\nu,z}\times Q)\cap\operatorname{supp}\varphi}\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1} w_{\nu,z,n}(y)\cdot\varphi(x,y)\,dy\,dx$$
$$= \sum_{z\in\mathbb{Z}^\nu_\varphi}\int_Q\int_{Q_{\nu,z}}\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}(w_{\nu,z,n}(y)-\hat{v}_{\nu,z,n}(y))\cdot\varphi(x,y)\,dy\,dx$$
$$+\sum_{z\in\mathbb{Z}^\nu_\varphi}\int_Q\int_{Q_{\nu,z}}\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}\hat{v}_{\nu,z,n}(y)\cdot\varphi(x,y)\,dy\,dx.$$

By (3.3), we have

$$\begin{aligned}&\left|\sum_{z\in\mathbb{Z}^\nu_\varphi}\int_Q\int_{Q_{\nu,z}}\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}(w_{\nu,z,n}(y)-\hat{v}_{\nu,z,n}(y))\cdot\varphi(x,y)\,dy\,dx\right|\\&\quad\le C\sum_{z\in\mathbb{Z}^\nu_\varphi}\|w_{\nu,z,n}(y)-\hat{v}_{\nu,z,n}(y)\|_{L^p(Q;\mathbb{R}^N)}\|\varphi\|_{L^\infty(\Omega\times Q;\mathbb{R}^d)},\end{aligned}$$

which by (4.43) converges to zero as $n\to+\infty$ (here we used the fact that the previous series is actually a finite sum for every $\nu\in\mathbb{N}$ fixed). On the other hand,

$$\begin{aligned}&\sum_{z\in\mathbb{Z}^\nu_\varphi}\int_Q\int_{Q_{\nu,z}}\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}\hat{v}_{\nu,z,n}(y)\cdot\varphi(x,y)\,dy\,dx\\&\quad=\sum_{z\in\mathbb{Z}^\nu_\varphi}\int_Q\int_{Q_{\nu,z}}\left[\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}T_{\frac{1}{\nu}}v_{\nu,n}\left(\frac{z}{\nu},y\right)-T_{\frac{1}{\nu}}\psi\left(\frac{z}{\nu},y\right)\right]\cdot\varphi(x,y)\,dy\,dx\\&\qquad+\sum_{z\in\mathbb{Z}^\nu_\varphi}\int_Q\int_{Q_{\nu,z}}T_{\frac{1}{\nu}}\psi\left(\frac{z}{\nu},y\right)\cdot\varphi(x,y)\,dy\,dx.\end{aligned}\tag{4.44}$$

By the periodicity of $\mathcal{A}$, the first term in the right-hand side of (4.44) satisfies

$$\begin{aligned}&\lim_{n\to+\infty}\sum_{z\in\mathbb{Z}^\nu_\varphi}\int_Q\int_{Q_{\nu,z}}\left[\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}T_{\frac{1}{\nu}}v_{\nu,n}\left(\frac{z}{\nu},y\right)-T_{\frac{1}{\nu}}\psi\left(\frac{z}{\nu},y\right)\right]\cdot\varphi(x,y)\,dy\,dx\\&=\lim_{n\to+\infty}\left(\frac{1}{\nu}\right)^N\sum_{z\in\mathbb{Z}^\nu_\varphi}\int_Q\int_Q\left[\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}v_{\nu,n}\left(\frac{z}{\nu}+\frac{y}{\nu}\right)-\psi\left(\frac{z}{\nu}+\frac{y}{\nu}\right)\right]\\&\quad\cdot\varphi\left(\frac{z}{\nu}+\frac{\eta}{\nu},y\right)dy\,d\eta\\&=\lim_{n\to+\infty}\sum_{z\in\mathbb{Z}^\nu_\varphi}\int_{Q_{\nu,z}}\int_Q\left[\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor\nu x\right)^{-1}v_{\nu,n}(x)-\psi(x)\right]\\&\quad\cdot\varphi\left(\frac{1}{\nu}\lfloor\nu x\rfloor+\frac{1}{\nu}\eta,\nu x\right)d\eta\,dx\\&=\lim_{n\to+\infty}\int_{\cup_{z\in\mathbb{Z}^\nu_\varphi}Q_{\nu,z}}\left[\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor\nu x\right)^{-1}v_{\nu,n}(x)-\psi(x)\right]\\&\quad\cdot\left(\int_Q\varphi\left(\frac{1}{\nu}\lfloor\nu x\rfloor+\frac{1}{\nu}\eta,\nu x\right)d\eta\right)dx.\end{aligned}$$

By (4.27), and recalling that ψ and $\{v_{\nu,n}\}$ have been extended to 0 outside Ω, we conclude that

$$\begin{aligned}&\lim_{n\to+\infty}\int_\Omega\int_Q\mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}w_{\nu,n}(x,y)\cdot\varphi(x,y)\,dy\,dx\\&\quad=\sum_{z\in\mathbb{Z}_\nu}\int_Q\int_{Q_{\nu,z}}T_{\frac{1}{\nu}}\psi\left(\frac{z}{\nu},y\right)\cdot\varphi(x,y)\,dx\,dy,\end{aligned}$$

and so

$$\mathcal{A}\left(\left\lfloor \frac{1}{\nu\varepsilon_n} \right\rfloor y\right)^{-1} w_{\nu,n}(x,y) \rightharpoonup \sum_{z\in\mathbb{Z}_\nu} \chi_{Q_{\nu,z}\cap\Omega}(x) T_{\frac{1}{\nu}}\psi\left(\frac{z}{\nu}, y\right)$$

weakly in $L^p(\Omega\times Q;\mathbb{R}^d)$, as $n\to+\infty$. Finally, we claim that

$$\sum_{z\in\mathbb{Z}_\nu} \chi_{Q_{\nu,z}\cap\Omega}(x) T_{\frac{1}{\nu}}\psi\left(\frac{z}{\nu}, y\right) \to \psi(x) \tag{4.45}$$

strongly in $L^p(\Omega;\mathbb{R}^d)$ as $\nu\to+\infty$.

Indeed, let $\varphi\in L^{p'}(\Omega\times Q;\mathbb{R}^d)$. Then by Holder's inequality

$$\begin{aligned}
&\left|\int_\Omega\int_Q \left(\sum_{z\in\mathbb{Z}_\nu} \chi_{Q_{\nu,z}\cap\Omega}(x) T_{\frac{1}{\nu}}\psi\left(\frac{z}{\nu}, y\right) - \psi(x)\right)\cdot\varphi(x,y)\,dy\,dx\right| \\
&= \left|\sum_{z\in\mathbb{Z}_\nu}\int_{Q_{\nu,z}\cap\Omega}\int_Q \left(T_{\frac{1}{\nu}}\psi\left(\frac{z}{\nu}, y\right) - \psi(x)\right)\cdot\varphi(x,y)\,dy\,dx\right| \\
&= \left|\sum_{z\in\mathbb{Z}_\nu}\int_{Q_{\nu,z}\cap\Omega}\int_Q \left(\psi\left(\frac{z}{\nu} + \frac{y}{\nu}\right) - \psi(x)\right)\cdot\varphi(x,y)\,dy\,dx\right| \\
&= \left|\sum_{z\in\mathbb{Z}_\nu}\int_{Q_{\nu,z}\cap\Omega}\int_Q \left(\psi\left(\frac{1}{\nu}\left\lfloor \nu x\right\rfloor + \frac{y}{\nu}\right) - \psi(x)\right)\cdot\varphi(x,y)\,dy\,dx\right| \\
&= \left|\int_\Omega\int_Q (T_{\frac{1}{\nu}}\psi(x,y) - \psi(x))\cdot\varphi(x,y)\,dy\,dx\right| \\
&\le \|T_{\frac{1}{\nu}}\psi(x,y) - \psi(x)\|_{L^p(\Omega\times Q;\mathbb{R}^d)}\|\varphi\|_{L^{p'}(\Omega\times Q;\mathbb{R}^d)}.
\end{aligned}$$

Property (4.45), and thus (4.41), follow in view of Proposition 2.7.

Step 5 By Steps 1–4 it follows that

$$\begin{aligned}
&\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right. \\
&\qquad\left. \text{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)\right\} \\
&\ge \inf_{w\in C_u}\inf\left\{\liminf_{\nu\to+\infty}\liminf_{n\to+\infty}\int_\Omega\int_Q g\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y, \int_Q \mathcal{A}(z)(u(x)\right.\right. \\
&\quad + w(x,z))\,dz + w_{\nu,n}(x,y)\Bigg)\,dy\,dx : \\
&\quad \mathrm{div}_y\, w_{\nu,n}(x,y) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x\in\Omega, \\
&\quad w_{\nu,n} \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega\times Q;\mathbb{R}^d) \text{ as } n\to+\infty, \\
&\quad \mathcal{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1} w_{\nu,n}(x,y) \rightharpoonup u(x) - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right)\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy \\
&\qquad \text{weakly in } L^p(\Omega;\mathbb{R}^d) \text{ as } n\to+\infty \text{ and } \nu\to+\infty, \text{ and} \\
&\quad \left.\int_Q w_{\nu,n}(x,y)\,dy = 0\right\}.
\end{aligned} \tag{4.46}$$

By a diagonalization argument, given $\{w_{\nu,n}\}$ as above we can construct $\{n(\nu)\}$ such that, setting

$$\varepsilon_\nu := \varepsilon_{n(\nu)} \quad w_\nu(x,y) := w_{\nu,n(\nu)},$$

we obtain the following inequality

$$\begin{aligned}
&\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right.\\
&\qquad\left.\text{and } \mathscr{A}_\varepsilon^{\rm div} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)\right\}\\
&\quad\geq \inf_{w\in C_u}\inf\left\{\liminf_{\nu\to+\infty}\int_\Omega\int_Q g\left(\left\lfloor\frac{1}{\nu\varepsilon_\nu}\right\rfloor y, \int_Q A(z)(u(x)+w(x,z))\,dz+w_\nu(x,y)\right)dy\,dx :\right.\\
&\quad \operatorname{div}_y w_\nu(x,y)=0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x\in\Omega,\\
&\quad w_\nu \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega\times Q;\mathbb{R}^d) \text{ as } \nu\to+\infty,\\
&\quad A\left(\left\lfloor\frac{1}{\nu\varepsilon_\nu}\right\rfloor y\right)^{-1} w_\nu(x,y) \rightharpoonup u(x) - \left(\int_Q A(z)^{-1}\,dz\right)\int_Q A(y)(u(x)+w(x,y))\,dy\\
&\qquad\left.\text{weakly in } L^p(\Omega;\mathbb{R}^d) \text{ as } \nu\to+\infty, \text{ and } \int_Q w_\nu(x,y)\,dy = 0\right\}.
\end{aligned}\tag{4.47}$$

Associating to every sequence $\{w_\nu\}$ as in (4.47) the maps

$$\phi_\nu(x,y) := \int_Q A(z)(u(x)+w(x,z))\,dz + w_\nu(x,y) \quad \text{for a.e. } x\in\Omega \text{ and } y\in Q,$$

inequality (4.47) can be rewritten as

$$\begin{aligned}
&\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right.\\
&\qquad\left.\text{and } \mathscr{A}_\varepsilon^{\rm div} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)\right\}\\
&\quad\geq \inf_{w\in C_u}\inf\left\{\liminf_{\nu\to+\infty}\int_\Omega\int_Q f\left(A\left(\left\lfloor\frac{1}{\nu\varepsilon_\nu}\right\rfloor y\right)^{-1}\phi_\nu(x,y)\right)dy\,dx :\right.\\
&\quad \operatorname{div}_y \phi_\nu(x,y)=0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x\in\Omega,\\
&\quad \phi_\nu \rightharpoonup \int_Q A(z)(u(x)+w(x,z))\,dz \quad \text{weakly in } L^p(\Omega\times Q;\mathbb{R}^d) \text{ as } \nu\to+\infty,\\
&\quad A\left(\left\lfloor\frac{1}{\nu\varepsilon_\nu}\right\rfloor y\right)^{-1}\phi_\nu(x,y) \rightharpoonup u(x) \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d) \quad \text{as } \nu\to+\infty, \text{ and}\\
&\quad\left.\int_Q \phi_\nu(x,z)\,dz = \int_Q A(z)(u(x)+w(x,z))\,dz\right\}.
\end{aligned}\tag{4.48}$$

Finally, for $\{\phi_\nu\}$ as above, considering the maps

$$v_\nu(x,y) := A\left(\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor y\right)^{-1}\phi_\nu(x,y) \quad \text{for a.e. } x\in\Omega \text{ and } y\in Q,$$

we deduce that

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right.$$

$$
\begin{aligned}
&\text{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)\Big\} \\
&\geq \inf_{w\in\mathcal{C}_u^{\mathscr{A}}} \inf \Big\{ \liminf_{\nu\to+\infty} \int_\Omega \int_Q f(v_\nu(x,y))\,dy\,dx : \\
&\operatorname{div}_y \left(\mathcal{A}\left(\left\lfloor \frac{1}{\nu\varepsilon_\nu}\right\rfloor y\right) v_\nu(x,y)\right) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x\in\Omega, \\
&\mathcal{A}\left(\left\lfloor \frac{1}{\nu\varepsilon_\nu}\right\rfloor y\right) v_\nu(x,y) \rightharpoonup \int_Q \mathcal{A}(z)(u(x) \\
&\quad + w(x,z))\,dz \quad \text{weakly in } L^p(\Omega\times Q;\mathbb{R}^d) \text{ as } \nu\to+\infty, \\
&v_\nu(x,y) \rightharpoonup u(x) \quad \text{weakly in } L^p(\Omega\times Q;\mathbb{R}^d) \text{ as } \nu\to+\infty, \text{ and} \\
&\int_Q \mathcal{A}\left(\left\lfloor \frac{1}{\nu\varepsilon_\nu}\right\rfloor z\right) v_\nu(x,z)\,dz = \int_Q \mathcal{A}(z)(u(x)+w(x,z))\,dz \Big\} \\
&\geq \inf \Big\{ \liminf_{\nu\to+\infty} \int_\Omega \int_Q f(u(x)+w_\nu(x,y))\,dy\,dx : \\
&\operatorname{div}_y \left(\mathcal{A}\left(\left\lfloor \frac{1}{\nu\varepsilon_\nu}\right\rfloor y\right)(u(x)+w_\nu(x,y))\right) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \quad \text{for a.e. } x\in\Omega, \\
&\operatorname{div}_x \int_Q \left(\mathcal{A}\left(\left\lfloor \frac{1}{\nu\varepsilon_\nu}\right\rfloor y\right)(u(x)+w_\nu(x,y))\right) dy = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^l), \text{ and} \\
&w_\nu(x,y) \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega\times Q;\mathbb{R}^d)\Big\}.
\end{aligned} \tag{4.49}
$$

By (4.49) it follows, in particular, that

$$
\begin{aligned}
&\inf \Big\{ \liminf_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d) \\
&\qquad \text{and } \mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)\Big\} \\
&\quad \geq \inf \Big\{ \liminf_{\nu\to+\infty} \int_\Omega \int_Q f(u_\nu(x)+w_\nu(x,y))\,dy\,dx : \\
&\qquad u_\nu \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\ w_\nu \in \mathcal{C}_{u^\nu}^{\mathscr{A}\left(\left\lfloor \frac{1}{\nu\varepsilon_\nu}\right\rfloor \cdot\right)}, \\
&\qquad w_\nu(x,y) \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega\times Q;\mathbb{R}^d)\Big\}.
\end{aligned} \tag{4.50}
$$

Fix $\{u_\nu\}$ and $\{w_\nu\}$ as in (4.50). Then there exists a constant r such that

$$
\sup_{\nu\in\mathbb{N}} \|w_\nu\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \leq r. \tag{4.51}
$$

Therefore, setting

$$
n_\nu := \left\lfloor \frac{1}{\nu\varepsilon_\nu}\right\rfloor,
$$

and

$$
u_n := \begin{cases} u_\nu & \text{if } n = n_\nu, \\ u & \text{otherwise,} \end{cases}
$$

we have

$$\liminf_{\nu\to+\infty}\int_\Omega\int_Q f(u_\nu(x)+w_\nu(x,y))\,dy\,dx \geq \liminf_{\nu\to+\infty}\mathcal{F}^r_{\mathscr{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_\nu}\right\rfloor\cdot\right)}(u_\nu)$$

$$= \liminf_{\nu\to+\infty}\overline{\mathcal{F}}^r_{\mathscr{A}\left(\left\lfloor\frac{1}{\nu\varepsilon_\nu}\right\rfloor\cdot\right)}(u_\nu) = \liminf_{\nu\to+\infty}\overline{\mathcal{F}}^r_{\mathscr{A}(n_\nu\cdot)}(u_{n_\nu}) \geq \liminf_{n\to+\infty}\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(u_n)$$

$$\geq \inf\left\{\liminf_{n\to+\infty}\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(u_n) : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right\}. \tag{4.52}$$

The thesis follows by taking the infimum with respect to r in the right-hand side of (4.52) and by invoking (4.50). □

Remark 4.10 We point out that the truncation by r in (4.51) and (4.52) will be used in a fundamental way. Infact it guarantees that the sequences constructed in the proof of the limsup inequality (see Proposition 4.12) are uniformly bounded in L^p, and hence it allows us to apply Attouch's diagonalization lemma (see [3, Lemma 1.15 and Corollary 1.16]) in Step 3 of the proof of Proposition 4.12.

Remark 4.11 In the case in which $A^i \in L^\infty(\mathbb{R}^N;\mathbb{M}^{l\times d})$, $i=1,\dots,N$, the previous proof yields the inequality

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right.$$
$$\left.\text{and } \mathscr{A}^{\mathrm{div}}_\varepsilon u_\varepsilon \to 0 \text{ strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \quad \text{for every } 1\leq q<p\right\}$$
$$\geq \mathscr{F}_{\mathscr{A}}(u).$$

To see that, arguing as in Step 1 we get

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right.$$
$$\left.\text{and } \mathscr{A}^{\mathrm{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \quad \text{for every } 1\leq q<p\right\}$$
$$\geq \inf_{w\in\mathcal{C}^{\mathscr{A}}_u}\inf\left\{\liminf_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, v_{\varepsilon_n}(x)\right)dx :\right.$$
$$v_{\varepsilon_n} \rightharpoonup \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),$$
$$\operatorname{div} v_{\varepsilon_n} \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \quad \text{for every } 1\leq q<p,$$
$$\left.\text{and } \mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} v_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right\}.$$

By Lemma 4.3 and Remark 4.4, inequality (4.17) is replaced by

$$\inf_{w\in\mathcal{C}^{\mathscr{A}}_u}\inf\left\{\liminf_{n\to+\infty}\int_\Omega g\left(\frac{x}{\varepsilon_n}, v_{\varepsilon_n}(x)\right)dx :\right.$$
$$v_{\varepsilon_n} \rightharpoonup \int_Q\int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),$$

$$
\begin{aligned}
&\operatorname{div} v_{\varepsilon_n} \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \quad \text{for every } 1 \le q < p,\\
&\text{and } \mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} v_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\Big\}\\
\ge & \inf_{w\in \mathcal{C}^{\mathscr{A}}_u} \inf \Big\{ \liminf_{n\to+\infty} \int_\Omega g\left(\frac{x}{\varepsilon_n}, \int_Q \mathcal{A}(y)(u(x)+w(x,y))\,dy + \tilde{v}_{\varepsilon_n}(x)\right) dx :\\
&\{\tilde{v}_{\varepsilon_n}\} \text{ is p-equiintegrable, } \tilde{v}_{\varepsilon_n} \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\\
&\operatorname{div} \tilde{v}_{\varepsilon_n} = 0 \text{ in } W^{-1,p}(\Omega;\mathbb{R}^l), \text{ and}\\
&\mathcal{A}\left(\frac{x}{\varepsilon_n}\right)^{-1} \tilde{v}_{\varepsilon_n} \rightharpoonup u(x) - \left(\int_Q \mathcal{A}(z)^{-1}\,dz\right)\int_Q \mathcal{A}(y)^{-1}(u(x)+w(x,y))\,dy\\
&\text{weakly in } L^p(\Omega;\mathbb{R}^d)\Big\}.
\end{aligned}
$$

The result now follows by arguing exactly as in the proof of Proposition 4.9.

We finally prove the limsup inequality in Theorem 4.2.

Proposition 4.12 *Under the assumptions of Theorem 4.2, for every $u \in \mathcal{C}^{\mathscr{A}}$ there exists a sequence $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d)$ such that*

$$u_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega;\mathbb{R}^d), \tag{4.53}$$

$$\mathscr{A}^{\text{div}}_\varepsilon u_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l), \tag{4.54}$$

$$\limsup_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon(x))\,dx \le \mathscr{F}_{\mathscr{A}}(u). \tag{4.55}$$

Proof We subdivide the proof into three steps.
Step 1 Fix $n \in \mathbb{N}$. We first show that for every $u \in \mathcal{C}^{\mathscr{A}(n\cdot)} \cap C^1(\Omega;\mathbb{R}^d)$ and $w \in \mathcal{C}^{\mathscr{A}(n\cdot)}_u \cap C^1(\Omega; C^1_{\text{per}}(\mathbb{R}^N;\mathbb{R}^d))$ there exists a sequence $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d)$ and a constant C independent of n and ε such that

$$u_\varepsilon \overset{2-s}{\to} u + w \quad \text{strongly two-scale in } L^p(\Omega\times Q;\mathbb{R}^d), \tag{4.56}$$

$$\mathscr{A}^{\text{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l), \tag{4.57}$$

$$\int_\Omega f(u_\varepsilon(x))\,dx \to \int_\Omega\int_Q f(u(x)+w(x,y))\,dy\,dx, \tag{4.58}$$

as $\varepsilon \to 0$, and

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^d)} \le C(\|u\|_{L^p(\Omega;\mathbb{R}^d)} + \|w\|_{L^p(\Omega\times Q;\mathbb{R}^d)}). \tag{4.59}$$

Define

$$u_\varepsilon(x) := u(x) + w\left(x, \frac{x}{n\varepsilon}\right) \quad \text{for a.e. } x \in \Omega.$$

By Proposition 2.3 we have

$$
\begin{aligned}
&u_\varepsilon \rightharpoonup \int_Q (u(x)+w(x,y))\,dy = u(x) \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\\
&u_\varepsilon \overset{2-s}{\to} u + w \quad \text{strongly two-scale in } L^p(\Omega\times Q;\mathbb{R}^d),\\
&f(u_\varepsilon) \rightharpoonup \int_Q f(u(x)+w(x,y))\,dy \quad \text{weakly in } L^1(\Omega),
\end{aligned}
$$

as $\varepsilon \to 0$. In particular, we obtain immediately (4.56) and (4.58). Property (4.59) follows by (4.56), Proposition 2.5 and Theorem 2.6. To prove (4.57), we notice that

$$\begin{aligned}\mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon &= \sum_{i=1}^N A^i\left(\frac{x}{\varepsilon}\right)\left(\frac{\partial u(x)}{\partial x_i} + \frac{\partial w}{\partial x_i}\left(x, \frac{x}{n\varepsilon}\right)\right)\\ &\quad + \frac{1}{n\varepsilon}\left(n\frac{\partial A^i}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\left(u(x) + w\left(x, \frac{x}{n\varepsilon}\right)\right) + A^i\left(\frac{x}{\varepsilon}\right)\frac{\partial w}{\partial y_i}\left(x, \frac{x}{n\varepsilon}\right)\right)\\ &= \sum_{i=1}^N A^i\left(n\frac{x}{n\varepsilon}\right)\left(\frac{\partial u(x)}{\partial x_i} + \frac{\partial w}{\partial x_i}\left(x, \frac{x}{n\varepsilon}\right)\right) \end{aligned} \tag{4.60}$$

where in the last equality we used the fact that $w \in \mathcal{C}_u^{\mathscr{A}(n\cdot)} \cap C^1(\Omega; C^1_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d))$. Applying Proposition 2.3 we obtain

$$\mathscr{A}_\varepsilon^{\mathrm{div}} u_\varepsilon \rightharpoonup \int_Q \sum_{i=1}^N A^i(ny)\left(\frac{\partial u(x)}{\partial x_i} + \frac{\partial w}{\partial x_i}(x, y)\right) dy \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d), \tag{4.61}$$

and hence strongly in $W^{-1,p}(\Omega; \mathbb{R}^d)$ by the compact embedding $L^p \hookrightarrow W^{-1,p}$. On the other hand, since $w \in \mathcal{C}_u^{\mathscr{A}(n\cdot)}$, there holds

$$\begin{aligned}\int_Q \sum_{i=1}^N A^i(ny)\left(\frac{\partial u(x)}{\partial x_i} + \frac{\partial w}{\partial x_i}(x, y)\right) dy &= \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(ny)(u(x) + w(x, y))\, dy\right)\\ &= 0.\end{aligned} \tag{4.62}$$

Combining (4.61) with (4.62) we deduce (4).

Step 2 We will now extend the construction in Step 1 to the general case where $u \in \mathcal{C}^{\mathscr{A}(n\cdot)}$ and $w \in \mathcal{C}_u^{\mathscr{A}(n\cdot)}$. Extend u and w by setting them equal to zero outside Ω and $\Omega \times Q$, respectively. We claim that we can find sequences $\{u^k\}$ and $\{w^k\}$ such that $u^k \in C^\infty(\bar{\Omega}; \mathbb{R}^d)$, $w^k \in C^\infty(\bar{\Omega}; C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d))$, and

$$u^k \to u \quad \text{strongly in } L^p(\Omega; \mathbb{R}^d), \tag{4.63}$$

$$w^k \to w \quad \text{strongly in } L^p(\Omega \times Q; \mathbb{R}^d), \tag{4.64}$$

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(ny)(u^k(x) + w^k(x, y))\, dy\right) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l), \tag{4.65}$$

and

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(A^i(ny)(u^k(x) + w^k(x, y))\right) \to 0 \quad \text{strongly in } L^p(\Omega; W^{-1,p}(Q; \mathbb{R}^l)). \tag{4.66}$$

Indeed, by first regularizing u and w with respect to the variable x, we construct two sequences $\{u^k\}$ and $\{\tilde{w}^k\}$ such that $u^k \in C^\infty(\bar{\Omega}; \mathbb{R}^d)$, $\tilde{w}^k \in C^\infty(\bar{\Omega}; L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d))$, and

$$u^k \to u \quad \text{strongly in } L^p(\Omega; \mathbb{R}^d),$$
$$\tilde{w}^k \to w \quad \text{strongly in } L^p(\Omega \times Q; \mathbb{R}^d),$$
$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(ny)(u^k(x) + \tilde{w}^k(x, y))\, dy\right) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l).$$

In addition,

$$\sum_{i=1}^{N} \frac{\partial}{\partial y_i}\Big(A^i(ny)(u^k(x) + \tilde{w}^k(x,y))\Big) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega. \tag{4.67}$$

Now, by regularizing with respect to y we construct $\{w^k\}$, such that $w^k \in C^\infty(\bar{\Omega}; C^\infty_{\text{per}}(\mathbb{R}^N;\mathbb{R}^d))$. It is immediate to see that $\{u^k\}$ and $\{w^k\}$ satisfy (4.63)–(4.65), and in particular

$$\tilde{w}^k - w^k \to 0 \quad \text{strongly in } L^p(\Omega; L^p_{\text{per}}(Q;\mathbb{R}^d)). \tag{4.68}$$

To prove (4.66), consider maps $\varphi \in L^{p'}(\Omega)$ and $\psi \in W_0^{1,p'}(\Omega;\mathbb{R}^l)$. By the regularity of the operators A^i and by (4.67) there holds

$$\begin{aligned}
&\left|\int_\Omega \left\langle \sum_{i=1}^{N} \frac{\partial}{\partial y_i}\Big(A^i(ny)(u^k(x) + w^k(x,y))\Big),\ \psi(y)\right\rangle \varphi(x)\,dx\right| \\
&= \left|\sum_{i=1}^{N}\int_\Omega\int_Q A^i(ny)(u^k(x)+w^k(x,y))\cdot\varphi(x)\frac{\partial\psi(y)}{\partial y_i}\,dy\,dx\right| \\
&= \left|\sum_{i=1}^{N}\int_\Omega\int_Q A^i(ny)(u^k(x)+w^k(x,y)-(u^k(x)+\tilde{w}^k(x,y)))\cdot\varphi(x)\frac{\partial\psi(y)}{\partial y_i}\,dy\,dx\right| \\
&\le C\|w^k - \tilde{w}^k\|_{L^p(\Omega\times Q;\mathbb{R}^d)}\|\psi\|_{W_0^{1,p'}(Q;\mathbb{R}^d)}\|\varphi\|_{L^{p'}(\Omega)}.
\end{aligned}$$

Property (4.66) follows now by (4.68).

Apply Lemma 3.3 and Remark 3.4 to the sequence $\{u^k + w^k\}$ to construct a sequence $\{v^k\} \subset C^1(\Omega; C^\infty_{\text{per}}(\mathbb{R}^N;\mathbb{R}^d))$ such that

$$v^k - (u^k + w^k) \to 0 \quad \text{strongly in } L^p(\Omega\times Q;\mathbb{R}^d), \tag{4.69}$$

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\int_Q A^i(ny)v^k(x,y)\,dy\right) = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^l) \tag{4.70}$$

and

$$\sum_{i=1}^{N} \frac{\partial}{\partial y_i}(A^i(ny)v^k(x,y)) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x \in \Omega. \tag{4.71}$$

Consider now the maps

$$v^k_\varepsilon(x) := v^k\left(x, \frac{x}{n\varepsilon}\right) \quad \text{for a.e. } x \in \Omega.$$

By Proposition 2.3, arguing as in the proof of (4.59), we observe that

$$\begin{aligned}
&v^k_\varepsilon \rightharpoonup \int_Q v^k(x,y)\,dy \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\\
&v^k_\varepsilon \xrightarrow{2-s} v^k \quad \text{strongly two-scale in } L^p(\Omega\times Q;\mathbb{R}^d),\\
&\int_\Omega f(v^k_\varepsilon(x))\,dx \to \int_\Omega\int_Q f(v^k(x,y))\,dy\,dx.
\end{aligned}$$

as $\varepsilon \to 0$, and there exists a constant C independent of ε and k such that

$$\|v_\varepsilon^k\|_{L^p(\Omega;\mathbb{R}^d)} \leq C\|v^k\|_{L^p(\Omega\times Q;\mathbb{R}^d)}$$

for every ε and k. Hence, by (4.63), (4.64), and (4.69), there exists a constant C independent of ε and k such that

$$\|v_\varepsilon^k\|_{L^p(\Omega;\mathbb{R}^d)} \leq C(\|u\|_{L^p(\Omega;\mathbb{R}^d)} + \|w\|_{L^p(\Omega\times Q;\mathbb{R}^d)})$$

for every ε and k. In addition, again by Proposition 2.3, proceeding as in the proof of we can establish the analogues of (4.60)–(4.62) and we conclude that

$$\mathscr{A}_\varepsilon^{\mathrm{div}} v_\varepsilon^k = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(A^i\left(\frac{x}{\varepsilon}\right)v_\varepsilon^k(x)\right) \to \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(ny)v^k(x,y)\,dy\right) = 0 \quad (4.72)$$

strongly in $W^{-1,p}(\Omega;\mathbb{R}^l)$, as $\varepsilon \to 0$. Now, by (4.63), (4.64) and (4.69),

$$v_\varepsilon^k \overset{2-s}{\to} u + w \quad \text{strongly two-scale in } L^p(\Omega \times Q;\mathbb{R}^d) \tag{4.73}$$

as $\varepsilon \to 0$ and $k \to +\infty$, in this order. Hence, by Theorem 2.6, (4.8), (4.63), (4.64), (4.69), (4.72) and (4.73),

$$\limsup_{k\to+\infty}\limsup_{\varepsilon\to 0}\Big\{\|T_\varepsilon v_\varepsilon^k - (u+w)\|_{L^p(\Omega\times Q;\mathbb{R}^d)} + \|\mathscr{A}_\varepsilon^{\mathrm{div}} v_\varepsilon^k\|_{W^{-1,p}(\Omega;\mathbb{R}^l)} + \left|\int_\Omega f(v_\varepsilon^k(x))\,dx - \int_\Omega\int_Q f(u(x)+w(x,y))\,dy\,dx\right|\Big\} = 0.$$

By Attouch's diagonalization lemma [3, Lemma 1.15 and Corollary 1.16] we can extract a sequence $\{k(\varepsilon)\}$ such that, setting

$$v^\varepsilon := v_\varepsilon^{k(\varepsilon)},$$

the sequence $\{v^\varepsilon\}$ satisfies (4.56)–(4.59).

Step 3 Let $u \in \mathcal{C}^{\mathscr{A}}$ and $\eta > 0$. Then $\mathscr{F}_{\mathscr{A}}(u) < +\infty$ and there exists $r_\eta > 0$ such that

$$\mathscr{F}_{\mathscr{A}}(u) + \eta > \inf\left\{\liminf_{n\to+\infty} \overline{\mathcal{F}}^{r_\eta}_{\mathscr{A}(n\cdot)}(u_n) : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d)\right\}.$$

In particular, there exists a sequence $\{u_n^\eta\} \subset L^p(\Omega;\mathbb{R}^d)$ with $u_n \in \mathcal{C}^{\mathscr{A}(n\cdot)}$ for every $n \in \mathbb{N}$, such that

$$u_n^\eta \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d) \tag{4.74}$$

as $n \to +\infty$, and

$$\mathscr{F}_{\mathscr{A}}(u) + \eta \geq \lim_{n\to+\infty} \overline{\mathcal{F}}^{r_\eta}_{\mathscr{A}(n\cdot)}(u_n^\eta) = \lim_{n\to+\infty} \mathcal{F}^{r_\eta}_{\mathscr{A}(n\cdot)}(u_n^\eta).$$

By the definition of $\mathcal{F}^{r_\eta}_{\mathscr{A}(n\cdot)}$, for every $n \in \mathbb{N}$ there exists $w_n^\eta \in C^{\mathscr{A}(n\cdot)}_{u_n^\eta}$ such that

$$\|w_n^\eta\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \leq r_\eta, \tag{4.75}$$

and

$$\mathcal{F}^{r_\eta}_{\mathscr{A}(n\cdot)}(u_n^\eta) \leq \int_\Omega\int_Q f(u_n^\eta(x) + w_n^\eta(x,y))\,dy\,dx \leq \mathcal{F}^{r_\eta}_{\mathscr{A}(n\cdot)}(u_n^\eta) + \frac{1}{n}.$$

Applying Steps 1 and 2 we construct sequences $\{v^\eta_{n,\varepsilon}\} \subset L^p(\Omega \times Q; \mathbb{R}^d)$ such that

$$\begin{aligned}
&\sup_{\varepsilon>0} \|v^\eta_{n,\varepsilon}\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \leq C\|u^\eta_n + w^\eta_n\|_{L^p(\Omega\times Q;\mathbb{R}^d)},\\
&v^\eta_{n,\varepsilon} \rightharpoonup u^\eta_n \quad \text{weakly in } L^p(\Omega \times Q; \mathbb{R}^d),\\
&\mathscr{A}^{\text{div}}_\varepsilon v^\eta_{n,\varepsilon} \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l),\\
&\int_\Omega f(v^\eta_{n,\varepsilon}(x))\,dx \to \int_\Omega \int_Q f(u^\eta_n(x) + w^\eta_n(x,y))\,dy\,dx,
\end{aligned} \tag{4.76}$$

as $\varepsilon \to 0$. In addition, by (4.74)–(4.76), the sequence $\{v^\eta_{n,\varepsilon}\}$ is uniformly bounded in $L^p(\Omega \times Q; \mathbb{R}^d)$. Therefore, by the metrizability of bounded sets in the weak L^p topology and Attouch's diagonalization lemma [3, Lemma 1.15 and Corollary 1.16] there exists a sequence $\{n(\varepsilon)\}$ such that, setting

$$u^\eta_\varepsilon := v^\eta_{n(\varepsilon),\varepsilon},$$

properties (4.53) and (4.54) are fulfilled, with

$$\limsup_{\varepsilon\to 0} \int_\Omega f(u^\eta_\varepsilon(x))\,dx \leq \mathscr{F}_{\mathscr{A}}(u) + \eta.$$

The thesis follows now by the arbitrariness of η. $\square$

5 Homogenization for measurable operators

Here we prove the main result of this paper, concerning the case in which $A^i \in L^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1, \ldots, N$.

Theorem 5.1 *Let $1 < p < +\infty$. Let $A^i \in L^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1, \ldots, N$, assume that the operator $\mathcal{A}$ satisfies the invertibility requirement in (3.3), and let $\mathscr{A}^{\text{div}}_\varepsilon$ be the operator defined in (3.1). Let $f : \mathbb{R}^d \to [0, +\infty)$ be a continuous function satisfying the growth condition (4.8). Then, for every $u \in L^p(\Omega; \mathbb{R}^d)$ there holds*

$$\begin{aligned}
&\inf\Big\{\liminf_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega; \mathbb{R}^d),\\
&\qquad \textit{and } \mathscr{A}^{\text{div}}_\varepsilon u_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \textit{ for every } 1 \leq q < p\Big\}\\
&= \inf\Big\{\limsup_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega; \mathbb{R}^d),\\
&\qquad \textit{and } \mathscr{A}^{\text{div}}_\varepsilon u_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \textit{ for every } 1 \leq q < p\Big\} = \mathscr{F}_{\mathscr{A}}(u).
\end{aligned}$$

The strategy of the proof consists in constructing a sequence of operators $\mathscr{A}_k$ with smooth coefficients which approximate the operator $\mathscr{A}$, so that Theorem 4.2 can be applied to each $\mathscr{A}_k$. Let $\{\rho_k\}$ be a sequence of mollifiers and consider the operators

$$\mathscr{A}_k : L^p(\Omega; \mathbb{R}^d) \to W^{-1,p}(\Omega; \mathbb{R}^l)$$

defined as

$$\mathscr{A}_k v(x) := \sum_{i=1}^N A^i_k(x) \frac{\partial v(x)}{\partial x_i}, \tag{5.1}$$

where $A^i_k := A^i * \rho_k$ for every $i = 1, \dots, N$, and for every k. Then $A^i_k \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1, \dots, N$, for every k,

$$A^i_k \to A^i \quad \text{strongly in } L^m(Q; \mathbb{M}^{l\times d}) \tag{5.2}$$

for $1 \le m < +\infty, \quad i = 1, \dots, N$,

$$\|A^i_k\|_{L^\infty(Q;\mathbb{M}^{l\times d})} \le \|A^i\|_{L^\infty(Q;\mathbb{M}^{l\times d})} \quad \text{for } i = 1, \dots, N, \tag{5.3}$$

and the operators $\mathscr{A}_k$ satisfy the uniform ellipticity condition

$$\mathbb{A}_k(x)\lambda \cdot \lambda \ge \alpha|\lambda|^2 \quad \text{for every } \lambda \in \mathbb{R}^d, \quad \text{for every } k. \tag{5.4}$$

We first prove two preliminary lemmas. The first one will allow us to approximate every element $u \in \mathcal{C}^{\mathscr{A}}$ by sequences $\{u^k\} \subset L^p(\Omega; \mathbb{R}^d)$ with $u^k \in \mathcal{C}^{\mathscr{A}_k}$ for every k.

Lemma 5.2 *Let $1 < p < +\infty$. Let $\mathscr{A}$ be as in Theorem 5.1 and let $\{\mathscr{A}_k\}$ be the sequence of operators with smooth coefficients constructed as above. Let $\mathcal{C}^{\mathscr{A}}$ be the class introduced in (3.23) and let $u \in \mathcal{C}^{\mathscr{A}}$. Then there exists a sequence $\{u^k\} \subset L^p(\Omega; \mathbb{R}^d)$ such that $u^k \in \mathcal{C}^{\mathscr{A}_k}$ for every k, and*

$$u^k \to u \quad \textit{strongly in } L^p(\Omega; \mathbb{R}^d).$$

Moreover, for every $w \in \mathcal{C}^{\mathscr{A}}_u$ there exists a sequence $\{w^k\} \subset L^p(\Omega \times Q; \mathbb{R}^d)$ such that $w^k \in \mathcal{C}^{\mathscr{A}_k}_{u^k}$ for every k and

$$w^k \to w \quad \textit{strongly in } L^p(\Omega \times Q; \mathbb{R}^d).$$

Proof Let $u \in \mathcal{C}^{\mathscr{A}}$ and let $w \in \mathcal{C}^{\mathscr{A}}_u$. We first construct a sequence $\{v^n\} \subset L^p(\Omega \times Q; \mathbb{R}^d)$, defined as

$$v^n(x) := (u(x) + w(x, y))\varphi^n(x) \quad \text{for a.e. } x \in \Omega, y \in Q,$$

where $\{\varphi_n\} \in C^\infty_c(\Omega; [0, 1])$ with $\varphi_n \nearrow 1$. Without loss of generality, up to a dilation and a translation we can assume that $\Omega \subset Q$. Extending each map v^n by zero in $Q\backslash\Omega$ and then periodically, and arguing as in the proof of Lemma 3.3, it is easy to see that

$$\begin{aligned}
&v^n \to u + w \quad \text{strongly in } L^p(Q \times Q; \mathbb{R}^d),\\
&\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i(y)v^n(x, y)\,dy\right) \to 0 \quad \text{strongly in } W^{-1,p}(Q; \mathbb{R}^l),\\
&\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(A^i(y)v^n(x, y)\right) = 0 \quad \text{in } L^p(Q; W^{-1,p}(Q; \mathbb{R}^l)).
\end{aligned} \tag{5.5}$$

By (5.2), (5.3) and the dominated convergence theorem, we also have

$$A^i_k(y)v^n(x, y) \to A^i(y)v^n(x, y) \quad \text{strongly in } L^p(Q \times Q; \mathbb{R}^l)$$

as $k \to +\infty$, for every n. Therefore,

$$\begin{aligned}
&\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A^i_k(y)v^n(x, y)\,dy\right) \to 0 \quad \text{strongly in } W^{-1,p}(Q; \mathbb{R}^l),\\
&\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(A^i_k(y)v^n(x, y)\right) \to 0 \quad \text{strongly in } L^p(Q; W^{-1,p}(Q; \mathbb{R}^l))
\end{aligned}$$

as $k \to +\infty$ and $n \to +\infty$, in this order. In particular,

$$\lim_{n\to+\infty}\lim_{k\to+\infty}\Bigg\{\|v^n-(u+w)\|_{L^p(Q\times Q;\mathbb{R}^d)} + \left\|\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A_k^i(y)v^n(x,y)\,dy\right)\right\|_{W^{-1,p}(Q;\mathbb{R}^l)} + \sum_{i=1}^N \|A_k^i(y)v^n(x,y)-A^i(y)(u(x)+w(x,y))\|_{L^p(Q\times Q;\mathbb{R}^d)} + \left\|\left\|\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(A_k^i(y)v^n(x,y)\right)\right\|_{W^{-1,p}(Q;\mathbb{R}^l)}\right\|_{L^p(Q)}\Bigg\}=0,$$

hence by Attouch's diagonalization lemma [3, Lemma 1.15 and Corollary 1.16], we can extract a subsequence $\{n(k)\}$ such that

$$v^{n(k)} \to u+w \quad \text{strongly in } L^p(\Omega\times Q;\mathbb{R}^d), \tag{5.6}$$

$$\begin{aligned} &A_k^i(y)v^{n(k)}(x,y) \to A^i(y)(u(x)+w(x,y)) \quad \text{strongly in } L^p(Q\times Q;\mathbb{R}^l),\\ &\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q A_k^i(y)v^{n(k)}(x,y)\,dy\right) \to 0 \quad \text{strongly in } W^{-1,p}(Q;\mathbb{R}^l),\\ &\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(A_k^i(y)v^{n(k)}(x,y)\right) \to 0 \quad \text{strongly in } L^p(Q;W^{-1,p}(Q;\mathbb{R}^l)) \end{aligned} \tag{5.7}$$

as $k \to +\infty$. Setting

$$R_k^i(x,y) := A_k^i(y)v^{n(k)}(x,y) \quad \text{for a.e. } (x,y)\in Q\times Q,$$

and defining the mappings $R^k \in L^p(Q;L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d))$ as

$$R_{ij}^k := (R_k^i)_j, \quad \text{for all } i=1,\dots,N,\ j=1\dots,l,$$

we have that

$$\begin{aligned} &R_k^i(x,y) \to A^i(y)(u(x)+w(x,y)),\ i=1,\dots,N \quad \text{strongly in } L^p(Q\times Q;\mathbb{R}^d),\\ &\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q R_k^i(x,y)\,dy\right) \to 0 \quad \text{strongly in } W^{-1,p}(Q;\mathbb{R}^l),\\ &\sum_{i=1}^N \frac{\partial}{\partial y_i}(R_k^i(x,y)) \to 0 \quad \text{strongly in } L^p(Q;W^{-1,p}(Q;\mathbb{R}^l)). \end{aligned}$$

Therefore, using Lemma 3.2 we argue as in Lemma 3.3 and construct a sequence $S^k \in L^p(Q;L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d))$, satisfying

$$S^k - R^k \to 0 \quad \text{strongly in } L^p(Q \times Q; \mathbb{R}^d), \tag{5.8}$$

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\int_Q S^i_k(x,y)\,dy\right) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l), \tag{5.9}$$

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}(S^i_k(x,y)) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^l), \quad \text{for a.e. } x \in Q. \tag{5.10}$$

Finally, setting

$$u^k(x) := \int_Q \mathcal{A}_k(y)^{-1} S^k(x,y)\,dy \quad \text{for a.e. } x \in \Omega$$

and

$$w^k(x,y) := \mathcal{A}_k(y)^{-1} S^k(x,y) - u^k(x) \quad \text{for a.e. } x \in \Omega \text{ and } y \in Q,$$

by (5.9) and (5.10) we deduce that $w^k \in \mathcal{C}^{\mathscr{A}_k}_{u^k}$, i.e. $u^k \in \mathcal{C}^{\mathscr{A}_k}$ for every k. Moreover, by (3.3), (5.6) and (5.8),

$$\begin{aligned}
\|u^k - u\|_{L^p(\Omega;\mathbb{R}^d)} &= \left\| \int_Q (u^k(x) + w^k(x,y) - (u(x) + w(x,y)))\,dy \right\|_{L^p(\Omega;\mathbb{R}^d)} \\
&\le C\|u^k + w^k - (u+w)\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \\
&\le \|u^k + w^k - v^{n(k)}\|_{L^p(\Omega\times Q;\mathbb{R}^d)} + \|v^{n(k)} - (u+w)\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \\
&= \|\mathcal{A}_k(y)^{-1}(S^k(x,y) - R^k(x,y))\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \\
&\quad + \|v^{n(k)} - (u+w)\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \to 0.
\end{aligned}$$

The convergence of w^k to w follows in a similar way. □

In view of Lemma 5.2 we can prove the analog of Proposition 3.5 in the case in which $A^i \in L^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{M}^{l\times d})$, $i = 1, \ldots, N$.

Lemma 5.3 *Under the assumptions of Theorem 5.1, there holds*

$$\begin{aligned}
\mathcal{C}^{\mathscr{A}} = \Big\{ & u \in L^p(\Omega;\mathbb{R}^d) : \textit{there exists a sequence } \{v_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d) \textit{ such that} \\
& v_\varepsilon \rightharpoonup u \quad \textit{weakly in } L^p(\Omega;\mathbb{R}^d) \\
& \textit{and } \mathscr{A}^{\text{div}}_\varepsilon v_\varepsilon \to 0 \quad \textit{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l), \textit{ for all } 1 \le q < p \Big\}.
\end{aligned} \tag{5.11}$$

Proof Let $\mathcal{D}$ be the set in the right-hand side of (5.11). The inclusion

$$\mathcal{D} \subset \mathcal{C}^{\mathscr{A}}$$

follows by Remark 3.6. To prove the opposite inclusion, let $u \in \mathcal{C}^{\mathscr{A}}$ and let $w \in \mathcal{C}^{\mathscr{A}}_u$. By Lemma 5.2 we construct sequences $\{u^k\} \subset L^p(\Omega;\mathbb{R}^d)$ and $\{w^k\} \subset L^p(\Omega \times Q;\mathbb{R}^d)$ such that $u^k \in \mathcal{C}^{\mathscr{A}_k}$ for every k, $w^k \in \mathcal{C}^{\mathscr{A}_k}_{u^k}$ for every k,

$$u^k \to u \quad \text{strongly in } L^p(\Omega;\mathbb{R}^d)$$

and

$$w^k \to w \quad \text{strongly in } L^p(\Omega \times Q;\mathbb{R}^d),$$

where $\{\mathscr{A}_k\}$ is the sequence of operators with smooth coefficients defined in (5.1). By Proposition 3.5, for every k there exists a sequence $\{v^k_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^d)$ such that

$$v^k_\varepsilon \overset{2-s}{\to} u^k + w^k \quad \text{strongly 2-scale in } L^p(\Omega \times Q;\mathbb{R}^d),$$
$$\mathscr{A}^{\text{div}}_{k,\varepsilon} v^k_\varepsilon := \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(A^i_k\left(\frac{x}{\varepsilon}\right) v^k_\varepsilon(x)\right) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)$$

as $\varepsilon \to 0$. Hence, in particular, by Theorem 2.6,

$$\lim_{k\to+\infty}\lim_{\varepsilon\to 0} \|T_\varepsilon v^k_\varepsilon - (u+w)\|_{L^p(\Omega\times Q;\mathbb{R}^d)} + \|\mathscr{A}^{\text{div}}_{k,\varepsilon} v^k_\varepsilon\|_{W^{-1,p}(\Omega;\mathbb{R}^l)} = 0.$$

By Attouch's diagonalization lemma ([3, Lemma 1.15 and Corollary 1.16]), we can extract a subsequence $\{k(\varepsilon)\}$ such that

$$v^{k(\varepsilon)}_\varepsilon \overset{2-s}{\to} u + w \quad \text{strongly 2-scale in } L^p(\Omega \times Q;\mathbb{R}^d),$$
$$\mathscr{A}_{k(\varepsilon),\varepsilon}{}^{\text{div}} v^{k(\varepsilon)}_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l)$$

as $\varepsilon \to 0$. A truncation argument analogous to [16, Lemma 2.15] yields a p-equiintegrable sequence $\{v_\varepsilon\}$ satisfying

$$v_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),$$
$$\mathscr{A}_{k(\varepsilon),\varepsilon}{}^{\text{div}} v_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l), \tag{5.12}$$

for every $1 \le q < p$.

To complete the proof, it remains to prove that

$$\mathscr{A}^{\text{div}}_\varepsilon v_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \quad \text{for every } 1 \le q < p. \tag{5.13}$$

To this purpose, we first notice that, by (5.2) and by Severini–Egoroff's theorem, there exists a sequence of measurable sets $\{E_n\} \subset Q$ such that $|E_n| \le \frac{1}{n}$ and

$$A^i_{k(\varepsilon)} \to A^i \quad \text{uniformly on } Q\backslash E_n \tag{5.14}$$

for every $n = 1, \dots, +\infty$. Let $\eta > 0$ and $1 \le q < p$ be fixed, and for $z \in \mathbb{Z}^N$, set

$$Q_{\varepsilon,z} := \varepsilon z + \varepsilon Q \quad \text{and } Z_\varepsilon := \{z \in \mathbb{Z}^N : Q_{\varepsilon,z} \cap \Omega \neq \emptyset\}.$$

By the p-equiintegrability of $\{v_\varepsilon\}$ and hence of $\{T_\varepsilon v_\varepsilon\}$ (see Proposition 2.7), and by (5.3), we can assume that

$$\|T_\varepsilon v_\varepsilon\|_{L^p((\cup_{z\in Z_\varepsilon} Q_{\varepsilon,z}\backslash\Omega)\times Q;\mathbb{R}^d)} \le \eta, \tag{5.15}$$

and n is such that

$$\sum_{i=1}^N \int_\Omega \int_{E_n} |A^i_{k(\varepsilon)}(y) - A^i(y)|^q |T_\varepsilon v_\varepsilon(x,y)|^q \, dydx$$
$$\le C \sum_{i=1}^N (\|A^i_{k(\varepsilon)}\|^q_{L^\infty(Q;\mathbb{M}^{l\times d})} + \|A^i\|^q_{L^\infty(Q;\mathbb{M}^{l\times d})}) \|T_\varepsilon v_\varepsilon(x,y)\|^q_{L^q(\Omega\times E_n;\mathbb{R}^d)} \le \eta. \tag{5.16}$$

We first notice that, by (3.1), there holds

$$\begin{aligned}\left\|\mathscr{A}_\varepsilon^{\mathrm{div}} v_\varepsilon\right\|^q_{W^{-1,q}(\Omega;\mathbb{R}^l)} &\leq \left\|\mathscr{A}_{k(\varepsilon),\varepsilon}{}^{\mathrm{div}} v_\varepsilon\right\|^q_{W^{-1,q}(\Omega;\mathbb{R}^l)}\\ &\quad+\sum_{i=1}^N \int_\Omega \left|A^i_{k(\varepsilon)}\left(\frac{x}{\varepsilon}\right)-A^i\left(\frac{x}{\varepsilon}\right)\right|^q |v_\varepsilon(x)|^q\,dx\\ &\leq \left\|\mathscr{A}_{k(\varepsilon),\varepsilon}{}^{\mathrm{div}} v_\varepsilon\right\|^q_{W^{-1,q}(\Omega;\mathbb{R}^l)}+\sum_{i=1}^N \int_{\cup_{z\in Z_\varepsilon}(Q_{\varepsilon,z}\cap\Omega)} \left|A^i_{k(\varepsilon)}\left(\frac{x}{\varepsilon}\right)-A^i\left(\frac{x}{\varepsilon}\right)\right|^q |v_\varepsilon(x)|^q\,dx.\end{aligned}$$

Therefore, by (5.12) we deduce

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\left\|\mathscr{A}_\varepsilon^{\mathrm{div}} v_\varepsilon\right\|^q_{W^{-1,q}(\Omega;\mathbb{R}^l)}\\ &\quad\leq \limsup_{\varepsilon\to 0}\sum_{i=1}^N \int_{\cup_{z\in Z_\varepsilon}(Q_{\varepsilon,z}\cap\Omega)} \left|A^i_{k(\varepsilon)}\left(\frac{x}{\varepsilon}\right)-A^i\left(\frac{x}{\varepsilon}\right)\right|^q |v_\varepsilon(x)|^q\,dx. \end{aligned}\tag{5.17}$$

Changing variables, using the periodicity of the operators, and extending v_ε to zero outside Ω, the right-hand side of (5.17) can be estimated as

$$\begin{aligned}&\sum_{i=1}^N \int_{\cup_{z\in Z_\varepsilon}(Q_{\varepsilon,z}\cap\Omega)} \left|A^i_{k(\varepsilon)}\left(\frac{x}{\varepsilon}\right)-A^i\left(\frac{x}{\varepsilon}\right)\right|^q |v_\varepsilon(x)|^q\,dx\\ &\quad\leq \varepsilon^N\sum_{i=1}^N\sum_{z\in Z_\varepsilon}\int_Q |A^i_{k(\varepsilon)}(y)-A^i(y)|^q|v_\varepsilon(\varepsilon z+\varepsilon y)|^q\,dy\\ &\quad=\sum_{i=1}^N\sum_{z\in Z_\varepsilon}\int_{Q_{\varepsilon,z}}\int_Q |A^i_{k(\varepsilon)}(y)-A^i(y)|^q\left|v_\varepsilon\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor+\varepsilon y\right)\right|^q\,dydx\\ &\quad=\sum_{i=1}^N\int_{\cup_{z\in Z_\varepsilon}Q_{\varepsilon,z}}\int_Q |A^i_{k(\varepsilon)}(y)-A^i(y)|^q|T_\varepsilon v_\varepsilon(x,y)|^q\,dydx. \end{aligned}\tag{5.18}$$

By Proposition 2.5, (5.15), and by (5.16), the right-hand side of (5.18) is bounded from above as follows

$$\begin{aligned}&\sum_{i=1}^N\int_{\cup_{z\in Z_\varepsilon}Q_{\varepsilon,z}}\int_Q |A^i_{k(\varepsilon)}(y)-A^i(y)|^q|T_\varepsilon v_\varepsilon(x,y)|^q\,dydx\\ &\quad\leq \sum_{i=1}^N\int_{\cup_{z\in Z_\varepsilon}Q_{\varepsilon,z}\setminus\Omega}\int_Q |A^i_{k(\varepsilon)}(y)-A^i(y)|^q|T_\varepsilon v_\varepsilon(x,y)|^q\,dydx\\ &\qquad+\sum_{i=1}^N\int_\Omega\int_{E_n} |A^i_{k(\varepsilon)}(y)-A^i(y)|^q|T_\varepsilon v_\varepsilon(x,y)|^q\,dydx\\ &\qquad+\sum_{i=1}^N\int_\Omega\int_{Q\setminus E_n} |A^i_{k(\varepsilon)}(y)-A^i(y)|^q|T_\varepsilon v_\varepsilon(x,y)|^q\,dydx\\ &\quad\leq \eta+C\sum_{i=1}^N(\|A^i_{k(\varepsilon)}\|^q_{L^\infty(Q;\mathbb{M}^{l\times d})}+\|A^i\|^q_{L^\infty(Q;\mathbb{M}^{l\times d})})\|T_\varepsilon v_\varepsilon\|^q_{L^q((\cup_{z\in Z_\varepsilon}Q_{\varepsilon,z}\setminus\Omega)\times Q;\mathbb{R}^d)}\end{aligned}$$

$$+\sum_{i=1}^{N}\|A^i_{k(\varepsilon)}-A^i\|^q_{L^\infty(Q\setminus E_n;\mathbb{M}^{l\times d})}\|v_\varepsilon\|^q_{L^q(\Omega;\mathbb{R}^d)}$$

$$\leq C\eta + C\sum_{i=1}^{N}\|A^i_{k(\varepsilon)}-A^i\|_{L^\infty(Q\setminus E_n;\mathbb{M}^{l\times d})}. \tag{5.19}$$

Finally, by (5.14) and collecting (5.17)–(5.19), we obtain

$$\limsup_{\varepsilon\to 0}\left\|\mathscr{A}^{\mathrm{div}}_\varepsilon v_\varepsilon\right\|^q_{W^{-1,q}(\Omega;\mathbb{R}^l)} \leq C\eta + \lim_{\varepsilon\to 0} C\sum_{i=1}^{N}\|A^i_{k(\varepsilon)}-A^i\|_{L^\infty(Q\setminus E_n;\mathbb{M}^{l\times d})} = C\eta$$

for every $\eta > 0$. By the arbitrariness of η we conclude (5.13). □

Proof of Theorem 4.2 We first notice that by Remark 3.6 the thesis is trivial if $u \notin \mathcal{C}^{\mathscr{A}}$. By Proposition 4.9 and Remark 4.11, for every $u \in \mathcal{C}^{\mathscr{A}}$ there holds

$$\inf\Bigg\{\liminf_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),$$
$$\text{and } \mathscr{A}^{\mathrm{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \text{ for every } 1\leq q<p\Bigg\} \geq \mathscr{F}_{\mathscr{A}}(u).$$

To complete the proof of the theorem, since

$$\inf\Bigg\{\limsup_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \qquad \text{weakly in } L^p(\Omega;\mathbb{R}^d),$$
$$\text{and } \mathscr{A}^{\mathrm{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \text{ for every } 1\leq q<p\Bigg\}$$
$$\leq \inf\Bigg\{\limsup_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\ \{u_\varepsilon\} \text{ p-equiintegrable},$$
$$\text{and } \mathscr{A}^{\mathrm{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \text{ for every } 1\leq q<p\Bigg\},$$

it suffices to show that

$$\inf\Bigg\{\limsup_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon(x))\,dx : u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^d),\ \{u_\varepsilon\} \text{ p-equiintegrable},$$
$$\text{and } \mathscr{A}^{\mathrm{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \text{ for every } 1\leq q<p\Bigg\} \leq \mathscr{F}_{\mathscr{A}}(u). \tag{5.20}$$

To prove (5.20), we argue by approximation. Let $\{\mathscr{A}_k\}$ be the sequence of operators constructed in (5.1) and satisfying (5.2)–(5.4). We first prove that for n, $u_n \in \mathcal{C}^{\mathscr{A}(n\cdot)}$ and $w_n \in \mathcal{C}^{\mathscr{A}(n\cdot)}_{u_n}$ fixed,

$$\inf\Bigg\{\liminf_{k\to+\infty}\mathcal{F}^{2\|w_n\|_{L^p(\Omega\times Q;\mathbb{R}^d)}}_{\mathscr{A}_k(n\cdot)}(u_n+\phi_k) : \phi_k \to 0 \quad \text{strongly in } L^p(\Omega;\mathbb{R}^d),$$
$$u_n+\phi_k \in \mathcal{C}^{\mathscr{A}_k(n\cdot)} \text{ for every } k\Bigg\}$$
$$\leq \int_\Omega\int_Q f(u_n(x)+w_n(x,y))\,dy\,dx. \tag{5.21}$$

Indeed, by Lemma 5.2 there exist sequences $\{u^k\}\subset L^p(\Omega;\mathbb{R}^d)$ and $\{w^k\}\subset L^p(\Omega\times Q;\mathbb{R}^d)$, such that $u^k\in\mathcal{C}^{\mathscr{A}_k(n\cdot)}$, $w^k\in\mathcal{C}^{\mathscr{A}_k(n\cdot)}_{u^k}$ for every k,

$$u^k\to u_n\quad\text{strongly in } L^p(\Omega;\mathbb{R}^d),$$

and

$$w^k\to w_n\quad\text{strongly in } L^p(\Omega\times Q;\mathbb{R}^d). \tag{5.22}$$

For k big enough,

$$\mathcal{F}^{2\|w_n\|_{L^p(\Omega\times Q;\mathbb{R}^d)}}_{\mathscr{A}_k(n\cdot)}(u_n+(u^k-u_n))\le\int_\Omega\int_Q f(u^k(x)+w^k(x,y))\,dy\,dx\quad\text{for every } k,$$

and

$$\limsup_{k\to+\infty}\mathcal{F}^{2\|w_n\|_{L^p(\Omega\times Q;\mathbb{R}^d)}}_{\mathscr{A}_k(n\cdot)}(u_n+(u^k(x)-u_n(x)))\le\int_\Omega\int_Q f(u_n(x)+w_n(x,y))\,dy\,dx,$$

which in turn implies (5.21).

Let now $u\in\mathcal{C}^{\mathscr{A}}$ and $\eta>0$ be fixed. Then, there exist $r_\eta>0$ such that

$$\mathscr{F}_{\mathscr{A}}(u)+\eta\ge\inf\left\{\liminf_{n\to+\infty}\overline{\mathcal{F}}^{r_\eta}_{\mathscr{A}(n\cdot)}(u_n):\,u_n\rightharpoonup u\quad\text{weakly in } L^p(\Omega;\mathbb{R}^d)\right\},$$

and sequences $\{u_n^\eta\}\in L^p(\Omega;\mathbb{R}^d)$, $\{w_n^\eta\}\in L^p(\Omega\times Q;\mathbb{R}^d)$ satisfying

$$\begin{aligned}&u_n^\eta\rightharpoonup u\quad\text{weakly in } L^p(\Omega;\mathbb{R}^d),\\&w_n^\eta\in\mathcal{C}^{\mathscr{A}(n\cdot)}_{u_n^\eta}\quad\text{for every } n,\\&\|w_n^\eta\|_{L^p(\Omega\times Q;\mathbb{R}^d)}\le r_\eta\quad\text{for every } n,\end{aligned}\tag{5.23}$$

and

$$\mathscr{F}_{\mathscr{A}}(u)+2\eta\ge\lim_{n\to+\infty}\int_\Omega\int_Q f(u_n^\eta(x)+w_n^\eta(x,y))\,dy\,dx.$$

In particular, by (5.21),

$$\begin{aligned}&\mathscr{F}_{\mathscr{A}}(u)+2\eta\\&\ge\limsup_{n\to+\infty}\inf\Bigg\{\liminf_{k\to+\infty}\mathcal{F}^{2\|w_n^\eta\|_{L^p(\Omega\times Q;\mathbb{R}^d)}}_{\mathscr{A}_k(n\cdot)}(u_n^\eta+\phi_k):\,\phi_k\to 0\quad\text{strongly in } L^p(\Omega;\mathbb{R}^d),\\&\qquad u_n^\eta+\phi_k\in\mathcal{C}^{\mathscr{A}_k(n\cdot)}\text{ for every } k\Bigg\}.\end{aligned}$$

For every n,k, let $w^\eta_{n,k}\in\mathcal{C}^{\mathscr{A}_k(n\cdot)}_{u_n^\eta+\phi_k}$ be such that

$$\|w^\eta_{n,k}\|_{L^p(\Omega\times Q;\mathbb{R}^d)}\le 2\|w_n^\eta\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \tag{5.24}$$

and

$$\begin{aligned}\mathcal{F}^{2\|w_n^\eta\|_{L^p(\Omega\times Q;\mathbb{R}^d)}}_{\mathscr{A}_k(n\cdot)}(u_n^\eta+\phi_k)&\le\int_\Omega\int_Q f(u_n^\eta(x)+\phi_k(x)+w^\eta_{n,k}(x,y))\,dy\,dx\\&\le\mathcal{F}^{2\|w_n^\eta\|_{L^p(\Omega\times Q;\mathbb{R}^d)}}_{\mathscr{A}_k(n\cdot)}(u_n^\eta+\phi_k)+\frac{1}{k}.\end{aligned}$$

Arguing as in Steps 1 and 2 of Proposition 4.12, for every n, k we construct a sequence $\{v^\eta_{\varepsilon,n,k}(x)\} \subset L^p(\Omega; \mathbb{R}^d)$ such that

$$
\begin{aligned}
&\|v^\eta_{\varepsilon,n,k}\|_{L^p(\Omega;\mathbb{R}^d)} \le C\|u^\eta_n + \phi_k + w^\eta_{n,k}\|_{L^p(\Omega\times Q;\mathbb{R}^d)},\\
&v^\eta_{\varepsilon,n,k} \rightharpoonup u^\eta_n + \phi_k \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d),\\
&\mathscr{A}^{\mathrm{div}}_{k,\varepsilon} v^\eta_{\varepsilon,n,k} \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l),\\
&\int_\Omega f(v^\eta_{\varepsilon,n,k}(x))\,dx \to \int_\Omega\int_Q f(u^\eta_n(x) + \phi_k(x) + w^\eta_{n,k}(x, y))\,dy\,dx,
\end{aligned} \tag{5.25}
$$

as $\varepsilon \to 0$, where the constant C is independent of ε, n and k. In particular, (5.23) and (5.24) yield that the sequence $\{v^\eta_{\varepsilon,n,k}\}$ is uniformly bounded in $L^p(\Omega; \mathbb{R}^d)$. By (5.25) the metrizability of bounded sets in the weak L^p topology and Attouch's diagonalization lemma ([3, Lemma 1.15 and Corollary 1.16]), we can extract subsequences $\{n(\varepsilon)\}$ and $\{k(\varepsilon)\}$ such that, setting

$$
\tilde{u}_\varepsilon := v^\eta_{\varepsilon,n(\varepsilon),k(\varepsilon)},
$$

there holds

$$
\begin{aligned}
&\tilde{u}_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d),\\
&\mathscr{A}^{\mathrm{div}}_{k(\varepsilon)} \tilde{u}_\varepsilon \to 0 \quad \text{strongly in } W^{-1,p}(\Omega; \mathbb{R}^l),\\
&\limsup_{\varepsilon\to 0} \int_\Omega f(\tilde{u}_\varepsilon(x))\,dx \le \mathscr{F}_{\mathscr{A}}(u) + 2\eta.
\end{aligned}
$$

In view of Lemma 4.6 and Proposition 4.7, we can construct a further sequence $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^d)$, p-equiintegrable and satisfying

$$
\begin{aligned}
&u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d),\\
&\mathscr{A}^{\mathrm{div}}_{k(\varepsilon)} u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \quad \text{for every } 1 \le q < p,\\
&\limsup_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon(x))\,dx \le \limsup_{\varepsilon\to 0} \int_\Omega f(\tilde{u}_\varepsilon(x))\,dx \le \mathscr{F}_{\mathscr{A}}(u) + 2\eta.
\end{aligned}
$$

Property (5.20) follows now by noticing that exactly the same argument as in the proof of (5.13) yields

$$
\mathscr{A}^{\mathrm{div}}_\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^l) \quad \text{for every } 1 \le q < p,
$$

and by the arbitrariness of η. □

5.1 The case of constant coefficients

In this subsection we show that the homogenized energy obtained in Theorem 4.2 reduces to the one identified by Braides, Fonseca and Leoni in [7], in the case in which the operators A^i, $i = 1, \ldots, N$, are constant, and the constant rank condition (1.3) is satisfied by the differential operator $\mathscr{A} : L^p(\Omega; \mathbb{R}^d) \to W^{-1,p}(\Omega; \mathbb{R}^l)$ defined as

$$
\mathscr{A} u := \sum_{i=1}^N A^i \frac{\partial u}{\partial x_i} \quad \text{for every } u \in L^p(\Omega; \mathbb{R}^d).
$$

In this case the classes $\mathcal{C}_u^{\mathscr{A}}$ and $\mathcal{C}^{\mathscr{A}}$ defined in (3.22) and (3.23) become, respectively

$$\mathcal{C}_u^{\text{const}} := \left\{ w \in L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)) : \int_Q w(x,y)\,dy = 0 \text{ and } \sum_{i=1}^N A^i \frac{\partial w}{\partial y_i}(x,y) = 0 \right\}$$

and

$$\mathcal{C}^{\text{const}} := \left\{ u \in L^p(\Omega; \mathbb{R}^d) : \mathscr{A}u = 0 \right\}.$$

Recall that

$$\mathscr{F}_{\mathscr{A}}(u) = \inf_{r>0} \inf \left\{ \liminf_{n\to+\infty} \overline{\mathcal{F}}^r_{\mathscr{A}}(u_n) : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right\}.$$

By definition of $\mathscr{A}$-quasiconvex envelope, for every $u \in \mathcal{C}^{\text{const}}$ and $r > 0$ we have

$$\mathcal{F}^r_{\mathscr{A}}(u) \geq \int_\Omega Q_{\mathscr{A}} f(u(x))\,dx.$$

By [16, Theorem3.7], the $\mathscr{A}$-quasiconvex envelope is lower semicontinuous with respect to the weak L^p convergence of $\mathscr{A}$-vanishing maps, hence

$$\inf \left\{ \liminf_{n\to+\infty} \overline{\mathcal{F}}^r_{\mathscr{A}}(u_n) : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right\} \geq \int_\Omega Q_{\mathscr{A}} f(u(x))\,dx$$

for every $r > 0$, which yields

$$\mathscr{F}_{\mathscr{A}}(u) \geq \int_\Omega Q_{\mathscr{A}} f(u(x))\,dx. \tag{5.26}$$

Conversely, since $\mathscr{A}$ is a differential operator with constant coefficients, for every $u \in \mathcal{C}^{\text{const}}$ the null map belongs to $\mathcal{C}_u^{\text{const}}$. Hence,

$$\int_\Omega f(u(x))\,dx \geq \mathcal{F}^r_{\mathscr{A}}(u) = \overline{\mathcal{F}}^r_{\mathscr{A}}(u).$$

By taking the lower semicontinuous envelope of both sides with respect to the weak L^p convergence of $\mathscr{A}$-vanishing maps we obtain (see [7, Theorem1.1])

$$\begin{aligned}\int_\Omega Q_{\mathscr{A}}(f(u(x)))\,dx &\geq \inf \left\{ \liminf_{n\to+\infty} \overline{\mathcal{F}}^r_{\mathscr{A}}(u_n) : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right\} \\ &\geq \mathscr{F}_{\mathscr{A}}(u).\end{aligned} \tag{5.27}$$

Combining (5.26) and (5.27) we deduce that

$$\mathscr{F}_{\mathscr{A}}(u) = \int_\Omega Q_{\mathscr{A}} f(u(x))\,dx.$$

5.2 The case of a convex energy density

In this subsection we show that if f is convex then the definition of $\mathscr{F}_{\mathscr{A}}$ (see (4.3)) simplifies and reduces to a single cell formula.

For $u \in L^p(\Omega;\mathbb{R}^d)$ we introduce the set

$$\mathcal{C}^{\mathscr{A}}_{z,u} := \Big\{\eta \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^d)) : \int_Q\int_Q \eta(x,y,z)\,dz\,dy = 0 \text{ for a.e. } x\in\Omega,$$
$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\int_Q\int_Q A^i(z)(u(x)+\eta(x,y,z))\,dz\,dy\Big) = 0 \text{ in } W^{-1,p}(\Omega;\mathbb{R}^l),$$
$$\sum_{i=1}^N \frac{\partial}{\partial z_i}\Big(A^i(z)(u(x)+\eta(x,y,z))\Big) = 0 \text{ in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x\in\Omega \text{ and } y\in Q\Big\}. \tag{5.28}$$

Remark 5.4 In view of the second differential constraint in (5.28), if $\eta \in \mathcal{C}^{\mathscr{A}}_{z,u}$ there holds

$$\sum_{i=1}^N \int_\Omega\int_Q\int_Q A^i(z)(u(x)+\eta(x,y,z))\varphi_1(x)\varphi_2(y)\cdot\frac{\partial\varphi_3(z)}{\partial z_i}\,dz\,dy\,dx = 0$$

for every $\varphi_1 \in C^\infty_c(\Omega)$, $\varphi_2 \in C^\infty_c(Q)$, and $\varphi_3 \in W^{1,p'}_0(Q;\mathbb{R}^l)$. A density argument yields

$$\sum_{i=1}^N \int_\Omega\int_Q\int_Q A^i(z)(u(x)+\eta(x,y,z))\varphi_1(x)\cdot\frac{\partial\varphi_3(z)}{\partial z_i}\,dz\,dy\,dx = 0$$

for every $\varphi_1 \in C^\infty_c(\Omega)$ and $\varphi_3 \in W^{1,p'}_0(Q;\mathbb{R}^l)$, namely

$$\sum_{i=1}^N \frac{\partial}{\partial z_i}\Big(\int_Q A^i(z)(u(x)+\eta(x,y,z))\,dy\Big) = 0 \text{ in } W^{-1,p}(Q;\mathbb{R}^l) \text{ for a.e. } x\in\Omega.$$

We recall the notion of "r-two-scale convergence" (reduced two-scale convergence) introduced by Neukamm in the framework of dimension reduction problems (see [21]).

Definition 5.5 Let $1<p<+\infty$. Let $\phi \in L^p(\Omega\times Q\times Q)$ and $\{\phi_n\}\subset L^p(\Omega\times Q)$. We say that $\{\phi_n\}$ converges *weakly r-two-scale* to ϕ in $L^p(\Omega\times Q\times Q)$, $\phi_n \overset{r-2-s}{\rightharpoonup} \phi$, if

$$\int_\Omega\int_Q \phi_n(x,y)\varphi(x,y,ny)\,dy\,dx \to \int_\Omega\int_Q\int_Q \phi(x,y,z)\varphi(x,y,z)\,dz\,dy\,dx$$

for every $\varphi \in C^\infty_c(\Omega\times Q; C^\infty_{\mathrm{per}}(Q))$.

We point out that "r-two-scale convergence" is just a particular case of classical two-scale convergence, and that the standard properties of two-scale convergence are still valid in this framework (see [21, Proposition 6.2.5]).

The following characterization holds true.

Theorem 5.6 *Let* $1<p<+\infty$. *Let* $A^i \in L^\infty(Q;\mathbb{M}^{l\times d})$, $i=1,\dots,N$, *and let* $f:\mathbb{R}^d\to[0,+\infty)$ *be convex and satisfying the growth condition*

$$0\le f(v)\le C(1+|v|^p) \quad \textit{for every } v\in\mathbb{R}^d \quad \textit{and some } C>0.$$

Then for every $u\in\mathcal{C}^{\mathscr{A}}$ *there holds*

$$\mathscr{F}_{\mathscr{A}}(u) = \inf_{\eta\in\mathcal{C}^{\mathscr{A}}_{z,u}} \int_\Omega\int_Q\int_Q f(u(x)+\eta(x,y,z))\,dz\,dy\,dx.$$

Proof Let $u \in \mathcal{C}^{\mathscr{A}}$. We subdivide the proof into two steps.
Step 1 We claim that

$$\mathscr{F}_{\mathscr{A}}(u) \geq \inf_{\eta \in \mathcal{C}^{\mathscr{A}}_{z,u}} \int_\Omega \int_Q \int_Q f(u(x) + \eta(x, y, z))\, dz\, dy\, dx. \tag{5.29}$$

Indeed, in view of (4.1)–(4.3) fix $r > 0$ and $\{u_n\} \in L^p(\Omega; \mathbb{R}^d)$ with $u_n \rightharpoonup u$ weakly in $L^p(\Omega; \mathbb{R}^d)$, and $u_n \in \mathcal{C}_r^{\mathscr{A}(n\cdot)}$ for every $n \in \mathbb{N}$. Let $\{w_n\} \subset L^p(\Omega \times Q; \mathbb{R}^d)$ be such that $w_n \in \mathcal{C}_{u_n}^{\mathscr{A}(n\cdot)}$ for every $n \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \|w_n\|_{L^p(\Omega\times Q;\mathbb{R}^d)} \leq r$, and

$$\int_\Omega \int_Q f(u_n(x) + w_n(x, y))\, dy\, dx \leq \mathcal{F}^r_{\mathscr{A}(n\cdot)}(u_n) + \frac{1}{n} \quad \text{for every } n \in \mathbb{N}.$$

By the uniform bound on the L^p norm of $\{w_n\}$ there exist $w_0 \in L^p(\Omega \times Q; \mathbb{R}^d)$ and $\psi_0 \in L^p(\Omega \times Q \times Q; \mathbb{R}^d)$ such that, up to the extraction of (not relabeled) subsequences, there holds (see Proposition 2.2)

$$w_n \rightharpoonup w_0 \quad \text{weakly in } L^p(\Omega \times Q; \mathbb{R}^d),$$

and

$$w_n \overset{r-2-s}{\rightharpoonup} \psi_0 \quad \text{weakly two-scale in } L^p(\Omega \times Q \times Q; \mathbb{R}^d), \tag{5.30}$$

with $\int_Q \psi_0(x, y, z)\, dz = w_0(x, y)$ for a.e. $x \in \Omega$ and $y \in Q$. In particular, since $w_n \in \mathcal{C}_{u_n}^{\mathscr{A}(n\cdot)}$ for every $n \in \mathbb{N}$, we have $\int_Q w_n(x, y)\, dy = 0$ for a.e. $x \in \Omega$, and for every $n \in \mathbb{N}$, which in turn yields

$$\int_Q w_0(x, y)\, dy = \int_Q \int_Q \psi_0(x, y, z)\, dz\, dy = 0 \quad \text{for a.e. } x \in \Omega.$$

Analogously, by the weak L^p convergence of $\{u_n\}$ there exists $\phi_0 \in L^2(\Omega \times Q \times Q)$ such that, up to the extraction of a (not relabeled) subsequence, there holds

$$u_n \overset{r-2-s}{\rightharpoonup} u + \phi_0 \tag{5.31}$$

with $\int_Q \int_Q \phi_0(x, y, z) = 0$ for a.e. $x \in \Omega$.

Setting $\eta_0 := \psi_0 + \phi_0$, an adaptation of the argument in Step 1 of Proposition 3.5 and Remark 3.6, together with the periodicity of the operators A^i, $i = 1, \ldots, N$, allows to pass to the limit as $n \to +\infty$ in the differential constraints defining the classes $\mathcal{C}_{u_n}^{\mathscr{A}(n\cdot)}$, and to deduce that $\eta_0 \in \mathcal{C}^{\mathscr{A}}_{z,u}$. Indeed, let $\varphi \in W_0^{1,p'}(\Omega; \mathbb{R}^l)$. Since $w_n \in \mathcal{C}_{u_n}^{\mathscr{A}(n\cdot)}$, we have

$$\begin{aligned} &\left\langle \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\int_Q A^i(ny)(u_n(x) + w_n(x, y))\, dy \right), \varphi \right\rangle \\ &= -\sum_{i=1}^N \int_\Omega \int_Q A^i(ny)(u_n(x) + w_n(x, y)) \cdot \frac{\partial \varphi(x)}{\partial x_i}\, dy\, dx = 0 \end{aligned}$$

for every $n \in \mathbb{N}$. In view of (5.30) and (5.31) we obtain

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega \int_Q A^i(ny)(u_n(x) + w_n(x, y)) \cdot \frac{\partial\varphi(x)}{\partial x_i}\, dy\, dx$$
$$= \sum_{i=1}^{N} \int_\Omega \int_Q \int_Q A^i(z)(u(x) + \eta_0(x, y, z)) \cdot \frac{\partial\varphi(x)}{\partial x_i}\, dz\, dy\, dx.$$

Thus, by the arbitrariness of φ,

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\int_Q \int_Q A^i(z)(u(x) + \eta_0(x, y, z))\, dy\, dz\right) = 0 \quad \text{in } W^{-1,p}(\Omega; \mathbb{R}^l).$$

The second differential constraint in (5.28) follows by a similar argument.

By the convexity of f we use its two-scale L^p lower-semicontinuity (see [26, Proposition1.3]) to deduce that

$$\liminf_{n\to+\infty} \mathcal{F}^r_{\mathscr{A}(n\cdot)}(u_n) \geq \lim_{n\to+\infty} \int_\Omega \int_Q f(u_n(x) + w_n(x, y))\, dy\, dx$$
$$\geq \int_\Omega \int_Q \int_Q f(u(x) + \eta_0(x, y, z))\, dz\, dy\, dx$$
$$\geq \inf_{\eta\in\mathcal{C}^{\mathscr{A}}_{z,u}} \int_\Omega \int_Q \int_Q f(u(x) + \eta(x, y, z))\, dz\, dy\, dx. \tag{5.32}$$

Since the same procedure applies to every sequence $\{u_n\}$ as above and to every $r > 0$, we obtain (5.29).

Step 2 We show that for every $\lambda > 0$ the following inequality holds true

$$\mathscr{F}_{\mathscr{A}}(u) \leq \inf_{\eta\in\mathcal{C}^{\mathscr{A}}_{z,u}} \int_\Omega \int_Q \int_Q f(u(x) + \eta(x, y, z))\, dz\, dy\, dx + \lambda. \tag{5.33}$$

Indeed, let $\eta_\lambda \in \mathcal{C}^{\mathscr{A}}_{z,u}$ be such that

$$\int_\Omega \int_Q \int_Q f(u(x) + \eta_\lambda(x, y, z))\, dz\, dy\, dx$$
$$\leq \inf_{\eta\in\mathcal{C}^{\mathscr{A}}_{z,u}} \int_\Omega \int_Q \int_Q f(u(x) + \eta(x, y, z))\, dz\, dy\, dx + \lambda, \tag{5.34}$$

and set

$$w_n(x, y) := \int_Q \eta_\lambda(x, z, ny)\, dz \quad \text{for every } x \in \Omega \text{ and } y \in Q.$$

The fact that $\eta_\lambda \in \mathcal{C}^{\mathscr{A}}_{z,u}$ yields

$$\int_Q w_n(x, y)\, dy = 0 \quad \text{for a.e. } x \in \Omega, \tag{5.35}$$

for every $n \in \mathbb{N}$. By the periodicity of η_λ and of the operators A^i, $i = 1, \dots, N$, there holds

$$\int_Q A^i(ny)(u(x) + w_n(x, y))\,dy = \int_Q \int_Q A^i(ny)(u(x) + \eta_\lambda(x, z, ny))\,dz\,dy$$
$$= \frac{1}{n^N} \int_{nQ} \int_Q A^i(y)(u(x) + \eta_\lambda(x, z, y))\,dz\,dy = \int_Q \int_Q A^i(y)(u(x) + \eta_\lambda(x, z, y))\,dz\,dy,$$

for a.e. $x \in \Omega$. Since $\eta_\lambda \in \mathcal{C}^{\mathscr{A}}_{z,u}$ we obtain

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\int_Q A^i(ny)(u(x) + w_n(x, y))\,dy\Big)$$
$$= \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\int_Q \int_Q A^i(y)(u(x) + \eta_\lambda(x, z, y))\,dz\,dy\Big) = 0 \tag{5.36}$$

in $W^{-1,p}(\Omega; \mathbb{R}^l)$.

As a consequence of Remark 5.4 we deduce that

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}\Big(A^i(ny)(u(x) + w_n(x, y)\Big)$$
$$= \sum_{i=1}^N \frac{\partial}{\partial y_i} \int_Q \Big(A^i(ny)(u(x) + \eta_\lambda(x, z, ny))\Big)\,dz = 0 \tag{5.37}$$

in $W^{-1,p}(Q; \mathbb{R}^l)$, for a.e. $x \in \Omega$. By Jensen's inequality we have

$$\liminf_{n\to+\infty} \int_\Omega \int_Q f(u(x) + w_n(x, y))\,dy\,dx$$
$$\leq \lim_{n\to+\infty} \int_\Omega \int_Q \int_Q f(u(x) + \eta_\lambda(x, z, ny))\,dz\,dy\,dx$$
$$= \int_\Omega \int_Q \int_Q f(u(x) + \eta_\lambda(x, z, y))\,dz\,dy\,dx,$$

where in the last equality we used the Riemann Lebesgue Lemma with respect to the variable y, and Lebesgue Dominated convergence Theorem, taking into account that, due to periodicity

$$\|\eta_\lambda(x, z, ny)\|_{L^p(\Omega\times Q\times Q;\mathbb{R}^d)} = \|\eta_\lambda(x, z, y)\|_{L^p(\Omega\times Q\times Q;\mathbb{R}^d)}$$

for all $n \in \mathbb{N}$. Now (5.34) yields

$$\mathscr{F}_{\mathscr{A}}(u) \leq \inf\Big\{\liminf_{n\to+\infty} \mathcal{F}^{K_\eta}_{\mathscr{A}(n\cdot)}(u_n) : u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d)\Big\}$$
$$\leq \liminf_{n\to+\infty} \mathcal{F}^{K_\eta}_{\mathscr{A}(n\cdot)}(u)$$
$$\leq \liminf_{n\to+\infty} \int_\Omega \int_Q f(u(x) + w_n(x, y))\,dy\,dx$$
$$\leq \int_\Omega \int_Q \int_Q f(u(x) + \eta_\lambda(x, z, y))\,dz\,dy\,dx$$

$$\leq \inf_{\eta\in\mathcal{C}^{\mathscr{A}}_{z,u}} \int_\Omega \int_Q \int_Q f(u(x)+\eta(x,y,z))\,dz\,dy\,dx+\lambda,$$

which in turn implies (5.33). The thesis follows by combining (5.29) and (5.33), and by the arbitrariness of λ. □

5.3 Nonlocality of the operator

We end this section with an example that illustrates that, in general, when the operators A^i are not constant then the functional $\mathscr{F}_{\mathscr{A}}$ in (4.3) can be nonlocal, even when the energy density f is convex.

Example 5.7 Let $N=d=p=2$ and $l=1$, and choose

$$\Omega=(0,1)\times(0,1).$$

Let $a\in C^\infty_{\mathrm{per}}(\mathbb{R})$, with period $(-\frac{1}{2},\frac{1}{2})$, $a>-1$, and satisfying

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} a(s)\,ds=0 \quad\text{and}\quad \int_{-\frac{1}{2}}^{\frac{1}{2}} a^2(s)\,ds=1, \tag{5.38}$$

and consider the operators $A^1, A^2:\mathbb{R}^2\to\mathbb{R}^2$ defined as

$$A^1(x):=(1+a(x_2)\quad 0)\quad\text{and } A^2(x):=(0\quad 1)$$

for $x\in\mathbb{R}^2$. We have that

$$\mathcal{A}(y)\xi=\begin{pmatrix}1+a(y_2) & 0\\ 0 & 1\end{pmatrix}\xi$$

for $\xi\in\mathbb{R}^2$ and $y\in Q$, therefore the operator $\mathcal{A}$ satisfies the uniform invertibility assumption (3.3).

Consider the function $f:\mathbb{R}^2\to[0,+\infty)$, defined as $f(\xi):=|\xi|^2$ for every $\xi\in\mathbb{R}^2$. By (5.38), for n fixed, $u_n\in\mathcal{C}^{\mathscr{A}(n\cdot)}$ if and only if there exists $w_n\in L^2(\Omega\times Q;\mathbb{R}^2)$ such that $\int_Q w_n(x,y)\,dy=0$, and the following two conditions are satisfied

$$\frac{\partial}{\partial x_1}\left(\int_Q a(ny_2)w_{1,n}(x,y)\,dy\right)+\operatorname{div}u_n(x)=0\quad\text{in } W^{-1,2}(\Omega), \tag{5.39}$$

$$(1+a(ny_2))\frac{\partial w_{1,n}(x,y)}{\partial y_1}+\frac{\partial w_{2,n}(x,y)}{\partial y_2}=0\quad\text{in } W^{-1,2}(Q)\quad\text{for a.e. } x\in\Omega. \tag{5.40}$$

Moreover, for every $\tilde u\in L^2(\Omega;\mathbb{R}^d)$ and $\tilde w\in L^2(\Omega\times Q;\mathbb{R}^2)$ with $\int_Q \tilde w(x,y)\,dy=0$, there holds

$$\int_\Omega\int_Q f(\tilde u(x)+\tilde w(x,y))\,dy=\int_\Omega|\tilde u(x)|^2\,dx+\int_\Omega\int_Q|\tilde w(x,y)|^2\,dy\,dx. \tag{5.41}$$

In view of (5.41), for every $r>0$, $n\in\mathbb{N}$, and $u_n\in\mathcal{C}^{\mathscr{A}(n\cdot)}$,

$$\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(u_n)\geq\int_\Omega|u_n(x)|^2\,dx,$$

hence by classical lower-semicontinuity results (see,e.g.,[15, Theorem 5.14]),

$$\mathscr{F}_{\mathscr{A}}(u)\geq\int_\Omega|u(x)|^2\,dx\quad\text{for every } u\in\mathcal{C}^{\mathscr{A}}. \tag{5.42}$$

If $0 \in \mathcal{C}_u^{\mathscr{A}}$, i.e., $\operatorname{div} u = 0$ (and hence $0 \in \mathcal{C}_u^{\mathscr{A}(n\cdot)}$ for every n), then

$$\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(u) = \int_\Omega |u(x)|^2 dx$$

for every n and r, and choosing $u_n = u$ for every n in (4.3), we deduce that (5.42) holds with equality.

Strict inequality holds in (5.42) if $u \in \mathcal{C}^{\mathscr{A}}$ but $\operatorname{div} u \neq 0$. Such fields exist, consider for example,

$$u(x) := \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} \quad \text{for } x \in \Omega.$$

Note that the map

$$w(x, y) := \begin{pmatrix} a(y_2)x_1 \\ 0 \end{pmatrix} \quad \text{for every } (x, y) \in \Omega \times Q$$

satisfies $w \in \mathcal{C}_u^{\mathscr{A}}$ by (5.38)

Assume by contradiction that there exists $u \in \mathcal{C}^{\mathscr{A}}$, with $\operatorname{div} u \neq 0$, such that

$$\mathscr{F}_{\mathscr{A}}(u) = \int_\Omega |u(x)|^2\, dx. \tag{5.43}$$

By the definition of $\mathscr{F}_{\mathscr{A}}$ there exists a sequence $\{r_m\}$ of real numbers such that

$$\mathscr{F}_{\mathscr{A}}(u) = \lim_{m\to+\infty} \inf \left\{ \liminf_{n\to+\infty} \overline{\mathcal{F}}^{r_m}_{\mathscr{A}(n\cdot)}(u_n) :\ u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^d) \right\}.$$

For every $m \in \mathbb{N}$, let $\{u_n^m\} \subset L^2(\Omega; \mathbb{R}^d)$ be such that

$$u_n^m \rightharpoonup u \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d)$$

as $n \to +\infty$, and

$$\inf \left\{ \liminf_{n\to+\infty} \overline{\mathcal{F}}^{r_m}_{\mathscr{A}(n\cdot)}(u_n) :\ u_n \rightharpoonup u \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d) \right\} + \frac{1}{m} \geq \liminf_{n\to+\infty} \overline{\mathcal{F}}^{r_m}_{\mathscr{A}(n\cdot)}(u_n^m).$$

Then

$$\mathscr{F}_{\mathscr{A}}(u) = \lim_{m\to+\infty} \liminf_{n\to+\infty} \overline{\mathcal{F}}^{r_m}_{\mathscr{A}(n\cdot)}(u_n^m),$$

and by a diagonal argument we can extract a subsequence $\{n(m)\}$ such that, setting $u^m := u^m_{n(m)}$,

$$\mathscr{F}_{\mathscr{A}}(u) = \lim_{m\to+\infty} \overline{\mathcal{F}}^{r_m}_{\mathscr{A}(n(m)\cdot)}(u^m). \tag{5.44}$$

By (5.41), for every m there exists $w_m \in \mathcal{C}_{u^m}^{\mathscr{A}(n(m)\cdot)}$ such that $\|w_m\|_{L^2(\Omega\times Q;\mathbb{R}^d)} \leq r_m$ and

$$\begin{aligned} \overline{\mathcal{F}}^{r_m}_{\mathscr{A}(n(m)\cdot)}(u^m) + \frac{1}{m} &\geq \int_\Omega \int_Q f(u^m(x) + w_m(x, y))\, dy\, dx \\ &= \int_\Omega |u^m(x)|^2\, dx + \int_\Omega \int_Q |w_m(x, y)|^2\, dy\, dx. \end{aligned} \tag{5.45}$$

Hence, in view of (5.43), (5.44) and (5.45),

$$\int_\Omega |u(x)|^2\, dx = \lim_{m\to+\infty} \left(\int_\Omega |u^m(x)|^2\, dx + \int_\Omega \int_Q |w_m(x, y)|^2\, dy\, dx \right),$$

and so

$$u^m \to u \quad \text{strongly in } L^2(\Omega;\mathbb{R}^d), \tag{5.46}$$

and

$$w_m \to 0 \quad \text{strongly in } L^2(\Omega\times Q;\mathbb{R}^d). \tag{5.47}$$

By (5.39) and the boundedness of the function a, properties (5.46) and (5.47) yield

$$\operatorname{div} u^m \to 0 \quad \text{strongly in } W^{-1,2}(\Omega)$$

which in turn implies that $\operatorname{div} u = 0$, contradicting the assumptions on u.

We conclude that if $u \in \mathcal{C}^{\mathscr{A}}$ satisfies $\operatorname{div} u = 0$, then

$$\mathscr{F}_{\mathscr{A}}(u) = \int_\Omega |u(x)|^2\,dx, \tag{5.48}$$

whereas if $u \in \mathcal{C}^{\mathscr{A}}$ satisfies $\operatorname{div} u \neq 0$, then

$$\mathscr{F}_{\mathscr{A}}(u) > \int_\Omega |u(x)|^2\,dx. \tag{5.49}$$

We now provide an explicit expression for the functional $\mathscr{F}_{\mathscr{A}}$. We claim that for every $u \in \mathcal{C}^{\mathscr{A}}$ there holds

$$\mathscr{F}_{\mathscr{A}}(u) = \int_\Omega |u(x)|^2\,dx + \int_\Omega |\phi_u^\Omega(x)|^2\,dx, \tag{5.50}$$

where $\phi_u^\Omega \in L^2(\Omega)$ is the unique function satisfying $\int_0^1 \phi_u^\Omega(x_1,x_2)\,dx_1 = 0$ for a.e. $x_2 \in (0,1)$, and $\frac{\partial\phi_u^\Omega(x)}{\partial x_1} = -\operatorname{div} u(x)$ in $W^{-1,2}(\Omega)$.

To prove (5.50) we first establish a preliminary lemma.

Lemma 5.8 *Let $n \in \mathbb{N}$, and let $v \in \mathcal{C}^{\mathscr{A}(n\cdot)}$. Then*

$$\begin{aligned}\inf_{w\in\mathcal{C}_{\mathscr{A}(n\cdot)}(v)} &\left[\int_\Omega |v(x)|^2\,dx + \int_\Omega\int_Q |w(x,y)|^2\,dx\,dy\right] \\ &= \int_\Omega |v(x)|^2\,dx + \int_\Omega |\phi_v^\Omega(x)|^2\,dx,\end{aligned} \tag{5.51}$$

where $\int_0^1 \phi_v^\Omega(x_1,x_2)\,dx_1 = 0$ for a.e. $x_2 \in (0,1)$, and $\frac{\partial\phi_v^\Omega(x)}{\partial x_1} = -\operatorname{div} v(x)$ in $W^{-1,2}(\Omega)$.

Proof We recall that by (5.39) and (5.40), $w \in \mathcal{C}_{\mathscr{A}(n\cdot)}(v)$ if and only if

$$\begin{aligned}&\int_Q w(x,y)\,dy = 0, \quad \text{for a.e. } x \in \Omega,\\ &\frac{\partial}{\partial x_1}\left(\int_Q a(ny_2)w_1(x,y)\,dy\right) + \operatorname{div} v(x) = 0 \quad \text{in } W^{-1,2}(\Omega),\\ &(1+a(ny_2))\frac{\partial w_1(x,y)}{\partial y_1} + \frac{\partial w_2(x,y)}{\partial y_2} = 0 \quad \text{in } W^{-1,2}(Q) \quad \text{for a.e. } x \in \Omega.\end{aligned} \tag{5.52}$$

By (5.52), the map ϕ_v^Ω defined in the statement of Lemma 5.8 satisfies

$$\phi_v^\Omega(x) = \int_Q a(ny_2)w_1(x,y)\,dy - \int_0^1\left(\int_Q a(ny_2)w_1(x,y)\,dy\right)dx_1 \quad \text{for a.e. } x \in \Omega.$$

Rewriting w_1 as

$$w_1(x,y) := a(ny_2)\int_Q a(ny_2)w_1(x,y)\,dy + \eta_w(x,y),$$

we have

$$\int_Q a(ny_2)\eta_w(x,y)\,dy = 0,$$

and hence

$$\begin{aligned}\int_\Omega\int_Q |w(x,y)|^2\,dx\,dy &= \int_\Omega\int_Q |w_1(x,y)|^2\,dx\,dy + \int_\Omega\int_Q |w_2(x,y)|^2\,dx\,dy\\ &\geq \int_\Omega\Big(\int_Q a(ny_2)w_1(x,y)\,dy\Big)^2\,dx + \int_\Omega\int_Q |\eta_w(x,y)|^2\,dx\,dy\\ &\geq \int_\Omega |\phi_v^\Omega(x)|^2\,dx + \int_\Omega\Big(\int_0^1\Big(\int_Q a(ny_2)w_1(x,y)\,dy\Big)\,dx_1\Big)^2\,dx\\ &\geq \int_\Omega |\phi_v^\Omega(x)|^2\,dx \quad \text{for every } w\in\mathcal{C}_{\mathscr{A}(n\cdot)}(v).\end{aligned}$$

In particular, we obtain the lower bound

$$\inf_{w\in\mathcal{C}_{\mathscr{A}(n\cdot)}(v)}\Big[\int_\Omega |v(x)|^2\,dx + \int_\Omega\int_Q |w(x,y)|^2\,dx\,dy\Big] \geq \int_\Omega |v(x)|^2\,dx + \int_\Omega |\phi_v^\Omega(x)|^2\,dx.$$

Property (5.51) follows by observing that (5.38) yields

$$\Big(a(ny_2)\phi_v^\Omega(x), 0\Big) \in \mathcal{C}_{\mathscr{A}(n\cdot)}(v).$$

□

Let $u \in \mathcal{C}^{\mathscr{A}}$. Arguing as in the proof of (5.49) there exist a sequence $\{n_m\}\subset\mathbb{N}$, with $n_m \to +\infty$ as $m\to+\infty$, and sequences $\{u^m\}\subset L^2(\Omega;\mathbb{R}^2)$ and $\{w^m\}\subset L^2(\Omega\times Q;\mathbb{R}^2)$, such that

$$\begin{aligned}&u^m \rightharpoonup u \quad \text{weakly in } L^2(\Omega;\mathbb{R}^2),\\ &w^m \in \mathcal{C}_{u^m}^{\mathscr{A}(n_m\cdot)},\end{aligned} \tag{5.53}$$

and

$$\mathscr{F}_{\mathscr{A}}(u) = \lim_{m\to+\infty}\Big\{\int_\Omega |u^m(x)|^2\,dx + \int_\Omega\int_Q |w^m(x,y)|^2\,dx\,dy\Big\}.$$

In view of Lemma 5.8,

$$\mathscr{F}_{\mathscr{A}}(u) \geq \limsup_{m\to+\infty}\Big\{\int_\Omega |u^m(x)|^2\,dx + \int_\Omega |\phi^m(x)|^2\,dx\Big\}, \tag{5.54}$$

where

$$\int_0^1 \phi^m(x_1,x_2)\,dx_1 = 0, \quad \text{for a.e. } x_2\in(0,1), \tag{5.55}$$

and

$$\frac{\partial \phi^m(x)}{\partial x_1} = -\operatorname{div} u_n^m(x) \quad \text{in } W^{-1,2}(\Omega). \tag{5.56}$$

Since $u \in \mathcal{C}^{\mathscr{A}}$, there holds $\mathscr{F}_{\mathscr{A}}(u) < +\infty$, and the sequence $\{\phi^m\}$ is uniformly bounded in $L^2(\Omega)$. Therefore there exists $\phi_u^\Omega \in L^2(\Omega)$ such that, up to the extraction of a (not relabeled) subsequence,

$$\phi^m \rightharpoonup \phi_u^\Omega \quad \text{weakly in } L^2(\Omega), \tag{5.57}$$

where, by (5.53), (5.55) and (5.56),

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_u^\Omega(x_1, x_2)\, dx_1 = 0 \quad \text{for a.e. } x_2 \in (0, 1)$$

and

$$\frac{\partial \phi_u^\Omega(x)}{\partial x_1} = -\operatorname{div} u(x) \quad \text{in } W^{-1,2}(\Omega).$$

In particular, by (5.53) and the lower semicontinuity of the L^2-norm,

$$\mathscr{F}_{\mathscr{A}}(u) \geq \int_\Omega |u(x)|^2\, dx + \int_\Omega |\phi_u^\Omega(x)|^2\, dx.$$

To prove the opposite inequality, choose $u_n := u$, $w_n := \left(a(ny_2)\phi_u^\Omega(x), 0\right)$ for every $n \in \mathbb{N}$. By Lemma 5.8, for r big enough there holds

$$\overline{\mathcal{F}}^r_{\mathscr{A}(n\cdot)}(u_n) = \int_\Omega |u(x)|^2\, dx + \int_\Omega |\phi_u^\Omega(x)|^2\, dx.$$

The characterization (5.50) follows now by the definition of $\mathscr{F}_{\mathscr{A}}$.

We conclude this example by showing that the functional $\mathscr{F}_{\mathscr{A}}$ is nonlocal. Indeed, assume by contradiction that $\mathscr{F}_{\mathscr{A}}$ is local. Then for every $u \in \mathcal{C}^{\mathscr{A}}$ we can associate to $\mathscr{F}_{\mathscr{A}}$ an additive set function $\mathscr{F}_{\mathscr{A}}(u, \cdot)$ on the class $\mathcal{O}(\Omega)$ of open subsets of Ω. In particular, for every pair $\Omega_1, \Omega_2 \subset \Omega$, with $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, and for every $u \in \mathcal{C}^{\mathscr{A}}$ in Ω, there holds

$$\mathscr{F}_{\mathscr{A}}(u, \Omega) \leq \mathscr{F}_{\mathscr{A}}(u, \Omega \backslash \bar{\Omega}_1) + \mathscr{F}_{\mathscr{A}}(u, \Omega_2). \tag{5.58}$$

Let $\xi_1, \xi_2 \in \mathbb{R}$, with $\xi_1 \neq \xi_2$, and let $\varepsilon, \delta > 0$ be such that $\varepsilon < \frac{1}{2}$ and $\delta < \frac{1}{2} - \varepsilon$. Define

$$\Omega_1 := \left(\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon\right) \times \left(\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon\right) \quad \text{and}$$
$$\Omega_2 := \left(\tfrac{1}{2} - \varepsilon - \delta, \tfrac{1}{2} + \varepsilon + \delta\right) \times \left(\tfrac{1}{2} - \varepsilon - \delta, \tfrac{1}{2} + \varepsilon + \delta\right).$$

Consider the function

$$u_0(x) := \begin{cases} \begin{pmatrix} -\xi_1 \\ 0 \end{pmatrix} & \text{if } x \in \Omega_1, \\ \begin{pmatrix} -\xi_2 \\ 0 \end{pmatrix} & \text{otherwise in } \Omega. \end{cases}$$

We observe u_0 belongs to $\mathcal{C}^{\mathscr{A}}$, since the map

$$w_0(x,y) := \begin{cases} \begin{pmatrix} a(y_2)\xi_1 \\ 0 \end{pmatrix} & \text{if } x \in \Omega_1,\ y \in Q, \\ \begin{pmatrix} a(y_2)\xi_2 \\ 0 \end{pmatrix} & \text{otherwise in } \Omega \times Q, \end{cases}$$

satisfies $w_0 \in \mathcal{C}^{\mathscr{A}}_{u_0}$. By (5.48), since $\operatorname{div} u_0 = 0$ in $\Omega\backslash\bar{\Omega}_1$, there holds

$$\mathscr{F}_{\mathscr{A}}(u_0; \Omega\backslash\bar{\Omega}_1) = \int_{\Omega\bar{\Omega}_1} |u_0(x)|^2\,dx = (1-4\varepsilon^2)\xi_2^2. \tag{5.59}$$

A direct computation yields

$$\phi^{\Omega}_{u_0} = \begin{cases} (1-2\varepsilon)(\xi_1-\xi_2) & \text{in } \Omega_1 \\ 2\varepsilon(\xi_2-\xi_1) & \text{in } [\left(0, \frac{1}{2}-\varepsilon\right) \cup \left(\frac{1}{2}+\varepsilon, 1\right)] \times \left(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right) \\ 0 & \text{otherwise in } \Omega, \end{cases}$$

and

$$\phi^{\Omega_2}_{u_0} = \begin{cases} \frac{\delta(\xi_1-\xi_2)}{\varepsilon+\delta} & \text{in } \Omega_1 \\ \frac{\varepsilon(\xi_2-\xi_1)}{\varepsilon+\delta} & \text{in } [\left(\frac{1}{2}-\varepsilon-\delta, \frac{1}{2}-\varepsilon\right) \cup \left(\frac{1}{2}+\varepsilon, \frac{1}{2}+\varepsilon+\delta\right)] \times \left(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right) \\ 0 & \text{otherwise in } \Omega_2. \end{cases}$$

Therefore,

$$\mathscr{F}_{\mathscr{A}}(u_0, \Omega) = \int_\Omega |u_0(x)|^2\,dx + \int_\Omega |\phi^{\Omega}_{u_0}(x)|^2\,dx \tag{5.60}$$

$$= 4\varepsilon^2\xi_1^2 + (1-4\varepsilon^2)\xi_2^2 + 4\varepsilon^2(1-2\varepsilon)(\xi_1-\xi_2)^2, \tag{5.61}$$

and

$$\mathscr{F}_{\mathscr{A}}(u_0, \Omega_2) = \int_{\Omega_2} |u_0(x)|^2\,dx + \int_{\Omega_2} |\phi^{\Omega}_{u_0}(x)|^2\,dx \tag{5.62}$$

$$= 4\varepsilon^2\xi_1^2 + 4\delta(\delta+2\varepsilon)\xi_2^2 + \frac{4\varepsilon^2\delta(\xi_1-\xi_2)^2}{\varepsilon+\delta}. \tag{5.63}$$

Now (5.58) becomes

$$4\varepsilon^2(1-2\varepsilon)(\xi_1-\xi_2)^2 \le 4\delta(\delta+2\varepsilon)\xi_2^2 + \frac{4\varepsilon^2\delta(\xi_1-\xi_2)^2}{\varepsilon+\delta}$$

for every $\varepsilon < \frac{1}{2}$ and $\delta < \frac{1}{2} - \varepsilon$. Letting $\delta \to 0$ we get

$$4\varepsilon^2(1-2\varepsilon)(\xi_1-\xi_2)^2 \le 0.$$

Since $\xi_1 \neq \xi_2$, this contradicts the subadditivity of $\mathscr{F}_{\mathscr{A}}(u_0, \cdot)$ and yields the nonlocality of $\mathscr{F}_{\mathscr{A}}$.

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Multiscale homogenization in Kirchhoff's nonlinear plate theory

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The interplay between multiscale homogenization and dimension reduction for nonlinear elastic thin plates is analyzed in the case in which the scaling of the energy corresponds to Kirchhoff's nonlinear bending theory for plates. Different limit models are deduced depending on the relative ratio between the thickness parameter h and the two homogenization scales ε and ε^2.

Keywords: Dimension reduction; homogenization; Kirchhoff's nonlinear plate theory; nonlinear elasticity; multiscale convergence.

AMS Subject Classification: 35B27, 49J45, 74B20, 74E30

1. Introduction

The search for lower dimensional models describing thin three-dimensional structures is a classical problem in mechanics of materials. Since the early 1990s it has been tackled successfully by means of variational techniques, and starting from the seminal papers in Refs. 1, 11, 12 and 17, hierarchies of limit models have been deduced by Γ-convergence, depending on the scaling of the elastic energy with respect to the thickness parameter.

The first homogenization results in nonlinear elasticity have been proved in Refs. 6 and 19. In these two papers, Braides and Müller assume p-growth of a stored energy density W that oscillates periodically. They show that as the periodicity scale goes to zero, the elastic energy W converges to a homogenized energy, whose density is obtained by means of an infinite-cell homogenization formula.

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In Refs. 4 and 7 the authors treat simultaneously homogenization and dimension reduction for thin plates, in the membrane regime and under p-growth assumptions of the stored energy density. More recently, in Refs. 16, 22 and 26 models for homogenized plates have been derived under physical growth conditions for the energy density. We briefly describe these results.

Let

$$\Omega_h := \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right)$$

be the reference configuration of a nonlinearly elastic thin plate, where ω is a bounded domain in $\mathbb{R}^2$, and $h > 0$ is the thickness parameter. Assume that the physical structure of the plate is such that an in-plane homogeneity scale $\varepsilon(h)$ arises, where $\{h\}$ and $\{\varepsilon(h)\}$ are monotone decreasing sequences of positive numbers, $h \to 0$, and $\varepsilon(h) \to 0$ as $h \to 0$. In Refs. 16, 22 and 26, the rescaled nonlinear elastic energy associated to a deformation $v \in W^{1,2}(\Omega_h; \mathbb{R}^3)$ is given by

$$\mathcal{I}^h(v) := \frac{1}{h} \int_{\Omega_h} W\left(\frac{x'}{\varepsilon(h)}, \nabla v(x)\right) dx,$$

where $x' := (x_1, x_2) \in \omega$, and the stored energy density W is periodic in its first argument and satisfies the commonly adopted assumptions in nonlinear elasticity, as well as a nondegeneracy condition in a neighborhood of the set of proper rotations.

In Ref. 22 the authors focus on the scaling of the energy corresponding to Von Kármán plate theory, that is they consider deformations $v^h \in W^{1,2}(\Omega_h; \mathbb{R}^3)$ such that

$$\limsup_{h \to 0} \frac{\mathcal{I}^h(v^h)}{h^4} < +\infty.$$

Under the assumption that the limit

$$\gamma_1 := \lim_{h \to 0} \frac{h}{\varepsilon(h)}$$

exists, different homogenized limit models are identified, depending on the value of $\gamma_1 \in [0, +\infty]$.

A parallel analysis is carried in Ref. 16, where the scaling of the energy associated to Kirchhoff's plate theory is studied, i.e. the deformations under consideration satisfy

$$\limsup_{h \to 0} \frac{\mathcal{I}^h(v^h)}{h^2} < +\infty.$$

In this situation a lack of compactness occurs when $\gamma_1 = 0$ (the periodicity scale tends to zero much more slowly than the thickness parameter). A partial solution to this problem, in the case in which

$$\gamma_2 := \lim_{h \to 0} \frac{h}{\varepsilon^2(h)} = +\infty$$

is proposed in Ref. 26, by means of a careful application of Friesecke, James and Müller's quantitative rigidity estimate, and a construction of piecewise constant rotations (see Theorem 4.1 in Ref. 11, Theorem 6 in Ref. 12, and Lemma 3.11 in Ref. 26). The analysis of simultaneous homogenization and dimension reduction for Kirchhoff's plate theory in the remaining regimes is still an open problem.

In this paper we deduce a multiscale version of the results in Refs. 16 and 26. We focus on the scaling of the energy which corresponds to Kirchhoff's plate theory, and we assume that the plate undergoes the action of two homogeneity scales — a coarser one and a finer one — i.e. the rescaled nonlinear elastic energy is given by

$$\mathcal{J}^h(v) := \frac{1}{h}\int_{\Omega_h} W\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla v(x)\right) dx$$

for every deformation $v \in W^{1,2}(\Omega_h;\mathbb{R}^3)$, where the stored energy density W is periodic in its first two arguments and, again, satisfies the usual assumptions in nonlinear elasticity, as well as the nondegeneracy condition (see Sec. 2) adopted in Refs. 16, 22 and 26. We consider sequences of deformations $\{v^h\} \subset W^{1,2}(\Omega_h;\mathbb{R}^3)$ verifying

$$\limsup_{h\to 0} \frac{\mathcal{J}^h(v^h)}{h^2} < +\infty, \tag{1.1}$$

and we seek to identify the effective energy associated to the rescaled elastic energies $\left\{\frac{\mathcal{J}^h(v^h)}{h^2}\right\}$ for different values of γ_1 and γ_2, i.e. depending on the interaction of the homogeneity scales with the thickness parameter.

As in Ref. 16, a sequence of deformations satisfying (1.1) converges, up to the extraction of a subsequence, to a limit deformation $u \in W^{1,2}(\omega;\mathbb{R}^3)$ satisfying the isometric constraint

$$\partial_{x_\alpha} u(x') \cdot \partial_{x_\beta} u(x') = \delta_{\alpha,\beta} \quad \text{for a.e. } x' \in \omega,\ \ \alpha,\beta \in \{1,2\}. \tag{1.2}$$

We will prove that the effective energy is given by

$$\mathcal{E}^{\gamma_1}(u) := \begin{cases} \dfrac{1}{12}\displaystyle\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\hom}(\Pi^u(x'))dx' & \text{if } u \text{ satisfies (1.2)}, \\ +\infty & \text{otherwise}, \end{cases}$$

where Π^u is the second fundamental form associated to u (see (4.4)), and $\overline{\mathscr{Q}}^{\gamma_1}_{\hom}$ is a quadratic form dependent on the value of γ_1, with explicit characterization provided in (5.2)–(5.4). To be precise, our main result is the following.

Theorem 1.1. *Let* $\gamma_1 \in [0,+\infty]$ *and let* $\gamma_2 = +\infty$. *Let* $\{v^h\} \subset W^{1,2}(\Omega_h;\mathbb{R}^3)$ *be a sequence of deformations satisfying the uniform energy estimate* (1.1). *There exists* *a map* $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *verifying* (1.2) *such that, up to the extraction of a* (*not relabeled*) *subsequence, there holds*

$$v^h(x', hx_3) - \fint_{\Omega_1} v^h(x', hx_3)dx \to u \quad \textit{strongly in } L^2(\Omega_1;\mathbb{R}^3),$$

$$\nabla_h v^h(x', hx_3) \to (\nabla' u | n_u) \quad \textit{strongly in } L^2(\Omega_1;\mathbb{M}^{3\times 3}),$$

with

$$n_u(x') := \partial_{x_1} u(x') \wedge \partial_{x_2} u(x') \quad \textit{for a.e. } x' \in \omega,$$

and

$$\liminf_{h\to 0} \frac{\mathcal{J}^h(v^h)}{h^2} \geq \mathcal{E}^{\gamma_1}(u). \tag{1.3}$$

Moreover, for every $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *satisfying* (1.2), *there exists a sequence* $\{v^h\} \subset W^{1,2}(\Omega_h;\mathbb{R}^3)$ *such that*

$$\limsup_{h\to 0} \frac{\mathcal{J}^h(v^h)}{h^2} \leq \mathcal{E}^{\gamma_1}(u). \tag{1.4}$$

We remark that our main theorem is consistent with the results proved in Refs. 16 and 26. Indeed, in the presence of a single homogeneity scale, it follows directly from (5.2)–(5.4) that $\overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}$ reduces to the effective energy identified in Refs. 16 and 26 for $\gamma_1 \in (0, +\infty]$ and $\gamma_1 = 0$, respectively. The main difference with respect to Refs. 16 and 26 is in the structure of the homogenized energy density $\overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}$, which is obtained by means of a double pointwise minimization, first with respect to the faster periodicity scale, and then with respect to the slower one and the x_3 variable (see (5.2)–(5.4)).

The quadratic behavior of the energy density around the set of proper rotations together with the linearization occurring due to the high scalings of the elastic energy yield a convex behavior for the homogenization problem, so that, despite the nonlinearity of the three-dimensional energies, the effective energy does not have an infinite-cell structure, in contrast with Ref. 19. The main techniques for the proof of the liminf inequality (1.3) are the notion of multiscale convergence introduced in Ref. 3, and its adaptation to dimension reduction (see Ref. 20). The proof of the limsup inequality (1.4) follows that of Theorem 2.4 in Ref. 16.

The crucial part of the paper is the characterization of the three-scale limit of the sequence of linearized elastic stresses (see Sec. 4). We deal with sequences having unbounded L^2-norms but whose oscillations on the scale ε or ε^2 are uniformly controlled. As in Lemmas 3.6–3.8 in Ref. 16, to enhance their multiple-scales oscillatory behavior we work with suitable oscillatory test functions having vanishing average in their periodicity cell.

The presence of three scales increases the technicality of the problem in all scaling regimes. For $\gamma_1 \in (0, +\infty]$, Friesecke, James and Müller's rigidity estimate (Theorem 4.1 in Ref. 11) leads us to work with sequences of rotations that are piecewise constant on cubes of size $\varepsilon(h)$ with centers in $\varepsilon(h)\mathbb{Z}^2$. However, in order to identify the three-scale limit of the linearized stresses, we must consider sequences oscillating on a scale $\varepsilon^2(h)$. This problem is solved in Step 1 of the proof of Theorem 4.1, by subdividing the cubes of size $\varepsilon^2(h)$, with centers in $\varepsilon^2(h)\mathbb{Z}^2$, into "good cubes" lying completely within a bigger cube of size $\varepsilon(h)$ and center in $\varepsilon(h)\mathbb{Z}^2$ and "bad cubes", and by showing that the measure of the intersection between ω and the set of "bad cubes" converges to zero faster than or comparable to $\varepsilon(h)$, as $h \to 0$.

The opposite problem arises in the case in which $\gamma_1 = 0$. By Friesecke, James and Müller's rigidity estimate (Theorem 4.1 in Ref. 11), it is natural to work with sequences of piecewise constant rotations which are constant on cubes of size $\varepsilon^2(h)$ having centers in the grid $\varepsilon^2(h)\mathbb{Z}^2$, whereas in order to identify the limit multiscale stress we need to deal with oscillating test functions with vanishing averages on a scale $\varepsilon(h)$. The identification of "good cubes" and "bad cubes" of size $\varepsilon^2(h)$ is thus not helpful in this latter framework as the contribution of the oscillating test functions on cubes of size $\varepsilon^2(h)$ is not negligible anymore. Therefore, we are only able to perform an identification of the multiscale limit in the case $\gamma_2 = +\infty$, extending to the multiscale setting the results in Ref. 26. The identification of the effective energy in the case in which $\gamma_1 = 0$ and $\gamma_2 \in [0, +\infty)$ remains an open question.

The paper is organized as follows: in Sec. 2 we set the problem and introduce the assumptions on the energy density. In Sec. 3 we recall a few compactness results and the definition and some properties of multiscale convergence. Sections 4 and 5 are devoted to the identification of the limit linearized stress and to the proof of the liminf inequality (1.3). In Sec. 6 we show the optimality of the lower bound deduced in Sec. 5, and we exhibit a recovery sequence satisfying (1.4).

1.1. *Notation*

In what follows, $Q := (-\frac{1}{2}, \frac{1}{2})^2$ denotes the unit cube in $\mathbb{R}^2$ centered at the origin and with sides parallel to the coordinate axes. We will write a point $x \in \mathbb{R}^3$ as

$$x = (x', x_3), \quad \text{where } x' \in \mathbb{R}^2 \text{ and } x_3 \in \mathbb{R},$$

and we will use the notation ∇' to denote the gradient with respect to x'. For every $r \in \mathbb{R}$, $\lceil r \rceil$ is its greatest integer part. With a slight abuse of notation, for every $x' \in \mathbb{R}^2$, $\lceil x' \rceil$ and $\lfloor x' \rfloor$ are the points in $\mathbb{R}^2$ whose coordinates are given by the greatest and least integer parts of the coordinates of x', respectively. Given a map $\phi \in W^{1,2}(\mathbb{R}^2)$, $(y \cdot \nabla')\phi(x')$ stands for

$$(y \cdot \nabla')\phi(x') := y_1 \partial_{x_1}\phi(x') + y_2 \partial_{x_2}\phi(x') \quad \text{for a.e. } x' \in \mathbb{R}^2 \text{ and } y \in Q.$$

We write $(\nabla')^{\perp}\phi$ to indicate the map

$$(\nabla')^{\perp}\phi(x') := (-\partial_{x_2}\phi, \partial_{x_1}\phi) \quad \text{for a.e. } x' \in \mathbb{R}^2.$$

We denote by $\mathbb{M}^{n\times m}$ the set of matrices with n rows and m columns and by $\mathrm{SO}(3)$ the set of proper rotations, that is

$$\mathrm{SO}(3) := \{R \in \mathbb{M}^{3\times 3} : R^T R = \mathrm{Id} \text{ and } \det R = 1\}.$$

Given a matrix $M \in \mathbb{M}^{3\times 3}$, M' stands for the 3×2 submatrix of M given by its first two columns. For every $M \in \mathbb{M}^{n\times n}$, $\operatorname{sym} M$ is the $n\times n$ symmetrized matrix defined as

$$\operatorname{sym} M := \frac{M + M^T}{2}.$$

Whenever a map $v \in L^2, C^\infty, \dots,$ is Q-periodic, that is

$$v(x + e_i) = v(x), \quad i = 1, 2,$$

for a.e. $x \in \mathbb{R}^2$, where $\{e_1, e_2\}$ is the orthonormal canonical basis of $\mathbb{R}^2$, we write $v \in L^2_{\mathrm{per}}, C^\infty_{\mathrm{per}}, \dots,$ respectively. We implicitly identify the spaces $L^2(Q)$ and $L^2_{\mathrm{per}}(\mathbb{R}^2)$. We denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$ by $|A|$.

We adopt the convention that C designates a generic constant, whose value may change from expression to expression in the same formula.

2. Setting of the Problem

Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain whose boundary is piecewise C^1. This regularity assumption is only needed in Sec. 6, while the results in Secs. 3–5 continue to hold for every bounded Lipschitz domain $\omega \subset \mathbb{R}^2$. We assume that the set

$$\Omega_h := \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right)$$

is the reference configuration of a nonlinearly elastic thin plate. In the sequel, $\{h\}$ and $\{\varepsilon(h)\}$ are monotone decreasing sequences of positive numbers, $h \to 0$, $\varepsilon(h) \to 0$ as $h \to 0$, such that the following limits exist

$$\gamma_1 := \lim_{h\to 0} \frac{h}{\varepsilon(h)} \quad \text{and} \quad \gamma_2 := \lim_{h\to 0} \frac{h}{\varepsilon^2(h)},$$

with $\gamma_1, \gamma_2 \in [0, +\infty]$. There are five possible regimes: $\gamma_1, \gamma_2 = +\infty$; $0 < \gamma_1 < +\infty$ and $\gamma_2 = +\infty$; $\gamma_1 = 0$ and $\gamma_2 = +\infty$; $\gamma_1 = 0$ and $0 < \gamma_2 < +\infty$; $\gamma_1 = 0$ and $\gamma_2 = 0$. We focus here on the first three regimes, that is on the cases in which $\gamma_2 = +\infty$.

For every deformation $v \in W^{1,2}(\Omega_h; \mathbb{R}^3)$, we consider its rescaled elastic energy

$$\mathcal{J}^h(v) := \frac{1}{h} \int_{\Omega_h} W\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla v(x)\right) dx,$$

where $W : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0, +\infty)$ represents the stored energy density of the plate, and $(y, z, F) \mapsto W(y, z, F)$ is measurable and Q-periodic in its first two variables, i.e. with respect to y and z. We also assume that for a.e. y and z, the map $W(y,z,\cdot)$ is continuous and satisfies the following assumptions:

(H1) $W(y,z,RF) = W(y,z,F)$ for every $F \in \mathbb{M}^{3\times 3}$ and for all $R \in SO(3)$ (frame indifference),
(H2) $W(y,z,F) \geq C_1 \operatorname{dist}^2(F;SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$ (nondegeneracy),
(H3) there exists $\delta > 0$ such that $W(y,z,F) \leq C_2 \operatorname{dist}^2(F;SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$ with $\operatorname{dist}(F;SO(3)) < \delta$,
(H4) $\lim_{|G|\to 0} \frac{W(y,z,\mathrm{Id}+G)-\mathscr{Q}(y,z,G)}{|G|^2} = 0$, where $\mathscr{Q}(y,z,\cdot)$ is a quadratic form on $\mathbb{M}^{3\times 3}$.

By assumptions (H1)–(H4) we obtain the following lemma, which guarantees the continuity of the quadratic map $\mathscr{Q}$ introduced in (H4).

Lemma 2.1. *Let* $W : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,+\infty)$ *satisfy* (H1)–(H4) *and let* $\mathscr{Q} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,+\infty)$ *be defined as in* (H4). *Then,*

(i) $\mathscr{Q}(y,z,\cdot)$ *is continuous for a.e.* $y,z \in \mathbb{R}^2$,
(ii) $\mathscr{Q}(\cdot,\cdot,F)$ *is* $Q \times Q$*-periodic and measurable for every* $F \in \mathbb{M}^{3\times 3}$,
(iii) *for a.e.* $y,z \in \mathbb{R}^2$, *the map* $\mathscr{Q}(y,z,\cdot)$ *is quadratic on* $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$, *and satisfies*

$$\frac{1}{C}|\mathrm{sym}\,\mathrm{F}|^2 \leq \mathscr{Q}(y,z,F) = \mathscr{Q}(y,z,\mathrm{sym}\,\mathrm{F}) \leq C|\mathrm{sym}\,\mathrm{F}|^2$$

for all $F \in \mathbb{M}^{3\times 3}$, *and some* $C > 0$. *In addition, there exists a monotone function*

$$r : [0,+\infty) \to [0,+\infty],$$

such that $r(\delta) \to 0$ *as* $\delta \to 0$, *and*

$$|W(y,z,\mathrm{Id}+F) - \mathscr{Q}(y,z,F)| \leq |F|^2 r(|F|)$$

for all $F \in \mathbb{M}^{3\times 3}$, *for a.e.* $y,z \in \mathbb{R}^2$.

We refer to Lemma 2.7 in Ref. 21 and to Lemma 4.1 in Ref. 22 for a proof of Lemma 2.1 in the case in which $\mathscr{Q}$ is independent of z. The proof in our setting is a straightforward adaptation.

As it is usual in dimension reduction analysis, we perform a change of variables in order to reformulate the problem on a domain independent of the varying thickness parameter. We set

$$\Omega := \Omega_1 = \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right),$$

and we consider the change of variables $\psi^h : \Omega \to \Omega^h$, defined as

$$\psi^h(x) = (x', hx_3) \quad \text{for every } x \in \Omega.$$

To every deformation $v \in W^{1,2}(\Omega_h;\mathbb{R}^3)$ we associate a function $u \in W^{1,2}(\Omega;\mathbb{R}^3)$, defined as $u := v \circ \psi^h$, whose elastic energy is given by

$$\mathcal{E}^h(u) = \mathcal{J}^h(v) = \int_\Omega W\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla_h u(x)\right) dx,$$

where

$$\nabla_h u(x) := \left(\nabla' u(x) \middle| \frac{\partial_{x_3} u(x)}{h}\right) \quad \text{for a.e. } x \in \Omega.$$

In this paper we focus on the asymptotic behavior of sequences of deformations $\{u^h\} \subset W^{1,2}(\Omega_h; \mathbb{R}^3)$ satisfying the uniform energy estimate

$$\mathcal{E}^h(u^h) := \int_\Omega W\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla_h u^h(x)\right) dx \le Ch^2 \quad \text{for every } h > 0. \tag{2.1}$$

We remark that in the case in which W is independent of y and z, such scalings of the energy lead to Kirchhoff's nonlinear plate theory, which was rigorously justified by means of Γ-convergence techniques in the seminal paper Ref. 11.

3. Compactness Results and Multiscale Convergence

In this section we present a few preliminary results which will allow us to deduce compactness for sequences of deformations satisfying the uniform energy estimate (2.1).

We first recall Theorem 4.1 in Ref. 11, which provides a characterization of limits of deformations whose scaled gradients are uniformly close in the L^2-norm to the set of proper rotations.

Theorem 3.1. *Let $\{u^h\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ be such that*

$$\limsup_{h\to 0} \frac{1}{h^2} \int_\Omega \operatorname{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3)) dx < +\infty. \tag{3.1}$$

Then, there exists a map $u \in W^{2,2}(\omega; \mathbb{R}^3)$ such that, up to the extraction of a (*not relabeled*) *subsequence,*

$$u^h - \fint_\Omega u^h(x) dx \to u \quad \textit{strongly in } L^2(\Omega; \mathbb{R}^3),$$

$$\nabla_h u^h \to (\nabla' u | n_u) \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}),$$

with

$$\partial_{x_\alpha} u(x') \cdot \partial_{x_\beta} u(x') = \delta_{\alpha,\beta} \quad \textit{for a.e. } x' \in \omega, \ \alpha, \beta \in \{1, 2\} \tag{3.2}$$

and

$$n_u(x') := \partial_{x_1} u(x') \wedge \partial_{x_2} u(x') \quad \textit{for a.e. } x' \in \omega. \tag{3.3}$$

A crucial point in the proof of the liminf inequality (1.3) (see Secs. 4 and 5) is to approximate the scaled gradients of deformations with uniformly small energies, by sequences of maps which are either piecewise constant on cubes of size comparable to the homogenization parameters with values in the set of proper rotations, or have Sobolev regularity and are close in the L^2-norm to piecewise constant rotations. The following lemma has been stated in Lemma 3.3 in Ref. 26, and its proof follows by combining Theorem 6 in Ref. 12 with the argument in the proof of Theorem 4.1 and Sec. 3 in Ref. 11. We remark that the additional regularity of the limit deformation u in Theorem 3.1 is a consequence of Lemma 3.1, and in particular of the approximation of scaled gradients by $W^{1,2}$ maps.

Lemma 3.1. *Let $\gamma_0 \in (0,1]$ and let $h, \delta > 0$ be such that*

$$\gamma_0 \leq \frac{h}{\delta} \leq \frac{1}{\gamma_0}.$$

There exists a constant C, depending only on ω and γ_0, such that for every $u \in W^{1,2}(\omega;\mathbb{R}^3)$ there exists a map $R : \omega \to \mathrm{SO}(3)$ piecewise constant on each cube $x + \delta Q$, with $x \in \delta\mathbb{Z}^2$, and there exists $\tilde{R} \in W^{1,2}(\omega;\mathbb{M}^{3\times 3})$ such that

$$\begin{aligned}\|\nabla_h u - R\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})} + \|R - \tilde{R}\|^2_{L^2(\omega;\mathbb{M}^{3\times3})} + h^2\|\nabla'\tilde{R}\|^2_{L^2(\omega;\mathbb{M}^{3\times3}\times\mathbb{M}^{3\times3})}\\ \leq C\|\mathrm{dist}(\nabla_h u;\mathrm{SO}(3))\|_{L^2(\Omega)}.\end{aligned}$$

Moreover, for every $\xi \in \mathbb{R}^2$ satisfying

$$|\xi|_\infty := \max\{|\xi\cdot e_1|, |\xi\cdot e_2|\} < h,$$

and for every $\omega' \subset \omega$, with $\mathrm{dist}(\omega',\partial\omega) > Ch$, there holds

$$\|R(x') - R(x'+\xi)\|_{L^2(\omega';\mathbb{M}^{3\times3})} \leq C\|\mathrm{dist}(\nabla_h u;\mathrm{SO}(3))\|^2_{L^2(\omega)}.$$

We now recall the definitions of "2-scale convergence" and "3-scale convergence". For a detailed treatment of 2-scale convergence we refer to, e.g. Refs. 2, 18 and 23. The main results on multiscale convergence may be found in Refs. 3, 5, 8 and 9.

Definition 3.1. Let D be an open set in $\mathbb{R}^N$ and let Y^N be the unit cube in $\mathbb{R}^N$,

$$Y^N := \left(-\frac{1}{2},\frac{1}{2}\right)^N.$$

Let $u \in L^2(D\times Y^N)$ and $\{u^h\} \in L^2(D)$. We say that $\{u^h\}$ *converges weakly* 2*-scale to* u in $L^2(D\times Y^N)$, and we write $u^h \xrightarrow{2-s} u$ if

$$\int_D u^h(\xi)\varphi\left(\xi,\frac{\xi}{\varepsilon(h)}\right)d\xi \to \int_D\int_{Y^N} u(\xi,\eta)\varphi(\xi,\eta)d\eta\,d\xi$$

for every $\varphi \in C_c^\infty(D;C_{\mathrm{per}}(Y^N))$.

Let $u \in L^2(D\times Y^N\times Y^N)$ and $\{u^h\} \in L^2(D)$. We say that $\{u^h\}$ *converges weakly* 3*-scale to* u in $L^2(D\times Y^N\times Y^N)$, and we write $u^h \xrightarrow{3-s} u$, if

$$\int_D u^h(\xi)\varphi\left(\xi,\frac{\xi}{\varepsilon(h)},\frac{\xi}{\varepsilon^2(h)}\right)d\xi \to \int_D\int_{Y^N}\int_{Y^N} u(\xi,\eta,\lambda)\varphi(\xi,\eta,\lambda)d\lambda\,d\eta\,d\xi$$

for every $\varphi \in C_c^\infty(D;C_{\mathrm{per}}(Y^N\times Y^N))$.

We say that $\{u^h\}$ *converges strongly* 3*-scale to* u in $L^2(D \times Y^N \times Y^N)$, and we write $u^h \xrightarrow{3\text{-}s} u$, if

$$u^h \xrightharpoonup{3-s} u \quad \text{weakly 3-scale}$$

and

$$\|u^h\|_{L^2(D)} \to \|u\|_{L^2(D\times Y^N \times Y^N)}.$$

In order to simplify the statement of Theorem 4.1 and its proof, we introduce the definition of "dr-3-scale convergence" (dimension reduction 3-scale convergence), i.e. 3-scale convergence adapted to dimension reduction, inspired by Neukamm's 2-scale convergence adapted to dimension reduction (see Ref. 20).

Definition 3.2. Let $u \in L^2(\Omega \times Q \times Q)$ and $\{u^h\} \in L^2(\Omega)$. We say that $\{u^h\}$ *converges weakly dr-*3*-scale to* u in $L^2(\Omega \times Q \times Q)$, and we write $u^h \xrightharpoonup{dr\text{-}3\text{-}s} u$, if

$$\int_\Omega u^h(x)\varphi\left(x, \frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}\right) dx \to \int_\Omega \int_Q \int_Q u(x,y,z)\varphi(x,y,z)dz\, dy\, dx$$

for every $\varphi \in C_c^\infty(\Omega; C_{\mathrm{per}}(Q \times Q))$.

Remark 3.1. We point out that "dr-3-scale convergence" is just a particular case of classical 3-scale convergence. Indeed, what sets apart "dr-3-scale convergence" from the classical 3-scale convergence is solely the fact that the test functions in Definition 3.2 depend on x_3 but oscillate only in the cross-section ω. In particular, if $\{u^h\} \in L^2(\Omega)$ and

$$u^h \xrightharpoonup{dr\text{-}3\text{-}s} u \quad \text{weakly dr-3-scale}$$

then $\{u^h\}$ is bounded in $L^2(\Omega)$. Therefore, by Theorem 1.1 in Ref. 3 there exists $\xi \in L^2(\Omega \times (Q \times (-\frac{1}{2}, \frac{1}{2})) \times (Q \times (-\frac{1}{2}, \frac{1}{2})))$ such that, up to the extraction of a (not relabeled) subsequence,

$$u^h \xrightharpoonup{3\text{-}s} \xi \quad \text{weakly 3-scale,}$$

that is u^h weakly 3-scale converge to ξ in $L^2(\Omega \times (Q \times (-\frac{1}{2}, \frac{1}{2})) \times (Q \times (-\frac{1}{2}, \frac{1}{2})))$ (in the sense of classical 3-scale convergence). Hence, the "dr-3-scale limit" u and the "classical 3-scale limit" ξ are related by

$$u(x,y,z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi(x,y,z,\eta,\tau)d\eta\, d\tau \quad \text{for a.e. } x \in \omega \text{ and } y,z \in Q.$$

We now state a theorem regarding the characterization of limits of scaled gradients in the multiscale setting adapted to dimension reduction. We omit its proof as it is a simple generalization of the arguments in Theorem 6.3.3 in Ref. 20.

Theorem 3.2. *Let $u, \{u^h\} \subset W^{1,2}(\Omega)$ be such that*

$$u^h \rightharpoonup u \quad \text{weakly in } W^{1,2}(\Omega)$$

and

$$\limsup_{h\to 0} \int_\Omega |\nabla_h u^h(x)|^2\, dx < \infty.$$

Then u is independent of x_3. Moreover, there exist $u_1 \in L^2(\Omega; W^{1,2}_{\text{per}}(Q))$, $u_2 \in L^2(\Omega \times Q; W^{1,2}_{\text{per}}(Q))$, and $\bar{u} \in L^2(\omega \times Q \times Q; W^{1,2}(-\frac{1}{2}, \frac{1}{2}))$ such that, up to the extraction of a (not relabeled) subsequence,

$$\nabla_h u^h \xrightharpoonup{dr\text{-}3\text{-}s} (\nabla' u + \nabla_y u_1 + \nabla_z u_2 | \partial_{x_3} \bar{u}) \quad \text{weakly } dr\text{-3-scale}.$$

Moreover,

(i) *if $\gamma_1 = \gamma_2 = +\infty$ (i.e. $\varepsilon(h) \ll h$), then $\partial_{y_i} \bar{u} = \partial_{z_i} \bar{u} = 0$, for $i = 1, 2$;*
(ii) *if $0 < \gamma_1 < +\infty$ and $\gamma_2 = +\infty$ (i.e. $\varepsilon(h) \sim h$), then*

$$\bar{u} = \frac{u_1}{\gamma_1};$$

(iii) *if $\gamma_1 = 0$ and $\gamma_2 = +\infty$ (i.e. $h \ll \varepsilon(h) \ll h^{\frac{1}{2}}$), then*

$$\partial_{x_3} u_1 = 0 \quad \text{and} \quad \partial_{z_i} \bar{u} = 0,\ i = 1, 2.$$

In the last part of this section we collect some properties of sequences having unbounded L^2-norms but whose oscillations on the scale ε or ε^2 are uniformly controlled. Arguing as in Lemmas 3.6–3.8 in Ref. 16, we highlight the multi-scale oscillatory behavior of our sequences by testing them against products of maps with compact support and oscillatory functions with vanishing average in their periodicity cell. In the proof of Theorem 4.1 we refer to Proposition 3.2 in Ref. 16 and Proposition 3.2 in Ref. 26, so for simplicity we introduce the notation needed in those papers.

Definition 3.3. Let $\tilde{f} \in L^2(\omega \times Q)$ be such that

$$\int_Q \tilde{f}(\cdot, y) dy = 0 \quad \text{a.e. in } \omega.$$

We write

$$f^h \xrightharpoonup{\text{osc},Y} \tilde{f}$$

if

$$\lim_{h\to 0} \int_\omega f^h(x')\varphi(x') g\left(\frac{x'}{\varepsilon(h)}\right) dx' = \int_\omega \int_Q \tilde{f}(x', y)\varphi(x') g(y) dy\, dx'$$

for every $\varphi \in C^\infty_c(\omega)$ and $g \in C^\infty_{\text{per}}(Q)$, with $\int_Q g(y)dy = 0$.

Let $\{f^h\} \subset L^2(\omega)$ and let $\tilde{\tilde{f}} \in L^2(\omega \times Q \times Q)$ be such that

$$\int_Q \tilde{\tilde{f}}(\cdot, \cdot, z)dz = 0 \quad \text{a.e. in } \omega \times Q.$$

We write

$$f^h \xrightarrow{\text{osc},Z} \tilde{\tilde{f}}$$

if

$$\lim_{h\to 0} \int_\omega f^h(x')\psi\left(x', \frac{x'}{\varepsilon(h)}\right)\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)dx' = \int_\omega \int_Q \int_Q \tilde{\tilde{f}}(x', y, z)\psi(x', y)\varphi(z)dz\,dy\,dx'$$

for every $\psi \in C_c^\infty(\omega; C_{\text{per}}^\infty(Q))$ and $\varphi \in C_{\text{per}}^\infty(Q)$, with $\int_Q \varphi(z)dz = 0$.

Remark 3.2. As a direct consequence of the definition of multiscale convergence and density arguments, if $\{f^h\} \subset L^2(\omega)$, then

$$f^h \xrightarrow{2-s} f \quad \text{weakly 2-scale}$$

if and only if

$$f^h(x) \xrightarrow{\text{osc},Y} f(x) - \int_Q f(x, y)dy.$$

Analogously,

$$f^h \xrightarrow{3-s} \tilde{f} \quad \text{weakly 3-scale}$$

if and only if

$$f^h(x) \xrightarrow{\text{osc},Z} \tilde{f} - \int_Q \tilde{f}(x, y, z)dz.$$

We recall finally Lemmas 3.7 and 3.8 in Ref. 16.

Lemma 3.2. *Let $\{f^h\} \subset L^\infty(\omega)$ and $f^0 \in L^\infty(\omega)$ be such that*

$$f^h \overset{*}{\rightharpoonup} f^0 \quad \textit{weakly-* in } L^\infty(\omega).$$

Assume that f^h are constant on each cube $Q(\varepsilon(h)z, \varepsilon(h))$, with $z \in \mathbb{Z}^2$. If $f^0 \in W^{1,2}(\omega)$, then

$$\frac{f^h}{\varepsilon(h)} \xrightarrow{\text{osc},Y} -(y \cdot \nabla')f^0.$$

Lemma 3.3. *Let $\{f^h\} \subset W^{1,2}(\omega)$, $f^0 \in W^{1,2}(\omega)$, and $\phi \in L^2(\omega; W_{\text{per}}^{1,2}(Q))$ be such that*

$$f^h \rightharpoonup f^0 \quad \textit{weakly in } W^{1,2}(\omega),$$

and

$$\nabla' f^h \xrightarrow{2-s} \nabla' f^0 + \nabla_y \phi \quad \textit{weakly 2-scale},$$

with $\int_Q \phi(x', y)\, dy = 0$ *for a.e.* $x' \in \omega$. *Then,*

$$\frac{f^h}{\varepsilon(h)} \xrightarrow{\mathrm{osc},Y} \phi.$$

4. Identification of the Limit Stresses

Due to the linearized behavior of the nonlinear elastic energy around the set of proper rotations, a key point in the proof of the liminf inequality (1.3) is to establish a characterization of the weak limit, in the sense of 3-scale-dr convergence, of the sequence of linearized elastic stresses

$$E^h := \frac{\sqrt{(\nabla_h u^h)^T \nabla_h u^h} - \mathrm{Id}}{h}.$$

We introduce the following classes of functions:

$$\begin{aligned}\mathcal{C}_{\gamma_1,+\infty} := \Big\{ & U \in L^2(\Omega \times Q \times Q; \mathbb{M}^{3\times 3}): \\ & \text{there exist } \phi_1 \in L^2\left(\omega; W^{1,2}\left(\left(-\frac{1}{2}, \frac{1}{2}\right); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)\right)\right) \\ & \text{and } \phi_2 \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)) \\ & \text{such that } U = \mathrm{sym}\left(\nabla_y \phi_1 \,\middle|\, \frac{\partial_{x_3}\phi_1}{\gamma_1}\right) + \mathrm{sym}(\nabla_z \phi_2 | 0)\Big\},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathcal{C}_{+\infty,+\infty} := \{ & U \in L^2(\Omega \times Q \times Q; \mathbb{M}^{3\times 3}): \\ & \text{there exist } d \in L^2(\Omega;\mathbb{R}^3),\ \phi_1 \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)) \\ & \text{and } \phi_2 \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)) \\ & \text{such that } U = \mathrm{sym}\,(\nabla_y \phi_1 | d) + \mathrm{sym}\,(\nabla_z \phi_2 | 0)\},\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\mathcal{C}_{0,+\infty} := \Bigg\{ & U \in L^2(\Omega \times Q \times Q; \mathbb{M}^{3\times 3}): \\ & \text{there exist } \xi \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^2)),\ \eta \in L^2(\omega; W^{2,2}_{\mathrm{per}}(Q)), \\ & g_i \in L^2(\Omega \times Y), i = 1,2,3, \text{ and } \phi \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)) \text{ such that} \\ & U = \mathrm{sym}\begin{pmatrix} \nabla_y \xi + x_3 \nabla_y^2 \eta & g_1 \\ & g_2 \\ g_1 \quad g_2 & g_3 \end{pmatrix} + \mathrm{sym}(\nabla_z \phi | 0)\Bigg\}.\end{aligned} \tag{4.3}$$

We now state the main result of this section.

Theorem 4.1. *Let $\gamma_1 \in [0, +\infty]$ and $\gamma_2 = +\infty$. Let $\{u^h\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ be a sequence of deformations satisfying* (3.1) *and converging to a deformation u in the sense of Theorem* 3.1*. Then there exist $E \in L^2(\Omega \times Q \times Q; \mathbb{M}^{3\times 3}_{\text{sym}})$, $B \in L^2(\omega; \mathbb{M}^{2\times 2})$, and $U \in C_{\gamma_1,+\infty}$, such that, up to the extraction of a (not relabeled) subsequence,*

$$E^h \xrightarrow{dr\text{-}3\text{-}s} E \quad \textit{weakly dr-3-scale},$$

where

$$E(x,y,z) = \begin{pmatrix} x_3 \Pi^u(x') + \operatorname{sym} B(x') & 0 \\ 0 & 0 \end{pmatrix} + U(x,y,z),$$

for almost every $(x,y,z) \in \Omega \times Q \times Q$, with

$$\Pi^u_{\alpha,\beta}(x') := -\partial^2_{\alpha,\beta} u(x') \cdot n_u(x') \quad \textit{for } \alpha, \beta = 1, 2, \tag{4.4}$$

and $n_u(x') := \partial_1 u(x') \wedge \partial_2 u(x')$ for every $x' \in \omega$.

Proof. Let $\{u^h\}$ be as in the statement of the theorem. By Theorem 3.1 the map $u \in W^{2,2}(\omega; \mathbb{R}^3)$ is an isometry, and

$$\nabla_h u^h \to (\nabla' u | n_u) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}). \tag{4.5}$$

For simplicity, we subdivide the proof into three cases, corresponding to the three regimes $0 < \gamma_1 < +\infty$, $\gamma_1 = +\infty$, and $\gamma_1 = 0$. Each case will be treated in multiple steps.

Case 1: $0 < \gamma_1 < +\infty$ and $\gamma_2 = +\infty$.

Applying Lemma 3.1 with $\delta(h) = \varepsilon(h)$, we construct two sequences $\{R^h\} \subset L^\infty(\omega; SO(3))$ and $\{\tilde{R}^h\} \subset W^{1,2}(\omega; \mathbb{M}^{3\times 3})$ such that R^h is piecewise constant on every cube of the form $Q(\varepsilon(h)z, \varepsilon(h))$, with $z \in \mathbb{Z}^2$, and

$$\begin{aligned} &\|\nabla_h u^h - R^h\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} + \|R^h - \tilde{R}^h\|^2_{L^2(\omega;\mathbb{M}^{3\times 3})} \\ &\quad + h^2 \|\nabla' \tilde{R}^h\|^2_{L^2(\omega;\mathbb{M}^{3\times 3}\times\mathbb{M}^{3\times 3})} \le C\|\operatorname{dist}(\nabla_h u^h; SO(3))\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.6}$$

By (3.1) and (4.6), there holds

$$\begin{aligned} \nabla_h u^h - R^h &\to 0 \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}), \\ R^h - \tilde{R}^h &\to 0 \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}), \end{aligned}$$

and $\{\tilde{R}^h\}$ is bounded in $W^{1,2}(\omega; \mathbb{M}^{3\times 3})$. Therefore, by (4.5) and the uniform boundedness of the sequence $\{R^h\}$ in $L^\infty(\omega; \mathbb{M}^{3\times 3})$, and in particular in $L^2(\omega; \mathbb{M}^{3\times 3})$,

$$R^h \to R \quad \text{strongly in } L^2(\omega; \mathbb{M}^{3\times 3}), \quad R^h \rightharpoonup^* R \quad \text{weakly* in } L^\infty(\omega; \mathbb{M}^{3\times 3}), \tag{4.7}$$

and

$$\tilde{R}^h \rightharpoonup R \quad \text{weakly in } W^{1,2}(\omega; \mathbb{M}^{3\times 3}), \tag{4.8}$$

where

$$R := (\nabla' u | n_u). \tag{4.9}$$

In order to identify the multiscale limit of the linearized stresses, we argue as in the proof of Proposition 3.2 in Ref. 16, and we introduce the scaled linearized strains

$$G^h := \frac{(R^h)^T \nabla_h u^h - \mathrm{Id}}{h}. \tag{4.10}$$

By (3.1) and (4.6) the sequence $\{G^h\}$ is uniformly bounded in $L^2(\Omega; \mathbb{M}^{3\times 3})$. By standard properties of 3-scale convergence (see Theorem 2.4 in Ref. 3) there exists $G \in L^2(\Omega \times Q \times Q; \mathbb{M}^{3\times 3})$ such that, up to the extraction of a (not relabeled) subsequence,

$$G^h \xrightarrow{3\text{-}s} G \quad \text{weakly 3-scale}. \tag{4.11}$$

By the identity

$$\sqrt{(\mathrm{Id} + hF)^T (\mathrm{Id} + hF)} = \mathrm{Id} + h \,\mathrm{sym}\, F + O(h^2),$$

and observing that

$$E^h = \frac{\sqrt{(\nabla_h u^h)^T \nabla_h u^h} - \mathrm{Id}}{h} = \frac{\sqrt{(\mathrm{Id} + hG^h)^T (\mathrm{Id} + hG^h)} - \mathrm{Id}}{h},$$

there holds

$$E = \mathrm{sym}\, G. \tag{4.12}$$

By (4.11), it follows that

$$G^h \xrightarrow{2\text{-}s} \int_Q G(x, y, z) dz \quad \text{weakly 2-scale}.$$

Therefore, by Proposition 3.2 in Ref. 16 there exist $B \in L^2(\omega; \mathbb{M}^{2\times 2})$ and $\phi_1 \in L^2(\omega; W^{1,2}((-\frac{1}{2}, \frac{1}{2}); W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^3)))$ such that

$$\begin{aligned} &\mathrm{sym} \int_Q G(x, y, \xi) d\xi \\ &\quad = \begin{pmatrix} x_3 \Pi^u(x') + \mathrm{sym}\, B(x') & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{sym} \left(\nabla_y \phi_1(x, y) \middle| \frac{\partial_{x_3} \phi_1(x, y)}{\gamma_1} \right) \end{aligned} \tag{4.13}$$

for a.e. $x \in \Omega$ and $y \in Y$. Thus, by (4.12) and (4.13) to complete the proof we only need to prove that

$$\mathrm{sym}\, G(x, y, z) - \mathrm{sym} \int_Q G(x, y, \xi) d\xi = \mathrm{sym}(\nabla_z \phi_2(x, y, z) | 0) \tag{4.14}$$

for some $\phi_2 \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^3))$.

Set

$$\bar{u}^h(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} u^h(x', x_3)dx_3 \quad \text{for a.e. } x' \in \omega, \tag{4.15}$$

and define $r^h \in W^{1,2}(\Omega;\mathbb{R}^3)$ as

$$u^h(x) =: \bar{u}^h(x') + hx_3\tilde{R}^h(x')e_3 + hr^h(x', x_3) \quad \text{for a.e. } x \in \Omega. \tag{4.16}$$

We remark that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} r^h(x', x_3)dx_3 = 0 \tag{4.17}$$

and

$$\frac{\nabla_h u^h - R^h}{h} = \left(\frac{\nabla'\bar{u}^h - (R^h)'}{h} + x_3\nabla'\tilde{R}^h e_3 \,\middle|\, \frac{(\tilde{R}^h - R^h)}{h} e_3 \right) + \nabla_h r^h. \tag{4.18}$$

We first notice that by (3.1), (4.6), (4.8), and (4.17), the sequence $\{r^h\}$ is uniformly bounded in $W^{1,2}(\Omega;\mathbb{R}^3)$. Hence, by Theorem 3.2(ii) there exist $r \in W^{1,2}(\omega;\mathbb{R}^3)$, $\hat{\phi}_1 \in L^2(\omega; W^{1,2}((-\frac{1}{2},\frac{1}{2}); W^{1,2}_{\text{per}}(Q;\mathbb{R}^3)))$ and $\hat{\phi}_2 \in L^2(\Omega \times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$ such that, up to the extraction of a (not relabeled) subsequence,

$$\nabla_h r^h \xrightarrow{dr\text{-}3\text{-}s} \left(\nabla' r + \nabla_y\hat{\phi}_1 + \nabla_z\hat{\phi}_2 \,\middle|\, \frac{\partial_{x_3}\hat{\phi}_1}{\gamma_1} \right) \quad \text{weakly dr-3-scale.} \tag{4.19}$$

By (3.1) and (4.6), and since R^h does not depend on x_3, $\left\{\frac{\nabla_h\bar{u}^h - (R^h)'}{h}\right\}$ is bounded in $L^2(\omega;\mathbb{M}^{3\times 2})$. Therefore by Theorem 2.4 in Ref. 3 there exists $V \in L^2(\omega \times Q \times Q;\mathbb{M}^{3\times 2})$ such that, up to the extraction of a (not relabeled) subsequence,

$$\frac{\nabla'\bar{u}^h - (R^h)'}{h} \xrightarrow{3\text{-}s} V \quad \text{weakly 3-scale.} \tag{4.20}$$

Case 1, Step 1: Characterization of V.

In view of (4.14), we provide a characterization of

$$V(x', y, z) - \int_Q V(x', y, \xi)d\xi.$$

We claim that there exists $v \in L^2(\omega \times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$ such that

$$V(x', y, z) - \int_Q V(x', y, \xi)d\xi = \nabla_z v(x', y, z) \quad \text{for a.e. } x' \in \omega, \text{ and } y, z \in Q. \tag{4.21}$$

Arguing as in the proof of Proposition 3.2 in Ref. 16, we first notice that by Lemma 3.7 in Ref. 3 to prove (4.21) it is enough to show that

$$\int_\omega \int_Q \int_Q \left(V(x', y, z) - \int_Q V(x', y, \xi) d\xi \right) : (\nabla')^\perp \varphi(z) \psi(x', y) dz\, dy\, dx' = 0 \tag{4.22}$$

for every $\varphi \in C^1_{\text{per}}(Q; \mathbb{R}^3)$ and $\psi \in C^\infty_c(\omega; C^\infty_{\text{per}}(Q))$. Fix $\varphi \in C^1_{\text{per}}(Q; \mathbb{R}^3)$ and $\psi \in C^\infty_c(\omega; C^\infty_{\text{per}}(Q))$. We set

$$\tilde{\varphi}^\varepsilon(x') := \varepsilon^2(h) \varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \quad \text{for every } x' \in \omega.$$

Then,

$$\begin{aligned}
&\int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \psi\left(x', \frac{x'}{\varepsilon(h)}\right) dx' \\
&\quad = \int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \tilde{\varphi}^\varepsilon(x') \psi\left(x', \frac{x'}{\varepsilon(h)}\right) dx' \\
&\quad = \int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \left[\tilde{\varphi}^\varepsilon(x') \psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right] dx' \\
&\qquad - \int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : \left[\tilde{\varphi}^\varepsilon(x') \otimes \left((\nabla')^\perp_x \psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right.\right. \\
&\qquad \left.\left. + \frac{1}{\varepsilon(h)} (\nabla')^\perp_y \psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right)\right] dx'.
\end{aligned} \tag{4.23}$$

The first term on the right-hand side of (4.23) is equal to zero, due to the definition of $(\nabla')^\perp$. Therefore we obtain

$$\begin{aligned}
&\int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \psi\left(x', \frac{x'}{\varepsilon(h)}\right) dx' \\
&\quad = -\frac{\varepsilon^2(h)}{h} \int_\omega \nabla' \bar{u}^h(x') : \left[\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \otimes (\nabla')^\perp_x \psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right] \\
&\qquad - \frac{\varepsilon(h)}{h} \int_\omega \nabla' \bar{u}^h(x') : \left[\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \otimes (\nabla')^\perp_y \psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right].
\end{aligned} \tag{4.24}$$

By (4.6), the regularity of the test functions, and since $\gamma_2 = +\infty$, we get

$$\frac{\varepsilon^2(h)}{h} \int_\omega \nabla' \bar{u}^h(x') : \left[\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \otimes (\nabla')^\perp_x \psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right] dx' \to 0, \tag{4.25}$$

while by (4.5), (4.9), and the regularity of the test functions,

$$\begin{aligned}
&\lim_{h \to 0} \frac{\varepsilon(h)}{h} \int_\omega \nabla' \bar{u}^h(x') : \left[\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \otimes (\nabla')^\perp_y \psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right] dx' \\
&\quad = \frac{1}{\gamma_1} \int_\omega \int_Q \int_Q R'(x') : (\varphi(z) \otimes (\nabla')^\perp_y \psi(x', y)) dz\, dy\, dx' = 0,
\end{aligned} \tag{4.26}$$

where the latter equality is due to the periodicity of ψ with respect to the y variable. Combining (4.23), (4.24), (4.25) and (4.26), we conclude that

$$\lim_{h\to 0}\int_\omega \frac{\nabla'\bar{u}^h(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx' = 0. \tag{4.27}$$

In view of (4.20), and since

$$\int_\omega\int_Q\int_Q\left(\int_Q V(x',y,\xi)d\xi\right) : (\nabla')^\perp\varphi(z)\psi(x',y)dz\,dy\,dx' = 0$$

by the periodicity of φ, (4.22) will be established once we show that

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx' = 0. \tag{4.28}$$

In order to prove (4.28), we adapt Lemma 3.8 in Ref. 16 to our framework.

Since $\psi \in C_c^\infty(\omega; C^\infty_{\text{per}}(Q))$ and $h \to 0$, we can assume, without loss of generality, that for h small enough

$$\operatorname{dist}(\operatorname{supp}\psi; \partial\omega \times Q) > \left(1 + \frac{3}{\gamma_1}\right)h.$$

We define

$$\mathbb{Z}^\varepsilon := \{z \in \mathbb{Z}^2 : Q(\varepsilon(h)z, \varepsilon(h)) \times Q \cap \operatorname{supp}\psi \neq \emptyset\}$$

and

$$Q_\varepsilon := \bigcup_{z\in\mathbb{Z}^\varepsilon} Q(\varepsilon(h)z, \varepsilon(h)).$$

Since $0 < \gamma_1 < +\infty$, for h small enough we have $\sqrt{2}\varepsilon(h) < \frac{2h}{\gamma_1}$, so that

$$\operatorname{dist}(Q_\varepsilon; \partial\omega) \geq \left(1 + \frac{3}{\gamma_1}\right)h - \sqrt{2}\varepsilon(h) \geq \left(1 + \frac{1}{\gamma_1}\right)h.$$

We subdivide

$$\mathcal{Q}_{\varepsilon^2} := \{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) : \lambda \in \mathbb{Z}^2 \text{ and } Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \cap Q_\varepsilon \neq \emptyset\}$$

into two subsets:

(a) "good cubes of size $\varepsilon^2(h)$", i.e. those which are entirely contained in a cube of size $\varepsilon(h)$ belonging to Q_ε, and where $(R^h)'$ is hence constant,
(b) "bad cubes of size $\varepsilon^2(h)$", i.e. those intersecting more than one element of Q_ε.

We observe that, as $\gamma_2 = +\infty$,

$$\mathrm{dist}(\mathcal{Q}_{\varepsilon^2}; \partial\omega) \geq \mathrm{dist}(Q_\varepsilon; \partial\omega) - \sqrt{2}\varepsilon^2(h) > h \tag{4.29}$$

for h small enough, and

$$\#\mathbb{Z}^\varepsilon \leq C\frac{|\omega|}{\varepsilon^2(h)}. \tag{4.30}$$

Moreover, if $z \in \mathbb{Z}^\varepsilon$, $\lambda \in \mathbb{Z}^2$, and

$$\varepsilon^2(h)\lambda \in Q(\varepsilon(h)z, \varepsilon(h) - \varepsilon^2(h)),$$

then $Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))$ is a "good cube", therefore the boundary layer of $Q(\varepsilon(h)z, \varepsilon(h))$, that could possibly intersect "bad cubes" has measure given by

$$\begin{aligned} &|Q(\varepsilon(h)z, \varepsilon(h))| - |Q(\varepsilon(h)z, \varepsilon(h) - \varepsilon^2(h))| \\ &\quad = \varepsilon(h)^2 - (\varepsilon(h) - \varepsilon(h)^2)^2 = 2\varepsilon(h)^3 - \varepsilon(h)^4. \end{aligned}$$

By (4.30) we conclude that the sum of all areas of "bad cubes" intersecting Q_ε is bounded from above by

$$C\frac{|\omega|}{\varepsilon^2(h)}(2\varepsilon^3(h) - \varepsilon^4(h)) \leq C\varepsilon(h). \tag{4.31}$$

We define the sets

$$\mathbb{Z}_g^\varepsilon := \{\lambda \in \mathbb{Z}^2 : \exists\, z \in \mathbb{Z}^\varepsilon \text{ s.t. } Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \subset Q(\varepsilon(h)z, \varepsilon(h))\}$$

and

$$\mathbb{Z}_b^\varepsilon := \{\lambda \in \mathbb{Z}^2 : Q(\varepsilon(h)^2\lambda, \varepsilon^2(h)) \cap Q_\varepsilon \neq \emptyset \text{ and } \lambda \notin \mathbb{Z}_g^\varepsilon\}$$

(where "g" and "b" stand for "good" and "bad", respectively). We rewrite (4.28) as

$$\begin{aligned} &\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx' \\ &\quad = \sum_{\lambda\in\mathbb{Z}_g^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx' \\ &\qquad + \sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx'. \end{aligned}$$

Since the maps $\{(R^h)'\}$ are piecewise constant on "good cubes", by the periodicity of φ we have

$$\begin{aligned} &\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx' \\ &\quad = \sum_{\lambda\in\mathbb{Z}_g^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h} \end{aligned}$$

$$
\begin{aligned}
&: (\nabla')^{\perp}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\left(\psi\left(x', \frac{x'}{\varepsilon(h)}\right) - \psi(\varepsilon^2(h)\lambda, \varepsilon(h)\lambda)\right) dx' \\
&+ \sum_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} \\
&: (\nabla')^{\perp}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\left(\psi\left(x', \frac{x'}{\varepsilon(h)}\right) - \psi(\varepsilon^2(h)\lambda, \varepsilon(h)\lambda)\right) dx' \\
&+ \sum_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^{\perp}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \psi(\varepsilon^2(h)\lambda, \varepsilon(h)\lambda) dx'.
\end{aligned}
\tag{4.32}
$$

We claim that

$$
\lim_{h\to 0}\left|\sum_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^{\perp}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \psi(\varepsilon^2(h)\lambda, \varepsilon(h)\lambda) dx'\right| = 0.
\tag{4.33}
$$

Indeed, by the periodicity of φ,

$$
\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (\nabla')^{\perp}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) dx' = 0 \quad \text{for every } \lambda \in \mathbb{Z}^2,
$$

and we have

$$
\begin{aligned}
&\left|\sum_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^{\perp}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \psi\big(\varepsilon^2(h)\lambda, \varepsilon(h)\lambda\big) dx'\right| \\
&\quad = \left|\sum_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)}{h}\right. \\
&\qquad \left. : (\nabla')^{\perp}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \psi\big(\varepsilon^2(h)\lambda, \varepsilon(h)\lambda\big) dx'\right|.
\end{aligned}
$$

Therefore, by Hölder's inequality,

$$
\begin{aligned}
&\left|\sum_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^{\perp}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \psi(\varepsilon^2(h)\lambda, \varepsilon(h)\lambda) dx'\right| \\
&\quad \leq \frac{C}{h} \int_{\bigcup_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} |(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)|\, dx' \\
&\quad \leq \frac{C}{h} \left|\bigcup_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))\right|^{\frac{1}{2}} \|(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)\|_{L^2\left(\bigcup_{\lambda\in\mathbb{Z}^{\varepsilon}_{b}} Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))\right)}.
\end{aligned}
\tag{4.34}
$$

Every cube $Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))$ in the previous sum intersects at most four elements of Q_ε. For every $\lambda \in \mathbb{Z}_b^\varepsilon$, let $Q(\varepsilon(h)z_i^\lambda, \varepsilon), i = 1, \ldots, 4$, be such cubes, where

$$\#\{z_i^\lambda : i = 1, \ldots, 4\} \leq 4.$$

Without loss of generality, for every $\lambda \in \mathbb{Z}_b^\varepsilon$ we can assume that

$$\varepsilon^2(h)\lambda \in Q(\varepsilon(h)z_4^\lambda, \varepsilon(h)),$$

so that

$$|(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)| = 0 \quad \text{a.e. in } Q(\varepsilon(h)z_4^\lambda, \varepsilon(h)).$$

Hence,

$$\sum_{\lambda \in \mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} |(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)|^2 dx'$$

$$= \sum_{\lambda \in \mathbb{Z}_b^\varepsilon} \sum_{i=1}^{3} \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \cap Q(\varepsilon(h)z_i^\lambda, \varepsilon(h))} |(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)|^2 dx'.$$

Since the maps $\{R^h\}$ are piecewise constant on each set

$$Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \cap Q(\varepsilon(h)z_i^\lambda, \varepsilon(h)),$$

there holds

$$|(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)| = |(R^h)'(x') - (R^h)'(x' + \xi)|$$

for some $\xi \in \{\pm\varepsilon^2(h)e_1, \pm\varepsilon^2(h)e_2, \pm\varepsilon^2(h)e_1 \pm \varepsilon^2(h)e_2\}$.

Therefore, by (4.29) and Lemma 3.1, and since $\gamma_1 \in (0, +\infty)$, we have

$$\sum_{\lambda \in \mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} |(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)|^2 dx'$$

$$\leq C\|\mathrm{dist}(\nabla_h u^h; \mathrm{SO}(3))\|_{L^2(\Omega)}^2. \tag{4.35}$$

Combining (3.1), (4.31), (4.34) and (4.35), we finally get the inequality

$$\left| \sum_{\lambda \in \mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^\perp \varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \psi(\varepsilon^2(h)\lambda, \varepsilon(h)\lambda) dx' \right|$$

$$\leq \frac{C}{h} \left| \bigcup_{\lambda \in \mathbb{Z}_b^\varepsilon} Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \right|^{\frac{1}{2}} \|\mathrm{dist}(\nabla_h u^h; \mathrm{SO}(3))\|_{L^2(\Omega)}$$

$$\leq C \left| \bigcup_{\lambda \in \mathbb{Z}_b^\varepsilon} Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \right|^{\frac{1}{2}} \leq C\sqrt{\varepsilon(h)},$$

and this concludes the proof of (4.33).

Estimates (4.32) and (4.33) yield

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x',\frac{x'}{\varepsilon(h)}\right)dx'$$

$$= \lim_{h\to 0}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h}$$

$$: (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\left(\psi\left(x',\frac{x'}{\varepsilon(h)}\right)-\psi(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda)\right)dx'$$

$$= \lim_{h\to 0}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h}$$

$$: (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\left(\int_0^1\frac{d}{dt}\phi_\varepsilon(\varepsilon^2(h)\lambda+t(x'-\varepsilon^2(h)\lambda))dt\right)dx',$$

where $\phi_\varepsilon(x') := \psi(x', \frac{x'}{\varepsilon(h)})$ for every $x'\in\omega$. Therefore, by the periodicity of φ:

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x',\frac{x'}{\varepsilon(h)}\right)dx'$$

$$= \lim_{h\to 0}\Bigg[\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^2(h)}{h}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}(R^h)'(x') : (\nabla')^\perp\varphi$$

$$\times\left(\frac{x'-\varepsilon^2(h)\lambda}{\varepsilon^2(h)}\right)\left(\int_0^1\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda+t(x'-\varepsilon^2(h)\lambda))\cdot\frac{(x'-\varepsilon^2(h)\lambda)}{\varepsilon^2(h)}\,dt\right)dx'\Bigg]. \tag{4.36}$$

Changing coordinates in (4.36) we get

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x',\frac{x'}{\varepsilon(h)}\right)dx'$$

$$= \lim_{h\to 0}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)$$

$$: (\nabla')^\perp\varphi(z)\left(\int_0^1\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda+t\varepsilon^2(h)z)dt\cdot z\right)dz$$

$$= \lim_{h\to 0}\Bigg[\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)$$

$$: (\nabla')^\perp\varphi(z)\left(\int_0^1(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda+t\varepsilon^2(h)z)-\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda))dt\cdot z\right)dz$$

$$+ \sum_{\lambda \in (\mathbb{Z}_b^\varepsilon \cup \mathbb{Z}_g^\varepsilon)} \frac{\varepsilon^6(h)}{h} \int_Q (R^h)'(\varepsilon^2(h)z + \varepsilon^2(h)\lambda)$$

$$\left. : (\nabla')^\perp \varphi(z)(\nabla' \phi_\varepsilon(\varepsilon^2(h)\lambda) \cdot z) dz \right]. \tag{4.37}$$

We notice that

$$\lim_{h\to 0} \left[\sum_{\lambda \in (\mathbb{Z}_b^\varepsilon \cup \mathbb{Z}_g^\varepsilon)} \frac{\varepsilon^6(h)}{h} \int_Q (R^h)'(\varepsilon^2(h)z + \varepsilon^2(h)\lambda) \right.$$

$$\left. : (\nabla')^\perp \varphi(z) \left(\int_0^1 (\nabla' \phi_\varepsilon(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z) - \nabla' \phi_\varepsilon(\varepsilon^2(h)\lambda))dt \right) \cdot z\, dz \right] = 0. \tag{4.38}$$

Indeed, since $\|(\nabla')^2 \phi_\varepsilon\|_{L^\infty(\omega\times Q;\mathbb{M}^{3\times 3})} \leq \frac{C}{\varepsilon^2(h)}$, we have

$$\left| \sum_{\lambda \in (\mathbb{Z}_b^\varepsilon \cup \mathbb{Z}_g^\varepsilon)} \frac{\varepsilon^6(h)}{h} \int_Q (R^h)'(\varepsilon^2(h)z + \varepsilon^2(h)\lambda) \right.$$

$$\left. : (\nabla')^\perp \varphi(z) \left(\int_0^1 (\nabla' \phi_\varepsilon(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z) - \nabla' \phi_\varepsilon(\varepsilon^2(h)\lambda))dt \cdot z \right) dz \right|$$

$$\leq C \frac{\varepsilon^6(h)}{h} \sum_{\lambda \in (\mathbb{Z}_b^\varepsilon \cup \mathbb{Z}_g^\varepsilon)} \int_Q |(R^h)'(\varepsilon^2(h)z + \varepsilon^2(h)\lambda)| \|(\nabla')^2 \phi_\varepsilon\|_{L^\infty(\Omega\times Q)} |\varepsilon^2(h)z| dz$$

$$\leq C \frac{\varepsilon^6(h)}{h} \sum_{\lambda \in (\mathbb{Z}_b^\varepsilon \cup \mathbb{Z}_g^\varepsilon)} \int_Q |(R^h)'(\varepsilon^2(h)z + \varepsilon^2(h)\lambda)| dz$$

$$= C \frac{\varepsilon^2(h)}{h} \sum_{\lambda \in (\mathbb{Z}_b^\varepsilon \cup \mathbb{Z}_g^\varepsilon)} \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} |(R^h)'(x')| dx' \leq C \frac{\varepsilon^2(h)}{h} \|(R^h)'\|_{L^1(\omega;\mathbb{M}^{3\times 3})}$$

which converges to zero by (4.7) and because $\gamma_2 = +\infty$.

By (4.38), estimate (4.37) simplifies as

$$\lim_{h\to 0} \int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp \varphi \left(\frac{x'}{\varepsilon^2(h)} \right) \psi \left(x', \frac{x'}{\varepsilon(h)} \right) dx'$$

$$= \lim_{h\to 0} \sum_{\lambda \in (\mathbb{Z}_b^\varepsilon \cup \mathbb{Z}_g^\varepsilon)} \frac{\varepsilon^6(h)}{h} \int_Q (R^h)'(\varepsilon^2(h)z + \varepsilon^2(h)\lambda)$$

$$: (\nabla')^\perp \varphi(z)(\nabla' \phi_\varepsilon(\varepsilon^2(h)\lambda) \cdot z) dz$$

$$= \lim_{h\to 0}\Bigg[\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q((R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)-(R^h)'(\varepsilon^2(h)\lambda))$$

$$: (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)dz$$

$$+\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q(R^h)'(\varepsilon^2(h)\lambda):(\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)dz\Bigg]. \tag{4.39}$$

We observe that

$$\lim_{h\to 0}\Bigg[\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q((R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)-(R^h)'(\varepsilon^2(h)\lambda))$$

$$: (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)dz\Bigg]=0. \tag{4.40}$$

Indeed, since $\varphi\in C^1_{\mathrm{per}}(\mathbb{R}^2;\mathbb{M}^{3\times 3})$ and $\|(\nabla')\phi_\varepsilon\|_{L^\infty(\omega\times Q)}\leq\frac{C}{\varepsilon(h)}$, recalling the definition of the sets $\mathbb{Z}_b^\varepsilon$ and $\mathbb{Z}_g^\varepsilon$, and applying Hölder's inequality, (3.1), (4.31), and (4.35), we obtain

$$\Bigg|\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q((R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)-(R^h)'(\varepsilon^2(h)\lambda))$$

$$: (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)dz\Bigg|$$

$$\leq C\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\int_Q|(R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)-(R^h)'(\varepsilon^2(h)\lambda)|dz$$

$$=\frac{C\varepsilon(h)}{h}\sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|dx'$$

$$\leq\frac{C\varepsilon(h)}{h}\Bigg|\bigcup_{\lambda\in\mathbb{Z}_b^\varepsilon}Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))\Bigg|^{\frac{1}{2}}\|\mathrm{dist}(\nabla_h u^h;\mathrm{SO}(3))\|_{L^2(\Omega)}\leq C\varepsilon(h)^{\frac{3}{2}}.$$

Collecting (4.39) and (4.40), we deduce that

$$\lim_{h\to 0}\int_\omega\frac{(R^h)'(x')}{h}:(\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x',\frac{x'}{\varepsilon(h)}\right)dx'$$

$$=\lim_{h\to 0}\Bigg[\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q(R^h)'(\varepsilon^2(h)\lambda):(\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)dz\Bigg]. \tag{4.41}$$

Since $0 < \gamma_1 < +\infty$ and $\gamma_2 = +\infty$, by (4.7) we have

$$\lim_{h\to 0} \sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)} \frac{\varepsilon^6(h)}{h} \int_Q \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (R^h)'(x') : (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)dx'\,dz$$

$$= \lim_{h\to 0} \frac{\varepsilon^2(h)}{h} \int_\omega \int_Q (R^h)'(x')$$

$$: (\nabla')^\perp\varphi(z) \left[\left(\nabla_x\psi\left(x', \frac{x'}{\varepsilon(h)}\right) + \frac{1}{\varepsilon(h)}\nabla_y\psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right)\cdot z\right] dz\,dx'$$

$$= \frac{1}{\gamma_1} \int_\omega \int_Q \int_Q R'(x') : (\nabla')^\perp\varphi(z)(\nabla_y\psi(x',y)\cdot z)dz\,dy\,dx' = 0,$$

by the periodicity of ψ with respect to y. We observe that if $\lambda \in \mathbb{Z}_g^\varepsilon$, then

$$\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (R^h)'(x') : (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)dx'$$

$$= (R^h)'(\varepsilon^2(h)\lambda) : \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)dx',$$

and we obtain

$$\lim_{h\to 0} \Bigg[\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)} \frac{\varepsilon^6(h)}{h} \int_Q (R^h)'(\varepsilon^2(h)\lambda) : (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)dz$$

$$- \sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)} \frac{\varepsilon^6(h)}{h} \int_Q \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (R^h)'(x')$$

$$: (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)dx'\,dz \Bigg]$$

$$= \lim_{h\to 0} \Bigg[\sum_{\lambda\in\mathbb{Z}_g^\varepsilon} \frac{\varepsilon^6(h)}{h} (R^h)'(\varepsilon^2(h)\lambda)$$

$$: \int_Q (\nabla')^\perp\varphi(z) \left[\left(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda) - \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \nabla'\phi_\varepsilon(x')\,dx'\right)\cdot z\right] dz$$

$$+ \sum_{\lambda\in\mathbb{Z}_b^\varepsilon} \frac{\varepsilon^6(h)}{h} \int_Q (R^h)'(\varepsilon^2(h)\lambda) : (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)dz$$

$$- \sum_{\lambda\in\mathbb{Z}_b^\varepsilon} \frac{\varepsilon^6(h)}{h} \int_Q \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (R^h)'(x') : (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)dx'dz \Bigg].$$

By the regularity of φ and ψ, and the boundedness of $\{R^h\}$ in $L^\infty(\omega;\mathbb{M}^{3\times 3})$,

$$\begin{aligned}&\Bigg|\sum_{\lambda\in\mathbb{Z}^\varepsilon_g}\frac{\varepsilon^6(h)}{h}(R^h)'(\varepsilon^2(h)\lambda)\\&\quad:\int_Q(\nabla')^\perp\varphi(z)\left[\left(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)-\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\nabla'\phi_\varepsilon(x')\,dx'\right)\cdot z\right]dz\Bigg|\\&\quad\leq C\frac{\varepsilon^2(h)}{h}\sum_{\lambda\in\mathbb{Z}^\varepsilon_g}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)-\nabla'\phi_\varepsilon(x')|dx'\\&\quad\leq C\frac{\varepsilon^4(h)}{h}\|\nabla^2\phi_\varepsilon\|_{L^\infty(\omega\times Q;\mathbb{M}^{3\times3})}\leq C\frac{\varepsilon^2(h)}{h},\end{aligned}\tag{4.42}$$

which converges to zero, because $\gamma_2=+\infty$. On the other hand,

$$\begin{aligned}&\sum_{\lambda\in\mathbb{Z}^\varepsilon_b}\frac{\varepsilon^6(h)}{h}\int_Q\Bigg[(R^h)'(\varepsilon^2(h)\lambda):(\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)dz\\&\qquad-\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}(R^h)'(x'):(\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)dx'\Bigg]dz\\&\quad=\sum_{\lambda\in\mathbb{Z}^\varepsilon_b}\frac{\varepsilon^6(h)}{h}\int_Q(R^h)'(\varepsilon^2(h)\lambda)\\&\qquad:(\nabla')^\perp\varphi(z)\left[\left(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)-\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\nabla'\phi_\varepsilon(x')dx'\right)\cdot z\right]dz\\&\qquad+\sum_{\lambda\in\mathbb{Z}^\varepsilon_b}\frac{\varepsilon^6(h)}{h}\int_Q\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}((R^h)'(\varepsilon^2(h)\lambda)-(R^h)'(x'))\\&\qquad:(\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)dx'\,dz.\end{aligned}\tag{4.43}$$

Therefore, arguing as in (4.42), the first term on the right-hand side of (4.43) is bounded by $C\frac{\varepsilon^2(h)}{h}$, whereas by (4.31) and the boundedness of $\{R^h\}$ in $L^\infty(\omega;\mathbb{M}^{3\times3})$,

$$\begin{aligned}&\Bigg|\sum_{\lambda\in\mathbb{Z}^\varepsilon_b}\frac{\varepsilon^6(h)}{h}\int_Q\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}((R^h)'(\varepsilon^2(h)\lambda)-(R^h)'(x'))\\&\qquad:(\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)dx'\,dz\Bigg|\end{aligned}$$

$$\leq C\frac{\varepsilon(h)}{h}\sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|dx'$$

$$\leq C\frac{\varepsilon^2(h)}{h}, \tag{4.44}$$

which converges to zero as $\gamma_2=+\infty$.

Combining (4.41)–(4.44) we conclude that

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x',\frac{x'}{\varepsilon(h)}\right)dx'=0. \tag{4.45}$$

By (4.20), (4.27), and (4.45), we obtain

$$\int_\omega\int_Q\int_Q\left(V(x',y,z)-\int_Z V(x',y,\xi)d\xi\right) : (\nabla')^\perp\varphi(z)\psi(x',y)dz\,dy\,dx'=0,$$

for all $\varphi\in C^1_{\text{per}}(Q;\mathbb{R}^3)$ and $\psi\in C_c^\infty(\omega;C^\infty_{\text{per}}(Q))$.

This completes the proof of (4.21).

Case 1, Step 2: Characterization of the limit linearized strain G.

In order to identify the multiscale limit of the sequence of linearized strains G^h, by (4.12), (4.14), (4.18)–(4.20) we now characterize the weak 3-scale limits of the sequences $\{x_3\nabla'\tilde{R}^h e_3\}$ and $\{\frac{1}{h}(\tilde{R}^h e_3 - R^h e_3)\}$.

By (4.8) and Theorem 1.2 in Ref. 3 there exist $S\in L^2(\omega;W^{1,2}_{\text{per}}(Q;\mathbb{M}^{3\times 3}))$ and $T\in L^2(\omega\times Q;W^{1,2}_{\text{per}}(Q;\mathbb{M}^{3\times 3}))$ such that

$$\nabla'\tilde{R}^h \xrightarrow{3\text{-}s} \nabla' R+\nabla_y S+\nabla_z T \quad \text{weakly 3-scale}, \tag{4.46}$$

where $\int_Q S(x',y)\,dy=0$ for a.e. $x'\in\omega$, and $\int_Q T(x',y,z)\,dz=0$ for a.e. $x'\in\omega$, and $y\in Y$. By (3.1) and (4.6), there exists $w\in L^2(\omega\times Q\times Q;\mathbb{R}^3)$ such that

$$\frac{1}{h}(\tilde{R}^h e_3 - R^h e_3)\xrightarrow{3\text{-}s} w \quad \text{weakly 3-scale}$$

and hence,

$$\frac{1}{h}(\tilde{R}^h e_3 - R^h e_3)\rightharpoonup w_0 \quad \text{weakly in } L^2(\omega;\mathbb{R}^3),$$

where

$$w_0(x'):=\int_Q\int_Q w(x',y,z)dy\,dz,$$

for a.e. $x'\in\omega$. We claim that

$$\frac{1}{h}(\tilde{R}^h e_3 - R^h e_3)\xrightarrow{3\text{-}s} w_0(x')+\frac{1}{\gamma_1}S(x',y)e_3+\frac{(y\cdot\nabla')R(x')e_3}{\gamma_1}, \tag{4.47}$$

weakly 3-scale. We first remark that the same argument as in the proof of (4.28) yields

$$\frac{R^h e_3}{h}\xrightarrow{\text{osc},Z} 0.$$

Moreover, since $\gamma_1 \in (0, +\infty)$, by (4.7), Lemmas 3.2 and 3.3, there holds

$$\frac{R^h e_3}{h} \xrightharpoonup{\text{osc},Y} -\frac{(y \cdot \nabla')Re_3}{\gamma_1}$$

and

$$\frac{\tilde{R}^h e_3}{h} \xrightharpoonup{\text{osc},Y} \frac{Se_3}{\gamma_1},$$

where in the latter property we used the fact that $\int_Q \nabla_z T(x', y, z)dz = 0$ for a.e. $x' \in \omega$ and $y \in Y$ by periodicity, and $\int_Q S(x', y)dy = 0$ for a.e. $x' \in \omega$. Therefore, by Remark 3.2, to prove (4.47) we only need to show that

$$\frac{\tilde{R}^h e_3}{h} \xrightharpoonup{\text{osc},Z} 0. \tag{4.48}$$

To this purpose, fix $\varphi \in C^\infty_{\text{per}}(Q)$, with $\int_Q \varphi(z)dz = 0$, and $\psi \in C^\infty_c(\Omega; C^\infty_{\text{per}}(Q))$, and let $g \in C^2(Q)$ be the unique periodic solution to

$$\begin{cases} \Delta g(z) = \varphi(z), \\ \displaystyle\int_Q g(z)dz = 0. \end{cases}$$

Set

$$g^\varepsilon(x') := \varepsilon^2(h) g\left(\frac{x'}{\varepsilon^2(h)}\right) \quad \text{for every } x' \in \omega, \tag{4.49}$$

so that

$$\Delta g^\varepsilon(x') = \frac{1}{\varepsilon^2(h)}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \quad \text{for every } x' \in \omega. \tag{4.50}$$

By (4.49) and (4.50), and for $i \in \{1, 2, 3\}$, we obtain

$$\begin{aligned} &\int_\omega \frac{\tilde{R}^h_{i3}(x')}{h}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx' \\ &\quad = \frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\Delta g^\varepsilon(x')\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx'. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} &\int_\omega \frac{\tilde{R}^h_{i3}(x')}{h}\varphi\left(\frac{x'}{\varepsilon^2(h)}\right)\psi\left(x', \frac{x'}{\varepsilon(h)}\right)dx' \\ &\quad = -\frac{\varepsilon^2(h)}{h}\int_\omega \nabla'\tilde{R}^h_{i3}(x') \cdot \nabla'\left(g^\varepsilon(x')\psi\left(x', \frac{x'}{\varepsilon(h)}\right)\right)dx' \\ &\quad\quad - \frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\left(2\nabla' g^\varepsilon(x') \cdot (\nabla_{x'}\psi)\left(x', \frac{x'}{\varepsilon(h)}\right)\right. \\ &\quad\quad \left. + g^\varepsilon(x')(\Delta_{x'}\psi)\left(x', \frac{x'}{\varepsilon(h)}\right)\right)dx' \end{aligned}$$

$$-\frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\left[2\nabla' g^\varepsilon(x')\cdot\nabla_y\psi\left(x',\frac{x'}{\varepsilon(h)}\right)\right.$$
$$\left.+\,2g^\varepsilon(x')(\operatorname{div}_y\nabla_{x'}\psi)\left(x',\frac{x'}{\varepsilon(h)}\right)\right]dx'$$
$$-\frac{1}{h\varepsilon(h)}\int_\omega \tilde{R}^h_{i3}(x')g^\varepsilon(x')\Delta_y\psi\left(x',\frac{x'}{\varepsilon(h)}\right)dx'. \tag{4.51}$$

Since $\nabla'(g^\varepsilon(\cdot)\psi(\cdot,\frac{\cdot}{\varepsilon(h)}))\in L^\infty(\omega;\mathbb{R}^2)$,

$$\lim_{h\to 0}\frac{\varepsilon^2(h)}{h}\int_\omega \nabla'\tilde{R}^h_{i3}(x')\cdot\nabla'\left(g^\varepsilon(x')\psi\left(x',\frac{x'}{\varepsilon(h)}\right)\right)dx' = 0, \tag{4.52}$$

where we used the fact that $\gamma_2=+\infty$, and similarly,

$$\lim_{h\to 0}\frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\left(2\nabla' g^\varepsilon(x')\cdot(\nabla_{x'}\psi)\left(x',\frac{x'}{\varepsilon(h)}\right)\right.$$
$$\left.+\,g^\varepsilon(x')(\Delta_{x'}\psi)\left(x',\frac{x'}{\varepsilon(h)}\right)\right)dx' = 0. \tag{4.53}$$

Regarding the third term on the right-hand side of (4.51), we write

$$\frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\left[2\nabla' g^\varepsilon(x')\cdot\nabla_y\psi\left(x',\frac{x'}{\varepsilon(h)}\right)\right.$$
$$\left.+\,2g^\varepsilon(x')(\operatorname{div}_y\nabla_{x'}\psi)\left(x',\frac{x'}{\varepsilon(h)}\right)\right]dx'$$
$$=2\frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\nabla' g\left(\frac{x'}{\varepsilon^2(h)}\right)\cdot\nabla_y\psi\left(x',\frac{x'}{\varepsilon(h)}\right)dx'$$
$$+\,\frac{2\varepsilon^3(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')g\left(\frac{x'}{\varepsilon^2(h)}\right)(\operatorname{div}_y\nabla_{x'}\psi)\left(x',\frac{x'}{\varepsilon(h)}\right)dx'. \tag{4.54}$$

By the regularity of g and ψ,

$$\nabla' g\left(\frac{x'}{\varepsilon^2(h)}\right)\cdot\nabla_y\psi\left(x',\frac{x'}{\varepsilon(h)}\right)\xrightarrow{3\text{-}s}\nabla g(z)\nabla_y\psi(x',y)\quad\text{strongly 3-scale.}$$

Therefore, by (4.8), and since $0<\gamma_1<+\infty$ and $\gamma_2=+\infty$, we obtain

$$\lim_{h\to 0}\left[\frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\left[2\nabla' g^\varepsilon(x')\cdot\nabla_y\psi\left(x',\frac{x'}{\varepsilon(h)}\right)\right.\right.$$
$$\left.\left.+\,2g^\varepsilon(x')(\operatorname{div}_y\nabla_{x'}\psi)\left(x',\frac{x'}{\varepsilon(h)}\right)\right]dx'\right]$$
$$=\frac{2}{\gamma_1}\int_\omega\int_Q\int_Q R_{i3}(x')\nabla g(z)\cdot\nabla_y\psi(x',y)\,dz\,dy\,dx' = 0, \tag{4.55}$$

where the last equality is due to the periodicity of ψ in the y variable.

Again by the regularity of g and ψ,

$$g\left(\frac{x'}{\varepsilon^2(h)}\right)\Delta_y\psi\left(x',\frac{x'}{\varepsilon(h)}\right)\xrightarrow{3\text{-}s} g(z)\Delta_y\psi(x',y)\quad\text{strongly 3-scale,}$$

hence, by (4.8), and since $0<\gamma_1<+\infty$ and $\psi\in C_c^\infty(\omega;C^\infty_{\mathrm{per}}(Q))$, the fourth term on the right-hand side of (4.51) satisfies

$$\lim_{h\to 0}\frac{1}{h\varepsilon(h)}\int_\omega \tilde{R}^h_{i3}(x')g^\varepsilon(x')\Delta_y\psi\left(x',\frac{x'}{\varepsilon(h)}\right)dx'$$
$$=\frac{1}{\gamma_1}\int_\omega\int_Q\int_Q R_{i3}(x')g(z)\Delta_y\psi(x',y)dz\,dy\,dx'=0. \tag{4.56}$$

Claim (4.48), and thus (4.47), follow now by combining (4.51) with (4.52)–(4.56).

Case 1, Step 3: Characterization of E.

By (4.7), and by collecting (4.18), (4.19), (4.20), (4.46), and (4.47), we deduce the characterization

$$\begin{aligned}R(x')G(x,y,z)=&\left(\nabla' r(x')+\nabla_y\hat{\phi}_1(x,y)+\nabla_z\hat{\phi}_2(x,y,z)\,\middle|\,\frac{1}{\gamma_1}\partial_{x_3}\hat{\phi}_1(x,y)\right)\\&+\left(V(x',y,z)\,\middle|\,w_0(x')+\frac{1}{\gamma_1}S(x',y)e_3+\frac{(y\cdot\nabla')R'(x')}{\gamma_1}e_3\right)\\&+x_3(\nabla' R(x')e_3+\nabla_y S(x',y)e_3+\nabla_z T(x',y,z)e_3|0)\end{aligned}$$

for a.e. $x\in\Omega$ and $y,z\in Q$, where $r\in W^{1,2}(\omega;\mathbb{R}^3)$, $\hat{\phi}_1\in L^2(\omega;W^{1,2}((-\frac{1}{2},\frac{1}{2});W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)))$, $w_0\in L^2(\omega;\mathbb{R}^3)$, $S\in L^2(\omega;W^{1,2}_{\mathrm{per}}(Q;\mathbb{M}^{3\times 3}))$, $V\in L^2(\omega\times Q\times Q;\mathbb{M}^{3\times 2})$, $\hat{\phi}_2\in L^2(\Omega\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, and $T\in L^2(\omega\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{M}^{3\times 3}))$. Therefore, by (4.21):

$$\begin{aligned}&\operatorname{sym}G(x,y,z)-\int_Q\operatorname{sym}G(x,y,\xi)d\xi\\&=\operatorname{sym}\left[R(x')^T\left(V(x',y,z)-\int_Q V(x',y,z)+\nabla_z\hat{\phi}_2(x,y,z)\,\middle|\,0\right)\right.\\&\left.\quad+x_3R(x')^T(\nabla_z T(x',y,z)e_3|0)\right]\\&=\operatorname{sym}\left[R(x')^T(\nabla_z v(x',y,z)+\nabla_z\hat{\phi}_2(x,y,z)+x_3\nabla_z T(x',y,z)e_3\|0)\right],\end{aligned}$$

where $Te_3,\tilde{\tilde{v}}\in L^2(\omega\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$. The thesis follows now by (4.12), (4.13), and by setting

$$\phi_2:=R^T(v+\hat{\phi}_2+x_3Te_3),$$

for a.e. $x\in\Omega$, and $y,z\in Q$.

Case 2: $\gamma_1=+\infty$ and $\gamma_2=+\infty$.

The proof is very similar to the first case where $0<\gamma_1<+\infty$. We only outline the main modifications.

Arguing as in the proof of Proposition 3.2 in Ref. 16, in order to construct the sequence $\{R^h\}$, we apply Lemma 3.1 with

$$\delta(h) := \left(2\left\lceil\frac{h}{\varepsilon(h)}\right\rceil + 1\right)\varepsilon(h).$$

This way,

$$\lim_{h\to 0}\frac{h}{\delta(h)} = \frac{1}{2}$$

and the maps R^h are piecewise constant on cubes of the form $Q(\delta(h)z, \delta(h))$, with $z \in \mathbb{Z}^2$. In particular, since $\left\{\frac{\delta(h)}{\varepsilon(h)}\right\}$ is a sequence of odd integers, by Lemma A.1 the maps R^h are piecewise constant on cubes of the form $Q(\varepsilon(h)z, \varepsilon(h))$ with $z \in \mathbb{Z}^2$, and (4.6) holds true. Defining $\{r^h\}$ as in (4.16), we obtain equality (4.18). By Theorem 3.2(i), there exist $r \in W^{1,2}(\omega;\mathbb{R}^3)$, $\hat{\phi}_1 \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, $\hat{\phi}_2 \in L^2(\Omega\times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, and $\bar{\phi} \in L^2(\omega; W^{1,2}((-\frac{1}{2},\frac{1}{2});\mathbb{R}^3))$ such that

$$\nabla_h r^h \xrightarrow{dr\text{-}3\text{-}s} (\nabla' r + \nabla_y\hat{\phi}_1 + \nabla_z\hat{\phi}_2 | \partial_{x_3}\bar{\phi}) \quad \text{weakly dr-3-scale.} \tag{4.57}$$

Moreover, (4.13) now becomes

$$\operatorname{sym}\int_Q G(x,y,\xi)d\xi = \begin{pmatrix} x_3\Pi^u(x') + \operatorname{sym} B(x') & 0 \\ 0 & 0\end{pmatrix} + \operatorname{sym}\left(\nabla_y\phi_1(x,y)\,\middle|\,\partial_{x_3}\bar{\phi}\right)$$

for a.e. $x \in \Omega$ and $y \in Q$, where $B \in L^2(\omega;\mathbb{M}^{2\times 2})$. Arguing as in Steps 1–3 of Case 1, we obtain the characterization

$$\begin{aligned} E(x,y,z) = &\begin{pmatrix} x_3\Pi^u(x') + \operatorname{sym} B(x') & 0 \\ 0 & 0\end{pmatrix} \\ &+ \operatorname{sym}(\nabla_y\phi_1(x,y)|d(x)) + \operatorname{sym}(\nabla_z\phi_2(x,y,z)|0), \end{aligned}$$

with $d := \partial_{x_3}\bar{\phi} \in L^2(\Omega;\mathbb{R}^3)$, $\phi_1 \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, and $\phi_2 \in L^2(\Omega\times Q; W^{1,2}_{\mathrm{per}} \times (Q;\mathbb{R}^3))$.

Case 3: $\gamma_1 = 0$ and $\gamma_2 = +\infty$.

The structure of the proof is similar to that of Cases 1 and 2, therefore we only outline the main steps and key points, leaving the details to the reader.

We first apply Lemma 3.1 with

$$\delta(h) := \left(2\left\lceil\frac{h}{\varepsilon^2(h)} + 1\right\rceil\right)\varepsilon^2(h),$$

and by Lemma A.1 we construct

$$\{R^h\} \subset L^\infty(\omega;SO(3)) \quad \text{and} \quad \{\tilde{R}^h\} \subset W^{1,2}(\omega;\mathbb{M}^{3\times 3}),$$

satisfying (4.6), and with R^h piecewise constant on every cube of the form

$$Q(\varepsilon^2(h)z, \varepsilon^2(h)), \quad \text{with } z \in \mathbb{Z}^2.$$

Arguing as in Case 1, we obtain the convergence properties in (4.7) and (4.8), and the identification of E reduces to establishing a characterization of the weak 3-scale limit G of the sequence $\{G^h\}$ defined in (4.10). In view of Proposition 3.2 in Ref. 26, there exist $B \in L^2(\omega; \mathbb{M}^{2\times 2})$, $\xi \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^2))$, $\eta \in L^2(\omega; W^{2,2}_{\mathrm{per}} \times (Q;\mathbb{R}^2))$, and $g_i \in L^2(\Omega \times Y)$, $i = 1, 2, 3$, such that

$$\begin{aligned}\int_Q E(x,y,z)dz &= \mathrm{sym} \int_Q G(x,y,z)dz \\ &= \begin{pmatrix} x_3\Pi^u(x') + \mathrm{sym}\, B(x') & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathrm{sym}\,\nabla_y \xi(x,y) + x_3 \nabla_y^2 \eta(x',y) & g_1(x,y) \\ & g_2(x,y) \\ g_1(x,y) \quad g_2(x,y) & g_3(x,y) \end{pmatrix} \end{aligned} \tag{4.58}$$

for a.e. $x \in \Omega$ and $y \in Y$. We consider the maps $\{\bar{u}^h\}$ and $\{r^h\}$ defined in (4.15) and (4.16), and we perform the decomposition in (4.18). By Theorem 3.2(iii) there exist maps $r \in W^{1,2}(\omega;\mathbb{R}^3)$, $\hat{\phi}_1 \in L^2(\omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, $\hat{\phi}_2 \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, and $\bar{\phi} \in L^2(\omega \times Q; W^{1,2}((-\frac{1}{2},\frac{1}{2});\mathbb{R}^3))$ such that

$$\nabla_h r^h \xrightarrow{dr\text{-}3\text{-}s} (\nabla' r + \nabla_y \hat{\phi}_1 + \nabla_z \hat{\phi}_2 | \partial_{x_3} \bar{\phi}) \quad \text{weakly dr-3-scale.}$$

Defining V as in (4.20), we first need to show that

$$V(x',y,z) - \int_Q V(x',y,z)dz = \nabla_z v(x',y,z) \tag{4.59}$$

for a.e. $x' \in \omega$, and $y, z \in Q$, for some $v \in L^2(\omega \times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$.

As in Case 1-Step 1, by Lemma 3.7 in Ref. 3 and by a density argument, to prove (4.59) it is enough to show that

$$\int_\omega \int_Q \int_Q \left(V(x',y,z) - \int_Q V(x',y,z)dz \right) : (\nabla')^\perp \varphi(z)\phi(y)\psi(x')dz\,dy\,dx' = 0 \tag{4.60}$$

for every $\varphi \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^3)$, $\phi \in C^\infty_{\mathrm{per}}(Q)$ and $\psi \in C^\infty_c(\omega)$.

Fix $\varphi \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^3)$, $\phi \in C^\infty_{\mathrm{per}}(Q)$, $\psi \in C^\infty_c(\omega)$, and set

$$\varphi^\varepsilon(x') := \varepsilon^2(h)\phi\left(\frac{x'}{\varepsilon(h)}\right)\varphi\left(\frac{x'}{\varepsilon^2(h)}\right) \quad \text{for every } x' \in \mathbb{R}^2.$$

Integrating by parts and applying Riemann–Lebesgue lemma (see Ref. 10) we deduce

$$\begin{aligned}&\lim_{h\to 0} \int_\omega \frac{\nabla_h \bar{u}^h - (R^h)'}{h} : (\nabla')^\perp \varphi^\varepsilon(x')\psi(x') \\ &\quad = \int_\omega \int_Q \int_Q V(x',y,z) : \nabla^\perp \varphi(z)\phi(y)\psi(x')dz\,dy\,dx'. \end{aligned} \tag{4.61}$$

In view of (4.61), (4.60) reduces to showing that

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp \varphi^\varepsilon(x')\psi(x')dx' = 0. \tag{4.62}$$

The key idea to prove (4.62) is to work on cubes $Q(\varepsilon^2(h)z, \varepsilon^2(h))$, with $z \in \mathbb{Z}^2$. Exploiting the periodicity of φ and the fact that $\{R^h\}$ is piecewise constant on such cubes, we add and subtract the values of ϕ and ψ in $\varepsilon^2(h)z$, and use the smoothness of the maps to control their oscillations on each cube $Q(\varepsilon^2(h)z, \varepsilon^2(h))$, for $z \in \mathbb{Z}^2$. Defining

$$\hat{\mathbb{Z}}^\varepsilon := \{z \in \mathbb{Z}^2 : Q(\varepsilon^2(h)z, \varepsilon^2(h)) \cap \operatorname{supp}\psi \neq \emptyset\},$$

a crucial point is to prove the equivalent of (4.41), that is to show that

$$\begin{aligned}\lim_{h\to 0}\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon} \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q \Big\{(R^h)'(x')\\ : (\nabla')^\perp\varphi(z)\left[\nabla'\phi\left(\frac{x'}{\varepsilon(h)}\right)\cdot z\right]\psi(x')\Big\}\,dz\,dx' = 0.\end{aligned} \tag{4.63}$$

This is achieved by adding and subtracting in (4.63) the function $\frac{\tilde{R}^h}{h}$, i.e.

$$\begin{aligned}&\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon} \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q (R^h)'(x') : (\nabla')^\perp\varphi(z)\left[\nabla'\phi\left(\frac{x'}{\varepsilon(h)}\right)\cdot z\right]\psi(x')dz\,dx'\\
&= \frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}\int_Q \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\Big\{((R^h)'(x') - (\tilde{R}^h)'(x'))\\
&\quad : (\nabla')^\perp\varphi(z)\left[\nabla'\phi\left(\frac{x'}{\varepsilon(h)}\right)\cdot z\right]\psi(x')\Big\}\,dz\,dx'\\
&\quad + \frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon} \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q (\tilde{R}^h)'(x')\\
&\quad : (\nabla')^\perp\varphi(z)\left[\nabla'\phi\left(\frac{x'}{\varepsilon(h)}\right)\cdot z\right]\psi(x')dz\,dx'.\end{aligned}$$

By (3.1), (4.6) and by the regularity of the test functions ϕ, φ, and ψ, we have

$$\begin{aligned}&\Bigg|\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon} \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q \Big\{((R^h)'(x') - (\tilde{R}^h)'(x'))\\
&\quad : (\nabla')^\perp\varphi(z)\left[\nabla'\phi\left(\frac{x'}{\varepsilon(h)}\right)\cdot z\right]\psi(x')\,dz\Big\}\,dx'\Bigg|\\
&\le C\varepsilon(h)\left\|\frac{(R^h)' - (\tilde{R}^h)'}{h}\right\|_{L^2(\omega;\mathbb{M}^{3\times 2})} \le C\varepsilon(h).\end{aligned} \tag{4.64}$$

Finally, by (4.8) and Theorem 1.2 in Ref. 3, there exist $S \in L^2(\omega; W^{1,2}_{\mathrm{per}}(Q; \mathbb{M}^{3\times 3}))$, and $T \in L^2(\omega \times Q; W^{1,2}_{\mathrm{per}}(Q; \mathbb{M}^{3\times 3}))$ such that

$$\nabla' \tilde{R}^h \xrightarrow{3\text{-}s} \nabla' R + \nabla_y S + \nabla_z T \quad \text{weakly 3-scale,} \tag{4.65}$$

where $\int_Q S(x', y)\, dy = 0$ for a.e. $x' \in \omega$, and $\int_Q T(x', y, z)\, dz = 0$ for a.e. $x' \in \omega$ and $y \in Q$. By Lemma 3.3,

$$\frac{\tilde{R}^h}{\varepsilon(h)} \xrightarrow{\text{osc},Y} S,$$

and hence

$$\lim_{h\to 0} \int_\omega \frac{(\tilde{R}^h)'(x')}{\varepsilon(h)} \nabla' \phi\left(\frac{x'}{\varepsilon(h)}\right) \psi(x') dx' = \int_\omega \int_Q S'(x', y) \nabla' \phi(y) \psi(x') dx'\, dy. \tag{4.66}$$

Since $\gamma_2 = +\infty$, (4.66) yields

$$\begin{aligned}
&\lim_{h\to 0} \frac{\varepsilon^5(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} \fint_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} \int_Q \Bigg\{ (\tilde{R}^h)'(x') \\
&\qquad : (\nabla')^\perp \varphi(z) \left[\nabla' \phi\left(\frac{x'}{\varepsilon(h)}\right) \cdot z \right] \psi(x') \Bigg\}\, dz\, dx' \\
&\quad = \lim_{h\to 0} \frac{\varepsilon^2(h)}{h} \int_\omega \int_Q \left\{ \frac{(\tilde{R}^h)'(x')}{\varepsilon(h)} : (\nabla')^\perp \varphi(z) \left[\nabla' \phi\left(\frac{x'}{\varepsilon(h)}\right) \cdot z \right] \psi(x') \right\} dz\, dx \\
&\quad = \frac{1}{\gamma_2} \int_\Omega \int_Q S'(x', y) : (\nabla')^\perp \varphi(z) [\nabla' \phi(y) \cdot z] \psi(x') dz\, dx' = 0
\end{aligned}$$

which, together with (4.64), implies (4.63).

Once the proof of (4.59) is completed, to identify E we need to characterize the weak 3-scale limit of the scaled linearized strains G^h (see (4.10), (4.11) and (4.12)). By (4.18) this reduces to study the weak 3-scale limit of the sequence

$$\left\{ \frac{R^h e_3 - \tilde{R}^h e_3}{h} \right\}.$$

By (3.1) and (4.6), there exists $w \in L^2(\omega \times Q \times Q; \mathbb{R}^3)$ such that

$$\frac{(\tilde{R}^h - R^h)}{h} \xrightarrow{3\text{-}s} w(x', y, z) \quad \text{weakly 3-scale.}$$

We claim that

$$w(x', y, z) - \int_Q w(x', y, z) dz = 0 \tag{4.67}$$

for a.e. $x' \in \omega$, and $y, z \in Q$. To prove (4.67), by Remark 3.2, we have to show that

$$\frac{\tilde{R}^h e_3 - R^h e_3}{h} \xrightarrow{\text{osc},Z} 0.$$

A direct application of the argument in the proof of (4.62) yields

$$\frac{R^h e_3}{h} \xrightarrow{\text{osc},Z} 0,$$

therefore (4.67) is equivalent to proving that

$$\frac{\tilde{R}^h e_3}{h} \xrightarrow{\text{osc},Z} 0$$

which follows arguing similarly to Case 1, Step 2, proof of (4.48).

Finally, with an argument similar to that of Case 1, Step 3, and combining (4.59) with (4.65), and (4.67), we obtain

$$\begin{aligned} R(x')G(x,y,z) - \int_Q R(x')G(x,y,z)dz \\ = (\nabla_z v(x',y,z) + \nabla_z \hat{\phi}_2(x,y,z) + x_3 \nabla_z T(x',y,z)e_3|0) \end{aligned}$$

for a.e. $x \in \Omega$, and $y, z \in Q$, where $v, Te_3 \in L^2(\omega \times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$, and $\hat{\phi}_2 \in L^2(\Omega \times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$.

By (4.12),

$$E(x,y,z) - \int_Q E(x,y,z)dz = \operatorname{sym}(\nabla_z \phi(x,y,z)|0)$$

for a.e. $x \in \Omega$, and $y, z \in Q$, where $\phi := R^T(v + \hat{\phi}_2 + x_3 T e_3)$. In view of (4.58) we conclude that

$$\begin{aligned} E(x,y,z) &= \begin{pmatrix} x_3 \Pi^u(x') + \operatorname{sym} B(x') & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \operatorname{sym} \nabla_y \xi(x,y) + x_3 \nabla_y^2 \eta(x',y) & & g_1(x,y) \\ & & g_2(x,y) \\ g_1(x,y) & g_2(x,y) & g_3(x,y) \end{pmatrix} + \operatorname{sym}(\nabla_z \phi(x,y,z)|0) \end{aligned}$$

for a.e. $x \in \Omega$, and $y, z \in Q$, where $B \in L^2(\omega; \mathbb{M}^{2\times 2})$, $\xi \in L^2(\Omega; W^{1,2}_{\text{per}}(Q;\mathbb{R}^2))$, $\eta \in L^2(\omega; W^{2,2}_{\text{per}}(Q))$, $g_i \in L^2(\Omega \times Y)$, $i = 1,2,3$, and $\phi \in L^2(\Omega \times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$. The thesis follows now by (4.3). □

5. The Γ-Liminf Inequality

With the identification of the limit linearized stress obtained in Sec. 4, we now find a lower bound for the effective limit energy associated to sequences of deformations with uniformly small three-dimensional elastic energies, satisfying (1.3).

Theorem 5.1. *Let* $\gamma_1 \in [0, +\infty]$ *and let* $\gamma_2 = +\infty$. *Let* $\{u^h\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *be a sequence of deformations satisfying the uniform energy estimate* (2.1) *and* *converging to* $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *as in Theorem* 3.1. *Then,*

$$\liminf_{h\to 0}\frac{\mathcal{E}^h(u^h)}{h^2}\geq \frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\hom}(\Pi^u(x'))dx',$$

where Π^u *is the map defined in* (4.4), *and:*

(a) *if* $\gamma_1=0$, *for every* $A\in\mathbb{M}^{2\times 2}_{\rm sym}$:

$$\begin{aligned}\overline{\mathscr{Q}}^{0}_{\hom}(A):=\inf\Bigg\{&\int_{(-\frac12,\frac12)\times Q}\mathscr{Q}_{\hom}\left(y,\begin{pmatrix}x_3A+B & 0\\ 0 & 0\end{pmatrix}\right.\\
&\left.+\,\mathrm{sym}\begin{pmatrix}\mathrm{sym}\,\nabla_y\xi(x_3,y)+x_3\nabla_y^2\eta(y) & g_1(x_3,y)\\ & g_2(x_3,y)\\ g_1(x_3,y)\quad g_2(x_3,y) & g_3(x_3,y)\end{pmatrix}\right):\\
&\xi\in L^2\left(\left(-\frac12,\frac12\right);W^{1,2}_{\rm per}(Q;\mathbb{R}^2)\right),\eta\in W^{2,2}_{\rm per}(Q),\\
&g_i\in L^2\left(\left(-\frac12,\frac12\right)\times Q\right), i=1,2,3,\ B\in\mathbb{M}^{2\times 2}_{\rm sym}\Bigg\};\end{aligned}\tag{5.1}$$

(b) *if* $0<\gamma_1<+\infty$, *for every* $A\in\mathbb{M}^{2\times 2}_{\rm sym}$:

$$\begin{aligned}\overline{\mathscr{Q}}^{\gamma_1}_{\hom}(A):=\inf\Bigg\{&\int_{(-\frac12,\frac12)\times Q}\mathscr{Q}_{\hom}\left(y,\begin{pmatrix}x_3A+B & 0\\ 0 & 0\end{pmatrix}\right.\\
&\left.+\,\mathrm{sym}\left(\nabla_y\phi_1(x_3,y)\,\middle|\,\frac{\partial_{x_3}\phi_1(x_3,y)}{\gamma_1}\right)\right):\\
&\phi_1\in W^{1,2}\left(\left(-\frac12,\frac12\right);W^{1,2}_{\rm per}(Q;\mathbb{R}^3)\right),\ B\in\mathbb{M}^{2\times 2}_{\rm sym}\Bigg\};\end{aligned}\tag{5.2}$$

(c) *if* $\gamma_1=+\infty$, *for every* $A\in\mathbb{M}^{2\times 2}_{\rm sym}$:

$$\begin{aligned}\overline{\mathscr{Q}}^{\infty}_{\hom}(A):=\inf\Bigg\{&\int_{(-\frac12,\frac12)\times Q}\mathscr{Q}_{\hom}\left(y,\begin{pmatrix}x_3A+B & 0\\ 0 & 0\end{pmatrix}\right.\\
&\left.+\,\mathrm{sym}(\nabla_y\phi_1(x_3,y)|d(x_3))\right):d\in L^2\left(\left(-\frac12,\frac12\right);\mathbb{R}^3\right),\\
&\phi_1\in L^2\left(\left(-\frac12,\frac12\right);W^{1,2}_{\rm per}(Q;\mathbb{R}^3)\right),\ \textit{and } B\in\mathbb{M}^{2\times 2}_{\rm sym}\Bigg\},\end{aligned}\tag{5.3}$$

where

$$\mathscr{Q}_{\hom}(y,C):=\inf\left\{\int_Q\mathscr{Q}(y,z,C+\mathrm{sym}(\nabla\phi_2(z)|0)):\phi_2\in W^{1,2}_{\rm per}(Q;\mathbb{R}^3)\right\}\tag{5.4}$$

for a.e. $y\in Q$, *and for every* $C\in\mathbb{M}^{3\times 3}_{\rm sym}$.

Proof. The proof is an adaptation of the proof of Theorem 2.4 in Ref. 16. For the convenience of the reader, we briefly sketch it in the case $0 < \gamma_1 < +\infty$. The proof in the cases $\gamma_1 = +\infty$ and $\gamma_1 = 0$ is analogous.

Without loss of generality, we can assume that $⨍_\Omega u^h(x)dx = 0$. By assumption (H2) and by Theorem 3.1, $u \in W^{2,2}(\omega;\mathbb{R}^3)$ is an isometry, with

$$u^h \to u \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3)$$

and

$$\nabla_h u^h \to (\nabla' u|n_u) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}),$$

where the vector n_u is defined according to (3.2) and (3.3). By Theorem 4.1 there exists $E \in L^2(\Omega \times Q \times Q;\mathbb{M}^{3\times 3})$ such that, up to the extraction of a (not relabeled) subsequence,

$$E^h := \frac{\sqrt{(\nabla_h u^h)^T \nabla_h u^h} - \mathrm{Id}}{h} \xrightarrow{dr\text{-}3\text{-}s} E \quad \text{weakly dr-3-scale},$$

with

$$\begin{aligned} E(x,y,z) &= \begin{pmatrix} \operatorname{sym} B(x') + x_3 \Pi^u(x') & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \operatorname{sym}\left(\nabla_y \phi_1(x,y) \,\middle|\, \frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1} \right) + \operatorname{sym}(\nabla_z \phi_2(x,y,z)|0), \end{aligned} \tag{5.5}$$

for a.e. $x' \in \omega$, and $y, z \in Q$, where $B \in L^2(\omega;\mathbb{M}^{2\times 2})$, $\phi_1 \in L^2(\omega; W^{1,2}((-\frac{1}{2},\frac{1}{2}); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, and $\phi_2 \in L^2(\omega \times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$. Arguing as in the proof of Theorem 6.1(i) in Ref. 11, by performing a Taylor expansion around the identity, and by Lemma A.2 we deduce that

$$\begin{aligned} \liminf_{h\to 0} \frac{\mathcal{E}^h(u^h)}{h^2} &\geq \liminf_{h\to 0} \int_\Omega \mathscr{Q}\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, E^h(x) \right) dx \\ &\geq \int_\Omega \int_Q \int_Q \mathscr{Q}(y,z,E(x,y,z))dz\,dy\,dx. \end{aligned}$$

By (5.2), (5.4), and (5.5), we finally conclude that

$$\begin{aligned} \liminf_{h\to 0} \frac{\mathcal{E}^h(u^h)}{h^2} &\geq \int_\Omega \int_Q \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} \operatorname{sym} B(x') + x_3\Pi^u(x') & 0 \\ 0 & 0 \end{pmatrix} \right. \\ &\quad \left. + \operatorname{sym}\left(\nabla_y \phi_1(x,y) \,\middle|\, \frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1} \right) \right) dy\,dx \\ &\geq \int_\Omega \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(x_3\Pi^u(x'))dx = \int_\Omega x_3^2 \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(\Pi^u(x'))dx \\ &= \frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(\Pi^u(x'))dx'. \end{aligned}$$

□

6. The Γ-Limsup Inequality: Construction of the Recovery Sequence

Let $W^{2,2}_R(\omega;\mathbb{R}^3)$ be the set of all $u\in W^{2,2}(\omega;\mathbb{R}^3)$ satisfying (3.2). Let $\mathcal{A}(\omega)$ be the set of all $u\in W^{2,2}_R(\omega;\mathbb{R}^3)\cap C^\infty(\bar\omega;\mathbb{R}^3)$ such that, for all $B\in C^\infty(\bar\omega;\mathbb{M}^{2\times 2}_{\rm sym})$ with $B=0$ in a neighborhood of

$$\{x'\in\omega:\Pi^u(x')=0\}$$

(where Π^u is the map defined in (4.4)), there exist $\alpha\in C^\infty(\bar\omega)$ and $g\in C^\infty(\bar\omega;\mathbb{R}^2)$ such that

$$B=\operatorname{sym}\nabla' g+\alpha\Pi^u. \tag{6.1}$$

Remark 6.1. Note that for $u\in W^{2,2}_R(\omega;\mathbb{R}^3)\cap C^\infty(\bar\omega;\mathbb{R}^3)$, condition (6.1) (see Lemmas 4.3 and 4.4 in Ref. 16), is equivalent to writing

$$B=\operatorname{sym}((\nabla' u)^T\nabla' V) \tag{6.2}$$

for some $V\in C^\infty(\bar\omega;\mathbb{R}^3)$ (see Lemmas 4.3 and 4.4 in Ref. 26).

Indeed, (6.2) follows from (6.1) setting

$$V:=(\nabla' u)g+\alpha n_u,$$

and in view of the cancellations due to (3.2). Conversely, (6.1) is obtained from (6.2) defining $g:=(\nabla' u)^T V$ and $\alpha:=V\cdot n_u$.

A key tool in the proof of the limsup inequality (1.4) is the following lemma, which has been proved in Lemma 4.3 in Ref. 16 (see also Refs. 13–15, 24 and 25). Again, the arguments in the previous sections of this paper continue to hold if ω is a bounded Lipschitz domain. The piecewise C^1-regularity of $\partial\omega$ is necessary for the proof of the limsup inequality (1.4) (although it can be slightly relaxed as in Ref. 14), since it is required in order to obtain the following density result.

Lemma 6.1. *The set $\mathcal{A}(\omega)$ is dense in $W^{2,2}_R(\omega;\mathbb{R}^3)$ in the strong $W^{2,2}$ topology.*

Before we prove the limsup inequality (1.4), we state a lemma and a corollary that guarantee the continuity of the relaxations (defined in (5.2)–(5.4)) of the quadratic map $\mathscr{Q}$ introduced in (H4). The proof of Lemma 6.2 is a combination of the proof of Lemma 4.2 in Ref. 16, the proof of Lemma 2.10 in Ref. 22 and Lemma 4.2 in Ref. 26. Corollary 6.1 is a direct consequence of Lemma 6.2.

Lemma 6.2. *Let $\overline{\mathscr{Q}}^{\gamma_1}_{\rm hom}$ and $\mathscr{Q}_{\rm hom}$ be the maps defined in* (5.1)–(5.4), *and let $\gamma_2=+\infty$.*

(i) *Let $0<\gamma_1<+\infty$. Then for every $A\in\mathbb{M}^{2\times 2}_{\rm sym}$ there exists a unique pair*

$$(B,\phi_1)\in\mathbb{M}^{2\times 2}_{\rm sym}\times W^{1,2}\left(\left(-\frac{1}{2},\frac{1}{2}\right);W^{1,2}_{\rm per}(Q;\mathbb{R}^3)\right)$$

with

$$\int_{(-\frac{1}{2},\frac{1}{2})\times Q} \phi_1(x_3,y)dy\,dx_3 = 0,$$

such that

$$\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(A) = \int_{(-\frac{1}{2},\frac{1}{2})\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} x_3A+B & 0 \\ 0 & 0 \end{pmatrix}\right.$$
$$\left. + \operatorname{sym}\left(\nabla_y\phi_1(x_3,y)\,\middle|\,\frac{\partial_{x_3}\phi_1(x_3,y)}{\gamma_1}\right)\right).$$

The induced mapping

$$A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \mapsto (B(A),\phi_1(A)) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \times W^{1,2}\left(\left(-\frac{1}{2},\frac{1}{2}\right); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)\right)$$

is bounded and linear.

(ii) *Let* $\gamma_1 = +\infty$. *Then for every* $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ *there exists a unique triple*

$$(B,d,\phi_1) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \times L^2\left(\left(-\frac{1}{2},\frac{1}{2}\right);\mathbb{R}^3\right) \times L^2\left(\left(-\frac{1}{2},\frac{1}{2}\right); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)\right)$$

with

$$\int_{(-\frac{1}{2},\frac{1}{2})\times Q} \phi_1(x_3,y)dy\,dx_3 = 0,$$

such that

$$\overline{\mathscr{Q}}^{\infty}_{\mathrm{hom}}(A) = \int_{(-\frac{1}{2},\frac{1}{2})\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} x_3A+B & 0 \\ 0 & 0 \end{pmatrix}\right.$$
$$\left. + \operatorname{sym}(\nabla_y\phi_1(x_3,y)|d(x_3))\right).$$

The induced mapping $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \mapsto (B(A),d(A),\phi_1(A)) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \times L^2((-\frac{1}{2},\frac{1}{2});\mathbb{R}^3) \times L^2((-\frac{1}{2},\frac{1}{2}); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$ *is bounded and linear.*

(iii) *Let* $\gamma_1 = 0$. *Then for every* $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ *there exists a unique* 6*-tuple*

$$(B,\xi,\eta,g_1,g_2,g_3)$$

with $B \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$, $\xi \in L^2((-\frac{1}{2},\frac{1}{2});\ W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^2))$, $\eta \in W^{2,2}_{\mathrm{per}}(Q)$, $g_i \in L^2((-\frac{1}{2},\frac{1}{2});\times Q), i = 1,2,3,$ *such that*

$$\overline{\mathscr{Q}}^{0}_{\mathrm{hom}}(A) = \int_{(-\frac{1}{2},\frac{1}{2})\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} x_3A+B & 0 \\ 0 & 0 \end{pmatrix}\right.$$
$$\left. + \operatorname{sym}\begin{pmatrix} \operatorname{sym}\nabla_y\xi(x_3,y) + x_3\nabla_y^2\eta(y) & g_1(x_3,y) \\ & g_2(x_3,y) \\ g_1(x_3,y) \quad g_2(x_3,y) & g_3(x_3,y) \end{pmatrix}\right).$$

The induced mapping

$$A \mapsto (B(A), \xi(A), \eta(A), g_1(A), g_2(A), g_3(A))$$

from $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ *to* $\mathbb{M}^{2\times 2}_{\mathrm{sym}} \times L^2((-\frac{1}{2}, \frac{1}{2}); \mathbb{R}^3) \times W^{2,2}_{\mathrm{per}}(Q) \times L^2((-\frac{1}{2}, \frac{1}{2}) \times Q; \mathbb{R}^3)$ *is bounded and linear.*

For a.e. $y \in Q$ *and for every* $C \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ *there exists a unique* $\phi_2 \in W^{1,2}_{\mathrm{per}} \times (Q; \mathbb{R}^3)$, *with* $\int_Q \phi_2(z)\, dz = 0$, *such that*

$$\mathscr{Q}_{\mathrm{hom}}(y, C) = \int_Q \mathscr{Q}(y, z, C + \mathrm{sym}(\nabla\phi_2(z)|0)).$$

The induced mapping

$$C \in \mathbb{M}^{3\times 3}_{\mathrm{sym}} \mapsto \phi_2(C) \in W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^3)$$

is bounded and linear. Furthermore, the induced operator

$$P : L^2\left(\left(-\frac{1}{2}, \frac{1}{2}\right) \times Q; \mathbb{M}^{3\times 3}\right) \to L^2\left(\left(-\frac{1}{2}, \frac{1}{2}\right) \times Q; W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^3)\right),$$

defined as

$$P(C) := \phi_2(C) \quad \text{for every} \quad C \in L^2\left(\left(-\frac{1}{2}, \frac{1}{2}\right) \times Q; \mathbb{M}^{3\times 3}\right)$$

is bounded and linear.

Corollary 6.1. *Let* $\gamma_1 \in [0, +\infty]$. *The map* $\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}$ *is continuous, and there exists* $c_1(\gamma_1) \in (0, +\infty)$ *such that*

$$\frac{1}{c_1}|F|^2 \le \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(F) \le c_1|F|^2$$

for every $F \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$.

(i) *If* $0 < \gamma_1 < +\infty$, *then for every* $A \in L^2(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *there exists a unique triple* $(B, \phi_1, \phi_2) \in L^2(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}}) \times L^2(\omega; W^{1,2}((-\frac{1}{2}, \frac{1}{2}); W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^3))) \times L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^3))$ *such that*

$$\begin{aligned}
\frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(A(x'))dx' &= \int_\Omega \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(x_3 A(x'))dx \\
&= \int_{\Omega\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix}\right. \\
&\qquad \left. + \,\mathrm{sym}\left(\nabla_y \phi_1(x, y) \,\middle|\, \frac{\partial_{x_3}\phi_1(x, y)}{\gamma_1}\right)\right) dy\, dx \\
&= \int_{\Omega\times Q\times Q} \mathscr{Q}\left(y, z, \begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix}\right. \\
&\qquad + \,\mathrm{sym}\left(\nabla_y \phi_1(x, y) \,\middle|\, \frac{\partial_{x_3}\phi_1(x, y)}{\gamma_1}\right) \\
&\qquad \left. + \,\mathrm{sym}(\nabla_z \phi_2(x, y, z)|0)\right) dz\, dy\, dx.
\end{aligned}$$

(ii) *If* $\gamma_1 = +\infty$, *then for every* $A \in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *there exists a unique 4-tuple* $(B, d, \phi_1, \phi_2) \in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}) \times L^2(\Omega;\mathbb{R}^3) \times L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)) \times L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$ *such that*

$$\begin{aligned}\frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\infty}_{\mathrm{hom}}(A(x'))\,dx' &= \int_\Omega \overline{\mathscr{Q}}^{\infty}_{\mathrm{hom}}(x_3 A(x'))\,dx' \\ &= \int_{\Omega\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0\end{pmatrix}\right. \\ &\quad \left. + \operatorname{sym}(\nabla_y \phi_1(x,y) | d(x))\right) dy\,dx \\ &= \int_{\Omega\times Q\times Q} \mathscr{Q}\left(y, z, \begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0\end{pmatrix}\right. \\ &\quad + \operatorname{sym}(\nabla_y \phi_1(x,y) | d(x)) \\ &\quad \left. + \operatorname{sym}(\nabla_z \phi_2(x,y,z) | 0)\right) dz\,dy\,dx.\end{aligned}$$

(iii) *If* $\gamma_1 = 0$, *then for every* $A \in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *there exists a unique 7-tuple* $(B, \xi, \eta, g_1, g_2, g_3, \phi) \in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}) \times L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^2)) \times L^2(\Omega; W^{2,2}_{\mathrm{per}}(Q)) \times L^2(\Omega \times Q;\mathbb{R}^3) \times L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, *such that*

$$\begin{aligned}\frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{0}_{\mathrm{hom}}(A(x'))dx' &= \int_\Omega \overline{\mathscr{Q}}^{0}_{\mathrm{hom}}(x_3 A(x'))dx' \\ &= \int_{\Omega\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0\end{pmatrix}\right. \\ &\quad \left. + \operatorname{sym}\begin{pmatrix} \operatorname{sym}\nabla_y \xi(x,y) + x_3 \nabla_y^2 \eta(x',y) & g_1(x,y) \\ & g_2(x,y) \\ g_1(x,y) \quad g_2(x,y) & g_3(x,y)\end{pmatrix}\right) \\ &= \int_{\Omega\times Q\times Q} \mathscr{Q}\left(y, z, \begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0\end{pmatrix}\right. \\ &\quad + \operatorname{sym}\begin{pmatrix} \operatorname{sym}\nabla_y \xi(x,y) + x_3 \nabla_y^2 \eta(x',y) & g_1(x,y) \\ & g_2(x,y) \\ g_1(x,y) \quad g_2(x,y) & g_3(x,y)\end{pmatrix} \\ &\quad \left. + \operatorname{sym}(\nabla_z \phi_2(x,y,z) | 0)\right) dz\,dy\,dx.\end{aligned}$$

We now prove that the lower bound obtained in Sec. 5 is optimal.

Theorem 6.1. *Let $\gamma_1 \in [0, +\infty]$. Let $\overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}$ and $\mathscr{Q}_{\text{hom}}$ be the maps defined in* (5.1)–(5.4), *let $u \in W^{2,2}_R(\omega; \mathbb{R}^3)$ and let Π^u be the map introduced in* (4.4). *Then there exists a sequence $\{u^h\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that*

$$\limsup_{h\to 0} \frac{\mathcal{E}^h(u^h)}{h^2} \leq \frac{1}{12} \int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}(\Pi^u(x'))dx'. \tag{6.3}$$

Proof. The proof is an adaptation of the proof of Theorem 2.4 in Ref. 16 and the proof of Theorem 2.4 in Ref. 26. We outline the main steps in the cases $0 < \gamma_1 < +\infty$ and $\gamma_1 = 0$ for the convenience of the reader. The proof in the case $\gamma_1 = +\infty$ is analogous.

Case 1: $0 < \gamma_1 < +\infty$ and $\gamma_2 = +\infty$.

By Lemma 6.1 and Corollary 6.1 it is enough to prove the theorem for $u \in \mathcal{A}(\omega)$. By Corollary 6.1 there exist $B \in L^2(\omega; \mathbb{M}^{2\times 2})$, $\phi_1 \in L^2(\omega; W^{1,2}((-\frac{1}{2}, \frac{1}{2}); W^{1,2}_{\text{per}}(Q; \mathbb{R}^3))$, and $\phi_2 \in L^2(\Omega \times Q; W^{1,2}_{\text{per}}(Q; \mathbb{R}^3))$ such that

$$\begin{aligned}
&\frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}(\Pi^u(x'))dx' \\
&\quad = \int_\Omega \int_Q \int_Q \mathscr{Q}\left(y, z, \begin{pmatrix} \operatorname{sym} B(x') + x_3\Pi^u(x') & 0 \\ 0 & 0 \end{pmatrix}\right. \\
&\qquad \left. + \operatorname{sym}\left(\nabla_y \phi_1(x,y) \middle| \frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1}\right) + \operatorname{sym}(\nabla_z \phi_2(x,y,z)|0)\right) dz\, dy\, dx.
\end{aligned}$$

Since B depends linearly on Π^u by Lemma 6.2, in particular there holds

$$\{x' : \Pi^u(x') = 0\} \subset \{x' : B(x') = 0\}.$$

By Lemma 6.2, we can argue by density and we can assume that $B \in C^\infty(\bar{\omega}; \mathbb{M}^{2\times 2})$, $B = 0$ in a neighborhood of $\{x' : \Pi^u(x') = 0\}$, $\phi_1 \in C^\infty_c(\omega; C^\infty((-\frac{1}{2}, \frac{1}{2}); C^\infty(Q; \mathbb{R}^3))$, and $\phi_2 \in C^\infty_c(\omega \times Q; C^\infty(Q; \mathbb{R}^3))$. In addition, since $u \in \mathcal{A}(\omega)$, by (6.1) there exist $\alpha \in C^\infty(\bar{\omega})$, and $g \in C^\infty(\bar{\omega}; \mathbb{R}^2)$ such that

$$B = \operatorname{sym} \nabla' g + \alpha \Pi^u.$$

Set

$$\begin{aligned}
v^h(x) &:= u(x') + h((x_3 + \alpha(x'))n_u(x') + (g(x') \cdot \nabla')y(x')), \\
R(x') &:= (\nabla' u(x')|n_u(x')), \\
b(x') &:= -\begin{pmatrix} \partial_{x_1}\alpha(x') \\ \partial_{x_2}\alpha(x') \end{pmatrix} + \Pi^u(x')g(x')
\end{aligned}$$

and let

$$u^h(x) := v^h(x') + h\varepsilon(h)\tilde{\phi}_1\left(x, \frac{x'}{\varepsilon(h)}\right) + h\varepsilon^2(h)\tilde{\phi}_2\left(x, \frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}\right)$$

for a.e. $x \in \Omega$, where

$$\tilde{\phi}_1 := R\left(\phi_1 + \gamma_1 x_3 \begin{pmatrix} b \\ 0 \end{pmatrix}\right) \quad \text{and} \quad \tilde{\phi}_2 := R\phi_2.$$

Arguing similarly to the proof of Theorem 2.4 (upper bound) in Ref. 16, it can be shown that (6.3) holds.

Case 2: $\gamma_1 = 0$ and $\gamma_2 = +\infty$.

By Lemma 6.1 and Corollary 6.1 it is enough to prove the theorem for $u \in \mathcal{A}(\omega)$. By Corollary 6.1 there exist $B \in L^2(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$, $\xi \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^2))$, $\eta \in L^2(\Omega; W^{2,2}_{\mathrm{per}}(Q))$, $g_i \in L^2(\Omega \times Y)$, $i = 1, 2, 3$, and $\phi \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^3))$ such that

$$\begin{aligned}
&\frac{1}{12}\int_\omega \overline{\mathscr{Q}}^0_{\mathrm{hom}}(\Pi^u(x'))dx' \\
&\quad = \int_{\Omega\times Q\times Q} \mathscr{Q}\left(y, z, \begin{pmatrix} x_3\Pi^u(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix}\right. \\
&\qquad + \mathrm{sym}\begin{pmatrix} \mathrm{sym}\,\nabla_y \xi(x,y) + x_3 \nabla_y^2 \eta(x', y) & g_1(x,y) \\ & g_2(x,y) \\ g_1(x,y) \quad g_2(x,y) & g_3(x,y) \end{pmatrix} \\
&\qquad \left. + \mathrm{sym}(\nabla_z \phi_2(x,y,z)|0)\right) dz\, dy\, dx.
\end{aligned}$$

By the linear dependence of B on Π^u, in particular there holds

$$\{x' : \Pi^u(x') = 0\} \subset \{x' : B(x') = 0\}.$$

By density, we can assume that $B \in C^\infty(\bar{\omega}; \mathbb{M}^{2\times 2})$, $\xi \in C^\infty_c(\omega; C^\infty_{\mathrm{per}}(Q; \mathbb{R}^2))$, $\eta \in C^\infty_c(\omega; C^\infty_{\mathrm{per}}(Q))$, and $g_i \in C^\infty_c(\omega; C^\infty_{\mathrm{per}}((-\frac{1}{2}, \frac{1}{2}) \times Q))$, $i = 1, 2, 3$. Since $u \in \mathcal{A}(\omega)$, by (6.2) there exists a displacement $V \in C^\infty(\bar{\omega}; \mathbb{R}^2)$ such that

$$B = \mathrm{sym}((\nabla' u)^T \nabla' V).$$

Set

$$\begin{aligned}
v^h(x) &:= u(x') + h x_3 n_u(x') = h(V(x') + h x_3 \mu(x')), \\
\mu(x') &:= (\mathrm{Id} - n_u(x') \otimes n_u(x'))(\partial_1 V(x') \wedge \partial_2 u(x') + \partial_1 u(x') \wedge \partial_2 V(x')), \\
R(x') &:= (\nabla' u(x') | n_u(x')),
\end{aligned}$$

and let

$$u^h(x) := v^h(x) - \varepsilon^2(h) n_u(x')\eta\left(x', \frac{x'}{\varepsilon(h)}\right)$$
$$+ h\varepsilon^2(h) x_3 R(x') \begin{pmatrix} \partial_{x_1}\eta\left(x', \frac{x'}{\varepsilon(h)}\right) + \frac{1}{\varepsilon(h)}\partial_{y_1}\eta\left(x', \frac{x'}{\varepsilon(h)}\right) \\ \partial_{x_2}\eta\left(x', \frac{x'}{\varepsilon(h)}\right) + \frac{1}{\varepsilon(h)}\partial_{y_2}\eta\left(x', \frac{x'}{\varepsilon(h)}\right) \end{pmatrix}$$
$$+ h\varepsilon(h) R(x') \begin{pmatrix} \xi\left(x', \frac{x'}{\varepsilon(h)}\right) \\ 0 \end{pmatrix}$$
$$+ h^2 \int_{-\frac{1}{2}}^{x_3} R(x') g\left(x', t, \frac{x'}{\varepsilon(h)}\right) dt + h\varepsilon^2(h) R(x')\phi\left(x, \frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}\right),$$

for a.e. $x \in \Omega$. The proof of (6.3) is a straightforward adaptation of the proof of Theorem 2.4 (upper bound) in Ref. 26. □

Proof of Theorem 1.1. Theorem 1.1 follows now by Theorems 5.1 and 6.1. □

7. Concluding Remarks

The rigorous identification of two-dimensional models for thin three-dimensional structures is a classical question in mechanics of materials. Recently, in Refs. 16, 22 and 26, simultaneous homogenization and dimension reduction for thin plates has been studied, under physical growth conditions for the energy density, and in the situation in which one periodic in-plane homogeneity scale arises.

In this paper we deduced a multiscale version of Refs. 16 and 26, extending the analysis to the case in which two periodic in-plane homogeneity scale are present, in the framework of Kirchhoff's nonlinear plate theory. Denoting by h the thickness of the plate, and by $\varepsilon(h)$ and $\varepsilon^2(h)$ the two periodicity scales, we provided a characterization of the effective energy in the regimes

$$\lim_{h\to 0} \frac{h}{\varepsilon(h)} := \gamma_1 \in [0, +\infty] \quad \text{and} \quad \lim_{h\to 0} \frac{h}{\varepsilon^2(h)}; = \gamma_2 = +\infty.$$

The analysis relies on multiscale convergence methods and on a careful study of the multiscale limit of the sequence of linearized three-dimensional stresses, based on Friesecke, James and Müller's rigidity estimate (see Theorem 4.1 in Ref. 11).

The identification of the reduced models for $\gamma_1 = 0$ and $\gamma_2 \in [0, +\infty)$ remains an open problem.

Appendix

In this section we collect a few results which played an important role in the proof of Theorem 1.1. We recall that in Case 2, we claimed that the maps R^h are piecewise constant on cubes of the form $Q(\varepsilon(h)z, \varepsilon(h))$, $z \in \mathbb{Z}^2$. Indeed, this holds if we show that for every $z \in \mathbb{Z}^2$ there exists $z' \in \mathbb{Z}^2$ such that

$$Q(\varepsilon(h)z, \varepsilon(h)) \subset Q(\delta(h)z', \delta(h))$$

or, equivalently, with $m := \frac{\delta(h)}{\varepsilon(h)} \in \mathbb{N}$,

$$\left(z - \frac{1}{2}, z + \frac{1}{2}\right) \subset m\left(z' - \frac{1}{2}, z' + \frac{1}{2}\right). \tag{A.1}$$

The next lemma attests that this holds provided m is odd.

Lemma A.1. *Let $a \in \mathbb{N}_0$. Then for every $z \in \mathbb{Z}$ there exists $z' \in \mathbb{Z}$ such that* (A.1) *holds with $m = 2a + 1$.*

Proof. Without loss of generality we may assume that $z \in \mathbb{N}_0$ (the case in which $z < 0$ is analogous). Solving (A.1) is equivalent to finding $z' \in \mathbb{Z}$ such that

$$\begin{cases} z - \dfrac{1}{2} \geq (2a+1)z' - \dfrac{(2a+1)}{2}, \\[2ex] z + \dfrac{1}{2} \leq (2a+1)z' + \dfrac{(2a+1)}{2}, \end{cases} \tag{A.2}$$

that is

$$\begin{cases} z \geq (2a+1)z' - a, \\ z \leq (2a+1)z' + a. \end{cases} \tag{A.3}$$

Let $n, l \in \mathbb{N}_0$ be such that $z = n(2a+1) + l$ and

$$l < 2a + 1. \tag{A.4}$$

Then (A.3) is equivalent to

$$\begin{cases} n(2a+1) + l + a \geq (2a+1)z', \\ n(2a+1) + l - a \leq (2a+1)z'. \end{cases} \tag{A.5}$$

Now, if $0 \leq l \leq a$ it is enough to choose $z' = n$. If $l > a$, the result follows setting $z' := n + 1$. Indeed, with $a + 1 > r > 1 \in \mathbb{N}$ such that $l = a + r$, (A.5) simplifies as

$$\begin{cases} n(2a+1) + 2a + r \geq (2a+1)(n+1), \\ n(2a+1) + r \leq (2a+1)(n+1), \end{cases}$$

that is

$$\begin{cases} 2a + r \geq 2a + 1, \\ r \leq 2a + 1, \end{cases}$$

which is trivially satisfied. □

Remark A.1. By Lemma A.1 it follows that, setting $p := \frac{\delta(h)}{\varepsilon^2(h)}$ and provided p is odd, for every $z \in \mathbb{Z}^2$ there exists $z' \in \mathbb{Z}^2$ such that

$$Q(\varepsilon^2(h)z, \varepsilon^2(h)) \subset Q(\delta(h)z, \delta(h)).$$

This observation allowed us to construct the sequence $\{R^h\}$ in Case 3 of the proof of Theorem 1.1.

Remark A.2. We point out that if m is even there may be $z \in \mathbb{Z}$ such that (A.1) fails to be true for every $z' \in \mathbb{Z}$, i.e.

$$\left(z - \frac{1}{2}, z + \frac{1}{2}\right) \not\subseteq \left(mz' - \frac{m}{2}, mz' + \frac{m}{2}\right).$$

Indeed, if m is even, then $z = \frac{3}{2}m \in \mathbb{N}$ and (A.2) becomes

$$\begin{cases} \dfrac{3}{2}m - \dfrac{1}{2} \geq mz' - \dfrac{m}{2}, \\ \dfrac{3}{2}m + \dfrac{1}{2} \leq mz' + \dfrac{m}{2}, \end{cases}$$

which in turn is equivalent to

$$z' \in \left[1 + \frac{1}{2m}, 2 - \frac{1}{2m}\right].$$

This last condition leads to a contradiction as

$$\left[1 + \frac{1}{2m}, 2 - \frac{1}{2m}\right] \cap \mathbb{Z} = \emptyset \quad \text{for every } m \in \mathbb{N}.$$

We conclude the Appendix with a result that played a key role in the identification of the limit elastic stress, and in the proof of the liminf and limsup inequalities (1.3) and (1.4). We omit its proof, as it follows by Lemma 4.3 in Ref. 22.

Lemma A.2. *Let* $\mathscr{Q} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0, +\infty)$ *be such that*:

(i) $\mathscr{Q}(y, z, \cdot)$ *is continuous for a.e.* $y, z \in \mathbb{R}^2$,
(ii) $\mathscr{Q}(\cdot, \cdot, F)$ *is* $Q \times Q$*-periodic and measurable for every* $F \in \mathbb{M}^{3\times 3}$,
(iii) *for a.e.* $y, z \in \mathbb{R}^2$, *the map* $\mathscr{Q}(y, z, \cdot)$ *is quadratic on* $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$, *and satisfies*

$$\frac{1}{C}|\mathrm{sym}\, F|^2 \leq \mathscr{Q}(y, z, F) = \mathscr{Q}(y, z, \mathrm{sym}\, F) \leq C|\mathrm{sym}\, F|^2$$

for all $F \in \mathbb{M}^{3\times 3}$, *and some* $C > 0$.

Let $\{E^h\} \subset L^2(\Omega; \mathbb{M}^{3\times 3})$ *and* $E \in L^2(\Omega \times Q \times Q; \mathbb{M}^{3\times 3})$ *be such that*

$$E^h \xrightarrow{dr\text{-}3\text{-}s} E \quad \textit{weakly dr-3-scale.}$$

Then

$$\liminf_{h\to 0} \int_\Omega \mathscr{Q}\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, E^h(x)\right) dx \geq \int_\Omega \int_Q \int_Q \mathscr{Q}(y, z, E(x, y, z)) dz\, dy\, dx.$$

If in addition

$$E^h \xrightarrow{dr\text{-}3\text{-}s} E \quad \textit{strongly dr-3-scale,}$$

then

$$\lim_{h\to 0} \int_\Omega \mathscr{Q}\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, E^h(x)\right) dx = \int_\Omega \int_Q \int_Q \mathscr{Q}(y, z, E(x, y, z)) dz\, dy\, dx.$$

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HOMOGENIZATION IN BV OF A MODEL FOR LAYERED COMPOSITES IN FINITE CRYSTAL PLASTICITY

ELISA DAVOLI, RITA FERREIRA, AND CAROLIN KREISBECK

Abstract. In this work, we study the effective behavior of a two-dimensional variational model within finite crystal plasticity for high-contrast bilayered composites. Precisely, we consider materials arranged into periodically alternating thin horizontal strips of an elastically rigid component and a softer one with one active slip system. The energies arising from these modeling assumptions are of integral form, featuring linear growth and non-convex differential constraints. We approach this non-standard homogenization problem via Gamma-convergence. A crucial first step in the asymptotic analysis is the characterization of rigidity properties of limits of admissible deformations in the space BV of functions of bounded variation. In particular, we prove that, under suitable assumptions, the two-dimensional body may split horizontally into finitely many pieces, each of which undergoes shear deformation and global rotation. This allows us to identify a potential candidate for the homogenized limit energy, which we show to be a lower bound on the Gamma-limit. In the framework of non-simple materials, we present a complete Gamma-convergence result, including an explicit homogenization formula, for a regularized model with an anisotropic penalization in the layer direction.

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1. Introduction

Metamaterials are artificially engineered composites whose heterogeneities are optimized to improve structural performances. Due to their special mechanical properties, arising as a result of complex microstructures, metamaterials play a key role in industrial applications and are an increasingly active field of research. Two natural questions when dealing with composite materials are how the effective material response is influenced by the geometric distribution of its components, and how the mechanical properties of the components impact the overall macroscopic behavior of the metamaterial.

In what follows, we investigate these questions for a special class of metamaterials with two characteristic features that are of relevance in a number of applications: (i) the material consists of two components arranged in a highly anisotropic way into periodically alternating layers, and (ii) the (elasto)plastic properties of the two components exhibit strong differences, in the sense that one is rigid, while the other one is considerably softer, allowing for large (elasto)plastic deformations. The analysis of variational models for such layered high-contrast materials was initiated in [13]. There, the authors derive a macroscopic description for a two-dimensional model in the context of geometrically nonlinear but rigid elasticity, assuming that the softer component can be deformed along a single active slip system with linear self-hardening.

These results have been extended to general dimensions, to energy densities with p-growth for $1 < p < +\infty$, and to the case with non-trivial elastic energies, which allows treating very stiff (but not necessarily rigid) layers, see [14, 12].

In this paper, we carry the ideas of [13] forward to a model for plastic composites without linear hardening, in the spirit of [18]. This change turns the variational problem in [13], having quadratic growth (cf. also [15, 16]), into one with energy densities that grow merely linearly.

The main novelty lies in the fact that the homogenization analysis must be performed in the class BV of functions of bounded variation (see [2]) to account for concentration phenomena. This gives rise to conceptual mathematical difficulties: on the one hand, the standard convolution techniques commonly used for density arguments in BV or SBV cannot be directly applied because they do not preserve the intrinsic constraints of the problem; on the other hand, constraint-preserving approximations in this weaker setting of BV are rather challenging, as one needs to simultaneously regularize the absolutely continuous part of the distributional derivative of the functions and accommodate their jump sets.

To state our results precisely, we first introduce the relevant model with its main modeling hypotheses. Throughout the article, we analyze two versions of the model, namely with and without regularization.

Let e_1 and e_2 be the standard unit vectors in $\mathbb{R}^2$, and let $x = (x_1, x_2)$ denote a generic point in $\mathbb{R}^2$. Unless specified otherwise, $\Omega \subset \mathbb{R}^2$ is an x_1-connected, bounded domain with Lipschitz boundary, that is, an open set whose slices in the x_1-direction are (possibly empty) open intervals (see Subsection 2.4 for the precise definition). For such a domain Ω, we set

$$a_\Omega := \inf_{x\in\Omega} x_2 \qquad \text{and} \qquad b_\Omega := \sup_{x\in\Omega} x_2, \tag{1.1}$$

as well as

$$c_\Omega := \inf_{x\in\Omega} x_1 \qquad \text{and} \qquad d_\Omega := \sup_{x\in\Omega} x_1. \tag{1.2}$$

Assume that Ω is the reference configuration of a body with heterogeneities in the form of periodically alternating thin horizontal layers. To describe the bilayered structure mathematically, consider the periodicity cell $Y := [0,1)^2$, which we subdivide into $Y = Y_{\text{soft}} \cup Y_{\text{rig}}$ with $Y_{\text{soft}} := [0,1) \times [0,\lambda)$ for $\lambda \in (0,1)$ and $Y_{\text{rig}} := Y \setminus Y_{\text{soft}}$. All sets are extended by periodicity to $\mathbb{R}^2$. The (small) parameter $\varepsilon > 0$ describes the thickness of a pair (one rigid, one softer) of fine layers, and can be viewed as the intrinsic length scale of the system. The collections of all rigid and soft layers in Ω can be expressed as $\varepsilon Y_{\text{rig}} \cap \Omega$ and $\varepsilon Y_{\text{soft}} \cap \Omega$, respectively. For an illustration of the geometrical assumptions, see Figure 1.

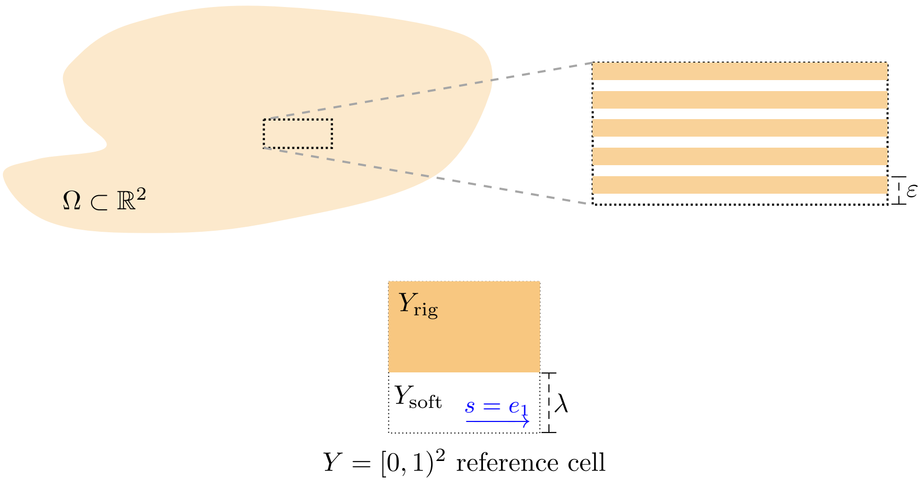


FIGURE 1. A bilayered x_1-connected domain Ω

Following the classical theory of elastoplasticity at finite strains (see, e.g., [31] for an overview), we assume that the gradient of any deformation $u : \Omega \to \mathbb{R}^2$ decomposes into the product of an elastic strain, F_{el}, and a plastic one, F_{pl}. In the literature, different models of finite plasticity have been proposed (see, e.g., [3, 22, 29, 30, 37]), as well as alternative descriptions via the theory of structured deformations (see [10, 11, 24, 6] and the references therein). Here, we adopt the classical model by Lee on finite crystal plasticity introduced in [33, 35, 34], according to which the deformation gradients satisfy

$$\nabla u = F_{\text{el}} F_{\text{pl}}. \tag{1.3}$$

In addition, we suppose that the elastic behavior of the body is purely rigid, meaning that

$$F_{\text{el}} \in SO(2) \text{ almost everywhere in } \Omega, \tag{1.4}$$

and that the plastic part satisfies

$$F_{\text{pl}} = \mathbb{I} + \gamma s \otimes m, \tag{1.5}$$

where $s \in \mathbb{R}^2$ with $|s| = 1$ is the slip direction of the slip system, $m = s^\perp$ is the normal to the slip plane, and the map γ measures the amount of slip. Denoting by $\mathcal{M}_s$ the set

$$\mathcal{M}_s := \{F \in \mathbb{R}^{2\times 2} : \det F = 1 \text{ and } |Fs| = 1\},$$

the multiplicative decomposition (1.3) (under assumptions (1.4) and (1.5)) is equivalent to $\nabla u \in \mathcal{M}_s$ almost everywhere in Ω. Whereas the material is free to glide along the slip system in the softer phase, it is required that γ vanishes on the layers consisting of a rigid material, i.e., $\gamma = 0$ in $\varepsilon Y_{\rm rig} \cap \Omega$.

Collecting the previous modeling assumptions, we define, for $\varepsilon > 0$, the class $\mathcal{A}_\varepsilon$ of *admissible layered deformations* by

$$\begin{aligned}\mathcal{A}_\varepsilon &:= \{u \in W^{1,1}(\Omega;\mathbb{R}^2) : \nabla u \in \mathcal{M}_s \text{ a.e. in } \Omega,\ \nabla u \in SO(2) \text{ a.e. in } \varepsilon Y_{\rm rig} \cap \Omega\}\\ &= \{u \in W^{1,1}(\Omega;\mathbb{R}^2) : \nabla u = R(\mathbb{I} + \gamma s \otimes m) \text{ a.e. in } \Omega,\\ &\qquad\qquad R \in L^\infty(\Omega; SO(2)) \text{ and } \gamma \in L^1(\Omega) \text{ with } \gamma = 0 \text{ a.e. in } \varepsilon Y_{\rm rig} \cap \Omega\}.\end{aligned} \tag{1.6}$$

The elastoplastic energy of a deformation $u \in L^1_0(\Omega;\mathbb{R}^2) := \{u \in L^1(\Omega;\mathbb{R}^2) : \int_\Omega u \,\mathrm{d}x = 0\}$, given by

$$E_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\gamma| \,\mathrm{d}x & \text{for } u \in \mathcal{A}_\varepsilon,\\ \infty & \text{otherwise in } L^1_0(\Omega;\mathbb{R}^2),\end{cases} \tag{1.7}$$

represents the internal energy contribution of the system during a single incremental step in a time-discrete variational description. This way of modeling excludes preexistent plastic distortions, and can be considered a reasonable assumption for the first time step of a deformation process. The elastoplastic energy can be complemented with terms modeling the work done by external body or surface forces.

The limit behavior of sequences $(u_\varepsilon)_\varepsilon$ of low energy states for $(E_\varepsilon)_\varepsilon$ gives information about the macroscopic material response of the layered composites. In the following, we focus the analysis of this asymptotic behavior on the $s = e_1$ case, when the slip direction is parallel to the orientation of the layers, cf. also Figure 1. Note that different slip directions can be treated similarly, but the arguments are technically more involved. In fact, for $s \notin \{e_1, e_2\}$, small-scale laminate microstructures on the softer layers need to be taken into account, which requires an extra relaxation step. We refer to [18] for the relaxation mechanism and to [13] for the strategy of how to apply it to layered structures.

An important first step towards identifying the limit behavior of the energies $(E_\varepsilon)_\varepsilon$ (in the sense of Γ-convergence) is the proof of a general statement of asymptotic rigidity for layered structures in the context of functions of bounded variation. The following result characterizes the weak* limits in BV of deformations whose gradients coincide pointwise with rotations on the rigid layers of the material. Note that no additional constraints are imposed on the softer components at this point.

Theorem 1.1 (Asymptotic rigidity of layered structures in BV). *Let $\Omega \subset \mathbb{R}^2$ be an x_1-connected domain. Assume that $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ is a sequence satisfying*

$$\nabla u_\varepsilon \in SO(2) \text{ a.e. in } \varepsilon Y_{\rm rig} \cap \Omega \text{ for all } \varepsilon, \tag{1.8}$$

and that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$ for some $u \in BV(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0$. Then,

$$u(x) = R(x_2)x + \psi(x_2) \text{ for } \mathcal{L}^2\text{- a.e. } x \in \Omega, \tag{1.9}$$

where $R \in BV(a_\Omega, b_\Omega; SO(2))$ and $\psi \in BV(a_\Omega, b_\Omega;\mathbb{R}^2)$ (cf. (1.1)).

Conversely, any function $u \in BV(\Omega;\mathbb{R}^2)$ as in (1.9) can be attained as weak*-limit in $BV(\Omega;\mathbb{R}^2)$ of a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ satisfying (1.8).

To prove the first part of Theorem 1.1, we adapt the arguments in [13] to the BV-setting. The second assertion follows from a tailored one-dimensional density result in BV, which involves approximating functions that are constant on the rigid layers (see Lemma 3.3 below). Up to minor adaptations, analogous statements hold in higher dimensions. We refer to Remark 3.4 for the specific assumptions on the geometry of the set Ω under which a higher-dimensional counterpart of Theorem 1.1 can be proved.

A natural potential candidate for the limiting behavior of $(E_\varepsilon)_\varepsilon$ in the sense of Γ-convergence (see [8, 20] for an introduction, as well as the references therein) is the functional $E : L^1_0(\Omega;\mathbb{R}^2) \to [0,\infty]$, given by

$$E(u) = \begin{cases} \displaystyle\int_\Omega |\psi' \cdot Re_1| \,\mathrm{d}x + |D^s u|(\Omega) & \text{if } u \in \mathcal{A},\\ \infty & \text{otherwise,}\end{cases} \tag{1.10}$$

where

$$\begin{aligned}\mathcal{A} := \{u \in BV(\Omega;\mathbb{R}^2)\colon\ & u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega \text{ with}\\ & R \in BV(a_\Omega, b_\Omega; SO(2)),\ \psi \in BV(a_\Omega, b_\Omega;\mathbb{R}^2),\ \text{and } \det \nabla u = 1 \text{ a.e. in } \Omega\}.\end{aligned} \tag{1.11}$$

We refer to Remark 5.1 for an alternative representation of the functional E.

The next theorem states that E provides indeed a lower bound for our homogenization problem.

Theorem 1.2 (Lower bound on the Γ-limit of $(E_\varepsilon)_\varepsilon$). *Let $\Omega \subset \mathbb{R}^2$ be an x_1-connected domain, and let E_ε and E be the functionals introduced in (1.7) and (1.10), respectively. Then, every sequence $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ with uniformly bounded energies, $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < \infty$, has a subsequence that converges weakly* in $BV(\Omega;\mathbb{R}^2)$ to some $u \in \mathcal{A} \cap L^1_0(\Omega;\mathbb{R}^2)$. Additionally,*

$$\Gamma(L^1)\text{-}\liminf_{\varepsilon\to 0} E_\varepsilon \geq E. \tag{1.12}$$

The proof of the first assertion is given in Proposition 4.3. It relies on Theorem 1.1 in combination with a technical argument about the weak continuity properties of Jacobian determinants (see Lemma 4.2). In Section 5, we exhibit two different proofs of (1.12): A first one relying on a Reshetnyak's lower semicontinuity theorem (see, e.g., [2, Theorem 2.38]), and an alternative one exploiting the properties of the admissible layered deformations. The identification of E as the Γ-limit of the sequence $(E_\varepsilon)_\varepsilon$, though, remains an open problem. Indeed, verifying the optimality of the lower bound in Theorem 1.2 is rather challenging, as it requires to approximate elements of $\mathcal{A}$ by means of sequences in $\mathcal{A}_\varepsilon$ at least in the sense of the strict convergence in BV. We refer to Remark 5.2 for a detailed discussion of the main difficulties. Even if the requirement on the convergence of the energies is dropped, recovering the jumps of maps in the effective domain of E under consideration of the non-standard differential inclusions in $\mathcal{A}_\varepsilon$ is by itself another challenging problem. Solving this problem requires delicate geometrical constructions, which are currently not available for all elements in $\mathcal{A}$.

Yet, there are two subclasses of physically relevant deformations in $\mathcal{A}$ for which we can find suitable approximations by sequences of admissible layered deformations. The precise statement is given in Theorem 1.3 below.

The first of these two subclasses is $\mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)$ (we refer to Subsection 2.3 for the definition of the set SBV_∞) whose jump sets are given by a union of finitely many lines. Heuristically, this subclass describes deformations that break Ω horizontally into a finite number of pieces, which may get sheared and rotated individually.

The second subclass is

$$\begin{aligned}\mathcal{A}^{\|} := \big\{u \in BV(\Omega;\mathbb{R}^2)\colon\ & u(x) = Rx + \vartheta(x_2)Re_1 + c \text{ for a.e. } x \in \Omega \text{ with}\\ & R \in SO(2),\ \vartheta \in BV(a_\Omega, b_\Omega),\ \text{and } c \in \mathbb{R}^2\big\}.\end{aligned} \tag{1.13}$$

In comparison with $\mathcal{A}$, functions in $\mathcal{A}^{\|}$ satisfy two additional constraints, namely the fact that the rotation R is constant and that the jumps of functions in $\mathcal{A}^{\|}$ are parallel to Re_1. With the notation $\mathcal{A}^{\|}$, we intend to highlight the second feature. The intuition behind maps in $\mathcal{A}^{\|}$ are non-trivial macroscopic deformations that (up to a global rotation) may make the material break along finite or infinitely many horizontal lines, induce sliding of the pieces relative to each other, and cause horizontal shearing within each individual piece. For an illustration of the two subclasses, see Figure 2.

Theorem 1.3 (Approximation of maps in $(\mathcal{A} \cap SBV_\infty) \cup \mathcal{A}^{\|}$). *Let $\Omega \subset \mathbb{R}^2$ be an x_1-connected domain and $u \in (\mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)) \cup \mathcal{A}^{\|}$. Then, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ such that $u_\varepsilon \in \mathcal{A}_\varepsilon$ for every ε, and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$.*

As a first step towards proving Theorem 1.3, we establish an admissible piecewise affine approximation for limiting deformations with a single jump line (see Lemma 4.5). The construction relies on the characterization of rank-one connections in $\mathcal{M}_{e_1}$ proved in [13, Lemma 3.1], with transition lines stretching over the full width of Ω to avoid triple junctions (see Remark 4.6). In Propositions 4.7 and 4.9, we extend the arguments to $\mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)$ and $\mathcal{A}^{\|}$, respectively.

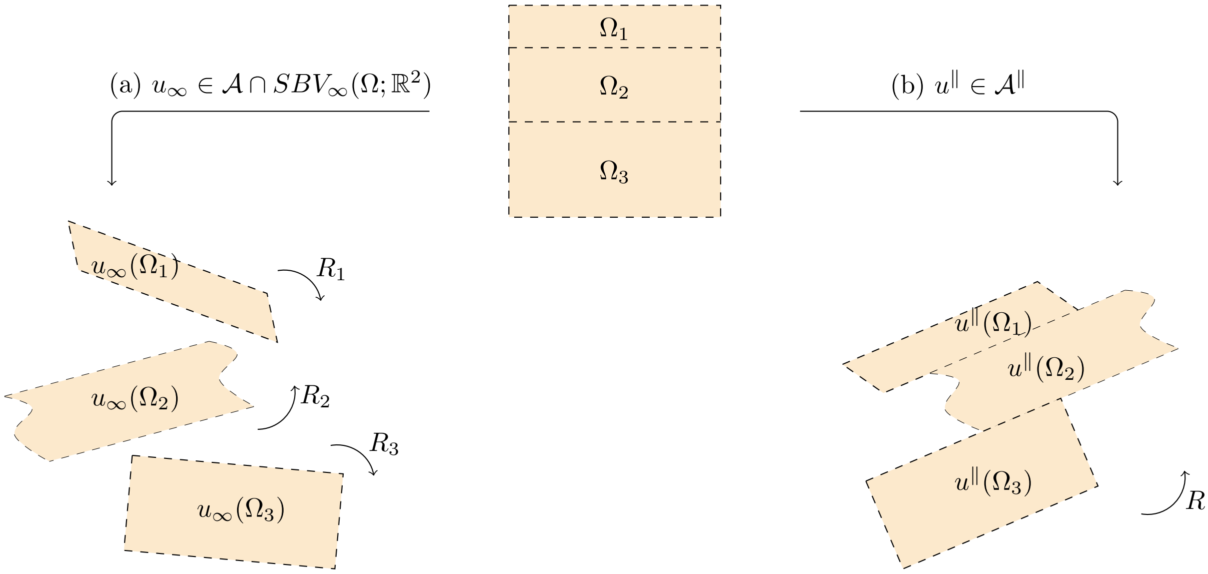


FIGURE 2. A typical deformation of a reference configuration $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ through maps in (a) $\mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)$ and (b) $\mathcal{A}^\parallel$.

Problems in finite crystal plasticity without additional regularizations are generally known to be challenging because of the oscillations of minimizing sequences arising as a byproduct of relaxation mechanisms in the slip systems. This phenomenon is one of the main reasons why a full relaxation theory in finite crystal plasticity is still missing (see [17, Remark 3.2]). In our setting, it hampers the full characterization of weak limits of sequences with uniformly bounded energies. The observation that regularizations can help overcome the above compensated-compactness issue (see also Remark 6.2) motivates the introduction of a penalized version of our problem. After a higher-order penalization of the energy in the layer direction, we obtain the following Γ-convergence result. The attained limit deformations are given by the class $\mathcal{A}^\parallel$.

Theorem 1.4 (Γ-convergence of the regularized energies). *Let $\Omega \subset \mathbb{R}^2$ be an x_1-connected domain and $\mathcal{A}_\varepsilon$ the set introduced in (1.6). Fix $p > 2$ and $\delta > 0$. For each $\varepsilon > 0$, let $E_\varepsilon^\delta : L_0^1(\Omega;\mathbb{R}^2) \to [0,\infty]$ be the functional defined by*

$$E_\varepsilon^\delta(u) := \begin{cases} \displaystyle\int_\Omega |\gamma| \,\mathrm{d}x + \delta\|\partial_1 u\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} & \text{for } u \in \mathcal{A}_\varepsilon, \\ \infty & \text{otherwise.} \end{cases} \tag{1.14}$$

Then, the family $(E_\varepsilon^\delta)_\varepsilon$ Γ-converges with respect to the strong L^1-topology to the functional $E^\delta : L_0^1(\Omega;\mathbb{R}^2) \to [0,\infty]$ given by

$$E^\delta(u) := \begin{cases} \displaystyle\int_\Omega |\vartheta'(x_2)| \,\mathrm{d}x + |D^s u|(\Omega) + \ \delta|\Omega| & \text{for } u \in \mathcal{A}^\parallel, \\ \infty & \text{otherwise,} \end{cases}$$

where ϑ' denotes the approximate differential of ϑ (cf. Section 2.2).

The penalization in (1.14) can be viewed in the spirit of non-simple materials [39, 40]. Working with stored energy densities that depend on the Hessian of the deformations has proved successful in overcoming lack of compactness in a variety of applications; see, e.g., [5, 21, 27, 36, 38]. Very recently, there has been an effort towards weakening higher-order regularizations: It is shown in [7] that the full norm of the Hessian can be replaced by a control of its minors (gradient polyconvexity) in the context of locking materials; for solid-solid phase transitions, an anisotropic second-order penalization is considered in [23]. Along these lines, we introduce the regularized energies in (1.13) that penalize the variation of deformations only in the layer direction. This is enough to deduce that the limiting rotation (as $\varepsilon \to 0$) is global and that it determines the direction of the limiting jump. In Section 6, we provide two alternative proofs of this result: A first one relying on Alberti's rank one theorem (see Section 2.1) in combination with the approximation result in Theorem 1.3, and a second one based on separate regularizations of the regular and the singular part of the limiting maps, and inspired by [19, Lemma 3.2].

This paper is organized as follows. In Section 2.1, we collect a few preliminaries, including some background on (special) functions of bounded variation. Section 3 is devoted to the analysis of asymptotic rigidity for layered structures in the setting of BV-functions. A characterization of limits of admissible layered deformations is provided in Section 4. Eventually, Sections 5 and 6 contain the proof of a lower bound for the homogenization problem without regularization (Theorem 1.2) and the full Γ-convergence analysis of the regularized problem (Theorem 1.4), respectively.

2. Preliminaries

2.1. **Notation.** In this section, unless mentioned otherwise, Ω is a bounded domain in $\mathbb{R}^N$ with $N \in \mathbb{N}$. Throughout the rest of the paper, we assume mostly that $N = 2$.

We represent by $\mathcal{L}^N$ the N-dimensional Lebesgue measure and by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. Whenever we write "a.e. in Ω", we mean "almost everywhere in Ω" with respect to $\mathcal{L}^N\lfloor\Omega$. To simplify the notation, we often omit the expression "a.e. in Ω" in mathematical relations involving Lebesgue measurable functions. Given a Lebesgue measurable set $B \subset \mathbb{R}^N$, we also use the shorter notation $|B| = \mathcal{L}^N(B)$ for the Lebesgue measure of B, while the characteristic function of B in $\mathbb{R}^N$ is denoted by $\mathbb{1}_B$ and takes values 0 and 1.

The set $SO(N) := \{R \in \mathbb{R}^{N\times N} : RR^T = \mathbb{I}, \det R = 1\}$, where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{N\times N}$, consists of all proper rotations. We recall that for $N = 2$, $R \in SO(2)$ if and only if there is $\theta \in [-\pi, \pi)$ such that

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

For two vectors $a, b \in \mathbb{R}^d$, $a \otimes b := ab^T$ stands for their tensor product. If $a = (a_1, a_2)^T \in \mathbb{R}^2$, we set $a^\perp := (-a_2, a_1)^T$.

We use the standard notation for spaces of vector-valued functions; namely, $L^p_\mu(\Omega;\mathbb{R}^d)$ with $p \in [1,\infty]$ and a positive measure μ for L^p-spaces, $W^{1,p}(\Omega;\mathbb{R}^d)$ with $p \in [1,\infty]$ for Sobolev spaces, $C(\Omega;\mathbb{R}^d)$ for the space of continuous functions, $C^\infty(\Omega;\mathbb{R}^d)$ and $C^\infty_c(\Omega;\mathbb{R}^d)$ for the spaces of smooth functions without and with compact support, and $C^{0,\alpha}(\Omega;\mathbb{R}^d)$ with $\alpha \in [0,1]$ for Hölder spaces. We denote by $C_0(\Omega;\mathbb{R}^d)$ the space of continuous functions that vanish on the boundary of Ω. Moreover, $\mathcal{M}(\Omega;\mathbb{R}^d)$ is the space of finite vector-valued Radon measures. In the case of scalar-valued functions and measures, we omit the codomain; for instance, we write $L^1(\Omega)$ instead of $L^1(\Omega;\mathbb{R})$.

The duality pairing between $C_0(\Omega;\mathbb{R}^d)$ and $\mathcal{M}(\Omega;\mathbb{R}^d)$ is represented by $\langle\mu,\zeta\rangle := \int_\Omega \zeta \,\mathrm{d}\mu$, and $\mu\otimes\nu$ denotes the product measure of two measures μ and ν.

Throughout this manuscript, ε stands for a small (positive) parameter, and is usually thought of as taking values on a positive sequence converging to zero.

2.2. **Functions of bounded variation.** We adopt the standard notations for the space $BV(\Omega;\mathbb{R}^d)$ of vector-valued functions of bounded variation, and refer the reader to [2] for a thorough treatment of this space. Here, we only recall some of its basic properties.

A function $u \in L^1(\Omega;\mathbb{R}^d)$ is called a *function of bounded variation*, written $u \in BV(\Omega;\mathbb{R}^d)$, if its distributional derivative Du satisfies $Du \in \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$. The space $BV(\Omega;\mathbb{R}^d)$ is a Banach space when endowed with the norm $\|u\|_{BV(\Omega;\mathbb{R}^d)} := \|u\|_{L^1(\Omega;\mathbb{R}^d)} + |Du|(\Omega)$, where $|Du| \in \mathcal{M}(\Omega)$ is the total variation of Du.

Let $D^a u$ and $D^s u$ denote the absolutely continuous and the singular part of the Radon–Nikodym decomposition of Du with respect to $\mathcal{L}^N\lfloor\Omega$, and let $D^j u$ and $D^c u$ be the jump and Cantor parts of Du. The following chain of equalities holds:

$$\begin{aligned} Du &= D^a u + D^s u = \nabla u\mathcal{L}^N\lfloor\Omega + D^s u = \nabla u\mathcal{L}^N\lfloor\Omega + D^j u + D^c u \\ &= \nabla u\mathcal{L}^N\lfloor\Omega + (u^+ - u^-)\otimes\nu_u\mathcal{H}^{N-1}\lfloor J_u + D^c u, \end{aligned} \tag{2.1}$$

where ∇u is the approximate differential of u (that is, the density of $D^a u$), u^+ and u^- are the approximate one-sided limits at the jump points, J_u is the jump set of u, and ν_u is the normal to J_u (cf. [2, Chapter 3]).

Following [2, p. 186], we can exploit the polar decomposition of a measure and the fact that all parts of the derivative of u in (2.1) are mutually singular to write $Du = g_u|Du|$ with a map $g_u \in L^1_{|Du|}(\Omega;\mathbb{R}^{d\times N})$

satisfying $|g_u| = 1$ for $|Du|$-a.e. $x \in \Omega$ and

$$D^a u = g_u|D^a u|, \quad D^s u = g_u|D^s u|, \quad D^j u = g_u|D^j u|, \quad D^c u = g_u|D^c u|.$$

Note that

$$g_u(x) = \frac{\nabla u(x)}{|\nabla u(x)|} \text{ for } \mathcal{L}^N\text{-a.e. } x \in \Omega \text{ such that } |\nabla u(x)| \neq 0,$$

$$g_u(x) = \frac{u(x^+) - u(x^-)}{|u(x^+) - u(x^-)|} \otimes \nu_u(x) \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in J_u, \tag{2.2}$$

$$g_u(x) = \bar{g}_u(x) \otimes n_u(x) \text{ for } |D^c u|\text{-a.e. } x \in \Omega \text{ with suitable Borel maps } \bar{g}_u : \Omega \to \mathbb{R}^d,\ n_u : \Omega \to \mathbb{R}^N. \tag{2.3}$$

The last equality relies on Alberti's rank-one theorem (see [1]).

Let $u \in BV(\Omega;\mathbb{R}^d)$ and $(u_j)_{j\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^d)$ be a sequence. One says that $(u_j)_{j\in\mathbb{N}}$ weakly* converges to u in $BV(\Omega;\mathbb{R}^d)$, written $u_j \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^d)$, if $u_j \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and $Du_j \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$. The sequence $(u_j)_{j\in\mathbb{N}}$ is said to converge strictly to u in $BV(\Omega;\mathbb{R}^d)$, written $u_j \overset{*}{\to} u$ in $BV(\Omega;\mathbb{R}^d)$, if $u_j \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and $|Du_j|(\Omega) \to |Du|(\Omega)$. We recall that strict convergence in $BV(\Omega;\mathbb{R}^d)$ implies weak* convergence in $BV(\Omega;\mathbb{R}^d)$. Moreover, from every bounded sequence in $BV(\Omega;\mathbb{R}^d)$ one can extract a weakly* convergent subsequence (see [2, Theorem 3.23]).

In the one-dimensional setting, i.e., for $\varphi \in BV(a,b;\mathbb{R}^d)$ with $\Omega = (a,b) \subset \mathbb{R}^N$ and $N = 1$, we write φ' in place of $\nabla\varphi$ to denote the approximate differential of φ. Accordingly, we use the notation $Du = \varphi'\mathcal{L}^1 + D^s\varphi$ for the decomposition of the distributional derivative of φ with respect to the Lebesgue measure.

A function $\varphi \in BV(a,b;\mathbb{R}^d)$ is called a jump or Cantor function if $D\varphi = D^j\varphi$ or $D\varphi = D^c\varphi$, respectively. We denote the sets of all jump and Cantor functions by $BV^j(a,b;\mathbb{R}^d)$ and $BV^c(a,b;\mathbb{R}^d)$, respectively. As shown in [2, Corollary 3.33], it is a special property of the one-dimensional setting that

$$BV(a,b;\mathbb{R}^d) = W^{1,1}(a,b;\mathbb{R}^d) + BV^j(a,b;\mathbb{R}^d) + BV^c(a,b;\mathbb{R}^d). \tag{2.4}$$

Throughout this paper, two-dimensional functions of the form

$$u(x) = R(x_2)x + \psi(x_2) \tag{2.5}$$

with $x = (x_1, x_2) \in \Omega = Q := (c,d) \times (a,b) \subset \mathbb{R}^2$, where $R \in BV(a,b;SO(2))$ and $\psi \in BV(a,b;\mathbb{R}^2)$, play a fundamental role. Maps u as in (2.5) satisfy $u \in BV(\Omega;\mathbb{R}^2)$. Denoting by $D_1 u := Du \otimes e_1$ and $D_2 u := Du \otimes e_2$, the first and second columns of Du, respectively, we have for all $\zeta \in C_0(\Omega)$ that

$$\begin{aligned}
\langle D_1 u, \zeta\rangle &= \int_\Omega \zeta(x)R(x_2)e_1 \,\mathrm{d}x_1\mathrm{d}x_2,\\
\langle D_2 u, \zeta\rangle &= \int_\Omega \big(\zeta(x)R(x_2)e_2 + R'(x_2)x + \psi'(x_2)\big)\,\mathrm{d}x_1\mathrm{d}x_2\\
&\quad + \int_\Omega \zeta(x)x_1 \,\mathrm{d}x_1\mathrm{d}D^sR(x_2)e_1 + \int_\Omega \zeta(x)x_2 \,\mathrm{d}x_1\mathrm{d}D^sR(x_2)e_2 + \int_\Omega \zeta(x)\,\mathrm{d}x_1\mathrm{d}D^s\psi(x_2).
\end{aligned}$$

Hence, $Du = D^a u + D^s u$ with

$$\begin{aligned}
D^a u &= \big(R + (R'x + \psi') \otimes e_2\big)\mathcal{L}^2\lfloor\Omega,\\
D^s u &= \Big(\big(x^T\mathcal{L}^1\lfloor(c,d)\otimes D^sR^T\big)^T + \mathcal{L}^1\lfloor(c,d)\otimes D^s\psi\Big) \otimes e_2,
\end{aligned} \tag{2.6}$$

where $\mathcal{L}^1\lfloor(c,d)\otimes D^sR^T$ and $\mathcal{L}^1\lfloor(c,d)\otimes D^s\psi$ denote the restrictions to the Borel σ-algebra on $\Omega = Q$ of the product measures between $\mathcal{L}^1\lfloor(c,d)$ and D^sR^T and $D^s\psi$, respectively.

We observe further that there exists $\theta \in BV(a,b;[-\pi,\pi])$ such that

$$R = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \quad \text{and} \quad R' = \theta'\begin{bmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{bmatrix}, \tag{2.7}$$

where the representation of R' follows from the chain rule in BV; see, e.g., [2, Theorem 3.96].

2.3. **Special functions of bounded variation.** A function $u \in BV(\Omega;\mathbb{R}^d)$ is said to be a *special function of bounded variation*, written $u \in SBV(\Omega;\mathbb{R}^d)$, if the Cantor part of its distributional derivative satisfies

$$D^c u = 0.$$

In particular, it holds for every $u \in SBV(\Omega;\mathbb{R}^d)$ that

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \lfloor J_u.$$

The space $SBV(\Omega;\mathbb{R}^d)$ is a proper subspace of $BV(\Omega;\mathbb{R}^d)$(c.f. [2, Corollary 4.3]).

Next, we recall the definition of the space $SBV_\infty(\Omega;\mathbb{R}^d)$ of special functions of bounded variation with bounded gradient and jump length, which is given by

$$SBV_\infty(\Omega;\mathbb{R}^d) := \{u \in SBV(\Omega;\mathbb{R}^d) \colon \nabla u \in L^\infty(\Omega;\mathbb{R}^{d\times N}) \text{ and } \mathcal{H}^{N-1}(J_u) < +\infty\}.$$

It is shown in [9] that the distributional curl of ∇u for $u \in SBV_\infty(\Omega;\mathbb{R}^d)$ is a measure concentrated on J_u.

Finally, we introduce the space

$$PC(a,b;\mathbb{R}^d) = SBV_\infty(a,b;\mathbb{R}^d) \cap \{u \in BV(a,b;\mathbb{R}^d) \colon D^a u = 0\}, \tag{2.8}$$

which contains piecewise constant one-dimensional functions with values in $\mathbb{R}^d$.

2.4. **Geometry of the domain.** In this section, we specify our main assumptions on the geometry of Ω, which, as mentioned in the Introduction, will mostly be a bounded Lipschitz domain in $\mathbb{R}^2$. Let us first recall from [14, Section 3] the definitions of *locally one-dimensional* and *one-dimensional* functions.

Definition 2.1 (Locally one-dimensional functions in the e_2-direction). *Let $\Omega \subset \mathbb{R}^2$ be open. A function $f : \Omega \to \mathbb{R}^d$ is* locally one-dimensional in the e_2-direction *if for every $x \in \Omega$, there exists an open cuboid $Q_x \subset \Omega$, containing x and with sides parallel to the standard coordinate axes, such that for all $y = (y_1, y_2), z = (z_1, z_2) \in Q_x$,*

$$f(y) = f(z) \qquad \text{if } y_2 = z_2. \tag{2.9}$$

We say that f is (globally) one-dimensional *in the e_2-direction if (2.9) holds for every $y, z \in \Omega$.*

Analogous arguments to those in [14, Section 3] show that a function $f \in BV(\Omega;\mathbb{R}^d)$ satisfying $D_1 f = 0$ is locally one-dimensional in the e_2-direction. The following geometrical requirement is the counterpart of [14, Definitions 3.6 and 3.7] in our setting.

Definition 2.2 (x_1-connectedness). *We say that an open set $\Omega \subset \mathbb{R}^2$ is* x_1-connected *if for every $t \in \mathbb{R}$, the set $\{x_2 = t\} \cap \Omega$ is a (possibly empty) interval.*

In what follows, we always assume that the set $\Omega \subset \mathbb{R}^2$ is an x_1-connected domain. Under this geometrical assumption, the notions of locally and globally one-dimensional functions in the e_2-direction coincide. We refer to [14, Section 3] for an extended discussion on the topic, as well as for some explicit geometrical examples.

3. Asymptotic rigidity of layered structures in BV

In this section, we prove Theorem 1.1, which characterizes the asymptotic behavior of deformations of bilayered materials that correspond to rigid body motions on the stiff layers, but do not experience any further structural constraints on the softer layers. This qualitative result is not just limited to applications in crystal plasticity, but can be useful for a larger class of layered composites where fracture may occur.

We start by introducing some notation. Assume that $\Omega \subset \mathbb{R}^2$ is an x_1-connected domain. For $\varepsilon > 0$, let

$$\mathcal{B}_\varepsilon := \{u \in W^{1,1}(\Omega;\mathbb{R}^2) : \nabla u \in SO(2) \text{ in } \varepsilon Y_{\text{rig}} \cap \Omega\} \tag{3.1}$$

represent the class of *layered deformations with rigid components,* and let

$$\mathcal{B}_0 := \{u \in BV(\Omega;\mathbb{R}^2) : \text{ there exists } (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \text{ with } u_\varepsilon \in \mathcal{B}_\varepsilon \text{ for all } \varepsilon \tag{3.2}$$
$$\text{such that } u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } BV(\Omega;\mathbb{R}^2)\}$$

be the associated set of asymptotically attainable deformations.

We aim at proving that $\mathcal{B}_0$ coincides with the set of *asymptotically rigid deformations* given by

$$\mathcal{B} := \big\{u \in BV(\Omega;\mathbb{R}^2)\colon\ u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega \\ \text{with } R \in BV(a_\Omega, b_\Omega; SO(2)) \text{ and } \psi \in BV(a_\Omega, b_\Omega;\mathbb{R}^2)\big\}, \tag{3.3}$$

cf. (1.1). This identity will be a consequence of Propositions 3.1 and 3.2 below.

Proposition 3.1 (Limiting behavior of maps in $\mathcal{B}_\varepsilon$). *Let $\Omega = (0,1)\times(-1,1)$. Then,*

$$\mathcal{B}_0 \subset \mathcal{B}, \tag{3.4}$$

where $\mathcal{B}_0$ and $\mathcal{B}$ are the sets introduced in (3.2) and (3.3), respectively.

Proof. The proof is inspired by and generalizes ideas from [13, Proposition 2.1]. Let $u \in \mathcal{B}_0$. Then, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ satisfying $\nabla u_\varepsilon \in SO(2)$ a.e. in $\varepsilon Y_{\rm rig}\cap\Omega$ for all ε, and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$.

Fix $0<\varepsilon<1$, and let $I_\varepsilon := \{i \in \mathbb{Z}\colon\ (\mathbb{R}\times\varepsilon(i-1,i))\cap\Omega \neq \emptyset\}$. For each $i \in I_\varepsilon$, we define a strip, P_ε^i, by setting

$$P_\varepsilon^i := (\mathbb{R}\times\varepsilon[i-1,i))\cap\Omega.$$

Note that if $i\in\mathbb{Z}$ is such that $|i| > 1+\lceil\frac{1}{\varepsilon}\rceil$, then $i \notin I_\varepsilon$. Moreover, defining $i_\varepsilon^+ := \max I_\varepsilon$ and $i_\varepsilon^- := \min I_\varepsilon$, then

i) for $i_\varepsilon^- < i < i_\varepsilon^+$, P_ε^i is the union of two neighboring connected components of $\varepsilon Y_{\rm rig}\cap\Omega$ and $\varepsilon Y_{\rm soft}\cap\Omega$;

ii) we may have $\varepsilon Y_{\rm soft}\cap P_\varepsilon^{i_\varepsilon^-} = \emptyset$ or $\varepsilon Y_{\rm rig}\cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$.

From Reshetnyak's theorem, we infer that on each nonempty rigid layer $\varepsilon Y_{\rm rig}\cap P_\varepsilon^i$ with $i \in I_\varepsilon$, the gradient ∇u_ε is constant and coincides with a rotation $R_\varepsilon^i \in SO(2)$. Moreover, there exists $b_\varepsilon^i \in \mathbb{R}^2$ such that $u_\varepsilon(x) = R_\varepsilon^i x + b_\varepsilon^i$ in $\varepsilon Y_{\rm rig}\cap P_\varepsilon^i$.

Using these rotations R_ε^i, we define a piecewise constant function, $\Sigma_\varepsilon : (-1,1)\to\mathbb{R}^{2\times2}$, by setting $\Sigma_\varepsilon(t) = \sum_{i\in I_\varepsilon} R_\varepsilon^i \mathbb{1}_{\varepsilon[i-1,1)}(t)$ for $t\in(-1,1)$, where $R_\varepsilon^{i_\varepsilon^+} := R_\varepsilon^{i_\varepsilon^+-1}$ if $\varepsilon Y_{\rm rig}\cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$. We claim that there exist a subsequence of $(\Sigma_\varepsilon)_\varepsilon$, which we do not relabel, and a function $R\in BV(-1,1;SO(2))$ such that

$$\Sigma_\varepsilon \to R \quad \text{in } L^1(-1,1;\mathbb{R}^{2\times2}). \tag{3.5}$$

To prove (3.5), we first observe that the total variation of the one-dimensional function Σ_ε coincides with its pointwise variation, and can be calculated to be

$$|D\Sigma_\varepsilon|(-1,1) = \sum_{i\in I_\varepsilon\setminus\{i_\varepsilon^-\}} |R_\varepsilon^i - R_\varepsilon^{i-1}| = \sqrt{2}\sum_{i\in I_\varepsilon\setminus\{i_\varepsilon^-\}} |R_\varepsilon^i e_1 - R_\varepsilon^{i-1}e_1|. \tag{3.6}$$

Next, we show that the right-hand side of (3.6) is uniformly bounded. By linear interpolation in the x_2-direction on the softer layers, it follows for all $i\in I_\varepsilon\setminus\{i_\varepsilon^-\}$ if $\varepsilon Y_{\rm rig}\cap P_\varepsilon^{i_\varepsilon^+}\neq\emptyset$ and $i\in I_\varepsilon\setminus\{i_\varepsilon^\pm\}$ if $\varepsilon Y_{\rm rig}\cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$ that

$$\begin{aligned}\int_{\varepsilon Y_{\rm soft}\cap P_\varepsilon^i} |\nabla u_\varepsilon e_2|\,\mathrm{d}x &= \int_0^1\int_{\varepsilon(i-1)}^{\varepsilon(i-1+\lambda)} |\partial_2 u_\varepsilon(x_1,x_2)|\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &\geq \int_0^1 |u_\varepsilon(x_1,\varepsilon(i-1+\lambda)) - u_\varepsilon(x_1,\varepsilon(i-1))|\,\mathrm{d}x_1\\ &= \int_0^1 |(R_\varepsilon^i e_1 - R_\varepsilon^{i-1}e_1)x_1 + b_\varepsilon^i - b_\varepsilon^{i-1}|\,\mathrm{d}x_1 \geq \frac{1}{4}|R_\varepsilon^i e_1 - R_\varepsilon^{i-1}e_1|.\end{aligned} \tag{3.7}$$

The first estimate is a consequence of Jensen's inequality, and optimization over translations yields the second one. To be more precise, the last estimate in (3.7) is based on the observation that for any given $a\in\mathbb{R}^2\setminus\{0\}$,

$$\min_{b\in\mathbb{R}^2}\int_0^1 |ta+b|\,\mathrm{d}t = \min_{\alpha,\,\beta\in\mathbb{R}}\int_0^1 |(t+\alpha)a+\beta a^\perp|\,\mathrm{d}t = |a|\min_{\alpha\in\mathbb{R}}\int_0^1|t+\alpha|\,\mathrm{d}t = \frac{|a|}{4}.$$

From (3.6) and (3.7), since $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ as a weakly* converging sequence is uniformly bounded in $BV(\Omega;\mathbb{R}^2)$, and recalling that $R_\varepsilon^{i_\varepsilon^+} = R_\varepsilon^{i_\varepsilon^+-1}$ if $\varepsilon Y_{\rm rig} \cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$, we conclude that

$$|D\Sigma_\varepsilon|(-1,1) \leq 4\sqrt{2}\int_\Omega |\nabla u_\varepsilon|\,\mathrm{d}x \leq C. \tag{3.8}$$

The convergence in (3.5) follows now from the weak* relative compactness of bounded sequences in $BV(-1,1;\mathbb{R}^{2\times 2})$ (see Section 2.2), together with the fact that strong L^1-convergence is length and angle preserving. The latter guarantees that the limit function $R \in BV(-1,1;\mathbb{R}^{2\times 2})$ takes values only in $SO(2)$.

Next, we show that there is $\psi \in BV(-1,1;\mathbb{R}^2)$ such that

$$u(x) = R(x_2)x + \psi(x_2) \tag{3.9}$$

for a.e. $x \in \Omega$, which implies that $u \in \mathcal{B}$ and concludes the proof. To this end, we define auxiliary functions σ_ε, $b_\varepsilon \in L^\infty(\Omega;\mathbb{R}^2)$ for $\varepsilon > 0$ by setting

$$\sigma_\varepsilon(x) = \sum_{i\in I_\varepsilon} (R_\varepsilon^i x)\mathbb{1}_{P_\varepsilon^i}(x) \qquad \text{and} \qquad b_\varepsilon(x) = \sum_{i\in I_\varepsilon} b_\varepsilon^i \mathbb{1}_{P_\varepsilon^i}(x)$$

for $x \in \Omega$, where $R_\varepsilon^{i_\varepsilon^+} := R_\varepsilon^{i_\varepsilon^+-1}$ and $b_\varepsilon^{i_\varepsilon^+} := b_\varepsilon^{i_\varepsilon^+-1}$ if $\varepsilon Y_{\rm rig} \cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$. Further, let $w_\varepsilon := \sigma_\varepsilon + b_\varepsilon$.

By Poincaré's inequality applied in the x_2-direction, we obtain

$$\begin{aligned}\int_\Omega |u_\varepsilon - w_\varepsilon|\,\mathrm{d}x &= \sum_{i\in I_\varepsilon:\, \varepsilon Y_{\rm soft}\cap P_\varepsilon^i \neq \emptyset} \int_0^1 \int_{\max\{\varepsilon(i-1),-1\}}^{\min\{\varepsilon(i-1+\lambda),1\}} |u_\varepsilon - w_\varepsilon|\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &\leq \varepsilon\lambda \sum_{i\in I_\varepsilon} \int_{\varepsilon Y_{\rm soft}\cap P_\varepsilon^i} |\partial_2 u_\varepsilon - R_\varepsilon^i e_2|\,\mathrm{d}x \leq \varepsilon\lambda(\|u_\varepsilon\|_{W^{1,1}(\Omega;\mathbb{R}^2)} + |\Omega|) \leq C\varepsilon.\end{aligned}$$

Consequently,

$$w_\varepsilon \to u \qquad \text{in } L^1(\Omega;\mathbb{R}^2). \tag{3.10}$$

Moreover, for $x \in \Omega$,

$$|\sigma_\varepsilon(x) - R(x_2)x| \leq \Big|\sum_{i\in I_\varepsilon} (R_\varepsilon^i - R(x_2))\mathbb{1}_{P_\varepsilon^i}(x)\Big||x| \leq \sqrt{2}|\Sigma_\varepsilon(x_2) - R(x_2)|,$$

which, together with (3.5), proves that

$$\sigma_\varepsilon \to \sigma \qquad \text{in } L^1(\Omega;\mathbb{R}^2), \tag{3.11}$$

where $\sigma(x) := R(x_2)x \in BV(\Omega;\mathbb{R}^2)$.

Finally, exploiting (3.10) and (3.11), we conclude that there exists $b \in BV(\Omega;\mathbb{R}^2)$ such that $b_\varepsilon \to b$ in $L^1(\Omega;\mathbb{R}^2)$. In view of the one-dimensional character of the stripes P_ε^i, we infer that $\partial_1 b = 0$. Eventually, identifying b with a function $\psi \in BV(-1,1;\mathbb{R}^2)$ yields (3.9).

□

Next, we prove that the converse inclusion of (3.4) holds. In the following, let $I_{\rm rig}$ be the projection of $Y_{\rm rig}$ onto the second component; that is, $I_{\rm rig}$ corresponds to the 1-periodic extension of the interval $[\lambda, 1)$. Analogously, we write $I_{\rm soft}$ for the 1-periodic extension of $[0, \lambda)$.

Proposition 3.2 (Approximation of maps in $\mathcal{B}$). *Let $\Omega = (0,1)\times(-1,1)$. Then,*

$$\mathcal{B}_0 \supset \mathcal{B}. \tag{3.12}$$

Here, $\mathcal{B}_0$ and $\mathcal{B}$ are the sets from (3.2) and (3.3), respectively.

Proof. Let $u \in \mathcal{B}$, and let $R \in BV(-1,1;SO(2))$ and $\psi \in BV(-1,1;\mathbb{R}^2)$ be such that

$$u(x) = R(x_2)x + \psi(x_2)$$

for a.e. $x \in \Omega$. Using Lemma 3.3 below, as well as the fact that strict convergence implies weak* convergence in BV, we construct sequences $(R_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1;SO(2))$ and $(\psi_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1;\mathbb{R}^2)$ such that

$$R_\varepsilon' = 0 \quad \text{and} \quad \psi_\varepsilon' = 0 \qquad \text{on } \varepsilon I_{\rm rig} \cap (-1,1), \tag{3.13}$$

$$R_\varepsilon \overset{*}{\rightharpoonup} R \text{ in } BV(-1,1;\mathbb{R}^{2\times 2}) \quad \text{and} \quad \psi_\varepsilon \overset{*}{\rightharpoonup} \psi \quad \text{in } BV(-1,1;\mathbb{R}^2). \tag{3.14}$$

Define $u_\varepsilon(x) := R_\varepsilon(x_2)x + \psi_\varepsilon(x_2)$ for $x \in \Omega$. Then, $u_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ for every ε, with

$$\nabla u_\varepsilon(x) = R_\varepsilon(x_2) + R'_\varepsilon(x_2)x \otimes e_2 + \psi'_\varepsilon(x_2) \otimes e_2$$

for a.e. $x \in \Omega$. In particular, $\nabla u_\varepsilon = R_\varepsilon \in SO(2)$ a.e. in $\varepsilon Y_{\text{rig}} \cap \Omega$ by (3.13); hence, $u_\varepsilon \in \mathcal{B}_\varepsilon$. Moreover, $\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} < \infty$ and $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ by (3.14), from which we conclude that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$. This completes the proof. □

The next lemma states a one-dimensional approximation result of BV-maps by Lipschitz functions that are constant on $\varepsilon I_{\text{rig}}$, which was an important ingredient in the previous proof.

Lemma 3.3 (1D-approximation by maps constant on $\varepsilon I_{\text{rig}}$). *Let $I = (a,b) \subset \mathbb{R}$ and $w \in BV(I;\mathbb{R}^d)$. Then, there exists a sequence $(w_\varepsilon)_\varepsilon \subset W^{1,\infty}(I;\mathbb{R}^d)$ with the following three properties:*

(i) $w_\varepsilon \to w$ *in* $L^1(I;\mathbb{R}^d)$;

(ii) $\displaystyle\int_I |w'_\varepsilon| \,\mathrm{d}t \to |Dw|(I)$;

(iii) $w'_\varepsilon = 0$ *on* $\varepsilon I_{\text{rig}} \cap I$.

Moreover, if w takes values in $SO(2)$ and $w \in BV(I;SO(2))$, then each w_ε may be taken in $W^{1,\infty}(I;SO(2))$.

Proof. Let $w \in BV(I;\mathbb{R}^d)$. By [2, Theorem 3.9, Remark 3.22], w can be approximated by a sequence of smooth functions $(v_\delta)_\delta \subset C^\infty(\bar{I};\mathbb{R}^d)$ in the sense of strict convergence in BV; that is,

$$v_\delta \to w \text{ in } L^1(I;\mathbb{R}^d) \quad \text{and} \quad \int_I |v'_\delta| \,\mathrm{d}t \to |Dw|(I) \tag{3.15}$$

as $\delta \to 0$. To obtain property (iii), we will reparametrize v_δ so that it is *stopped* on the set $\varepsilon I_{\text{rig}}$ and *accelerated* otherwise, and eventually apply a diagonalization argument.

We start by introducing for every $\varepsilon > 0$ a Lipschitz function $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_\varepsilon(t) := \begin{cases} \frac{1}{\lambda}(t - i\varepsilon) + i\varepsilon & \text{if } i\varepsilon \le t \le i\varepsilon + \lambda\varepsilon, \\ (i+1)\varepsilon & \text{if } i\varepsilon + \lambda\varepsilon \le t < \varepsilon(i+1), \end{cases}$$

for each $i \in \mathbb{Z}$ and $t \in \varepsilon[i, i+1)$. For all $t \in \mathbb{R}$, we have $t \le \varphi_\varepsilon(t) \le t + \varepsilon(1-\lambda)$ and $\varphi'_\varepsilon(t) = \psi(\frac{t}{\varepsilon})$, where ψ is the 1-periodic function such that $\psi(t) = \frac{1}{\lambda}$ if $0 \le t \le \lambda$, and $\psi(t) = 0$ if $\lambda < t < 1$. By the Riemann–Lebesgue lemma on weak convergence of periodically oscillating sequences, it follows that $\psi(\frac{\cdot}{\varepsilon}) \overset{*}{\rightharpoonup} 1$ in $L^\infty(\mathbb{R})$. Thus, $\varphi_\varepsilon \overset{*}{\rightharpoonup} \varphi$ in $W^{1,\infty}_{loc}(\mathbb{R})$, where $\varphi(t) := t$. In particular, φ_ε converges uniformly to φ on every compact set $K \subset \mathbb{R}$.

Next, we define for $\varepsilon > 0$ a Lipschitz function $\tilde{\varphi}_\varepsilon : \bar{I} \to \bar{I}$ by setting

$$\tilde{\varphi}_\varepsilon(t) := \begin{cases} \varphi_\varepsilon(t) & \text{if } a \le t \le b_\varepsilon, \\ b & b_\varepsilon \le t \le b, \end{cases}$$

where $b_\varepsilon \in (a,b]$ is such that $\varphi_\varepsilon(b_\varepsilon) = b$. Note that by definition of φ_ε, there exists at least one such b_ε. We claim that $b_\varepsilon \to b$ as $\varepsilon \to 0$. In fact, extracting a subsequence if necessary, we have $b_\varepsilon \to c$ for some $c \in [a,b]$. Then,

$$|b - c| = |\varphi_\varepsilon(b_\varepsilon) - \varphi(c)| \le |\varphi_\varepsilon(b_\varepsilon) - \varphi_\varepsilon(c)| + |\varphi_\varepsilon(c) - \varphi(c)| \le \tfrac{1}{\lambda}|b_\varepsilon - c| + |\varphi_\varepsilon(c) - \varphi(c)|,$$

from which we infer that $b = c$ by letting $\varepsilon \to 0$. Because the limit does not depend on the subsequence, the whole sequence $(b_\varepsilon)_\varepsilon$ converges to b. Consequently, $\tilde{\varphi}_\varepsilon(t) \to \varphi(t) = t$ for all $t \in \bar{I}$, and since also $\|\tilde{\varphi}_\varepsilon\|_{W^{1,\infty}(I)} = O(1)$ as $\varepsilon \to 0$, we deduce that

$$\tilde{\varphi}_\varepsilon \overset{*}{\rightharpoonup} \varphi \text{ in } W^{1,\infty}(I) \quad \text{and} \quad \|\tilde{\varphi}_\varepsilon - \varphi\|_{L^\infty(I)} \to 0. \tag{3.16}$$

Finally, we set $w_{\varepsilon,\delta} := v_\delta \circ \tilde{\varphi}_\varepsilon \in W^{1,\infty}(I;\mathbb{R}^d)$, and observe that

$$\|w_{\varepsilon,\delta} - w\|_{L^1(I;\mathbb{R}^d)} \le \|v_\delta \circ \tilde{\varphi}_\varepsilon - v_\delta\|_{L^1(I;\mathbb{R}^d)} + \|v_\delta - w\|_{L^1(I;\mathbb{R}^d)} \quad \text{and} \quad \int_I |w'_{\varepsilon,\delta}| \,\mathrm{d}t = \int_I |v'_\delta \circ \tilde{\varphi}_\varepsilon| \, \tilde{\varphi}'_\varepsilon \,\mathrm{d}t.$$

Hence, by (3.15), (3.16), the boundedness of each v_δ and v'_δ, and a weak-strong convergence argument, it follows that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\|w_{\varepsilon,\delta}-w\|_{L^1(I;\mathbb{R}^d)}=0, \tag{3.17}$$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_I |w'_{\varepsilon,\delta}|\,\mathrm{d}t=\lim_{\delta\to 0}\int_I |v'_\delta\circ\varphi|\,\varphi'\,\mathrm{d}t=\lim_{\delta\to 0}\int_I |v'_\delta|\,\mathrm{d}t=|Dw|(I). \tag{3.18}$$

In view of (3.17) and (3.18), we apply Attouch's diagonalization lemma [4] to find a sequence $(w_\varepsilon)_\varepsilon\subset W^{1,1}(I;\mathbb{R}^d)$ with $w_\varepsilon:=w_{\varepsilon,\delta(\varepsilon)}$ satisfying (i) and (ii). We observe further that each w_ε satisfies (iii) by construction.

To conclude, we address the issue of constraint-preserving approximations for $w\in BV(I;SO(2))$. In this case, we argue as above, but replace the density argument leading to (3.15) by its analogue for BV functions with values on manifolds, see [28, Theorem 1.2]. This allows us to assume that $v_\delta\in C^\infty(\bar I;SO(2))$, and eventually yields $w_\varepsilon\in W^{1,\infty}(I;SO(2))$. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. In view of the discussion on locally and globally one-dimensional functions in Section 2.4, it suffices to prove the statement on rectangles with sides parallel to the axes. A simple modification of the proofs of Propositions 3.1 and 3.2 shows that these results hold for any such rectangle. Then, Theorem 1.1 follows by extension and exhaustion arguments in the spirit of [14, Lemma A.2]. □

Remark 3.4 (The higher dimensional setting). We point out that the results of Theorem 1.1 continue to hold for domains $\Omega\subset\mathbb{R}^N$, $N\in\mathbb{N}$, satisfying the flatness and cross-connectedness assumptions in [14, Definitions 3.6 and 3.7]. We omit the proof here as it follows from that of Theorem 1.1 up to minor adaptations. Notice in particular that [13, Lemma A1] provides a higher-dimensional version of (3.7).

We conclude this section by characterizing two special subsets of $\mathcal{B}$ (see (3.3)), which will be useful in the following. Using (2.6), it can be checked that

$$\mathcal{B}\cap W^{1,1}(\Omega;\mathbb{R}^2)=\big\{u\in W^{1,1}(\Omega;\mathbb{R}^2)\colon u(x)=R(x_2)x+\psi(x_2)\text{ for a.e. }x\in\Omega,\\ \text{with }R\in W^{1,1}(a_\Omega,b_\Omega;SO(2))\text{ and }\psi\in W^{1,1}(a_\Omega,b_\Omega;\mathbb{R}^2)\big\} \tag{3.19}$$

and

$$\mathcal{B}\cap SBV(\Omega;\mathbb{R}^2)=\big\{u\in SBV(\Omega;\mathbb{R}^2)\colon u(x)=R(x_2)x+\psi(x_2)\text{ for a.e. }x\in\Omega,\\ \text{with }R\in SBV(a_\Omega,b_\Omega;SO(2))\text{ and }\psi\in SBV(a_\Omega,b_\Omega;\mathbb{R}^2)\big\}. \tag{3.20}$$

By definition, and accounting for the fact that R takes values in $SO(2)$, the jump set of $u\in\mathcal{B}\cap SBV(\Omega;\mathbb{R}^2)$ is related to the jump sets of R and ψ via

$$J_u=[(c_\Omega,d_\Omega)\times(J_R\cup J_\psi)]\cap\Omega,$$

cf. (1.2).

4. Asymptotic behavior of admissible layered deformations

In this section, we prove Theorem 1.3, which characterizes the asymptotic behavior of deformations of bilayered materials that coincide with rigid body rotations on the stiffer layers, and are subject to a single slip constraint on the softer layers. The latter is described with the help of the set

$$\begin{aligned}\mathcal{M}_{e_1}&=\{F\in\mathbb{R}^{2\times 2}\colon \det F=1\text{ and }|Fe_1|=1\}\\&=\{F\in\mathbb{R}^{2\times 2}\colon F=R(\mathbb{I}+\gamma e_1\otimes e_2)\text{ with }R\in SO(2)\text{ and }\gamma\in\mathbb{R}\}.\end{aligned} \tag{4.1}$$

As in the previous section, we consider $\Omega=(0,1)\times(-1,1)$ for simplicity. The results for general x_1-connected domains follow as in the proof of Theorem 1.1.

Using the representations of $\mathcal{M}_{e_1}$ in (4.1) and recalling the sets $\mathcal{B}_\varepsilon$ introduced in (3.1), the sets of admissible layered deformations defined in (1.6) admit the equivalent representations

$$\begin{aligned}\mathcal{A}_\varepsilon&=\mathcal{B}_\varepsilon\cap\{u\in W^{1,1}(\Omega;\mathbb{R}^2):\nabla u\in\mathcal{M}_{e_1}\text{ a.e. in }\Omega\}\\&=\{u\in W^{1,1}(\Omega;\mathbb{R}^2):\nabla u=R(\mathbb{I}+\gamma e_1\otimes e_2)\text{ with }R\in L^\infty(\Omega;SO(2))\text{ and}\\&\qquad\gamma\in L^1(\Omega)\text{ such that }\gamma=0\text{ in }\varepsilon Y_{\mathrm{rig}}\cap\Omega\}.\end{aligned} \tag{4.2}$$

In the sequel, according to the context, we will always adopt the most convenient representation.

In analogy with $\mathcal{B}_0$ defined in (3.2), we introduce the set

$$\mathcal{A}_0 := \{u \in BV(\Omega;\mathbb{R}^2) : \text{there exists } (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \text{ with } u_\varepsilon \in \mathcal{A}_\varepsilon \text{ for all } \varepsilon \\ \text{such that } u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } BV(\Omega;\mathbb{R}^2)\} \tag{4.3}$$

of *asymptotically admissible deformations*. We aim at characterizing $\mathcal{A}_0$, or suitable subclasses thereof, in terms of the set $\mathcal{A}$ introduced in (1.11). Note that

$$\mathcal{A} = \mathcal{B} \cap \{u \in BV(\Omega;\mathbb{R}^2) : \det \nabla u = 1 \text{ a.e. in } \Omega\}, \tag{4.4}$$

where $\mathcal{B}$ is given by (3.3). Moreover, recalling the notation for the distributional derivative of one-dimensional BV-functions discussed in Section 2.2, we can equivalently express $\mathcal{A}$ as follows.

Proposition 4.1. *Let $\Omega = (0,1)\times(-1,1)$. Then, $\mathcal{A}$ from (1.11) admits these two alternative representations:*

$$\mathcal{A} = \{u \in BV(\Omega;\mathbb{R}^2)\colon \nabla u(x) = R(x_2)(\mathbb{I} + \gamma(x_2)e_1\otimes e_2) \text{ for a.e. } x \in \Omega, \text{ with} \\ R \in BV(-1,1;SO(2)),\ \gamma \in L^1(-1,1), \text{ and } (D^s u)e_1 = 0\} \tag{4.5}$$

and

$$\mathcal{A} = \{u \in BV(\Omega;\mathbb{R}^2)\colon u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega, \text{ with } R \in BV(-1,1;SO(2)) \\ \text{and } \psi \in BV(-1,1;\mathbb{R}^2) \text{ such that } \psi'\cdot Re_2 = 0 \text{ and } R' = 0 \text{ a.e. in } (-1,1)\}. \tag{4.6}$$

Proof. Let $\tilde{A}$ and $\hat{\mathcal{A}}$ denote the sets on the right-hand side of (4.5) and (4.6), respectively. We will show that $\mathcal{A} \subset \tilde{\mathcal{A}} \cap \hat{\mathcal{A}}$, $\hat{\mathcal{A}} \subset \mathcal{A}$, and $\tilde{\mathcal{A}} \subset \hat{\mathcal{A}}$, from which (4.5) and (4.6) follow.

We start by proving that $\mathcal{A} \subset \tilde{\mathcal{A}} \cap \hat{\mathcal{A}}$. Fix $u \in \mathcal{A}$. Due to (2.6), we have $(D^s u)e_1 = 0$ and

$$\nabla u = R + (R'x + \psi')\otimes e_2 = R(\mathbb{I} + R^T(R'x+\psi')\otimes e_2). \tag{4.7}$$

We first observe that the condition $\det\nabla u = 1$ becomes $1 + R^T(R'x+\psi')\cdot e_2 = 1$ or, equivalently, $(R'x+\psi')\cdot Re_2 = 0$. This condition, together with the independence of R, R', and ψ' on x_1, yields

$$R'e_1 \cdot Re_2 = 0 \qquad \text{and} \qquad (x_2R'e_2 + \psi')\cdot Re_2 = 0. \tag{4.8}$$

Let $\theta \in BV(-1,1;[-\pi,\pi])$ be as in (2.7). Then, the first condition in (4.8) gives $\theta' = 0$; consequently, also $R' = 0$. Thus, the second equation in (4.8) becomes $\psi'\cdot Re_2 = 0$, which shows that $u \in \hat{\mathcal{A}}$. Moreover, $\psi'\cdot Re_2 = 0$ is equivalent to $R^T\psi'\cdot e_2 = 0$; hence, $u \in \tilde{\mathcal{A}}$ with $\gamma := Re_1\cdot\psi'$. Thus, $\mathcal{A} \subset \tilde{\mathcal{A}} \cap \hat{\mathcal{A}}$.

Next, we observe that if $u \in \hat{\mathcal{A}}$, then, using (4.7), we have

$$\det\nabla u = 1 + R^T(R'x+\psi')\cdot e_2 = 1 + R^T\psi'\cdot e_2 = 1 + \psi'\cdot Re_2 = 1.$$

Hence, $u \in \mathcal{A}$, which shows that $\hat{\mathcal{A}} \subset \mathcal{A}$.

Finally, we prove that $\tilde{\mathcal{A}} \subset \hat{\mathcal{A}}$. Let $u \in \tilde{\mathcal{A}}$. Then, $(Du)e_1 = (\nabla u)e_1\mathcal{L}^2\lfloor\Omega + (D^su)e_1 = Re_1\mathcal{L}^2\lfloor\Omega$. By this identity and the Du Bois-Reymond lemma (see [32], for instance), we can find $\phi \in BV(-1,1;\mathbb{R}^2)$ such that

$$u(x) = R(x_2)x_1e_1 + \phi(x_2).$$

In particular, $\nabla u(x) = R(x_2)e_1\otimes e_1 + (R'(x_2)x_1e_1 + \phi'(x_2))\otimes e_2$. Consequently, using the expression for ∇u given by the definition of $\tilde{\mathcal{A}}$, together with the independence of R, R', γ, and ϕ' on x_1, we conclude that

$$R' = 0 \quad \text{and} \quad \phi' = Re_2 + \gamma Re_1.$$

Finally, set $\psi(x_2) := \phi(x_2) - R(x_2)x_2e_2$ for $x_2 \in (-1,1)$. Then, we have $\psi \in BV(-1,1;\mathbb{R}^2)$, which satisfies $\psi'\cdot Re_2 = \gamma Re_1\cdot Re_2 = 0$, because $R \in SO(2)$ in $(-1,1)$, and also $u(x) = R(x_2)x + \psi(x_2)$. Thus, $u \in \hat{\mathcal{A}}$, which implies $\tilde{\mathcal{A}} \subset \hat{\mathcal{A}}$. □

The following lemma on weak continuity of Jacobian determinants for gradients in $W^{1,1}(\Omega;\mathbb{R}^2)$ with suitable additional properties will be instrumental in the proof of the inclusion $\mathcal{A}_0 \subset \mathcal{A}$.

Lemma 4.2 (Weak continuity properties of Jacobian determinants). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and let $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ be a uniformly bounded sequence satisfying* $\det \nabla u_\varepsilon = 1$ *a.e. in Ω for all ε and*

$$\|\partial_1 u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^2)} \leq C, \tag{4.9}$$

where C is a positive constant independent of ε. If $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ for some $u \in BV(\Omega;\mathbb{R}^2)$, then $\det \nabla u = 1$ *a.e. in Ω.*

Proof. The claim in Lemma 4.2 would be an immediate consequence of [26, Theorem 2] if in place of (4.9), we required

$$(\operatorname{adj} \nabla u_\varepsilon)_\varepsilon \subset L^2(\Omega;\mathbb{R}^{2\times 2}), \tag{4.10}$$

which, because of the structure of the adjoint matrix in this two-dimensional setting, is equivalent to $\nabla u_\varepsilon \in L^2(\Omega;\mathbb{R}^{2\times 2})$ for all ε. Even though we are not assuming this here, it is still possible to validate the arguments of [26, Proof of Theorem 2] in our context, as we detail next.

Since $|\operatorname{adj} \nabla u_\varepsilon| = |\nabla u_\varepsilon|$, it can be checked that in order to mimic the proof of [26, Theorem 2] with $N = 2$, we are only left to prove the following: If $(\varphi_j)_{j\in\mathbb{N}}$ is a sequence of standard mollifiers and Ω' is an arbitrary open set compactly contained in Ω, then $(\det \nabla u_{\varepsilon,j})_{j\in\mathbb{N}}$ converges to $\det \nabla u_\varepsilon$ in $L^1(\Omega')$ as $j \to \infty$ for all ε, where $u_{\varepsilon,j} := \varphi_j * u_\varepsilon$.

In Step 4 of the proof of [26, Theorem 2], this convergence is a consequence of the Vitali–Lebesgue lemma using (4.10), the bound $|\det A| \leq |\operatorname{adj} A|^2$ for all $A \in \mathbb{R}^{2\times 2}$ (see [26, (7)]), and well-known properties of mollifiers.

Here, similar arguments can be invoked, but instead of the estimate $|\det A| \leq |\operatorname{adj} A|^2$ for $A \in \mathbb{R}^{2\times 2}$, we use the fact that (4.9) yields

$$|\det \nabla u_{\varepsilon,j}| = |(\partial_1 u_{\varepsilon,j})^\perp \cdot \partial_2 u_{\varepsilon,j}| \leq C|\partial_2 u_{\varepsilon,j}| \leq C|\nabla u_{\varepsilon,j}|$$

a.e. in Ω. Hence, since $u_{\varepsilon,j} \to u_\varepsilon$ in $W^{1,1}(\Omega';\mathbb{R}^2)$ and pointwise a.e. in Ω as $j \to \infty$, we conclude that $(\det \nabla u_{\varepsilon,j})_{j\in\mathbb{N}}$ converges to $\det \nabla u_\varepsilon$ in $L^1(\Omega')$ as $j \to \infty$ for all ε by the Vitali–Lebesgue lemma. □

We obtain from the following proposition that weak* limits of sequences in $\mathcal{A}_\varepsilon$ belong to $\mathcal{A}$.

Proposition 4.3 (Asymptotic behavior of sequences in $\mathcal{A}_\varepsilon$). *Let $\Omega = (0,1) \times (-1,1)$. Then,*

$$\mathcal{A}_0 \subset \mathcal{A}, \tag{4.11}$$

where $\mathcal{A}_0$ and $\mathcal{A}$ are the sets introduced in (4.3) and (1.11), respectively.

Proof. The statement follows from the inclusion $\mathcal{A}_\varepsilon \subset \mathcal{B}_\varepsilon$ (see (4.2)) and the identity (4.4) in conjunction with Proposition 3.1 and Lemma 4.2, observing that the condition $\nabla u_\varepsilon \in \mathcal{M}_{e_1}$ a.e. in Ω guarantees $|\partial_1 u_\varepsilon| = |\nabla u_\varepsilon e_1| = 1$ a.e. in Ω, and hence $\|\partial_1 u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^2)} = 1$ for any ε. □

The question whether the set $\mathcal{A}$ can be further identified as limiting set for sequences in $\mathcal{A}_\varepsilon$, namely, whether the equality $\mathcal{A}_0 = \mathcal{A}$ is true, cannot be answered at this point. However, as stated in Theorem 1.3, the inclusions $\mathcal{A}_0 \supset \mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)$ and $\mathcal{A}_0 \supset \mathcal{A}^\parallel$ hold. Before proving these inclusions, we discuss a further characterization of some special subsets of $\mathcal{A}$.

Remark 4.4 (Structure of subsets of $\mathcal{A}$). Similarly to (3.19) and (3.20), using fine properties of one-dimensional BV functions, the sets $\mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2)$, $\mathcal{A} \cap SBV(\Omega;\mathbb{R}^2)$, and $\mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)$ can be characterized as follows.

(a) In view of (2.6) and (4.6), one observes that

$$\begin{aligned}
\mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2) = \{u \in W^{1,1}(\Omega;\mathbb{R}^2)\colon\ & u(x) = Rx + \theta(x_2)Re_1 + c \text{ for a.e. } x \in \Omega,\\
& \text{with } R \in SO(2),\ \theta \in W^{1,1}(-1,1),\ c \in \mathbb{R}^2\}\\
= \{u \in W^{1,1}(\Omega;\mathbb{R}^2)\colon\ & \nabla u(x) = R(\mathbb{I} + \gamma(x_2)e_1 \otimes e_2) \text{ for a.e. } x \in \Omega,\\
& \text{with } R \in SO(2),\ \gamma \in L^1(-1,1)\}.
\end{aligned}$$

Additionally, as a consequence of the construction of the recovery sequence in the Γ-convergence homogenisation result [13, Theorem 1.1], we also know that

$$\mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2) = \{u \in W^{1,1}(\Omega;\mathbb{R}^2)\colon \text{ there exists } (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \text{ with } u_\varepsilon \in \mathcal{A}_\varepsilon \text{ for all } \varepsilon$$

such that $u_\varepsilon \rightharpoonup u$ in $W^{1,1}(\Omega;\mathbb{R}^2)\}$.

(b) Using (2.6) and (4.6) once more, we have

$$\begin{aligned}\mathcal{A}\cap SBV(\Omega;\mathbb{R}^2) = \{u \in SBV(\Omega;\mathbb{R}^2) \colon\, & u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega,\\ & \text{with } R \in SBV(-1,1;SO(2)) \text{ and } \psi \in SBV(-1,1;\mathbb{R}^2)\\ & \text{such that } R' = 0 \text{ and } \psi' \cdot Re_2 = 0 \text{ a.e. in } (-1,1)\}.\end{aligned}$$

Note that both J_R and J_ψ are given by an at most countable union of points in $(-1,1)$, which implies that J_u consists of at most countably many segments parallel to e_1. It is not possible to conclude that the functions R are piecewise constant according to [2, Definition 4.21], as we have, a priori, no control on $\mathcal{H}^0(J_R)$ (cf. [2, Example 4.24]).

(c) With (b) and [2, Theorem 4.23], and recalling (2.8), it follows that

$$\begin{aligned}\mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2) = \{u \in SBV_\infty(\Omega;\mathbb{R}^2) \colon\, & u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega,\\ & \text{with } R \in PC(-1,1;SO(2)) \text{ and } \psi \in SBV_\infty(-1,1;\mathbb{R}^2)\\ & \text{such that } \psi' \cdot Re_2 = 0 \text{ a.e. in } (-1,1)\}.\end{aligned}$$

Here, both J_R and J_ψ are finite sets of points in $(-1,1)$, and J_u is given by a finite union of segments parallel to e_1. Alternatively, one can express $\mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2)$ with the help of a Caccioppoli partition of Ω into finitely many horizontal strips; precisely,

$$\begin{aligned}\mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2) = \{u \in SBV_\infty(\Omega;\mathbb{R}^2) \colon\, & \nabla u|_{E_i} = R_i(\mathbb{I} + \gamma_i e_1 \otimes e_2), \text{ with } \{E_i\}_{i=1}^n \text{ a partition of } \Omega\\ & \text{such that } E_i = (\mathbb{R}\times I_i)\cap\Omega \text{ with } I_i \subset (-1,1) \text{ for } i = 1,\dots,n,\\ & R_i \in SO(2) \text{ and } \gamma_i \in L^1(E_i) \text{ with } \partial_1\gamma_i = 0 \text{ for } i = 1,\dots,n\}.\end{aligned}$$

In the following lemma, we construct an admissible piecewise affine approximation for basic limit deformations in $\mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2)$ with a non-trivial jump along the horizontal line at $x_2 = 0$. Based on this construction, we will then establish the inclusion $\mathcal{A}_0 \supset \mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2)$ in Proposition 4.7 below.

Lemma 4.5 (Approximation of maps in $\mathcal{A}\cap SBV_\infty$ with a single jump). *Let $\Omega = (0,1)\times(-1,1)$, and let $u \in \mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2)$ be such that $u(x) = R(x_2)x + \psi(x_2)$ for a.e. $x \in \Omega$, where*

$$R(t) := \begin{cases} R^+ & \text{if } t \in [0,1)\\ R^- & \text{if } t \in (-1,0)\end{cases} \quad \text{and} \quad \psi(t) := \begin{cases} \psi^+ & \text{if } t \in [0,1)\\ \psi^- & \text{if } t \in (-1,0)\end{cases} \qquad \text{for } t \in (-1,1),$$

with some $R^\pm \in SO(2)$ and $\psi^\pm \in \mathbb{R}^2$. Then, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ with $\int_\Omega u_\varepsilon \,\mathrm{d}x = \int_\Omega u \,\mathrm{d}x$ and $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε, and such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$.

Proof. We start by observing that for u as in the statement of the lemma, there holds

$$Du = R\mathcal{L}^2\lfloor\Omega + [(R^+ - R^-)e_1x_1 + (\psi^+ - \psi^-)] \otimes e_2\mathcal{H}^1\lfloor\big((0,1)\times\{0\}\big). \tag{4.12}$$

Let $S \in SO(2)$ be such that *(i)* $S \neq R^\pm$; *(ii)* Se_1 and R^+e_1 are linearly independent; *(iii)* $\theta^\pm \in (-\pi,\pi)\setminus\{0\}$ is the rotation angle of $S^TR^\pm$, cf. (2.7). Due to *(ii)*, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\psi^+ - \psi^- = \alpha R^+e_1 + \beta Se_1. \tag{4.13}$$

For each $\varepsilon > 0$, set

$$\gamma_\varepsilon^+ := \frac{4\alpha}{\varepsilon\lambda},\ \ \gamma_\varepsilon^- := \frac{4\beta}{\varepsilon\lambda},\ \ \mu_\varepsilon^\pm := \pm\frac{4}{\varepsilon\lambda} + \tan\Big(\frac{\theta^\pm}{2}\Big),\ \ \tilde\mu_\varepsilon^\pm := \pm\frac{4}{\varepsilon\lambda} - \tan\Big(\frac{\theta^\pm}{2}\Big), \tag{4.14}$$

and let $V_\varepsilon \in L^1(\Omega;\mathbb{R}^{2\times 2})$ be the function defined by

$$V_\varepsilon(x) = \begin{cases} R^+ & \text{if } x \in (0,1)\times(\varepsilon\lambda, 1),\\ R^+(\mathbb{I}+\gamma_\varepsilon^+ e_1\otimes e_2) & \text{if } x \in (0,1)\times(\frac{3\varepsilon\lambda}{4}, \varepsilon\lambda),\\ R^+(\mathbb{I}+\mu_\varepsilon^+ e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (-\frac{\varepsilon\lambda}{4}x_1 + \frac{3\varepsilon\lambda}{4}, \frac{3\varepsilon\lambda}{4}),\\ S(\mathbb{I}+\tilde{\mu}_\varepsilon^+ e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (\frac{\varepsilon\lambda}{2}, -\frac{\varepsilon\lambda}{4}x_1 + \frac{3\varepsilon\lambda}{4}),\\ S(\mathbb{I}+\gamma_\varepsilon^- e_1\otimes e_2) & \text{if } x \in (0,1)\times(\frac{\varepsilon\lambda}{4}, \frac{\varepsilon\lambda}{2}),\\ S(\mathbb{I}+\tilde{\mu}_\varepsilon^- e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (\frac{\varepsilon\lambda}{4}x_1, \frac{\varepsilon\lambda}{4}),\\ R^-(\mathbb{I}+\mu_\varepsilon^- e_1\otimes e_2) & \text{if } x \in (0,1) \text{ and } x_2 \in (0, \frac{\varepsilon\lambda}{4}x_1),\\ R^- & \text{if } x \in (0,1)\times(-1,0), \end{cases} \tag{4.15}$$

see Figure 3.

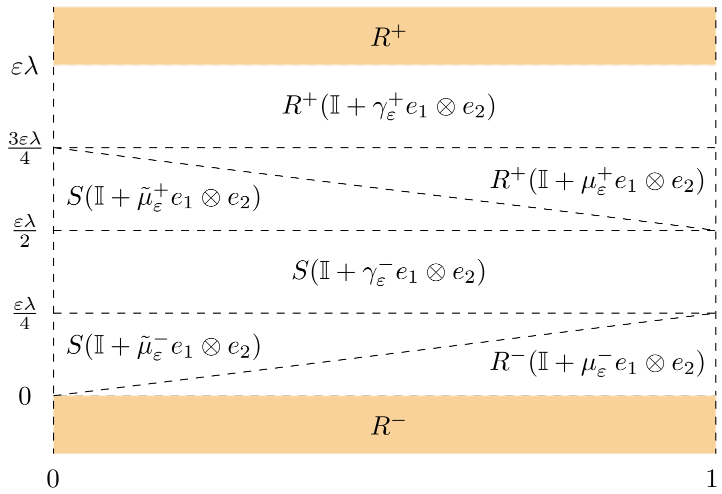


FIGURE 3. Construction of V_ε.

By construction, each function V_ε takes values only in $\mathcal{M}_{e_1}$, and its piecewise definition is chosen such that neighboring matrices in Figure 3 are rank-one-connected along their separating lines according to [13, Lemma 3.1]. Hence, there exists a Lipschitz function $u_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that $\nabla u_\varepsilon = V_\varepsilon$. By adding a suitable constant, we may assume that $\int_\Omega u_\varepsilon \,\mathrm{d}x = \int_\Omega u \,\mathrm{d}x$. In view of the Poincaré–Wirtinger inequality and (4.15), $(u_\varepsilon)_\varepsilon$ is a uniformly bounded sequence in $W^{1,1}(\Omega;\mathbb{R}^2)$ satisfying $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε (cf. (4.2)).

To prove that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$, it suffices to show that

$$Du_\varepsilon \overset{*}{\rightharpoonup} Du \text{ in } \mathcal{M}(\Omega;\mathbb{R}^{2\times 2}), \tag{4.16}$$

or, equivalently, in view of (4.12), that for every $\varphi \in C_0(\Omega;\mathbb{R}^2)$,

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \int_\Omega R(x_2)\varphi(x)\,\mathrm{d}x + \int_0^1 [(R^+ - R^-)e_1 x_1 + (\psi^+ - \psi^-)]\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1. \tag{4.17}$$

Clearly,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{(0,1)\times[(-1,0)\cup(\varepsilon\lambda,1)]}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{(0,1)\times[(-1,0)\cup(\varepsilon\lambda,1)]} R(x_2)\varphi(x)\,\mathrm{d}x\\ &= \int_\Omega R(x_2)\varphi(x)\,\mathrm{d}x. \end{aligned}\tag{4.18}$$

Moreover, using (4.14), a change of variables, and Lebegue's dominated convergence theorem together with the continuity and boundedness of φ, we have

$$\lim_{\varepsilon\to 0}\int_{(0,1)\times(0,\frac{\varepsilon\lambda}{4}x_1)}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\int_0^1\int_0^{\frac{\varepsilon\lambda}{4}x_1} R^-\big(\mathbb{I}+\tan\big(\tfrac{\theta^-}{2}\big)e_1\otimes e_2-\tfrac{4}{\varepsilon\lambda}e_1\otimes e_2\big)\varphi(x)\,\mathrm{d}x_2\,\mathrm{d}x_1\\
&= \lim_{\varepsilon\to 0}\int_0^1\int_0^{x_1} R^-\big(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\tfrac{\varepsilon\lambda}{4}\tan\big(\tfrac{\theta^-}{2}\big)e_1\otimes e_2-e_1\otimes e_2\big)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1\\
&= -\int_0^1\int_0^{x_1} R^-e_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}z\,\mathrm{d}x_1 = -\int_0^1 x_1R^-e_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1. \qquad (4.19)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{(0,1)\times(\frac{\varepsilon\lambda}{4}x_1,\frac{\varepsilon\lambda}{4})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x\\
&\quad= \lim_{\varepsilon\to 0}\int_0^1\int_{x_1}^1 S\big(\tfrac{\varepsilon\lambda}{4}\mathbb{I}-\tfrac{\varepsilon\lambda}{4}\tan\big(\tfrac{\theta^-}{2}\big)e_1\otimes e_2-e_1\otimes e_2\big)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1\\
&\quad= \int_0^1 (x_1-1)Se_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1, \qquad (4.20)\\
&\lim_{\varepsilon\to 0}\int_{(0,1)\times(\frac{\varepsilon\lambda}{4},\frac{\varepsilon\lambda}{2})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x\\
&\quad= \lim_{\varepsilon\to 0}\int_0^1\int_1^2 S\big(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\beta e_1\otimes e_2\big)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1 = \int_0^1 \beta Se_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1, \qquad (4.21)\\
&\lim_{\varepsilon\to 0}\int_{(0,1)\times(\frac{\varepsilon\lambda}{2},-\frac{\varepsilon\lambda}{4}x_1+\frac{3\varepsilon\lambda}{4})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x\\
&\quad= \lim_{\varepsilon\to 0}\int_0^1\int_2^{3-x_1} S\big(\tfrac{\varepsilon\lambda}{4}\mathbb{I}-\tfrac{\varepsilon\lambda}{4}\tan\big(\tfrac{\theta^+}{2}\big)e_1\otimes e_2+e_1\otimes e_2\big)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1\\
&\quad= \int_0^1 (1-x_1)Se_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1, \qquad (4.22)\\
&\lim_{\varepsilon\to 0}\int_{(0,1)\times(-\frac{\varepsilon\lambda}{4}x_1+\frac{3\varepsilon\lambda}{4},\frac{3\varepsilon\lambda}{4})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x\\
&\quad= \lim_{\varepsilon\to 0}\int_0^1\int_{3-x_1}^3 R^+\big(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\tfrac{\varepsilon\lambda}{4}\tan\big(\tfrac{\theta^+}{2}\big)e_1\otimes e_2+e_1\otimes e_2\big)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1\\
&\quad= \int_0^1 x_1R^+e_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1, \qquad (4.23)
\end{aligned}
$$

and

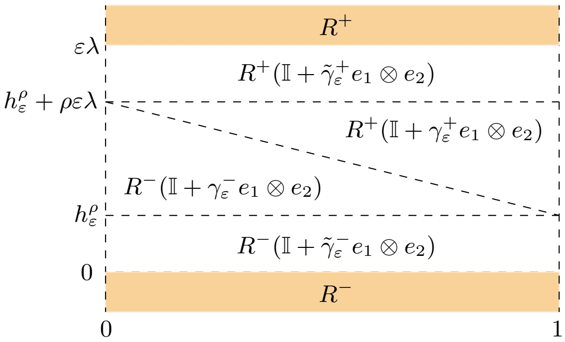
$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{(0,1)\times(\frac{3\varepsilon\lambda}{4},\varepsilon\lambda)}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x\\
&\quad= \lim_{\varepsilon\to 0}\int_0^1\int_3^4 R^+\big(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\alpha e_1\otimes e_2\big)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1 = \int_0^1 \alpha R^+e_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1. \qquad (4.24)
\end{aligned}
$$

Combining (4.18)–(4.24) and (4.13), we finally obtain (4.17). □

Remark 4.6 (On the construction in Lemma 4.5). Notice that the main idea of the construction in the proof of Lemma 4.5 for dealing with jumps is to use piecewise affine functions that are as simple as possible to accommodate them. Since triple junctions where two of the three angles add up to π are not compatible (compare with [13, Lemma 3.1]), we work with inclined interfaces that stretch over the full width of Ω.

Let $u \in \mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2)$ be as in Lemma 4.5, and assume that either $R^+\neq \pm R^-$ or $R^+=R^-$. In these cases, we can simplify the construction of $(u_\varepsilon)_\varepsilon$ in the previous proof. We focus here on stating the counterparts of Figure 3 and (4.14), and omit the detailed calculations, which are very similar to (4.18)–(4.24). Note further that these constructions are not just simpler, but also energetically more favorable, see Remark 5.2 below for more details.

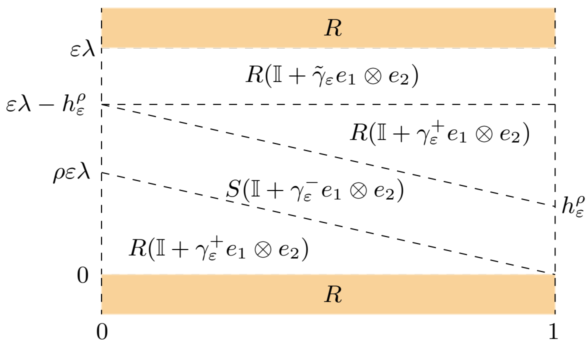
(i) If $R^+\neq \pm R^-$, we may replace the construction depicted in Figure 3 by:



$$\psi^+ - \psi^- = \alpha R^+ e_1 + \beta R^- e_1$$
$$\theta \in (-\pi, \pi) \setminus \{0\} \text{ rotation angle of } (R^-)^T R^+$$
$$\rho \in (0,1), \ \ h_\varepsilon^\rho := \tfrac{\varepsilon\lambda - \rho\varepsilon\lambda}{2}$$
$$\gamma_\varepsilon^+ := \tfrac{1}{\rho\varepsilon\lambda} + \tan(\tfrac{\theta}{2}), \ \ \gamma_\varepsilon^- := \tfrac{1}{\rho\varepsilon\lambda} - \tan(\tfrac{\theta}{2})$$
$$\tilde{\gamma}_\varepsilon^+ \text{ satisfies } \alpha = \lim_{\varepsilon\to 0} \tilde{\gamma}_\varepsilon^+ (\varepsilon\lambda - h_\varepsilon^\rho - \rho\varepsilon\lambda)$$
$$\tilde{\gamma}_\varepsilon^- \text{ satisfies } \beta - 1 = \lim_{\varepsilon\to 0} \tilde{\gamma}_\varepsilon^- h_\varepsilon^\rho$$

FIGURE 4. Alternative construction of V_ε if $R^+ \neq \pm R^-$.

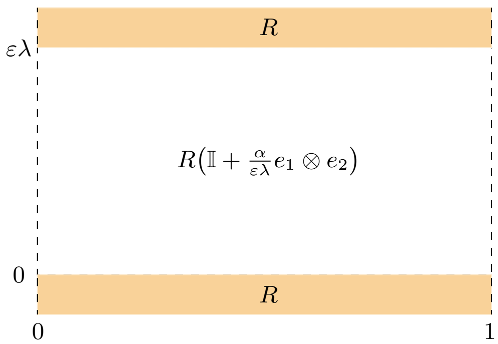
(ii) If R is constant, i.e., $R^+ = R^-$, and $\psi^+ - \psi^-$ is not parallel to Re_1, the construction in Figure 3 can be replaced by:



$$S \in SO(2)\colon Re_1 \text{ and } Se_1 \text{ are linearly independent}$$
$$\psi^+ - \psi^- = \alpha Re_1 + \beta Se_1, \ \ \beta \neq 0, \ \ \iota := \operatorname{sign}(\beta)$$
$$\theta \in (-\pi, \pi) \setminus \{0\} \text{ rotation angle of } R^T S$$
$$\rho := \tfrac{\iota}{2\beta + \iota} \in (0,1), \ \ h_\varepsilon^\rho := \tfrac{\varepsilon\lambda - \rho\varepsilon\lambda}{2}$$
$$\gamma_\varepsilon^+ := \iota \tfrac{1}{\rho\varepsilon\lambda} + \tan(\tfrac{\theta}{2}), \ \ \gamma_\varepsilon^- := \iota \tfrac{1}{\rho\varepsilon\lambda} - \tan(\tfrac{\theta}{2})$$
$$\tilde{\gamma}_\varepsilon \text{ satisfies } \alpha - \iota = \lim_{\varepsilon\to 0} \tilde{\gamma}_\varepsilon h_\varepsilon^\rho$$

FIGURE 5. Alternative construction of V_ε if R is constant and $\psi^+ - \psi^-$ is not parallel to Re_1.

(iii) If R is constant, i.e., $R^+ = R^-$, and $\psi^+ - \psi^-$ is parallel to Re_1, then we can use the following construction in place of Figure 3:



$$\psi^+ - \psi^- = \alpha Re_1$$
$$\alpha = \iota|\psi^+ - \psi^-|, \ \ \iota := \operatorname{sign}((\psi^+ - \psi^-) \cdot Re_1)$$

FIGURE 6. Alternative construction of V_ε if R is constant and $\psi^+ - \psi^-$ is parallel to Re_1.

Note that in case (i), the slope ρ of the interfaces can attain any value between 0 and 1, while in (ii), ρ is determined by the value of β. In terms of the energies, the construction in case (iii) provides an optimal approximation, which will be detailed in Section 6.

We proceed by extending Lemma 4.5 to arbitrary functions $u \in \mathcal{A} \cap SBV_\infty(\Omega; \mathbb{R}^2)$.

Proposition 4.7. *Let* $\Omega = (0,1)\times(-1,1)$. *Then, for every* $u \in \mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2)$, *there exists a sequence* $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ *with* $\int_\Omega u_\varepsilon \,\mathrm{d}x = \int_\Omega u \,\mathrm{d}x$ *and* $u_\varepsilon \in \mathcal{A}_\varepsilon$ *for all* ε, *and such that* $u_\varepsilon \overset{*}{\rightharpoonup} u$ *in* $BV(\Omega;\mathbb{R}^2)$ *or, in other words,*

$$\mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2)\subset \mathcal{A}_0,$$

cf. (4.3).

Proof. In view of Remark 4.4 (c), it holds that $J_u = \bigcup_{i=1}^{\ell}(0,1)\times\{a_i\}$ for some $\ell\in\mathbb{N}$ and $a_i\in(-1,1)$ with $a_1<a_2<\cdots<a_\ell$, and setting $a_0 := -1$ and $a_{\ell+1} := 1$, gives

$$\begin{aligned} Du = {}&\sum_{i=0}^{\ell} R_i(\mathbb{I}+\gamma e_1\otimes e_2)\mathcal{L}^2\lfloor\big((0,1)\times(a_i,a_{i+1})\big)\\ &+\sum_{i=1}^{\ell}[(R_i-R_{i-1})x_1e_1+(R_ia_ie_2+\psi_i^+-R_{i-1}a_ie_2-\psi_i^-)]\otimes e_2\mathcal{H}^1\lfloor\big((0,1)\times\{a_i\}\big), \end{aligned} \tag{4.25}$$

where $\gamma\in L^1(-1,1)$, and $R_i\in SO(2)$ and $\psi_i\in\mathbb{R}^2$ for $i=0,...,\ell$.

We now perform a similar construction as in Lemma 4.5 in a convenient softer layer *near* each a_i, accounting for the possibility that one or more of the jump lines may not intersect $\varepsilon Y_{\mathrm{soft}}\cap\Omega$, and replacing R^+ by R_i, R^- by R_{i-1}, ψ^+ by $R_ia_ie_2+\psi_i^+$, and ψ^- by $R_{i-1}a_ie_2+\psi_i^-$.

To be precise, fix $\varepsilon>0$ and $i\in\{1,...,\ell\}$. Let $S_i\in SO(2)$ be such that *(i)* $S_i\notin\{R_{i-1},R_i\}$; *(ii)* S_ie_1 and R_ie_1 are linearly independent; *(iii)* θ_i^-, $\theta_i^+\in(-\pi,\pi)\setminus\{0\}$ are the rotation angles of $S_i^TR_{i-1}$ and $S_i^TR_i$, respectively. By *(ii)*, there exist α_i, $\beta_i\in\mathbb{R}$ such that

$$R_ia_ie_2+\psi_i^+-R_{i-1}a_ie_2-\psi_i^- = \alpha_iR_ie_1+\beta_iS_ie_1. \tag{4.26}$$

Moreover, we set

$$\gamma_{\varepsilon,i}^+ := \frac{4\alpha_i}{\varepsilon\lambda},\;\; \gamma_{\varepsilon,i} := \frac{4\beta_i}{\varepsilon\lambda},\;\; \mu_{\varepsilon,i}^\pm := \pm\frac{4}{\varepsilon\lambda}+\tan\Big(\frac{\theta_i^\pm}{2}\Big),\;\; \tilde\mu_{\varepsilon,i}^\pm := \pm\frac{4}{\varepsilon\lambda}-\tan\Big(\frac{\theta_i^\pm}{2}\Big),$$

and let $\kappa_\varepsilon^i\in\mathbb{Z}$ be the unique integer such that $a_i\in\varepsilon[\kappa_\varepsilon^i,\kappa_\varepsilon^i+1)$. Observing that $a_i\neq a_j$ for $i,j\in\{1,\dots,\ell\}$ with $i\neq j$ and $a_i\in(-1,1)$ for all $i\in\{1,\dots,\ell\}$, we may assume that the sets $\{\varepsilon[\kappa_\varepsilon^i,\kappa_\varepsilon^i+1)\}_{i=1,\dots,\ell}$ are pairwise disjoint, and that $\bigcup_{i=1}^{\ell}\varepsilon[\kappa_\varepsilon^i,\kappa_\varepsilon^i+1]\subset(-1,1)$ (this is true for sufficiently small $\varepsilon>0$). Finally, with $\kappa_\varepsilon^0 := -\lambda-\frac{1}{\varepsilon}$ and $\kappa_\varepsilon^{\ell+1} := \frac{1}{\varepsilon}$, let $V_\varepsilon\in L^1(\Omega;\mathbb{R}^{2\times2})$ be the function defined by

$$V_\varepsilon(x) := \begin{cases} R_i(\mathbb{I}+\frac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}}e_1\otimes e_2) & \text{if } x\in(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i,\varepsilon\kappa_\varepsilon^{i+1}) \text{ for some } i\in\{0,..,\ell\},\\ R_i(\mathbb{I}+\gamma_{\varepsilon,i}^+e_1\otimes e_2) & \text{if } x\in(0,1)\times(\frac{3\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i,\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i) \text{ for some } i\in\{1,..,\ell\},\\ R_i(\mathbb{I}+\mu_{\varepsilon,i}^+e_1\otimes e_2) & \text{if } x_1\in(0,1) \text{ and } x_2\in(-\frac{\varepsilon\lambda}{4}x_1+\frac{3\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i,\frac{3\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i)\\ & \text{for some } i\in\{1,..,\ell\},\\ S_i(\mathbb{I}+\tilde\mu_{\varepsilon,i}^+e_1\otimes e_2) & \text{if } x_1\in(0,1) \text{ and } x_2\in(\frac{\varepsilon\lambda}{2}+\varepsilon\kappa_\varepsilon^i,-\frac{\varepsilon\lambda}{4}x_1+\frac{3\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i)\\ & \text{for some } i\in\{1,..,\ell\},\\ S_i(\mathbb{I}+\gamma_{\varepsilon,i}e_1\otimes e_2) & \text{if } x\in(0,1)\times(\frac{\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i,\frac{\varepsilon\lambda}{2}+\varepsilon\kappa_\varepsilon^i) \text{ for some } i\in\{1,..,\ell\},\\ S_i(\mathbb{I}+\tilde\mu_{\varepsilon,i}^-e_1\otimes e_2) & \text{if } x_1\in(0,1) \text{ and } x_2\in(\frac{\varepsilon\lambda}{4}x_1+\varepsilon\kappa_\varepsilon^i,\frac{\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i) \text{ for some } i\in\{1,..,\ell\},\\ R_{i-1}(\mathbb{I}+\mu_{\varepsilon,i}^-e_1\otimes e_2) & \text{if } x_1\in(0,1) \text{ and } x_2\in(\varepsilon\kappa_\varepsilon^i,\frac{\varepsilon\lambda}{4}x_1+\varepsilon\kappa_\varepsilon^i) \text{ for some } i\in\{1,..,\ell\}. \end{cases}$$

As in the proof of Lemma 4.5, invoking [13, Lemma 3.1] on rank-one connections in $\mathcal{M}_{e_1}$, we find that V_ε is a gradient field, meaning that there is $u_\varepsilon\in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that $\nabla u_\varepsilon = V_\varepsilon$. Adding a suitable constant allows us to assume that $\int_\Omega u_\varepsilon\,\mathrm{d}x = \int_\Omega u\,\mathrm{d}x$. By construction, $(u_\varepsilon)_\varepsilon$ is a uniformly bounded sequence in $W^{1,1}(\Omega;\mathbb{R}^2)$ such that $u_\varepsilon\in\mathcal{A}_\varepsilon$ for all ε (see (4.2)). To prove that $u_\varepsilon\overset{*}{\rightharpoonup}u$ in $BV(\Omega;\mathbb{R}^2)$, it suffices to show that

$$Du_\varepsilon \overset{*}{\rightharpoonup} Du \text{ in } \mathcal{M}(\Omega;\mathbb{R}^{2\times2}). \tag{4.27}$$

The proof of (4.27) follows along the lines of (4.16). For this reason, we only highlight the main differences. First, note that the conditions $\varepsilon\kappa_\varepsilon^0 = -\varepsilon\lambda-1 = -\varepsilon\lambda+a_0$, $\varepsilon\kappa_\varepsilon^{\ell+1} = 1 = a_{\ell+1}$, and $\varepsilon\kappa_\varepsilon^i\le a_i\le\varepsilon(\kappa_\varepsilon^i+1)$ yield

$$\lim_{\varepsilon\to0}\varepsilon\kappa_\varepsilon^i = a_i \quad \text{for all } i\in\{0,...,\ell+1\}.$$

Hence, $\mathbb{1}_{(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i,\varepsilon\kappa_\varepsilon^{i+1})} \to \mathbb{1}_{(0,1)\times(a_i,a_{i+1})}$ and $\gamma\mathbb{1}_{(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i,\varepsilon\kappa_\varepsilon^{i+1})} \to \gamma\mathbb{1}_{(0,1)\times(a_i,a_{i+1})}$ in $L^1(\Omega)$ for $i\in\{0,...,\ell+1\}$. On the other hand, by the Riemann–Lebesgue lemma, we have $\mathbb{1}_{\varepsilon Y_{\rm soft}} \overset{*}{\rightharpoonup} \lambda$ in $L^\infty(\mathbb{R}^2)$; thus,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i,\varepsilon\kappa_\varepsilon^{i+1})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_\Omega R_i(\mathbb{I}+\tfrac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\rm soft}}e_1\otimes e_2)\mathbb{1}_{(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i,\varepsilon\kappa_\varepsilon^{i+1})}\varphi(x)\,\mathrm{d}x\\ &= \int_{(0,1)\times(a_i,a_{i+1})}R_i(\mathbb{I}+\gamma(x_2)e_1\otimes e_2)\varphi(x)\,\mathrm{d}x\end{aligned}$$

for all $i\in\{0,...,\ell\}$ and $\varphi\in C_0(\Omega)$. Arguing as in (4.19) with the change of variables $z=\frac{4}{\varepsilon\lambda}(x_2-\varepsilon\kappa_\varepsilon^i)$, leads to

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{(0,1)\times(\varepsilon\kappa_\varepsilon^i,\frac{\varepsilon\lambda}{4}x_1+\varepsilon\kappa_\varepsilon^i)}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x\\ &=\lim_{\varepsilon\to 0}\int_0^1\int_{\varepsilon\kappa_\varepsilon^i}^{\frac{\varepsilon\lambda}{4}x_1+\varepsilon\kappa_\varepsilon^i}R_{i-1}\big(\mathbb{I}+\tan\big(\tfrac{\theta_i^-}{2}\big)e_1\otimes e_2-\tfrac{4}{\varepsilon\lambda}e_1\otimes e_2\big)\varphi(x)\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &=\lim_{\varepsilon\to 0}\int_0^1\int_0^{x_1}R_{i-1}\big(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\tfrac{\varepsilon\lambda}{4}\tan\big(\tfrac{\theta_i^-}{2}\big)e_1\otimes e_2-e_1\otimes e_2\big)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z+\varepsilon\kappa_\varepsilon^i)\,\mathrm{d}z\,\mathrm{d}x_1\\ &=-\int_0^1\int_0^{x_1}R_{i-1}e_1\otimes e_2\varphi(x_1,a_i)\,\mathrm{d}z\,\mathrm{d}x_1=-\int_0^1R_{i-1}x_1e_1\otimes e_2\varphi(x_1,a_i)\,\mathrm{d}x_1\end{aligned}$$

for all $i\in\{1,...,\ell\}$ and $\varphi\in C_0(\Omega)$. Similarly, one can calculate the counterparts to (4.20)–(4.24) in the present setting. In view of (4.25) and (4.26), we deduce (4.27), which ends the proof. □

Remark 4.8 (On the construction in Proposition 4.7). We observe that the sequence of Lipschitz functions $(u_\varepsilon)_\varepsilon$ constructed in Proposition 4.7 to approximate a given $u\in\mathcal{A}\cap SBV_\infty(\Omega;\mathbb{R}^2)$ is such that

$$\lim_{\varepsilon\to 0}\int_\Omega|\nabla u_\varepsilon|\,\mathrm{d}x\sim|Du|(\Omega)+2\ell.$$

In other words, the asymptotic behavior of the total variation of $(u_\varepsilon)_\varepsilon$ incorporates a positive term that is proportional to the number of jumps of the limit function. This fact prevents us from bootstrapping the argument in Proposition 4.7 to generalize it to an arbitrary function in $\mathcal{A}\cap SBV(\Omega;\mathbb{R}^2)$.

An analogous statement to Proposition 4.7 holds in $\mathcal{A}^\|$.

Proposition 4.9. *Let $\Omega=(0,1)\times(-1,1)$. If $u\in\mathcal{A}^\|$, then there exists a sequence $(u_\varepsilon)_\varepsilon\subset W^{1,1}(\Omega;\mathbb{R}^2)$ such that $u_\varepsilon\in\mathcal{A}_\varepsilon$ for all ε and $u_\varepsilon\overset{*}{\rightharpoonup}u$ in $BV(\Omega;\mathbb{R}^2)$; that is,*

$$\mathcal{A}^\|\subset\mathcal{A}_0.$$

Proof. Let $u\in\mathcal{A}^\|$. Based on (1.13) and (2.4), we can split u into $u=v+w$, where

$$v(x):=Rx+\vartheta_a(x_2)Re_1+c\quad\text{and}\quad w(x):=\vartheta_s(x_2)Re_1\qquad\text{for } x\in\Omega,\tag{4.28}$$

with $R\in SO(2)$, $c\in\mathbb{R}^2$, $\vartheta_a\in W^{1,1}(-1,1)$, and $\vartheta_s\in BV(-1,1)$ such that $\vartheta_s'=0$. By construction, we have that $v\in W^{1,1}(\Omega;\mathbb{R}^2)$ with $\nabla v(x)=R(\mathbb{I}+\vartheta_a'(x_2)e_1\otimes e_2)$.

For every $\varepsilon>0$, let $v_\varepsilon\in W^{1,1}(\Omega;\mathbb{R}^2)$ be the function satisfying $\int_\Omega v_\varepsilon\,\mathrm{d}x=\int_\Omega v\,\mathrm{d}x$ and

$$\nabla v_\varepsilon(x)=R\Big(\mathbb{I}+\frac{\vartheta_a'(x_2)}{\lambda}\mathbb{1}_{\varepsilon Y_{\rm soft}}(x)e_1\otimes e_2\Big).\tag{4.29}$$

By the Riemann–Lebesgue lemma,

$$v_\varepsilon\rightharpoonup v\qquad\text{in } W^{1,1}(\Omega;\mathbb{R}^{2\times 2}).\tag{4.30}$$

On the other hand, applying Lemma 3.3 to ϑ_s, we can find a sequence $(\vartheta_\varepsilon)_\varepsilon\subset W^{1,\infty}(-1,1)$ such that $\vartheta_\varepsilon\overset{*}{\rightharpoonup}\vartheta_s$ in $BV(-1,1)$ and $\vartheta_\varepsilon'=0$ on $\varepsilon I_{\rm rig}\cap(-1,1)$. Then, setting $w_\varepsilon(x):=\vartheta_\varepsilon(x_2)Re_1+\int_\Omega(w-\vartheta_\varepsilon(x_2)Re_1)\,\mathrm{d}x$ yields

$$\nabla w_\varepsilon(x)=\vartheta_\varepsilon'(x_2)Re_1\otimes e_2=\vartheta_\varepsilon'(x_2)\mathbb{1}_{\varepsilon Y_{\rm soft}}Re_1\otimes e_2\tag{4.31}$$

and

$$w_\varepsilon \overset{*}{\rightharpoonup} w \quad \text{in } BV(\Omega;\mathbb{R}^2). \tag{4.32}$$

We define the maps $u_\varepsilon := v_\varepsilon + w_\varepsilon$ in $W^{1,1}(\Omega;\mathbb{R}^2)$ for every ε,
and infer from (4.29) and (4.31) that

$$\nabla u_\varepsilon = R(\mathbb{I} + \gamma_\varepsilon e_1 \otimes e_2),$$

where $\gamma_\varepsilon(x) := \big(\frac{\vartheta_\alpha'(x_2)}{\lambda} + \vartheta_\varepsilon'(x_2)\big)\mathbb{1}_{\varepsilon Y_{\rm soft}}(x)$ is a function in $L^1(\Omega)$ satisfying $\gamma_\varepsilon = 0$ in $\varepsilon Y_{\rm rig} \cap \Omega$. In particular, $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε.

Combining (4.30) and (4.32) shows that $u_\varepsilon \overset{*}{\rightharpoonup} v + w = u$ in $BV(\Omega;\mathbb{R}^2)$, which finishes the proof. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. In view of the discussion in Section 2.4, it suffices to prove the statement on a rectangle of the form $(c_\Omega, d_\Omega) \times (a_\Omega, b_\Omega)$, where we recall (1.1) and (1.2). A simple modification of the proofs of Propositions 4.3, 4.7, and 4.9 shows that these results hold for any such rectangles, from which Theorem 1.3 follows. □

5. A lower bound on the homogenized energy

In this section, we present partial results for the homogenization problem for layered composites with rigid components discussed in the Introduction. More precisely, we establish a lower bound estimate on the asymptotic behavior of the sequence of energies $(E_\varepsilon)_\varepsilon$ (see (1.7)), and highlight the main difficulties in the construction of matching upper bounds. Note that the following analysis is restricted to the case $s = e_1$.

As a start, we first give alternative representations for the involved energies, which will be useful in the sequel.

Remark 5.1 (Equivalent formulations for E_ε and E). In view of the definition of $\mathcal{A}_\varepsilon$ (see (1.6)), it is straightforward to check that the functional E_ε in (1.7) satisfies

$$E_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \sqrt{|\partial_2 u|^2 - 1}\,\mathrm{d}x & \text{if } u \in \mathcal{A}_\varepsilon, \\ \infty & \text{otherwise,} \end{cases} = \begin{cases} \displaystyle\int_\Omega \sqrt{|\nabla u|^2 - 2\det \nabla u}\,\mathrm{d}x & \text{if } u \in \mathcal{A}_\varepsilon, \\ \infty & \text{otherwise,} \end{cases}$$

for $u \in L^1_0(\Omega;\mathbb{R}^2)$. Similarly, according to Proposition 4.1, the functional E from (1.10) can be expressed as

$$E(u) = \begin{cases} \displaystyle\int_\Omega |\gamma|\,\mathrm{d}x + |D^s u|(\Omega) & \text{if } u \in \mathcal{A}, \\ \infty & \text{otherwise,} \end{cases}$$

for $u \in L^1_0(\Omega;\mathbb{R}^2)$.

We can now provide a bound from below on Γ-$\liminf_{\varepsilon\to 0} E_\varepsilon$ and prove Theorem 1.2.

Proof of Theorem 1.2. For clarity, we subdivide the proof into two steps. In the first one, we establish the compactness property. In the second step, we provide two alternative proofs of (1.12). The first proof is based on a Reshetnyak's lower semicontinuity result, while the second version is more elementary, relying on the weak* lower semicontinuity of the total variation of a measure. Either of the arguments highlights a different feature of the representation of $\mathcal{A}$.

Step 1: Compactness. Assume that $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ is such that $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < \infty$. Then, $u_\varepsilon \in \mathcal{A}_\varepsilon$ and $\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} < \infty$. Hence, using the Poincaré–Wirtinger inequality, there exist a subsequence $(u_{\varepsilon_j})_{j\in\mathbb{N}}$ and $u \in L^1_0(\Omega;\mathbb{R}^2) \cap BV(\Omega;\mathbb{R}^2)$ such that $u_{\varepsilon_j} \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$. By Proposition 4.3, we conclude that $u \in L^1_0(\Omega;\mathbb{R}^2) \cap \mathcal{A}$.

Step 2: Lower bound. Let $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ and $u \in L^1_0(\Omega;\mathbb{R}^2)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$. We want to show that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq E(u). \tag{5.1}$$

To prove (5.1), one may assume without loss of generality that the limit inferior on the right-hand side of (5.1) is actually a limit and that this limit is finite. Then, $u_\varepsilon \in \mathcal{A}_\varepsilon$ and $E_\varepsilon(u_\varepsilon) < C$ for all ε, where $C > 0$ is a constant independent of ε. Hence, by Step 1, $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$ and $u \in \mathcal{A}$.

Step 2a: Version I. We observe that the map $\mathbb{R}^{2\times2} \ni F \mapsto \sqrt{|F|^2 - 2\det F}$ is convex (see [18]) and one-homogeneous. Consequently, it follows from Remark 5.1 and Reshetnyak's lower semicontinuity theorem (see [2, Theorem 2.38]), under consideration of our notation for the polar decomposition $Du = g_u|Du|$ introduced in Section 2.2, that

$$\liminf_{\varepsilon\to0} E_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to0} \int_\Omega \sqrt{|\nabla u_\varepsilon|^2 - 2\det\nabla u_\varepsilon}\,\mathrm{d}x \geq \int_\Omega \sqrt{|g_u|^2 - 2\det g_u}\,\mathrm{d}|Du|. \tag{5.2}$$

Since $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in BV(\Omega; SO(2))$ and $(D^s u)e_1 = 0$ (see (4.5)), we have $|\nabla u|^2 - 2\det\nabla u = |\gamma|^2$ for $\mathcal{L}^2$-a.e. in Ω and $\det g_u = 0$ for $|D^s u|$-a.e. in Ω. Thus,

$$\begin{aligned}
&\int_\Omega \sqrt{|g_u|^2 - 2\det g_u}\,\mathrm{d}|Du| \\
&\quad= \int_\Omega \sqrt{|\nabla u|^2 - 2\det\nabla u}\,\mathrm{d}x + \int_\Omega \sqrt{|g_u|^2 - 2\det g_u}\,\mathrm{d}|D^s u| \\
&\quad= \int_\Omega |\gamma|\,\mathrm{d}x + |D^s u|(\Omega) = E(u),
\end{aligned} \tag{5.3}$$

where we also used that the relation $|g_u| = 1$ holds $|D^s u|$-a.e. in Ω.

From (5.2) and (5.3), we deduce (5.1).

Step 2b: Version II. By the definition of $\mathcal{A}_\varepsilon$ and (4.1),

$$\nabla u_\varepsilon = R_\varepsilon + \gamma_\varepsilon R_\varepsilon e_1 \otimes e_2$$

with $R_\varepsilon \in L^\infty(\Omega; SO(2))$ and $\gamma_\varepsilon \in L^1(\Omega)$. Since $|\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2| = |\gamma_\varepsilon|$ due to $|R_\varepsilon e_1| = 1$, the estimate $E_\varepsilon(u_\varepsilon) = \int_\Omega |\gamma_\varepsilon|\,\mathrm{d}x < C$ implies that $(\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2)_\varepsilon$ is uniformly bounded in $L^1(\Omega;\mathbb{R}^{2\times2})$. Hence, after extracting a subsequence if necessary (not relabeled),

$$(\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2)\mathcal{L}^2 \lfloor \Omega \overset{*}{\rightharpoonup} \nu \qquad \text{in } \mathcal{M}(\Omega;\mathbb{R}^{2\times2})$$

for some $\nu \in \mathcal{M}(\Omega;\mathbb{R}^{2\times2})$. Note further that the convergence $\nabla u_\varepsilon \mathcal{L}^2\lfloor\Omega \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega;\mathbb{R}^{2\times2})$ along with (4.5) yields also $R_\varepsilon \overset{*}{\rightharpoonup} R$ in $L^\infty(\Omega;\mathbb{R}^{2\times2})$, where $R \in L^\infty(\Omega; SO(2))$ satisfies in particular that $(\nabla u)e_1 = Re_1$. Hence, we have

$$\nu = Du - R\mathcal{L}^2\lfloor\Omega = (\gamma R e_1 \otimes e_2)\mathcal{L}^2\lfloor\Omega + D^s u,$$

where the last equality follows again from (4.5), and by the lower semicontinuity of the total variation,

$$\begin{aligned}
\liminf_{\varepsilon\to0} E_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to0} \int_\Omega |\gamma_\varepsilon|\,\mathrm{d}x = \liminf_{\varepsilon\to0} \int_\Omega |\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2|\,\mathrm{d}x \\
&\geq |\nu|(\Omega) = \int_\Omega |\gamma R e_1 \otimes e_2|\,\mathrm{d}x + |D^s u|(\Omega) = \int_\Omega |\gamma|\,\mathrm{d}x + |D^s u|(\Omega) = E(u).
\end{aligned}$$
□

Remark 5.2 (Discussion regarding optimality of the lower bound). (a) The lower bound (1.12) is optimal in $\mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$ and, more generally (cf. also Remark 4.4), in the set $\mathcal{A}^\parallel \cap L^1_0(\Omega;\mathbb{R}^2)$ introduced in (1.13). Precisely, we have

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to0} E_\varepsilon(u) = E(u) \tag{5.4}$$

for all $u \in \mathcal{A}^\parallel \cap L^1_0(\Omega;\mathbb{R}^2)$. In view of (1.12), the proof of (5.4) is directly related to the ability to construct a recovery sequence. We detail two alternative constructions for $u \in \mathcal{A}^\parallel$ in Section 6 below. For illustration, we treat here the simpler special case where $u \in \mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$.

If $u \in \mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$, then $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ for some $R \in SO(2)$ and $\gamma \in L^1(\Omega)$ such that $\partial_1\gamma = 0$ (see Remark 4.4 (a)). As in the proof of Proposition 4.9, we take $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$ such that $\nabla u_\varepsilon = R(\mathbb{I} + \frac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}} e_1 \otimes e_2)$ for all ε. Then, by the Riemann–Lebesgue lemma, $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$ and $\lim_{\varepsilon\to0} E_\varepsilon(u_\varepsilon) = E(u)$.

(b) The question whether (5.4) holds for a larger class than $\mathcal{A}^\parallel$ is open at this point. We observe that the gradient-based constructions in Lemma 4.5, Remark 4.6 (i)–(ii), and Proposition 4.7 yield upper bounds on the Γ-$\limsup$, which, however, do not match the lower bound of Theorem 1.2. This indicates that, in general, a more tailored approach will be necessary.

(c) The upper bounds on the Γ-$\limsup$ of $(E_\varepsilon)_\varepsilon$ resulting from Lemma 4.5, Remark 4.6 (i)–(ii), and Proposition 4.7 can be quantified. As previously mentioned, the constructions in Remark 4.6 (iii) and Proposition 4.9 are even recovery sequences. This is not the case for the general construction in Lemma 4.5 and for those highlighted in Remark 4.6 (i)–(ii). In the following, we suppose that $u \in \mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$ has a single jump as in the statement of Lemma 4.5; i.e.,

$$u(x) = \mathbb{1}_{(0,1)\times(0,1)}(x)(R^+(x_2)x + \psi^+(x_2)) + \mathbb{1}_{(0,1)\times(-1,0)}(x)(R^-(x_2)x + \psi^-(x_2))$$

with $R^\pm \in SO(2)$ and $\psi^\pm \in \mathbb{R}^2$. Then,

$$E(u) = \int_0^1 |(R^+ - R^-)e_1 x_1 + (\psi^+ - \psi^-)| \, \mathrm{d}x_1,$$

which can be estimated from above by

$$E(u) \le |R^+e_1 - R^-e_1| \int_0^1 x_1 \, \mathrm{d}x_1 + |\psi^+ - \psi^-| \le 1 + |\psi^+ - \psi^-|. \tag{5.5}$$

For the sequence $(u_\varepsilon)_\varepsilon$ constructed in Lemma 4.5 (and Lemma 4.7), we obtain, recalling (4.13), that

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + |\beta| + 2 > |\alpha| + |\beta| + 1 \ge E(u).$$

Regarding the construction of $(u_\varepsilon)_\varepsilon$ in Remark 4.6 (i), it follows that

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + |\beta - 1| + 1.$$

This limit is strictly greater than $E(u)$ as we will show next. If $|\beta - 1| > |\beta|$ (i.e., if $\beta < \frac{1}{2}$), this is an immediate consequence of (5.5). For $\frac{1}{2} \le \beta < 1$, we use that $\psi^+ - \psi^- = \alpha R^+ e_1 + \beta R^- e_1$ yields

$$E(u) \le \int_0^1 |x_1 + \alpha| \, \mathrm{d}x_1 + \int_0^\beta (\beta - x_1) \, \mathrm{d}x_1 + \int_\beta^1 (x_1 - \beta) \, \mathrm{d}x_1 \le 1 + |\alpha| + \beta(\beta - 1) < 1 + |\alpha| + |\beta - 1|.$$

If $\beta \ge 1$, we note that $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + \beta$, and subdivide the estimate of $E(u)$ into three cases. Recalling the assumption $R^+ \ne \pm R^-$, we set $c := R^+e_1 \cdot R^-e_1 \in (-1,1)$ to obtain

$$E(u) = \int_0^1 \sqrt{(x_1+\alpha)^2 + (\beta - x_1)^2 + 2c(x_1+\alpha)(\beta - x_1)} \, \mathrm{d}x_1.$$

Then, we have for $\alpha \ge 0$ that

$$E(u) < \int_0^1 \sqrt{(x_1+\alpha)^2 + (\beta - x_1)^2 + 2(x_1+\alpha)(\beta - x_1)} \, \mathrm{d}x_1 = |\alpha + \beta| \le |\alpha| + \beta,$$

for $\alpha \le -1$ that

$$\begin{aligned} E(u) &< \int_0^1 \sqrt{(x_1+\alpha)^2 + (\beta - x_1)^2 - 2(x_1+\alpha)(\beta - x_1)} \, \mathrm{d}x_1 = \int_0^1 (-2x_1 - \alpha + \beta) \, \mathrm{d}x_1 \\ &= -1 - \alpha + \beta < -\alpha + \beta = |\alpha| + \beta, \end{aligned}$$

and for $-1 < \alpha < 0$ that

$$E(u) < \int_0^{-\alpha} (-2x_1 - \alpha + \beta) \, \mathrm{d}x_1 + \int_{-\alpha}^1 |\alpha + \beta| \, \mathrm{d}x_1 = \alpha + \beta + \alpha^2 < -\alpha + \beta = |\alpha| + \beta.$$

Summing up, we have shown that in the context of Remark 4.6 (i),

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) > E(u).$$

Finally, we consider the sequence $(u_\varepsilon)_\varepsilon$ constructed in Remark 4.6 (ii). Then,

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |\alpha - \iota| + |\beta| + 1,$$

and since $R^+ = R^-$ in this case,

$$E(u) = \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta Re_1 \cdot Se_1}.$$

Using the fact that $Re_1 \cdot Se_1 \in (-1,1)$, it can be checked that, also here, we have

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) > E(u).$$

6. Homogenization of the regularized problem

This section is devoted to the proof of our main Γ-convergence result, Theorem 1.4. We first provide an alternative characterization of the class $\mathcal{A}^{\parallel}$ of *restricted asymptotically admissible deformations* introduced in (1.13).

Lemma 6.1. *Let $\Omega = (0,1)\times(-1,1)$. Then, $\mathcal{A}^{\parallel}$ as in (1.13) admits the representation*

$$\begin{aligned}\mathcal{A}^{\parallel} = \{u \in BV(\Omega;\mathbb{R}^2)\colon \nabla u = R(\mathbb{I}+\gamma e_1\otimes e_2) \text{ with } R\in SO(2),\ \gamma\in L^1(\Omega) \text{ such that } \partial_1\gamma = 0,\\ D^s u = (\varrho\otimes e_2)|D^s u| \text{ with } \varrho \in L^1_{|D^s u|}(\Omega;\mathbb{R}^2) \text{ such that}\\ |\varrho| = 1 \text{ and } \varrho || Re_1 \text{ for } |D^s u|\text{-a.e. in } \Omega\}.\end{aligned}\tag{6.1}$$

Proof. Let $\tilde{\mathcal{A}}^{\parallel}$ denote the set on the right-hand side of (6.1). Arguing as in the beginning of the proof of Proposition 4.9 (precisely, with the notation of (1.13), we set $\gamma(x) = \vartheta_a'(x_2)$ for $x\in\Omega$, and observe that $(D^s u)e_2 = \mathcal{L}^1\lfloor(0,1)\otimes D^s\vartheta_s Re_1)$ and exploiting the polar decomposition of measures (cf. (2.2) and (2.3)) gives rise to $\mathcal{A}^{\parallel}\subset\tilde{\mathcal{A}}^{\parallel}$. Conversely, the inclusion $\tilde{\mathcal{A}}^{\parallel}\subset\mathcal{A}$, which follows from (4.5), along with (4.6) yields that $\tilde{\mathcal{A}}^{\parallel}\subset\mathcal{A}^{\parallel}$. □

We are now in a position to prove the Γ-convergence of the energies $(E^{\varepsilon}_{\delta})_\varepsilon$ in (1.14) as $\varepsilon\to 0$.

Proof of Theorem 1.4. As before in the proofs of Theorems 1.1 and 1.3, one may assume without loss of generality that $\Omega = (0,1)\times(-1,1)$. We proceed in three steps.

Step 1: Compactness. Let $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)\cap L^1_0(\Omega;\mathbb{R}^2)$ be a sequence such that $E^{\delta}_{\varepsilon}(u_\varepsilon)\le C$ for all $\varepsilon>0$. Then, because $u_\varepsilon\in\mathcal{A}_\varepsilon$ for all ε,

$$\nabla u_\varepsilon = R_\varepsilon(\mathbb{I}+\gamma_\varepsilon e_1\otimes e_2)\in L^1(\Omega;\mathbb{R}^{2\times 2}),\tag{6.2}$$

and $\|\gamma_\varepsilon\|_{L^1(\Omega)}\le C$ for every $\varepsilon>0$. Additionally, since each map R_ε takes value in the set of proper rotations, it holds that $\|R_\varepsilon\|^2_{L^\infty(\Omega;\mathbb{R}^{2\times2})} = 2$ for all $\varepsilon>0$. Consequently, along with the Poincaré-Wirtinger inequality,

$$\|u_\varepsilon\|_{W^{1,1}(\Omega;\mathbb{R}^2)}\le C.$$

We further know that $\|\partial_1 u_\varepsilon\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} = \|R_\varepsilon e_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)}\le C/\delta$ for any ε. Thus, after extracting subsequences if necessary, one can find $u\in BV(\Omega;\mathbb{R}^2)$, $\gamma\in\mathcal{M}(\Omega)$, and $R\in W^{1,p}(\Omega;\mathbb{R}^{2\times2})$ such that

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } BV(\Omega;\mathbb{R}^2),\tag{6.3}$$

$$\gamma_\varepsilon\mathcal{L}^2 \overset{*}{\rightharpoonup}\gamma \quad \text{in } \mathcal{M}(\Omega),$$

$$R_\varepsilon\rightharpoonup R \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^{2\times2}).\tag{6.4}$$

Recalling the compact embedding $W^{1,p}(\Omega)\hookrightarrow\hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for some $0<\alpha<1-\frac{2}{p}$, it follows even that $R\in W^{1,p}(\Omega;SO(2))\cap C^{0,\alpha}(\overline{\Omega};\mathbb{R}^{2\times2})$ and

$$R_\varepsilon\to R \quad \text{in } L^\infty(\Omega;\mathbb{R}^{2\times2}).\tag{6.5}$$

As a consequence of Proposition 4.3, it holds that $u\in\mathcal{A}$. From Proposition 4.1 and Alberti's rank one theorem (cf. Section 2.1), we can further infer that $R\in SO(2)$, $\gamma\in L^1(\Omega)$ with $\partial_1\gamma=0$, and that Du satisfies

$$\nabla u = R(\mathbb{I}+\gamma e_1\otimes e_2) \quad\text{and}\quad D^s u = (\varrho\otimes e_2)|D^s u|,\tag{6.6}$$

where $\varrho\in L_{|D^s u|}(\Omega;\mathbb{R}^2)$ with $|\varrho|=1$ for $|D^s u|$-a.e. in Ω. To conclude that $u\in\mathcal{A}^{\parallel}$, in view of Lemma 6.1, it remains to show that

$$\varrho || Re_1 \quad |D^s u|\text{-a.e. in } \Omega.\tag{6.7}$$

To prove (6.7), we first observe that for every ε, the identity $(\nabla u_\varepsilon)e_2 = R_\varepsilon e_2 + \gamma_\varepsilon R_\varepsilon e_1$, which follows from $u_\varepsilon\in\mathcal{A}_\varepsilon$, yields

$$\int_\Omega [(\nabla u_\varepsilon)e_2\cdot R_\varepsilon e_2 - 1]\varphi \,\mathrm{d}x = 0\tag{6.8}$$

for all $\varphi \in C_c^\infty(\Omega)$. Thus, by (6.3) and (6.5) in combination with a weak-strong convergence argument, taking the limit $\varepsilon \to 0$ in (6.8) leads to

$$\int_\Omega \varphi \,\mathrm{d}x = \int_\Omega \varphi Re_2 \cdot \,\mathrm{d}((Du)e_2) = \int_\Omega \varphi Re_2 \cdot (\nabla u)e_2 \,\mathrm{d}x + \int_\Omega \varphi Re_2 \cdot \,\mathrm{d}((D^s u)e_2)$$

for every $\varphi \in C_c^\infty(\Omega)$. Next, we plug in the identities $(\nabla u)e_2 = Re_2 + \gamma Re_1$ and $(D^s u)e_2 = \varrho |D^s u|$ (see (6.6)) to derive that

$$0 = \int_\Omega \varphi Re_2 \cdot \,\mathrm{d}((D^s u)e_2) = \int_\Omega \varphi Re_2 \cdot \varrho \,\mathrm{d}|D^s u|$$

for every $\varphi \in C_c^\infty(\Omega)$, which completes the proof of (6.7).

Step 2: Lower bound. Let $(u_\varepsilon) \subset L_0^1(\Omega;\mathbb{R}^2)$ and $u \in L_0^1(\Omega;\mathbb{R}^2)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$. We want to show that

$$E^\delta(u) \le \liminf_{\varepsilon\to 0} E_\varepsilon^\delta(u_\varepsilon). \tag{6.9}$$

To prove (6.9), we proceed as in the proof of (5.1), observing in addition that

$$\liminf_{\varepsilon\to 0} \delta\|\partial_1 u_\varepsilon\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} = \liminf_{\varepsilon\to 0} \delta\|R_\varepsilon e_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} \ge \delta\|Re_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} = \delta|\Omega|$$

due to (6.2) and (6.4) with $R \in SO(2)$.

Step 3: Upper bound. Let $u \in L_0^1(\Omega;\mathbb{R}^2) \cap \mathcal{A}^{\|}$. We want to show that there is a sequence $(u_\varepsilon) \subset L_0^1(\Omega;\mathbb{R}^2)$ such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$, and

$$E^\delta(u) \ge \limsup_{\varepsilon\to 0} E_\varepsilon^\delta(u_\varepsilon). \tag{6.10}$$

Let $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \cap L_0^1(\Omega;\mathbb{R}^2)$ be the sequence constructed in the proof of Proposition 4.9, that is, $u_\varepsilon \in \mathcal{A}_\varepsilon$ for every ε with

$$\nabla u_\varepsilon(x) = R\Big(\mathbb{I} + \Big(\frac{\vartheta_a'(x_2)}{\lambda} + \vartheta_\varepsilon'(x_2)\Big)\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}}(x) e_1 \otimes e_2\Big),$$

where $(\vartheta_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1)$ satisfies

$$\lim_{\varepsilon\to 0} \int_{-1}^1 |\vartheta_\varepsilon'| \,\mathrm{d}x_2 = |D^s\vartheta_s|(-1,1) = |D^s u|(\Omega),$$

and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$. Recalling that $\vartheta' = \vartheta_a' + \vartheta_s' = \vartheta_a'$, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0} E_\varepsilon^\delta(u_\varepsilon) &\le \lim_{\varepsilon\to 0}\bigg(\int_\Omega \frac{|\vartheta_a'(x_2)|}{\lambda}\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}}(x)\,\mathrm{d}x + \int_{-1}^1 |\vartheta_\varepsilon'(x_2)|\,\mathrm{d}x_2 + \delta\|Re_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)}\bigg)\\ &= \int_\Omega |\vartheta'(x_2)|\,\mathrm{d}x + |D^s u|(\Omega) + \delta|\Omega| = E^\delta(u),\end{aligned}$$

which proves (6.10) and completes the proof of the theorem. □

Remark 6.2 (On compensated compacteness). We point out that if $u_\varepsilon \in \mathcal{A}_\varepsilon$, with $\nabla u_\varepsilon = R_\varepsilon(\mathbb{I} + \gamma_\varepsilon e_1 \otimes e_2)$ for $R_\varepsilon \in L^\infty(\Omega;SO(2))$ and $\gamma_\varepsilon \in L^1(\Omega)$ with $\gamma_\varepsilon = 0$ on $\varepsilon Y_{\mathrm{rig}} \cap \Omega$, is such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$, and if in addition,

$$R_\varepsilon \to R \quad \text{in } C(\Omega;\mathbb{R}^{2\times 2}),$$

then a weak-strong convergence argument implies that

$$\gamma_\varepsilon \mathcal{L}^2 \lfloor \Omega = \big[(\nabla u_\varepsilon)e_2 \cdot R_\varepsilon e_1\big]\mathcal{L}^2\lfloor\Omega \overset{*}{\rightharpoonup} (Du)e_2 \cdot Re_1 \qquad \text{in } \mathcal{M}(\Omega).$$

However, if continuity and uniform convergence of R_ε fail, the limit representation above may no longer be true in general, even if $R \in C(\Omega;SO(2))$. To see this, let us consider the basic construction in Remark 4.6 (ii). In this case,

$$\gamma_\varepsilon \mathcal{L}^2\lfloor\Omega \overset{*}{\rightharpoonup} (\alpha+\beta)\mathcal{H}^1\lfloor\big((0,1)\times\{0\}\big) \qquad \text{in } \mathcal{M}(\Omega), \tag{6.11}$$

whereas

$$(Du)e_2 \cdot Re_1 = [(\psi^+ - \psi^-)\cdot Re_1]\mathcal{H}^1\lfloor\big((0,1)\times\{0\}\big). \tag{6.12}$$

Recalling that $\psi^+ - \psi^- = \alpha Re_1 + \beta Se_1$, the quantities in (6.11) and (6.12) can only match if $Re_1 \| Se_1$, which contradicts the assumption that Re_1 and Se_1 are linearly independent.

The role of the higher-order regularization in (1.14) is exactly that it helps overcome the issue discussed above. In fact, it guarantees the desired compactness properties for sequences of deformations with equibounded energies.

To conclude, we present an alternative construction for the recovery sequence in Step 3 of the proof of Theorem 1.4.

Alternative proof of Theorem 1.4. As before, we may assume without loss of generality that $\Omega = (0,1)\times(-1,1)$. Moreover, the compactness property and lower bound can be studied exactly as in the proof of Theorem 1.4 above.

We are then left to show that given $u \in L^1_0(\Omega;\mathbb{R}^2)\cap\mathcal{A}^{\|}$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ satisfying $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and (6.10). We will proceed in three steps, building up complexity.

Step 1. We assume first that $u \in L^1_0(\Omega;\mathbb{R}^2)\cap\mathcal{A}^{\|}$ is an SBV-function with a single, constant jump line at $x_2 = 0$.

This case can be treated as highlighted in Remark 4.6 (iii). Let $R \in SO(2)$, $\gamma \in L^1(\Omega)$ with $\partial_1\gamma = 0$, and ψ^+, $\psi^- \in \mathbb{R}^2$ with $(\psi^+ - \psi^-)||Re_1$ be such that

$$Du = R(\mathbb{I} + \gamma e_1\otimes e_2)\mathcal{L}^2\lfloor\Omega + (\psi^+ - \psi^-)\otimes e_2\mathcal{H}^1\lfloor((0,1)\times\{0\}).$$

Note that setting $\iota := \operatorname{sign}((\psi^+ - \psi^-)\cdot Re_1) \in \{\pm1\}$, we have $\psi^+ - \psi^- = \iota|\psi^+ - \psi^-|Re_1$ and

$$|Du|(\Omega) = |D^a u|(\Omega)| + |D^s u|(\Omega)| = |D^a u|(\Omega) + |D^j u|(\Omega) = \int_\Omega |R(\mathbb{I} + \gamma e_1\otimes e_2)|\,\mathrm{d}x + |\psi^+ - \psi^-|.$$

For each $\varepsilon > 0$, set $\tau_\varepsilon := \iota\frac{|D^j u|(\Omega)}{\lambda\varepsilon} = \iota\frac{|\psi^+ - \psi^-|}{\lambda\varepsilon}$. Arguing as, for instance, in the proof of Lemma 4.5, we can find $u_\varepsilon \in L^1_0(\Omega;\mathbb{R}^2)\cap\mathcal{A}_\varepsilon$ such that

$$\nabla u_\varepsilon = \begin{cases} R(\mathbb{I} + \tau_\varepsilon e_1\otimes e_2) & \text{if } x \in (0,1)\times(0,\lambda\varepsilon),\\ R(\mathbb{I} + \frac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}\cap\Omega} e_1\otimes e_2) & \text{otherwise,}\end{cases}$$

and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$. Next, we show that this construction yields convergence of energies. Indeed, we have

$$\begin{aligned}\lim_{\varepsilon\to0} E^\delta_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon\to0}\left(\int_{(0,1)\times(0,\lambda\varepsilon)} |\tau_\varepsilon|\,\mathrm{d}x + \int_{\Omega\setminus(0,1)\times(0,\lambda\varepsilon)} \left|\frac{\gamma}{\lambda}\right|\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}}\,\mathrm{d}x + \delta\|Re_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)}\right)\\ &= |\psi^+ - \psi^-| + \int_\Omega |\gamma|\,\mathrm{d}x + \delta|\Omega| = |D^s u|(\Omega) + \int_\Omega |\gamma|\,\mathrm{d}x + \delta|\Omega| = E^\delta(u).\end{aligned}$$

Step 2. We assume next that $u \in L^1_0(\Omega;\mathbb{R}^2)\cap\mathcal{A}^{\|}$ is an SBV-function with a finite number of horizontal jump lines and with constant upper and lower approximate limits on each jump line.

In this setting, $\nabla u = R(\mathbb{I} + \gamma e_1\otimes e_2)$ with $R \in SO(2)$ and $\gamma \in L^1(\Omega)$ with $\partial_1\gamma = 0$, $J_u = \bigcup_{i=1}^\ell (0,1)\times\{a_i\}$ with $\ell \in \mathbb{N}$ and $-1 < a_1 < a_2 < \cdots < a_\ell < 1$, $D^j u = \sum_{i=1}^\ell (\psi_i^+ - \psi_i^-)\otimes e_2\mathcal{H}^1\lfloor((0,1)\times\{a_i\})$ with $\psi_i^\pm \in \mathbb{R}^2$ satisfying $(\psi_i^+ - \psi_i^-)||Re_1$ for all $i \in \{1,...,\ell\}$, and $D^c u = 0$. Hence,

$$Du = R(\mathbb{I} + \gamma e_1\otimes e_2)\mathcal{L}^2\lfloor\Omega + \sum_{i=1}^\ell(\psi_i^+ - \psi_i^-)\otimes e_2\mathcal{H}^1\lfloor((0,1)\times\{a_i\}) \tag{6.13}$$

and

$$|D^s u|(\Omega) = \sum_{i=1}^\ell |\psi_i^+ - \psi_i^-|.$$

As in the proof of Proposition 4.7, the idea is to perform a construction similar to that in Step 1 around each jump line but accounting for the possibility that one or more of the jump lines may not intersect $\varepsilon Y_{\mathrm{soft}}\cap\Omega$.

Fix $i \in \{1,...,\ell\}$ and $\varepsilon > 0$, and let $\kappa^i_\varepsilon \in \mathbb{Z}$ be the integer such that $a_i \in \varepsilon[\kappa^i_\varepsilon, \kappa^i_\varepsilon + 1)$. Since $a_i \neq a_j$ if $i \neq j$, we may assume that the sets $\{\varepsilon[\kappa^i_\varepsilon, \kappa^i_\varepsilon + 1)\}_i$ are pairwise disjoint for all $\varepsilon > 0$ (this is true for all $\varepsilon > 0$ sufficiently small). Then, we take $u_\varepsilon \in L^1_0(\Omega;\mathbb{R}^2)\cap\mathcal{A}_\varepsilon$ such that

$$\nabla u_\varepsilon = \begin{cases} R(\mathbb{I} + \tau^i_\varepsilon e_1\otimes e_2) & \text{in } (0,1)\times\varepsilon(\kappa^i_\varepsilon, \kappa^i_\varepsilon + \lambda),\\ R(\mathbb{I} + \frac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}\cap\Omega} e_1\otimes e_2) & \text{otherwise,}\end{cases}$$

where $\tau_\varepsilon^i = \iota_i \frac{|\psi_i^+ - \psi_i^-|}{\lambda\varepsilon}$ with $\iota_i := \mathrm{sign}((\psi_i^+ - \psi_i^-)\cdot Re_1) \in \{\pm 1\}$. As in the proof of Proposition 4.7, we obtain that

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla u_\varepsilon \varphi \,\mathrm{d}x = \sum_{i=1}^{\ell}\int_0^1 \iota_i|\psi_i^+ - \psi_i^-|(Re_1\otimes e_2)\varphi(x_1,a_i)\,\mathrm{d}x_1 + \int_\Omega R(\mathbb{I}+\gamma e_1\otimes e_2)\varphi\,\mathrm{d}x \tag{6.14}$$

for all $\varphi \in C_0(\Omega)$. Recalling (6.13) and the equalities $\psi_i^+ - \psi_i^- = \iota_i|\psi_i^+ - \psi_i^-|Re_1$ for $i\in\{1,...,\ell\}$, (6.14) shows that $Du_\varepsilon \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega;\mathbb{R}^{2\times 2})$. Hence, $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$.

Finally, proceeding exactly as in Step 1, we conclude that this construction also yields convergence of the energies. This ends Step 2.

Step 3. We consider now the general case $u \in L_0^1(\Omega;\mathbb{R}^2)\cap\mathcal{A}^{\|}$.

Similarly to the beginning of the proof of Proposition 4.9 (see (4.28)), we can write

$$u(x) = x_1 Re_1 + \phi_a(x_2) + \phi_s(x_2), \quad x\in\Omega,$$

where $\phi_a(x_2) := x_2 Re_2 + \vartheta_a(x_2)Re_1 + c$ and $\phi_s(x_2) := \vartheta_s(x_2)Re_1$. Note that $\phi_a \in W^{1,1}(-1,1;\mathbb{R}^2)$ and $\phi_s \in BV(-1,1;\mathbb{R}^2)$ is the sum of a jump function and a Cantor function; in particular, $\vartheta' = \vartheta_a'$ and $D\phi_s = D^s\phi_s$ (see (2.4)). Moreover,

$$\begin{aligned}
&\nabla u = Re_1\otimes e_1 + \nabla\phi_a\otimes e_2 = R(\mathbb{I}+\vartheta_a' e_1\otimes e_2) = R(\mathbb{I}+\vartheta' e_1\otimes e_2),\\
&D^s u = \mathcal{L}^1\lfloor(0,1)\otimes D\phi_s,\\
&|D^s u|\lfloor\Omega = \mathcal{L}^1\lfloor(0,1)\otimes|D\phi_s|.
\end{aligned} \tag{6.15}$$

By Lemma 6.1, there exists $\varrho \in L^1_{|D^s u|}(-1,1;\mathbb{R}^2)$ with $|\varrho| = 1$ such that

$$D^s u = (\varrho\otimes e_2)|D^s u| \quad \text{and} \quad \varrho = (\varrho\cdot Re_1)Re_1. \tag{6.16}$$

Let $\varrho_h \in C^\infty([-1,1])$ be such that

$$\lim_{h\to\infty}\int_\Omega |\varrho_h(x_2) - \varrho(x_2)|\,\mathrm{d}|D^s u|(x) = 0. \tag{6.17}$$

Since $|\varrho| = 1$, we can choose such a sequence so that $|\varrho_h| \le 1$.

Due to the properties of good representatives (see [2, (3.24)]) and [19, Lemma 3.2], for each $n\in\mathbb{N}$, there exists a piecewise constant function $\phi_n \in BV(-1,1;\mathbb{R}^2)$, of the form

$$\phi_n = \sum_{i=0}^{\ell_n} b_i^n \chi_{A_i^n},$$

where $\ell_n\in\mathbb{N}$, $(b_i^n)_{i=0}^{\ell_n} \subset \mathbb{R}^2$, and $(A_i^n)_{i=0}^{\ell_n}$ is a partition of $(-1,1)$ into intervals with $\sup A_i^n = \inf A_{i+1}^n$, satisfying

$$J_{\phi_n} = \bigcup_{i=1}^{\ell_n}\{a_i^n\} \text{ with } a_i^n := \sup A_{i-1}^n,$$

$$\lim_{n\to\infty}\|\phi_n - \phi_s\|_{L^1(-1,1;\mathbb{R}^2)} = 0, \tag{6.18}$$

$$\lim_{n\to\infty}|D\phi_n|(-1,1) = \lim_{n\to\infty}|D^j\phi_n|(-1,1) = |D\phi_s|(-1,1) = |D^s u|(\Omega). \tag{6.19}$$

Indeed, (6.18) and (6.19) mean that $(\phi_n)_{n\in\mathbb{N}}$ converges strictly to ϕ_s in $BV(-1,1;\mathbb{R}^2)$, which implies that

$$|D\phi_n| \overset{*}{\rightharpoonup} |D\phi_s| \quad \text{in } \mathcal{M}(-1,1), \tag{6.20}$$

see [2, Proposition 3.5].

Finally, for $n\in\mathbb{N}$, we define

$$u_n(x) := x_1 Re_1 + \phi_a(x_2) + \phi_n(x_2) + c_n, \ \ x\in\Omega,$$

where $c_n \in \mathbb{R}^2$ are constants chosen so that $\int_\Omega u_n \,\mathrm{d}x = 0$. Note that $c_n \to 0$ as $n\to\infty$ by (6.18). Moreover, for each $n\in\mathbb{N}$, the map $u_n \in L_0^1(\Omega;\mathbb{R}^2)$ has the same structure as in Step 2 apart from the condition $(u_n^+ - u_n^-)||Re_1$ on J_{u_n}, which a priori is not satisfied. Choosing $\iota_i^n := \varrho_h(a_i^n)\cdot Re_1$, we can invoke Step 2 up to, and including, (6.14) to construct a sequence $(u^{n,h}_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)\cap W^{1,1}(\Omega;\mathbb{R}^2)$ that satisfies for all $\varphi \in C_0(\Omega)$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega \nabla u^{n,h}_\varepsilon \varphi \,\mathrm{d}x = \sum_{i=1}^{\ell_n}\int_0^1 &(\varrho_h(a^n_i)\cdot Re_1)|b^n_i - b^n_{i-1}|(Re_1\otimes e_2)\varphi(x_1,a^n_i)\,\mathrm{d}x_1 \\ &+ \int_\Omega R(\mathbb{I}+\vartheta'_a(x_2)e_1\otimes e_2)\varphi\,\mathrm{d}x.\end{aligned} \tag{6.21}$$

We conclude from (6.15), (6.16), (6.17), (6.18), (6.20), and the Lebesgue dominated convergence theorem that

$$\begin{aligned}&\lim_{h\to\infty}\lim_{n\to\infty}\sum_{i=1}^{\ell_n}\int_0^1 (\varrho_h(a^n_i)\cdot Re_1)|b^n_i - b^n_{i-1}|(Re_1\otimes e_2)\varphi(x_1,a^n_i)\,\mathrm{d}x_1 \\ &= \lim_{h\to\infty}\lim_{n\to\infty}\int_0^1\int_{-1}^1 (\varrho_h(x_2)\cdot Re_1)(Re_1\otimes e_2)\varphi(x_1,x_2)\,\mathrm{d}|D\phi_n|(x_2)\,\mathrm{d}x_1 \\ &= \lim_{h\to\infty}\int_0^1\int_{-1}^1 (\varrho_h(x_2)\cdot Re_1)(Re_1\otimes e_2)\varphi(x_1,x_2)\,\mathrm{d}|D\phi_s|(x_2)\,\mathrm{d}x_1 \\ &= \int_\Omega (\varrho(x_2)\cdot Re_1)(Re_1\otimes e_2)\varphi\,\mathrm{d}|D^s u| = \int_\Omega (\varrho(x_2)\otimes e_2)\varphi\,\mathrm{d}|D^s u| = \int_\Omega \varphi\,\mathrm{d}D^s u.\end{aligned} \tag{6.22}$$

Recalling that $|\varrho_h(a^n_i)\cdot Re_1|\leq 1$, we can further argue as in Steps 1 and 2 regarding the convergence of the energies to get

$$\limsup_{\varepsilon\to 0} E^\delta_\varepsilon(u^{n,h}_\varepsilon) \leq E^\delta(u_n) = \int_\Omega |\vartheta'_a(x_2)|\,\mathrm{d}x + |D^s\phi_n|(-1,1) + \delta|\Omega| \tag{6.23}$$

$$= \int_\Omega |\vartheta'(x_2)|\,\mathrm{d}x + |D^j\phi_n|(-1,1) + \delta|\Omega|. \tag{6.24}$$

Letting $n\to\infty$ and $h\to\infty$ in (6.21) and (6.23), from (6.22), (6.19), and (6.15), we conclude that for all $\varphi\in C_0(\Omega)$,

$$\lim_{h\to\infty}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega \nabla u^{n,h}_\varepsilon\varphi\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}Du, \tag{6.25}$$

$$\limsup_{h\to\infty}\limsup_{n\to\infty}\lim_{\varepsilon\to 0} E^\delta_\varepsilon(u^{n,h}_\varepsilon) \leq \int_\Omega |\vartheta'(x_2)|\,\mathrm{d}x + |D^s u|(\Omega) + \delta|\Omega| = E^\delta(u). \tag{6.26}$$

Owing to the separability of $C_0(\Omega)$ and (6.25)–(6.26), we can use a diagonalization argument as that in [25, proof of Proposition 1.11 (p.449)] to find sequences $(h_\varepsilon)_\varepsilon$ and $(n_\varepsilon)_\varepsilon$ such that h_ε, $n_\varepsilon\to\infty$ as $\varepsilon\to 0$ and $\tilde u_\varepsilon := u^{n_\varepsilon,h_\varepsilon}_\varepsilon \in L^1_0(\Omega;\mathbb{R}^2)\cap W^{1,1}(\Omega;\mathbb{R}^2)$ has all the desired properties. □

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Chapter 2

Wulff-shape emergence in crystallization problems

This chapter consists of the following papers:

1) E. Davoli, P. Piovano, U. Stefanelli.
   Wulff shape emergence in graphene.
   Math. Models Methods Appl. Sci. **26** (2016), 2277–2310.

2) E. Davoli, P. Piovano, U. Stefanelli.
   Sharp $N^{3/4}$ law for minimizers of the edge-isoperimetric problem on the triangular lattice.
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Wulff shape emergence in graphene

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Graphene samples are identified as minimizers of configurational energies featuring both two- and three-body atomic-interaction terms. This variational viewpoint allows for a detailed description of ground-state geometries as connected subsets of a regular hexagonal lattice. We investigate here how these geometries evolve as the number n of carbon atoms in the graphene sample increases. By means of an equivalent characterization of minimality via a discrete isoperimetric inequality, we prove that ground states converge to the ideal hexagonal Wulff shape as $n \to \infty$. Precisely, ground states deviate from such hexagonal Wulff shape by at most $Kn^{3/4} + \mathrm{o}(n^{3/4})$ atoms, where both the constant K and the rate $n^{3/4}$ are sharp.

Keywords: Ground state; hexagonal lattice; isoperimetric inequality; Wulff shape.

AMS Subject Classification: 82D25

1. Introduction

The recent realization of high crystalline quality graphene samples at room temperature can be regarded as a breakthrough in materials science and has led to the attribution of the 2010 Nobel Prize in physics to Geim and Novoselov. The fascinating electronic and mechanical properties of single-atom-thick carbon surfaces are believed to offer unprecedented opportunities for innovative applications, ranging from next-generation electronics to pharmacology, and including batteries and solar cells. New findings are emerging at an always increasing pace and involve thousands of researchers worldwide cutting across materials science, physics, and chemistry, extending from fundamental science to novel applications.

The stand of the mathematical understanding of graphene is comparably less developed. All available results are extremely recent and concern the modeling of transport properties of electrons in graphene sheets,[3,6,11,14,23,33,34] homogenization,[7,32] atomistic-to-continuum passage for nanotubes,[12] geometry of monolayers under Gaussian perturbations,[10] external charges[25] or magnetic fields,[9] combinatorial description of graphene patches,[20] and numerical simulation of dynamics via nonlocal elasticity theory.[43] Remarkably, the determination of the equilibrium shapes and the Wulff shapes of graphene samples and graphene nanostructures is still a challenging problem.[1,5,17]

Graphene ideally corresponds to a regular, two-dimensional, hexagonal arrangement of carbon atoms. In the bulk of a graphene sample each carbon atom is covalently bonded to three neighbors. These covalent bonds are of sp^2-hybridized type and ideally form $2\pi/3$ angles in a plane. In order to describe these bonds, some phenomenological interaction energies, including two- and three-body interaction terms, have been presented and partially validated.[37,38] The arrangement of carbon atoms in the two-dimensional crystal emerges then as the global effect of the combination of local atomic interactions, and can be seen as the result of a *geometric optimization* process: by identifying the *configuration* of n carbon atoms with their positions $\{x_1, \ldots, x_n\} \subset \mathbb{R}^2$, one minimizes a given *configurational energy* $E : \mathbb{R}^{2n} \to \mathbb{R} \cup \{\infty\}$ and proves that the minimizers are indeed subsets of a regular hexagonal lattice. The configurational energies for carbon feature a decomposition $E = E_2 + E_3$ where E_2 corresponds to an attractive–repulsive two-body interaction, favoring some preferred spacing of the atoms, and E_3 encodes three-body interactions, expressing the specific geometry of sp^2-covalent bonding in carbon.

The above-variational viewpoint brings the study of graphene geometries into the realm of the so-called *crystallization problems*. A first analysis in this direction is in Ref. 24, where E_2 is assumed to be of Lennard-Jones type and E_3 favors $2\pi/3$ and $4\pi/3$ bond angles. The focus of Ref. 24 is on the *thermodynamic limit*: as $n \to \infty$ the minimal energy density is proven to converge to a finite value, corresponding indeed to the configuration in which the atoms arrange themselves in a suitably stretched hexagonal lattice. Analogous thermodynamic-limit results are obtained in Ref. 13, where nonetheless the term E_2 favors π bond angles. The crystallization problem for a *finite number* of carbon atoms is studied in Ref. 31 where the periodicity of ground states as well as the exact quantification of the ground-state energy is obtained, together with the discussion of carbon nanostructures such as fullerenes and nanotubes, see also Refs. 27, 28, 31. The reader is referred to Refs. 18, 40 and 41 for one-dimensional crystallization results, to Refs. 22, 35, 39 and 42 for the two-dimensional case either in the finite and in the thermodynamic-limit case, and to Refs. 29 and 30 for crystallization in the square lattice. Results on the three-dimensional thermodynamic limit are available in Refs. 15 and 16, and a recent review on the crystallization problem can be found in Ref. 4.

Our analysis moves from the consideration that, as the configurational energy favors bonding, ground states are expected to have minimal perimeter, since boundary atoms have necessarily less neighbors. These heuristics are here made precise by providing a new characterization of ground states based on a crystalline isoperimetric inequality. Indeed, we prove in Proposition 3.4 below that ground states correspond to isoperimetric extremizers and we determine the exact isoperimetric constant. Analogous results are obtained in Refs. 29 and 30 for the square lattice, and in Ref. 8 for the triangular lattice.

The minimality of the ground-state perimeter gives rise to the emergence of large polygonal clusters as the number of atoms n increases. Indeed, one is interested in identifying a so-called *Wulff shape* to which all properly-rescaled ground states converge. This has been successfully obtained for both the triangular[2,8,36] and the square lattice,[29,30] where ground states approach a hexagon and a square, respectively, as $n \to \infty$. Quite remarkably, in both the triangular and the square case it has been proved that ground states differ from the Wulff shape by at most $\mathrm{O}(n^{3/4})$ atoms, this bound being sharp. This is what it is usually referred to as the $n^{3/4}$-law.

The central aim of this paper is to establish the Wulff shape emergence for graphene samples and to investigate the $n^{3/4}$-law in this setting. Precisely, we provide sharp quantitative convergence results for ground states G_n to the correspondingly rescaled Wulff shape, in terms both of the Hausdorff distance and of the flat distance of the empirical measures μ_{G_n}, to the measure with density $\frac{4}{\sqrt{3}}\chi_W$, i.e. the rescaled characteristic function of the (rescaled) hexagonal Wulff shape.

With respect to previous contributions to this subject the novelty of our paper is three-fold. First, we provide a complete characterization of ground states, for all numbers of atoms, as well as a detailed description of their geometry. In particular, as a byproduct of our isoperimetric characterization we are able to investigate the edge geometry of graphene patches. Graphene atoms tend to naturally arrange themselves into *hexagonal samples* whose edges can have, roughly speaking, two shapes: they can either form *zigzag* or *armchair* structures (see Refs. 5, 19, 26 and below).

We prove here that hexagonal configurations having armchair edges do not satisfy the isoperimetric equality, whereas those with zigzag edges do (see Definition 4.1). Namely, we have the following.

Theorem 1.1. (Zigzag-edge selectivity) *Zigzag hexagons are ground states, armchair hexagons are not.*

This provides an analytical counterpart to the experimental results in Ref. 26, confirming the zigzag-edge selectivity in the growth process of graphene samples.

The second main result of the paper is the discussion of the Wulff shape emergence in the hexagonal system, which is not a simple Bravais lattice but rather a so-called multilattice. We relate the Wulff shape emergence with the isoperimetric nature of ground states. Our result reads as follows.

Theorem 1.2. (Emergence of the Wulff shape) *Let G_n be a sequence of ground states in the hexagonal lattice. Let W_n be the zigzag hexagon centered in the origin and with side r_n (see* (1.6) *below). Then, there exists a suitable translation G'_n of G_n such that*

$$|G'_n \backslash W_n| \leq K n^{3/4} + \mathrm{o}(n^{3/4}), \tag{1.1}$$

where $|\cdot|$ is the cardinality of the set, and

$$K := \frac{2^{7/4}}{3^{1/4}}. \tag{1.2}$$

In addition, there holds:

$$d_{\mathcal{H}}(G'_n, W_n) \leq \mathrm{O}(n^{1/4}),$$

$$\|\mu_{G'_n} - \mu_{W_n}\| \leq K n^{-1/4} + \mathrm{o}(n^{-1/4}), \tag{1.3}$$

$$\|\mu_{G'_n} - \mu_{W_n}\|_F \leq K n^{-1/4} + \mathrm{o}(n^{-1/4}), \tag{1.4}$$

$$\mu_{G'_n} \rightharpoonup^* \frac{4}{\sqrt{3}} \chi_W \quad \textit{weakly* in the sense of measures,}$$

and

$$\left\| \mu_{G'_n} - \frac{4}{\sqrt{3}} \chi_W \right\|_{\mathrm{F}} \leq 2K n^{-1/4} + \mathrm{o}(n^{-1/4}), \tag{1.5}$$

where $d_{\mathcal{H}}$ is the Hausdorff distance, $\|\cdot\|$ is the total variation, and $\|\cdot\|_F$ is the flat norm (see (2.4)*).*

Our third main result concerns the sharpness of the $n^{3/4}$-law (1.1). We show not only the sharpness of the convergence ratio, but also of the constant K in front of the leading term. We have the following.

Theorem 1.3. (Sharpness of the $n^{3/4}$-law) *There exists a sequence of integers n_j such that for every sequence of ground states $\{G_{n_j}\}$ properties* (1.1), (1.3) *and* (1.4) *hold with equalities.*

Our proof strategy differs from that of Refs. 29 and 36, as it is not based on configuration rearrangements. The argument here moves from the control of the radius r_{G_n} of the maximal hexagon H_{G_n} contained in a ground state G_n with n atoms. In particular, we define

$$r_n := \min\{r_{G_n} : G_n \text{ is a ground state with } n \text{ atoms}\}, \tag{1.6}$$

and we show that every ground state (up to translation) consists of the n-Wulff shape W_n with comparably few extra atoms, see Sec. 6. Precisely, we prove a delicate estimate of the form $r_n \sim n^{1/2}$ which entails that the atoms of G_n which do not belong to W_n are at most $\mathrm{O}(n^{3/4})$. An outcome of our proof is that the convergence rates and the constants above are sharp. Indeed, we explicitly construct a sequence of integers such that every corresponding sequence of ground states attains the right-hand sides of (1.1), (1.3) and (1.4).

In the triangular lattice, the existence of a sequence of ground states whose deviation from the Wulff shape is exactly of order $n^{3/4}$ was exhibited in Ref. 36 with no specific control on the convergence constants. With a different implementation of the method discussed here, we revisited the triangular-lattice case in Ref. 8, obtaining explicit, sharp convergence constants.

The paper is organized as follows. In Sec. 2, we introduce some notation and a few definitions. In Sec. 3, we highlight the isoperimetric nature of ground states. Section 4 contains a discussion of the equilibrium shapes of graphene samples, and a proof of the fact that *armchair hexagons* are not ground states. In particular, we prove there Theorem 1.1. In Sec. 5, we provide delicate lower and upper bounds for r_n. Section 6 is eventually devoted to the proofs of Theorems 1.2 and 1.3.

2. Notation and Setting of the Problem

Let the *hexagonal lattice* be given by

$$\mathcal{L} := \{mt_1 + nt_2 + cw : m, n \in \mathbb{Z},\ c \in \{0, 1\}\},$$

with

$$t_1 := \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}, \quad t_2 := \begin{pmatrix} \sqrt{3}/2 \\ 3/2 \end{pmatrix}, \quad \text{and} \quad w := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that the minimal distance between points in $\mathcal{L}$ is 1 (see Fig. 1).

We denote a *configuration* of n atoms by $C_n := \{x_1, \ldots, x_n\} \in \mathbb{R}^{2n}$, the distance between two atoms, x_i and x_j, by ℓ_{ij}, and the counterclockwise-oriented angle between the two segments $x_i - x_j$ and $x_k - x_j$ by θ_{ijk}. The *energy* of a configuration C_n is defined as

$$E(C_n) := E_2(C_n) + E_3(C_n) = \frac{1}{2} \sum_{i \neq j} v_2(\ell_{ij}) + \sum_{(i,j,k) \in \mathcal{A}} v_3(\theta_{ijk}), \tag{2.1}$$

where $v_2 : [0, \infty) \to [-1, \infty]$ and $v_3 : [0, 2\pi] \to [0, \infty)$ are the two-body and the three-body interaction potentials. We notice that the energy is invariant under rotations and translations. Two atoms x_i and x_j are said to be *bonded*, or there is an (*active*) *bond* between x_i and x_j, if $1 \leq \ell_{ij} < \sqrt{2}$. The index set $\mathcal{A}$ in (2.1)

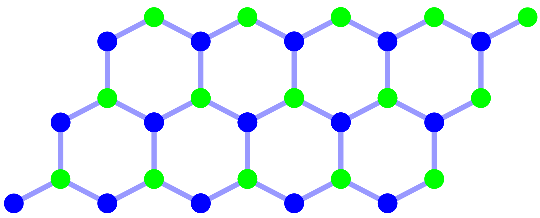

Fig. 1. An example of a subset of $\mathcal{L}$.

is defined as the set of all triples (i,j,k) for which the angle θ_{ijk} separates two active bonds. We will always assume that $v_2(1) = -1$ and that $v_2(\ell)$ vanishes for $\ell \geq \sqrt{2}$ (see below). We work under the assumption that v_3 reaches its minimum value only at the angles $\pi/3$ and $2\pi/3$, and that ground states are subsets of the hexagonal lattice. We use the standard notation for the right- and left-continuous integer-parts: $\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leq x\}$ and $\lceil x \rceil := \min\{z \in \mathbb{Z} : x \leq z\}$, respectively.

Under suitable assumptions on the potentials v_2 and v_3, it was established in Ref. 31 that all ground-state configurations are (isometric to) subsets of the hexagonal lattice $\mathcal{L}$ and that the value of the energy for every ground state with n atoms is given by

$$e_n := -\left\lfloor \frac{3n}{2} - \sqrt{\frac{3n}{2}} \right\rfloor. \tag{2.2}$$

From this point on all configurations are hence seen as subsets of $\mathcal{L}$.

The *bond graph* of a configuration C_n is the graph consisting of all its vertices and active bonds. For every atom $x_i \in C_n$, we indicate by $b(x_i)$ the number of active bonds of C_n with an endpoint in x_i. Denoting by $B(C_n)$ the total number of bonds in C_n, there holds

$$B(C_n) = \frac{1}{2}\sum_{i=1}^{n} b(x_i).$$

A configuration C_n is said to be *connected* if for every two atoms $y_1, y_2 \in C_n$ there exists a collection of atoms $x_1, \ldots, x_i \in C_n$ such that y_1 is bonded to x_1, x_i is bonded to y_2, and every atom x_j, with $2 \leq j \leq i-1$ is bonded both to x_{j-1} and to x_{j+1}. We call *minimal simple cycles* of a configuration all the simple cycles in the graph that are hexagons of side 1.

The *area* $A(C_n)$ of a configuration C_n is given by the number of minimal simple cycles of C_n. Denoting by $F(C_n) \subset \mathbb{R}^2$ the closure of the union of the regions enclosed by the minimal simple cycles of C_n, and by $G(C_n) \subset \mathbb{R}^2$ the union of all bonds which are not included in $F(C_n)$, the *perimeter* P of a configuration C_n is defined as

$$P(C_n) := \mathcal{H}^1(\partial F(C_n)) + 2\mathcal{H}^1(G(C_n)),$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. As already observed in Ref. 29, there holds

$$P(C_n) = \lim_{\varepsilon \searrow 0} \mathcal{H}^1(\partial(\partial F(C_n) \cup G(C_n) + B_\varepsilon)),$$

where $B_\varepsilon = \{y \in \mathbb{R}^2 \mid |y| \leq \varepsilon\}$.

With a slight abuse of notation, the symbol $|\cdot|$ will denote, according to the context, both the absolute value of a real number and the cardinality of a set.

We will often use the notion of *edge boundary* Θ of a configuration with respect to a reference lattice: this is the union of unit segments in the reference lattice that are not included in the graph of C_n but share one and only one endpoint with C_n,

$$\Theta(C_n) := \big\{(x,y) \in (\mathcal{L})^2 : x \in C_n,\, y \notin C_n\big\}.$$

The *edge perimeter* of a configuration C_n will be defined as the number of segments belonging to its edge boundary.

For every configuration $C_n := \{x_1, \ldots, x_n\}$ in $\mathcal{L}$, we denote by μ_{C_n} the *empirical measure associated to the rescaled configuration* $\{x_1/\sqrt{n}, \ldots, x_n/\sqrt{n}\}$, that is,

$$\mu_{C_n} := \frac{1}{n}\sum_i \delta_{x_i/\sqrt{n}}. \tag{2.3}$$

Given a Lebesgue measurable set $A \subset \mathbb{R}^2$, we will designate by $\mathcal{L}^2(A)$ its two-dimensional Lebesgue measure. For any bounded Radon measure μ, the symbol $\|\mu\|$ will represent its total variation in $\mathbb{R}^2$, whereas $\|\mu\|_F$ will be the *flat norm* of μ, defined as

$$\|\mu\|_F := \sup\left\{\int_{\mathbb{R}^2} \varphi\, d\mu : \varphi \text{ is Lipschitz with } \|\varphi\|_{W^{1,\infty}(\mathbb{R}^2)} \le 1\right\}. \tag{2.4}$$

The set of bounded Radon measures on $\mathbb{R}^2$ will be denoted by $M_b(\mathbb{R}^2)$.

3. Discrete Isoperimetric Inequality

In this section, we prove that connected configurations satisfy a discrete isoperimetric inequality, and we characterize ground states as configurations realizing the isoperimetric equality. We first deduce some preliminary relations between the area, the perimeter, the edge perimeter, and the energy of configurations. Let C_n be a configuration. Then

$$E(C_n) = -B(C_n) = -\frac{1}{2}\sum_{i=1}^{n} b(x_i).$$

Since every atom in $\mathcal{L}$ has exactly three bonds, we have

$$|\Theta(C_n)| = \sum_{i=1}^{n}(3 - b(x_i)), \tag{3.1}$$

and the energy and the edge perimeter of configurations are related by

$$E(C_n) = -\frac{3}{2}n + \frac{1}{2}|\Theta(C_n)|. \tag{3.2}$$

Recalling that every minimal simple cycle of C_n consists of six bonds, we have

$$\begin{aligned}6A(C_n) &= 2B(C_n \cap F(C_n)) - B(C_n \cap \partial F(C_n))\\ &= -2E(C_n \cap F(C_n)) - \mathcal{H}^1(\partial F(C_n)).\end{aligned}$$

On the other hand,

$$\mathcal{H}^1(G(C_n)) = B(C_n \cap G(C_n)) = -E(C_n \cap G(C_n)).$$

Hence, we obtain

$$\begin{aligned}P(C_n) &= \mathcal{H}^1(\partial F(C_n)) + 2\mathcal{H}^1(G(C_n))\\ &= -2E(C_n \cap F(C_n)) - 6A(C_n) - 2E(C_n \cap G(C_n))\\ &= -2E(C_n) - 6A(C_n),\end{aligned}$$

that is

$$E(C_n) = -3A(C_n) - \frac{1}{2}P(C_n). \tag{3.3}$$

In conclusion, we can express the energy of a hexagonal configuration C_n as a linear combination of its area and its perimeter. Likewise, in view of (3.2), the edge perimeter satisfies

$$|\Theta(C_n)| = 3n - 6A(C_n) - P(C_n).$$

The following result is a direct corollary of Theorem 7.3 (p. 142) in Ref. 21.

Proposition 3.1. *There exists a total order* $\tau : \mathbb{N} \to \mathcal{L}$ *such that for all* $n \in \mathbb{N}$ *the configuration* D_n *defined by* $D_n := \{x_{\tau(1)}, \ldots, x_{\tau(n)}\}$ (*which we call* daisy) *minimizes* E *over all configurations with* n *atoms, i.e.*

$$E(D_n) = \min_{C_n \subset \mathcal{L}} E(C_n) = e_n, \tag{3.4}$$

where e_n *is the quantity defined in* (2.2).

The total order in Proposition 3.1 is nonunique. For the sake of definiteness we fix here our attention on a specific order τ, as described in Ref. 31. For $n = 6k^2$, $k \in \mathbb{N}$, the sequence $\{D_{6k^2}\}$ is defined inductively as follows: D_6 is a minimal simple cycle in $\mathcal{L}$, and D_{24} is obtained by externally attaching to all bonds of D_6 another hexagon. D_{6k^2} is then defined recursively (see Fig. 2).

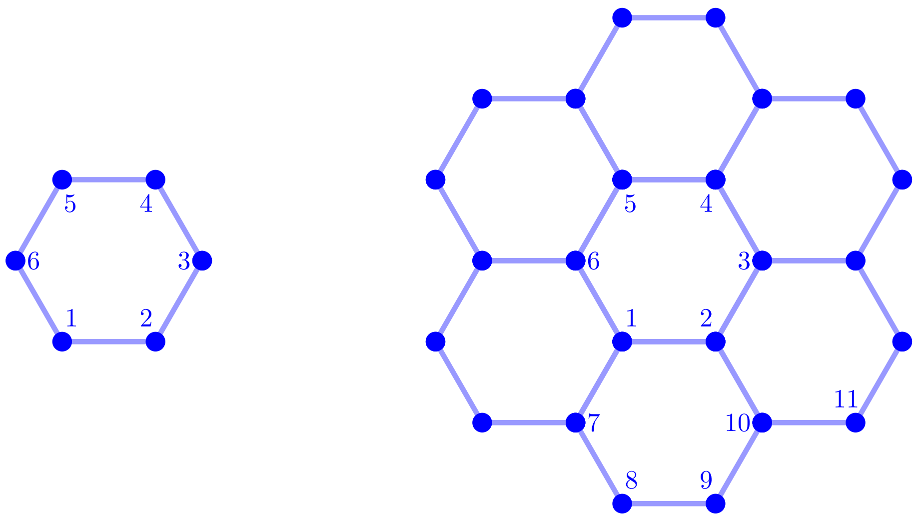

Fig. 2. The daisies D_6 and D_{24} and the total order τ.

For $k, m \in \mathbb{N}$, and $0 < m < 12k(k+1)$, D_{6k^2+m} is constructed as in the proof of Proposition 5.1 (Step 1) in Ref. 31. In view of Proposition 3.1, it is always possible to add one atom to a daisy D_n so that the new configuration with $n+1$ points is still a ground state.

To every configuration $C_n \subset \mathcal{L}$, we associate a weight function

$$\Delta_{C_n} : C_n \to \{0, 1, 2\},$$

defined as

$$\Delta_{C_n}(x) := |\{y \in C_n : (x, y) \in C_n \times C_n,\ y <_\tau x\}|,$$

for every $x \in C_n$, and we rewrite C_n as the union

$$C_n = \bigcup_{k=0}^{2} C_n^k,$$

where

$$C_n^k := \{x \in C_n : \Delta_{C_n}(x) = k\}$$

for $k = 0, \ldots, 2$. In particular, $|C_n^0|$ corresponds to the number of *connected components* of C_n.

The next proposition allows us to express the energy, the perimeter, the edge perimeter, and the area of a configuration C_n as a function of the cardinality of the sets C_n^k.

Proposition 3.2. *Let C_n be a configuration in $\mathcal{L}$. Then,*

$$E(C_n) = -|C_n^1| - 2|C_n^2|, \tag{3.5}$$

$$A(C_n) = |C_n^2|, \tag{3.6}$$

$$P(C_n) = 2|C_n^1| - 2|C_n^2|, \tag{3.7}$$

$$|\Theta(C_n)| = 3|C_n^0| + |C_n^1| - |C_n^2| \tag{3.8}$$

for every $n > 1$. Moreover,

$$E(C_n) = -3A(C_n) - |\Theta(C_n)| + 3|C_n^0|. \tag{3.9}$$

Proof. We first observe that

$$E(C_n) = -\sum_{i=1}^{n} \Delta_{C_n}(x_i).$$

For $i = 0, \ldots, n-1$, let C_i be the subset of C_n containing its first i points according to the total order τ. If $x_{\tau(i)} \in C_n^0$, then

$$A(C_i) - A(C_{i-1}) = 0, \quad P(C_i) - P(C_{i-1}) = 0, \quad |\Theta(C_i)| - |\Theta(C_{i-1})| = 3; \tag{3.10}$$

if $x_{\tau(i)} \in C_n^1$, then

$$A(C_i) - A(C_{i-1}) = 0, \quad P(C_i) - P(C_{i-1}) = 2, \quad |\Theta(C_i)| - |\Theta(C_{i-1})| = 1; \tag{3.11}$$

whereas, if $x_{\tau(i)} \in C_n^2$, we have

$$A(C_i) - A(C_{i-1}) = 1, \quad P(C_i) - P(C_{i-1}) = -2, \quad |\Theta(C_i)| - |\Theta(C_{i-1})| = -1. \tag{3.12}$$

Properties (3.5)–(3.8) follow from (3.10)–(3.12). Claim (3.9) is a direct consequence of (3.3), (3.7) and (3.8). □

In view of Proposition 3.2 we obtain the following.

Proposition 3.3. *The following assertions are equivalent and hold true for every connected hexagonal configuration C_n:*

(i) $|\Theta(D_n)| \leq |\Theta(C_n)|$;
(ii) $P(D_n) \leq P(C_n)$;
(iii) $A(D_n) \geq A(C_n)$.

Proof. The first assertion is a direct consequence of (3.2) and (3.4), and is equivalent to (ii) by (3.7) and (3.8). The equivalence between (ii) and (iii) follows by (3.3) and (3.4). □

We are now in a position to characterize connected ground states as extremizers of a discrete isoperimetric problem.

Proposition 3.4. *Every connected configuration C_n satisfies*

$$\sqrt{A(C_n)} \leq k_n P(C_n), \tag{3.13}$$

where

$$k_n := \frac{\sqrt{\lfloor (\alpha_n)^2 - \alpha_n \rfloor - n + 1}}{4(\alpha_n)^2 - 4\lfloor (\alpha_n)^2 - \alpha_n \rfloor - 6} \tag{3.14}$$

and $\alpha_n := \sqrt{3n/2}$.

Moreover, connected ground states correspond to those configurations for which (3.13) *holds with equality, and, equivalently, to those configurations that attain the maximum area*

$$a_n := -n + \lfloor (\alpha_n)^2 - \alpha_n \rfloor + 1,$$

and the minimum perimeter

$$p_n := 4(\alpha_n)^2 - 4\lfloor (\alpha_n)^2 - \alpha_n \rfloor - 6.$$

Proof. We claim that

$$\sqrt{A(D_n)} = k_n P(D_n). \tag{3.15}$$

In fact, in view of (3.5) and Theorem 3.1, there holds

$$e_n = E(D_n) = -|D_n^1| - 2|D_n^2|,$$

whereas by (3.2) and (3.8),

$$3n + 2e_n = |\Theta(D_n)| = 3 + |D_n^1| - |D_n^2|,$$

where e_n is the ground-state energy defined in (2.2). Solving the previous system of equations we deduce

$$|D_n^1| = 2n + e_n - 2, \tag{3.16}$$

and

$$|D_n^2| = -n - e_n + 1. \tag{3.17}$$

Claim (3.15) follows from (3.6), (3.7), (3.16) and (3.17), by observing that

$$\begin{aligned}\sqrt{A(D_n)} &= \sqrt{|D_n^2|} = \sqrt{-n - e_n + 1} \\ &= k_n(6n + 4e_n - 6) = k_n(2|D_n^1| - 2|D_n^2|) = k_n P(D_n).\end{aligned}$$

Inequality (3.13) is a direct consequence of (3.15) and Proposition 3.3. By Proposition 3.3 and (3.2), connected ground states G_n satisfy

$$|\Theta(G_n)| = e_n + \frac{3}{2}n$$

and attain the maximum area and the minimum perimeter. The values of a_n and p_n follow from (3.8), (3.7) and (3.9). □

4. Equilibrium Shapes of Graphene Samples

In this section, we characterize the edge geometry of graphene samples. We first introduce a few definitions.

Definition 4.1. For every $s \in \mathbb{N}$ we define the set $\mathcal{H}_s^Z$ of *zigzag hexagons of side* s as

$$\mathcal{H}_s^Z := \{D_{6s^2} + q : q \in \mathcal{L}\}$$

(for all $s \in \mathbb{N}$, the configuration D_{6s^2} is a complete *hexagon* of hexagons). For $s \in \mathbb{N}$, $s \geq 3$, the set $\mathcal{H}_s^A$ of *armchair hexagons of side* s is defined as

$$\mathcal{H}_s^A := \{A_s + q : q \in \mathcal{L}\}.$$

In the expression above, A_3 is given by the union of D_{24} with six extra minimal simple cycles, glued externally to the center of each side of D_{24} (see Fig. 3). For $s > 3$, A_s is defined recursively by adding an extra armchair layer of minimal simple cycles to A_{s-1}. We point out that the construction is different for s even and s odd (see Fig. 3).

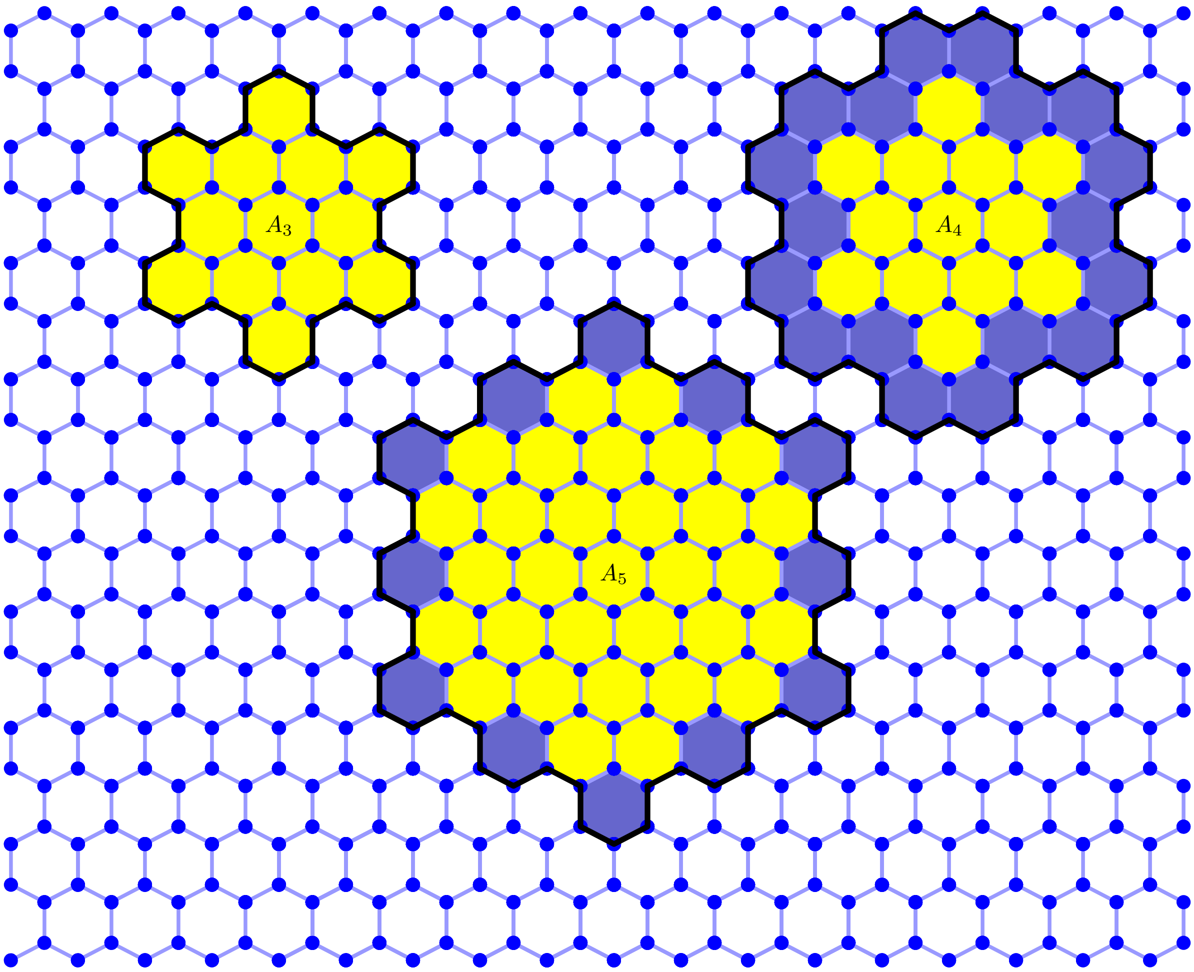

Fig. 3. (Color online) The armchair hexagons A_3, A_4 and A_5. Each armchair hexagon A_s for $s > 3$ is obtained by adding a layer of extra minimal simple cycles (in blue) to the corresponding armchair hexagon A_{s-1} (in yellow). Notice the different structure for s even and for s odd.

4.1. *Proof of Theorem* 1.1

The optimality of zigzag hexagons follows in view of Definition 4.1 and Proposition 3.1.

We claim that for every $s \in \mathbb{N}$, $s \geq 3$, there holds

$$P(A_s) > p_{|A_s|}. \tag{4.1}$$

Indeed, by the definition of A_s we have $|A_3| = 42$, and for $s > 3$:

$$|A_s| = |A_{s-1}| + \begin{cases} 6(2s-1) & \text{if } s \text{ is even,} \\ 6s & \text{if } s \text{ is odd,} \end{cases}$$

that is

$$|A_s| = \frac{9}{2}s^2 + \begin{cases} 3s & \text{if } s \text{ is even,} \\ \dfrac{3}{2} & \text{if } s \text{ is odd.} \end{cases}$$

On the other hand, the perimeter of each armchair hexagon A_s is given by

$$P(A_s) = 12s - 6. \tag{4.2}$$

For s odd, we have

$$\alpha_{|A_s|} = \sqrt{\frac{3|A_s|}{2}} = \frac{3}{2}s\sqrt{3 + \frac{1}{s^2}}.$$

Hence,

$$\alpha^2_{|A_s|} = \frac{9}{4}(3s^2 + 1) \in \mathbb{N},$$

and

$$\alpha^2_{|A_s|} - \lfloor \alpha^2_{|A_s|} - \alpha_{|A_s|} \rfloor = \left\lceil \frac{3}{2}s\sqrt{3 + \frac{1}{s^2}} \right\rceil.$$

In view of Proposition 3.4 there holds

$$\begin{align} p_{|A_s|} &= 4\alpha^2_{|A_s|} - 4\lfloor \alpha^2_{|A_s|} - \alpha_{|A_s|} \rfloor - 6 \tag{4.3}\\ &\leq 4\left(\frac{3}{2}s\sqrt{3 + \frac{1}{s^2}}\right) - 2 \tag{4.4}\\ &= 6\sqrt{3}s + \frac{6}{s\left(\sqrt{3} + \sqrt{3 + \frac{1}{s^2}}\right)} - 2 \tag{4.5}\\ &< 6\sqrt{3}s + \frac{\sqrt{3}}{s} - 2 < 12s - 6 \tag{4.6} \end{align}$$

for $s \geq 3$.

By combining (4.2) and (4.6) we obtain claim (4.1) for s odd. The result for s even, $s \geq 4$ follows via analogous computations. In view of Proposition 3.4 and (4.1) armchair hexagons are not extremizers of the isoperimetric inequality, and hence are not ground states.

5. The Radius of the n-Wulff Shape

For simplicity in what follows we will refer to the elements of $\mathcal{H}^Z_s$ as *hexagons of side* s, omitting the word *zigzag*. We first introduce the notion of *maximal hexagon* associated to a ground state.

Let G_n be a ground state in the hexagonal lattice $\mathcal{L}$. Let

$$r_{G_n} := \max\{s \in \mathbb{N} : \text{there exists a point } q \in \mathcal{L} \text{ such that } D_{6s^2} + q \subseteq G_n\}.$$

For every $q \in \mathcal{L}$ such that $D_{6r^2_{G_n}} + q \subset G_n$, we will refer to the set

$$H_{G_n} := D_{6r^2_{G_n}} + q,$$

as a *maximal hexagon associated to* G_n. We recall that

$$G_n = \bigcup_{k=0}^{2} G_n^k,$$

where

$$G_n^k := \{x \in G_n : \Delta_{G_n}(x) = k\}.$$

Let us preliminary check that maximal hexagons are non-degenerate for $n > 6$. We recall that the n-Wulff shape W_n is the zigzag hexagon centered in the origin with side r_n (see (1.6)), i.e.

$$W_n := D_{6r_n^2}.$$

Proposition 5.1. *The radius* r_n *of the* n*-Wulff shape* W_n *(see* (1.6) *and Theorem* 1.2*) with* $n > 6$*, satisfies* $r_n \geq 1$.

Proof. Let $n \in \mathbb{N}$ be such that there exists a ground state G_n with $r_{G_n} = 0$. Then G_n does not contain any set of the form $D_6 + q$ with $q \in \mathcal{L}$, that is, for every $x \in G_n$, there holds (see (3.12)):

$$x \notin G_n^2. \tag{5.1}$$

By (3.11) and (3.12), property (5.1) is equivalent to the claim that every element of $G_n \backslash G_n^0$ contributes to the overall perimeter of G_n, and the contribution of each element is exactly 2. Since we are assuming that G_n is connected (i.e. $|G_n^0| = 1$), this implies that

$$P(G_n) \geq 2(n-1).$$

By Proposition 3.4 it follows that

$$4(\alpha_n)^2 - 4\lfloor(\alpha_n)^2 - \alpha_n\rfloor - 6 = P(D_n) = p_n \geq 2(n-1),$$

which in turn implies

$$n - 1 \geq \lfloor(\alpha_n)^2 - \alpha_n\rfloor \geq (\alpha_n)^2 - \alpha_n,$$

and finally yields $n - \sqrt{6n} \leq 0$, that is $0 \leq n \leq 6$. $\square$

Fix $n \in \mathbb{N}$ and let G_n be a connected ground state. We aim at proving an estimate from below on the radius r_{G_n} of H_{G_n} in terms of the number n of atoms. We first introduce some definitions.

Definition 5.1. (Zigzag path) Let ℓ be a line orthogonal to one of the three diameters of a minimal simple cycle of the lattice and intersecting $\mathcal{L}$. The *zigzag path* identified by ℓ is the union of points $p \in \mathcal{L}$ such that either $p \in \ell$ or there exists a minimal simple cycle H of $\mathcal{L}$ such that p belongs to H, and the two atoms in H bonded to P are in ℓ. Note that each point of a zigzag path divides it into two *half-zigzag paths* (see Fig. 5).

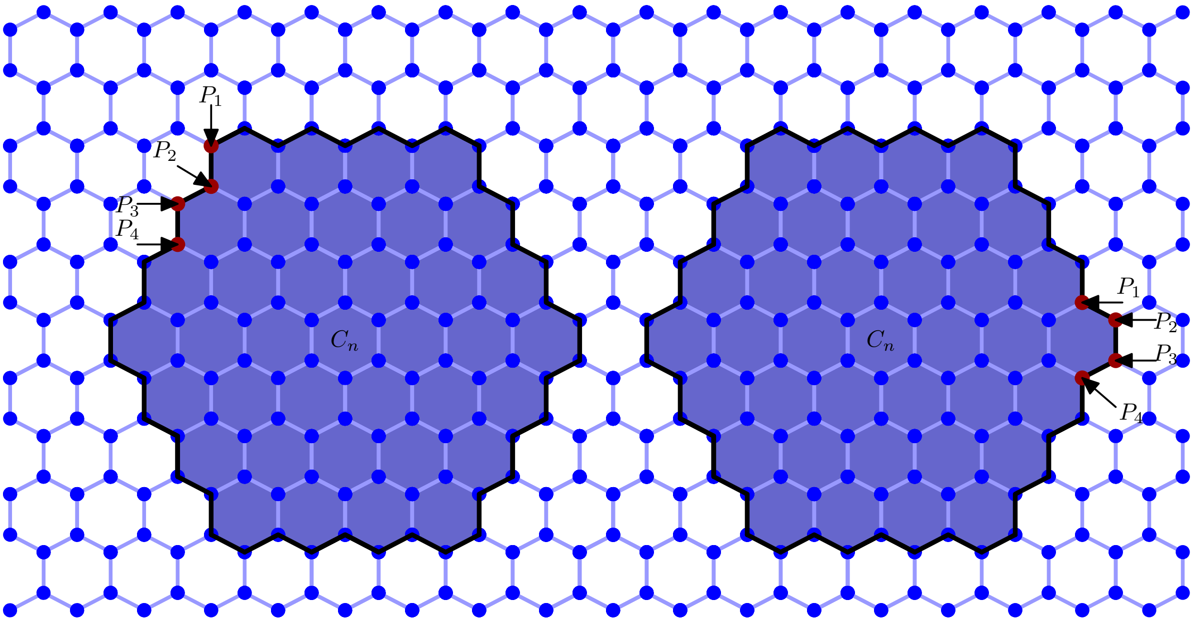

Fig. 4. On the left, the points $P_1, \ldots, P_4$ belong to a *side* of C_n, on the right the segment joining P_2 and P_3 is a *corner edge* of C_n.

Let $P_1, \ldots, P_4 \in \mathcal{L} \cap \partial F(C_n)$ be such that P_1 is bonded to P_2, and for $i = 2, 3$ the point P_i is bonded both to P_{i-1} and P_{i+1}. If there exists a unique zigzag path passing through all the points $P_1, \ldots, P_4$ we will say that this zigzag path is a *side* of C_n. If two different (nonparallel) zigzag paths intersect in the unitary segment joining P_2 and P_3 we will refer to this segment as a *corner edge of* C_n (see Fig. 4).

We will say that C_n has an *angle* α in a corner edge v (or in a point P) if the two lines ℓ^1_α and ℓ^2_α, identifying the sides of C_n and passing through v (respectively, P), intersect forming an angle of width α. The choice of α or $2\pi - \alpha$ will be clear

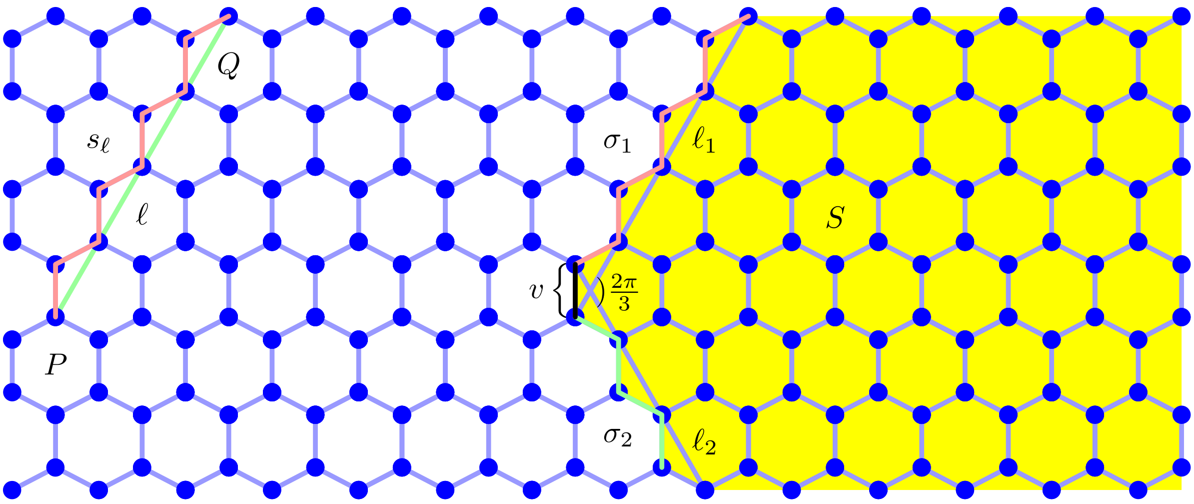

Fig. 5. (Color online) On the left, the zigzag path s_ℓ originated by the line ℓ. On the right, the two zigzag paths σ_1 and σ_2 intersect in the corner edge v, forming an angle $2\pi/3$. The associated angular sector S is marked in yellow.

from the context. Alternatively, we will say that the zigzag paths associated to ℓ^1_α and ℓ^2_α *form an angle* α *with* (*or in*) v (*respectively*, P).

Finally, if $S \subset \mathcal{L}$ is such that $\partial(F(S)) \cap \mathcal{L}$ (see Sec. 2) is the union of two zigzag paths forming an angle α, we will call S an *angular sector of width* α, see Fig. 5.

By Proposition 5.1 we can assume that $r_{G_n} \geq 1$. Let $v_0, \ldots, v_5$ be the corner edges of H_{G_n}, where v_0 is assumed to be lying on the x-axis (without loss of generality), and $v_1, \ldots, v_5$ are numbered counterclockwise starting from v_0. For $k = 0, \ldots, 4$, let s_k be the zigzag path joining v_k and v_{k+1}, and let s_5 be the zigzag path joining v_5 and v_0. Let l_k be the line identifying the path s_k, and denote by ν_k the unit normal to l_k pointing toward the exterior of H_{G_n}. We define

$$\lambda_k := \max\{j \in \mathbb{N} : s^j_k \cap G_n \neq 0\}, \tag{5.2}$$

where

$$s^j_k := s_k + j\nu_k$$

for $j \in \mathbb{N}$. Let π_k be also the subset of $\mathcal{L}$ such that $\partial F(\pi_k) = s_k$ and $\mathring{F}(\pi_k) \cap H_{G_n} = \emptyset$.

We show now that ground states satisfy a connectedness property with respect to zigzag paths.

Definition 5.2. (Hex-connectedness) Let $\mathcal{S}$ be a subset of $\mathcal{L}$ and let $P \in \mathcal{L}$. We say that P *disconnects a zigzag path in* $\mathcal{S}$ if $P \notin \mathcal{S}$ and there exist $P_a, P_b \in \mathcal{S}$ such that P_a and P_b are joined by a zigzag path passing through P.

Let $\mathcal{S}$ be a subset of $\mathcal{L}$. We say that $\mathcal{S}$ is *hex-connected* if every $P \in \mathcal{L}$ disconnects at most one zigzag path in $\mathcal{S}$.

Notice that from every point $P \in \mathcal{L}$ there are exactly three nonparallel lines which depart from P and identify a zigzag path (see Definition 5.1).

Proposition 5.2. *Ground states are hex-connected.*

Proof. For the sake of contradiction assume that there exists a ground state G_n which is not hex-connected. Then there exists a point $P \in \mathcal{L}$ which disconnects two zigzag paths in G_n. In particular, there exists a line ℓ_0 orthogonal to one of the diameters of a minimal simple cycle of the lattice, and intersecting $\mathcal{L}$, such that the two half-zigzag paths starting from P and identified by ℓ_0 are both intersecting G_n. Let $\ell_1, \ldots, \ell_m$ be the lines parallel to ℓ_0, intersecting G_n, and such that for every $i = 1, \ldots, m$, the distance between ℓ_i and ℓ_0 is given by $3n_i/2$, where $n_i \in \mathbb{N}$. For $i = 0, \ldots, m$, let c_i be the number of points of G_n contained in the zigzag path identified by ℓ_i.

We first rearrange the set $\{c_i\}$ in a decreasing order, constructing another set $\{d_i\}$ with the property that $d_0 \geq d_1 \geq \cdots \geq d_m$. Then, we separate the elements of $\{d_i\}$ having odd indexes from those having even indexes and we consider a new family $\{f_i\}$ obtained by first taking into account the elements of $\{d_i\}$ with even indexes, in decreasing order with respect to their indexes, and then the elements of $\{d_i\}$ having odd indexes, with increasing order with respect to their indexes. In particular we define:

$$f_i := \begin{cases} d_{m-2i} & i = 0, \dots, \dfrac{m}{2} \\ d_{2i-m-1} & i = \dfrac{m}{2} + 1, \dots, m \end{cases} \qquad \text{if } m \text{ is even, and}$$

$$f_i := \begin{cases} d_{m-1-2i} & i = 0, \dots, \dfrac{(m-1)}{2} \\ d_{2i-m} & i = \dfrac{(m+1)}{2}, \dots, m \end{cases} \qquad \text{if } m \text{ is odd.}$$

The set $\{f_i\}$ constructed above has the property that its central elements have the maximum value, and the values of the elements decrease in an alternated way by moving from the center of $\{f_i\}$ toward $i = 0$ and $i = m$. Let $\bar{\imath}$ and $\bar{\imath} + 1$ be the indexes corresponding to the two central elements of the set $\{f_i\}$, if m is odd, and to the central element of $\{f_i\}$ and the maximum between its two neighbors, if m is even. As an example, if we start with a set $\{c_i\} = \{3, 4, 7, 8, 2, 2, 8\}$, the family $\{d_i\}$ is given by $\{8, 8, 7, 4, 3, 2, 2\}$ and the set $\{f_i\}$ by $\{2, 3, 7, 8, 8, 4, 2\}$. Here $\bar{\imath} = 4$.

Fix two points $P_{\bar{\imath}}, P_{\bar{\imath}+1} \in \mathcal{L}_h$ such that the segment $P_{\bar{\imath}}P_{\bar{\imath}+1}$ has length one and is orthogonal to ℓ_0. Let σ_1 and σ_2 be two half-zigzag paths, starting from $P_{\bar{\imath}}$ and $P_{\bar{\imath}+1}$, respectively, forming an angle $2\pi/3$ with $P_{\bar{\imath}}P_{\bar{\imath}+1}$, and such that there exists a convex region S of the plane whose boundary is given by σ_1, σ_2, and $P_{\bar{\imath}}P_{\bar{\imath}+1}$.

Consider the points $P_0, \dots, P_{\bar{\imath}-1} \in \sigma_1$, defined as

$$|P_{\bar{\imath}} - P_j| = (\bar{\imath} - j)\sqrt{3}, \quad j = 0, \dots, \bar{\imath} - 1.$$

Analogously, consider the points $P_{\bar{\imath}+2}, \dots, P_m \in \sigma_2$, satisfying

$$|P_{\bar{\imath}+1} - P_j| = (j - \bar{\imath})\sqrt{3}, \quad j = \bar{\imath} + 2, \dots, m.$$

For $j = 0, \dots, m$, let $\tilde{\ell}_j$ be the line parallel to ℓ_0 and passing through P_j. To construct the set $\tilde{G}_n$ we consider f_j consecutive points in S on the zigzag path identified by each line $\tilde{\ell}_j$, starting from P_j (see Figs. 5 and 6). The set $\tilde{G}_n$ is clearly hex-connected, the number of bonds in the zigzag paths identified by the lines parallel to ℓ_0 has increased, and the number of bonds between parallel zigzag paths has not decreased. Hence

$$E(\tilde{G}_n) < E(G_n),$$

providing a contradiction to the optimality of the ground state G_n. □

As a corollary of Proposition 5.2 it follows that ground states have no *vacancies*.

Proposition 5.3. *Let G_n be a ground state. Then $F(G_n)$ is simply connected.*

Proof. By contradiction, if $F(G_n)$ is not simply connected then there exists a point in $\mathcal{L}$ that disconnects three zigzag paths in G_n. Therefore G_n is not hex-connected. □

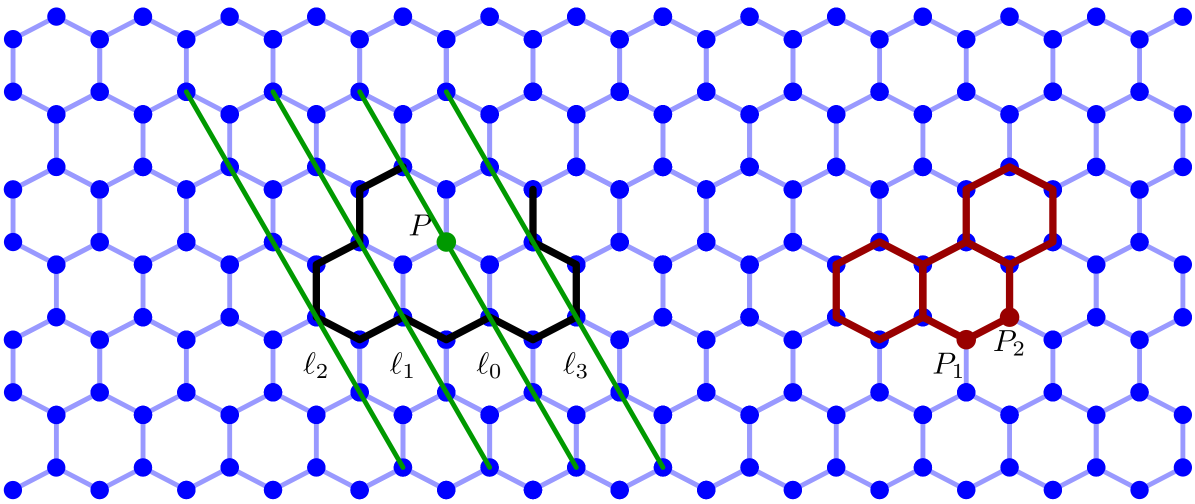

Fig. 6. (Color online) A configuration C_n before (in black) and after (in red) the rearrangement described in Proposition 5.2.

We conclude this overview on connectedness properties of ground states with the following proposition.

Proposition 5.4. *Ground states are connected.*

We omit the proof of this result as it follows by adapting the proof of Proposition 5.2.

In view of Propositions 5.2–5.4, the quantity λ_k defined in (5.2) provides the number of nonempty parallel zigzag paths of atoms in $G_n \cap \pi_k$. By the definition of τ, each partially full level of atoms around H_{G_n} is characterized by the fact that the difference between the number of points on the level having weight one and those having weight two is strictly positive. To be precise,

$$\sum_{k=0}^{5} \lambda_k \leq |G_n^1 \backslash H_{G_n}| - |G_n^2 \backslash H_{G_n}|.$$

On the other hand, by (3.11) and (3.12),

$$|G_n^1 \backslash H_{G_n}| - |G_n^2 \backslash H_{G_n}| = \frac{P(G_n) - P(H_{G_n})}{2} = \frac{p_n}{2} - 6r_{G_n} + 3,$$

thus yielding

$$\sum_{k=0}^{5} \lambda_k \leq \frac{p_n}{2} - 6r_{G_n} + 3. \tag{5.3}$$

Given a ground state G_n and its maximal hexagon $H_{G_n} := D_{6r_{G_n}^2} + q$, denote by $H_{G_n}^+$ and $H_{G_n}^{++}$ the sets $H_{G_n}^+ := D_{6(r_{G_n}+1)^2} + q$, and $H_{G_n}^{++} := D_{6(r_{G_n}+2)^2} + q$, respectively. Denote by v_i' and v_i'', $i = 0, \ldots, 5$, the corner edges of $H_{G_n}^+$ and $H_{G_n}^{++}$, respectively, with the convention that both v_i' and v_i'' are parallel to v_i. For $i = 0, \ldots, 5$, let V_i^1 and V_i^2 be the two extrema of v_i, numbered counterclockwise. Let $(V_i')^1$, $(V_i')^2$, $(V_i'')^1$, $(V_i'')^2$, s_i' and s_i'' be defined accordingly.

In the remaining part of this subsection we provide a characterization of the geometry of $G_n\backslash H^+_{G_n}$, by subdividing this set into *good* polygons P_k and *bad* polygons T_k, and by showing that the cardinality of $G_n\backslash H^+_{G_n}$ is, roughly speaking, the same as the one of the union of *good* polygons.

We first prove that, by the optimality of H_{G_n}, there exists a corner edge of $H^{++}_{G_n}$ which does not intersect G_n.

Proposition 5.5. *Let G_n be a ground state, and H_{G_n} be its maximal hexagon. Then there exists a corner edge v''_j of $H^{++}_{G_n}$, $j = 0, \ldots, 5$, which does not intersect G_n.*

Proof. By the maximality of H_{G_n}, there exists a point $P \in \partial F(H^+_{G_n})$ such that $P \notin G_n$.

If P does not disconnect s'_i then either v'_i or v'_{i+1} do not intersect G_n. By the hex-connectedness of G_n (see Proposition 5.2) then also the corresponding corner edge of $H^{++}_{G_n}$ does not intersect G_n, and we obtain the thesis.

Assume now that P disconnects s'_i. Since G_n is hex-connected, the point P does not disconnect any other zigzag path. Therefore there exists an angular sector S centered in P and of width $\pi/3$ such that

$$S \cap G_n = \emptyset. \tag{5.4}$$

Assume by contradiction that all corner edges of $H^{++}_{G_n}$ intersect G_n. In view of (5.4), the set $(G_n\backslash H_{G_n}) \cap \pi_i$ is subdivided into two components. Denoting them by Γ_1 and Γ_2, we have that $\Gamma_j \cap s^1_i \neq \emptyset$ and $\Gamma_j \cap s^2_i \neq \emptyset$, for $j = 1, 2$. Without loss of generality we can assume that $\Gamma_1 \cap s^{\lambda_i}_i \neq \emptyset$. Consider now the set $M := \Gamma_1 \cap (s^{\lambda_i}_i \cup s^{\lambda_i - 1}_i)$. We claim that we can construct a new set $\tilde{G}_n$, by rearranging the atoms of M and by leaving the other elements of G_n fixed, such that

$$E(\tilde{G}_n) < E(G_n). \tag{5.5}$$

There are three possible scenarios.

Case 1: Γ_1 contains at least two points P_a and P_b with the property that for each of them there is no minimal cycle passing through it and entirely contained in G_n. We proceed by moving the two points to $s^1_i \cap (\mathcal{L}\backslash G_n)$ in such a way that P_a is bonded to Γ_2. If possible, we move also P_b to $s^1_i \cap (\mathcal{L}\backslash G_n)$ so that P_a and P_b are bonded. If this is not possible because $s_i \cap (\mathcal{L}\backslash G_n)$ contains only one element, then we already created an extra bond. With this procedure we lose two bonds when removing P_a and P_b from Γ_1, but we gain at least three bonds when we attach them to Γ_2, therefore the total energy strictly decreases.

Case 2: In Γ_1 there exists exactly one point P_a with the property that there is no minimal cycle containing it and entirely contained in G_n. We argue moving this single point to $s^1_i \cap (\mathcal{L}\backslash G_n)$ in such a way that P_a is bonded to Γ_2. Afterward, we move iteratively all the (remaining) points in $s^{\lambda_i}_i \cap \Gamma_1$ to $s^1_i \cap (\mathcal{L}\backslash G_n)$ (in the same way as described in Case 1 for P_b). If after moving P_a there are no remaining points in $s_i^{\lambda_i} \cap \Gamma_1$, we apply the same rearrangement to $s_i^{\lambda_i - 1} \cap \Gamma_1$ (note that $\lambda_i \geq 2$ because all corner edges of $H_{G_n}^{++}$ intersect G_n). As a result of the procedure described above, the energy is strictly decreased. If at any moment during the process of attaching points to Γ_2 we create a bond between Γ_1 and Γ_2, we stop the rearrangement as the number of bonds has strictly increased.

Case 3: Every point of Γ_1 belongs to a minimal cycle entirely contained in G_n. In this case we first move all points in $s_i^{\lambda_i} \cap \Gamma_1$ but one, in the same way as described in Cases 1 and 2. As a result of this procedure, either we already created an extra bond (and hence there is nothing left to prove) or we are now in the same situation described in Case 2. The thesis follows then arguing exactly as in Case 2. $\square$

We proceed by showing that for every hexagon of side $r_{G_n} + 2$ there exists an angular sector of width $\pi/3$, and centered in one of its corner edges, which does not intersect G_n.

Proposition 5.6. *Let G_n be a ground state, and H_{G_n} be its maximal hexagon. Then*:

(i) *There exists a corner edge v_i'' of $H_{G_n}^{++}$, $i = 0, \ldots, 5$, and an angular sector S of width $\pi/3$, centered in $(V_i'')^1$ or $(V_i'')^2$, and such that $S \cap G_n = \emptyset$.*
(ii) *Every hexagon in $\mathcal{L}$ with side $r_{G_n} + 2$ has a corner edge and a corresponding angular sector of width $\pi/3$ which do not intersect G_n.*

Proof. By Proposition 5.5 we can assume that v_0'' does not intersect G_n. Assume first that both $(V_0'')^1$ and $(V_0'')^2$ do not disconnect any zigzag path. Consider the two half-zigzag paths in which v_0'' divides s_0''. Then at least one of them does not intersect G_n. Analogously, at least one of the two half-zigzag paths in which v_0' divides s_5' does not intersect G_n. Finally, the two half-zigzag paths, departing from $(V_0')^1$ and $(V_0')^2$, not parallel to s_0 and s_5, and in the opposite direction with respect to the center of H_{G_n}, do not intersect G_n. According to the geometric position of the four half-zigzag paths identified beforehand, and using again the hex-connectedness of G_n we obtain (i), the sector S being of width $2\pi/3$. The case in which at least one between $(V_0'')^1$ and $(V_0'')^2$ disconnects one zigzag path (see Proposition 5.2) follows accordingly, yielding a sector S of width $\pi/3$. The proof of (ii) is an adaptation of the proof of (i). $\square$

Without loss of generality, in view of Proposition 5.6 we can assume that $v_0'' \not\subseteq G_n$. For $k = 2, 3, 4, 5$, let π_k' be the subset of $\mathcal{L}$ such that

$$\begin{cases} \pi_k' = F(\pi_k') \cap \mathcal{L}, \\ \partial F(\pi_k') \cap \mathcal{L} = s_k^{\lambda_k}, \\ H_{G_n} \subset \pi_k'. \end{cases}$$

Consider the set $R := \left(\bigcap_{k=0}^{5} \pi_k' \right) \Big\backslash H_{G_n}^{+}$.

By construction, $G_n \subset H^+_{G_n} \cup R$, and for every $x \in R$ and $k = 0, \dots, 5$ there exists

$$j_k \in \left[-\lambda_{\left(\frac{k+3}{6} - \lfloor \frac{k+3}{6} \rfloor\right)} - 2r_{G_n} - 2, \lambda_k \right],$$

such that $x \in s_k^{j_k}$. In particular, every $x \in R$ is uniquely determined by a pair of indexes $(j_k, j_{k'})$, with $k' \neq k + 3$ in $\mathbb{Z}_6$.

We subdivide the region R into disjoint polygons as

$$R = \left(\bigcup_{j=0}^{5} P_j \right) \cup \left(\bigcup_{j=0}^{5} T_j \right). \tag{5.6}$$

For $a \in [-2(r_{G_n} + 1), 2(r_{G_n} + 1)]$, denote by $P_k^1(a)$ the subset of $\mathcal{L}$ enclosed by s_k^2, s_k^a, s_{k+1}^1, $s_{k+1}^{-r_{G_n}}$; and by $P_k^2(a)$ the set delimited by s_k^a, $s_k^{\lambda_k}$, $s_{k-1}^{\lambda_{k-1} - r_{G_n}}$, $s_{k-1}^{\lambda_{k-1}}$. For $k = 0, \dots, 5$, the sets P_k in (5.6) are defined as follows:

$$P_k := \begin{cases} P_k^1(\lambda_k) \cap R & \text{if } \lambda_k \leq \lambda_{k-1} + 1, \\ (P_k^1(\lambda_k - \lambda_{k-1} + 1) \cap R) \cup P_k^2(\lambda_k - \lambda_{k-1} + 1) & \text{if } \lambda_k > \lambda_{k-1} + 1, \end{cases}$$

with the convention that $\lambda_{-1} := \lambda_5$. Note that $|P_k| = 2(r_{G_n} + 1)(\lambda_k - 1)$ for every $k = 0, \dots, 5$.

The sets T_k are given, roughly speaking, by the "portions of $\mathcal{L}$" between P_{k-1} and P_k. To be precise,

$$T_k := \Big\{ x \in R : x \in s_{k-1}^{j_{k-1}} \cap s_k^{j_k}, \text{ with } 2 \leq j_{k-1} \leq \lambda_{k-1},\ 2 \leq j_k \leq \lambda_k,\ j_{k-1} \geq j_k$$
$$\text{and, if } \lambda_{k-1} > \lambda_{k-2} + 1,\ j_{k-1} \leq j_k + \lambda_{k-1} - \lambda_{k-2} \Big\}, \tag{5.7}$$

see Fig. 7.

We have that

$$n \leq |H^+_{G_n}| + |R| - |R \backslash G_n|,$$

where $|H^+_{G_n}| = 6(r_{G_n} + 1)^2$. We observe that

$$|R| = \sum_{j=0}^{5} |P_j| + \sum_{j=0}^{5} |T_j| = 2(r_{G_n} + 1) \sum_{j=0}^{5} (\lambda_j - 1) + \sum_{j=0}^{5} |T_j| - 1.$$

We proceed now in counting the points in R which do not belong to the ground state G_n. In particular, we prove a lower bound for such number in terms of the cardinality of

$$\mathcal{H} := \{ H \subset \mathcal{L} \cap (H^+_{G_n} \cup R) : H \text{ is a hexagon of side } r_{G_n} + 2 \}.$$

Proposition 5.7.

$$|R \backslash G_n| \geq 2|\mathcal{H}|.$$

Proof. Set $M := |\mathcal{H}|$. We show by induction on $m = 1, \dots, M$ that for every family $\mathcal{H}_m \subset \mathcal{H}$ with $|\mathcal{H}_m| = m$, there exists a collection of pairs of bonded atoms

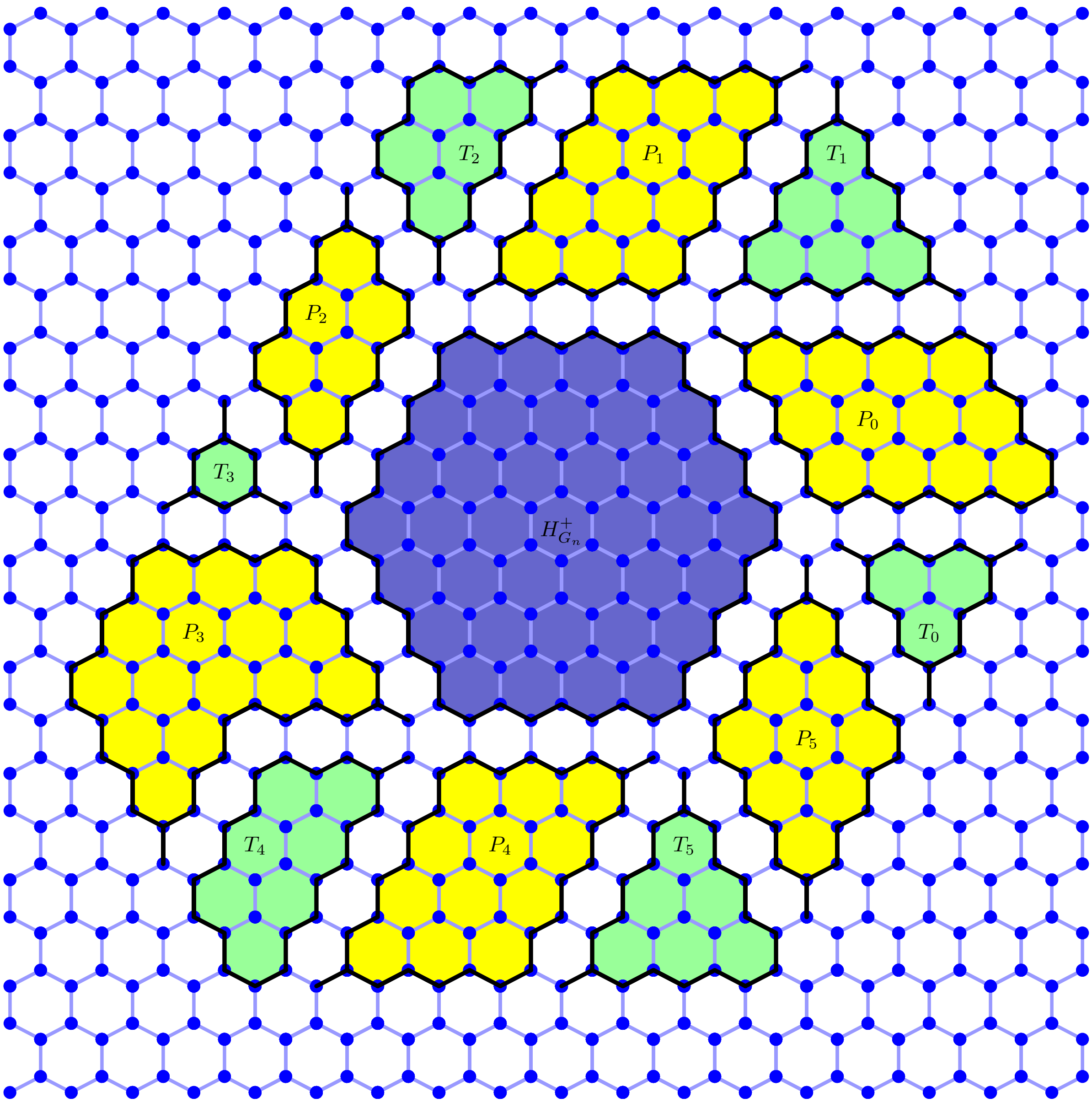

Fig. 7. The structure of a ground state G_n. In the figure above, $r_{G_n} = 3$, $\lambda_0 = 6$, $\lambda_1 = 6$, $\lambda_2 = 4$, $\lambda_3 = 7$, $\lambda_4 = 6$, and $\lambda_5 = 5$. The blue shapes outlined in black, the yellow shapes outlined in black, and the green shapes outlined in black correspond to the closed subsets of the plane associated to $H^+_{G_n}$, to the parallelograms P_j, and to the triangles T_j, respectively.

$V_{\mathcal{H}_m} \subset R\backslash G_n$ with $|V_{\mathcal{H}_m}| = m$ satisfying the following property: identifying each segment with its extrema, the correspondence associating to each pair $(\nu_1, \nu_2) \in V_{\mathcal{H}_m}$ a hexagon $H \in \mathcal{H}_m$ having a corner edge in (ν_1, ν_2) is a bijection.

We remark that the thesis will follow once we prove the assertion for $m = M$. For $m = 1$ the claim holds by Proposition 5.6. Assume now that the claim is satisfied for $m = \bar{m}$. Consider a family $\mathcal{H}_{\bar{m}+1} = \{H_1, \ldots, H_{\bar{m}+1}\} \subset \mathcal{H}$, and the polygon $\mathcal{P}_{\bar{m}+1} := \bigcup_{i=1}^{\bar{m}+1} H_i \subset R$. We subdivide the remaining part of the proof into four steps.

Step 1: There exists a corner edge $(\tilde{\nu}_1, \tilde{\nu}_2)$ of $\mathcal{P}_{\bar{m}+1}$ such that $\tilde{\nu}_i \in \mathcal{P}_{\bar{m}+1}\backslash G_n$ for $i = 1, 2$.

Indeed, assume by contradiction that every corner edge of $\mathcal{P}_{\bar{m}+1}$ belongs to G_n. Then every corner edge of $H_{\bar{m}+1}$ in $\mathring{F}(\mathcal{P}_{\bar{m}+1})$ would belong to G_n by Proposition 5.3. Thus all corner edges of $H_{\bar{m}+1}$ would belong to G_n, contradicting Proposition 5.5.

Step 2: By Proposition 5.6(ii) there exists an angular sector S of width at least $\pi/3$, centered in $\tilde{\nu}_1$, $\tilde{\nu}_2$, or in $(\tilde{\nu}_1, \tilde{\nu}_2)$, and such that $\bar{S} \cap G_n = \emptyset$. Denote by σ_1 and σ_2 the two zigzag paths forming its boundary.

Step 3: We claim that there exists a corner edge (ν_1, ν_2) of $\mathcal{P}_{\bar{m}+1}$ such that $(\nu_1, \nu_2) \subset \mathcal{P}_{\bar{m}+1} \backslash G_n$, and (ν_1, ν_2) is associated to an angle $2\pi/3$ of $\mathcal{P}_{\bar{m}+1}$.

Observe that $\mathcal{P}_{\bar{m}+1}$ can have corner edges with angles $2\pi/3$, $4\pi/3$ or $5\pi/3$. If the corner edge $(\tilde{\nu}_1, \tilde{\nu}_2)$ found in Step 2 is associated to an angle $2\pi/3$, there is nothing to prove. If $(\tilde{\nu}_1, \tilde{\nu}_2)$ corresponds to an angle $4\pi/3$ or $5\pi/3$, then there are two possible cases:

Case 1: $\mathring{F}(S) \cap \mathcal{P}_{\bar{m}+1} = \emptyset$. Then, for every $j = 1, 2$, there exists $\hat{\nu}_j \in \sigma_j$ such that $\hat{\nu}_j$ is one of the two extrema of a corner edge of $\mathcal{P}_{\bar{m}+1}$ entirely contained in S, and at least one among the zigzag paths from $\hat{\nu}_1$ to $\tilde{\nu}_1$ and from $\hat{\nu}_2$ to $\tilde{\nu}_2$ is contained in $\partial F(\mathcal{P}_{\bar{m}+1})$. In addition, the corner edge associated to such $\tilde{\nu}_j$ does not intersect G_n (because it is a subset of S), and is associated to an angle $2\pi/3$ (since $\mathring{F}(S) \cap \mathcal{P}_{\bar{m}+1} = \emptyset$). The proof follows by considering the corner edge associated to $\hat{\nu}_j$.

Case 2: $\mathring{F}(S) \cap \mathcal{P}_{\bar{m}+1} \neq \emptyset$. Let ℓ_1 and ℓ_2 be the lines generating σ_1 and σ_2, and let n_1 and n_2 be the unit normal vectors to ℓ_1 and ℓ_2, respectively, pointing outside S. Define

$$\sigma_1^k := \sigma_1 - \frac{3}{2} k n_1 \quad \text{and} \quad \sigma_2^k := \sigma_2 - \frac{3}{2} k n_2,$$

for $k \in \mathbb{N}$. Since $\mathcal{P}_{\bar{m}+1} \cap S$ is bounded, we can find

$$k_1 := \max\{k \in \mathbb{N} : \sigma_1^k \cap \mathcal{P}_{\bar{m}+1} \cap S \neq \emptyset\}$$

and

$$k_2 := \max\{k \in \mathbb{N} : \sigma_2^k \cap \mathcal{P}_{\bar{m}+1} \cap S \neq \emptyset\}.$$

For $j = 1, 2$, the intersection $\sigma_j^{k_j} \cap \partial F(\mathcal{P}_{\bar{m}+1}) \cap S$ is either a corner edge of $\mathcal{P}_{\bar{m}+1}$ associated to an angle $2\pi/3$, or a zigzag path forming an angle $2\pi/3$ with a corner edge of $\mathcal{P}_{\bar{m}+1}$.

Step 4: Let (ν_1, ν_2) be the corner edge provided by Step 3. Then, there exists a unique $H_{\bar{j}} \in \mathcal{H}_{\bar{m}+1}$ having a corner edge identified by (ν_1, ν_2). Thus, by the induction hypothesis on $\{H_1, \ldots, H_{\bar{m}+1}\} \backslash \{H_{\bar{j}}\}$, there exists a family of corner edges

$$\{(\nu_1^j, \nu_2^j)\}_{j=1,\ldots,\bar{m}+1,\, j \neq \bar{j}} \subset R \backslash G_n$$

such that, for every j, (ν_1^j, ν_2^j) is a corner edge of H_j, and for every $i \neq j$ (ν_1^j, ν_2^j) is not a corner edge of H_i. The thesis follows by setting $(\nu_1^{\bar{j}}, \nu_2^{\bar{j}}) = (\nu_1, \nu_2)$, and taking $V_{\mathcal{H}_{\bar{m}+1}} = \{(\nu_1^1, \nu_2^1), \ldots, (\nu_1^{\bar{m}+1}, \nu_2^{\bar{m}+1})\}$. □

The next step consists in estimating $|\mathcal{H}|$ from below, in terms of the cardinality of the sets T_j and the number of levels λ_j.

Proposition 5.8.

$$2|\mathcal{H}| \geq \sum_{j=0}^{5} |T_j| - 2\lambda_1 - 4\lambda_2 - 4\lambda_3 - 4\lambda_4 - 2\lambda_5 + 18.$$

Proof. For $k = 2, 3, 4, 5$, let U_k be the region of $\mathcal{L}$ containing $H^+_{G_n}$ and delimited by the zigzag paths s^1_{k+1}, s^1_{k+2} and s^1_{k+3} (with $s^1_6 := s^1_0$, $s^1_7 := s^1_1$, and $s^1_8 := s^1_2$). Let $\mathcal{H}_k := \{H \in \mathcal{H} : H \subset U_k$ and has a vertex in $T_k\}$, $k = 2, 3, 4, 5$.

We claim that

$$|\mathcal{H}_k| \geq \frac{|T_k| - 2(\lambda_k + \lambda_{k-1} - 3)}{2}. \tag{5.8}$$

Indeed, let $(\tilde{x}, \hat{x}) \in T_k$ and consider $(\tilde{j}_k, \tilde{j}_{k-1}, \tilde{j}_{k-2})$ such that $\tilde{x} \in s^{\tilde{j}_k}_k \cap s^{\tilde{j}_{k-1}}_{k-1} \cap s^{\tilde{j}_{k-2}}_{k-2}$ and $\hat{x} \in s^{\tilde{j}_k}_k \cap s^{\tilde{j}_{k-1}}_{k-1} \cap s^{\tilde{j}_{k-2}-1}_{k-2}$. We identify $\tilde{x}$ and $\hat{x}$ with the triple of indexes $(\tilde{j}_k, \tilde{j}_{k-1}, \tilde{j}_{k-2})$ and $(\tilde{j}_k, \tilde{j}_{k-1}, \tilde{j}_{k-2} - 1)$, and we write $\tilde{x} = (\tilde{j}_k, \tilde{j}_{k-1}, \tilde{j}_{k-2})$ and $\hat{x} = (\tilde{j}_k, \tilde{j}_{k-1}, \tilde{j}_{k-2} - 1)$. Let $H_{\tilde{x},\hat{x}}$ be the hexagon with corner edges identified by the pairs $(\tilde{x}, \hat{x})$, and the pairs:

$$\begin{aligned}
w_1 &:= [(\tilde{j}_k - (r_{G_n} + 2), \tilde{j}_{k-1}, \tilde{j}_{k-2} + (r_{G_n} + 1)), (\tilde{j}_k - (r_{G_n} + 1), \\
&\qquad \tilde{j}_{k-1}, \tilde{j}_{k-2} + (r_{G_n} + 1))], \\
w_2 &:= [(\tilde{j}_k - (2r_{G_n} + 3), \tilde{j}_{k-1} - (r_{G_n} + 2), \tilde{j}_{k-2} + (r_{G_n} + 1)), \\
&\qquad (\tilde{j}_k - (2r_{G_n} + 3), \tilde{j}_{k-1} - (r_{G_n} + 1), \tilde{j}_{k-2} + (r_{G_n} + 1))], \\
w_3 &:= [(\tilde{j}_k - (2r_{G_n} + 3), \tilde{j}_{k-1} - (2r_{G_n} + 3), \tilde{j}_{k-2} - 1), \\
&\qquad (\tilde{j}_k - (2r_{G_n} + 3), \tilde{j}_{k-1} - (2r_{G_n} + 3), \tilde{j}_{k-2})], \\
w_4 &:= [(\tilde{j}_k - (r_{G_n} + 1), \tilde{j}_{k-1} - (2r_{G_n} + 3), \tilde{j}_{k-2} - (r_{G_n} + 2)), \\
&\qquad (\tilde{j}_k - (r_{G_n} + 2), \tilde{j}_{k-1} - (2r_{G_n} + 3), \tilde{j}_{k-2} - (r_{G_n} + 2))], \\
w_5 &:= [(\tilde{j}_k, \tilde{j}_{k-1} - (r_{G_n} + 2), \tilde{j}_{k-2} - (r_{G_n} + 2)), \\
&\qquad (\tilde{j}_k, \tilde{j}_{k-1} - (r_{G_n} + 1), \tilde{j}_{k-2} - (r_{G_n} + 2))].
\end{aligned}$$

We observe that $H_{\tilde{x},\hat{x}}$ is contained in U_k and has a corner edge in T_k if the following inequalities are satisfied:

$$\begin{aligned}
\tilde{j}_k - (2r_{G_n} + 3) &\geq -2r_{G_n} - 1, \quad \tilde{j}_k \leq \lambda_k, \\
\tilde{j}_{k-1} - (2r_{G_n} + 3) &\geq -2r_{G_n} - 1, \quad \tilde{j}_{k-1} \leq \lambda_{k-1}, \\
\tilde{j}_{k-2} - (r_{G_n} + 2) &\geq -2r_{G_n} - 1, \quad \tilde{j}_{k-2} + r_{G_n} + 1 \leq \lambda_{k-2}.
\end{aligned}$$

Hence, if $(\tilde{j}_k, \tilde{j}_{k-1}, \tilde{j}_{k-2})$ is such that:

$$2 \le \tilde{j}_k \le \lambda_k,$$
$$2 \le \tilde{j}_{k-1} \le \lambda_{k-1},$$
$$-r_{G_n} + 1 \le \tilde{j}_{k-2} \le \lambda_{k-2} - r_{G_n} - 1,$$

then $H_{\tilde{x},\hat{x}} \subset U_k$ and has a corner edge in T_k. By the definition of the sets T_k (see (5.7)), the previous properties are fulfilled by every $x \in T_k$, apart from those points belonging to the portion of $\partial F(T_k)$ which is adjacent either to P_{k-1} or to P_k. Denoting by $\tilde{T}_k$ this latter set, claim (5.8) follows once we observe that

$$|\tilde{T}_k| = \frac{|T_k| - 2(\lambda_k + \lambda_{k-1} - 3)}{2}. \qquad \square$$

Combining Propositions 5.7 and 5.8 we estimate from above and from below the radius r_{G_n} of the maximal hexagon H_{G_n}.

Proposition 5.9.

$$\rho_n \le r_n \le R_n \le \rho_n + \frac{2}{3}\sqrt{[(\alpha_n)^2 - \lfloor(\alpha_n)^2 - \alpha_n\rfloor]^2 - (\alpha_n)^2 + 39},$$

where r_n *is the quantity defined in* (1.6), $R_n := \max\{r_{G_n} : G_n$ *is a ground state with* n *atoms*$\}$, *and*

$$\rho_n := \frac{(\alpha_n)^2}{3} - \frac{\lfloor(\alpha_n)^2 - \alpha_n\rfloor}{3} - 3 - \frac{1}{3}\sqrt{[(\alpha_n)^2 - \lfloor(\alpha_n)^2 - \alpha_n\rfloor]^2 - (\alpha_n)^2 + 39}, \tag{5.9}$$

with $\alpha_n = \sqrt{3n/2}$.

Proof. *By Propositions* 5.7 *and* 5.8, *we have*

$$|R \backslash G_n| \ge \sum_{j=0}^{5} |T_j| - 2\lambda_1 - 4\lambda_2 - 4\lambda_3 - 4\lambda_4 - 2\lambda_5 + 18.$$

Therefore, by (5.3) *we obtain*

$$\begin{aligned}
n &\le |H^+_{G_n}| + |R| - |R\backslash G_n| \\
&\le 6(r_{G_n}+1)^2 + \sum_{j=0}^{5}|P_j| + \sum_{j=0}^{5}|T_j| - \sum_{j=0}^{5}|T_j| + 2\lambda_1 + 4\lambda_2 + 4\lambda_3 + 4\lambda_4 + 2\lambda_5 - 18 \\
&\le 6(r_{G_n}+1)^2 + (2r_{G_n}+6)\left(\sum_{k=0}^{5}\lambda_k - 1\right) + 2 \\
&= 6(r_{G_n}+1)^2 + (r_{G_n}+3)(p_n - 12r_{G_n} - 6) + 2 \\
&= -6(r_{G_n}+1)^2 + (p_n - 18)(r_{G_n}+1) + 2p_n + 14.
\end{aligned}$$

The thesis follows by solving the inequality with respect to $r_{G_n}+1$ *and using the definitions of* r_n, p_n *and* α_n *(see Proposition* 3.4*).* $\square$

We conclude this section with a refinement of the estimate from above on r_n.

Proposition 5.10.

$$r_n \leq \rho_n + \mathrm{O}(1).$$

Proof. For every $n \in \mathbb{N}$, $n > 6$, let

$$\tilde{\rho}_n := \left\lceil \frac{(\alpha_n)^2}{3} - \frac{\lfloor (\alpha_n)^2 - \alpha_n \rfloor}{3} - \frac{1}{3}\sqrt{[(\alpha_n)^2 - \lfloor (\alpha_n)^2 - \alpha_n \rfloor]^2 - (\alpha_n)^2} \right\rceil. \tag{5.10}$$

Let $H_{\tilde{\rho}_n} := D_{6\tilde{\rho}_n^2}$, and

$$h_n := \frac{p_n - P(H_{\tilde{\rho}_n})}{4} = \frac{p_n - 6(2\tilde{\rho}_n - 1)}{4} = \frac{p_n}{4} - 3\tilde{\rho}_n + \frac{3}{2}.$$

Consider the hexagonal configurations C_c given by the union of the hexagons $H_{\tilde{\rho}_n}$ with the "parallelograms" of height h_n constructed on two consecutive sides of $H_{\tilde{\rho}_n}$. To be precise, denoting by $s_0^n, \ldots, s_5^n$ the zigzag paths passing through the sides of $H_{\tilde{\rho}_n}$, and setting

$$s_k^{n,j} := s_k^n + j\mathbf{e}_k, \quad k = 0, \ldots, 5,$$

for every $n \in \mathbb{N}$, define the set C_c to be the portion of $\mathcal{L}$ enclosed by the zigzag paths $s_0^n, s_1^{n,h_n}, s_2^{n,h_n}, s_3^n, s_4^n, s_5^n$.

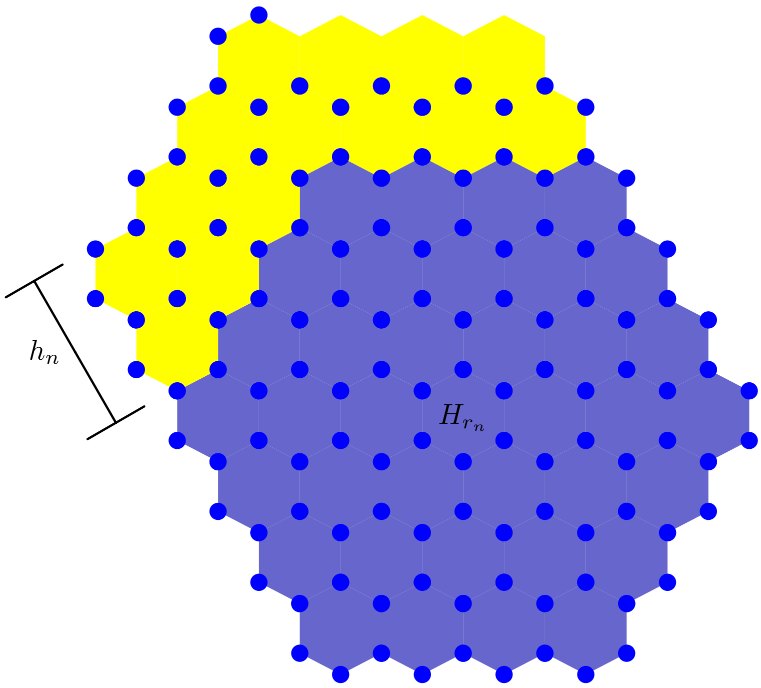

Fig. 8. (Color online) In the figure above, $n = 120$, $\tilde{\rho}_n = 4$ and $h_n = 2$. The configuration C_c is defined as the union of $H_{\tilde{\rho}_n}$ with the two yellow parallelograms of height h_n, constructed on the sides of $H_{\tilde{\rho}_n}$. The ground state G_n (given by the collection of the blue atoms) satisfies $H_{\tilde{\rho}_n} \subseteq G_n \subseteq C_c$, and $|C_c \backslash G_n| \leq 2\tilde{\rho}_n - 1$.

By construction, the perimeter of C_c satisfies

$$P(C_c) = p_n.$$

We claim that for n big enough there exists a ground state G_n such that $H_{\tilde{\rho}_n} \subseteq G_n \subseteq C_c$, and $|C_c \backslash G_n| \leq 2\tilde{\rho}_n - 1$. Indeed,

$$\begin{aligned} |C_c| = |H_{\tilde{\rho}_n}| + 4\tilde{\rho}_n h_n &= 6\tilde{\rho}_n^2 + \tilde{\rho}_n[p_n - 6(2\tilde{\rho}_n - 1)] \\ &= -6\tilde{\rho}_n^2 + (p_n + 6)\tilde{\rho}_n. \end{aligned} \tag{5.11}$$

A direct computation shows that

$$6s^2 - (p_n + 6)s + n \leq 0$$

for every s satisfying

$$\begin{aligned} s \in \Bigg[&\frac{(\alpha_n)^2}{3} - \frac{\lfloor (\alpha_n)^2 - \alpha_n \rfloor}{3} - \frac{1}{3}\sqrt{[(\alpha_n)^2 - \lfloor (\alpha_n)^2 - \alpha_n \rfloor]^2 - (\alpha_n)^2}, \\ &\cdot \frac{(\alpha_n)^2}{3} - \frac{\lfloor (\alpha_n)^2 - \alpha_n \rfloor}{3} + \frac{1}{3}\sqrt{[(\alpha_n)^2 - \lfloor (\alpha_n)^2 - \alpha_n \rfloor]^2 - (\alpha_n)^2} \Bigg], \end{aligned} \tag{5.12}$$

whereas

$$6s^2 - (4 + p_n)s + n - 1 \geq 0$$

for every $s \in \mathbb{R}$ such that

$$\begin{aligned} s \leq -\frac{1}{6} &+ \frac{(\alpha_n)^2}{3} - \frac{\lfloor (\alpha_n)^2 - \alpha_n \rfloor}{3} \\ &- \frac{1}{3}\sqrt{[(\alpha_n)^2 - \lfloor (\alpha_n)^2 - \alpha_n \rfloor]^2 - (\alpha_n)^2 - \frac{1}{4} - p_n} \end{aligned}$$

or

$$\begin{aligned} s \geq -\frac{1}{6} &+ \frac{(\alpha_n)^2}{3} - \frac{\lfloor (\alpha_n)^2 - \alpha_n \rfloor}{3} \\ &+ \frac{1}{3}\sqrt{[(\alpha_n)^2 - \lfloor (\alpha_n)^2 - \alpha_n \rfloor]^2 - (\alpha_n)^2 - \frac{1}{4} - p_n}. \end{aligned} \tag{5.13}$$

In particular, (5.12) and (5.13) hold for $s = \tilde{\rho}_n$, yielding

$$0 \leq |C_c| - n \leq 2\tilde{\rho}_n - 1. \tag{5.14}$$

The claim follows by (5.14), and by observing that by the definition of C_c it is possible to remove up to $2\tilde{\rho}_n - 1$ points from $C_c \backslash H_{\tilde{\rho}_n}$ without changing the perimeter of the configuration. In particular, $H_{G_n} = H_{\tilde{\rho}_n}$. The thesis is thus a direct consequence of (1.6), (5.9) and (5.10). $\square$

6. Sharp Convergence to the Wulff Shape

In this section we prove that as the number n of atoms tends to infinity, ground states differ from a hexagonal Wulff shape by at most $\mathrm{O}(n^{3/4})$ points and we show that this estimate is sharp. The proof strategy consists in exploiting Proposition 5.9 to deduce an upper bound on the number of points belonging to G_n but not to the n-Wulff shape W_n.

Let W be the regular hexagon defined as the convex hull of the vectors

$$\left\{\pm\frac{1}{\sqrt{6}}t_1,\ \pm\frac{1}{\sqrt{6}}t_2,\ \pm\frac{1}{\sqrt{6}}(t_2-t_1)\right\},$$

and let χ_W be its characteristic function. Denote by μ the measure

$$\mu := \frac{4}{\sqrt{3}}\chi_W.$$

6.1. *Proof of Theorem* 1.2

We subdivide the proof into two steps.

Step 1: Let G_n be a ground state. Without loss of generality, assume that $n > 6$, and hence, by Proposition 5.1, that the maximal hexagon H_{G_n} is not degenerate and $r_n \geq 1$. Let $q_n \in \mathcal{L}$ be such that $H_{G_n} = D_{6r^2_{G_n}} + q_n$. We claim that

$$d_{\mathcal{H}}(G'_n, W_n) \leq \mathrm{O}(n^{1/4}), \tag{6.1}$$

and

$$|G'_n \backslash W_n| = K_n n^{3/4} + \mathrm{o}(n^{3/4}), \tag{6.2}$$

where

$$G'_n := G_n - q_n$$

and

$$K_n := \frac{4\alpha_n}{3n^{3/4}}\sqrt{\left((\alpha_n)^2 - \lfloor(\alpha_n)^2-\alpha_n\rfloor\right)^2 - (\alpha_n)^2}, \tag{6.3}$$

with $\alpha_n = \sqrt{3n/2}$.

Indeed, we first observe that

$$d_{\mathcal{H}}(G_n, H_{G_n}) \leq \max_{i=0,\ldots,5} \lambda_i.$$

In view of (5.3) and of Proposition 5.9, we obtain the upper bounds

$$\begin{aligned} d_{\mathcal{H}}(G_n, H_{G_n}) &\leq 2(\alpha_n^2) - 2\lfloor(\alpha_n)^2 - \alpha_n\rfloor - 6\rho_n \\ &\leq 18 + 2\sqrt{[(\alpha_n)^2 - \lfloor(\alpha_n)^2-\alpha_n\rfloor]^2 - (\alpha_n)^2 + 39}, \end{aligned}$$

and

$$d_{\mathcal{H}}(H_{G'_n}, W_n) \le r_{G_n} - r_n \le \frac{2}{3}\sqrt{[(\alpha_n)^2 - \lfloor(\alpha_n)^2 - \alpha_n\rfloor]^2 - (\alpha_n)^2 + 39}.$$

On the other hand, Propositions 5.9 and 5.10 yield the equality

$$\begin{aligned}
|G'_n \backslash W_n| &= n - 6r_n^2 \\
&= n - 6\rho_n^2 - 6(r_n + \rho_n)(r_n - \rho_n) \\
&= n - 6\rho_n^2 + \mathrm{O}(n^{1/2}) \\
&= n - 6\Bigg(\frac{\alpha_n}{3} + \frac{(\alpha_n)^2 - \alpha_n}{3} - \frac{\lfloor(\alpha_n)^2 - \alpha_n\rfloor}{3} - 3 \\
&\qquad - \frac{1}{3}\sqrt{[(\alpha_n)^2 - \lfloor(\alpha_n)^2 - \alpha_n\rfloor]^2 - (\alpha_n)^2 + 39}\Bigg)^2 + \mathrm{o}(n^{3/4}) \\
&= n - 6\left(\frac{(\alpha_n)^2}{9} - \frac{2\alpha_n}{9}\sqrt{[(\alpha_n)^2 - \lfloor(\alpha_n)^2 - \alpha_n\rfloor]^2 - (\alpha_n)^2 + 39}\right) + \mathrm{o}(n^{3/4}) \\
&= \frac{4\alpha_n}{3}\sqrt{[(\alpha_n)^2 - \lfloor(\alpha_n)^2 - \alpha_n\rfloor]^2 - (\alpha_n)^2} + \mathrm{o}(n^{3/4}). \qquad (6.4)
\end{aligned}$$

Claims (6.1) and (6.2) follow now by the definition of α_n and by the observation that

$$\sqrt{[(\alpha_n)^2 - \lfloor(\alpha_n)^2 - \alpha_n\rfloor]^2 - (\alpha_n)^2} = \sqrt{2\eta_n\alpha_n + \eta_n^2} \le 1 + \sqrt{\alpha_n} = \mathrm{O}(n^{1/4}),$$

where $\eta_n := (\alpha_n)^2 - \alpha_n - \lfloor(\alpha_n)^2 - \alpha_n\rfloor$.

Step 2: Step 1 yields the equality

$$\|\mu_{G'_n} - \mu_{W_n}\| = \frac{|G'_n \Delta W_n|}{n} = K_n n^{-1/4} + \mathrm{o}(n^{-1/4}), \qquad (6.5)$$

where $\mu_{G'_n}$ and μ_{W_n} are the empirical measures associated to G'_n and W_n, respectively (see (2.3)). Let $\mu_n := \mu_{W_n}$.

For every $x_i \in W_n$, denote by Ω_i its Voronoi cell in $\mathcal{L}$, that is the equilateral triangle centered in x_i, of side $\sqrt{3}$ and with edges orthogonal to the three lattice directions. Finally, define Ω_i^n as the set

$$\Omega_i^n := \{x/\sqrt{n} : x \in \Omega_i\}.$$

Let $\varphi \in W^{1,\infty}(\mathbb{R}^2)$. We observe that

$$\left\| \frac{x_i}{\sqrt{n}} - x \right\|_{L^\infty(\Omega_i^n)} \le \sqrt{\frac{3}{n}},$$

and by (6.4),

$$\begin{aligned}
\mathcal{L}^2\left(\left(\bigcup_{i=1}^{6r_n^2} \Omega_i^n\right) \Delta W\right) &= \left|\frac{3\sqrt{3}}{2}\left(\frac{\sqrt{3}r_n}{\sqrt{n}}\right)^2 - \frac{3\sqrt{3}}{4}\right| = \frac{3\sqrt{3}}{4}\left|\frac{6r_n^2}{n} - 1\right| \\
&= \frac{3\sqrt{3}}{4}\left|\frac{n - K_n n^{3/4}}{n} - 1\right| = \frac{3\sqrt{3}}{4}K_n n^{-1/4}.
\end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
\left|\int_{\mathbb{R}^2} \varphi \, d\mu_n - \int_{\mathbb{R}^2} \varphi \, d\mu\right| &= \left|\frac{1}{n}\sum_{i=1}^{6r_n^2} \varphi\left(\frac{x_i}{\sqrt{n}}\right) - \frac{4}{3\sqrt{3}}\int_W \varphi \, dx\right| \\
&= \frac{4}{3\sqrt{3}}\left|\sum_{i=1}^{6r_n^2} \varphi\left(\frac{x_i}{\sqrt{n}}\right)\mathcal{L}^2(\Omega_i^n) - \int_W \varphi \, dx\right| \\
&\leq \frac{4}{3\sqrt{3}}\left|\sum_{i=1}^{6r_n^2} \int_{\Omega_i^n}\left(\varphi\left(\frac{x_i}{\sqrt{n}}\right) - \varphi(x)\right) dx\right| \\
&\quad + \frac{4}{3\sqrt{3}}\|\varphi\|_{L^\infty(\mathbb{R}^2)}\mathcal{L}^2\left(\left(\bigcup_{i=1}^{6r_n^2}\Omega_i^n\right)\Delta W\right) \\
&\leq \frac{4}{3\sqrt{3}}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}\sum_{i=1}^{6r_n^2}\int_{\Omega_i^n}\left|\frac{x_i}{\sqrt{n}} - x\right| dx \\
&\quad + \frac{4}{3\sqrt{3}}\|\varphi\|_{L^\infty(\mathbb{R}^2)}\mathcal{L}^2\left(\left(\bigcup_{i=1}^{6r_n^2}\Omega_i^n\right)\Delta W\right) \\
&\leq \frac{4}{3\sqrt{n}}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}\mathcal{L}^2\left(\bigcup_{i=1}^{6r_n^2}\Omega_i^n\right) \\
&\quad + \frac{4}{3\sqrt{3}}\|\varphi\|_{L^\infty(\mathbb{R}^2)}\mathcal{L}^2\left(\left(\bigcup_{i=1}^{6r_n^2}\Omega_i^n\right)\Delta W\right) \\
&= \|\varphi\|_{W^{1,\infty}(\mathbb{R}^2)}\mathrm{o}(n^{-1/4}) + \|\varphi\|_{L^\infty(\mathbb{R}^2)}K_n n^{-1/4}. \qquad (6.6)
\end{aligned}
$$

Denoting by G_n' the set $G_n' := G_n - q_n$, and by $\mu_{G_n'}$ its associated empirical measure, inequalities (6.5) and (6.6) yield

$$
\mu_{(G_n)'} \rightharpoonup^* \mu, \quad \text{weakly* in } M_b(\mathbb{R}^2), \tag{6.7}
$$

and

$$
\|\mu_{(G_n)'} - \mu\|_{\mathrm{F}} \leq 2K_n n^{-1/4} + \mathrm{o}(n^{-1/4}). \tag{6.8}
$$

We notice that $K_n = 0$ for every $n \in \mathbb{N}$ such that $n = 6k^2$ for some $k \in \mathbb{N}$. This reflects the fact that for those n the daisy D_n is the unique ground state, whose maximal hexagon is the daisy itself.

In view of the definition of α_n, a direct computation shows that

$$K_n = \frac{2^{7/4}}{3^{1/4}} \sqrt{\left(\frac{3n}{2} - \sqrt{\frac{3n}{2}}\right) - \left(\left\lfloor \frac{3n}{2} - \sqrt{\frac{3n}{2}} \right\rfloor\right)} + \mathrm{o}(1).$$

Hence, in particular,

$$\limsup_{n\to+\infty} K_n \leq \frac{2^{7/4}}{3^{1/4}} = K.$$

This completes the proof of Theorem 1.2.

6.2. *Proof of Theorem* 1.3

The proof consists in finding a sequence $\{n_i\}$, $i \in \mathbb{N}$, such that

$$K_{n_i} \to K \tag{6.9}$$

as $i \to +\infty$. Indeed, in view of (6.2), (6.5), (6.8), (6.9), for every $\{n_i\}$ verifying (6.9), and for every sequence of ground states $\{G_{n_i}\}$, there exist suitable translations $\{G'_{n_i}\}$ such that:

$$|G'_{n_i} \backslash W_{n_i}| = K n_i^{3/4} + \mathrm{o}(n_i^{3/4}),$$

$$\|\mu_{G'_{n_i}} - \mu_{W_{n_i}}\| = K n_i^{-1/4} + \mathrm{o}(n_i^{-1/4}),$$

and

$$\|\mu_{G'_{n_i}} - \mu_{W_{n_i}}\|_F = K n_i^{-1/4} + \mathrm{o}(n_i^{-1/4}).$$

A possible choice is to consider

$$n_i := 2 + 6i^2.$$

In fact we have

$$\frac{3n_i}{2} - \sqrt{\frac{3n_i}{2}} = 9i^2 + 3 - \sqrt{9i^2+3} = 9i^2 + 3 - 3i - \frac{1}{i\left(1 + \sqrt{1 + \frac{1}{3i^2}}\right)},$$

and hence

$$\left(\frac{3n_i}{2} - \sqrt{\frac{3n_i}{2}}\right) - \left\lfloor \left(\frac{3n_i}{2} - \sqrt{\frac{3n_i}{2}}\right) \right\rfloor = 1 - \frac{1}{i\left(1 + \sqrt{1 + \frac{1}{3i^2}}\right)} \to 1$$

as $i \to +\infty$, which in turn yields (6.9). This completes the proof of Theorem 1.3.

Before closing this section let us comment on the fact that, as a byproduct of our construction, we also obtain sharp estimates on the distance of any sequence $\{G_n\}$ of (translated) ground states from the n-Wulff shape, in terms of the constant K_n defined in (6.3) (see (6.2), (6.5) and (6.8)). Let us finally stress the nonuniqueness of the n-dimensional Wulff shape W_n: any zigzag hexagon $D_{6\tilde{r}_n^2}$, with radius $\tilde{r}_n = r_n + \mathrm{O}(1)$ (e.g. $\tilde{r}_n = \tilde{\rho}_n$) would in fact lead to the same sharp results.

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Sharp $N^{3/4}$ Law for the Minimizers of the Edge-Isoperimetric Problem on the Triangular Lattice

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Abstract We investigate the edge-isoperimetric problem (EIP) for sets of n points in the triangular lattice by emphasizing its relation with the emergence of the Wulff shape in the crystallization problem. By introducing a suitable notion of perimeter and area, EIP minimizers are characterized as extremizers of an isoperimetric inequality: they attain maximal area and minimal perimeter among connected configurations. The maximal area and minimal perimeter are explicitly quantified in terms of n. In view of this isoperimetric characterizations, EIP minimizers M_n are seen to be given by hexagonal configurations with some extra points at their boundary. By a careful computation of the cardinality of these extra points, minimizers M_n are estimated to deviate from such hexagonal configurations by at most $K_t\, n^{3/4} + \mathrm{o}(n^{3/4})$ points. The constant K_t is explicitly determined and shown to be sharp.

Keywords Edge-isoperimetric problem · Edge perimeter · Triangular lattice · Isoperimetric inequality · Wulff shape · $N^{3/4}$ law

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1 Introduction

This paper is concerned with the *edge-isoperimetric problem* (EIP) in the triangular lattice

$$\mathcal{L}_t := \{m\boldsymbol{t}_1 + n\boldsymbol{t}_2 \; : \; m, n \in \mathbb{Z}\} \quad \text{for } \boldsymbol{t}_1 := (1,0) \text{ and } \boldsymbol{t}_2 := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Let $\mathcal{C}_n$ be the family of sets C_n containing n distinct elements $x_1, \ldots, x_n$ in $\mathcal{L}_t$. The *edge perimeter* $|\Theta(C_n)|$ of a set $C_n \in \mathcal{C}_n$ is the cardinality of the *edge boundary* Θ of C_n defined by

$$\Theta(C_n) := \{(x_i, x_j) \; : \; |x_i - x_j| = 1, x_i \in C_n \text{ and } x_j \in \mathcal{L}_t \setminus C_n\}. \tag{1}$$

Note that, with a slight abuse of notation, the symbol $|\cdot|$ denotes, according to the context, both the cardinality of a set and the euclidean norm in $\mathbb{R}^2$. The EIP over the family $\mathcal{C}_n$ consists in characterizing the solutions to the minimum problem:

$$\theta_n := \min_{C_n \in \mathcal{C}_n} |\Theta(C_n)|. \tag{2}$$

Our main aim is to provide a characterization of the minimizers M_n of (2) as extremizers of a suitable isoperimetric inequality (see Theorem 1.1) and to show that there exists a *hexagonal Wulff shape* in $\mathcal{L}_t$ from which M_n differs by at most

$$K_t\, n^{3/4} + \mathrm{o}(n^{3/4}) \tag{3}$$

points (see Theorem 1.2). A crucial issue of our analysis is that both the exponent and the constant in front of the leading term in (3) are explicitly determined and optimal (see Theorem 1.4).

The EIP is a classical combinatorial problem. We refer to Bezrukov (1999), Harper (2004) for the description of this problem in various settings and for a review of the corresponding results available in the literature. The importance of the EIP is however not only theoretical, since the edge perimeter (and similar notions) bears relevance in problems from *machine learning*, such as classification and clustering (see Trillos and Slepcev 2016 and references therein). Note, however, that in this other more statistical setting the edge perimeter is not defined for configurations contained in a specific lattice, but for point clouds obtained as random samples.

We shall emphasize the link between the EIP and the *Crystallization Problem* (CP). For this reason, we will often refer to the sets $C_n \in \mathcal{C}_n$ as *configurations of particles* in $\mathcal{L}_t$ and to minimal configurations as *ground states*. The CP consists in analytically explaining why particles at low temperature arrange in periodic lattices by proving that the minima of a suitable *configurational energy* are subsets of a regular lattice. At low temperatures, particle interactions are expected to be essentially determined by particle positions. In this classical setting, all available CP results in the literature with respect to a finite number n of particles are in two dimensions for a phenomenological energy E defined from $\mathbb{R}^{2n}$, the set of possible particle positions, to $\mathbb{R} \cup \{+\infty\}$. In Heitmann and Radin (1980), Radin (1981) the energy E takes the form

$$E(\{y_1, \ldots, y_n\}) := \frac{1}{2} \sum_{i \neq j} v_2(|y_i - y_j|) \tag{4}$$

for specific potentials $v_2 : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ representing two-body interactions. Additional three-body interaction terms have been included in the energy in Mainini and Stefanelli (2014), Mainini et al. (2014a, b). We also refer the reader to E and Li (2009), Flatley and Theil (2015), Theil (2006) for results in the *thermodynamic limit* with a Lennard-Jones-like potential v_2 not vanishing at a certain distance and to Blanc and Lewin (2015) for a general review on the CP.

The link between the EIP on $\mathcal{L}_t$ and the CP resides on the fact that when only two-body and short-ranged interactions are considered, the minima of E are expected to be subsets of a triangular lattice. The fact that ground states are subsets of $\mathcal{L}_t$ has been analytically shown in Heitmann and Radin (1980) and Radin (1981), respectively, with $v_2 := v_{\text{sticky}}$, where v_{sticky} is the *sticky-disk* potential, i.e.,

$$v_{\text{sticky}}(\ell) := \begin{cases} +\infty & \text{if } \ell \in [0, 1) \\ -1 & \text{if } \ell = 1 \\ 0 & \text{if } \ell > 1\,, \end{cases} \tag{5}$$

and $v_2 := v_{\text{soft}}$, where v_{soft} is the *soft-disk* potential, i.e.,

$$v_{\text{soft}}(\ell) := \begin{cases} +\infty & \text{if } \ell \in [0, 1) \\ 24\ell - 25 & \text{if } \ell \in [1, 25/24] \\ 0 & \text{if } \ell > 25/24\,. \end{cases} \tag{6}$$

In particular with both the choices (5) and (6) for v_2, we have that

$$E(C_n) = -|B(C_n)| \tag{7}$$

for every $C_n \in \mathcal{C}_n$. Here, the set

$$B(C_n) := \{(x_i, x_j) \;:\; |x_i - x_j| = 1, i < j, \text{ and } x_i, x_j \in C_n\} \tag{8}$$

represents the *bonds* of $C_n \in \mathcal{C}_n$. Note that the definition of $B(C_n)$ in (8) is independent of the order in which the elements of C_n are labeled. The number of bonds of C_n with an endpoint in x_i will be instead denoted by

$$b(x_i) = |\{j \in \{1, \ldots, n\} \;:\; (x_i, x_j) \in B(C_n) \text{ or } (x_j, x_i) \in B(C_n)\}| \tag{9}$$

for every $x_i \in C_n$. The link between the EIP and the CP consists in the fact that by (1), (7), and (9) we have that

$$|\Theta(C_n)| = \sum_{i=1}^{n}(6 - b(x_i)) = 6n - \sum_{i=1}^{n} b(x_i)$$
$$= 6n - 2|B(C_n)| = 6n + 2E(C_n) \tag{10}$$

for every $C_n \in \mathcal{C}_n$, since the degree of $\mathcal{L}_t$ is 6.

In view of (10) minimizing E among configurations in $\mathcal{C}_n$ is equivalent to the EIP (2), and since for both the choices (5) and (6) for v_2 by Heitmann and Radin (1980), Radin (1981) ground states belong to $\mathcal{C}_n$, the ground states of the CP correspond to the minimizers of the EIP. Furthermore, in Heitmann and Radin (1980), Radin (1981) the energy of ground states with n particles has been also explicitly quantified in terms of n to be equal to

$$e_n := -\lfloor 3n - \sqrt{12n-3} \rfloor = -3n + \lceil \sqrt{12n-3} \rceil \tag{11}$$

where $\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leq x\}$ and $\lceil x \rceil := \min\{z \in \mathbb{Z} : x \leq z\}$ denote the standard right- and left-continuous functions, respectively. Therefore, (10) and (11) entails also a characterization of θ_n in terms of n, i.e.,

$$\theta_n = 6n + 2e_n = 2\lceil \sqrt{12n-3} \rceil. \tag{12}$$

A first property of the minimizers of (2) has been provided in Harper (2004), Theorem 7.2 where it is shown that the EIP has the *nested-solution property*, i.e., there exists a total order $\tau : \mathbb{N} \to \mathcal{L}_t$ such that for all $n \in \mathbb{N}$ the configuration

$$D_n := \{x_{\tau(1)}, \ldots, x_{\tau(n)}\}$$

is a solution of (2) (see Proposition 2.1 and the discussion below for the definition of τ). Given the symmetry of the configurations D_n, we will refer to them as *daisies* in the following. Since solutions of the EIP are in general nonunique, the aim of this paper is to characterize them all.

In this paper, we provide a first characterization of the minimizers M_n of the EIP by introducing an isoperimetric inequality in terms of suitable notions of *area* and *perimeter* of configurations in $\mathcal{C}_n$ and by showing that the connected minimizers M_n of the EIP are optimal with respect to it. We refer here the reader to (25) and (26) for the definition of the area $A(C_n)$ and the perimeter $P(C_n)$ of a configuration $C_n \in \mathcal{C}_n$. Note also that we say that a configuration C_n is connected if given any two points $x_i, x_j \in C_n$ then there exists a sequence y_k of points in C_n with $k = 1, \ldots, K$ for some $K \in \mathbb{N}$ such that $y_1 = x_i$, $y_K = x_j$, and either (y_k, y_{k+1}) or (y_{k+1}, y_k) is in $B(C_n)$ for every $k = 1, \ldots, K-1$. It easily follows that minimizers of the EIP need to be connected. Our isoperimetric characterization reads as follows.

Theorem 1.1 (Isoperimetric characterization) *Every connected configuration $C_n \in \mathcal{C}_n$ satisfies*

$$\sqrt{A(C_n)} \le k_n P(C_n), \tag{13}$$

where

$$k_n := \frac{\sqrt{-2\theta_n + 8n + 4}}{\theta_n - 6}. \tag{14}$$

Moreover, connected minimizers $M_n \in \mathcal{C}_n$ of the EIP correspond to those configurations for which (13) holds with the equality. Finally, connected minimizers attain the maximal area $a_n := -\theta_n/2 + 2n + 1$ and the minimal perimeter $p_n := \theta_n/2 - 3$.

Notice that a similar isoperimetric result has been already achieved in the square lattice in Mainini et al. (2014a) with a different method, based on introducing a rearrangement of the configurations. Theorem 1.1 is instead proved by assigning to each element x of a configuration $C_n \in \mathcal{C}_n$ a weight $\omega_{C_n}(x)$ that depends on C_n and on the above-mentioned order τ [see (32)].

Furthermore, we observe that the isoperimetric constant k_n given by (14) satisfies

$$k_n \le \frac{1}{\sqrt{6}} \quad \text{for every } n \in \mathbb{N},$$

with $k_n = 1/\sqrt{6}$ if and only if $n = 1 + 3s + 3s^2$ for some $s \in \mathbb{N}$. Note that for $n = 1 + 3s + 3s^2$, as already observed in Harper (2004), the hexagonal daisy $D_{1+3s+3s^2}$ is the unique minimizer of the EIP.

In the following, we will often refer to lattice translations of $D_{1+3s+3s^2}$ as *hexagonal configurations with radius* $s \in \mathbb{N}$ since each configuration $D_{1+3s+3s^2}$ can be seen as the intersection of $\mathcal{L}_t$ and a regular hexagon with side s. In order to further characterize the solutions of the EIP, we associate to every minimizer M_n a maximal hexagonal configuration $H_{r_{M_n}}$ that is contained in M_n and we evaluate how much M_n differs from $H_{r_{M_n}}$ (see Sect. 3).

In view of the isoperimetric characterization of the ground states provided by Theorem 1.1, we are able to sharply estimate the *distance* of M_n to $H_{r_{M_n}}$ both in terms of the cardinality of $M_n \setminus H_{r_{M_n}}$ and by making use of empirical measures. We associate to every configuration $C_n = \{x_1, \ldots, x_n\}$ the empirical measure denoted by $\mu_{C_n} \in M_b(\mathbb{R}^2)$ (where $M_b(\mathbb{R}^2)$ is the set of bounded Radon measures in $\mathbb{R}^2$) of the rescaled configuration $\{x_1/\sqrt{n}, \ldots, x_n/\sqrt{n}\}$, i.e.,

$$\mu_{C_n} := \frac{1}{n} \sum_i \delta_{x_i/\sqrt{n}},$$

and we denote by $\|\cdot\|$ and $\|\cdot\|_{\mathrm{F}}$ the total variation norm and the *flat norm*, respectively (see Whitney 1957 and (72) for the definition of flat norm). Our second main result is the following.

Theorem 1.2 (Convergence to the Wulff shape). *For every sequence of minimizers M_n in $\mathcal{L}_t$, there exists a sequence of suitable translations M_n' such that*

$$\mu_{M_n'} \rightharpoonup^* \frac{2}{\sqrt{3}}\chi_W \quad \textit{weakly* in the sense of measures},$$

where χ_W is the characteristic function of the regular hexagon W defined as the convex hull of the vectors

$$\left\{\pm\frac{1}{\sqrt{3}}\boldsymbol{t}_1,\ \pm\frac{1}{\sqrt{3}}\boldsymbol{t}_2,\ \pm\frac{1}{\sqrt{3}}(\boldsymbol{t}_2-\boldsymbol{t}_1)\right\}.$$

Furthermore, the following assertions hold true:

$$\left|M_n \setminus H_{r_{M_n}}\right| \le K_t n^{3/4} + \mathrm{o}(n^{3/4}), \tag{15}$$

$$\left\|\mu_{M_n} - \mu_{H_{r_{M_n}}}\right\| \le K_t n^{-1/4} + \mathrm{o}(n^{-1/4}), \tag{16}$$

$$\left\|\mu_{M_n'} - \mu_{H_{r_{M_n}}}\right\|_{\mathrm{F}} \le K_t n^{-1/4} + \mathrm{o}(n^{-1/4}), \tag{17}$$

and

$$\left\|\mu_{M_n'} - \frac{2}{\sqrt{3}}\chi_W\right\|_{\mathrm{F}} \le 2K_t n^{-1/4} + \mathrm{o}(n^{-1/4}), \tag{18}$$

where $H_{r_{M_n}}$ is the maximal hexagon associated to M_n, and

$$K_t := \frac{2}{3^{1/4}}. \tag{19}$$

The proof of Theorem 1.2 is based on the isoperimetric characterization of the minimizers provided by Theorem 1.1 and relies in a fundamental way on the maximality of the radius r_{M_n} of the maximal hexagonal configuration $H_{r_{M_n}}$. The latter is essential to carefully estimate the number of particles of M_n that reside outside $H_{r_{M_n}}$ in terms of r_{M_n} itself and the minimal perimeter p_n. Thanks to this fine estimate we are able to find a lower bound on r_{M_n} in terms of n only [see (69)]. In particular, the method provides a lower bound for the radius r_{M_n} that allows us also to estimate from above the discrepancy between the sets M_n and $H_{r_{M_n}}$ in the Hausdorff distance that is defined by

$$d_{\mathcal{H}}(S_1, S_2) = \max\left\{\sup_{x\in S_1}\inf_{y\in S_2}|x-y|,\ \sup_{y\in S_2}\inf_{x\in S_1}|x-y|\right\}$$

for nonempty sets $S_1, S_2 \subset \mathbb{R}^2$.

Corollary 1.3 (Hausdorff distance) *For any minimizer M_n and its associated maximal hexagon $H_{r_{M_n}}$ there holds*

$$d_{\mathcal{H}}\left(M_n, H_{r_{M_n}}\right) \leq 2 \cdot 3^{1/4} n^{1/4} + \mathrm{O}(1). \tag{20}$$

We observe that in view also of Theorem 1.1 estimates (15)–(18) and (20) provide a measure in different topologies of the fluctuation of the isoperimetric configurations in $\mathcal{L}_t$ with respect to corresponding maximal hexagons. Similar estimates have been studied in the context of isoperimetric Borel sets with finite Lebesgue measure in $\mathbb{R}^d$, $d \geq 2$. We refer the reader to Fusco et al. (2008) for the first complete proof of the quantitative isoperimetric inequality in such setting, and to Cicalese and Leonardi (2012), Figalli et al. (2010) for subsequent proofs employing different techniques.

Moreover, Theorem 1.2 appears to be an extension of analogous results obtained in Au Yeung et al. (2012), Schmidt (2013) by using a completely different method hinged on Γ-convergence. In that context, the set W is the asymptotic *Wulff shape* and we will also often refer to W in this way. More precisely the minimization problem (4) is reformulated in Au Yeung et al. (2012), Schmidt (2013) in terms of empirical measures by introducing the energy functional

$$\mathcal{E}_n(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^2 \setminus \mathrm{diag}} \frac{n}{2} v_2(\sqrt{n}|x-y|)\, \mathrm{d}\mu \otimes \mathrm{d}\mu & \mu = \mu_{C_n} \text{ for some } C_n \in \mathcal{C}_n, \\ \infty & \text{otherwise} \end{cases} \tag{21}$$

defined on the set of nonnegative Radon measures in $\mathbb{R}^2$ with mass 1, where v_2 is (a quantified small perturbation of) the sticky-disk potential (Heitmann and Radin 1980). In Au Yeung et al. (2012), Schmidt (2013) it is proved that the rescaled sequence of functionals $n^{-1/2}(2\mathcal{E}_n + 6n)$ Γ-converges with respect to the weak* convergence of measures to the anisotropic perimeter

$$\mathcal{P}(\mu) := \begin{cases} \displaystyle\int_{\partial^* S} \varphi(\nu_S)\, \mathrm{d}\mathcal{H}^1 & \text{if } \mu = \frac{2}{\sqrt{3}} \chi_S \text{ for some set } S \text{ of finite perimeter} \\ & \text{and such that } \mathcal{L}^2(S) := \sqrt{3}/2, \\ \infty & \text{otherwise} \end{cases} \tag{22}$$

where $\partial^* S$ is the reduced boundary of S, ν_S is the outward-pointing normal vector to S, $\mathcal{L}^2(S)$ is the two-dimensional Lebesgue measure of S, $\mathcal{H}^1$ is the one-dimensional measure, and the anisotropic density φ is defined by

$$\varphi(\nu) := 2\left(\nu_2 - \frac{\nu_1}{\sqrt{3}}\right)$$

for every $\nu = (\nu_1, \nu_2)$ with $\nu_1 = -\sin\alpha$ and $\nu_2 = \cos\alpha$ for $\alpha \in [0, \pi/6]$.

Let us note here that the Γ-convergence result provided in Au Yeung et al. (2012) can be restated as a Γ-convergence result for the edge perimeter. In fact, since the energy functional $\mathcal{E}_n$ is such that

$$\mathcal{E}_n(\mu_{C_n}) = E(C_n) \tag{23}$$

for every $C_n \in \mathcal{C}_n$, by (10) we have that the functional $\mathcal{T}_n := \mathcal{E}_n(\mu) + 6n$ is such that

$$\mathcal{T}_n(\mu_{C_n}) = |\Theta(C_n)|$$

and $n^{-1/2}\mathcal{T}_n$ Γ-converges with respect to the weak* convergence of measures to the anisotropic perimeter $\mathcal{P}(\mu)$.

Besides the completely independent method, the main achievement of this paper with respect to Au Yeung et al. (2012), Schmidt (2013) is that of sharply estimating the constant K_t in formulas (15), (16), and (17). The deviation of the minimizers from the Wulff shape of order $n^{3/4}$ was exhibited in Schmidt (2013) and referred to as the $n^{3/4}$-*law*. Here we sharpen the result from Schmidt (2013) by determining the optimal constant in estimates (15), (16), and (17). We have the following.

Theorem 1.4 (Sharpness of the estimates) *A sequence of minimizers M_{n_i} satisfying* (15)–(17) *with equalities can be explicitly constructed for $n_i := 2 + 3i + 3i^2$ with $i \in \mathbb{N}$.*

The proof of Theorem 1.4 is based on the estimate:

$$|M_n \setminus H_{r_{M_n}}| \leq K_n n^{3/4} + \mathrm{o}(n^{3/4}) \tag{24}$$

which holds true for the explicitly determined constant K_n introduced in (73). Estimate (24) is a consequence of the lower bound for the radius r_{M_n} established in the proof of Theorem 1.2, see (69). In fact, a sequence of minimizers $\bar{M}_n$ satisfying (24) with equality can be explicitly constructed. Note that such configurations $\bar{M}_n$ are singled out among configurations that present extra elements outside their maximal hexagon $H_{\bar{M}_n}$ in correspondence of only two consecutive faces of $H_{\bar{M}_n}$ (see Fig. 6). Therefore, to establish Theorem 1.4 is enough to show that

$$\limsup_{n\to\infty} K_n = K_t$$

and to exhibit a subsequence n_i that realizes the limit.

Finally, we notice that our method appears to be implementable in other settings possibly including three-body interactions. This is done for the crystallization problem in the hexagonal lattice $\mathcal{L}_h$ in a companion paper (Davoli et al. 2016). Furthermore, we observe that analogous results to Theorem 1.2 were obtained in the context of the crystallization problem in the square lattice in Mainini et al. (2014a, b) with a substantially different method (even though also based on an isoperimetric characterization of the minimizers) resulting only in suboptimal estimates.

The paper is organized as follows. In Sect. 2, we introduce the notions of area A and perimeter P of configurations $C_n \in \mathcal{C}_n$, we define the order τ in $\mathcal{L}_t$, and we introduce the notion of weight ω_{C_n}. Furthermore, in Sect. 2.1 we provide the proof of Theorem 1.1. In Sect. 3, we introduce the notion of maximal hexagons $H_{r_{M_n}}$ associated to minimizers M_n of (2) and we carefully estimate r_{M_n} from below in terms of n. In Sect. 4, we use the latter lower bound in order to study the convergence to the Wulff shape by providing the proof of Theorems 1.2 and 1.4 in Sects. 4.1 and 4.2, respectively.

2 Isoperimetric Inequality

In this section, we introduce the notion of area and perimeter of a configuration in $\mathcal{C}_n$ and we deduce various relations between its area, perimeter, energy and its edge boundary including a isoperimetric inequality.

We define the area A of a configuration $C_n \in \mathcal{C}_n$ by

$$A(C_n) := |T(C_n)| \tag{25}$$

where $T(C_n)$ is the family of ordered triples of elements in C_n forming triangles with unitary edges, i.e.,

$$T(C_n) := \{(x_{i_1}, x_{i_2}, x_{i_3}) : \ x_{i_1}, x_{i_2}, x_{i_3} \in C_n, i_1 < i_2 < i_3, \ \text{and } |x_{i_j} - x_{i_k}| = 1 \text{ for } j \neq k\}.$$

The definition of $A(C_n)$ is invariant with respect to any relabeling of the particles of C_n.

In order to introduce the perimeter of a configuration in $\mathcal{C}_n$ let us denote by $F(C_n) \subset \mathbb{R}^2$ the closure of the union of the regions enclosed by the triangles with vertices in $T(C_n)$, and by $G(C_n) \subset \mathbb{R}^2$ the union of all bonds which are not included in $F(C_n)$. The *perimeter* P of a regular configuration $C_n \in \mathcal{C}_n$ is defined as

$$P(C_n) := \mathcal{H}^1(\partial F(C_n)) + 2\mathcal{H}^1(G(C_n)), \tag{26}$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. Note in particular that

$$P(C_n) = \lim_{\varepsilon \searrow 0} \mathcal{H}^1\Big(\partial\big(\partial F(C_n) \cup G(C_n) + B_\varepsilon\big)\Big)$$

where $B_\varepsilon = \{y \in \mathbb{R}^2 \ : \ |y| \leq \varepsilon\}$.

Since every triangle with vertices in $T(C_n)$ contributes with 3 bonds to $B(C_n)$, by (7) and (25) we have that

$$\begin{aligned} 3A(C_n) &= 2\,|B(C_n \cap F(C_n))| \ - \ |B(C_n \cap \partial F(C_n))| \\ &= -2\,E(C_n \cap F(C_n)) - \mathcal{H}^1(\partial F(C_n)). \end{aligned} \tag{27}$$

Thus, by recalling (26) and (27) the equality

$$\mathcal{H}^1(G(C_n)) = |B(C_n \cap G(C_n))| = -E(C_n \cap G(C_n))$$

yields

$$\begin{aligned} P(C_n) &= -2E(C_n \cap F(C_n)) - 3A(C_n) - 2E(C_n \cap G(C_n)) \\ &= -2E(C_n) - 3A(C_n), \end{aligned}$$

and we conclude that

$$E(C_n) = -\frac{3}{2}A(C_n) - \frac{1}{2}P(C_n). \tag{28}$$

Notice that (28) allows to express the energy of a configuration C_n as a linear combinations of its area and its perimeter, and that by (10) an analogous relation can be deduced for the edge boundary, namely

$$|\Theta(C_n)| = 6n - 3A(C_n) - P(C_n). \tag{29}$$

As already discussed in the introduction, in view of (10) we are able to combine the exact quantification of the ground-state energy E established in Heitmann and Radin (1980), Radin (1981) with the nested-solution property provided by Harper (2004), Theorem 7.2. We record this fact in the following result that we state here without proof.

Proposition 2.1 *There exists a total order $\tau : \mathbb{N} \to \mathcal{L}_t$ such that for all $n \in \mathbb{N}$ the configuration D_n defined by $D_n := \{x_{\tau(1)}, \ldots, x_{\tau(n)}\}$ which we refer to as* daisy *with n points is a solution of* (2)*, i.e.,*

$$|\Theta(D_n)| = \min_{C_n \in \mathcal{L}_t} |\Theta(C_n)| = \theta_n, \tag{30}$$

where θ_n is given by (12).

We remark that the sequence of daisy ground states $\{D_n\}$ satisfies the property that

$$D_{n+1} = D_n \cup \{x_{\tau(n+1)}\}.$$

In particular, within the class of daisy configurations one can pass from a ground state to another by properly adding atoms at the right place, determined by the order τ.

The total order provided by Theorem 2.1 is not unique. We will consider here the total order τ on $\mathcal{L}_t$ defined by moving clockwise on concentric daisies centered at a fixed point, as the radius of the daisies increases. To be precise, let $x_{\tau(1)}$ be the origin $(0, 0)$ and let $x_{\tau(2)}$ be a point in $\mathcal{L}_t$ such that there is an active bond between $x_{\tau(2)}$ and $x_{\tau(1)}$. For $i = 3, \ldots, 7$, we define the points $x_{\tau(i)} \in \mathcal{L}_t$ as the vertices of the hexagon H_k with center $x_{\tau(1)}$ and radius 1, numbered clockwise starting from $x_{\tau(2)}$. We then consider the regular hexagons H_k that are centered at $x_{\tau(1)}$, and have radius k and one side parallel to the vector $x_{\tau(2)} - x_{\tau(1)}$, and proceed by induction on the radius $k \in \mathbb{N}$. To this aim, notice that the number of points of $\mathcal{L}_t$ contained in H_k is $n_k := 1 + 3k + 3k^2$. Assume that all the points $x_{\tau(i)}$, with $i \leq n_k$, have been

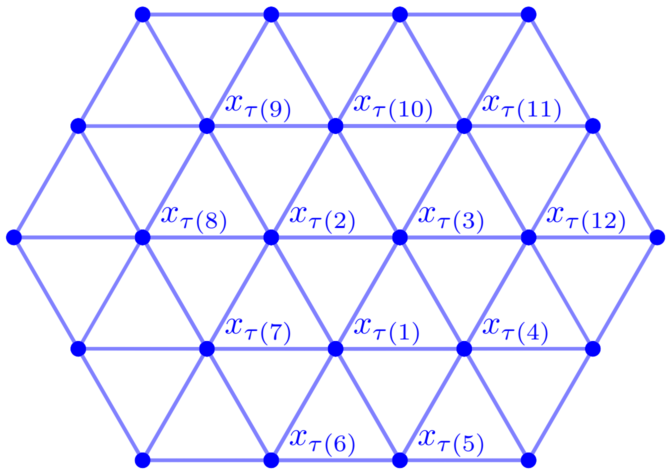


Fig. 1 The total order τ is defined by considering the concentric hexagons centered in $x_{\tau(1)}$ with increasing radii, and by ordering the points clockwise within each hexagon

identified. We define $x_{\tau(1+n_k)}$ as the point $p \in \mathcal{L}_t \cap \ell_k$ such that $|p - x_{\tau(n_k)}| = 1$ and $p \neq x_{\tau(n_k-1)}$, where ℓ_k denotes the line parallel to the vector $x_{\tau(2)} - x_{\tau(1)}$, and passing through the point $x_{\tau(n_k)}$. For $i \in (n_k + 1, n_{k+1}]$, we then define $x_{\tau(i)}$ by clockwise numbering the points of $\mathcal{L}_t$ on the boundary of H_k (see Fig. 1).

We will write $x <_\tau y$ referring to the total order τ described above. A weight function ω is defined on $\mathcal{L}_t$ by the following

$$\omega(x) := |\{y \in \mathcal{L}_t \ : \ |x - y| = 1 \text{ and } y <_\tau x\}|,$$

for every $x \in \mathcal{L}_t$. We observe that ω assumes value 0 at the point $x_{\tau(1)}$, value 1 at $x_{\tau(2)}$ (that is a point bonded to $x_{\tau(1)}$), and values 2 or 3 at all the other points in $\mathcal{L}_t$ (see Fig. 2). Furthermore, we have that

$$E(D_n) = -\sum_{i=1}^{n} \omega(x_{\tau(i)}) \quad \text{for every } n \in \mathbb{N}. \tag{31}$$

and that $\mathcal{L}_t = \{x_{\tau(1)}, x_{\tau(2)}\} \cup \Omega_2 \cup \Omega_3$ with

$$\Omega_2 := \{x \in \mathcal{L}_t \ : \ \omega(x) = 2\} \quad \text{and} \quad \Omega_3 := \{x \in \mathcal{L}_t \ : \ \omega(x) = 3\}.$$

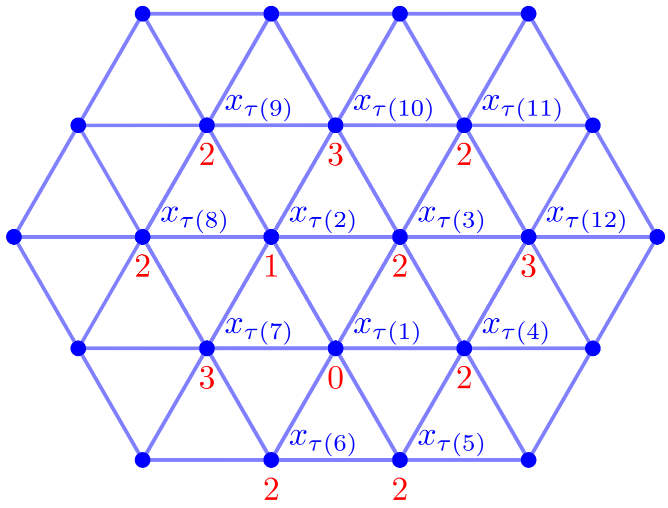
Moreover, for every configuration C_n we introduce a weight function ω_{C_n} defined by

$$\omega_{C_n}(x) := |\{y \in C_n \ : \ |x - y| = 1 \text{ and } y <_\tau x\}|, \tag{32}$$

for every $x \in C_n$ (and thus depending on C_n). In this way C_n can be rewritten as the union

$$C_n = \bigcup_{k=0}^{3} C_n^k$$

Fig. 2 The first elements of $\mathcal{L}_t$ with respect to the order τ are shown with their weight assigned by the value of the function ω appearing below them



where

$$C_n^k := \{x \in C_n \,:\, \omega_{C_n}(x) = k\} \tag{33}$$

for $k = 0, \ldots, 3$. We notice that $\omega_{C_n}(x) \leq \omega(x)$ for every $x \in C_n$ and that $|C_n^0|$ is the number of connected components of C_n.

In order to prove the isoperimetric inequality (13), we first express the energy, the perimeter, the edge perimeter, and the area of a regular configuration C_n as a function of the cardinality of the sets C_n^k.

Proposition 2.2 *Let C_n be a regular configuration in $\mathcal{L}_t$. Then*

$$E(C_n) = -|C_n^1| - 2|C_n^2| - 3|C_n^3|, \tag{34}$$
$$A(C_n) = |C_n^2| + 2|C_n^3|, \tag{35}$$
$$P(C_n) = 2|C_n^1| + |C_n^2|, \tag{36}$$
$$|\Theta(C_n)| = 6|C_n^0| + 4|C_n^1| + 2|C_n^2|, \tag{37}$$

for every $n \in \mathbb{N}$.

Proof Fix $n \in \mathbb{N}$, and let C_n be a regular configuration in $\mathcal{L}_t$. In analogy to (31) there holds

$$E(C_n) = -\sum_{i=1}^{n} \omega_{C_n}(x_i).$$

For $i = 0, \ldots, n-1$, denote by C_i the subset of C_n containing the first i points of C_n, according to the total order τ. If $x_{\tau(i)} \in C_n^0$, then

$$A(C_i) - A(C_{i-1}) = 0, \quad P(C_i) - P(C_{i-1}) = 0 \quad \text{and} \quad |\Theta(C_i)| - |\Theta(C_{i-1})| = 6; \tag{38}$$

if $x_{\tau(i)} \in C_n^1$, then

$$A(C_i) - A(C_{i-1}) = 0, \quad P(C_i) - P(C_{i-1}) = 2 \quad \text{and} \quad |\Theta(C_i)| - |\Theta(C_{i-1})| = 4; \tag{39}$$

if $x_{\tau(i)} \in C_n^2$, then

$$A(C_i) - A(C_{i-1}) = 1, \quad P(C_i) - P(C_{i-1}) = 1 \quad \text{and} \quad |\Theta(C_i)| - |\Theta(C_{i-1})| = 2; \tag{40}$$

whereas, if $x_{\tau(i)} \in C_n^3$, we have

$$A(C_i) - A(C_{i-1}) = 2, \quad P(C_i) - P(C_{i-1}) = 0 \quad \text{and} \quad |\Theta(C_i)| - |\Theta(C_{i-1})| = 0. \tag{41}$$

In view of (38)–(41), we obtain (34)–(37). □

We notice that from (34),(35), and (36) we also recover (28), which in turn, together with (37), yields

$$E(C_n) = -\frac{3}{2}A(C_n) - \frac{1}{4}|\Theta(C_n)| + \frac{3}{2}|C_n^0| \tag{42}$$

for every configuration C_n. Moreover, from the equality

$$\sum_{i=0}^{3} |C_n^i| = n,$$

(35), and (36) it follows that

$$A(C_n) = 2n - 2|C_n^0| - P(C_n). \tag{43}$$

Note that in particular if $C_n = D_n$ then $\omega_{C_n}(x) = \omega(x)$. Furthermore, $D_n^0 = \{x_{\tau(1)}\}$, $D_n^1 = \{x_{\tau(2)}\}$, $D_n^2 = \Omega_2 \cap D_n$, and $D_n^3 = \Omega_3 \cap D_n$. Therefore, (34)–(42) yield

$$E(D_n) = -1 - 2|\Omega_2 \cap D_n| - 3|\Omega_3 \cap D_n|, \tag{44}$$
$$A(D_n) = |\Omega_2 \cap D_n| + 2|\Omega_3 \cap D_n|, \tag{45}$$
$$P(D_n) = 2 + |\Omega_2 \cap D_n|, \tag{46}$$
$$|\Theta(D_n)| = 10 + 2|\Omega_2 \cap D_n|, \tag{47}$$

and by (42) and (43) we obtain

$$E(D_n) = -\frac{3}{2}A(D_n) - \frac{1}{4}|\Theta(D_n)| + \frac{3}{2},$$

and

$$A(D_n) = 2n - 2 - P(D_n)$$

for every $n > 1$.

Proposition 2.3 *The following assertions are equivalent and hold true for every connected configuration C_n:*

(i) $|\Theta(D_n)| \leq |\Theta(C_n)|$;
(ii) $P(D_n) \leq P(C_n)$;
(ii) $A(D_n) \geq A(C_n)$.

Proof The first assertion follows directly from (30) and is equivalent to the second by (36) and (37). The second assertion is equivalent to the third by (29) and (30). □

2.1 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1 by characterizing the minimizers of EIP as the solutions of a discrete isoperimetric problem. We proceed in two steps.
Step 1 We claim that

$$\sqrt{A(D_n)} = k_n P(D_n). \tag{48}$$

Indeed, by (11), (12), (30), (44), there holds

$$\frac{\theta_n}{2} - 3n = e_n = E(D_n) = -1 - 2|\Omega_2 \cap D_n| - 3|\Omega_3 \cap D_n|. \tag{49}$$

Equalities (12) and (47) yield

$$\theta_n = |\Theta(D_n)| = 10 + 2|\Omega_2 \cap D_n|. \tag{50}$$

Theorefore, by (49) and (50), we have

$$|\Omega_2 \cap D_n| = \frac{\theta_n}{2} - 5, \tag{51}$$

and

$$|\Omega_3 \cap D_n| = -\frac{\theta_n}{2} + n + 3. \tag{52}$$

Claim (48) follows now by (45), (46), (51) and (52), and by observing that

$$\begin{aligned}\sqrt{A(D_n)} &= \sqrt{|\Omega_2 \cap D_n| + 2|\Omega_3 \cap D_n|} = \sqrt{\theta_n/2 - 5 + 2(-\theta_n/2 + n + 3)} \\ &= \sqrt{-\theta_n/2 + 2n + 1} = k_n(\theta_n/2 - 3) = k_n(|\Omega_2 \cap D_n| + 2) = k_n P(D_n).\end{aligned}$$

Inequality (13) is a direct consequence of (48) and Proposition 2.3. By Proposition 2.3 we also deduce that the maximal area and the minimal perimeter among connected configurations are realized by $A(D_n) = -\theta_n/2 + 2n + 1$ and $P(D_n) = \theta_n/2 - 3$, respectively.

Step 2 We prove the characterization statement of Theorem 1.1. Let C_n be a connected configuration satisfying

$$\sqrt{A(C_n)} = k_n P(C_n). \tag{53}$$

We claim that C_n is a minimizer. In fact, the claim follows from

$$\begin{aligned} |\Theta(D_n)| \le |\Theta(C_n)| &= 6n - 3A(C_n) \; - \; P(C_n) \\ &= 6n \; - \; 3(k_n)^2 (P(C_n))^2 \; - \; P(C_n) \\ &\le 6n - 3(k_n)^2 (P(D_n))^2 \; - \; P(D_n) \\ &= 6n \; - \; 3A(D_n) - P(D_n) = |\Theta(D_n)| \end{aligned}$$

where we used (30) in the first inequality, (29) in the first and last equality, (28) in the second, (53) in the third, Proposition 2.3 in the second inequality, and (48) in the third equality.

Viceversa, let M_n be a connected minimizer. By (10), (36), and (37), $P(M_n) = P(D_n)$; by (28), $A(M_n) = A(D_n)$. Thus (13) holds with the equality by (48). This concludes the proof of the theorem.

3 Maximal Hexagons Associated to EIP Minimizers

In this section, we introduce the notion of maximal hexagons $H_{r_{M_n}}$ associated to minimizers M_n and we provide a uniform lower estimate of r_{M_n} in terms of n [see (69)].

Fix a minimizer M_n. Let $\mathcal{H}_s^{M_n}$ be the family of the configurations contained in M_n that can be seen as translations in $\mathcal{L}_t$ of daisy configurations $D_{1+3s+3s^2}$ for some $s \in \mathbb{N} \cup \{0\}$, i.e.,

$$\mathcal{H}_s^{M_n} := \{H_s \subset \mathcal{L}_t \; : \; H_s := D_{1+3s+3s^2} + q \text{ for some } q \in \mathcal{L}_t \text{ and } H_s \subset M_n\}, \tag{54}$$

and choose $H_{r_{M_n}}$ to be a configuration in $\mathcal{H}_{r_{M_n}}^{M_n}$ where

$$r_{M_n} := \max\{s \in \mathbb{N} \cup \{0\} \; : \; \mathcal{H}_s^{M_n} \neq \emptyset\}. \tag{55}$$

We will refer to $H_{r_{M_n}}$ as the *maximal hexagon associated to* M_n. Notice that the number of atoms of M_n contained in $H_{r_{M_n}}$ is

$$n(r_{M_n}) := 1 + 3\, r_{M_n} + 3\left(r_{M_n}\right)^2. \tag{56}$$

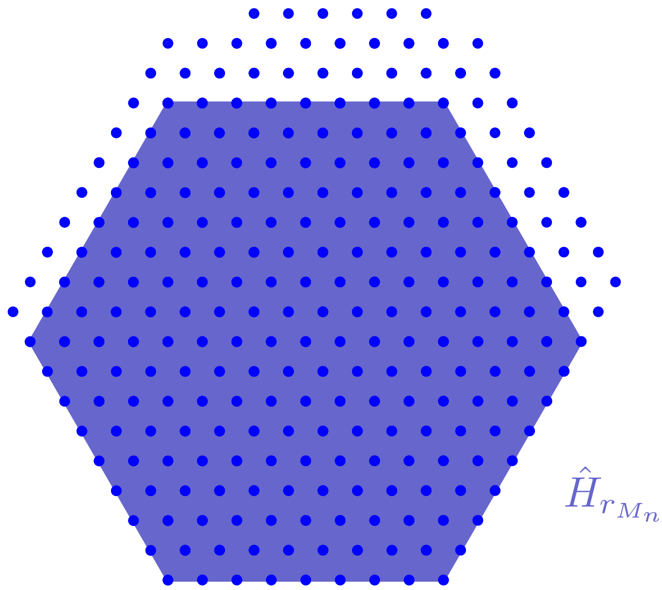


Fig. 3 A minimizer M_n is represented by the set of *dots* and its maximal hexagon $H_{r_{M_n}}$ is given by the intersection of M_n with the regular hexagon $\hat{H}_{r_{M_n}}$ which is drawn in dark color (*blue*) (Color figure online)

In the following, we will often denote the minimal regular hexagon containing $H_{r_{M_n}}$ by $\hat{H}_{r_{M_n}}$ (see Fig. 3), i.e.,

$$\hat{H}_{r_{M_n}} := F(H_{r_{M_n}})$$

Following the notation introduced in Sect. 2 in (33), we decompose M_n as

$$M_n = \bigcup_{k=0}^{3} M_n^k.$$

In the following proposition, we observe that if $n > 6$, then there exists a non-degenerate maximal hexagon for every minimizer.

Proposition 3.1 *For $n \leq 6$, then the maximal hexagon $H_{r_{M_n}}$ is degenerate for every minimizer M_n of* (2). *If $n > 6$, then the maximal radius r_{M_n} of every minimizer M_n of* (2) *satisfies $r_{M_n} \geq 1$.*

Proof It is immediate to check that for $n = 1$, $|M_n^1| = 0$, and for $n = 2$ or $n = 3$, $|M_n^1| = 1$. A direct analysis of the cases in which $n = 4, 5, 6$, shows that $2 \geq |M_n^1| \geq 1$. It is also straightforward to observe that for $n = 0, \ldots, 6$, there holds $r = 0$.

We claim that for $n \geq 7$ the radius r_{M_n} satisfies $r_{M_n} \geq 1$. Indeed, assume that M_n is such that $r_{M_n} = 0$. Then M_n does not contain any hexagon with radius 1 and hence, for every $x \in M_n$ we have that

$$b(x) \leq 5. \tag{57}$$

Property (57) is equivalent to claiming that every element of M_n contributes to the overall perimeter of M_n, and the contribution of each element is at least 1. Therefore,

$$P(M_n) \geq n.$$

By Theorem 1.1, it follows that

$$\frac{\theta_n}{2} - 3 \geq n, \tag{58}$$

which in turn by (12) implies

$$\sqrt{12n-3} - 2 \geq \lceil \sqrt{12n-3} \rceil - 3 \geq n,$$

that is

$$n^2 - 8n + 7 \leq 0,$$

which finally yields $1 \leq n \leq 7$. To conclude, it is enough to notice that for $n = 7$, $\theta_n/2 - 3 = 6$, thus contradicting (58). □

In view of Proposition 3.1 for every minimizer M_n with $n > 6$, we can fix a vertex V_0 of its (non-degenerate) hexagon $\hat{H}_{r_{M_n}}$ and denote by $V_1, \dots, V_5$ the other vertices of $\hat{H}_{r_{M_n}}$ numbered counterclockwise starting from V_0. For $k = 0, \dots, 4$, let us also denote by s_k the line passing through the side of $\hat{H}_{r_{M_n}}$ with endpoints V_k and V_{k+1}, and let s_5 be the line passing through V_5 and V_0.

In the following we will need to consider the number of *levels* of atoms in $\mathcal{L}_t$ around $H_{r_{M_n}}$ containing at least one element of M_n. Denote by $\mathbf{e}_k$ the outer unit normal to the side s_k of $\hat{H}_{r_{M_n}}$ and define

$$\lambda_k := \max\{j \in \mathbb{N} : s_k^j \cap M_n \neq \emptyset\} \tag{59}$$

where s_k^j are the lines of the lattice $\mathcal{L}_t$ parallel to s_k and not intersecting $H_{r_{M_n}}$, namely

$$s_k^j := s_k + \frac{\sqrt{3}}{2} j \mathbf{e}_k$$

for $j \in \mathbb{Z}$. Let also π_k be the open half-plane with boundary s_k and not intersecting the interior of $\hat{H}_{r_{M_n}}$.

We first show that M_n satisfies a connectedness property with respect to the directions determined by the lattice $\mathcal{L}_t$. To this purpose, we introduce the notion of *3-convexity with respect to* $\mathcal{L}_t$.

Definition 3.2 We recall that

$$\boldsymbol{t}_1 := (1, 0), \quad \boldsymbol{t}_2 := \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right), \quad \text{and define} \quad \boldsymbol{t}_3 := \boldsymbol{t}_2 - \boldsymbol{t}_1.$$

We say that a set $S \subset \mathcal{L}_t$ is *3-convex* if for every $p, q \in S$ such that $q := m\boldsymbol{t}_i + p$ for some $m \in \mathbb{N}$ and $i \in \{1, 2, 3\}$ one has that $q' := m'\boldsymbol{t}_i + p \in S$ for every integer $m' \in (0, m)$. Furthermore, we refer to the lines

$$\ell_i^p := \{q \in \mathbb{R}^2 \, : \, q = r\boldsymbol{t}_i + p \text{ for some } r \in \mathbb{R}\}$$

as the *lines of the lattice* $\mathcal{L}_t$ *at* p.

Note that by Definition 3.2 a set S is 3-convex if there is no line ℓ_i^p of the lattice $\mathcal{L}_t$ at a point $p \in \mathcal{L}_t \setminus S$ that is separated by p in two half-lines both containing points of the set S.

Proposition 3.3 *Let M_n be a minimizer. Then M_n is 3-convex.*

Proof For the sake of contradiction assume that the minimizer M_n is a not 3-convex. Then there exist a point $p \in \mathcal{L}_t \setminus M_n$ and $i \in \{1, 2, 3\}$ such that the line ℓ_i^p (see Definition 3.2) is divided by p in two half-lines both containing points of M_n. We claim that we can rearrange the n points of M_n in a new 3-convex configuration $\tilde{M}_n$ such that $|\Theta(\tilde{M}_n)| < |\Theta(M_n)|$ thus contradicting optimality.

Denote for simplicity $\ell_0 := \ell_i^p$ and let $\ell_1, \ldots, \ell_m$ be all the other lines parallel to ℓ_0 that intersect M_n. Furthermore, let $c_k = |M_n \cap \ell_k|$ for $k = 1, \ldots, m$. Starting from the elements of the sequence $\{c_k\}$, we rearrange them in a decreasing order, constructing another set $\{d_k\}$ with the property that $d_0 \geq d_1 \geq \cdots \geq d_m$. Finally, we separate the elements of $\{d_k\}$ having odd indexes from those having even indexes and we rearrange them in a new set $\{f_k\}$ obtained by first considering the elements of $\{d_k\}$ with even indexes, in decreasing order with respect to their indexes, and then the elements of $\{d_k\}$ having odd indexes, with increasing order with respect to their indexes. The set $\{f_k\}$ constructed as above has the property that the two central elements have the maximal value, and the values of the elements decrease in an alternated way by moving toward the sides of the ordered set. Let $\bar{k}$ be the index corresponding to the central element of the set $\{f_k\}$, if m is even, and to the maximum between the two central elements of $\{f_k\}$, if m is odd.

As an example, if we start with a set $\{c_k\} = \{9, 4, 2, 5, 3, 1, 17\}$, the sequence $\{d_k\}$ is given by $\{17, 9, 5, 4, 3, 2, 1\}$ and the sequence $\{f_k\}$ by $\{1, 3, 5, 17, 9, 4, 2\}$. Here $\bar{k} = 4$.

Fix a point $P_{\bar{k}} \in \mathcal{L}_t$ and an angular sector S of amplitude $2\pi/3$, with vertex in $P_{\bar{k}}$, whose sides σ_1 and σ_2 lay on the two lines departing from $P_{\bar{k}}$ which are not parallel to ℓ_0. Consider the points $P_0, \ldots, P_{\bar{k}-1} \in \sigma_1 \cap M_n$, such that

$$|P_k - P_{\bar{k}}| = \bar{k} - k \quad \text{for } k = 0, \ldots \bar{k} - 1.$$

Analogously, consider the points $P_{\bar{k}+1}, \ldots, P_m \in \sigma_2 \cap M_n$, satisfying

$$|P_k - P_{\bar{k}}| = k - \bar{k} \quad \text{for } k = \bar{k} + 1, \ldots, m.$$

For $k = 0, \ldots, m$, let $\tilde{\ell}_k$ be the line parallel to ℓ_0 and passing through P_k. To construct the set $\tilde{M}_n$, we consider f_k consecutive points on each line $\tilde{\ell}_k$, starting from P_k. We note that $|\tilde{M}_n| = |M_n| = n$, the number of bonds in each line parallel to ℓ_0 has increased. On the other hand, the number of bonds between different lines has not decreased. Indeed, given two parallel lines with a and b points, respectively, the maximal number of bonds between these two lines is either $2a$ if $a < b$, or $2a - 1$ if $a = b$. This maximal value is achieved by construction by the modified configuration. Hence,

$$|\Theta(\tilde{M}_n)| < |\Theta(M_n)|,$$

providing a contradiction to the optimality of M_n. □

Since every minimizer M_n is 3-convex, the quantity λ_k introduced in (59) for $k = 0, \dots, 5$ provides the number of non-empty levels of atoms in $M_n \cap \pi_k$ for $n > 6$. In fact, by the definition of τ each partially full level contains at least one point in $(M_n^1 \cup M_n^2) \setminus H_{r_{M_n}}$. Hence,

$$\sum_{k=0}^{5} \lambda_k \leq |M_n^1 \setminus H_{r_{M_n}}| + |M_n^2 \setminus H_{r_{M_n}}|. \tag{60}$$

On the other hand,

$$2|M_n^1 \setminus H_{r_{M_n}}| + |M_n^2 \setminus H_{r_{M_n}}| = P(M_n) - P(H_{r_{M_n}}) = p_n - 6\, r_{M_n}. \tag{61}$$

Therefore, by (60) and (61),

$$\sum_{k=0}^{5} \lambda_k \leq p_n - 6\, r_{M_n}. \tag{62}$$

In the remaining part of this section, we provide a characterization of the geometry of $M_n \setminus H_{r_{M_n}}$ for $n > 6$, by subdividing this set into *good* polygons P_k and *bad* polygons T_k, and by showing that the cardinality of $M_n \setminus H_{r_{M_n}}$ is, roughly speaking, of the same order of magnitude as the one of the union of *good* polygons.

Given a minimizer M_n and its maximal hexagon $H_{r_{M_n}}$, we denote by $H_{r_{M_n}+1}$ the hexagon with side $r_{M_n} + 1$ and having the same center as $H_{r_{M_n}}$. In the following, we denote the hexagon containing $H_{r_{M_n}+1}$ by

$$\hat{H}_{r_{M_n}+1} := F(H_{r_{M_n}+1}).$$

We first show that, by the optimality of $H_{r_{M_n}}$, there exists an angular sector of $2\pi/3$, and centered in one of the vertices of $\hat{H}_{r_{M_n}+1}$, which does not intersect M_n. To this end, we denote by V_i', $i = 0, \dots, 5$ the vertices of the hexagon $\hat{H}_{r_{M_n}+1}$, with the convention that V_i' lies on the half-line starting from the center of $H_{r_{M_n}}$ and passing through V_i.

Lemma 3.4 *Let M_n be a minimizer with $r_{M_n} > 0$. Then*

(*i*) *The hexagon $\hat{H}_{r_{M_n}+1}$ presents at least a vertex, say V_j' with $j \in \{0, \dots, 5\}$, that does not belong to M_n.*

(ii) *There exists* $k \in \{0, \ldots, 5\}$ *such that the open angular sector* S_k *of amplitude* $2\pi/3$, *centered in* V'_k, *and with sides* s^1_k *and* s^1_{k-1} (*with the convention that* $s^1_{-1} := s^1_5$) *is such that* $\overline{S_k} \cap M_n = \emptyset$.

(iii) *Every translation* $\hat{H}$ *of* $\hat{H}_{r_{M_n}+1}$ *by a vector* $\boldsymbol{t} := n\boldsymbol{t}_1 + m\boldsymbol{t}_2$ *with* $n, m \in \mathbb{Z}$ *that has a vertex* $v \notin M_n$ *admits a vertex* $w \notin M_n$ *(possibly different from* v*) and an open angular sector* S *of amplitude* $2\pi/3$ *and centered in* w *such that* $\overline{S} \cap M_n = \emptyset$.

Proof We begin by showing assertion (i). In view of the maximality of $H_{r_{M_n}}$ there exists a point $p \in \mathcal{L}_t$ on the boundary of $\hat{H}_{r_{M_n}+1}$ such that $p \notin M_n$. Either p is already a vertex of $\hat{H}_{r_{M_n}+1}$ or p is an internal point on the side of $\hat{H}_{r_{M_n}+1}$ parallel to s_j for some j. In this latter case, by the 3-convexity of M_n, either V'_j or V'_{j+1} does not belong to M_n and hence, also in this case assertion (i) holds true.

We now denote by V'_j the missing vertex of the hexagon $\hat{H}_{r_{M_n}+1}$ and prove assertion (ii). Let us consider the two half-lines in which V'_j divides the line s^1_j. By the 3-convexity of M_n, at least one of them does not intersect M_n. Analogously, if we consider the two half-lines in which V'_j divides the line s^1_{j-1}, by the 3-convexity of M_n at least one of them does not intersect M_n. Finally, if we consider the line s' passing through the center of $H_{r_{M_n}}$ and V'_j, the 3-convexity of M_n implies that the points of s' whose distance from the center of $H_{r_{M_n}}$ is bigger than $r_{M_n} + 1$ do not belong to M_n. In view of the geometric position of such three half-lines departing from V'_j, we can conclude that the claim holds true by using once again the 3-convexity of M_n.

Let us conclude by observing that assertion (iii) follows by a similar argument to the one employed to prove assertion (ii). If the center of $\hat{H}$ is in M_n, then the same argument works and we can chose $w = v$. If the center of $\hat{H}$ is not in M_n, then the line passing through the missing vertex v and the center of $\hat{H}$ does not intersect M_n outside $\hat{H}$ either for v or for the opposite vertex w with respect to the center of $\hat{H}$. □

In the following, we assume without loss of generality that the vertex V_0 has been chosen so that the index k in assertion (ii) of Lemma 3.4 is 0. Therefore, by assertion (ii) of Lemma 3.4 we obtain that the open angular sector S_0 of $2\pi/3$, centered in V'_0, and with sides s^1_0 and s^1_5 is such that $\overline{S_0} \cap M_n = \emptyset$.

Let us use the definition of the levels λ_k for $k = 0, \ldots, 5$ introduced in (59) to define a region $\hat{R}$ that contains all extra points of M_n, i.e., points of M_n not contained in $H_{r_{M_n}}$. We already know that we can take $\hat{R} \subset (\mathbb{R}^2 \setminus \hat{H}_{r_{M_n}}) \cap (\mathbb{R}^2 \setminus S_0)$. We define the region $\hat{R}$ as follows (see Fig. 4):

$$\hat{R} := \Big(\bigcup_{j=0}^{5} \hat{P}_j\Big) \bigcup \Big(\bigcup_{j=1}^{5} \hat{T}_j\Big) \tag{63}$$

The set $\hat{P}_0$ in (63) is the polygon delimited by the lines $s_5, s^1_0, s^{\lambda_0}_0, s^{-r+1}_5$ and the sets $\hat{P}_k$ in (63) is defined by

$$\hat{P}_k := \begin{cases} \hat{P}^1_k(\lambda_k) & \text{if } \lambda_k \le \lambda_{k-1} + 1, \\ \hat{P}^1_k(\lambda_k - \lambda_{k-1} + 1) \cup \hat{P}^2_k(\lambda_k - \lambda_{k-1} + 1) & \text{if } \lambda_k > \lambda_{k-1} + 1, \end{cases}$$

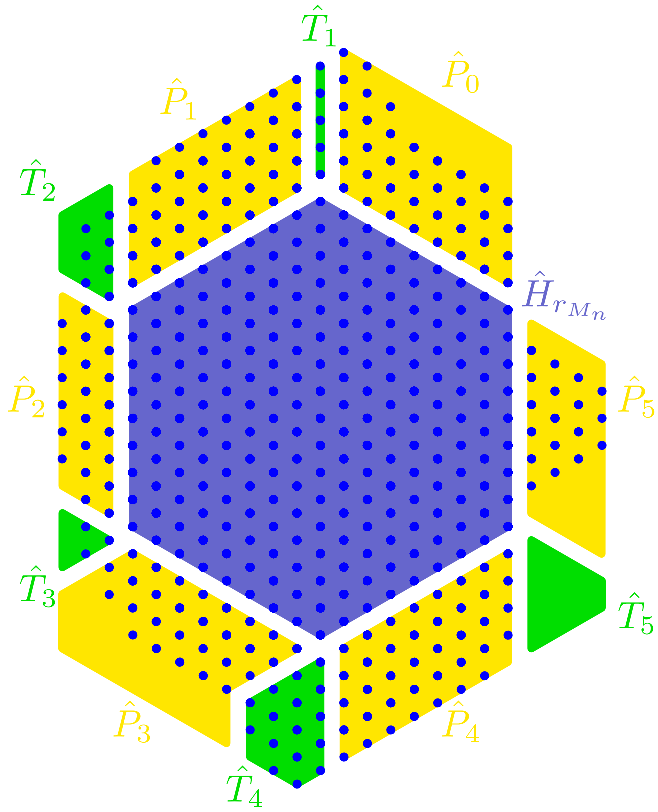


Fig. 4 Representation of the region $\hat{R}$ given by the union of the polygons $\hat{P}_j$ with $j = 0, \ldots, 5$ drawn in the lightest color (*yellow*) and the polygons $\hat{T}_j$ with $j = 1, \ldots, 5$ drawn in the middle color (*green*). Note that this picture has a mere illustrative purpose (the configuration is not a EIP minimizer) (Color figure online)

for every $k = 1, \ldots, 5$, where for every $a \in [-2\,r_{M_n}, 2\,r_{M_n}]$ we denote by $\hat{P}_k^1(a)$ the polygon contained between s_k^1, s_k^a, s_{k+1}, s_{k+1}^{-r+1}, and by $\hat{P}_k^2(a)$ the set delimited by s_k^a, $s_k^{\lambda_k}$, $s_{k-1}^{\lambda_{k-1}-r+1}$, $s_{k-1}^{\lambda_{k-1}}$. Finally the sets $\hat{T}_k$ are the *region* between $\hat{P}_{k-1}$ and $\hat{P}_k$ or, more precisely,

$$\hat{T}_k := \{x \in R : x \in s_{k-1}^{j_{k-1}} \cap s_k^{j_k},\ \text{with } 1 \leq j_{k-1} \leq \lambda_{k-1},\ 1 \leq j_k \leq \lambda_k,\ j_{k-1} \geq j_k \\ \text{and, if } \lambda_{k-1} > \lambda_{k-2} + 1,\ j_{k-1} \leq j_k + \lambda_{k-1} - \lambda_{k-2}\}. \tag{64}$$

Note that $\hat{T}_1$ by definition (64) reduces to a segment contained in the line $s_2^{-r_{M_n}}$ such that

$$|T_1| = \min\{\lambda_0, \lambda_1\}. \tag{65}$$

Furthermore, we consider the configurations $P_k := \hat{P}_k \cap \mathcal{L}_t$ for $k = 0, \ldots, 5$, $T_k := \hat{T}_k \cap \mathcal{L}_t$ for $k = 1, \ldots, 5$, and $R := \hat{R} \cap \mathcal{L}_t$. We notice that $M_n \subset H_{r_{M_n}} \cup R$ and that

$$n = |H_{r_{M_n}}| + |R| - |R \setminus M_n|,$$

where $|H_{r_{M_n}}| = 1 + 3\,r_{M_n} + 3\left(r_{M_n}\right)^2$, and

$$|R| = \sum_{k=0}^{5} |P_k| + \sum_{k=1}^{5} |T_k| = r_{M_n} \sum_{k=0}^{5} \lambda_k + \sum_{k=1}^{5} |T_k|$$

where in the last equality we used that $|P_k| = r_{M_n}\lambda_k$ for $k = 0, \ldots, 5$. Furthermore, for every $x \in R$ and every $k = 0, \ldots, 5$ there exists $j_k \in [-\lambda_{k'} - 2r, \lambda_k]$ with $k' := (k+3)_{\mathrm{mod}\,6}$ and $k' \in \{0, \ldots, 5\}$ such that $x \in s_k^{j_k}$. Hence, in particular, every $x \in R$ is uniquely determined by a pair of indexes $(j_k, j_{k'})$, with $k' \neq k+3$ in $\mathbb{Z}_6$.

Proposition 3.5 *Let $\mathcal{H}$ be the family of the configurations that can be seen as translations in $\mathcal{L}_t$ of the daisy configuration $D_{1+3s+3s^2}$ for $s := r_{M_n} + 1$ and that are contained in $H_{r_{M_n}} \cup R$, i.e.,*

$$\mathcal{H} := \{H \subset H_{r_{M_n}} \cup R \,:\, H = D_{1+3s+3s^2} + q \text{ for } s := r_{M_n} + 1 \text{ and some } q \in \mathcal{L}_t\}.$$

Then there holds

$$|R \setminus M_n| \geq |\mathcal{H}|.$$

Proof Let $h := |\mathcal{H}|$. We show by induction on $m = 1, \ldots, h$ that for every family $\mathcal{H}_m \subset \mathcal{H}$ with $|\mathcal{H}_m| = m$, there exists a set $V_{\mathcal{H}_m} \subset R \setminus M_n$ with $|V_{\mathcal{H}_m}| = m$, such that the correspondence that associates to each $v \in V_{\mathcal{H}_m}$ a hexagon $H \in \mathcal{H}_m$ if v is a vertex of $\hat{H} := F(H)$, is a bijection.

We remark that the thesis will follow once we prove the assertion for $m = h$. The claim holds for $m = 1$ by reasoning in the same way as in the first assertion of Lemma 3.4. Assume now that the claim is satisfied for $m = \bar{m}$. Consider a family $\mathcal{H}_{\bar{m}+1} = \{H_1, \ldots, H_{\bar{m}+1}\} \subset \mathcal{H}$, and the polygon

$$\mathcal{P}_{\bar{m}+1} := \bigcup_{i=1}^{\bar{m}+1} H_i \subset H_{r_{M_n}} \cup R.$$

Furthermore, let us define

$$\hat{\mathcal{P}}_{\bar{m}+1} := F(\mathcal{P}_{\bar{m}+1}).$$

We subdivide the remaining part of the proof into 4 steps.

Step 1 There exists a vertex $\tilde{v}$ of $\hat{\mathcal{P}}_{\bar{m}+1}$ that is not in M_n. Indeed, if all vertices of $\hat{\mathcal{P}}_{\bar{m}+1}$ belong to M_n, by 3-convexity$\mathcal{P}_{\bar{m}+1} \subset M_n$, and hence $H_{\bar{m}+1} \subset \mathcal{P}_{\bar{m}+1} \subset M_n$, which would contradict the maximality of r_{M_n}.

Step 2 By assertion (iii) of Lemma 3.4 there exists a vertex w of $\hat{\mathcal{P}}_{\bar{m}+1}$ not in M_n and an open angular sector S centered in w, amplitude $2\pi/3$, and sides $\sigma_1, \sigma_2 \subset \mathcal{L}_t$ such that $\bar{S} \cap M_n = \emptyset$.

Step 3 There exists a vertex v of $\hat{\mathcal{P}}_{\bar{m}+1}$ that is not in M_n and that corresponds to an interior angle of $\hat{\mathcal{P}}_{\bar{m}+1}$ of $2\pi/3$. In fact, $\hat{\mathcal{P}}_{\bar{m}+1}$ can have vertices with angles of $2\pi/3$, $4\pi/3$, and $5\pi/3$ only. If the vertex w detected in Step 2 corresponds to an angle of $2\pi/3$, there is nothing to prove. If w corresponds to an angle of $4\pi/3$ or $5\pi/3$, then we have two cases.

Case 1 The intersection between S and the closure of $\hat{\mathcal{P}}_{\bar{m}+1}$ is empty. Then, for every $j = 1, 2$, there exists $v_j \in \sigma_j$ such that the segment with endpoints w and v_j denoted by wv_j is contained in $\partial\hat{\mathcal{P}}_{\bar{m}+1}$ and v_j is a vertex of $\hat{\mathcal{P}}_{\bar{m}+1}$. Furthermore, $v_j \notin M_n$ because $v_j \in S$, and v_j is associated to an angle of $2\pi/3$, since $S \cap \bar{\mathcal{P}}_{\bar{m}+1} = \emptyset$. The proof follows by taking $v = v_1$.

Case 2 The intersection between S and the closure of $\hat{\mathcal{P}}_{\bar{m}+1}$ is nonempty. Without loss of generality, we can assume that the two sides of the angular sector S are given by

$$\sigma_1 = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta = \alpha \boldsymbol{t}_1 + w,\ \alpha > 0\right\}$$

and

$$\sigma_2 = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta = -\alpha \boldsymbol{t}_2 + w,\ \alpha > 0\right\}.$$

Define

$$\sigma_1^k := \sigma_1 - \frac{\sqrt{3}}{2} k\,(0, 1) \quad \text{and} \quad \sigma_2^k := \sigma_2 + k\,\boldsymbol{t}_1,$$

for $k \in \mathbb{N}$. Since $\mathcal{P}_{\bar{m}+1} \cap S$ is bounded, we can find

$$k_1 := \max\{k \in \mathbb{N} : \sigma_1^k \cap \mathcal{P}_{\bar{m}+1} \cap S \neq \emptyset\}$$

and

$$k_2 := \max\{k \in \mathbb{N} : \sigma_2^k \cap \mathcal{P}_{\bar{m}+1} \cap S \neq \emptyset\}.$$

For $j = 1, 2$, the intersection $\sigma_j^{k_j} \cap \partial\hat{\mathcal{P}}_{\bar{m}+1} \cap S$ is a segment with at least one endpoint $v \in S$ corresponding to a vertex of $\partial\hat{\mathcal{P}}_{\bar{m}+1}$ associated to an angle of $2\pi/3$.

Step 4 Let v be the vertex provided by Step 3. Then, there exists a unique $\hat{H}_{\bar{j}} \in \mathcal{H}_{\bar{m}+1}$ having v among its vertices. By the induction hypothesis on $\{\hat{H}_1, \ldots, \hat{H}_{\bar{m}+1}\} \setminus \{\hat{H}_{\bar{j}}\}$ there exists a family of vertices $\{v_j\}_{j=1,\ldots,\bar{m}+1,\ j\neq\bar{j}} \subset R \setminus M_n$ such that v_j is a vertex of $\hat{H}_j$ and for every $i \neq j$, v_j is not a vertex of $\hat{H}_i$. The thesis follows then by setting $v_{\bar{j}} = v$, and by taking $V_{\mathcal{H}_{\bar{m}+1}} = \{v_1, \ldots, v_{\bar{m}+1}\}$. □

In view of Proposition 3.5 in order to estimate from below the cardinality of $R \setminus M_n$, it suffices to estimate the cardinality of $\mathcal{H}$. To this end, we denote in the following by

$\hat{U}_k$ the closure of the region in $\mathbb{R}^2$ containing $H_{r_{M_n}}$ and delimited, respectively, by s_3, s_4, and s_5 for $k = 2$, s_4, s_5, and s_0 for $k = 3$, s_5, s_0, and s_1 for $k = 4$, and s_0, s_1, and s_2 for $k = 5$. Notice that $T_k \subset \hat{U}_k$ (see Fig. 5).

Lemma 3.6 *There holds*

$$|\mathcal{H}| \geq \sum_{j=2}^{5} |T_j| - \lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 + 4. \tag{66}$$

Proof For notational simplicity we will omit in the rest of this proof the dependence of the radius r_{M_n} on the minimizer M_n. We begin by noticing that

$$|\mathcal{H}| \geq \sum_{k=2}^{5} |\mathcal{H}_k| \tag{67}$$

where

$$\mathcal{H}_k := \{H \in \mathcal{H} : \ H \subset \hat{U}_k \text{ and has a vertex in } T_k\}$$

for $k = 2, 3, 4, 5$. We claim that

$$|\mathcal{H}_k| \geq |T_k| - \lambda_k - \lambda_{k-1} + 1 \tag{68}$$

and we observe that (66) directly follows from (67) and (68).

The rest of the proof is devoted to show (68). Let $x \in T_k$ and consider (j_k, j_{k-1}, j_{k-2}) such that $x \in s_k^{j_k} \cap s_{k-1}^{j_{k-1}} \cap s_{k-2}^{j_{k-2}}$. In the following, we identify x with the triple of indexes (j_k, j_{k-1}, j_{k-2}), and we write $x = (j_k, j_{k-1}, j_{k-2})$. Let H_x be the hexagon with vertices x,

$$\begin{aligned}
v_1 &:= (j_k - (r+1), j_{k-1}, j_{k-2} + (r+1)),\\
v_2 &:= (j_k - 2(r+1), j_{k-1} - (r+1), j_{k-2} + (r+1)),\\
v_3 &:= (j_k - 2(r+1), j_{k-1} - 2(r+1), j_{k-2}),\\
v_4 &:= (j_k - (r+1), j_{k-1} - 2(r+1), j_{k-2} - (r+1)),\\
v_5 &:= (j_k, j_{k-1} - (r+1), j_{k-2} - (r+1))
\end{aligned}$$

(see Fig. 5 for an example of an hexagon $H_x \in \mathcal{H}_2$ with $x \in T_2$).

H_x is contained in $\hat{U}_k$ if for every $j = 0, \ldots, 5$ there holds $v_j \in \hat{U}_k$. This latter condition is equivalent to checking that the following inequalities are satisfied

$$\begin{aligned}
&j_k - 2(r+1) \geq -2r, \quad j_k \leq \lambda_k,\\
&j_{k-1} - 2(r+1) \geq -2r, \quad j_{k-1} \leq \lambda_{k-1},\\
&j_{k-2} - (r+1) \geq -2r, \quad j_{k-2} + (r+1) \leq \lambda_{k-2}.
\end{aligned}$$

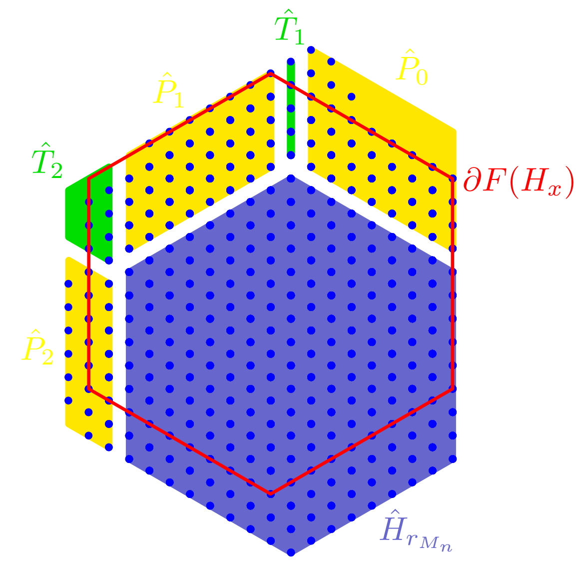


Fig. 5 The region $\hat{U}_2$ is shown and the boundary $\partial F(H_x)$ of a hexagon $H_x \in \mathcal{H}_2$ with vertex $x \in T_2$ is represented by a continuous (*red*) line. Note that this picture has a mere illustrative purpose (the configuration is not a EIP minimizer) (Color figure online)

Hence, if $x = (j_j, j_{k-1}, j_{k-2}) \in T_k$ is such that

$$\begin{aligned} 2 \leq j_k &\leq \lambda_k, \\ 2 \leq j_{k-1} &\leq \lambda_{k-1}, \\ -r+1 \leq j_{k-2} &\leq \lambda_{k-2} - (r+1), \end{aligned}$$

then $H_x \subset \hat{U}_k$. By the definition of the sets T_k [see (64)], the previous properties are fulfilled by every $x \in T_k$, apart from those points belonging to the portion of the boundary of $\hat{T}_k$ which is adjacent either to $\hat{P}_{k-1}$ or to $\hat{P}_k$. Denoting by $\tilde{T}_k$ this latter set, claim (68) follows once we observe that

$$|\tilde{T}_k| = |T_k| - \lambda_k - \lambda_{k-1} + 1.$$

□

Moving from Proposition 3.5 and Lemma 3.6, we deduce the lower estimate on the maximal radii r_{M_n} of the minimizers M_n of (2).

Proposition 3.7 *Let M_n be a minimizer of* (2) *with maximal radius r_{M_n}. Then*

$$r_{M_n} \geq \frac{\lceil \alpha_n \rceil}{6} - 2 - \frac{1}{6}\sqrt{\lceil \alpha_n \rceil^2 - (\alpha_n)^2 + 75} \tag{69}$$

with

$$\alpha_n := \sqrt{12n - 3}\,. \tag{70}$$

Proof For the sake of notational simplicity, we will omit in the rest of this proof the dependence of the maximal radius r_{M_n} from M_n. By Proposition 3.5 and Lemma 3.6 we have

$$|R \setminus M_n| \geq \sum_{j=2}^{5} |T_j| - \lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 + 4,$$

and so, by (62) and (65), we obtain

$$\begin{aligned}
n &= |H_{r_{M_n}}| + |R| - |R \setminus M_n| \\
&\leq 1 + 3r^2 + 3r + \sum_{j=0}^{5} |P_j| + \sum_{j=1}^{5} |T_j| - \sum_{j=2}^{5} |T_j| + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 - 4 \\
&\leq 1 + 3r^2 + 3r + r \sum_{j=0}^{5} \lambda_j + |T_1| + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 - 4 \\
&\leq 1 + 3r^2 + 3r + (r+2) \sum_{j=0}^{5} \lambda_j - 4 \\
&\leq 1 + 3r^2 + 3r + (r+2)(p_n - 6r) - 4 = -3r^2 + (p_n - 9)r + 2p_n.
\end{aligned}$$

Thus, the maximal radius satisfies the following inequality:

$$3r^2 - (p_n - 9)r + n - 2p_n \leq 0. \tag{71}$$

Estimate (69) follows from (71) by solving (71) with respect to r and recalling that $p_n = \theta_n/2 - 3$ by Theorem 1.1 and $\theta_n = 2\lceil \alpha_n \rceil$ by (12). □

A direct consequence of (69) is the upper bound on the Hausdorff distance between the sets M_n and $H_{r_{M_n}}$ introduced in Corollary 1.3.

Proof of Corollary 1.3 Let M_n be a minimizer. We assume with no loss of generality that $n > 6$ so that by Proposition 3.1 the maximal hexagon $H_{r_{M_n}}$ is not degenerate. Then

$$d_{\mathcal{H}}(M_n, H_{r_{M_n}}) \leq \max_{i=0,\dots,5} \lambda_i.$$

Therefore, by (62) and (70) we obtain that

$$\begin{aligned}
d_{\mathcal{H}}(M_n, H_{r_{M_n}}) &\leq p_n - 6r_{M_n} \\
&\leq 9 + \sqrt{\lceil \alpha_n \rceil^2 - (\alpha_n)^2 + 75} \\
&= \sqrt{\lceil \alpha_n \rceil^2 - (\alpha_n)^2} + \mathrm{O}(1) \\
&\leq \sqrt{2\lceil \alpha_n \rceil} + \mathrm{O}(1) \leq \sqrt{2}\sqrt{\sqrt{12n-3} + 1} + \mathrm{O}(1) \\
&\leq 2 \cdot 3^{1/4} n^{1/4} + \mathrm{O}(1)
\end{aligned}$$

where we used Proposition 3.7 in the second inequality. □

4 Convergence to the Wulff Shape

In this section, we use the lower bound (69) on the maximal radius r_{M_n} associated to each minimizer M_n of (2) to study the convergence of minimizers to the hexagonal asymptotic shape as the number n of points tends to infinity.

To this end, we recall from the introduction that W is the regular hexagon defined as the convex hull of the vectors

$$\left\{\pm\frac{1}{\sqrt{3}}\boldsymbol{t}_1,\ \pm\frac{1}{\sqrt{3}}\boldsymbol{t}_2,\ \pm\frac{1}{\sqrt{3}}\boldsymbol{t}_3\right\},$$

where $\mathbf{t_i}$ are defined in Definition 3.2 for $i = 1, 2, 3$. Furthermore, in the following μ will denote the measure

$$\mu := \frac{2}{\sqrt{3}}\chi_W,$$

where χ_W is the characteristic function of W. We recall that by $\|\cdot\|$ we denote the total variation norm and by $\|\cdot\|_{\mathrm{F}}$ the flat norm defined by

$$\|\mu\|_{\mathrm{F}} := \sup\left\{\int_{\mathbb{R}^2}\varphi\, d\mu\ :\ \varphi \text{ is Lipschitz with } \|\varphi\|_{W^{1,\infty}(\mathbb{R}^2)} \le 1\right\} \tag{72}$$

for every $\mu \in M_b(\mathbb{R}^2)$ (see Whitney 1957).

4.1 Proof of Theorem 1.2

In this subsection, we prove Theorem 1.2.

Step 1 We start by considering

$$K_n := \frac{\lceil\alpha_n\rceil}{6n^{3/4}}\sqrt{\lceil\alpha_n\rceil^2 - (\alpha_n)^2}, \tag{73}$$

where $\alpha_n := \sqrt{12n-3}$, see (70). In view of the definition of $H_{r_{M_n}}$, we observe that

$$\begin{aligned}
|M_n \setminus H_{r_{M_n}}| &= n - (1 + 3\,(r_{M_n})^2 + 3\,r_{M_n}) \\
&\le n - 1 - 3\left(\frac{\lceil\alpha_n\rceil}{6} - 2 - \frac{1}{6}\sqrt{\lceil\alpha_n\rceil^2 - (\alpha_n)^2 + 33}\right)^2 \\
&\quad - 3\left(\frac{\lceil\alpha_n\rceil}{6} - 2 - \frac{1}{6}\sqrt{\lceil\alpha_n\rceil^2 - (\alpha_n)^2 + 33}\right) \\
&= n - \frac{\lceil\alpha_n\rceil^2}{12} + \frac{\lceil\alpha_n\rceil}{6}\sqrt{\lceil\alpha_n\rceil^2 - (\alpha_n)^2} + \mathrm{o}(n^{3/4}) \\
&= \frac{\lceil\alpha_n\rceil}{6}\sqrt{\lceil\alpha_n\rceil^2 - (\alpha_n)^2} + \mathrm{o}(n^{3/4})
\end{aligned} \tag{74}$$

where we used Proposition 3.7 in the inequality. Therefore, by (73) and (74) we obtain estimate (24), i.e.,

$$|M_n \setminus H_{r_{M_n}}| \leq K_n n^{3/4} + \mathrm{o}(n^{3/4}).$$

Furthermore, since

$$\left\| \mu_{M_n} - \mu_{H_{r_{M_n}}} \right\| = \frac{\left| M_n \triangle H_{r_{M_n}} \right|}{n}$$

and $H_{r_{M_n}} \subset M_n$, by (24) we also obtain that

$$\left\| \mu_{M_n} - \mu_{H_{r_{M_n}}} \right\| \leq K_n n^{-1/4} + \mathrm{o}(n^{-1/4}). \tag{75}$$

We now define

$$d_n := 1 + 3\, r_{M_n} + 3 \left(r_{M_n} \right)^2$$

and consider the empirical measure $\mu_{D_{d_n}}$ associated to the daisy D_{d_n}. For every point $x_i \in D_{d_n}$, we denote by Z_i the Voronoi cell in $\mathcal{L}_t$ related to x_i that is the regular hexagon centered in x_i with side $1/\sqrt{3}$ and edges orthogonal to the three lattice directions. Furthermore, let $Z_i^n := \{x/\sqrt{n} \,:\, x \in Z_i\}$. We observe that

$$\left\| \frac{x_i}{\sqrt{n}} - x \right\|_{L^\infty(Z_i^n)} \leq \frac{1}{\sqrt{3n}}, \tag{76}$$

and

$$\mathcal{L}^2 \left(\left(\bigcup_{i=1}^{d_n} Z_i^n \right) \Delta W \right) = \frac{\sqrt{3}}{2} K_n n^{-1/4}. \tag{77}$$

For every $\varphi \in W^{1,\infty}(\mathbb{R}^2)$, we obtain that

$$\begin{aligned}
\Big| \int_{\mathbb{R}^2} \varphi \, d\mu_{D_n} - \int_{\mathbb{R}^2} \varphi \, d\mu \Big| &= \Big| \frac{1}{n} \sum_{i=1}^{d_n} \varphi\Big(\frac{x_i}{\sqrt{n}}\Big) - \frac{2}{\sqrt{3}} \int_W \varphi \, dx \Big| \\
&= \frac{2}{\sqrt{3}} \Big| \sum_{i=1}^{d_n} \varphi\Big(\frac{x_i}{\sqrt{n}}\Big) \mathcal{L}^2(Z_i^n) - \int_W \varphi \, dx \Big| \\
&\leq \frac{2}{\sqrt{3}} \Big| \sum_{i=1}^{d_n} \int_{Z_i^n} \Big(\varphi\Big(\frac{x_i}{\sqrt{n}}\Big) - \varphi(x) \Big) dx \Big| + \frac{2}{\sqrt{3}} \|\varphi\|_{L^\infty(\mathbb{R}^2)} \mathcal{L}^2 \left(\left(\bigcup_{i=1}^{d_n} Z_i^n \right) \Delta W \right) \\
&\leq \frac{2}{\sqrt{3}} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} \sum_{i=1}^{d_n} \int_{Z_i^n} \Big| \frac{x_i}{\sqrt{n}} - x \Big| dx + \frac{2}{\sqrt{3}} \|\varphi\|_{L^\infty(\mathbb{R}^2)} \mathcal{L}^2 \left(\left(\bigcup_{i=1}^{d_n} Z_i^n \right) \Delta W \right)
\end{aligned}$$

$$\leq \frac{2}{3\sqrt{n}}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}\mathcal{L}^2\left(\bigcup_{i=1}^{d_n} Z_i^n\right) + \frac{2}{\sqrt{3}}\|\varphi\|_{L^\infty(\mathbb{R}^2)}\mathcal{L}^2\left(\left(\bigcup_{i=1}^{d_n} Z_i^n\right)\Delta W\right)$$
$$\leq \frac{2}{3\sqrt{n}}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}\frac{\mathcal{L}^2(\hat{H}_{r_{M_n}+1})}{n} + \frac{2}{\sqrt{3}}\|\varphi\|_{L^\infty(\mathbb{R}^2)}\mathcal{L}^2\left(\left(\bigcup_{i=1}^{d_n} Z_i^n\right)\Delta W\right)$$
$$\leq \|\varphi\|_{W^{1,\infty}(\mathbb{R}^2)}\mathrm{O}(n^{-1/2}) + \|\varphi\|_{L^\infty(\mathbb{R}^2)}K_n n^{-1/4}, \tag{78}$$

where we used (76) and (77) in the third and the last inequality, respectively.
By combining (75) with (78), we obtain that

$$\mu_{M'_n} \rightharpoonup^* \mu \quad \text{weakly* in } M_b(\mathbb{R}^2), \tag{79}$$

and

$$\left\|\mu_{M'_n} - \mu\right\|_{\mathrm{F}} \leq 2K_n n^{-1/4} + \mathrm{o}(n^{-1/4}), \tag{80}$$

where $M'_n := M_n - q_n$, with $q_n \in \mathcal{L}_t$ such that $H_{r_{M_n}} = D_{1+3r_{M_n}+3r^2_{M_n}} + q_n$.
Step 2 Assertions (15)–(18) directly follow from (24), (75), and (80) since by (70) and (73) a direct computation shows that

$$K_n = \frac{\lceil\alpha_n\rceil}{6n^{3/4}}\sqrt{\lceil\alpha_n\rceil^2 - (\alpha_n)^2}$$
$$= \frac{2}{3^{1/4}}\sqrt{\left\lceil\sqrt{12n-3}\right\rceil - \sqrt{12n-3} + \mathrm{o}(1)}$$
$$= K_t\sqrt{\left\lceil\sqrt{12n-3}\right\rceil - \sqrt{12n-3} + \mathrm{o}(1)} \tag{81}$$

We notice here that Theorem 1.2 implies in particular the convergence (up to translations) of the empirical measures associated with the minimizers to the measure μ not only with respect to the weak$*$-converge of measures, but also with respect to the flat norm [see (72)].

We remark that an alternative approach to the one adopted in Theorem 1.2 is that of defining a unique n-configurational Wulff shape W_n for all the minimizer with n atoms. For example, we could define

$$W_n := \hat{W}_n \cap \mathcal{L}_t,$$

where $\hat{W}_n$ is the hexagon with side $p_n/6$ and center $x_{\tau(1)}$. We remark that the $\mathrm{O}(n^{1/4})$ estimate on the Hausdorff distance and the $\mathrm{O}(n^{3/4})$-law still hold true by replacing the maximal hexagon $H_{r_{M_n}}$ with W_n.

More precisely, by Proposition 3.7 we have that

$$d_{\mathcal{H}}(W_n, H_{r_{M_n}}) \leq 6\left|\frac{p_n}{6} - r\right| \leq \sqrt{\lceil\alpha_n\rceil^2 - (\alpha_n)^2} + \mathrm{O}(1) \tag{82}$$

and that

$$
\begin{aligned}
|W_n \setminus H_{r_{M_n}}| &\leq \left|3\left(\left\lfloor \frac{p_n}{6} \right\rfloor\right)^2 + 3\left(\left\lfloor \frac{p_n}{6} \right\rfloor\right) - 3\left(r_{M_n}\right)^2 - 3\,r_{M_n}\right| \\
&= 3\left(\left\lfloor \frac{p_n}{6} \right\rfloor + r_{M_n} + 1\right)\left|\left\lfloor \frac{p_n}{6} \right\rfloor - r_{M_n}\right| \\
&\leq \frac{p_n}{6}\sqrt{\lceil \alpha_n \rceil^2 - (\alpha_n)^2} + \mathrm{o}(n^{3/4}) \\
&= \frac{\lceil \alpha_n \rceil}{6}\sqrt{\lceil \alpha_n \rceil^2 - (\alpha_n)^2} + \mathrm{o}(n^{3/4}) \qquad (83)
\end{aligned}
$$

for every minimizer M_n. Therefore, we obtain that

$$
d_{\mathcal{H}}(M_n', W_n) \leq \mathrm{O}(n^{1/4})
$$

by (20) and (82), and

$$
\left|M_n' \triangle W_n\right| \leq \mathrm{O}(n^{3/4}) \qquad (84)
$$

by (24) and (84), with $M_n' := M_n - q_n$ where $q_n \in \mathcal{L}_t$ are chosen in such a way that

$$
H_{r_{M_n}} = D_{1+3r_{M_n}+3r_{M_n}^2} + q_n.
$$

Furthermore, from (84) it follows that

$$
\left\|\mu_{M_n'} - \mu_{W_n}\right\| = \frac{\left|M_n' \triangle W_n\right|}{n} \leq \mathrm{O}(n^{-1/4}).
$$

4.2 Proof of Theorem 1.4

In this subsection, we prove that the estimates (15)–(17) are sharp.
Step 1 In this step, we show that there exists a sequence of minimizers $\overline{M}_n$ such that, denoting by $H_{r_{\overline{M}_n}}$ their maximal hexagons,

$$
|\overline{M}_n \setminus H_{r_{\overline{M}_n}}| = K_n n^{3/4} + \mathrm{o}(n^{3/4}). \qquad (85)
$$

We will explicitly construct the minimizers $\overline{M}_n$. To this end, we denote by $\hat{H}_{r_n}$ the closure of the regular hexagon in $\mathbb{R}^2$ with center in $x_{\tau(1)}$ and side r_n defined by

$$
r_n := \left\lceil \frac{\lceil \alpha_n \rceil}{6} - \frac{1}{6}\sqrt{\lceil \alpha_n \rceil^2 - (\alpha_n)^2} \right\rceil,
$$

and we introduce $H_{r_n} := \hat{H}_{r_n} \cap \mathcal{L}_t$. Furthermore, we define

$$
h_n := \frac{p_n}{2} - 3r_n
$$

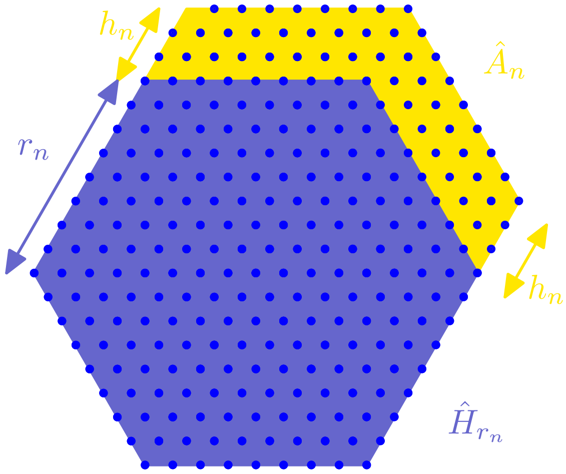


Fig. 6 The form of a minimizer $\overline{M}_n$ constructed in the proof of Theorem 1.4 is shown. The configuration $\overline{M}_n$ is contained in the union of the hexagon $\hat{H}_{r_n}$ drawn in the darkest color (*blue*) and the region $\hat{A}_n$ constructed on two of its sides drawn in the lightest color (*yellow*) (Color figure online)

and we consider the region

$$\hat{A}_n := \{x + h_n \boldsymbol{t}_2 \ : \ x \in \hat{H}_{r_n}\} \setminus \hat{H}_{r_n}$$

that consists of two parallelograms of height h_n constructed on two consecutive sides of H_{r_n} (see Fig. 6).

Let $c := |(\hat{H}_{r_n} \cup \hat{A}_n) \cap \mathcal{L}_t|$. We denote by C_c the configuration defined by

$$C_c := \left(\hat{H}_{r_n} \cup \hat{A}_n\right) \cap \mathcal{L}_t$$

and we observe that, by construction, the perimeter of C_c satisfies

$$P(C_c) = p_n. \tag{86}$$

We subdivide the remaining proof of the claim into two substeps.
Substep 1.1. We claim that for every n big enough there exists a minimizer $\overline{M}_n$ such that

$$H_{r_n} \subseteq \overline{M}_n \subseteq C_c$$

and $|C_c \setminus \overline{M}_n| \leq 2r_n - 1$.

We begin by observing that

$$\begin{aligned} c := |C_c| &= |H_{r_n}| + (2r_n + 1)h_n \\ &= 1 + 3r_n^2 + 3r_n + \left(r_n + \frac{1}{2}\right)(p_n - 6r_n) \\ &= -3r_n^2 + p_n r_n + 1 + \frac{p_n}{2}. \end{aligned} \tag{87}$$

Then, a direct computation shows that

$$3s^2 - p_n s - 1 - \frac{p_n}{2} \geq 0 \tag{88}$$

for every $s \in \left[\frac{\lceil\alpha_n\rceil}{6} - 3 - \frac{1}{6}\sqrt{\lceil\alpha_n\rceil^2 + 3}, \frac{\lceil\alpha_n\rceil}{6} - 3 + \frac{1}{6}\sqrt{\lceil\alpha_n\rceil^2 + 3}\right]$, and, for n big enough,

$$3s^2 + (2 - p_n)s - 2 - \frac{p_n}{2} + n \geq 0 \tag{89}$$

for every $s \in \mathbb{R}$. In particular, (88) and (89) hold for $s = r_n$ and for n sufficiently large, yielding

$$0 \leq c - n \leq 2r_n - 1. \tag{90}$$

We now observe that by the definition of C_c it is possible to remove up to $2r_n - 1$ points from $C_c \setminus H_{r_n}$ without changing the perimeter of the configuration. In view of (90), we construct $\overline{M}_n$ by removing in such a way $c - n$ points from C_c. It follows from (86) that $P(\overline{M}_n) = p_n$ and hence, the claim holds true.
Substep 1.2. Let $\overline{M}_n$ be the sequence of ground states constructed in the previous substep. In view of (90), and of the definition of α_n and p_n, there holds

$$\begin{aligned} |C_n \setminus H_{r_n}| &= (2r_n + 1)h_n \\ &= -6(r_n)^2 - 3r_n + p_n r_n + 1 + \frac{p_n}{2} \\ &= \frac{\lceil\alpha_n\rceil}{6}\sqrt{\lceil\alpha_n\rceil^2 - (\alpha_n)^2} + \mathrm{o}(n^{3/4}). \end{aligned} \tag{91}$$

Moreover, by the definition of $\overline{M}_n$ we have that

$$|C_n \setminus \overline{M}_n| \leq 2r_n - 1 = \mathrm{O}(n^{1/2}) = \mathrm{o}(n^{3/4}). \tag{92}$$

The thesis follows from combining (91) and (92) since H_{r_n} is by construction the maximal hexagon of $\overline{M}_n$.
Step 2 In this last step, we remark that

$$\limsup_{n\to+\infty} K_n = K_t \limsup_{n\to+\infty} \sqrt{\left\lceil\sqrt{12n - 3}\right\rceil - \sqrt{12n - 3}} \leq K_t,$$

and that for those $n_j \in \mathbb{N}$ of the form $n_j = 2 + 3j + 3j^2$ there holds

$$K_{n_j} \to \frac{2}{3^{1/4}} =: K_t \tag{93}$$

as $j \to +\infty$.

In fact, we have that

$$
\begin{aligned}
\sqrt{12n_j - 3} &= \sqrt{12(1+3j+3j^2)+9} \\
&= (6j+3)\sqrt{1+\frac{12}{(6j+3)^2}} \\
&= 6j+3+\frac{12}{(6j+3)\left[1+\sqrt{1+\frac{12}{(6j+3)^2}}\right]},
\end{aligned}
$$

which in turn yields

$$
\left\lceil \sqrt{12n_j-3} \right\rceil - \sqrt{12n_j-3} = 1 - \frac{12}{(6j+3)\left[1+\sqrt{1+\frac{12}{(6j+3)^2}}\right]} \to 1
$$

as $j \to +\infty$. □

It is remarkable that the leading terms in the estimates (24), (75), and (80) established in Step 1 of Theorem 1.2 are optimal for every $n \in \mathbb{N}$ as it follows from Step 1 of the proof of Theorem 1.4.

Finally, we notice that the bounded quantities K_n defined in (73) are 0 for every $n \in \mathbb{N}$ that can be written as $n = 1 + 3k + 3k^2$ for some $k \in \mathbb{N}$. This reflects the fact that for those n the daisy D_n is the unique minimizer, whose maximal hexagon $H_{r_{D_n}}$ is the daisy itself. Therefore, Theorem 1.4 also entails that, by adding a point to every EIP (2) with $n = 1 + 3i + 3i^2$ for some $i \in \mathbb{N}$, we pass not only from a problem characterized by uniqueness of solutions to a problem with nonuniqueness, but also from a situation of zero deviation of the minimizer from its maximal hexagon to the situation in which minimizers include one that attains the maximal deviation.

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Chapter 3

Time-evolving inelastic phenomena

This chapter consists of the following papers:

1) E. Davoli, U. Stefanelli.
   Dynamic perfect plasticity as convex minimization.
   SIAM Journal on Mathematical Analysis **51** (2019), 672–730.

2) E. Davoli, T. Roubíček, U. Stefanelli.
   Dynamic perfect plasticity and damage in viscoelastic solids.
   ZAMM - Zeitschrift für Angewandte Mathematik und Mechanik (2019), to appear.

SIAM J. MATH. ANAL.
Vol. 51, No. 2, pp. 672–730

DYNAMIC PERFECT PLASTICITY AS CONVEX MINIMIZATION*

ELISA DAVOLI† AND ULISSE STEFANELLI‡

Abstract. We present a novel variational approach to dynamic perfect plasticity. This is based on minimizing over entire trajectories parameter-dependent convex functionals of weighted-inertia-dissipation-energy (WIDE) type. Solutions to the system of dynamic perfect plasticity are recovered as limits of minimizing trajectories as the parameter goes to zero. The crucial compactness is achieved by means of a time discretization and a variational convergence argument.

Key words. weighted-inertia-dissipation-energy, dynamic perfect plasticity, elliptic regularization, time discretization, functions of bounded deformation

AMS subject classifications. 70H03, 70H30, 74C10, 74G65

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1. Introduction.

Plasticity is the macroscopic, inelastic behavior of a metal resulting from the accumulation of slip defects at its microscopic, crystalline level. As a result of these dislocations, the behavior of the material remains purely elastic (and hence reversible) as long as the magnitude of the stress remains *small*, and becomes irreversible as soon as a given stress-threshold is reached. When that happens, a plastic flow is developed such that, after unloading, the material remains permanently plastically deformed [27].

We refer the reader to [22, 34] for an overview on plasticity models; here we focus on *dynamic perfect plasticity* in the form of the classical *Prandtl–Reuss* model [16],

$$\rho\ddot{u} - \nabla\cdot\sigma = 0, \tag{1.1}$$

$$\sigma = \mathbb{C}(Eu - p), \tag{1.2}$$

$$\partial H(\dot{p}) \ni \sigma_D, \tag{1.3}$$

describing the basics of plastic behavior in metals [20]. Here $u(t) : \Omega \to \mathbb{R}^3$ denotes the (time-dependent) *displacement* of a body with reference configuration $\Omega \subset \mathbb{R}^3$ and density $\rho > 0$, and $\sigma(t) : \Omega \to \mathbb{M}^{3\times3}_{\rm sym}$ is its *stress*. In particular, relation (1.1) expresses the conservation of momenta. The constitutive relation (1.2) relates the stress $\sigma(t)$ to the *linearized strain* $Eu(t) = (\nabla u(t)+\nabla u(t)^\top)/2 : \Omega \to \mathbb{M}^{3\times3}_{\rm sym}$ and the *plastic strain* $p(t) : \Omega \to \mathbb{M}^{3\times3}_D$ (deviatoric tensors) via the fourth-order *elasticity tensor* $\mathbb{C}$. Finally, (1.3) expresses the plastic-flow rule: $H : \mathbb{M}^{3\times3}_D \to [0,+\infty)$ is a positively 1-homogeneous, convex *dissipation* function, σ_D stands for the deviatoric part of the stress, and the symbol ∂ is the subdifferential in the sense of convex analysis [9]. The system will be driven by imposing a nonhomogeneous boundary displacement. Details on notation and modeling are given in section 2.

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The focus of this paper is on recovering weak solutions to the dynamic perfect plasticity system (1.1)–(1.3) by minimizing parameter-dependent convex functionals over entire trajectories, and by passing to the parameter limit. In particular, we consider the *weighted-inertia-dissipation-energy (WIDE)* functional of the form

$$I_\varepsilon(u,p) = \int_0^T \int_\Omega \exp\left(-\frac{t}{\varepsilon}\right) \left(\frac{\rho\varepsilon^2}{2}|\ddot{u}|^2 + \varepsilon H(\dot{p}) + \frac{1}{2}(Eu-p) : \mathbb{C}(Eu-p)\right) dx\, dt, \tag{1.4}$$

to be defined on suitable admissible classes of entire trajectories $t \in [0,T] \mapsto (u(t), p(t))$: $\Omega \to \mathbb{R}^3 \times \mathbb{M}^{3\times 3}_D$ fulfilling given boundary-displacement and initial conditions (on u and p, respectively). The functional bears its name from the sum of the inertial term $\rho|\ddot{u}|^2/2$, the dissipative term $H(\dot{p})$, and the energy term $(Eu-p) : \mathbb{C}(Eu-p)/2$, weighted by different powers of ε as well as the function $\exp(-t/\varepsilon)$.

For all $\varepsilon > 0$ one can prove that (a suitable relaxation of) the convex functional I_ε admits minimizers $(u^\varepsilon, p^\varepsilon)$, which indeed approximate solutions to the dynamic perfect plasticity system (1.1)–(1.3). In particular, by computing the corresponding Euler–Lagrange equations, one finds that the minimizers $(u^\varepsilon, p^\varepsilon)$ weakly solve the elliptic-in-time approximating relations,

$$\varepsilon^2 \rho \ddddot{u}^\varepsilon - 2\varepsilon^2 \rho \dddot{u}^\varepsilon + \rho \ddot{u}^\varepsilon - \nabla \cdot \sigma^\varepsilon = 0, \tag{1.5}$$

$$\sigma^\varepsilon = \mathbb{C}(Eu^\varepsilon - p^\varepsilon), \tag{1.6}$$

$$-\varepsilon(\partial H(\dot{p}^\varepsilon))^\cdot + \partial H(\dot{p}^\varepsilon) \ni \sigma^\varepsilon_D, \tag{1.7}$$

along with Neumann conditions at the final time T.

The dynamic perfect plasticity system (1.1)–(1.3) is formally recovered by taking $\varepsilon \to 0$ in system (1.5)–(1.7). The main result of this paper consists of making this intuition rigorous, resulting in a new approximation theory for dynamic perfect plasticity.

The interest in this variational-approximation approach is threefold. First, the differential problem (1.1)–(1.3) is reformulated on purely variational grounds. This opens the possibility of applying the powerful tools of the calculus of variations to the problem, such as the direct method, relaxation, and Γ-convergence [15].

Second, by addressing a time-discrete analogue of this approach we contribute a novel numerical strategy in order to approximate dynamic perfect plasticity by means of space-time optimization and collocation methods. We believe this to be of potential interest in combination with global constraints or noncylindrical domains.

Eventually, the variational formulation via WIDE functionals is easily open to generalization by including more refined material effects, especially in terms of additional internal-variable descriptions. This indeed has been one of the main motivations for advancing the WIDE method; see, in particular, [10, 26] for applications in materials science. Details of the method in the case of dynamic perfect plasticity could then serve as the basis for developing complete theories for evolutionary dissipative processes, such as those involving damage or fracture effects.

As a by-product of our analysis, we obtain a new proof of existence of weak solutions to dynamic perfect plasticity. Note that existence results for (1.1)–(1.3) are indeed quite classical. In the quasi-static case, in which the inertial term is neglected, they date back to Suquet [50] and have been subsequently reformulated by Dal Maso, DeSimone, and Mora [11] and Francfort and Giacomini [18] within the theory of rate-independent processes (see the recent monograph [39]). In the dynamic case both the first existence results due to Anzellotti and Luckhaus [6, 35] and the subsequent revisiting of these results by Babadjian and Mora [7] are based on viscosity techniques. Dimension reduction has been tackled in both the quasi-static and the dynamic case in [13, 28, 29] and [36], respectively. Finally, in [12] convergence of solutions of the dynamic problem to solutions of the quasi-static problem has been shown. With respect to the available existence theories our approach is new, for it does not rely on viscous approximation but rather a global variational method.

Before moving on, let us review here the available literature on WIDE variational methods. At the level of Euler–Lagrange equations, elliptic-regularization techniques are classical and can be traced back to Lions [32, 33] and Oleĭnik [43]. Their variational version via global functionals was already mentioned in the classical textbook by Evans [17, Problem 3, p. 487] and has been used by Ilmanen [24] in the context of Brakke mean-curvature flow of varifolds and by Hirano [23] in connection with periodic solutions to gradient flows.

The formalism has then been applied in the context of rate-independent systems by Mielke and Ortiz [38]; see also the follow-up paper [40]. Viscous dynamics have been considered in many different settings, including gradient flows [41], curves of maximal slopes in metric spaces [44, 45], mean-curvature flow [48], doubly nonlinear equations [1, 2, 3, 4, 5], reaction-diffusion systems [37], and quasi-linear parabolic equations [8].

The dynamic case has been the object of a long-standing conjecture by De Giorgi on semilinear waves [14]. The conjecture was solved affirmatively [49] for finite-time intervals and by Serra and Tilli [46] for the whole time semiline, that is, in its original formulation. De Giorgi himself pointed out in [14] the interest in extending the method to other dynamic problems. The task has then been taken up in [31] for mixed hyperbolic-parabolic equations, in [30] for Lagrangian mechanics, and in [47] for other hyperbolic problems. The present paper delivers, in its main result (Theorem 2.3), the first realization of De Giorgi's suggestion in the context of continuum mechanics.

We briefly outline the main steps of the WIDE approach, and of the proof of Theorem 2.3, in our setting. First, we perform a time discretization of the WIDE functional. By choosing suitable test functions in the discrete Euler–Lagrange equations, and by performing time-discrete integration by parts, we prove in Theorem 4.8 a first a priori estimate for minimizers of the time-discrete WIDE functionals. A crucial point of the argument is to guarantee that the estimate above is uniform with respect to both the WIDE parameter ε and the width of the time-discretization step. Second, we show via a Γ-convergence type of argument that the same uniform a priori estimate is fulfilled by suitable minimizers of the WIDE functional at the time-continuous level (see Corollary 5.3). This latter estimate guarantees compactness of sequences of minimizers as ε tends to zero, and it allows us to recover conditions (1.1) and (1.2) in the limit. The third step (see Propositions 6.4 and 6.6) consists of deducing both an energy inequality at the ε-level fulfilled by minimizers, and a corresponding integrated-in-time counterpart. Finally, we pass to the limit in the energy inequality and show that the flow rule in (1.3) is attained in weak form (see subsection 2.8).

An alternative approach to deducing a uniform energy estimate analogous to that in Corollary 5.3 could be to try performing directly some very careful energy estimates in the equations, along the lines of [46, 47]. We have decided here to proceed instead as in [49], namely by first performing a discretization in time, establishing a uniform a priori estimate at the time-discrete level, and eventually showing that this estimate transfers to the time-continuous setting.

We have chosen to adopt this latter strategy for three main reasons. First, the existence results in the literature for solutions to both the quasi-static and the dynamic problem in perfect plasticity are classically proven by resorting to time discretization. Adopting the analogous strategy might allow us to gain further insight into the relation between classical approximations via viscosity solutions and those provided by the WIDE approach; this is currently an open question. Second, the WIDE approach for rate-independent processes has been developed by relying on a time-discrete analysis. In this regard, our analysis shows that the same methodology can be used to discuss both the quasi-static and the dynamic case. Finally, the establishment of the time-discrete a priori estimates might prove useful for advancing the study of the numerics of the problem.

The motivation for choosing dynamic perfect plasticity as a test for the WIDE methodology is threefold. First, the existence theory in this setting has already been fully characterized in both the quasi-static and the dynamic case. This provides a solid starting point for our analysis that might be not available in different frameworks. Second, we are interested in checking whether the WIDE methodology is amenable also to solving dynamic problems calling for very weak formulations in spaces of measures. Third, we intend to proceed along the line proposed by De Giorgi in his seminal paper [14, Conj. 4, Rem. 1] of extending the reach of the WIDE methodology beyond semilinear waves. This has partly succeeded in the case of additional superlinear dissipation [31, 47]. Our goal is then to check whether a similar analysis applies to perfect plasticity, in which the dynamic of the system is characterized by linear dissipation instead.

The paper is organized as follows. We introduce notation and state our main result, namely Theorem 2.3, in section 2. Then, we discuss in section 3 the existence of minimizers of the WIDE functionals. In section 4 a time discretization of the minimization problem is addressed. Its time-continuous limit is discussed in section 5 by means of variational convergence arguments. A parameter-dependent energy inequality is derived in section 6 and used in section 7 in order to pass to the limit as $\varepsilon \to 0$ and prove Theorem 2.3.

2. Statement of the main result. We devote this section to the specification of the material model and its mathematical setting. Some notions from measure theory need to be recalled, and we introduce the notation and assumptions to be used throughout the article. The specific form of the WIDE functionals is eventually introduced in subsection 2.9, and we conclude by stating our main result, namely Theorem 2.3.

2.1. Tensors. In what follows, for any map $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ we will denote by $\dot{f}$ its time derivative and by ∇f its spatial gradient. The set of 3×3 real matrices will be denoted by $\mathbb{M}^{3\times3}$. Given $M, N \in \mathbb{M}^{3\times3}$, we will denote their scalar product by $M : N := \mathrm{tr}(M^\top N)$, where tr denotes the trace and the superscript stands for transposition, and we will adopt the notation M_D to identify the deviatoric part of M, namely $M_D := M - \mathrm{tr}(M)\mathrm{Id}/3$, where Id is the identity matrix. The symbol $\mathbb{M}^{3\times3}_{\mathrm{sym}}$ will stand for the set of symmetric 3×3 matrices, whereas $\mathbb{M}^{3\times3}_D$ will be the subset of $\mathbb{M}^{3\times3}_{\mathrm{sym}}$ given by symmetric matrices having null trace.

2.2. Measures. Given a Borel set $B \subset \mathbb{R}^N$, the symbol $\mathcal{M}_b(B;\mathbb{R}^m)$ denotes the space of all bounded Borel measures on B with values in $\mathbb{R}^m$ ($m \in \mathbb{N}$). When $m = 1$ we will simply write $\mathcal{M}_b(B)$. We will endow $\mathcal{M}_b(B;\mathbb{R}^m)$ with the norm $\|\mu\|_{\mathcal{M}_b(B;\mathbb{R}^m)} := |\mu|(B)$, where $|\mu| \in \mathcal{M}_b(B)$ is the total variation of the measure μ.

If the relative topology of B is locally compact, then by the Riesz representation theorem the space $\mathcal{M}_b(B;\mathbb{R}^m)$ can be identified with the dual of $C_0(B;\mathbb{R}^m)$, which is the space of all continuous functions $\varphi : B \to \mathbb{R}^m$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $\mathcal{M}_b(B;\mathbb{R}^m)$ is defined using this duality.

2.3. Functions with bounded deformation. Let U be an open set of $\mathbb{R}^3$. The space $BD(U)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(U;\mathbb{R}^3)$ whose symmetric gradient $Eu := \operatorname{sym} Du := (Du + Du^T)/2$ (in the sense of distributions) belongs to $\mathcal{M}_b(U;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$. It is easy to see that $BD(U)$ is a Banach space endowed with the norm

$$\|u\|_{L^1(U;\mathbb{R}^3)} + \|Eu\|_{\mathcal{M}_b(U;\mathbb{M}^{3\times 3}_{\mathrm{sym}})}.$$

A sequence $\{u^k\}$ is said to converge to u weakly* in $BD(U)$ if $u^k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^3)$ and $Eu^k \rightharpoonup Eu$ weakly* in $\mathcal{M}_b(U;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. If U is bounded and has a Lipschitz boundary, then $BD(U)$ can be embedded into $L^{3/2}(U;\mathbb{R}^3)$ and every function $u \in BD(U)$ has a trace, still denoted by u, which belongs to $L^1(\partial U;\mathbb{R}^3)$. If Γ is a nonempty open subset of ∂U in the relative topology of ∂U, then there exists a constant $C > 0$, depending on U and Γ such that

$$\|u\|_{L^1(U;\mathbb{R}^3)} \leq C\|u\|_{L^1(\Gamma;\mathbb{R}^3)} + C\|Eu\|_{\mathcal{M}_b(U;\mathbb{M}^{3\times 3}_{\mathrm{sym}})}. \tag{2.1}$$

(See [51, Chapter II, Proposition 2.4 and Remark 2.5].) For the general properties of the space $BD(U)$ we refer the reader to [51].

2.4. The elasticity tensor. Let $\mathbb{C}$ be the *elasticity tensor*, which is considered a symmetric positive-definite linear operator $\mathbb{C} : \mathbb{M}^{3\times 3}_{\mathrm{sym}} \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$, and let $Q : \mathbb{M}^{3\times 3}_{\mathrm{sym}} \to [0,+\infty)$ be the quadratic form associated with $\mathbb{C}$, given by

$$Q(\xi) := \frac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}.$$

Let the two constants $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, with $0 < \alpha_{\mathbb{C}} \leq \beta_{\mathbb{C}}$, be such that

$$\alpha_{\mathbb{C}}|\xi|^2 \leq Q(\xi) \leq \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{\mathrm{sym}} \tag{2.2}$$

and

$$|\mathbb{C}\xi| \leq 2\beta_{\mathbb{C}}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}. \tag{2.3}$$

2.5. The reference configuration. Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\partial_{\partial\Omega}\Gamma_0$ is a connected, one-dimensional C^2 manifold. In the following we will assume that Ω is the reference configuration of our material and that Γ_0 is the Dirichlet portion of $\partial\Omega$, where time-dependent boundary conditions are prescribed.

2.6. The dissipation potential. Let K be a closed convex set of $\mathbb{M}^{3\times 3}_D$ such that there exist two constants r_K and R_K, with $0 < r_K \leq R_K$, satisfying

$$\{\xi \in \mathbb{M}^{3\times 3}_D : \ |\xi| \leq r_K\} \subset K \subset \{\xi \in \mathbb{M}^{3\times 3}_D : \ |\xi| \leq R_K\}.$$

The boundary of K is interpreted as the *yield surface*. The *plastic dissipation potential* is given by the support function $H : \mathbb{M}^{3\times 3}_D \to [0,+\infty)$ of K, defined as

$$H(\xi) := \sup_{\sigma\in K} \sigma : \xi.$$

Note that $K = \partial H(0)$ is the subdifferential of H at 0 (see, e.g., [9, section 1.4]). The function H is convex and positively 1-homogeneous, with

$$r_K|\xi| \le H(\xi) \le R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_D. \tag{2.4}$$

In particular, H satisfies the triangle inequality

$$H(\xi+\zeta) \le H(\xi) + H(\zeta) \quad \text{for every } \xi, \zeta \in \mathbb{M}^{3\times 3}_D. \tag{2.5}$$

For every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$ let $d\mu/d|\mu|$ be the Radon–Nikodým derivative of μ with respect to its variation $|\mu|$.

According to the theory of convex functions of measures [19], we introduce the nonnegative Radon measure $H(\mu) \in \mathcal{M}_b(\Omega \cup \Gamma_0)$, defined by

$$H(\mu)(A) := \int_A H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|,$$

for every Borel set $A \subset \Omega \cup \Gamma_0$. We also consider the functional

$$\mathcal{H} : \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D) \to [0, +\infty),$$

defined by

$$\mathcal{H}(\mu) := H(\mu)(\Omega \cup \Gamma_0) = \int_{\Omega \cup \Gamma_0} H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|,$$

for every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$. Notice that $\mathcal{H}$ is lower semicontinuous on $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$ with respect to weak* convergence. The following lemma is a consequence of [19, Theorem 4] and [51, Chapter II, Lemma 5.2] (see also [11, subsection 2.2]).

LEMMA 2.1. *Setting* $\mathcal{K}_D(\Omega) := \{\tau \in C_0(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D) : \tau(x) \in K$ *for every* $x \in \Omega\}$*, there holds*

$$\mathcal{H}(\mu) = \sup\{\langle \tau, \mu \rangle : \tau \in \mathcal{K}_D(\Omega)\}$$

for every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$.

2.7. The $\mathcal{H}$-dissipation. Let $s_1, s_2 \in [0, T]$ with $s_1 \le s_2$. For every function $\mu : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$, we define the *$\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$\begin{aligned} &D_{\mathcal{H}}(\mu; s_1, s_2) \\ &\quad := \sup\left\{ \sum_{j=1}^{n} \mathcal{H}(\mu(t_j) - \mu(t_{j-1})) : \ s_1 = t_0 \le t_1 \le \cdots \le t_n = s_2,\ n \in \mathbb{N} \right\}. \end{aligned} \tag{2.6}$$

Denoting by V_{tot} the pointwise variation of $t \to \mu(t)$, that is,

$$\begin{aligned} &V_{\mathrm{tot}}(\mu; s_1, s_2) \\ &:= \sup\left\{ \sum_{j=1}^{n} \|\mu(t_j) - \mu(t_{j-1})\|_{\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)} : s_1 = t_0 \le \cdots \le t_n = s_2, n \in \mathbb{N} \right\}, \end{aligned}$$

by (2.4) there holds

$$r_K V_{\mathrm{tot}}(\mu; s_1, s_2) \le D_{\mathcal{H}}(\mu; s_1, s_2) \le R_K V_{\mathrm{tot}}(\mu; s_1, s_2). \tag{2.7}$$

As in [38, section 4.2], for every nonincreasing and positive $a \in C([0,T])$ we define the *a-weighted $\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$D_{\mathcal{H}}(a;\mu;s_1,s_2) := \sup\left\{ \sum_{j=1}^{n} a(t_j)\mathcal{H}(\mu(t_j)-\mu(t_{j-1})) :\ t_0, t_n \in [s_1,s_2],\right.$$
$$\left. t_0 \le t_1 \le \cdots \le t_n,\ n \in \mathbb{N} \right\}, \tag{2.8}$$

and for every $b \in C([0,T])$ we introduce the *b-weighted $\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$\hat{D}_{\mathcal{H}}(b;\mu;s_1,s_2) := PMS \int_{s_1}^{s_2} b(t)\, dD_{\mathcal{H}}(\mu;0,t), \tag{2.9}$$

namely, as the Pollard–Moore–Stieltjes integral (see [21, sections 3 and 4]) of b with respect to the function of bounded variation

$$[0,T] \ni t \mapsto D_{\mathcal{H}}(\mu;0,t) \in [0, D_{\mathcal{H}}(\mu;0,T)].$$

Note that the integral above is well defined, owing to [21, Theorems 5.31 and 5.32], and that if b is nonincreasing and positive, then

$$\hat{D}_{\mathcal{H}}(b;\mu;s_1,s_2) = D_{\mathcal{H}}(b;\mu;s_1,s_2). \tag{2.10}$$

An adaptation of [11, Theorem 7.1] yields that if μ is absolutely continuous in time, then

$$D_{\mathcal{H}}(\mu;s_1,s_2) = \int_{s_1}^{s_2} \mathcal{H}(\dot{\mu})\, dt$$

and

$$D_{\mathcal{H}}(a;\mu;s_1,s_2) = \int_{s_1}^{s_2} a(t)\mathcal{H}(\dot{\mu})\, dt$$

for every nonincreasing and positive $a \in C([0,T])$.

2.8. The equations of dynamic perfect plasticity. On Γ_0 for every $t \in [0,T]$ we prescribe a boundary datum $w(t) \in W^{1/2,2}(\Gamma_0;\mathbb{R}^3)$. With a slight abuse of notation we also denote by $w(t)$ a $W^{1,2}(\Omega;\mathbb{R}^3)$-extension of the boundary condition to the set Ω.

The *set of admissible displacements and strains* for the boundary datum $w(t)$ is given by

$$\mathscr{A}(w(t)) := \Big\{ (u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{3\times 3}_D) :$$
$$Eu = e + p \text{ in } \Omega, \quad p = (w(t)-u) \odot \nu \mathcal{H}^2 \text{ on } \Gamma_0 \Big\}, \tag{2.11}$$

where $\odot$ stands for the symmetrized tensor product, namely,

$$a \odot b := (a \otimes b + b \otimes a)/2 \quad \forall\, a, b \in \mathbb{R}^3,$$

ν is the outer unit normal to $\partial\Omega$, and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. The function u represents the *displacement* of the body, while e and p are called the *elastic* and *plastic strain*, respectively. Note that the two equalities in (2.11) hold in the sense of $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$.

We point out that the constraint

$$p = (w(t) - u) \odot \nu \mathcal{H}^2 \text{ on } \Gamma_0 \tag{2.12}$$

is a relaxed formulation of the boundary condition $u = w(t)$ on Γ_0 (see also [42]). As remarked in [11], the mechanical meaning of (2.12) is that whenever the boundary datum is not attained a plastic slip develops, whose amount is directly proportional to the difference between the displacement u and the boundary condition $w(t)$.

Let $w \in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$. A *solution to the equations of dynamic perfect plasticity* is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $(L^2(\Omega;\mathbb{R}^3) \cap BD(\Omega)) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{3\times 3}_D)$ with $u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$ $\cap \operatorname{Lip}(0,T;BD(\Omega))$, $e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$, $p \in \operatorname{Lip}(0,T;\mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{3\times 3}_D))$ such that for every $t \in [0,T]$ there holds $(u(t), e(t), p(t)) \in \mathscr{A}(w(t))$, and for almost every $t \in [0,T]$ the following conditions are satisfied:

(c1) *equation of motion:* $\rho \ddot{u}(t) - \operatorname{div} \sigma(t) = 0$ in Ω and $\sigma(t)\nu = 0$ on $\partial\Omega \setminus \Gamma_0$ in the sense of Remark 4.4, where $\sigma(t) := \mathbb{C}e(t)$ is the stress tensor, and $\rho > 0$ is the constant density;

(c2) *stress constraint:* $\sigma_D(t) \in K$;

(c3) *energy inequality:*

$$\begin{aligned}\int_\Omega Q(e(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot{u}(t)|^2\,dx + \int_0^t \mathcal{H}(\dot{p}(t))\,dt \le \int_\Omega Q(e(0))\,dx \\ + \frac{\rho}{2}\int_\Omega |\dot{u}(0)|^2\,dx + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s) + \rho \ddot{u}(s)\cdot \dot{w}(s)\,dx\,ds.\end{aligned}$$

We remark that condition (c3) guarantees that the sum of the elastic and kinetic energies with the plastic dissipation at each time t is always less than or equal to the sum of the initial energy with the work due to the time-dependent boundary condition.

Under suitable assumptions, when (c1) and (c2) are satisfied, condition (c3) is indeed an equality, and it is equivalent to the following *flow rule:*

(c3′) $\dot{p}(t) = 0$ if $\sigma_D(t) \in \operatorname{int} K$, while $\dot{p}(t)$ belongs to the normal cone to K at $\sigma_D(t)$ if $\sigma_D(t) \in \partial K$.

A detailed analysis of the equivalence between (c1)–(c3) and (c1)–(c2), (c3′) has been performed in [11, section 6]. An adaptation of the argument yields the analogous statements in the dynamic setting.

The following existence and uniqueness result holds true (see [36, Theorem 3.1 and Remark 3.2]).

THEOREM 2.2 (existence of the evolution). *Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\partial_{\partial\Omega}\Gamma_0$ is a connected, one-dimensional C^2 manifold.*

Let $w \in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$ and $(u^0, e^0, p^0) \in \mathscr{A}(w(0))$ be such that $\operatorname{div} \mathbb{C}e^0 = 0$ *a.e. in Ω, $(\mathbb{C}e^0)\nu = 0$ $\mathcal{H}^2$-a.e. on $\partial\Omega \setminus \Gamma_0$, and $(\mathbb{C}e^0)_D \in K$ a.e. in Ω. Eventually, let $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$.*

Then there exist unique $u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap$ $\operatorname{Lip}(0,T;BD(\Omega))$, *$e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$, and* $p \in \operatorname{Lip}(0,T;\mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{3\times 3}_D))$ *solving* (c1), (c2), *and* (c3), *with $(u(0), e(0), p(0)) = (u^0, e^0, p^0)$, and $\dot{u}(0) = u^1$.*

2.9. The WIDE functional. Let the boundary datum

$$w \in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$$

be given. By reformulating the expression in (1.4) for the triple (u,e,p) one would be tempted to introduce the functional

$$(u,e,p) \mapsto \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}|^2\,dx + \varepsilon\mathcal{H}(\dot{p}) + \int_\Omega Q(e)\,dx\Big)\,dt,$$

to be defined on the set $\mathcal{V}$, given by the class of triples (u,e,p) such that the following conditions are fulfilled:

$\mathcal{V}$1. $u \in (W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^1(0,T;BD(\Omega)))$ with $u(0)=u^0$, $\dot{u}(0)=u^1$;

$\mathcal{V}$2. $p \in BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ and attains the initial datum $p(0)=p^0$;

$\mathcal{V}$3. $e(t) := Eu(t) - p(t)$ in $\mathcal{D}'(\Omega;\mathbb{M}^{3\times3}_{\rm sym})$ for every $t \in [0,T], e \in L^2((0,T)\times\Omega;\mathbb{M}^{3\times3}_{\rm sym})$ and $e(0)=e^0$;

$\mathcal{V}$4. $(u(t),e(t),p(t)) \in \mathscr{A}(w(t))$ for a.e. $t\in[0,T]$,

where $(u^0,e^0,p^0)\in\mathscr{A}(w(0))$, and $u^1 \in W^{1,2}(\Omega;\mathbb{R}^3)$ is such that $u^1=\dot{w}(0)$ on Γ_0.

We observe that if $(u,e,p)\in\mathcal{V}$, then $Eu \in W^{2,2}(0,T;W^{-1,2}(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))$. Thus, $e(t)$ is defined for every $t\in[0,T]$ as a map in $W^{-1,2}(\Omega;\mathbb{M}^{3\times3}_{\rm sym}) + \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)$, and the initial condition $e(0)=e^0$ is well justified. We stress that $BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ denotes here the set of maps μ such that $\mu(t)\in\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)$ for every $t\in[0,T]$, $\mu\in L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$, and $V_{\rm tot}(\mu;0,T)<+\infty$ (see also [11, Appendix]).

On the other hand, one readily sees that the term

$$\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}(\dot{p})\,dt$$

is not well defined in case p is not absolutely continuous with respect to time (see [11, Theorem 7.1]). We hence need to relax the form of the WIDE functional as

(2.13)

$$\begin{aligned} &I_\varepsilon(u,e,p)\\ &:= \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega|\ddot{u}|^2\,dx + \int_\Omega Q(e)\,dx\Big)\,dt + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T)\end{aligned}$$

for every $(u,e,p)\in\mathcal{V}$. As pointed out in subsection 2.7, an adaptation of [11, Theorem 7.1] yields

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T) = \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}(\dot{p})\,dt$$

whenever p is absolutely continuous with respect to time.

2.10. Main result. We are now ready to state the main result of the paper.

THEOREM 2.3 (dynamic perfect plasticity as convex minimization). *Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\partial_{\partial\Omega}\Gamma_0$ is a connected, one-dimensional C^2 manifold. Let $w\in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))\cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$, and let $(u^0,e^0,p^0)\in\mathscr{A}(w(0))$ be such that* $\operatorname{div}\mathbb{C}e^0=0$ *a.e. in Ω, $(\mathbb{C}e^0)\nu=0$ $\mathcal{H}^2$-a.e. on $\partial\Omega\setminus\Gamma_0$, and $(\mathbb{C}e^0)_D\in K$ a.e. in Ω. Eventually, let $(u^1,e^1,0)\in\mathscr{A}(\dot{w}(0))$.*

For every $\varepsilon > 0$ there exists $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\} \subset \mathcal{V}$ solving

$$I_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) = \min_{(u,e,p)\in\mathcal{V}} I_\varepsilon(u, e, p), \tag{2.14}$$

such that for $\varepsilon \to 0$ there holds

$$\begin{aligned} u^\varepsilon &\rightharpoonup u \quad \textit{weakly in } W^{1,2}(0,T; L^2(\Omega;\mathbb{R}^3)),\\ e^\varepsilon &\rightharpoonup e \quad \textit{weakly in } L^2(0,T; L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})). \end{aligned}$$

Additionally, for every $t \in [0,T]$ we have

$$p^\varepsilon(t) \rightharpoonup^* p(t) \quad \textit{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D),$$

and for a.e. $t \in [0,T]$ there exists a t-dependent subsequence $\{\varepsilon_t\}$ such that

$$\begin{aligned} u^{\varepsilon_t}(t) &\rightharpoonup^* u(t) \quad \textit{weakly* in } BD(\Omega),\\ e^{\varepsilon_t}(t) &\rightharpoonup e(t) \quad \textit{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}), \end{aligned}$$

where $u \in W^{2,\infty}(0,T; L^2(\Omega;\mathbb{R}^3)) \cap W^{1,\infty}(0,T;BD(\Omega))$, $e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$, and $p \in W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ is the unique solution to the dynamic perfect plasticity problem (c1), (c2), *and* (c3), *with $(u(0), e(0), p(0)) = (u^0, e^0, p^0)$ and $\dot u(0) = u^1$.*

The rest of the paper is devoted to the proof of Theorem 2.3. Our argument runs as follows: in section 3 we prove that minimizers $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$ of problem (2.14) exist. Then we devise an ε-independent a priori estimate on $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$ first in a discrete and then in a continuous setting (section 4) by means of a Γ-convergence argument (section 5). Finally, we derive an energy inequality at level $\varepsilon > 0$ (section 6) which allows for discussing the limit $\varepsilon \to 0$ in section 7 and for recovering condition (c3) in the limit.

We point out that the C^2 regularity of $\partial\Omega$ is needed in Theorem 2.3 in order to introduce a duality between stresses and plastic strains, along the lines of [25, Proposition 2.5]. For technical reasons it is not possible to use here the results in [18] and to consider the case of a Lipschitz $\partial\Omega$. We refer the reader to Remark 4.6 for some discussion of this point.

3. Minimizers of the WIDE functional. We start by focusing here on problem (2.14) and show that the functional I_ε admits a minimizer in $\mathcal{V}$.

PROPOSITION 3.1 (existence of minimizers). *For every $\varepsilon > 0$ there exists a triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{V}$ such that*

$$I_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) = \inf_{(u,e,p)\in\mathcal{V}} I_\varepsilon(u, e, p). \tag{3.1}$$

Proof. Fix $\varepsilon > 0$, and let $\{(u_n, e_n, p_n)\} \subset \mathcal{V}$ be a minimizing sequence for I_ε. We first observe that the triple

$$t \to (u^0 + tu^1 + w(t) - w(0) - t\dot w(0), e^0 + te^1 + Ew(t) - Ew(0) - tE\dot w(0), p^0)$$

belongs to $\mathcal{V}$. Hence,

$$\lim_{n\to+\infty} I_\varepsilon(u_n, e_n, p_n) \le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big) \cdot \Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot w|^2\,dx + \int_\Omega Q(e^0 + te^1 + Ew(t) - Ew(0) - tE\dot w(0))\,dx\Big)\,dt \le C,$$

thus yielding the uniform bound

$$\sup_{n\in\mathbb{N}}\Big\{\|\ddot{u}_n\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}+D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p_n;0,T) \\ +\|e_n\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}))}\Big\}\le C. \tag{3.2}$$

Since $(u_n,e_n,p_n)\in\mathcal{V}$, there holds $p_n(0)=p^0$ for every $n\in\mathbb{N}$. In view of (2.7) and (2.8),

$$r_K\exp(-T/\varepsilon)V_{\rm tot}(p_n;0,T)\le\exp(-T/\varepsilon)D_{\mathcal{H}}(p_n;0,T)\le D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p_n;0,T).$$

Therefore, we are in a position to apply the variant of Helly's theorem in [11, Lemma 7.2] and to deduce the existence of a subsequence, still denoted by $\{p_n\}$, and a map $p^\varepsilon\in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ such that

$$p_n(t)\rightharpoonup^* p^\varepsilon(t)\text{ weakly* in }\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\quad\text{for every }t\in[0,T], \tag{3.3}$$

and we have the lower semicontinuity of the $\mathcal{H}$-dissipation,

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,T)\le\liminf_{n\to+\infty}D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p_n;0,T). \tag{3.4}$$

By (3.2), there exist $e^\varepsilon\in L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))$ and $u^\varepsilon\in W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ such that, up to the extraction of a (non–relabeled) subsequence,

$$e_n\rightharpoonup e^\varepsilon\quad\text{weakly in }L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})) \tag{3.5}$$

and

$$u_n\rightharpoonup u^\varepsilon\quad\text{weakly in }W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{3.6}$$

This implies that $u^\varepsilon(0)=u^0$ and $\dot{u}^\varepsilon(0)=u^1$. By (3.3), (3.5), and (3.6) it follows that

$$e_n(t)\rightharpoonup e^\varepsilon(t)\quad\text{weakly in }\mathcal{D}'(\Omega;\mathbb{M}^{3\times3}_{\rm sym}) \tag{3.7}$$

for every $t\in[0,T]$, and hence $e^\varepsilon(0)=e^0$. In view of (3.5) and Fatou's lemma, there holds

$$\int_0^T\liminf_{n\to+\infty}\int_\Omega|e_n|^2\,dx\,dt\le\liminf_{n\to+\infty}\int_0^T\int_\Omega|e_n|^2\,dx\,dt\le C.$$

Thus, by (3.7) for a.e. $t\in[0,T]$ there exists a t-dependent subsequence $\{n_t\}$ such that

$$e_{n_t}(t)\rightharpoonup e^\varepsilon(t)\quad\text{weakly in }L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}). \tag{3.8}$$

Finally, by (2.1), (3.3), and (3.7), up to subsequences there holds

$$u_{n_t}(t)\rightharpoonup^* u^\varepsilon(t)\quad\text{weakly* in }BD(\Omega)\quad\text{for a.e. }t\in[0,T].$$

The fact that p^ε satisfies the Dirichlet condition on Γ_0 for a.e. $t\in[0,T]$ follows, arguing as in [11, Lemma 2.1]. The minimality of the limit triple $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ is a direct consequence of the lower semicontinuity of I_ε with respect to the convergences in (3.4), (3.5), and (3.6). ∎

We conclude this section with a result stating the uniqueness of the displacement for a given plastic strain.

PROPOSITION 3.2 (uniqueness of minimizers given the plastic strain). *Let (u_a, e_a, p_a) and (u_b, e_b, p_b) be two minimizers of I_ε in $\mathcal{V}$. Then there exists a constant C such that*

$$\varepsilon^2\|u_a - u_b\|^2_{W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))} + \|e_a - e_b\|^2_{L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}))} \le C\varepsilon \exp\Big(\frac{T}{\varepsilon}\Big)V_{\mathrm{tot}}(p_a - p_b; 0, T). \tag{3.9}$$

Proof. Arguing as in [11, Theorem 3.8], we set $v = u_a - u_b$, $f = e_a - e_b$, and $q = p_a - p_b$. Since $(v(t), f(t), q(t)) \in \mathscr{A}(0)$ for almost every $t \in [0, T]$, it follows that $(u_a, e_a, p_a) + \lambda(v, f, q) \in \mathcal{V}$ for every $\lambda \in \mathbb{R}$. Thus,

$$\begin{aligned} I_\varepsilon(u_a, e_a, p_a) &\le I_\varepsilon((u_a, e_a, p_a) + \lambda(v, f, q)) \\ &= \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}_a + \lambda\ddot{v}|^2\,dx + \int_\Omega Q(e_a + \lambda f)\,dx\Big)\,dt \\ &\quad + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_a + \lambda q; 0, T). \end{aligned}$$

By the arbitrariness of λ we deduce the inequality

$$\begin{aligned} -\,\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); q; 0, T) &\le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega \big(\varepsilon^2\rho\ddot{u}_a\ddot{v} + \mathbb{C}e_a : f\big)\,dx\,dt \\ &\le \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); -q; 0, T). \end{aligned} \tag{3.10}$$

Arguing analogously, the minimality of (u_b, e_b, p_b) yields

$$\begin{aligned} -\,\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); -q; 0, T) &\le -\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega \big(\varepsilon^2\rho\ddot{u}_b\ddot{v} + \mathbb{C}e_b : f\big)\,dx\,dt \\ &\le \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); q; 0, T). \end{aligned} \tag{3.11}$$

Summing (3.10) and (3.11) we obtain

$$\begin{aligned} &-\,\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_a - p_b; 0, T) - \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_b - p_a; 0, T) \\ &\le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega \big(\varepsilon^2\rho|\ddot{u}_a - \ddot{u}_b|^2 + 2Q(e_a - e_b)\big)\,dx\,dt \\ &\le \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_a - p_b; 0, T) + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_b - p_a; 0, T). \end{aligned}$$

The thesis follows now by (2.2), (2.7), and (2.8). ∎

We point out that minimizers of I_ε are, in general, nonunique. The proof of the approximation result in Theorem 2.3 will in fact rely on a selection of minimizers of I_ε performed via a Γ-convergence type of argument (see Corollary 5.3).

4. Discrete energy estimate. With the aim of establishing an a priori estimate on $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$ independent of ε we start by analyzing a time-discrete version of the problem. Fix $n \in \mathbb{N}$, set $\tau := T/n$, and consider the time partition

$$0 = t_0 < t_1 < \cdots < t_n = T, \qquad t_i := i\tau.$$

We define $w_0 := w(0)$, $w_1 := w_0 + \tau\dot{w}(0)$, and, for $i = 2, \dots, n$, we set $w_i := w(t_i)$. Our analysis will be set in the space

$$\begin{aligned} \mathscr{U}_\tau := \Big\{&(u_0, e_0, p_0), \dots, (u_n, e_n, p_n) \\ &\in \big((BD(\Omega) \cap L^2(\Omega; \mathbb{R}^3)) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)\big)^{n+1} : \\ &(u_i, e_i, p_i) \in \mathscr{A}(w_i) \quad \text{for } i = 1, \dots, n\Big\}. \end{aligned} \tag{4.1}$$

We define the *discrete energy functional* $I_{\varepsilon\tau} : \mathscr{U}_\tau \to [0, +\infty)$ as

$$
\begin{aligned}
(4.2)\qquad & I_{\varepsilon\tau}\big((u_0, e_0, p_0), \dots, (u_n, e_n, p_n)\big) \\
&:= \frac{\varepsilon^2\rho}{2}\sum_{i=2}^{n} \tau\eta_{\tau,i}\int_\Omega |\delta^2 u_i|^2\,dx + \sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_i)\,dx \\
&+ \varepsilon\tau\sum_{i=1}^{n-1}\eta_{\tau,i+1}\mathcal{H}(\delta p_i),
\end{aligned}
$$

where, given a vector $v = (v_1, \dots, v_n)$, the operator δ denotes its discrete derivative,

$$\delta v_i := \frac{v_i - v_{i-1}}{\tau}, \qquad \delta^k v_i := \frac{\delta^{k-1}v_i - \delta^{k-1}v_{i-1}}{\tau},$$

for $k \in \mathbb{N}$, $k > 1$, and where the Pareto weights

$$(4.3)\qquad \eta_{\tau,i} := \left(\frac{\varepsilon}{\varepsilon + \tau}\right)^i, \quad i = 0, \dots, n,$$

are a discretization of the map $t \to \exp\big(-t/\varepsilon\big)$. Define the set

$$
\begin{aligned}
(4.4)\qquad \mathscr{K}_\tau(u^0, e^0, p^0, u^1) := \{&(u_0, e_0, p_0), \dots, (u_n, e_n, p_n) \in \mathscr{U}_\tau : \\
&u_0 = u^0,\ e_0 = e^0,\ p_0 = p^0,\ \delta u_1 = u^1\}.
\end{aligned}
$$

Arguing as in Proposition 3.1 we obtain the following result.

LEMMA 4.1. *There exists an $(n{+}1)$-tuple of triples $(u_k^\varepsilon, e_k^\varepsilon, p_k^\varepsilon)$ such that $\big((u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)\big) \in \mathscr{K}_\tau(u^0, e^0, p^0, u^1)$ and*

$$
\begin{aligned}
(4.5)\quad & I_{\varepsilon\tau}\big((u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)\big) \\
&= \min_{\big((u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\big)\in\mathscr{K}_\tau(u^0,e^0,p^0,u^1)} I_{\varepsilon\tau}\big((u_0, e_0, p_0), \dots, (u_n, e_n, p_n)\big).
\end{aligned}
$$

4.1. Discrete Euler–Lagrange equations. We first compute the discrete Euler–Lagrange equations satisfied by a minimizing $(n{+}1)$-tuple $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$.

PROPOSITION 4.2 (discrete Euler–Lagrange equations). *Let $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$ be a solution to* (4.5)*. Then*

$$(4.6)\qquad \sum_{i=2}^{n}\varepsilon^2\rho\eta_{\tau,i}\int_\Omega \delta^2 u_i^\varepsilon \cdot \delta^2\varphi_i\,dx + \sum_{i=2}^{n-2}\eta_{\tau,i+2}\int_\Omega \mathbb{C}e_i^\varepsilon : E\varphi_i\,dx = 0$$

for every $\varphi_i \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that $\varphi_i = 0$ $\mathcal{H}^2$-a.e. on Γ_0, $i = 2, \dots, n$. In addition,

$$(4.7)\quad -\left(\frac{\varepsilon}{\varepsilon+\tau}\right)\mathcal{H}(\xi) - \mathcal{H}(-\xi) \le \left(\frac{\tau}{\varepsilon+\tau}\right)\int_\Omega \mathbb{C}e_i^\varepsilon : \xi\,dx \le \mathcal{H}(\xi) + \left(\frac{\varepsilon}{\varepsilon+\tau}\right)\mathcal{H}(-\xi)$$

for every $\xi \in L^2(\Omega;\mathbb{M}^{3\times3}_D)$, $i = 2, \dots, n-2$.

Proof. Let

$$(v_0, f_0, q_0), \dots, (v_n, f_n, q_n) \in \big(BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\big)^{n+1}$$

be such that $(v_i, f_i, q_i) \in \mathscr{A}(0)$ for $i = 1, \dots, n$, with $v_0 = \delta v_1 = 0$, and $f_0 = q_0 = 0$. Consider the $(n+1)$-tuple

$$(u_0^\varepsilon \pm \lambda v_0, e_0^\varepsilon \pm \lambda f_0, p_0^\varepsilon \pm \lambda q_0), \dots, (u_n^\varepsilon \pm \lambda v_n, e_n^\varepsilon \pm \lambda f_n, p_n^\varepsilon \pm \lambda q_n),$$

with $\lambda > 0$. By the minimality of $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$, there holds

$$\frac{1}{\lambda} I_{\varepsilon\tau}\big((u_0^\varepsilon \pm \lambda v_0, e_0^\varepsilon \pm \lambda f_0, p_0^\varepsilon \pm \lambda q_0), \dots, (u_n^\varepsilon \pm \lambda v_n, e_n^\varepsilon \pm \lambda f_n, p_n^\varepsilon \pm \lambda q_n)\big) \\ - \frac{1}{\lambda} I_{\varepsilon\tau}\big((u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)\big) \geq 0.$$

Therefore, by (2.5) and (4.2) we deduce the inequality

$$\begin{aligned} &- \varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(\delta q_i) \\ &\leq \varepsilon^2 \rho \sum_{i=2}^{n} \tau\eta_{\tau,i} \int_\Omega \delta^2 u_i^\varepsilon \cdot \delta^2 v_i \, dx + \sum_{i=2}^{n-2} \tau\eta_{\tau,i+2} \int_\Omega \mathbb{C} e_i^\varepsilon : f_i \, dx \\ &\leq \varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(-\delta q_i). \end{aligned} \tag{4.8}$$

For $i = 0, \dots, n$, let $\varphi_i \in W^{1,2}(\Omega; \mathbb{R}^3)$ with $\varphi_i = 0$ $\mathcal{H}^2$-a.e. on Γ_0, and let $\xi_i \in L^2(\Omega; \mathbb{M}_D^{3\times3})$. Choosing $v_i = \varphi_i$, $f_i = E\varphi_i$, and $q_i = 0$ for $i = 2, \dots, n$, by (4.8) we obtain

$$\varepsilon^2 \rho \sum_{i=2}^{n} \tau\eta_{\tau,i} \int_\Omega \delta^2 u_i^\varepsilon \cdot \delta^2 \varphi_i \, dx + \sum_{i=2}^{n-2} \tau\eta_{\tau,i+2} \int_\Omega \mathbb{C} e_i^\varepsilon : E\varphi_i \, dx = 0$$

for every $\varphi_1, \dots, \varphi_n \in W^{1,2}(\Omega; \mathbb{R}^3)$, $\varphi_i = 0$ $\mathcal{H}^2$-a.e. on Γ_0, $i = 0, \dots, n$, and hence (4.6). Choosing $v_i = 0$, $f_i = \xi_i$, and $q_i = -\xi_i$ for $i = 1, \dots, n$, estimate (4.8) yields

$$-\varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(-\delta\xi_i) \leq \sum_{i=2}^{n-2} \tau\eta_{\tau,i+2} \int_\Omega \mathbb{C} e_i^\varepsilon : \xi_i \, dx \leq \varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(\delta\xi_i)$$

for every $\xi_1, \dots, \xi_n \in L^2(\Omega; \mathbb{M}_D^{3\times3})$, and thus (4.7). ∎

We observe that it follows from (4.7) that $(\mathbb{C} e_i^\varepsilon)_D \in L^\infty(\Omega; \mathbb{M}_D^{3\times3})$ for every i and ε, although the bound is not uniform with respect to τ or ε. Indeed, for every B Borel subset of Ω and for every $M \in \mathbb{M}_D^{3\times3}$ we can choose $\xi = M\chi_B$ in (4.7), where χ_B denotes the characteristic function of B. We have

$$-\Big(\frac{\varepsilon}{\varepsilon+\tau}\Big) H(M) - H(-M) \leq \Big(\frac{\tau}{\varepsilon+\tau}\Big) \mathbb{C} e_i^\varepsilon(x) : M \leq H(M) + \Big(\frac{\varepsilon}{\varepsilon+\tau}\Big) H(-M) \tag{4.9}$$

for $i = 2, \dots, n-2$ and a.e. $x \in \Omega$, which by (2.4) imply

$$-2R_K |M| \leq \Big(\frac{\tau}{\varepsilon+\tau}\Big) \mathbb{C} e_i^\varepsilon(x) : M \leq 2R_K |M|$$

for $i = 2, \dots, n-2$, and every $M \in \mathbb{M}_D^{3\times3}$ for a.e. $x \in \Omega$. Thus, we get the estimate

$$\|(\mathbb{C} e_i^\varepsilon)_D\|_{L^\infty(\Omega; \mathbb{M}_D^{3\times3})} \leq 2\Big(\frac{\varepsilon+\tau}{\tau}\Big) R_K \tag{4.10}$$

for $i = 2, \dots, n-2$.

As a consequence of inequality (4.9), the deviatoric parts of the discrete stresses $\sigma_i^\varepsilon := \mathbb{C}e_i^\varepsilon$, $i = 2, \dots, n-2$, belong to the subdifferential in 0 of suitable convex and positively 1-homogeneous functions. Indeed, by (4.9) we have

$$\Big(\frac{\tau}{\varepsilon+\tau}\Big)\sigma_i^\varepsilon(x) \in \partial F_H^\varepsilon(0) \quad \text{for a.e. } x \in \Omega,\ i = 2, \dots, n-2,$$

where $F_H^\varepsilon : \mathbb{M}_D^{3\times 3} \to [0, +\infty)$ is defined as

$$F_H^\varepsilon(M) := H(M) + \left(\frac{\varepsilon}{\varepsilon+\tau}\right) H(-M)$$

for every $M \in \mathbb{M}_D^{3\times 3}$. The convexity and positive 1-homogeneity of F_H^ε follow directly by the corresponding properties of H.

By means of a discrete integration by parts in time, (4.6) can be equivalently reformulated in the following useful form.

Proposition 4.3 (discrete Euler–Lagrange part 2). *Let $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$ be a solution to (4.5). Then*

$$\delta^2 u_n^\varepsilon = \delta^3 u_n^\varepsilon = 0, \tag{4.11}$$

$$\int_\Omega \big[\rho(\varepsilon^2\delta^4 u_{i+2}^\varepsilon - 2\varepsilon\delta^3 u_{i+1}^\varepsilon + \delta^2 u_i^\varepsilon)\cdot\varphi + \mathbb{C}e_i^\varepsilon : E\varphi\big]\,dx = 0 \tag{4.12}$$

for $i = 2, \dots, n-2$ and for every $\varphi \in W^{1,2}(\Omega; \mathbb{R}^3)$ with $\varphi = 0$ $\mathcal{H}^2$-a.e. on Γ_0.

We omit the proof of this proposition as it follows by arguing exactly as in [49, subsection 2.3]. In view of (4.12), there holds

$$\begin{cases} \operatorname{div}\mathbb{C}e_i^\varepsilon = \rho(\varepsilon^2\delta^4 u_{i+2}^\varepsilon - 2\varepsilon\delta^3 u_{i+1}^\varepsilon + \delta^2 u_i^\varepsilon) & \text{a.e. in } \Omega,\\ \mathbb{C}e_i^\varepsilon\nu = 0 & \mathcal{H}^2\text{-a.e. on } \partial\Omega\setminus\Gamma_0, \end{cases} \tag{4.13}$$

and hence $\operatorname{div}\mathbb{C}e_i^\varepsilon \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, $i = 2, \dots, n-2$.

4.2. Stress-strain duality. In order to establish a uniform discrete energy estimate we need to introduce a preliminary notion of duality for the discrete stresses σ_i^ε and the plastic strains p_i^ε.

We work along the lines of [25] and [11, subsection 2.3]. Define the set

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}_{\mathrm{sym}}^{3\times 3}) : \sigma_D \in L^\infty(\Omega; \mathbb{M}_D^{3\times 3}) \text{ and } \operatorname{div}\sigma \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)\}. \tag{4.14}$$

By [25, Proposition 2.5], for every $\sigma \in \Sigma(\Omega)$ there holds

$$\sigma \in L^6(\Omega; \mathbb{M}_{\mathrm{sym}}^{3\times 3})$$

and

$$\|\mathrm{tr}\sigma\|_{L^6(\Omega;\mathbb{M}_{\mathrm{sym}}^{3\times 3})} \le C\big(\|\sigma\|_{L^1(\Omega;\mathbb{M}_{\mathrm{sym}}^{3\times 3})} + \|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}_D^{3\times 3})} + \|\operatorname{div}\sigma\|_{L^2(\Omega;\mathbb{R}^3)}\big).$$

In addition, we can introduce the trace $[\sigma\nu] \in W^{-1/2,2}(\partial\Omega;\mathbb{R}^3)$ (see, e.g., [51, Chapter I, Theorem 1.2]) as

$$\langle[\sigma\nu],\,\psi\rangle_{\partial\Omega} := \int_\Omega \operatorname{div}\sigma\cdot\psi\,dx + \int_\Omega \sigma : E\psi\,dx$$

for every $\psi \in W^{1,2}(\Omega;\mathbb{R}^3)$. Defining the normal and the tangential part of $[\sigma\nu]$ as

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu \quad \text{and} \quad [\sigma\nu]_\nu^\perp := [\sigma\nu] - ([\sigma\nu]\cdot\nu)\nu,$$

by [25, Lemma 2.4] we have that $[\sigma\nu]_\nu^\perp \in L^\infty(\partial\Omega;\mathbb{R}^3)$ and

$$\|[\sigma\nu]_\nu^\perp\|_{L^\infty(\partial\Omega;\mathbb{R}^3)} \le \frac{1}{\sqrt{2}}\|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)}.$$

Let $\sigma \in \Sigma(\Omega)$, and let $u \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, with $\operatorname{div} u \in L^2(\Omega)$. We define the distribution $[\sigma_D : E_D u]$ on Ω as

$$\langle[\sigma_D : E_D u], \varphi\rangle := -\int_\Omega \varphi \operatorname{div}\sigma\cdot u\,dx - \frac{1}{3}\int_\Omega \varphi \operatorname{tr}\sigma\cdot\operatorname{div} u\,dx - \int_\Omega \sigma : (u\odot\nabla\varphi)\,dx \tag{4.15}$$

for every $\varphi \in C_c^\infty(\Omega)$. By [25, Theorem 3.2] it follows that $[\sigma_D : E_D u]$ is a bounded Radon measure on Ω, whose variation satisfies

$$|[\sigma_D : E_D u]| \le \|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)}|E_D u| \quad \text{in } \Omega.$$

Let $\Pi_{\Gamma_0}(\Omega)$ be the set of admissible plastic strains, namely the set of maps $p \in \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$ such that there exist $u \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, $e \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\text{sym}})$, and $w \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $(u,e,p) \in \mathscr{A}(w)$. Note that the additive decomposition $Eu = e + p$ implies that $\operatorname{div} u \in L^2(\Omega)$.

It is possible to define a duality between elements of $\Sigma(\Omega)$ and $\Pi_{\Gamma_0}(\Omega)$. To be precise, given $p \in \Pi_{\Gamma_0}(\Omega)$ and $\sigma \in \Sigma(\Omega)$, we fix (u,e,w) such that $(u,e,p) \in \mathscr{A}(w)$, with $u \in L^2(\Omega;\mathbb{R}^3)$, and we define the measure $[\sigma_D : p] \in \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$ as

$$[\sigma_D : p] := \begin{cases} [\sigma_D : E_D u] - \sigma_D : e_D & \text{in } \Omega, \\ [\sigma\nu]_\nu^\perp\cdot(w-u)\mathcal{H}^2 & \text{on } \Gamma_0, \end{cases}$$

so that

$$\int_{\Omega\cup\Gamma_0} \varphi\,d[\sigma_D : p] = \int_\Omega \varphi\,d[\sigma_D : E_D u] - \int_\Omega \varphi\sigma_D : e_D\,dx + \int_{\Gamma_0} \varphi[\sigma\nu]_\nu^\perp\cdot(w-u)\,d\mathcal{H}^2$$

for every $\varphi \in C(\bar\Omega)$. Arguing as in [11, section 2], one can prove that the definition of $[\sigma_D : p]$ is independent of the choice of (u,e,w), and that if $\sigma_D \in C(\bar\Omega;\mathbb{M}^{3\times 3}_D)$ and $\varphi \in C(\bar\Omega)$, then

$$\int_{\Omega\cup\Gamma_0} \varphi\,d[\sigma_D : p] = \int_{\Omega\cup\Gamma_0} \varphi\sigma_D : \,dp.$$

Remark 4.4 (Neumann condition). We are now in a position to make the meaning of the Neumann condition in (c1) precise. The functional $[\sigma\nu] \in H^{-1/2}(\partial\Omega)$ is a distribution. As such, one can define its restriction $[\sigma\nu]|_A$ to the set $A := \partial\Omega\setminus\overline{\Gamma}_0$, which is open in the relative topology of $\partial\Omega$, as

$$\langle[\sigma\nu]|_A, \varphi\rangle := \langle[\sigma\nu], \tilde\varphi\rangle \quad \forall\varphi \in C_c^\infty(A), \tag{4.16}$$

where $\tilde\varphi \in C^\infty(\partial\Omega)$ is the trivial extension of φ to the whole of $\partial\Omega$. Condition (4.12) entails that $[\sigma\nu]|_A = 0$ as distribution. Hence, it is indeed a function, and $[\sigma\nu]|_A = 0$ almost everywhere.

We finally rewrite [11, Proposition 2.2] in our framework.

Proposition 4.5. *Let $\sigma \in \Sigma(\Omega)$, $w \in W^{1,2}(\Omega;\mathbb{R}^3)$, and $(u,e,p) \in \mathscr{A}(w)$, with $u \in L^2(\Omega;\mathbb{R}^3)$. Assume additionally that $[\sigma\nu] = 0$ on $\partial\Omega \setminus \overline{\Gamma}_0$ in the sense of Remark 4.4. Then*

$$[\sigma_D : p](\Omega \cup \Gamma_0) + \int_\Omega \sigma : (e - Ew)\,dx = -\int_\Omega \operatorname{div}\sigma \cdot (u - w)\,dx.$$

Remark 4.6. We point out that the C^2 regularity of $\partial\Omega$ is needed here in order to apply [25, Proposition 2.5]. It is not possible to use here the results in [18] and extend the analysis to the case in which $\partial\Omega$ is Lipschitz, as (4.13) only implies that $\operatorname{div}\mathbb{C}e_i^\varepsilon \in L^2(\Omega;\mathbb{R}^3)$, whereas [18, Proposition 6.1] requires $\operatorname{div}\mathbb{C}e_i^\varepsilon \in L^3(\Omega;\mathbb{R}^3)$.

4.3. Discrete energy estimate. This subsection is devoted to the proof of a uniform energy estimate at a time-discrete level. The formal proof strategy can be summarized as follows:

- We first test (1.1) against the map $t \to \dot{u}(t) - u^1 - \dot{w}(t) + \dot{w}(0)$, and (1.3) against $t \to \dot{p}(t)$. This provides an estimate of the form $\int_0^T F_\varepsilon(u,\dot{u},\ddot{u})\,dt \le C$, for a suitable function F_ε dependent on ε, and for a constant C dependent on the initial and boundary data;
- We then estimate the quantity

$$\int_0^T F_\varepsilon(u,\dot{u},\ddot{u})\,dt + \int_0^T \int_0^t F_\varepsilon(u,\dot{u},\ddot{u})\,ds,$$

  and perform the analogous strategy for (1.3), using the final conditions at time T.

The rigorous implementation of the methodology highlighted above relies on testing (4.13) against the map $\varphi = \tau(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))$, and on summing the resulting expression with its corresponding integrated-in-time counterpart. Before moving to the proof of the discrete energy estimate, we establish a preliminary lower bound on the mass of the measures $[(\mathbb{C}e_i^\varepsilon)_D : q]$, $i = 2,\dots,n-2$, where $q \in \Pi_{\Gamma_0}(\Omega)$ is such that there exist $v \in BD(\Omega) \cap L^2(\Omega;\mathbb{R}^3)$ and $f \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ satisfying $(v,f,q) \in \mathscr{A}(0)$. By choosing $q = \delta p_i^\varepsilon$, this will indeed allow us to estimate the quantities $[(\mathbb{C}e_i^\varepsilon)_D : \delta p_i^\varepsilon]$ from below in terms of $\mathcal{H}(\delta p_i^\varepsilon)$ for $i = 2,\dots,n-2$.

Caveat on notation. In the following we use the symbol C to indicate a generic constant, possibly depending on data and varying from line to line.

The following estimate holds true.

Proposition 4.7. *Let $q \in \Pi_{\Gamma_0}(\Omega)$, $v \in BD(\Omega) \cap L^2(\Omega;\mathbb{R}^3)$, and $f \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ be such that $(v,f,q) \in \mathscr{A}(0)$. Then, if $(u_0^\varepsilon,e_0^\varepsilon,p_0^\varepsilon),\dots,(u_n^\varepsilon,e_n^\varepsilon,p_n^\varepsilon)$ is a solution to (4.5), it satisfies*

$$\tau[(\mathbb{C}e_i^\varepsilon)_D : q](\Omega \cup \Gamma_0) + (\varepsilon+\tau)\mathcal{H}(\delta p_i^\varepsilon - q) + \varepsilon\mathcal{H}(q) \ge (\varepsilon+\tau)\mathcal{H}(\delta p_i^\varepsilon) \tag{4.17}$$

for every $i = 2,\dots,n-2$.

Proof. Let q be as in the statement of the proposition. By (4.10) and (4.13) it follows that $\mathbb{C}e_i^\varepsilon \in \Sigma(\Omega)$, $i = 2,\dots,n-2$. In view of the triangular inequality (2.5), since $(u_0^\varepsilon,e_0^\varepsilon,p_0^\varepsilon),\dots,(u_n^\varepsilon,e_n^\varepsilon,p_n^\varepsilon)$ is a solution to (4.5), it also solves the implicit minimum problem

$$\begin{aligned}
&I_{\varepsilon\tau}\big((u_0^\varepsilon,e_0^\varepsilon,p_0^\varepsilon),\dots,(u_n^\varepsilon,e_n^\varepsilon,p_n^\varepsilon)\big)\\
&\qquad= \min_{(u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\in\mathscr{K}_\tau(u^0,e^0,p^0,u^1)} J_{\varepsilon\tau}\big((u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\big),
\end{aligned}$$

where

$$J_{\varepsilon\tau}\big((u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\big) := \frac{\varepsilon^2\rho}{2}\sum_{j=2}^{n}\tau\eta_{\tau,j}\int_\Omega |\delta^2 u_j|^2\,dx$$
$$+\sum_{j=2}^{n-2}\tau\eta_{\tau,j+2}\int_\Omega Q(e_j)\,dx + \varepsilon\tau\sum_{j=1}^{n-1}\eta_{\tau,j+1}\left[\mathcal{H}\Big(\frac{p_j-p^\varepsilon_{j-1}}{\tau}\Big)+\mathcal{H}\Big(\frac{p^\varepsilon_{j-1}-p_{j-1}}{\tau}\Big)\right].$$

Arguing as in Proposition 4.2 we compute the Euler–Lagrange equations associated to the minimum problem above, and we perform variations $(u_0^\varepsilon \pm \lambda v_0, e_0^\varepsilon \pm \lambda f_0, p_0^\varepsilon \pm \lambda q_0),\dots,(u_n^\varepsilon \pm \lambda v_n, e_n^\varepsilon \pm \lambda f_n, p_n^\varepsilon \pm \lambda q_n)$, with $\lambda>0$, and $(v_0,f_0,q_0),\dots,(v_n,f_n,q_n)\in \big(BD(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})\times \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)\big)^{n+1}$ such that $(v_i,f_i,q_i)\in\mathscr{A}(0)$ for $i=1,\dots,n$, with $v_0=\delta v_1=0$, and $f_0=q_0=0$. The convexity of $\mathcal{H}$ yields

$$\varepsilon^2\rho\sum_{j=2}^{n}\tau\eta_{\tau,j}\int_\Omega \delta^2 u_j^\varepsilon\cdot\delta^2 v_j\,dx+\sum_{j=2}^{n-2}\tau\eta_{\tau,j+2}\int_\Omega \mathbb{C}e_j^\varepsilon : f_j\,dx$$
$$+\varepsilon\tau\sum_{j=1}^{n-1}\eta_{\tau,j+1}\left[\mathcal{H}\Big(\delta p_j^\varepsilon+\frac{q_j}{\tau}\Big)-\mathcal{H}(\delta p_j^\varepsilon)+\mathcal{H}\Big(-\frac{q_{j-1}}{\tau}\Big)\right]\geq 0.$$

By combining Proposition 4.5 with the Euler–Lagrange equation (4.13), and performing the discrete integration by parts in [49, subsection 2.3], we have

$$-\sum_{j=2}^{n-2}\tau\eta_{\tau,j+2}[(\mathbb{C}e_j^\varepsilon)_D : q_j](\Omega\cup\Gamma_0)$$
$$+\varepsilon\tau\sum_{j=1}^{n-1}\eta_{\tau,j+1}\left[\mathcal{H}\Big(\delta p_j^\varepsilon+\frac{q_j}{\tau}\Big)-\mathcal{H}(\delta p_j^\varepsilon)+\mathcal{H}\Big(-\frac{q_{j-1}}{\tau}\Big)\right]\geq 0.$$

The thesis follows by choosing $q_j=-\tau q$ for $j=i$, and $q_j=0$ otherwise. ∎

Given a vector $(w_0,\dots,w_n)$ we denote by $\bar w_\tau$ and w_τ its backward piecewise-constant and its piecewise-affine interpolants on the partition, that is,

$$\bar w_\tau(0)=w_\tau(0)=w_0,\quad \bar w_\tau(t)=w_i,\quad w_\tau(t):=\alpha_\tau(t)w_i+(1-\alpha_\tau(t))w_{i-1} \tag{4.18}$$

for $t\in((i-1)\tau,i\tau]$, $i=1,\dots,n$, where

$$\alpha_\tau(t):=\frac{(t-(i-1)\tau)}{\tau}\quad \text{for } t\in((i-1)\tau,i\tau],\quad i=1,\dots,n.$$

In particular, $\dot w_\tau(t)=\overline{\delta w}_\tau(t)$ for almost every $t\in(0,T)$. Analogously, we define the piecewise-constant maps

$$\bar\eta_\tau(t):=\eta_{\tau,i}\quad\text{for } t\in((i-1)\tau,i\tau],\quad i=1,\dots,n.$$

In addition, as in [49, subsection 2.5.1] we denote by $\tilde w_\tau$ the piecewise-quadratic interpolants, defined via

$$\begin{aligned}&\tilde w_\tau(t):=w_\tau(t)\quad\text{in } [0,\tau],\ \tilde w_\tau(i\tau)=w_i \text{ for } i=1,\dots,n,\\ &\dot{\tilde w}_\tau(t)=\alpha_\tau(t)\dot w_\tau(t)+(1-\alpha_\tau(t))\dot w_\tau(t-\tau)\quad\text{in } (\tau,T].\end{aligned}\tag{4.19}$$

Notice that

$$\dot{\tilde w}_\tau(t)=\dot w_\tau(t-\tau)+\tau\alpha_\tau(t)\ddot{\tilde w}_\tau(t)\quad\text{for a.e. } t\in(\tau,T].$$

THEOREM 4.8 (discrete energy estimate). *Let $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$ be a solution to (4.5). Assume in addition that $p^1 = 0$. Let $(\bar{u}_\tau^\varepsilon, \bar{e}_\tau^\varepsilon, \bar{p}_\tau^\varepsilon)$, $(u_\tau^\varepsilon, e_\tau^\varepsilon, p_\tau^\varepsilon)$, and $(\tilde{u}_\tau^\varepsilon, \tilde{e}_\tau^\varepsilon, \tilde{p}_\tau^\varepsilon)$ be the triples of associated piecewise-constant, piecewise-affine, and piecewise-quadratic interpolants, respectively. Then there exists a constant C (independent of ε and τ) such that*

$$
\begin{aligned}
(4.20)\qquad & \varepsilon\rho \int_{2\tau}^{T-2\tau} \int_\Omega |\ddot{\tilde{u}}_\tau^\varepsilon|^2 \,dx\,dt + \varepsilon\rho \int_{2\tau}^{T-2\tau} \int_{2\tau}^{t} \int_\Omega |\ddot{\tilde{u}}_\tau^\varepsilon|^2 \,dx\,ds\,dt \\
& + \rho \int_{\tau}^{T-2\tau} \int_\Omega |\dot{u}_\tau^\varepsilon|^2 \,dx\,dt + \int_{\tau}^{T-2\tau} \int_\Omega Q(\bar{e}_\tau^\varepsilon) \,dx\,dt + \int_{\tau}^{T-2\tau} \mathcal{H}(\dot{p}_\tau^\varepsilon)\,dt \\
& \le C\Big(1 + \frac{\tau}{\varepsilon}\Big).
\end{aligned}
$$

Proof. Take the map $\varphi = \tau(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))$ as test function in (4.13). For $k = 2, \dots, n-2$ we obtain

$$
\begin{aligned}
(4.21)\qquad & \varepsilon^2\rho \sum_{i=2}^{k} \tau \int_\Omega \delta^4 u_{i+2}^\varepsilon \cdot (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx \\
& - 2\varepsilon\rho \sum_{i=2}^{k} \tau \int_\Omega \delta^3 u_{i+1}^\varepsilon \cdot (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx \\
& + \rho \sum_{i=2}^{k} \tau \int_\Omega \delta^2 u_i^\varepsilon \cdot (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx \\
& - \sum_{i=2}^{k} \tau \int_\Omega \operatorname{div} \mathbb{C} e_i^\varepsilon \cdot (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx = 0.
\end{aligned}
$$

Arguing as in [49, subsection 2.4] we perform an integration by parts in time at the time-discrete level, and we estimate the first term in the left-hand side of (4.21) from below as

$$
\begin{aligned}
(4.22)\qquad & \varepsilon^2\rho \sum_{i=2}^{k} \tau \int_\Omega \delta^4 u_{i+2}^\varepsilon \cdot (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx \\
& \ge \frac{\varepsilon^2\rho}{2} \int_\Omega |\delta^2 u_2^\varepsilon|^2\,dx + \varepsilon^2\rho \int_\Omega \delta^3 u_{k+2}^\varepsilon \cdot (\delta u_k^\varepsilon - u^1 - \delta w_k + \dot{w}(0))\,dx \\
& - \frac{\varepsilon^2\rho}{2} \int_\Omega |\delta^2 u_{k+1}^\varepsilon|^2\,dx + \frac{\varepsilon^2\rho}{2} \sum_{i=2}^{k} \int_\Omega |\delta^2 u_{i+1}^\varepsilon - \delta^2 u_i^\varepsilon|^2\,dx \\
& + \varepsilon^2\rho \int_\Omega \delta^2 u_{k+1}^\varepsilon \cdot \delta^2 w_k\,dx - \varepsilon^2\rho \int_\Omega \delta^2 u_2^\varepsilon \cdot \delta^2 w_2\,dx \\
& - \frac{\varepsilon^2\rho}{2} \sum_{i=3}^{k} \tau \int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx - \frac{\varepsilon^2\rho}{2} \sum_{i=3}^{k} \tau \int_\Omega |\delta^3 w_i|^2\,dx.
\end{aligned}
$$

Analogously, the second and third terms in the left-hand side of (4.21) are bounded from below by

$$
\begin{aligned}
(4.23)\qquad & -2\varepsilon\rho\sum_{i=2}^{k}\tau\int_\Omega \delta^3 u^\varepsilon_{i+1}\cdot(\delta u^\varepsilon_i - u^1 - \delta w_i + \dot w(0))\,dx\\
&\geq -\varepsilon\rho\sum_{i=2}^{k}\tau\int_\Omega |\delta^2 w_i|^2\,dx - 2\varepsilon\rho\int_\Omega \delta^2 u^\varepsilon_{k+1}\cdot(\delta u^\varepsilon_k - u^1 - \delta w_k + \dot w(0))\,dx\\
&\quad + \varepsilon\rho\sum_{i=2}^{k}\tau\int_\Omega |\delta^2 u^\varepsilon_i|^2\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
(4.24)\quad & \rho\sum_{i=2}^{k}\tau\int_\Omega \delta^2 u^\varepsilon_i\cdot(\delta u^\varepsilon_i - u^1 - \delta w_i + \dot w(0))\,dx\\
&= \frac{\rho}{4}\int_\Omega |\delta u^\varepsilon_k - u^1|^2\,dx + \frac{\rho}{2}\sum_{i=2}^{k}\int_\Omega |\delta u^\varepsilon_i - \delta u^\varepsilon_{i-1}|^2\,dx\\
&\quad - \rho\sum_{i=2}^{k}\int_\Omega (\delta u^\varepsilon_i - \delta u^\varepsilon_{i-1})\cdot(\delta w_i - \dot w(0))\,dx\\
&\geq \frac{\rho}{2}\int_\Omega |\delta u^\varepsilon_k - u^1|^2\,dx + \frac{\rho}{2}\sum_{i=2}^{k}\int_\Omega |\delta u^\varepsilon_i - \delta u^\varepsilon_{i-1}|^2\,dx - \frac{\rho}{16}\sum_{i=2}^{k-1}\tau\int_\Omega |\delta u^\varepsilon_i|^2\,dx\\
&\quad - \rho\int_\Omega |\dot w(0)|^2\,dx - \rho\int_\Omega \delta u^\varepsilon_k\cdot\delta w_k\,dx\\
&\quad + \rho\int_\Omega \delta u^\varepsilon_1\cdot\delta w_2\,dx - 4\rho\sum_{i=3}^{k}\tau\int_\Omega |\delta^2 w_i|^2\,dx.
\end{aligned}
$$

Regarding the fourth term in the left-hand side of (4.21), by (4.10) and (4.13) there holds $\mathbb{C}e^\varepsilon_i \in \Sigma(\Omega)$ for $i = 2,\ldots,n-2$ (see (4.14)). Therefore, in view of Proposition 4.5 and (4.13), we have

$$
\begin{aligned}
&-\sum_{i=2}^{k}\tau\int_\Omega \operatorname{div}\mathbb{C}e^\varepsilon_i : (\delta u^\varepsilon_i - u^1 - \delta w_i + \dot w(0))\,dx\\
&= \sum_{i=2}^{k}\tau\int_\Omega \mathbb{C}e^\varepsilon_i : (\delta e^\varepsilon_i - e^1 - E\delta w_i + E\dot w(0))\,dx + \sum_{i=2}^{k}\tau[(\mathbb{C}e^\varepsilon_i)_D : \delta p^\varepsilon_i](\Omega\cup\Gamma_0)
\end{aligned}
$$

for $k = 2,\ldots,n-2$. On the one hand,

$$
\begin{aligned}
&\sum_{i=2}^{k}\tau\int_\Omega \mathbb{C}e^\varepsilon_i : (-E\delta w_i + E\dot w(0))\,dx\\
&\qquad \geq -\frac{1}{4}\sum_{i=2}^{k}\tau\int_\Omega Q(e^\varepsilon_i)\,dx - 4\sum_{i=2}^{k}\tau\int_\Omega Q(E\delta w_i - E\dot w(0))\,dx,
\end{aligned}
$$

and on the other hand,

$$
\sum_{i=2}^{k}\tau\int_\Omega \mathbb{C}e^\varepsilon_i : (\delta e^\varepsilon_i - e^1)\,dx \geq \int_\Omega Q(e^\varepsilon_k)\,dx - \int_\Omega Q(e^1)\,dx - \sum_{i=2}^{k}\tau\int_\Omega \mathbb{C}e^\varepsilon_i : e^1\,dx.
$$

By Proposition 4.7 we infer that

$$\sum_{i=2}^{k} \tau[(\mathbb{C}e_i^\varepsilon)_D : \delta p_i^\varepsilon](\Omega \cup \Gamma_0) \geq \sum_{i=2}^{k} \tau \mathcal{H}(\delta p_i^\varepsilon).$$

Therefore,

$$\begin{aligned} &- \sum_{i=2}^{k} \tau \int_\Omega \operatorname{div} \mathbb{C}e_i^\varepsilon : (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\, dx \\ &\geq \int_\Omega Q(e_k^\varepsilon)\, dx - \int_\Omega Q(e^1)\, dx - \sum_{i=2}^{k} \tau \int_\Omega \mathbb{C}e_i^\varepsilon : e^1\, dx \\ &\quad - \frac{1}{4} \sum_{i=2}^{k} \tau \int_\Omega Q(e_i^\varepsilon)\, dx - 4 \sum_{i=2}^{k} \tau \int_\Omega Q(E\delta w_i - E\dot{w}(0))\, dx + \sum_{i=2}^{k} \tau \mathcal{H}(\delta p_i^\varepsilon). \end{aligned} \tag{4.25}$$

By combining (4.22)–(4.25), equality (4.21) yields

(4.26)

$$\begin{aligned} &\varepsilon^2 \rho \int_\Omega \delta^3 u_{k+2}^\varepsilon \cdot (\delta u_k^\varepsilon - u^1 - \delta w_k + \dot{w}(0))\, dx - \frac{\varepsilon^2 \rho}{2} \int_\Omega |\delta^2 u_{k+1}^\varepsilon|^2\, dx + \frac{\varepsilon^2 \rho}{4} \int_\Omega |\delta^2 u_2^\varepsilon|^2\, dx \\ &\quad + \frac{\varepsilon^2 \rho}{2} \sum_{i=2}^{k} \int_\Omega |\delta^2 u_{i+1}^\varepsilon - \delta^2 u_i^\varepsilon|^2\, dx + \varepsilon^2 \rho \int_\Omega \delta^2 u_{k+1}^\varepsilon \cdot \delta^2 w_k\, dx + \frac{\rho}{4} \int_\Omega |\delta u_k^\varepsilon - u^1|^2\, dx \\ &\quad - 2\varepsilon\rho \int_\Omega \delta^2 u_{k+1}^\varepsilon \cdot (\delta u_k^\varepsilon - u^1 - \delta w_k + \dot{w}(0))\, dx + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho \sum_{i=3}^{k} \tau \int_\Omega |\delta^2 u_i^\varepsilon|^2\, dx \\ &\quad + \frac{\rho}{2} \sum_{i=2}^{k} \int_\Omega |\delta u_i^\varepsilon - \delta u_{i-1}^\varepsilon|^2\, dx - \rho \int_\Omega \delta u_k^\varepsilon \cdot \delta w_k\, dx + \rho \int_\Omega \delta u_1^\varepsilon \cdot \delta w_2\, dx \\ &\quad - \frac{\rho}{16} \sum_{i=2}^{k-1} \tau \int_\Omega |\delta u_i^\varepsilon|^2\, dx + \int_\Omega Q(e_k^\varepsilon)\, dx + \sum_{i=2}^{k} \tau \mathcal{H}(\delta p_i^\varepsilon) \\ &\quad\quad \leq \varepsilon^2 \rho \int_\Omega |\delta^2 w_2|^2\, dx + \frac{\varepsilon^2 \rho}{2} \sum_{i=3}^{k} \tau \int_\Omega |\delta^3 w_i|^2\, dx + \varepsilon\rho \sum_{i=2}^{k} \tau \int_\Omega |\delta^2 w_i|^2\, dx \\ &\quad\quad\quad + \rho \int_\Omega |\dot{w}(0)|^2\, dx + 4\rho \sum_{i=3}^{k} \tau \int_\Omega |\delta^2 w_i|^2\, dx + 4 \sum_{i=2}^{k} \tau \int_\Omega Q(E\delta w_i - E\dot{w}(0))\, dx \\ &\quad\quad\quad + \int_\Omega Q(e^1)\, dx + \sum_{i=2}^{k} \tau \int_\Omega \mathbb{C}e_i^\varepsilon : e^1\, dx + \frac{1}{4} \sum_{i=2}^{k} \tau \int_\Omega Q(e_i^\varepsilon)\, dx. \end{aligned}$$

Since $w \in W^{2,2}(0,T; W^{1,2}(\Omega;\mathbb{R}^3)) \cap C^3([0,T]; L^2(\Omega;\mathbb{R}^3))$, by Hölder's inequality there holds

$$\begin{aligned} \varepsilon^2 \rho \int_\Omega |\delta^2 w_2|^2\, dx &= \varepsilon^2 \rho \int_\Omega \Big| \frac{w(t_2) - 2\tau \dot{w}(0) - w(0)}{\tau^2} \Big|^2\, dx \\ &= \varepsilon^2 \rho \int_\Omega \Big| \frac{1}{\tau^2} \int_0^{2\tau} \int_0^\xi \ddot{w}(\lambda)\, d\lambda\, d\xi \Big|^2\, dx \leq C\varepsilon^2 \rho, \end{aligned} \tag{4.27}$$

as well as

$$\sum_{i=2}^{k}\tau\int_\Omega|\delta^2 w_i|^2\,dx=\sum_{i=3}^{k}\tau\int_\Omega\Big|\int_{(i-1)\tau}^{i\tau}\frac{\dot w(t)-\dot w(t-\tau)}{\tau^2}\,dt\Big|^2\,dx+C\tau \tag{4.28}$$
$$\le\frac{1}{\tau}\sum_{i=3}^{k}\int_\Omega\int_{(i-1)\tau}^{i\tau}\int_{t-\tau}^{t}|\ddot w(\xi)|^2\,d\xi\,dt\,dx+C\tau\le C\int_0^T\int_\Omega|\ddot w|^2\,dx\,dt+C\tau.$$

In addition, we have that

$$\sum_{i=3}^{k}\tau\int_\Omega|\delta^3 w_i|^2\,dx=\frac{1}{\tau^5}\sum_{i=3}^{k}\int_\Omega\Big|\int_{(i-1)\tau}^{i\tau}\int_{\xi-\tau}^{\xi}(\ddot w(s)-\ddot w(s-\tau))\,ds\,d\xi\Big|^2\,dx \tag{4.29}$$
$$\le C\int_\Omega\int_0^T|\dddot w|^2\,dt\,dx.$$

Finally, in view of Jensen's inequality, we compute

$$4\sum_{i=2}^{k}\tau\int_\Omega Q(E\delta w_i-E\dot w(0))\,dx \tag{4.30}$$
$$\le 4\tau(k-2)\int_\Omega Q(E\dot w(0))\,dx+8\sum_{i=2}^{k}\tau\int_\Omega Q\Big(\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}E\dot w(\xi)\,d\xi\Big)$$
$$+8\tau\int_\Omega Q\Big(\frac{1}{\tau}\int_0^{\tau}(E\dot w(\xi)-E\dot w(0))\,d\xi\Big)\,dx$$
$$\le 4\tau n\int_\Omega Q(E\dot w(0))\,dx+8\int_\Omega\int_0^T Q(Ew)\,dt\,dx$$
$$+8\int_\Omega\int_0^{\tau}Q(E\dot w(t)-E\dot w(0))\,dt\,dx.$$

By (4.27)–(4.30), the first two rows of the right-hand side of (4.26) are uniformly bounded in terms of the boundary datum w, independently of τ and ε. Therefore, we obtain the estimate

(4.31)

$$\varepsilon^2\rho\int_\Omega\delta^2 u^\varepsilon_{k+1}\cdot\delta^2 w_k\,dx+\varepsilon^2\rho\int_\Omega\delta^3 u^\varepsilon_{k+2}\cdot(\delta u^\varepsilon_k-u^1-\delta w_k+\dot w(0))\,dx+\frac{\varepsilon^2\rho}{4}\int_\Omega|\delta^2 u^\varepsilon_2|^2\,dx$$
$$-\frac{\varepsilon^2\rho}{2}\int_\Omega|\delta^2 u^\varepsilon_{k+1}|^2dx-2\varepsilon\rho\int_\Omega\delta^2 u^\varepsilon_{k+1}\cdot(\delta u^\varepsilon_k-u^1-\delta w_k+\dot w(0))dx+\frac{\rho}{4}\int_\Omega|\delta u^\varepsilon_k-u^1|^2dx$$
$$+\Big(\varepsilon-\frac{\varepsilon^2}{2}\Big)\rho\sum_{i=3}^{k}\tau\int_\Omega|\delta^2 u^\varepsilon_i|^2\,dx+\int_\Omega Q(e^\varepsilon_k)\,dx+\tau\sum_{i=2}^{k}\mathcal{H}(\delta p^\varepsilon_i)-\rho\int_\Omega\delta u^\varepsilon_k\cdot\delta w_k\,dx$$
$$+\rho\int_\Omega\delta u^\varepsilon_1\cdot\delta w_2\,dx-\frac{\rho}{16}\sum_{i=2}^{k-1}\tau\int_\Omega|\delta u^\varepsilon_i|^2\,dx$$
$$\le C+\int_\Omega Q(e^1)\,dx+\sum_{i=2}^{k}\tau\int_\Omega\mathbb{C}e^\varepsilon_i:e^1\,dx+\frac{1}{4}\sum_{i=2}^{k}\tau\int_\Omega Q(e^\varepsilon_i)\,dx.$$

Multiplying the previous inequality by τ and summing for $k = 2, \dots, n-2$, one obtains

(4.32)

$$\frac{\varepsilon^2\rho}{4}\tau(n-3)\int_\Omega |\delta^2 u_2^\varepsilon|^2\,dx + \frac{\rho}{4}\sum_{k=2}^{n-2}\tau\int_\Omega |\delta u_k^\varepsilon - u^1|^2\,dx$$
$$+\,\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^2 u_{k+1}^\varepsilon\cdot\delta^2 w_k\,dx + \varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^3 u_{k+2}^\varepsilon\cdot(\delta u_k^\varepsilon - u^1 - \delta w_k + \dot w(0))\,dx$$
$$-\,\frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega |\delta^2 u_{k+1}^\varepsilon|^2\,dx - 2\varepsilon\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^2 u_{k+1}^\varepsilon\cdot(\delta u_k^\varepsilon - u^1 - \delta w_k + \dot w(0))\,dx$$
$$+\,\Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho\sum_{k=3}^{n-2}\sum_{i=3}^{k}\tau^2\int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx + \sum_{k=2}^{n-2}\tau\int_\Omega Q(e_k^\varepsilon)\,dx + \sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\mathcal{H}(\delta p_i^\varepsilon)$$
$$-\,\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta u_k^\varepsilon\cdot\delta w_k\,dx + \rho\tau(n-3)\int_\Omega \delta u_1^\varepsilon\cdot\delta w_2\,dx - \frac{\rho}{16}\sum_{k=2}^{n-2}\sum_{i=2}^{k-1}\tau^2\int_\Omega |\delta u_i^\varepsilon|^2\,dx$$
$$\le C + \tau(n-3)\int_\Omega Q(e^1)\,dx + \sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\int_\Omega \mathbb{C}e_i^\varepsilon : e^1\,dx + \frac{(n-3)}{4}\sum_{i=2}^{n-2}\tau^2\int_\Omega Q(e_i^\varepsilon)\,dx.$$

By choosing $k = n-2$ in (4.31), and by observing that (4.11) yields $\delta^2 u_{n-1}^\varepsilon = 0$, we have

(4.33)

$$\frac{\varepsilon^2\rho}{4}\int_\Omega |\delta^2 u_2^\varepsilon|^2\,dx + \frac{\rho}{4}\int_\Omega |\delta u_{n-2}^\varepsilon - u^1|^2\,dx + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho\sum_{i=3}^{n-2}\tau\int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx + \int_\Omega Q(e_{n-2}^\varepsilon)\,dx$$
$$+\sum_{i=2}^{n-2}\tau\mathcal{H}(\delta p_i^\varepsilon) - \rho\int_\Omega \delta u_{n-2}^\varepsilon\cdot\delta w_{n-2}\,dx + \rho\int_\Omega \delta u_1^\varepsilon\cdot\delta w_2\,dx - \frac{\rho}{16}\sum_{i=2}^{n-3}\tau\int_\Omega |\delta u_i^\varepsilon|^2\,dx$$
$$\le C + \int_\Omega Q(e^1)\,dx + \sum_{i=2}^{n-2}\tau\int_\Omega \mathbb{C}e_i^\varepsilon : e^1\,dx + \frac{1}{4}\sum_{i=2}^{n-2}\tau\int_\Omega Q(e_i^\varepsilon)\,dx.$$

In view of (4.11) and (4.28), using again that $\delta^2 u_{n-1}^\varepsilon = 0$, we deduce the lower bounds

$$\begin{aligned}
\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^3 u_{k+2}^\varepsilon\cdot(\delta u_k^\varepsilon - u^1 - \delta w_k + \dot w(0))\,dx \qquad\qquad & \\
= -\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^2 u_{k+1}^\varepsilon\cdot(\delta^2 u_k^\varepsilon - \delta^2 w_k)\,dx & \\
\ge -\frac{3\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega |\delta^2 u_k^\varepsilon|^2\,dx - \frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega |\delta^2 w_k|^2\,dx & \\
\ge -\frac{3\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega |\delta^2 u_k^\varepsilon|^2\,dx - C &
\end{aligned} \tag{4.34}$$

and, analogously,

$$
\begin{aligned}
(4.35)\qquad & \varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^2 u^\varepsilon_{k+1}\cdot\delta^2 w_k\,dx \\
&\geq -\frac{\varepsilon^2\rho}{2}\sum_{k=3}^{n-2}\tau\int_\Omega|\delta^2 u^\varepsilon_k|^2\,dx-\frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta^2 w_k|^2\,dx \\
&\geq -\frac{\varepsilon^2\rho}{2}\sum_{k=3}^{n-2}\tau\int_\Omega|\delta^2 u^\varepsilon_k|^2\,dx - C.
\end{aligned}
$$

In addition, arguing as in [49, subsection 2.4], by expanding the term $\delta^2 u^\varepsilon_{k+1}$, we obtain

(4.36)

$$
\begin{aligned}
&-2\varepsilon\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^2 u^\varepsilon_{k+1}\cdot(\delta u^\varepsilon_k - u^1-\delta w_k+\dot w(0))\,dx \\
&= \varepsilon\rho\sum_{k=2}^{n-3}\int_\Omega|\delta u^\varepsilon_{k+1}-\delta u^\varepsilon_k|^2\,dx-\varepsilon\rho\int_\Omega|\delta u^\varepsilon_{n-1}-u^1|^2\,dx+\varepsilon\rho\int_\Omega|\delta u^\varepsilon_2-u^1|^2\,dx \\
&\quad -2\varepsilon\rho\int_\Omega(\delta u^\varepsilon_{n-1}-\delta u^\varepsilon_2)\cdot\dot w(0)\,dx+2\varepsilon\rho\sum_{k=2}^{n-2}\int_\Omega(\delta u^\varepsilon_{k+1}-\delta u^\varepsilon_k)\cdot\delta w_k\,dx \\
&\quad\geq -2\varepsilon\rho\int_\Omega|\delta u^\varepsilon_{n-1}-u^1|^2\,dx-\frac{\varepsilon\rho}{4}\sum_{k=3}^{n-1}\tau\int_\Omega|\delta^2 u^\varepsilon_k|^2\,dx - C,
\end{aligned}
$$

where we used (4.11), (4.28), and the estimate

$$
\begin{aligned}
&-2\varepsilon\rho\int_\Omega(\delta u^\varepsilon_{n-1}-\delta u^\varepsilon_2)\cdot\dot w(0)\,dx+2\varepsilon\rho\sum_{k=2}^{n-2}\int_\Omega(\delta u^\varepsilon_{k+1}-\delta u^\varepsilon_k)\cdot\delta w_k\,dx \\
&= -2\varepsilon\rho\int_\Omega(\delta u^\varepsilon_{n-1}-u^1)\cdot\dot w(0)\,dx-2\varepsilon\rho\int_\Omega(u^1-\delta u^\varepsilon_2)\cdot\dot w(0)\,dx \\
&\quad +2\varepsilon\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta^2 u^\varepsilon_{k+1}\cdot\delta w_k\,dx \\
&\quad\geq -\varepsilon\rho\int_\Omega|\delta u^\varepsilon_{n-1}-u^1|^2\,dx-2\varepsilon\rho\int_\Omega|\dot w(0)|^2\,dx-\varepsilon\rho\int_\Omega|u^1-\delta u^\varepsilon_2|^2\,dx \\
&\quad -\frac{\varepsilon\rho}{4}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta^2 u^\varepsilon_{k+1}|^2\,dx-4\varepsilon\rho\sum_{k=2}^{n-2}\tau\int_\Omega|\delta w_k|^2\,dx.
\end{aligned}
$$

Finally, using the elementary inequality

$$|\delta u^\varepsilon_i|^2\leq 2|\delta u^\varepsilon_i-u^1|^2+2|u^1|^2\quad\text{a.e. in }\Omega\ \text{ for every } i,$$

we deduce that

$$
\begin{aligned}
(4.37)\qquad & -2\varepsilon\rho\int_\Omega |\delta u^\varepsilon_{n-1}-u^1|^2\,dx-\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta u^\varepsilon_k\cdot\delta w_k\,dx\\
&+\rho\tau(n-3)\int_\Omega \delta u^\varepsilon_1\cdot\delta w_2\,dx-\frac{\rho}{16}\sum_{k=2}^{n-2}\sum_{i=2}^{k-1}\tau^2\int_\Omega|\delta u^\varepsilon_i|^2\,dx\\
&-\rho\int_\Omega \delta u^\varepsilon_{n-2}\cdot\delta w_{n-2}\,dx+\rho\int_\Omega\delta u^\varepsilon_1\cdot\delta w_2\,dx-\frac{\rho}{16}\sum_{i=2}^{n-3}\tau\int_\Omega|\delta u^\varepsilon_i|^2\,dx\\
&\geq-\Big(\frac14+2\varepsilon\Big)\rho\int_\Omega|\delta u^\varepsilon_{n-1}-u^1|^2\,dx-\frac{\rho}{15}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u^\varepsilon_k|^2\,dx\\
&-\frac{\rho}{16}\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\int_\Omega|\delta u^\varepsilon_i|^2\,dx-C\\
&\geq-\Big(\frac14+3\varepsilon\Big)\rho\int_\Omega|\delta u^\varepsilon_{n-1}-u^1|^2\,dx-\frac{2\rho}{15}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u^\varepsilon_k-u^1|^2\,dx\\
&-\frac{\rho}{8}\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\int_\Omega|\delta u^\varepsilon_i-u^1|^2\,dx-C.
\end{aligned}
$$

Summing (4.32) with (4.33), in view of (4.11), estimates (4.34)–(4.37) yield the inequality

(4.38)

$$
\begin{aligned}
&\frac{\rho}{8}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u^\varepsilon_k-u^1|^2\,dx+\Big(\frac12-3\varepsilon\Big)\rho\int_\Omega|\delta u^\varepsilon_{n-1}-u^1|^2\,dx\\
&+\Big(\frac{3\varepsilon}{4}-3\varepsilon^2\Big)\rho\sum_{k=3}^{n-2}\tau\int_\Omega|\delta^2u^\varepsilon_k|^2\,dx\\
&+\frac{\varepsilon^2\rho(1+\tau(n-3))}{4}\int_\Omega|\delta^2u^\varepsilon_2|^2\,dx+\Big(\varepsilon-\frac{\varepsilon^2}{2}\Big)\rho\sum_{k=3}^{n-2}\sum_{i=3}^{k}\tau^2\int_\Omega|\delta^2u^\varepsilon_i|^2\,dx\\
&+\sum_{k=2}^{n-2}\tau\int_\Omega Q(e^\varepsilon_k)\,dx+\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\mathcal{H}(\delta p^\varepsilon_i)+\int_\Omega Q(e^\varepsilon_{n-2})\,dx+\tau\sum_{i=2}^{n-2}\mathcal{H}(\delta p^\varepsilon_i)\\
&\leq(1+\tau(n-3))\int_\Omega Q(e^1)\,dx+\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\int_\Omega\mathbb{C}e^\varepsilon_i:e^1\,dx+\frac14\sum_{i=2}^{n-2}\tau\int_\Omega Q(e^\varepsilon_i)\,dx\\
&+\frac{(n-3)}{4}\sum_{i=2}^{n-2}\tau^2\int_\Omega Q(e^\varepsilon_i)\,dx+\sum_{i=2}^{n-2}\tau\int_\Omega\mathbb{C}e^\varepsilon_i:e^1\,dx+C.
\end{aligned}
$$

For τ and ε small enough we eventually obtain

$$\begin{aligned}(4.39)\quad &\varepsilon\rho\sum_{k=3}^{n-2}\tau\int_\Omega|\delta^2u_k^\varepsilon|^2\,dx+\varepsilon\rho\sum_{k=3}^{n-2}\sum_{i=3}^{k}\tau^2\int_\Omega|\delta^2u_i^\varepsilon|^2\,dx+\rho\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u_k^\varepsilon-u^1|^2\,dx\\&+\sum_{k=2}^{n-2}\tau\int_\Omega Q(e_k^\varepsilon)\,dx+\sum_{k=2}^{n-2}\tau\mathcal{H}(\delta p_k^\varepsilon)\le C,\end{aligned}$$

and the assertion follows. ∎

5. Γ-convergence from discrete to continuous. In this section we prove that for fixed $\varepsilon>0$ the sequence of discrete energy functionals $\{I_{\varepsilon\tau}\}$ (see (4.2)) converges, as the time step τ tends to zero, to the functional I_ε. This will allow us to pass to the limit $\tau\to 0$ in the discrete energy estimate (4.20) in order to obtain its continuous analogue; see (5.42) below.

In order to state the convergence result we need to introduce a few auxiliary spaces and to extend the energy functionals I_ε and $I_{\varepsilon\tau}$. Let

$$\begin{aligned}\mathscr{U}:=&\{(u,e,p)\in(W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))\cap L^1(0,T;BD(\Omega)))\\&\times L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))\times L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))\}\end{aligned}$$

and

$$\begin{aligned}&\mathscr{U}_\tau^{\rm affine}\\&:=\{(u,e,p):[0,T]\to(BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3))\times L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})\times\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\\&\quad\text{piecewise affine on the time partition of step }\tau\text{ on }[0,T],\\&\quad\text{and such that }(u(0),e(0),p(0)),\ (u(\tau),e(\tau),p(\tau)),\dots,\\&\quad(u(T),e(T),p(T))\in\mathscr{K}_\tau(u^0,e^0,p^0,u^1)\},\end{aligned}$$

where $\mathscr{K}_\tau$ is the class defined in (4.4). We set

$$G_\varepsilon(u,e,p):=\begin{cases}I_\varepsilon(u,e,p) & \text{if }(u,e,p)\in\mathcal{V},\\ +\infty & \text{otherwise in }\mathscr{U}\end{cases}$$

(where $\mathcal{V}$ is the space defined in subsection 2.9) and

$$\begin{aligned}&G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau)\\&:=\begin{cases}I_{\varepsilon\tau}\big((u_\tau(0),e_\tau(0),p_\tau(0)),\ (u_\tau(\tau),e_\tau(\tau),p_\tau(\tau)),\dots,(u_\tau(T),e_\tau(T),p_\tau(T))\big)\\ \qquad\text{if }(u_\tau,e_\tau,p_\tau)\in\mathscr{U}_\tau^{\rm affine},\\ +\infty\qquad\text{otherwise in }\mathscr{U}.\end{cases}\end{aligned}$$

We now show that the sequence of energies $\{G_{\varepsilon\tau}\}$ converges to G_ε in the sense of Γ-convergence in $\mathscr{U}$ as $\tau\to 0$.

THEOREM 5.1 (liminf inequality). *Let $\{(u_\tau,e_\tau,p_\tau)\}\subset\mathscr{U}_\tau^{\rm affine}$ and $(u,e,p)\in\mathscr{U}$ be such that*

$$(5.1)\qquad u_\tau\rightharpoonup u\quad\textit{weakly in }W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)),$$

$$(5.2)\qquad p_\tau(t)\rightharpoonup^* p(t)\quad\textit{weakly* in }\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\textit{ for every }t\in[0,T],$$

$$(5.3)\qquad \bar e_\tau\rightharpoonup e\quad\textit{weakly in }L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})).$$

Then, we have that

$$G_\varepsilon(u,e,p) \le \liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau).$$

Proof. Let $\{(u_\tau,e_\tau,p_\tau)\}$ and (u,e,p) be as in the statement of the theorem. If $\liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau)=+\infty$, there is nothing to prove, and therefore without loss of generality we can assume that

$$\liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau) = \liminf_{\tau\to 0}\Big[\frac{\varepsilon^2\rho}{2}\sum_{i=2}^{n}\tau\eta_{\tau,i}\int_\Omega |\delta^2 u_\tau(i\tau)|^2\,dx + \sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx + \varepsilon\tau\sum_{i=1}^{n-1}\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big] < +\infty. \tag{5.4}$$

In view of (5.1) and (5.2), it follows that $u(0)=u^0$ and $p(0)=p^0$. Denoting by $\bar u_\tau$ and $\tilde u_\tau$ the piecewise-constant and piecewise-quadratic interpolants associated to u_τ (see (4.18) and (4.19)), respectively, by (5.4), up to the extraction of a (not relabeled) subsequence, we have

$$\lim_{\tau\to 0}\Big[\frac{\varepsilon^2\rho}{2}\int_\tau^T \bar\eta_\tau\int_\Omega |\ddot{\tilde u}_\tau|^2\,dx\,dt + \int_\tau^{T-2\tau}\bar\eta_\tau(\cdot+2\tau)\int_\Omega Q(\bar e_\tau)\,dx\,dt \tag{5.5}$$

$$+\,\varepsilon\int_0^T \bar\eta_\tau(\cdot+\tau)\mathcal{H}(\dot p_\tau)\,dt\Big] < +\infty. \tag{5.6}$$

In view of (5.6) and (4.3), by Hölder's inequality we obtain the estimate

$$\begin{aligned}\eta_{\tau,i}\int_\Omega |\delta u_\tau(i\tau)|^2\,dx &\le 2\eta_{\tau,i}\int_\Omega\Big|\sum_{k=1}^{i-1}\tau\delta^2 u_\tau((k+1)\tau)\Big|^2 dx + 2\int_\Omega |u^1|^2\,dx\\ &\le C\tau\eta_{\tau,i}\int_\Omega\sum_{k=1}^{i-1}|\delta^2 u_\tau((k+1)\tau)|^2\,dx + 2\int_\Omega |u^1|^2\,dx\\ &\le C\tau\int_\Omega\sum_{k=1}^{i-1}\eta_{\tau,k}|\delta^2 u_\tau((k+1)\tau)|^2\,dx + 2\int_\Omega |u^1|^2\,dx.\end{aligned}$$

Thus, for τ small there holds

$$\liminf_{\tau\to 0}\Big[\frac{\varepsilon^2\rho}{2}\int_\tau^T\int_\Omega(|\ddot{\tilde u}_\tau|^2+|\dot u_\tau|^2)\,dx\,dt + \int_\tau^{T-2\tau}\int_\Omega Q(\bar e_\tau)\,dx\,dt + \varepsilon\int_0^T \mathcal{H}(\dot p_\tau)\,dt\Big] < +\infty. \tag{5.7}$$

Therefore, there exists a map $v\in W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ such that

$$\tilde u_\tau \rightharpoonup v \quad \text{weakly in } W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{5.8}$$

Arguing as in [49, subsection 2.5.1], we obtain that $u=v$ and $\dot u(0)=u^1$.

By (5.4) we deduce the upper bound

$$\lim_{\tau\to 0} D_{\mathcal{H}}(\bar p_\tau;0,T)\le C. \tag{5.9}$$

Since $\bar{p}_\tau(0) = p^0$ for every τ, by [11, Lemma 7.2] there exists a map $q \in BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))$ such that

$$\bar{p}_\tau(t) \rightharpoonup^* q(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D) \quad \text{for every } t\in[0,T], \tag{5.10}$$

and

$$D_{\mathcal{H}}(q;0,T) \le \liminf_{\tau} D_{\mathcal{H}}(\bar{p}_\tau;0,T).$$

By (5.7) and Fatou's lemma, for a.e. $t\in[0,T]$ there exist $f^t \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ and a t-dependent subsequence τ_t such that

$$\bar{e}_{\tau_t}(t) \rightharpoonup f^t \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}). \tag{5.11}$$

By (5.10) and (5.11), for a.e. $t\in[0,T]$, the sequence $\{E\bar{u}_{\tau_t}(t)\}$ is bounded in $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$ (see [11, Theorem 3.3]). This implies that for a.e. $t\in[0,T]$ there exists a map $v^t \in BD(\Omega)$ such that

$$\bar{u}_{\tau_t}(t) \rightharpoonup^* v^t \quad \text{weakly* in } BD(\Omega), \tag{5.12}$$

$$Ev^t = f^t + q(t), \tag{5.13}$$

$$q(t) = (w(t)-v^t)\odot \nu\mathcal{H}^2 \quad \text{on } \Gamma_0. \tag{5.14}$$

In view of (5.1), there holds

$$u_\tau(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3) \quad \text{for every } t\in[0,T]. \tag{5.15}$$

In addition, for fixed $i\in\mathbb{N}$ and for $t\in((i-1)\tau,i\tau]$, we have

$$\bar{u}_\tau(t) - u_\tau(t) = (i\tau - t)\dot{u}_\tau(t).$$

Thus, by (5.7) we obtain the estimate

$$\|\bar{u}_\tau - u_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} = \frac{\tau}{\sqrt{3}}\|\dot{u}_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le C\tau,$$

which in turn by (5.15) implies that

$$\bar{u}_\tau(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3) \quad \text{for a.e. } t\in[0,T]. \tag{5.16}$$

By (5.12) we conclude that

$$v^t = u(t) \quad \text{for a.e. } t\in[0,T]. \tag{5.17}$$

By (5.3) and (5.8), since $\bar{\eta}_\tau$, $\bar{\eta}_\tau(\cdot+2\tau) \to \exp(-\frac{\cdot}{\varepsilon})$ strongly in $L^\infty(0,T)$, we obtain that

$$\chi_{[\tau,T-2\tau]}\sqrt{\bar{\eta}_\tau(\cdot+2\tau)}\bar{e}_\tau \rightharpoonup \exp\Big(-\frac{\cdot}{\varepsilon}\Big)e \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})) \tag{5.18}$$

and

$$\chi_{[\tau,T-2\tau]}\sqrt{\bar{\eta}_\tau}\ddot{\bar{u}}_\tau \rightharpoonup \exp\Big(-\frac{\cdot}{\varepsilon}\Big)\ddot{u} \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.19}$$

where $\chi_{[\tau,T-\tau]}$ and $\chi_{[\tau,T-2\tau]}$ are the characteristic functions of the sets $[\tau,T-\tau]$ and $[\tau,T-2\tau]$, respectively. Additionally,

$$\bar{\eta}_\tau(\cdot+\tau) \to \exp(-t/\varepsilon) \quad \text{strongly in } L^\infty(0,T) \tag{5.20}$$

as $\tau\to 0$.

Fix $i \in \mathbb{N}$ and $t \in ((i-1)\tau, i\tau]$. Then,

$$\|\bar{p}_\tau(t) - p_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)} = \|(t-i\tau)\dot{p}_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}. \tag{5.21}$$

Therefore, by (2.4) one has

$$\begin{aligned}\|\bar{p}_\tau - p_\tau\|_{L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))} &= \frac{\tau}{2}\|\dot{p}_\tau\|_{L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))}\\ &\le \frac{\tau}{2r_K}\int_0^T \mathcal{H}(\dot{p}_\tau)\,dt \le C\tau,\end{aligned} \tag{5.22}$$

where the last inequality is due to (5.7). In view of (5.22),

$$\|\bar{p}_\tau(t) - p_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)} \to 0 \quad \text{for a.e. } t \in [0,T].$$

Thus, by (5.2) and (5.10) we deduce that

$$p(t) = q(t) \quad \text{for a.e. } t \in [0,T]. \tag{5.23}$$

By (5.13), (5.17), and (5.23) we conclude that $f^t = e(t)$ for a.e. $t \in [0,T]$ and $(u,e,p) \in \mathcal{V}$. Therefore, by (5.18) and (5.19) one has that

$$\begin{aligned}&\frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega |\ddot{u}(t)|^2\,dx\,dt + \frac{1}{2}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q(e(t))\,dx\,dt\\
&\le \frac{1}{2}\liminf_{\tau\to0}\int_0^T \Big[\varepsilon^2\rho\bar{\eta}_\tau(t)\chi_{[\tau,T-\tau]}(t)\int_\Omega |\ddot{\tilde{u}}_\tau(t)|^2\,dx\\
&\qquad + \bar{\eta}_\tau(t+2\tau)\chi_{[\tau,T-2\tau]}(t)\int_\Omega Q(\bar{e}_\tau(t))\,dx\Big]\,dt\\
&= \liminf_{\tau\to0}\Big[\frac{\varepsilon^2\rho}{2}\int_\tau^T \bar{\eta}_\tau\int_\Omega |\ddot{\tilde{u}}_\tau|^2\,dx\,dt + \frac{1}{2}\int_\tau^{T-2\tau}\bar{\eta}_\tau(\cdot+2\tau)\int_\Omega Q(\bar{e}_\tau)\,dx\,dt\Big]\\
&= \liminf_{\tau\to0}\Big[\frac{\varepsilon^2\rho}{2}\sum_{i=2}^n \tau\eta_{\tau,i}\int_\Omega |\delta^2 u_\tau(i\tau)|^2\,dx + \sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx\Big].\end{aligned} \tag{5.24}$$

To conclude we need to prove a liminf inequality for the plastic dissipation. For this purpose, let $0 \le r_0 < r_1 < \cdots < r_m \le T$. In view of (5.10) and (5.23), we have

$$\sum_{i=1}^m \exp\Big(-\frac{r_i}{\varepsilon}\Big)\mathcal{H}(p(r_i) - p(r_{i-1})) \le \liminf_{\tau\to0}\Big[\sum_{i=1}^m \exp\Big(-\frac{r_i}{\varepsilon}\Big)\mathcal{H}(\bar{p}_\tau(r_i) - \bar{p}_\tau(r_{i-1}))\Big].$$

On the other hand, since $\bar{p}_\tau$ only jumps in the points $i\tau$, $i = 1, \ldots, n$, we have

$$\begin{aligned}
&\sum_{i=1}^{m} \exp\Big(-\frac{r_i}{\varepsilon}\Big)\mathcal{H}(\bar{p}_\tau(r_i) - \bar{p}_\tau(r_{i-1}))\\
&\quad= \sum_{i=1}^{m} \exp\Big(-\frac{1}{\varepsilon}\Big\lfloor\frac{r_i}{\tau}\Big\rfloor\tau\Big)\mathcal{H}(\bar{p}_\tau(r_i) - \bar{p}_\tau(r_{i-1}))\\
&\qquad+ \sum_{i=1}^{m} \Big[\exp\Big(-\frac{r_i}{\varepsilon}\Big) - \exp\Big(-\frac{1}{\varepsilon}\Big\lfloor\frac{r_i}{\tau}\Big\rfloor\tau\Big)\Big]\mathcal{H}(\bar{p}_\tau(r_i) - \bar{p}_\tau(r_{i-1}))\\
&\quad\le \sum_{i=1}^{n} \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}(\bar{p}_\tau(i\tau) - \bar{p}_\tau((i-1)\tau))\\
&\qquad+ \frac{1}{\varepsilon}\sum_{i=1}^{m} \Big|r_i - \Big\lfloor\frac{r_i}{\tau}\Big\rfloor\tau\Big|\mathcal{H}(\bar{p}_\tau(i\tau) - \bar{p}_\tau((i-1)\tau))\\
&\quad\le \sum_{i=1}^{n} \tau\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}(\delta p_\tau(i\tau)) + \frac{\tau}{\varepsilon}D_{\mathcal{H}}(\bar{p}_\tau; 0, T).
\end{aligned}$$

By (5.9) there holds

$$\lim_{\tau\to 0}\frac{\tau}{\varepsilon}D_{\mathcal{H}}(\bar{p}_\tau; 0, T) = 0.$$

Thus, we obtain

$$\begin{aligned}
&\sum_{i=1}^{m}\exp\Big(-\frac{r_i}{\varepsilon}\Big)\mathcal{H}(p(r_i) - p(r_{i-1})) \le \liminf_{\tau\to 0}\Big[\sum_{i=1}^{m}\exp\Big(-\frac{r_i}{\varepsilon}\Big)\mathcal{H}(\bar{p}_\tau(r_i) - \bar{p}_\tau(r_{i-1}))\Big]\\
&\quad\le \liminf_{\tau\to 0}\Big[\tau\sum_{i=1}^{n}\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big]\\
&\quad\le \liminf_{\tau\to 0}\Big[\tau\sum_{i=1}^{n}\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big]\\
&\qquad+ \lim_{\tau\to 0}\tau\Big|\sum_{i=1}^{n}\Big(\exp\Big(-\frac{i\tau}{\varepsilon}\Big) - \eta_{\tau,i+1}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big|.
\end{aligned}$$

Since $\bar{p}_\tau$ only jumps in the points $i\tau$, $i = 1, \ldots, n$, we deduce

$$\tau\sum_{i=1}^{n}\mathcal{H}(\delta p_\tau(i\tau)) = D_{\mathcal{H}}(\bar{p}_\tau; 0, T).$$

Therefore, by (5.9) and (5.20), we obtain

$$\begin{aligned}
&\lim_{\tau\to 0}\tau\Big|\sum_{i=1}^{n}\Big(\exp\Big(-\frac{i\tau}{\varepsilon}\Big) - \eta_{\tau,i+1}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big|\\
&\quad\le \lim_{\tau\to 0}\Big\|\exp\Big(-\frac{t}{\varepsilon}\Big) - \bar{\eta}_\tau(t+\tau)\Big\|_{L^\infty(0,T)}\tau\sum_{i=1}^{n}\mathcal{H}(\delta p_\tau(i\tau))\\
&\quad= \lim_{\tau\to 0}\Big\|\exp\Big(-\frac{t}{\varepsilon}\Big) - \bar{\eta}_\tau(t+\tau)\Big\|_{L^\infty(0,T)}D_{\mathcal{H}}(\bar{p}_\tau; 0, T)\\
&\quad\le \lim_{\tau\to 0}C\Big\|\exp\Big(-\frac{t}{\varepsilon}\Big) - \bar{\eta}_\tau(t+\tau)\Big\|_{L^\infty(0,T)} = 0.
\end{aligned}$$

Thus, we have checked that

$$\sum_{i=1}^{m} \exp\Big(-\frac{r_i}{\varepsilon}\Big)\mathcal{H}(p(r_i)-p(r_{i-1})) \le \liminf_{\tau\to 0}\Big[\sum_{i=1}^{m}\exp\Big(-\frac{r_i}{\varepsilon}\Big)\mathcal{H}(\bar{p}_\tau(r_i)-\bar{p}_\tau(r_{i-1}))\Big]$$
$$\le \liminf_{\tau\to 0}\Big[\tau\sum_{i=1}^{n}\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big].$$

The arbitrariness of the time partition $\{t_j\}_{j=0,\dots,m}$ yields that

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p;0,T) \le \liminf_{\tau}\Big[\tau\sum_{i=1}^{n-1}\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big]. \tag{5.25}$$

The thesis follows now by combining (5.24) and (5.25). ∎

We now prove that the lower bound identified in Theorem 5.1 is optimal.

THEOREM 5.2 (limsup inequality). *Let $(u,e,p)\in\mathcal{V}$. There exists a sequence of triples $(u_\tau,e_\tau,p_\tau)\in\mathscr{U}_\tau^{\mathrm{affine}}$ such that*

$$u_\tau \to u \quad \textit{strongly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.26}$$
$$p_\tau(t) \rightharpoonup^* p(t) \quad \textit{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D) \textit{ for a.e. } t\in[0,T], \tag{5.27}$$
$$\bar{e}_\tau \to e \quad \textit{strongly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})), \tag{5.28}$$

and

$$\limsup_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau) \le G_\varepsilon(u,e,p). \tag{5.29}$$

Proof. Let u_τ be defined as the affine-in-time interpolant of the values

$$\begin{cases} u_\tau(0)=u^0,\\ u_\tau(\tau)=u^0+\tau u^1,\\ u_\tau(i\tau)=M_\tau(u)(i\tau) \quad \text{for every } i=2,\dots,n,\end{cases}$$

where M_τ is the backward mean operator,

$$M_\tau(u)(t) := \frac{1}{\tau}\int_{t-\tau}^{t} u(s)\,ds \quad \text{for every } t>\tau.$$

Define e_τ accordingly, let $\bar{e}_\tau$ be its associated piecewise-constant interpolant, and let p_τ be the piecewise-affine-in-time interpolant of the measure satisfying

$$\begin{cases} p_\tau(0)=p^0,\\ p_\tau(\tau)=p^0+\tau p^1,\\ p_\tau(i\tau)=M_\tau(p)(i\tau) \quad \text{for every } i=2,\dots,n,\end{cases}$$

where

$$\langle\varphi,\, M_\tau(p)(i\tau)\rangle := \frac{1}{\tau}\int_{t-\tau}^{t}\int_{\Omega\cup\Gamma_0}\varphi : \,dp(s)\,ds \quad \text{for every } \varphi\in C_0(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D).$$

The triple (u_τ, e_τ, p_τ) satisfies $(u_\tau, e_\tau, p_\tau) \in \mathscr{U}_\tau^{\mathrm{affine}}$, and (5.26) is obtained by arguing as in [49, subsection 2.5.2]. Property (5.28) follows by the Lebesgue differentiation theorem once we observe that

$$\int_\Omega |e(t)-\bar{e}_\tau(t)|^2\,dx \le \frac{1}{\tau}\int_{(i-2)\tau}^{i\tau}\int_\Omega |e(t)-e(s)|^2\,dx\,ds \le \frac{1}{\tau}\int_{t-2\tau}^{t+2\tau}\int_\Omega |e(t)-e(s)|^2\,dx\,ds$$

for every $t \in (2\tau, T]$.

Regarding the plastic strains, fix $t \in (0,T]$. For τ small enough, there exists $i > 2$ such that $t \in ((i-1)\tau, i\tau]$. Thus, for every $\varphi \in C_0(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$, there holds

$$\begin{aligned} &\Big|\int_{\Omega\cup\Gamma_0} \varphi\,dp_\tau(t) - \int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big| \\ &\quad= \frac{1}{\tau}\Big|\Big(\frac{t-(i-1)\tau}{\tau}\Big)\int_{(i-1)\tau}^{i\tau}\Big(\int_{\Omega\cup\Gamma_0}\varphi\,dp(s) - \int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big)\,ds \\ &\qquad+\Big(1-\Big(\frac{t-(i-1)\tau}{\tau}\Big)\Big)\int_{(i-2)\tau}^{(i-1)\tau}\Big(\int_{\Omega\cup\Gamma_0}\varphi\,dp(s) - \int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big)\,ds\Big| \\ &\quad\le \frac{\|\varphi\|_{L^\infty(\Omega\cup\Gamma_0)}}{\tau}\int_{t-2\tau}^{t+2\tau}\|p(s)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\,ds. \end{aligned} \tag{5.30}$$

In particular, for τ small enough we have

$$\|p_\tau(t)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)} \le \frac{1}{\tau}\int_{t-2\tau}^{t+2\tau}\|p(s)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\,ds.$$

Since $t \mapsto \|p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}$ is $L^1(0,T)$, in view of the Lebesgue differentiation theorem, we obtain that

$$p_\tau(t) \to p(t) \quad \text{strongly in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D) \quad \text{for a.e. } t\in[0,T]. \tag{5.31}$$

In addition, by the definition of p_τ there holds

(5.32)
$$D_{\mathcal{H}}(p_\tau;0,T) \le D_{\mathcal{H}}(p;0,T) + \tau\|p^1\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)} + 2\int_0^T \|p\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\,dt \le C.$$

Arguing as in [49, subsection 2.5.2] we obtain the inequality

$$\begin{aligned} &\limsup_{\tau\to0}\Big[\frac{\varepsilon^2\rho}{2}\sum_{i=2}^n \tau\eta_{\tau,i}\int_\Omega |\delta^2 u_\tau(i\tau)|^2\,dx + \sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx\Big] \\ &\le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}|^2\,dx + \int_\Omega Q(e)\,dx\Big)\,dt. \end{aligned}$$

To prove (5.29) it remains only to show that

$$\limsup_{\tau\to0}\Big[\tau\sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big] \le D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T). \tag{5.33}$$

We first observe that

$$
\begin{aligned}
(5.34)\qquad \tau \sum_{i=1}^{n} \eta_{\tau,i+1} \mathcal{H}(\delta p_\tau(i\tau)) &= \sum_{i=1}^{n} \eta_{\tau,i+1} \mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big) \\
&= \sum_{i=1}^{n} \Big(\eta_{\tau,i+1} - \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\Big)\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big) \\
&\quad + \sum_{i=1}^{n} \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big).
\end{aligned}
$$

By (5.32) the first term in the right-hand side of (5.34) can be bounded from above as

$$
\begin{aligned}
(5.35)\qquad &\Big|\sum_{i=1}^{n} \Big(\eta_{\tau,i+1} - \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\Big)\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big)\Big| \\
&\le \sum_{i=1}^{n} \mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big)\|\bar\eta_\tau(\cdot+\tau) - \exp(-\cdot/\varepsilon)\|_{L^\infty(0,T)} \\
&\le D_{\mathcal{H}}(p_\tau; 0, T)\|\bar\eta_\tau(\cdot+\tau) - \exp(-\cdot/\varepsilon)\|_{L^\infty(0,T)} \\
&\le C\|\bar\eta_\tau(\cdot+\tau) - \exp(-\cdot/\varepsilon)\|_{L^\infty(0,T)}
\end{aligned}
$$

and converges to zero as $\tau \to 0$.

To study the second term in the right-hand side of (5.34) we remark that for $i > 2$,

$$
(5.36)\qquad \mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big) \le \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau} \mathcal{H}(p(t) - p(s))\,ds\,dt.
$$

Indeed, for every $\varphi \in C_0(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D) \cap \mathcal{K}_D(\Omega)$, by Lemma 2.1 there holds

$$
\begin{aligned}
&\langle \varphi,\, p_\tau(i\tau) - p_\tau((i-1)\tau)\rangle \\
&= \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}\int_{\Omega\cup\Gamma_0} \varphi \cdot\, dp(t)\,dt - \frac{1}{\tau}\int_{(i-2)\tau}^{(i-1)\tau}\int_{\Omega\cup\Gamma_0} \varphi \cdot\, dp(s)\,ds \\
&= \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau}\int_{\Omega\cup\Gamma_0} \varphi \cdot\, d(p(t) - p(s))\,ds\,dt \\
&\le \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau} \mathcal{H}(p(t) - p(s))\,ds\,dt.
\end{aligned}
$$

A further application of Lemma 2.1 indeed yields (5.36). Analogously,

$$
(5.37)\quad \mathcal{H}\big(p_\tau(2\tau) - p_\tau(\tau)\big) \le \frac{1}{\tau}\int_{\tau}^{2\tau} \mathcal{H}(p(t) - p^0)\,dt + \tau\mathcal{H}(p^1) \le D_{\mathcal{H}}(p; 0, 2\tau) + \tau\mathcal{H}(p^1).
$$

In view of (5.36) and (5.37), we obtain

$$
\begin{aligned}
(5.38)\quad & \sum_{i=1}^{n} \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big) \\
&\le \sum_{i=2}^{n} \exp\Big(-\frac{i\tau}{\varepsilon}\Big) D_{\mathcal{H}}(p;0,i\tau) + 2\tau\mathcal{H}(p^1) \\
&\le \sum_{i=2}^{n} \exp\Big(-\frac{i\tau}{\varepsilon}\Big) \sup\Big\{\sum_{j=1}^{m} \mathcal{H}(p(s_j) - p(s_{j-1})) : 0 \le s_1 < \cdots < s_m \le i\tau\Big\} \\
&+ 2\tau\mathcal{H}(p^1) \le D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p; 0, T) + 2\tau\mathcal{H}(p^1).
\end{aligned}
$$

Estimate (5.33) follows now by combining (5.34)–(5.38). ∎

As a corollary of Theorems 5.1 and 5.2, we obtain a uniform energy estimate for minimizers of G_ε.

COROLLARY 5.3 (uniform energy estimate). *Let $p^1 = 0$. For every $\tau > 0$, let $(u_\tau, e_\tau, p_\tau) \in \mathscr{U}_\tau^{\rm affine}$ be a minimizer of $G_{\varepsilon\tau}$. Then, there exists a minimizer $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ of G_ε in $\mathcal{V}$ such that*

$$(5.39)\qquad \tilde u_\tau \rightharpoonup u^\varepsilon \quad \textit{weakly in } W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)),$$

$$(5.40)\qquad p_\tau(t) \rightharpoonup^* p^\varepsilon(t) \quad \textit{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D) \textit{ for a.e. } t\in[0,T],$$

$$(5.41)\qquad \bar e_\tau \rightharpoonup e^\varepsilon \quad \textit{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})),$$

where $\tilde u_\tau$ and $\bar e_\tau$ are the piecewise-quadratic and piecewise-constant interpolants of u_τ and e_τ, respectively (see (4.18) *and* (4.19)*). In addition, there exists a constant C, independent of ε, and such that*

$$
\begin{aligned}
(5.42)\qquad & \varepsilon\rho\int_0^T\int_0^t\int_\Omega |\ddot u^\varepsilon|^2\,dx\,ds\,dt + \varepsilon\rho\int_0^T\int_\Omega |\ddot u^\varepsilon|^2\,dx\,dt \\
&+ \rho\int_0^T\int_\Omega |\dot u^\varepsilon|^2\,dx\,dt + \int_0^T\int_\Omega Q(e^\varepsilon)\,dx\,dt + D_{\mathcal{H}}(p^\varepsilon;0,T) \le C.
\end{aligned}
$$

Proof. Let $\{(u_\tau, e_\tau, p_\tau)\}$ be as in the statement of the theorem. Let w_τ be the piecewise-affine-in-time interpolant associated with the maps $\{w_0,\dots,w_n\}$ (see (4.1)). Since $(u^0+tu^1-w(0)-t\dot w(0)+w_\tau(t), e^0+te^1-Ew(0)-tE\dot w(0)+Ew_\tau(t), p^0) \in \mathscr{U}_\tau^{\rm affine}$ for every $\tau > 0$, there holds

$$
\begin{aligned}
(5.43)\quad & G_{\varepsilon\tau}(u_\tau, e_\tau, p_\tau) \\
&\le G_{\varepsilon\tau}(u^0+tu^1-w(0)-t\dot w(0)+w_\tau(t), e^0+te^1 - Ew(0)-tE\dot w(0)+Ew_\tau(t), p^0) \\
&= \sum_{i=2}^{n-2} \tau\eta_{\tau,i+2}\int_\Omega Q(e^0 + i\tau e^1 - Ew(0) - i\tau E\dot w(0) + Ew_i)\,dx \le C
\end{aligned}
$$

for every $\tau > 0$. Arguing as in the proof of Theorem 5.1, in view of (5.43) there exists $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{V}$ such that (5.39)–(5.41) hold true, with

$$(5.44)\quad \chi_{[\tau,T-2\tau]}\sqrt{\bar\eta_\tau(\cdot+2\tau)}\,\bar e_\tau \rightharpoonup \exp\Big(-\frac{\cdot}{\varepsilon}\Big)e^\varepsilon \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})),$$

and

$$\frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega |\ddot{u}^\varepsilon(t)|^2\,dx\,dt \tag{5.45}$$
$$\leq \frac{\varepsilon^2\rho}{2}\liminf_{\tau\to 0}\int_0^T \bar{\eta}_\tau(t)\chi_{[\tau,T-\tau]}(t)\int_\Omega |\tilde{\ddot{u}}_\tau(t)|^2\,dx,$$

$$\frac{1}{2}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q(e^\varepsilon(t))\,dx\,dt \tag{5.46}$$
$$\leq \frac{1}{2}\liminf_{\tau\to 0}\int_0^T \bar{\eta}_\tau(t+2\tau)\chi_{[\tau,T-2\tau]}(t)\int_\Omega Q(\bar{e}_\tau(t))\,dx\,dt,$$

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,T) \tag{5.47}$$
$$\leq \liminf_{\tau}\Big[\tau\sum_{i=1}^{n-1}\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big].$$

Hence,

$$G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)\leq \liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau). \tag{5.48}$$

Let now $(v,f,q)\in\mathcal{V}$. By Theorem 5.2 there exist maps $(v_\tau,f_\tau,q_\tau)\in\mathscr{U}_\tau^{\rm affine}$ such that

$$\limsup_{\tau\to 0} G_{\varepsilon\tau}(v_\tau,f_\tau,q_\tau)\leq G_\varepsilon(v,f,q). \tag{5.49}$$

The minimality of $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ follows then by the minimality of (u_τ,e_τ,p_τ) and by combining (5.48) with (5.49). Using again Theorem 5.2 we get the existence of a sequence $\{(\hat{u}_\tau,\hat{e}_\tau,\hat{p}_\tau)\}\subset\mathscr{U}_\tau^{\rm affine}$ such that

$$\limsup_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau)\leq \limsup_{\tau\to 0} G_{\varepsilon\tau}(\hat{u}_\tau,\hat{e}_\tau,\hat{p}_\tau)\leq G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon). \tag{5.50}$$

Combining (5.48) with (5.50), we conclude that

$$\lim_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau)=G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon).$$

In view of Theorem 4.8, by (5.39) and (5.41) we have

$$\varepsilon\rho\int_0^T\int_0^t\int_\Omega |\ddot{u}^\varepsilon|^2\,dx\,ds\,dt+\varepsilon\rho\int_0^T\int_\Omega |\ddot{u}^\varepsilon|^2\,dx\,dt+\rho\int_0^T\int_\Omega |\dot{u}^\varepsilon|^2\,dx\,dt \tag{5.51}$$
$$+\int_0^T\int_\Omega Q(e^\varepsilon)\,dx\,dt\leq C.$$

In addition, by (5.40), the lower semicontinuity of $\mathcal{H}$, and Theorem 4.8,

$$\sup_{a>0} D_{\mathcal{H}}(p^\varepsilon;a,T-a)\leq \sup_{a>0}\liminf_{\tau\to 0} D_{\mathcal{H}}(p_\tau;a,T-a)\leq C. \tag{5.52}$$

The thesis follows by combining (5.51) and (5.52). □

6. Energy inequality at level ε. The central results of this section are Propositions 6.4 and 6.6, delivering an energy inequality at the level $\varepsilon > 0$ fulfilled by minimizers, and its integrated-in-time counterpart (see (6.17)). The proof strategy follows closely that of [47, Theorem 2.5 (c)]. The additional difficulties in our setting are due to the fact that the dissipation potential satisfies linear growth conditions from above, and the triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ is required to fulfill the constraint $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{V}$. Another crucial difference is that our analysis is performed on the finite interval $[0, T]$ instead of in the entire semiline $t \geq 0$. The methodology relies on the notion of *approximate energy* (see (6.3)). This consists, roughly speaking, of the sum of the kinetic and elastic energies with the plastic dissipation potential, suitably weighted by a rescaled ε-dependent probability kernel. The structure of the proof will be the following: first, in Lemma 6.1 we will exploit the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ to perform some internal variations, by considering as competitors the composition of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ with reparametrizations of the time interval $[0, T]$. In Corollaries 6.2 and 6.3 we will establish some properties of the approximate energy, which in turn will be the starting points of the proofs of Propositions 6.4 and 6.6. An additional characterization of minimizing triples will be provided in Proposition 6.7.

As in [47, section 4] we first introduce some auxiliary quantities. Throughout this section we assume that $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$, and we consider a minimizer $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ of G_ε. We set

$$\mathcal{K}_\varepsilon(t) := \frac{\rho\varepsilon^2}{2} \int_\Omega |\dot{u}^\varepsilon(t)|^2 \, dx \quad \text{and} \quad \mathcal{H}_\varepsilon(t) := \varepsilon D_{\mathcal{H}}(p^\varepsilon; 0, t),$$

for every $t \in [0, T]$, and set as well

$$\mathcal{W}_\varepsilon(t) := \int_\Omega Q(e^\varepsilon(t)) \, dx \quad \text{and} \quad \mathcal{D}_\varepsilon(t) := \frac{\rho\varepsilon^2}{2} \int_\Omega |\ddot{u}^\varepsilon(t)|^2 \, dx$$

for a.e. $t \in [0, T]$, and we define the *locally integrable Lagrangian*

$$\mathcal{L}_\varepsilon(t) := \mathcal{D}_\varepsilon(t) + \mathcal{W}_\varepsilon(t) + \frac{\mathcal{H}_\varepsilon(t)}{\varepsilon} \quad \text{for a.e. } t \in [0, T].$$

Note that $\mathcal{K}_\varepsilon \in W^{1,1}(0, T)$, with

$$\dot{\mathcal{K}}_\varepsilon(t) = \rho\varepsilon^2 \int_\Omega \ddot{u}^\varepsilon(t) \cdot \dot{u}^\varepsilon(t) \, dx \quad \text{for a.e. } t \in [0, T]. \tag{6.1}$$

For $f : [0, T] \to [0, +\infty]$ measurable we consider the operator

$$\mathcal{A}f(t) := \int_t^T \exp\Big(\frac{t - s}{\varepsilon}\Big) f(s) \, ds \quad \text{for every } t \in [0, T].$$

We point out that if

$$\mathcal{A}f(0) = \int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big) f(s) \, ds < +\infty,$$

then $f \in L^1([0, T])$, and

$$\dot{\mathcal{A}}(t) = \frac{\mathcal{A}f(t)}{\varepsilon} - f(t) \quad \text{for a.e. } t \in [0, T] \tag{6.2}$$

satisfies $\dot{\mathcal{A}} \in L^1(0,T)$. In other words, if $\mathcal{A}f(0) < +\infty$, then $\mathcal{A}f \in W^{1,1}([0,T])$.

A direct computation yields

$$\mathcal{A}^2 f(t) := \mathcal{A}(\mathcal{A}f)(t) = \int_t^T \exp\Big(\frac{t-s}{\varepsilon}\Big)(s-t)f(s)\,ds \quad \text{for every } t \in [0,T]$$

and

$$\mathcal{A}^2 f(0) = \int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big) s f(s)\,ds.$$

For every $\varepsilon > 0$ the *approximate energy* $\mathcal{E}_\varepsilon$ is defined as

$$\mathcal{E}_\varepsilon(t) := \mathcal{K}_\varepsilon(t) + \mathcal{A}^2\mathcal{W}_\varepsilon(t) + \frac{1}{\varepsilon}\mathcal{A}^2\mathcal{H}_\varepsilon(t) \quad \text{for every } t \in [0,T]. \tag{6.3}$$

While the Lagrangian $\mathcal{L}_\varepsilon$ is given by the sum of the inertial term, the elastic energy, and the dissipation, the approximate energy features the kinetic energy associated with the model, and an integrated version of the sum of the elastic energy and the plastic dissipation potential, weighted by a suitably rescaled probability kernel. The presence of the third term in the right-hand side of (6.3) is a key difference with respect to [47] and is needed due to the linear growth assumptions on the plastic dissipation potential in our setting. Indeed, in the case in which the dissipation potential is quadratic, the associated estimates simplify, and it is thus possible to control this quantity without adding the dissipative term to the approximate energy (see Definition 4.2 and Proposition 4.4 in [47]). This is not the case in the situation in which the dissipation potential grows only linearly (see (4.8) in [47]). The term $\mathcal{A}^2\mathcal{H}_\varepsilon$ is added to the approximate energy in order to overcome this technical difficulty.

We start by proving a preliminary inequality involving the quantities $\mathcal{D}_\varepsilon$, $\mathcal{K}_\varepsilon$, and $\mathcal{L}_\varepsilon$.

LEMMA 6.1. *Let $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$, and let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε. Then,*

$$\begin{aligned}
&\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)(\varepsilon\dot{g}(s) - g(s))\mathcal{L}_\varepsilon(s)\,ds - 4\varepsilon\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)\dot{g}(s)\mathcal{D}_\varepsilon(s)\,ds\\
&\quad - \varepsilon\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)\ddot{g}(s)\dot{\mathcal{K}}_\varepsilon(s)\,ds + \varepsilon^3\rho\int_0^T\int_\Omega \exp\Big(-\frac{s}{\varepsilon}\Big)\ddot{u}^\varepsilon(s)\cdot(\dot{w}(s)g(s))^{\cdot\cdot}\,dx\,ds\\
&\quad + \varepsilon\int_0^T\int_\Omega \exp\Big(-\frac{s}{\varepsilon}\Big)\mathbb{C}e^\varepsilon(s) : E\dot{w}(s)g(s)\,dx\,ds\\
&\quad + \varepsilon\int_0^T\int_\Omega \exp\Big(-\frac{s}{\varepsilon}\Big)\mathbb{C}e^\varepsilon(s) : (e^1 - E\dot{w}(0))s\dot{g}(0)\,dx\,ds \ge 0
\end{aligned}$$

for every $g \in C^2([0,T])$ such that $g(0) = 0$ and $g(t) \ge 0$ for every $t \in [0,T]$.

Proof. We argue as in [47, Proposition 4.4], and for every $\delta > 0$ we consider the map

$$\varphi_\delta(t) := t - \delta\varepsilon g(t) \quad \text{for every } t \in [0,T].$$

For δ small, φ_δ is a C^2 diffeomorphism from $[0,T]$ to $[0,\varphi_\delta(T)]$, with inverse ψ_δ : $[0,\varphi_\delta(T)] \to [0,T]$ satisfying

$$\psi_\delta(t) := t + \delta\varepsilon g(\psi_\delta(t)) \quad \text{for every } t \in [0,T].$$

We define the triple

$$\tilde{u}^\varepsilon(t) := u^\varepsilon(\varphi_\delta(t)) + t\delta\varepsilon\dot{g}(0)u^1 + w(t) - w(\varphi_\delta(t)) - t\delta\varepsilon\dot{g}(0)\dot{w}(0),$$
$$\tilde{p}^\varepsilon(t) := p^\varepsilon(\varphi_\delta(t))$$

for every $t \in [0, T]$, and

$$\tilde{e}^\varepsilon(t) := e^\varepsilon(\varphi_\delta(t)) + t\delta\varepsilon\dot{g}(0)e^1 + Ew(t) - Ew(\varphi_\delta(t)) - t\delta\varepsilon\dot{g}(0)E\dot{w}(0)$$

for every $t \in [0, T]$. Since $(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) \in \mathcal{V}$, by the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ there holds

$$\limsup_{\delta\to 0} \frac{G_\varepsilon(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) - G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon)}{\delta} \geq 0. \tag{6.4}$$

We make the preliminary observation that

$$\dot{\tilde{u}}^\varepsilon(t) = \dot{u}^\varepsilon(\varphi_\delta(t))\dot{\varphi}_\delta(t) + \delta\varepsilon\dot{g}(0)u^1 + \dot{w}(t) - \dot{w}(\varphi_\delta(t))\dot{\varphi}_\delta(t) - \delta\varepsilon\dot{g}(0)\dot{w}(0)$$

for every $t \in [0, T]$, and

$$\ddot{\tilde{u}}^\varepsilon(t) = \ddot{u}^\varepsilon(\varphi_\delta(t))(\dot{\varphi}_\delta(t))^2 + \dot{u}^\varepsilon(\varphi_\delta(t))\ddot{\varphi}_\delta(t) + \ddot{w}(t) - \ddot{w}(\varphi_\delta(t))(\dot{\varphi}_\delta(t))^2 - \dot{w}(\varphi_\delta(t))\ddot{\varphi}_\delta(t)$$

for a.e. $t \in [0, T]$. Therefore, a change of variable in inequality (6.4) yields

$$\begin{aligned}
\limsup_{\delta\to 0} \frac{1}{\delta}\Big\{ &\int_0^{\varphi_\delta(T)} \dot{\psi}_\delta(t) \exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}^\varepsilon(t)(\dot{\varphi}_\delta(\psi_\delta(t)))^2 \\
&+ \dot{u}^\varepsilon(t)\ddot{\varphi}_\delta(\psi_\delta(t)) + \ddot{w}(\psi_\delta(t)) - \ddot{w}(t)(\dot{\varphi}_\delta(\psi_\delta(t)))^2 - \dot{w}(t)\ddot{\varphi}_\delta(\psi_\delta(t))|^2\, dx \\
&+ \int_\Omega Q\big(e^\varepsilon(t) + \psi_\delta(t)\delta\varepsilon\dot{g}(0)e^1 + Ew(\psi_\delta(t)) - Ew(t) - \psi_\delta(t)\delta\varepsilon\dot{g}(0)E\dot{w}(0)\big)\, dx\Big]\, dt \\
&- \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}^\varepsilon(t)|^2\, dx + \int_\Omega Q(e^\varepsilon(t))\, dx\Big]\, dt \\
&+ \varepsilon(D_\mathcal{H}(\exp(-\,\cdot\,/\varepsilon); \tilde{p}^\varepsilon; 0, T) - D_\mathcal{H}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, T))\Big\} \geq 0,
\end{aligned}$$

which in turn, since $\varphi_\delta(T) \leq T$, implies

$$\begin{aligned}
\limsup_{\delta\to 0} \frac{1}{\delta}\Big\{ &\int_0^{\varphi_\delta(T)} \dot{\psi}_\delta(t) \exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}^\varepsilon(t)(\dot{\varphi}_\delta(\psi_\delta(t)))^2 \\
&+ \dot{u}^\varepsilon(t)\ddot{\varphi}_\delta(\psi_\delta(t)) + \ddot{w}(\psi_\delta(t)) - \ddot{w}(t)(\dot{\varphi}_\delta(\psi_\delta(t)))^2 - \dot{w}(t)\ddot{\varphi}_\delta(\psi_\delta(t))|^2\, dx \\
&+ \int_\Omega Q(e^\varepsilon(t) + \psi_\delta(t)\delta\varepsilon\dot{g}(0)e^1 + Ew(\psi_\delta(t)) - Ew(t) - \psi_\delta(t)\delta\varepsilon\dot{g}(0)E\dot{w}(0))\, dx\Big]\, dt \\
&- \int_0^{\varphi_\delta(T)} \exp\Big(-\frac{t}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}^\varepsilon(t)|^2\, dx + \int_\Omega Q(e^\varepsilon(t))\, dx\Big]\, dt \\
&+ \varepsilon(D_\mathcal{H}(\exp(-\,\cdot\,/\varepsilon); \tilde{p}^\varepsilon; 0, T) - D_\mathcal{H}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, \varphi_\delta(T)))\Big\} \geq 0.
\end{aligned} \tag{6.5}$$

The inertial terms satisfy

$$
\begin{aligned}
(6.6)\quad & \limsup_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^{\varphi_\delta(T)}\dot\psi_\delta(t)\exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega|\ddot u^\varepsilon(t)(\dot\varphi_\delta(\psi_\delta(t)))^2\\
&+\dot u^\varepsilon(t)\ddot\varphi_\delta(\psi_\delta(t))+\ddot w(\psi_\delta(t))-\ddot w(t)(\dot\varphi_\delta(\psi_\delta(t)))^2-\dot w(t)\ddot\varphi_\delta(\psi_\delta(t))|^2\Big]\,dx\,dt\\
&-\frac{\varepsilon^2\rho}{2}\int_0^{\varphi_\delta(T)}\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\Big\}\\
&=\frac{\varepsilon^2\rho}{2}\int_0^T(\varepsilon\dot g(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\\
&+\varepsilon^3\rho\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega\ddot u^\varepsilon(t)\cdot\Big(-2\ddot u^\varepsilon(t)\dot g(t)-\dot u^\varepsilon(t)\ddot g(t)+\ddot w(t)g(t)\\
&+2\ddot w(t)\dot g(t)+\dot w(t)\ddot g(t)\Big)\,dx\,dt.
\end{aligned}
$$

This latter inequality follows by the dominated convergence theorem, by the fact that the left-hand side coincides with the integral between 0 and $\varphi_\delta(T)$ of the incremental ratio between 0 and δ of the function

$$
\begin{aligned}
&\dot\psi_\delta(t)\exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega|\ddot u^\varepsilon(t)(\dot\varphi_\delta(\psi_\delta(t)))^2+\dot u^\varepsilon(t)\ddot\varphi_\delta(\psi_\delta(t))+\ddot w(\psi_\delta(t))\\
&\quad-\ddot w(t)(\dot\varphi_\delta(\psi_\delta(t)))^2-\dot w(t)\ddot\varphi_\delta(\psi_\delta(t))|^2\Big]\,dx,
\end{aligned}
$$

and by the identities

$$
\begin{aligned}
&\frac{\partial}{\partial\delta}\Big(\dot\psi_\delta(t)\exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big)\Big|_{\delta=0}=(\varepsilon\dot g(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big),\\
&\frac{\partial}{\partial\delta}\psi_\delta(t)\Big|_{\delta=0}=\varepsilon g(t),\\
&\frac{\partial}{\partial\delta}(\dot\varphi_\delta(\psi_\delta(t)))^2\Big|_{\delta=0}=-2\varepsilon\dot g(t),\\
&\frac{\partial}{\partial\delta}\ddot\varphi_\delta(\psi_\delta(t))\Big|_{\delta=0}=-\varepsilon\ddot g(t)
\end{aligned}
$$

for every $t\in[0,T]$, and

$$
\begin{aligned}
&\frac{\partial}{\partial\delta}\Big(\int_\Omega|\ddot u^\varepsilon(t)(\dot\varphi_\delta(\psi_\delta(t)))^2+\dot u^\varepsilon(t)\ddot\varphi_\delta(\psi_\delta(t))+\ddot w(\psi_\delta(t))\\
&\quad-\ddot w(t)(\dot\varphi_\delta(\psi_\delta(t)))^2-\dot w(t)\ddot\varphi_\delta(\psi_\delta(t))|^2\,dx\Big)\Big|_{\delta=0}\\
&\quad=2\varepsilon\int_\Omega\ddot u^\varepsilon(t)\cdot\Big(-2\ddot u^\varepsilon(t)\dot g(t)-\dot u^\varepsilon(t)\ddot g(t)+\ddot w(t)g(t)+2\ddot w(t)\dot g(t)+\dot w(t)\ddot g(t)\Big)\,dx
\end{aligned}
$$

for a.e. $t\in[0,T]$. Analogously,

$$
\begin{aligned}
&\frac{\partial}{\partial\delta}\int_\Omega Q(e^\varepsilon(t)+\psi_\delta(t)\delta\varepsilon\dot g(0)e^1+Ew(\psi_\delta(t))-Ew(t)-\psi_\delta(t)\delta\varepsilon\dot g(0)E\dot w(0))\,dx\Big|_{\delta=0}\\
&\quad=\varepsilon\int_\Omega\mathbb{C}e^\varepsilon(t):\Big(E\dot w(t)g(t)+t\dot g(0)e^1-t\dot g(0)E\dot w(0)\Big)\,dx,
\end{aligned}
$$

for a.e. $t \in [0,T]$, and hence

$$\limsup_{\delta\to 0} \frac{1}{\delta}\Big\{ \int_0^{\varphi_\delta(T)} \dot{\psi}_\delta(t) \exp\Big(- \frac{\psi_\delta(t)}{\varepsilon}\Big) \int_\Omega Q\Big(e^\varepsilon(t) + \psi_\delta(t)\delta\varepsilon\dot{g}(0)e^1 + Ew(\psi_\delta(t))$$

$$- Ew(t) - \psi_\delta(t)\delta\varepsilon\dot{g}(0)E\dot{w}(0)\Big)\,dx\,dt - \int_0^{\varphi_\delta(T)} \exp\Big(- \frac{t}{\varepsilon}\Big) \int_\Omega Q(e^\varepsilon(t))\,dx\,dt\Big\}$$

$$= \int_0^T (\varepsilon\dot{g}(t) - g(t)) \exp\Big(- \frac{t}{\varepsilon}\Big) \int_\Omega Q(e^\varepsilon(t))\,dx\,dt$$

(6.7)

$$+ \varepsilon \int_0^T \exp\Big(- \frac{t}{\varepsilon}\Big) \int_\Omega \mathbb{C}e^\varepsilon(t) : \Big(E\dot{w}(t)g(t) + t\dot{g}(0)e^1 - t\dot{g}(0)E\dot{w}(0)\Big)\,dx\,dt.$$

To complete the proof of the lemma it remains to study the asymptotic behavior of the dissipation as $\delta \to 0$. Fix $\lambda > 0$, and let $0 \le t_0 < t_1 < \cdots < t_m \le T$ be such that

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); \tilde{p}^\varepsilon; 0, T) \le \sum_{i=1}^m \exp\left(-\frac{t_i}{\varepsilon}\right) \mathcal{H}(\tilde{p}^\varepsilon(t_i) - \tilde{p}^\varepsilon(r_{i-1})) + \lambda.$$

For $i = 1, \dots, m$, let $s_i \in [0, \varphi_\delta(T)]$ be such that $t_i = \psi_\delta(s_i)$. There holds

$$\sum_{i=1}^m \exp\left(-\frac{t_i}{\varepsilon}\right) \mathcal{H}(\tilde{p}^\varepsilon(t_i) - \tilde{p}^\varepsilon(r_{i-1})) = \sum_{i=1}^m \exp\left(-\frac{\psi_\delta(s_i)}{\varepsilon}\right) \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1}))$$

$$= \sum_{i=1}^m \exp\Big(-\frac{s_i}{\varepsilon}\Big) \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1}))$$

$$+ \sum_{i=1}^m \exp\Big(-\frac{s_i}{\varepsilon}\Big) \Big[\exp\left(-\delta g(\psi_\delta(s_i))\right) - 1\Big] \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1}))$$

$$\le D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); p^\varepsilon; 0, \varphi_\delta(T))$$

$$- \delta \sum_{i=1}^m \exp\Big(-\frac{s_i}{\varepsilon}\Big) g(\psi_\delta(s_i)) \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1})) + \mathrm{O}(\delta^2)$$

$$= D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); p^\varepsilon; 0, \varphi_\delta(T))$$

$$- \delta \sum_{i=1}^m \exp\Big(-\frac{s_i}{\varepsilon}\Big) g(s_i) \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1})) + \mathrm{O}(\delta^2).$$

Thus,

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); \tilde{p}^\varepsilon; 0, T) \le D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); p^\varepsilon; 0, \varphi_\delta(T)) + \lambda$$

$$- \delta \sum_{i=1}^m \exp\Big(-\frac{s_i}{\varepsilon}\Big) g(s_i) \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1})) + \mathrm{O}(\delta^2).$$

By considering finer and finer refinements of $\{t_0, \dots, t_m\}$, in view of the definition of $\hat{D}_{\mathcal{H}}$ (see (2.9)), and by the arbitrariness of λ we conclude that

(6.8)

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); \tilde{p}^\varepsilon; 0, T)$$

$$\le D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); p^\varepsilon; 0, \varphi_\delta(T)) + \delta\hat{D}_{\mathcal{H}}(-\exp(-\cdot/\varepsilon)g(\cdot); 0, \varphi_\delta(T)) + \mathrm{O}(\delta^2).$$

By (6.8) and [21, Theorem 4.5] we obtain

$$
\begin{aligned}
(6.9)\qquad &\limsup_{\delta\to 0}\frac{\varepsilon}{\delta}\big(D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);\tilde{p}^\varepsilon;0,T)-D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,\varphi_\delta(T))\big)\\
&\le \limsup_{\delta\to 0}\varepsilon\hat{D}_{\mathcal{H}}(-\exp(-\,\cdot\,/\varepsilon)g(\cdot);0,\varphi_\delta(T))=\limsup_{\delta\to 0}\\
&-\varepsilon PMS\int_0^{\varphi_\delta(T)}\exp\Big(-\frac{t}{\varepsilon}\Big)g(t)\,dD_{\mathcal{H}}(p^\varepsilon;0,t)\\
&=-\liminf_{\delta\to 0}\Big\{\varepsilon g(\varphi_\delta(T))\exp\Big(-\frac{\varphi_\delta(T)}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,\varphi_\delta(T))\\
&-\int_0^{\varphi_\delta(T)}(\varepsilon\dot{g}(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,t)\,dt\Big\}\\
&\le\int_0^T(\varepsilon\dot{g}(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,t)\,dt.
\end{aligned}
$$

The thesis follows by combining (6.1), (6.5)–(6.7), and (6.9) and by the definitions of $\mathcal{K}_\varepsilon$, $\mathcal{D}_\varepsilon$, and $\mathcal{L}_\varepsilon$. ∎

Setting

$$
(6.10)\qquad R_\varepsilon(t):=-\varepsilon\int_\Omega \mathbb{C}e^\varepsilon(t):\big(e^1-E\dot{w}(0)\big)\,dx,
$$

$$
(6.11)\qquad \tilde{R}_\varepsilon(t):=-\varepsilon^3\rho\int_\Omega \ddot{u}^\varepsilon(t)\cdot\ddot{w}(t)\,dx-\varepsilon\int_\Omega\mathbb{C}e^\varepsilon(t):E\dot{w}(t)\,dx,
$$

$$
(6.12)\qquad \hat{R}_\varepsilon(t):=-2\varepsilon^3\rho\int_\Omega \ddot{u}^\varepsilon(t)\cdot\ddot{w}(t)\,dx,
$$

$$
(6.13)\qquad \mathring{R}_\varepsilon(t):=-\varepsilon^3\rho\int_\Omega \ddot{u}^\varepsilon(t)\cdot\dot{w}(t)\,dx,
$$

for a.e. $t\in[0,T]$, and choosing $g(t)=t$ in Lemma 6.1, the same approximation argument as in [47, Corollary 4.5] yields the following.

Corollary 6.2. *Let $(u^1,e^1,0)\in\mathscr{A}(\dot{w}(0))$, and let $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ be a minimizer of G_ε. Then*

$$
\varepsilon\mathcal{A}\mathcal{L}_\varepsilon(0)-\mathcal{A}^2\mathcal{L}_\varepsilon(0)-4\varepsilon\mathcal{A}\mathcal{D}_\varepsilon(0)\ge\mathcal{A}^2R_\varepsilon(0)+\mathcal{A}^2\tilde{R}_\varepsilon(0)+\mathcal{A}\hat{R}_\varepsilon(0).
$$

Finally, by considering the sequence of maps $g_\delta:[0,T]\to[0,+\infty)$ defined as

$$
g_\delta(s):=\begin{cases}0 & \text{if } s\le t,\\ \frac{(s-t)^2}{2\delta} & \text{if } t<s<t+\delta,\\ s-t-\frac{\delta}{2} & \text{if } s\ge t+\delta\end{cases}
$$

in Lemma 6.1, and by letting δ go to zero, we deduce the following inequality.

Corollary 6.3. *Let $(u^1,e^1,0)\in\mathscr{A}(\dot{w}(0))$, and let $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ be a minimizer of G_ε. Then*

$$
\varepsilon\mathcal{A}\mathcal{L}_\varepsilon(t)-\mathcal{A}^2\mathcal{L}_\varepsilon(t)-4\varepsilon\mathcal{A}\mathcal{D}_\varepsilon(t)-\varepsilon\dot{\mathcal{K}}_\varepsilon(t)\ge\mathcal{A}^2\tilde{R}_\varepsilon(t)+\mathcal{A}\hat{R}_\varepsilon(t)+\mathring{R}_\varepsilon(t)
$$

for a.e. $t\in[0,T]$.

We are now in a position to prove an energy inequality at the level $\varepsilon > 0$.

PROPOSITION 6.4 (energy inequality). *Let $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$, and let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε. Then*

$$\begin{aligned}
&\frac{\mathcal{E}_\varepsilon(t)}{\varepsilon^2} - \rho \int_\Omega \dot{u}^\varepsilon(t) \cdot \dot{w}(t)\, dx + \frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{A}\hat{R}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{A}\tilde{R}_\varepsilon(t)}{\varepsilon} \\
&\quad \le \frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} - \rho \int_\Omega u^1 \cdot \dot{w}(0)\, dx + \frac{\mathcal{A}^2 \tilde{R}_\varepsilon(0)}{\varepsilon^2} + \frac{\mathcal{A}\hat{R}_\varepsilon(0)}{\varepsilon^2} + \frac{\mathcal{A}\tilde{R}_\varepsilon(0)}{\varepsilon} \\
&\qquad - \rho \int_0^t \int_\Omega \dot{u}^\varepsilon(s) \cdot \ddot{w}(s)\, dx\, ds + \int_0^t \frac{\tilde{R}_\varepsilon(s)}{\varepsilon}\, ds - \int_0^t \frac{\hat{R}_\varepsilon(s)}{\varepsilon^2}\, ds.
\end{aligned}$$

Proof. By the definition of the approximate energy (see (6.3)) there holds

$$\mathcal{E}_\varepsilon(t) := \mathcal{K}_\varepsilon(t) + \mathcal{A}^2(\mathcal{L}_\varepsilon - \mathcal{D}_\varepsilon)(t)$$

for every $t \in [0, T]$, which by (6.2) implies

$$\dot{\mathcal{E}}_\varepsilon(t) = \dot{\mathcal{K}}_\varepsilon(t) + \frac{\mathcal{A}^2 \mathcal{L}_\varepsilon(t)}{\varepsilon} - \frac{\mathcal{A}^2 \mathcal{D}_\varepsilon(t)}{\varepsilon} - \mathcal{A}\mathcal{L}_\varepsilon(t) + \mathcal{A}\mathcal{D}_\varepsilon(t)$$

for a.e. $t \in [0, T]$. On the one hand, in view of Corollary 6.3, we obtain the estimate

$$\begin{aligned}
\dot{\mathcal{E}}_\varepsilon(t) &\le -\frac{\mathcal{A}^2 \mathcal{D}_\varepsilon(t)}{\varepsilon} - 3\mathcal{A}\mathcal{D}_\varepsilon(t) - \frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon} - \frac{\mathcal{A}\hat{R}_\varepsilon(t)}{\varepsilon} - \frac{\mathring{R}_\varepsilon(t)}{\varepsilon} \\
&\le -\frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon} - \frac{\mathcal{A}\hat{R}_\varepsilon(t)}{\varepsilon} - \frac{\mathring{R}_\varepsilon(t)}{\varepsilon}
\end{aligned} \tag{6.14}$$

for a.e. $t \in [0, T]$. On the other hand, by (6.2),

$$-\frac{\mathring{R}_\varepsilon(t)}{\varepsilon^3} = \rho \Big(\int_\Omega \dot{u}^\varepsilon(t) \cdot \dot{w}(t)\, dx \Big)^{\bullet} - \rho \int_\Omega \dot{u}^\varepsilon(t) \cdot \ddot{w}(t)\, dx, \tag{6.15}$$

and

$$\begin{aligned}
-\frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^3} - \frac{\mathcal{A}\hat{R}^\varepsilon(t)}{\varepsilon^3} &= \Big(-\frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^2} - \frac{\mathcal{A}\hat{R}_\varepsilon(t)}{\varepsilon^2} \Big)^{\bullet} - \frac{\mathcal{A}\tilde{R}_\varepsilon(t)}{\varepsilon^2} - \frac{\hat{R}_\varepsilon(t)}{\varepsilon^2} \\
&= \Big(-\frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^2} - \frac{\mathcal{A}\hat{R}_\varepsilon(t)}{\varepsilon^2} - \frac{\mathcal{A}\tilde{R}_\varepsilon(t)}{\varepsilon} \Big)^{\bullet} - \frac{\tilde{R}_\varepsilon(t)}{\varepsilon} - \frac{\hat{R}_\varepsilon(t)}{\varepsilon^2}
\end{aligned} \tag{6.16}$$

for a.e. $t \in [0, T]$. By combining (6.14)–(6.16) we deduce

$$\begin{aligned}
&\Big(\frac{\mathcal{E}_\varepsilon(t)}{\varepsilon^2} - \rho \int_\Omega \dot{u}^\varepsilon(t) \cdot \dot{w}(t)\, dx + \frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{A}\hat{R}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{A}\tilde{R}_\varepsilon(t)}{\varepsilon} \Big)^{\bullet} \\
&\quad \le -\rho \int_\Omega \dot{u}^\varepsilon(t) \cdot \ddot{w}(t)\, dx + \frac{\tilde{R}_\varepsilon(t)}{\varepsilon} - \frac{\hat{R}_\varepsilon(t)}{\varepsilon^2}
\end{aligned}$$

for a.e. $t \in [0, T]$. An integration in time in $[0, T]$ yields the thesis. $\square$

The same argument in [47, Lemma 6.1] provides the following technical result.

LEMMA 6.5. *Let ℓ and m be two nonnegative functions in $L^1(0,T)$ such that*

$$\mathcal{A}^2\ell(t) \le m(t) \quad \text{for a.e. } t \in [0,T].$$

Then, for every $a > 0$ and every $\delta \in (0,1)$, there holds

$$\Big(\int_0^{\delta a} s \exp\Big(-\frac{s}{\varepsilon}\Big)\,ds\Big)\int_{t+\delta a}^{t+a} \ell(s)\,ds \le \int_t^{t+a} m(s)\,ds$$

for every $t \in [0, T-a]$.

In view of Proposition 6.4 and Lemma 6.5, we obtain an integrated-in-time version of the ε-energy inequality.

PROPOSITION 6.6 (integral energy inequality). *Let $(u^1, e^1, 0) \in \mathscr{A}(\dot w(0))$, and let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε. Then, for every $a > 0$ and $\delta \in (0,1)$, there holds*

$$\begin{aligned}
&\Big(\frac{1}{\varepsilon^2}\int_0^{\delta a} s\exp\Big(-\frac{s}{\varepsilon}\Big)\,ds\Big)\int_{t+\delta a}^{t+a}\Big(\int_\Omega Q(e^\varepsilon(s))\,dx + D_{\mathcal{H}}(p^\varepsilon;0,s)\Big)\,ds\\
&\quad + \frac{\rho}{2}\int_t^{t+a}\int_\Omega |\dot u^\varepsilon(s)|^2\,dx\,ds\\
&\quad - \rho\int_t^{t+a}\int_\Omega \dot u^\varepsilon(s)\cdot\dot w(s)\,dx\,ds \le -\int_t^{t+a}\Big(\frac{\mathcal{A}^2\tilde R_\varepsilon(s)}{\varepsilon^2} + \frac{\mathcal{A}\hat R_\varepsilon(s)}{\varepsilon^2} + \frac{\mathcal{A}\tilde R_\varepsilon(s)}{\varepsilon}\Big)\,ds\\
&\quad + \frac{\mathcal{E}_\varepsilon(0)a}{\varepsilon^2} - \rho a\int_\Omega u^1\cdot\dot w(0)\,dx + a\frac{\mathcal{A}^2\tilde R_\varepsilon(0)}{\varepsilon^2} + a\frac{\mathcal{A}\hat R_\varepsilon(0)}{\varepsilon^2} + a\frac{\mathcal{A}\tilde R_\varepsilon(0)}{\varepsilon}\\
&\quad - \rho\int_t^{t+a}\int_0^\xi\int_\Omega \dot u^\varepsilon(s)\cdot\ddot w(s)\,dx\,ds\,d\xi + \int_t^{t+a}\int_0^\xi \frac{\tilde R_\varepsilon(s)}{\varepsilon}\,ds\,d\xi - \int_t^{t+a}\int_0^\xi\frac{\hat R_\varepsilon(s)}{\varepsilon^2}\,ds\,d\xi
\end{aligned} \tag{6.17}$$

for every $t \in [0,T]$.

Proof. Owing to Proposition 6.4 we can apply Lemma 6.5, with

$$\ell(t) := \frac{\mathcal{W}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{H}_\varepsilon(t)}{\varepsilon^3},$$

and with

$$\begin{aligned}
m(t) := &-\frac{\mathcal{K}_\varepsilon(t)}{\varepsilon^2} + \rho\int_\Omega \dot u^\varepsilon(t)\cdot\dot w(t)\,dx - \frac{\mathcal{A}^2\tilde R_\varepsilon(t)}{\varepsilon^2} - \frac{\mathcal{A}\hat R_\varepsilon(t)}{\varepsilon^2} - \frac{\mathcal{A}\tilde R_\varepsilon(t)}{\varepsilon}\\
&+ \frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} - \rho\int_\Omega u^1\cdot\dot w(0)\,dx + \frac{\mathcal{A}^2\tilde R_\varepsilon(0)}{\varepsilon^2} + \frac{\mathcal{A}\hat R_\varepsilon(0)}{\varepsilon^2} + \frac{\mathcal{A}\tilde R_\varepsilon(0)}{\varepsilon}\\
&- \rho\int_0^t\int_\Omega \dot u^\varepsilon(s)\cdot\ddot w(s)\,dx\,ds + \int_0^t\frac{\tilde R_\varepsilon(s)}{\varepsilon}\,ds - \int_0^t\frac{\hat R_\varepsilon(s)}{\varepsilon^2}\,ds
\end{aligned}$$

for a.e. $t \in [0,T]$. The thesis follows by the definitions of $\mathcal{W}_\varepsilon$, $\mathcal{H}_\varepsilon$, and $\mathcal{K}_\varepsilon$. ∎

We conclude this section by showing a further characterization of ε-minimizers.

PROPOSITION 6.7 (weak energy equality). *Let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε.*

Then,

(6.18)

$$
\begin{aligned}
&\int_0^T \dot\varphi(t)\Big[\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx + 2\varepsilon\rho\int_0^t\int_\Omega |\ddot u^\varepsilon(s)|^2\,dx\,ds + \frac{\rho}{2}\int_\Omega |\dot u^\varepsilon(t)|^2\,dx \\
&+ D_{\mathcal{H}}(p^\varepsilon;0,t)\Big]\,dt = \int_0^T \dot\varphi(t)\int_0^t\int_\Omega \mathbb{C}e^\varepsilon(s) : E\dot w(s)\,dx\,ds\,dt \\
&- \frac{3\varepsilon^2\rho}{2}\int_0^T \ddot\varphi(t)\int_0^t\int_\Omega |\ddot u^\varepsilon(s)|^2\,dx\,ds\,dt - \varepsilon^2\rho\int_0^T \ddot\varphi(t)\int_\Omega \ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt \\
&+ \rho\int_0^T\int_\Omega \dot u^\varepsilon(t)\cdot\partial_t[\dot w(t)(\varphi(t)+2\varepsilon\dot\varphi(t))]\,dx\,dt - \varepsilon^2\rho\int_0^T\int_\Omega \ddot u^\varepsilon(t)\cdot\ddot w(t)\varphi(t)\,dx\,dt \\
&- \varepsilon\rho\int_0^T\int_\Omega \ddot\varphi(t)|\dot u^\varepsilon(t)|^2\,dx\,dt + 2\varepsilon\rho\int_0^T\int_\Omega \dot u^\varepsilon(t)\cdot\partial_t[\ddot w(t)(\varphi(t)+\varepsilon\dot\varphi(t))]\,dx\,dt
\end{aligned}
$$

for every $\varphi \in C_c^\infty(0,T)$.

This last result of this section relies on the argument developed in [38, Proposition 4.1], which consists of comparing the energy associated to a minimizer $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ of G_ε with that of a suitably rescaled triple $(\tilde u^\varepsilon, \tilde e^\varepsilon, \tilde p^\varepsilon)$, obtained by composing $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ with a diffeomorphic reparametrization of $[0,T]$. We postpone the proof of Proposition 6.7 to Appendix A.

7. Proof of Theorem 2.3. For the reader's convenience, we subdivide the proof into four steps. In Step 1, we deduce some first compactness properties for sequences of minimizers of G_ε satisfying the uniform energy estimate (5.42). In Step 2, we show that the limit triples identified in Step 1 satisfy conditions (c1) and (c2). Step 3 relies on the inequalities at level $\varepsilon > 0$ proven in section 6, and it is devoted to the proof of the energy inequality (c3). Finally, in Step 4 we prove that the limit triples satisfy the first-order initial condition $\dot u(0) = u^1$.

Step 1. Having established the uniform estimate (5.42), we are now ready to prove Theorem 2.3. For every $\varepsilon > 0$, let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε satisfying (5.42). Since $p^\varepsilon(0) = p^0$ for every $\varepsilon > 0$, by a generalization of Helly's theorem [11, Theorem 7.2] there exists $p \in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ such that

$$p^\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D) \quad \text{for every } t\in[0,T], \tag{7.1}$$

$$D_{\mathcal{H}}(p;0,T) \le \liminf_{\varepsilon\to0} D_{\mathcal{H}}(p^\varepsilon;0,T). \tag{7.2}$$

In addition, (5.42) yields the existence of maps $u \in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ and $e \in L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}))$ such that, up to subsequences,

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{7.3}$$

$$e^\varepsilon \rightharpoonup e \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})). \tag{7.4}$$

In particular, by (7.3) and the embedding of $W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ into $C_w([0,T];L^2(\Omega;\mathbb{R}^3))$, there holds

$$u^\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3) \quad \text{for every } t\in[0,T] \tag{7.5}$$

and $u(0) = u^0$. In view of (7.1), (7.4), and (7.5), we obtain that

$$e^\varepsilon(t) \rightharpoonup e(t) \quad \text{weakly in } \mathcal{D}'(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \quad \text{for every } t\in[0,T]. \tag{7.6}$$

By (7.1), (7.5), (7.6), and Fatou's lemma for a.e. $t \in [0,T]$, there exists a t-dependent subsequence $\{\varepsilon_t\}$ such that

$$e^{\varepsilon_t}(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}), \tag{7.7}$$

$$u^{\varepsilon_t}(t) \rightharpoonup^* u(t) \quad \text{weakly* in } BD(\Omega). \tag{7.8}$$

The fact that p satisfies the Dirichlet condition on Γ_0 for a.e. $t \in [0,T]$ follows by arguing as in [11, Lemma 2.1].

Step 2. Let $v \in C^\infty_c((0,T)\times\Omega;\mathbb{R}^3)$. For $\lambda > 0$, we have that

$$\Big(u^\varepsilon + \lambda \exp\Big(\frac{t}{\varepsilon}\Big)v,\, e^\varepsilon + \lambda \exp\Big(\frac{t}{\varepsilon}\Big)Ev,\, p^\varepsilon\Big) \in \mathcal{V},$$

and thus by the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$,

$$\frac{1}{\lambda}\left(G_\varepsilon\left(u^\varepsilon + \lambda\exp\left(\frac{t}{\varepsilon}\right)v,\, e^\varepsilon + \lambda\exp\left(\frac{t}{\varepsilon}\right)Ev,\, p^\varepsilon\right) - G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon)\right) \geq 0.$$

By the arbitrariness of λ, considering the limsup of the inequality above as $\lambda \to 0$, we deduce

$$\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon\cdot(v + 2\varepsilon\dot{v} + \varepsilon^2\ddot{v})\,dx\,dt + \int_0^T\int_\Omega \mathbb{C}e^\varepsilon : Ev\,dx\,dt \geq 0$$

for every $v \in C^\infty_c((0,T)\times\Omega;\mathbb{R}^3)$. Analogously, by considering variations of the form

$$\Big(u^\varepsilon - \lambda \exp\Big(\frac{t}{\varepsilon}\Big)v,\, e^\varepsilon - \lambda \exp\Big(\frac{t}{\varepsilon}\Big)Ev,\, p^\varepsilon\Big),$$

for $\lambda > 0$ and $v \in C^\infty_c((0,T)\times\Omega;\mathbb{R}^3)$, we obtain

$$\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon\cdot(v + 2\varepsilon\dot{v} + \varepsilon^2\ddot{v})\,dx\,dt + \int_0^T\int_\Omega \mathbb{C}e^\varepsilon : Ev\,dx\,dt = 0 \tag{7.9}$$

for every $v \in C^\infty_c((0,T)\times\Omega;\mathbb{R}^3)$. Integrating by parts with respect to time, (7.3) and (7.4) yield

$$-\rho\int_0^T\int_\Omega \dot{u}\cdot\dot{v}\,dx\,dt + \int_0^T\int_\Omega \mathbb{C}e : Ev\,dx\,dt = 0$$

for every $v \in C^\infty_c((0,T)\times\Omega;\mathbb{R}^3)$, that is,

$$\rho\ddot{u} - \operatorname{div}\mathbb{C}e = 0 \tag{7.10}$$

in the sense of distributions. Since the same procedure applies to every $v \in C^\infty_c(0,T;C^\infty(\bar{\Omega};\mathbb{R}^3))$ with $v = 0$ on Γ_0 for every $t \in [0,T]$, we also obtain

$$\mathbb{C}e\nu = 0 \quad \text{on } \partial\Omega\setminus\Gamma_0. \tag{7.11}$$

Let now $q \in C^\infty_c(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_D))$, $\lambda > 0$, and consider the test triple

$$\Big(u^\varepsilon,\, e^\varepsilon - \lambda\exp\Big(\frac{t}{\varepsilon}\Big)q, p^\varepsilon + \lambda\exp\Big(\frac{t}{\varepsilon}\Big)q\Big).$$

On the one hand, by the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$,

$$\frac{1}{\lambda}\left(G_\varepsilon\left(u^\varepsilon,\, e^\varepsilon - \lambda\exp\left(\frac{t}{\varepsilon}\right)q, p^\varepsilon + \lambda\exp\Big(\frac{t}{\varepsilon}\Big)q\right) - G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon)\right) \geq 0. \tag{7.12}$$

On the other hand,

$$\frac{1}{\lambda}\left(D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^{\varepsilon} + \lambda \exp(\cdot/\varepsilon) q; 0, T) - D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^{\varepsilon}; 0, T)\right)$$
$$\leq D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); \exp(\cdot/\varepsilon) q; 0, T),$$

and by the in-time regularity of q,

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); \exp(\cdot/\varepsilon) q; 0, T) = \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}\Big(\frac{1}{\varepsilon}\exp\Big(\frac{t}{\varepsilon}\Big) q(t) + \exp\Big(\frac{t}{\varepsilon}\Big)\dot{q}(t)\Big)$$
$$\leq \frac{1}{\varepsilon}\int_0^T \mathcal{H}(q(t))\,dt + \int_0^T \mathcal{H}(\dot{q}(t))\,dt.$$

Thus, (7.12) can be rewritten as

$$-\int_0^T \int_\Omega \mathbb{C} e^{\varepsilon} : q\,dx\,dt + \int_0^T \mathcal{H}(q(t))\,dt + \varepsilon \int_0^T \mathcal{H}(\dot{q}(t))\,dt \geq 0$$

for every $q \in C_c^\infty(0, T; L^2(\Omega; \mathbb{M}^{3\times 3}_D))$, and by (7.4),

$$\int_0^T \int_\Omega \mathbb{C} e : q\,dx\,dt \leq \int_0^T \mathcal{H}(q(t))\,dt$$

for every $q \in C_c^\infty(0, T; L^2(\Omega; \mathbb{M}^{3\times 3}_D))$. By approximation, the previous inequality holds in particular by choosing $q = M\chi_I\chi_B$ with $M \in \mathbb{M}^{3\times 3}_D$, I and B Borel subsets of $(0, T)$ and $\Omega \cup \Gamma_0$, respectively. Hence, we deduce that

$$(\mathbb{C} e(t))_D \in \partial H(0) \tag{7.13}$$

for a.e. $t \in [0, T]$ and $x \in \Omega$.

Step 3. It remains to show that the limit triple satisfies the energy inequality (c3). We first fix $a > 0$ and $\delta \in (0, 1)$, and we argue by passing to the limit as $\varepsilon \to 0$ in (6.17). Since

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_0^{\delta a} s \exp\Big(-\frac{s}{\varepsilon}\Big)\,ds = 1,$$

by (7.2), (7.3), and (7.4), we have

(7.14)

$$\int_{t+\delta a}^{t+a}\Big(\int_\Omega Q(e(s))\,dx + D_{\mathcal{H}}(p; 0, s)\Big)\,ds + \frac{\rho}{2}\int_t^{t+a}\int_\Omega |\dot{u}(s)|^2\,dx\,ds$$
$$-\rho\int_t^{t+a}\int_\Omega \dot{u}(s)\cdot\dot{w}(s)\,dx\,ds \leq -\rho a\int_\Omega u^1\cdot\dot{w}(0)\,dx$$
$$-\rho\int_t^{t+a}\int_0^\xi\int_\Omega \dot{u}(s)\cdot\ddot{w}(s)\,dx\,ds\,d\xi$$
$$+\limsup_{\varepsilon\to 0}\Big\{-\int_t^{t+a}\Big(\frac{\mathcal{A}^2\tilde{R}_\varepsilon(s)}{\varepsilon^2} + \frac{\mathcal{A}\hat{R}_\varepsilon(s)}{\varepsilon^2} + \frac{\mathcal{A}\tilde{R}_\varepsilon(s)}{\varepsilon}\Big)\,ds + \frac{\mathcal{E}_\varepsilon(0)a}{\varepsilon^2} + a\frac{\mathcal{A}^2\tilde{R}_\varepsilon(0)}{\varepsilon^2}$$
$$+ a\frac{\mathcal{A}\hat{R}_\varepsilon(0)}{\varepsilon^2} + a\frac{\mathcal{A}\tilde{R}_\varepsilon(0)}{\varepsilon} + \int_t^{t+a}\int_0^\xi \frac{\tilde{R}_\varepsilon(s)}{\varepsilon}\,ds\,d\xi - \int_t^{t+a}\int_0^\xi \frac{\hat{R}_\varepsilon(s)}{\varepsilon^2}\,ds\,d\xi\Big\},$$

where $\tilde{R}_\varepsilon$ and $\hat{R}_\varepsilon$ are the quantities defined in (6.11) and (6.12), respectively. By (5.42) there holds

(7.15)
$$\Big|\int_t^{t+a}\int_0^\xi \frac{\hat{R}_\varepsilon(s)}{\varepsilon^2}\,ds\,d\xi\Big| \leq Ca\varepsilon\|\ddot{u}^\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\|\ddot{w}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leq Ca\sqrt{\varepsilon}$$

and, analogously,

$$\Big|\int_t^{t+a}\int_0^\xi \frac{\tilde{R}_\varepsilon(s)}{\varepsilon}\,ds\,d\xi + \int_t^{t+a}\int_0^\xi\int_\Omega \mathbb{C}e^\varepsilon(s):E\dot{w}(s)\,dx\,ds\,d\xi\Big| \leq Ca\varepsilon\sqrt{\varepsilon} \tag{7.16}$$

for every $t\in[0,T]$. Thus, by (7.4)

(7.17)
$$\lim_{\varepsilon\to 0}\int_t^{t+a}\int_0^\xi \frac{\tilde{R}_\varepsilon(s)}{\varepsilon}ds\,d\xi - \int_t^{t+a}\int_0^\xi \frac{\hat{R}_\varepsilon(s)}{\varepsilon^2}ds\,d\xi = -\int_t^{t+a}\int_0^\xi\int_\Omega \mathbb{C}e(s):E\dot{w}(s)dx\,ds\,d\xi.$$

Arguing as in (7.16) and using again (5.42), we deduce

$$\begin{aligned}\Big|a\frac{\mathcal{A}^2\tilde{R}^\varepsilon(0)}{\varepsilon^2}\Big| &\leq Ca\sqrt{\varepsilon} + a\Big|\frac{1}{\varepsilon}\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)s\int_\Omega \mathbb{C}e^\varepsilon(s):E\dot{w}(s)\,dx\,ds\Big| \\ &\leq Ca\sqrt{\varepsilon} + C\frac{a}{\varepsilon}\Big\|\exp\Big(-\frac{s}{\varepsilon}\Big)s\Big\|_{L^2(0,T)} \leq Ca\sqrt{\varepsilon}\end{aligned} \tag{7.18}$$

and

$$\begin{aligned}\Big|a\frac{\mathcal{A}\tilde{R}^\varepsilon(0)}{\varepsilon}\Big| &\leq Ca\varepsilon\sqrt{\varepsilon} + a\Big|\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)\int_\Omega \mathbb{C}e^\varepsilon(s):E\dot{w}(s)\,dx\,ds\Big| \\ &\leq Ca\varepsilon\sqrt{\varepsilon} + Ca\Big\|\exp\Big(-\frac{s}{\varepsilon}\Big)\Big\|_{L^2(0,T)} \leq Ca\sqrt{\varepsilon}.\end{aligned} \tag{7.19}$$

The same argument yields

$$\Big|\frac{\mathcal{A}^2\tilde{R}^\varepsilon(t)}{\varepsilon^2}\Big| + \Big|\frac{\mathcal{A}\tilde{R}^\varepsilon(t)}{\varepsilon}\Big| \leq C\sqrt{\varepsilon} \quad \text{for every } t\in[0,T]. \tag{7.20}$$

Finally, estimates analogous to (7.15) imply

$$\Big|\frac{\mathcal{A}\hat{R}^\varepsilon(t)}{\varepsilon^2}\Big| \leq C\sqrt{\varepsilon} \quad \text{for every } t\in[0,T]. \tag{7.21}$$

By combining (7.14) with (7.17), (7.20), and (7.21) we conclude that

$$\begin{aligned}&\int_{t+\delta a}^{t+a}\Big(\int_\Omega Q(e(s))\,dx + D_{\mathcal{H}}(p;0,s)\Big)\,ds + \frac{\rho}{2}\int_t^{t+a}\int_\Omega |\dot{u}(s)|^2\,dx\,ds \\ &\quad - \rho\int_t^{t+a}\int_\Omega \dot{u}(s)\cdot\dot{w}(s)\,dx\,ds \leq -\rho a\int_\Omega u^1\cdot\dot{w}(0)\,dx \\ &\quad - \rho\int_t^{t+a}\int_0^\xi\int_\Omega \dot{u}(s)\cdot\ddot{w}(s)\,dx\,ds\,d\xi \\ &\quad - \int_t^{t+a}\int_0^\xi\int_\Omega \mathbb{C}e(s):E\dot{w}(s)\,dx\,ds\,d\xi + a\limsup_{\varepsilon\to 0}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2}\end{aligned} \tag{7.22}$$

for every $a > 0$ and $\delta \in (0,1)$. In particular, letting $\delta \to 0$, dividing by a, and letting $a \to 0$, by the Lebesgue differentiation theorem we deduce the inequality

$$\begin{aligned}
&\int_\Omega Q(e(t))\,dx + D_{\mathcal{H}}(p;0,t) + \frac{\rho}{2}\int_\Omega |\dot u(t)|^2\,dx - \rho\int_\Omega \dot u(t)\cdot\dot w(t)\,dx \\
&\quad\le -\rho\int_\Omega u^1\cdot\dot w(0)\,dx - \rho\int_0^t\int_\Omega \dot u(s)\cdot\ddot w(s)\,dx\,ds \\
&\quad - \int_0^t\int_\Omega \mathbb{C}e(s) : E\dot w(s)\,dx\,ds + \limsup_{\varepsilon\to 0}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2}
\end{aligned} \tag{7.23}$$

for a.e. $t \in [0,T]$. In order to complete the proof of the energy inequality (c3) it remains to estimate from above the quantity $\limsup_{\varepsilon\to 0}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2}$. To this end, we observe that, by the definition of the approximate energy, by Corollary 6.2, and by (7.20) and (7.21) there holds

$$\begin{aligned}
\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} &= \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{\mathcal{A}^2\mathcal{W}_\varepsilon(0)}{\varepsilon^2} + \frac{\mathcal{A}^2\mathcal{H}_\varepsilon(0)}{\varepsilon^3} \le \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{\mathcal{A}^2\mathcal{L}_\varepsilon(0)}{\varepsilon^2} \\
&\le \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{\mathcal{A}\mathcal{L}_\varepsilon(0)}{\varepsilon} - \frac{\mathcal{A}^2 R_\varepsilon(0)}{\varepsilon^2} - \frac{\mathcal{A}^2\tilde R_\varepsilon(0)}{\varepsilon^2} - \frac{\mathcal{A}\hat R_\varepsilon(0)}{\varepsilon^2} \\
&\le \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{\mathcal{A}\mathcal{L}_\varepsilon(0)}{\varepsilon} - \frac{\mathcal{A}^2 R_\varepsilon(0)}{\varepsilon^2} + C\sqrt{\varepsilon} \\
&= \frac{\rho}{2}\int_\Omega |u^1|^2\,dx \\
&+ \frac{1}{\varepsilon}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot u^\varepsilon(t)|^2\,dx + \int_\Omega Q(e^\varepsilon(t))\,dx + D_{\mathcal{H}}(p^\varepsilon;0,t)\Big)\,dt \\
&- \frac{\mathcal{A}^2 R_\varepsilon(0)}{\varepsilon^2} + C\sqrt{\varepsilon}.
\end{aligned}$$

By (5.42),

$$\begin{aligned}
\Big|\frac{\mathcal{A}^2 R_\varepsilon(0)}{\varepsilon^2}\Big| &= \frac{1}{\varepsilon}\Big|\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)s\int_\Omega \mathbb{C}e^\varepsilon(s) : \big(e^1 - E\dot w(0)\big)\,dx\,ds\Big| \\
&\le \frac{C}{\varepsilon}\Big\|\exp\Big(-\frac{s}{\varepsilon}\Big)s\Big\|_{L^2(0,T)} \le C\sqrt{\varepsilon}.
\end{aligned}$$

In view of [21, Theorem 4.5], we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,t)\,dt = -\exp\Big(-\frac{T}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,T) \\
&+ PMS\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\,dD_{\mathcal{H}}(p^\varepsilon;0,t) \\
&\quad\le D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,T).
\end{aligned}$$

Thus, we obtain

$$\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} \le \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon} + \frac{1}{\varepsilon}G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon).$$

By the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$, and since the triple

$$t \to (u^0 + tu^1 + w(t) - w(0) - t\dot w(0), e^0 + te^1 + Ew(t) - Ew(0) - tE\dot w(0), p^0)$$

belongs to $\mathcal{V}$, we deduce the upper bound

$$\begin{aligned}
\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} &\le \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon} + \frac{1}{\varepsilon}G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon)\\
&\le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\frac{\varepsilon\rho}{2}\int_\Omega |\ddot{w}(t)|^2\,dx\,dt\\
&+ \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\frac{1}{\varepsilon}\int_\Omega Q(e^0 + te^1 + Ew(t) - Ew(0) - tE\dot{w}(0))\,dx\,dt\\
&+ \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon}\\
&\le \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon} + \frac{1}{\varepsilon}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega (Q(e^0) + Ct)\,dx\,dt,
\end{aligned}$$

which in turn implies

$$\limsup_{\varepsilon\to 0}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} \le \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \int_\Omega Q(e^0)\,dx. \tag{7.24}$$

By combining (7.23) with (7.24) we have

$$\begin{aligned}
&\int_\Omega Q(e(t))\,dx + D_{\mathcal{H}}(p;0,t) + \frac{\rho}{2}\int_\Omega |\dot{u}(t)|^2\,dx - \rho\int_\Omega \dot{u}(t)\cdot\dot{w}(t)\,dx\\
&\quad\le \int_\Omega Q(e^0)\,dx + \frac{\rho}{2}\int_\Omega |u^1|^2\,dx - \rho\int_\Omega u^1\cdot\dot{w}(0)\,dx\\
&\quad- \int_0^t\int_\Omega \dot{u}(s)\cdot\ddot{w}(s)\,dx\,ds - \int_0^t\int_\Omega \mathbb{C}e(s) : E\dot{w}(s)\,dx\,ds
\end{aligned}$$

for a.e. $t\in[0,T]$. This completes the proof of condition (c3).

Step 4. In order to show that u satisfies the first-order initial condition

$$\dot{u}(0) = u^1 \quad \text{in } W^{-1,2}(\Omega;\mathbb{R}^3), \tag{7.25}$$

we argue as in [46, Theorem 4.2], and we claim that there exists a sequence $\varepsilon_n \to 0$ such that

$$\dot{u}^{\varepsilon_n}(t) \rightharpoonup \dot{u}(t) \quad \text{weakly in } W^{-1,2}(\Omega;\mathbb{R}^3) \tag{7.26}$$

for every $t\in[0,T]$.

To prove claim (7.26), we first observe that the minimality of the triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ yields the Euler–Lagrange equation

$$\varepsilon^2\rho\int_0^T\int_\Omega \exp\Big(-\frac{t}{\varepsilon}\Big)\ddot{u}^\varepsilon(t)\cdot\ddot{\phi}(t)\,dx\,dt + \int_0^T\int_\Omega \exp\Big(-\frac{t}{\varepsilon}\Big)\mathbb{C}e^\varepsilon(t) : E\phi(t)\,dx\,dt = 0 \tag{7.27}$$

for every $\phi\in W^{2,2}(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))$ satisfying $\phi(0)=\phi'(0)=0$. Let $\varepsilon_n\to 0$, and let S be a countable dense subset of $W^{1,2}_0(\Omega;\mathbb{R}^3)$. Let $I\subset(0,T)$ be defined as the set of points $t_0\in(0,T)$ such that

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\exp\Big(-\frac{t}{\varepsilon_n}\Big)\int_\Omega \ddot{u}^{\varepsilon_n}(t)\cdot h(x)\,dx\,dt = \exp\Big(-\frac{t_0}{\varepsilon_n}\Big)\int_\Omega \ddot{u}^{\varepsilon_n}(t_0)\cdot h(x)\,dx\,dt \tag{7.28}$$

for every $n \in \mathbb{N}$ and for every $h \in S$. Note that by the Lebesgue differentiation theorem the set $[0,T] \setminus I$ is negligible.

Fix $t_0 \in I$, and let $\varphi_{\delta n} \in C^{1,1}(\mathbb{R})$ be defined as

$$\varphi_{\delta n}(t) := \begin{cases} 0 & t \leq t_0, \\ \dfrac{(t-t_0)^2}{\delta \varepsilon_n^2} & t \in (t_0, t_0+\delta), \\ 2\dfrac{(t-t_0)}{\varepsilon_n^2} - \dfrac{\delta}{\varepsilon_n^2} & t \geq t_0 + \delta. \end{cases}$$

We observe that

$$\varphi''_{\delta n}(t) = \frac{2}{\delta \varepsilon_n^2} \chi_{(t_0, t_0+\delta)}(t),$$

where $\chi_{(t_0,t_0+\delta)}$ is the characteristic function of $(t_0, t_0+\delta)$. In addition,

$$|\varphi_{\delta n}(t)| \leq \frac{2}{\varepsilon_n^2}(t-t_0)^+ \quad \text{and} \quad \varphi_{\delta n}(t) \to \frac{2}{\varepsilon_n^2}(t-t_0)^+$$

as $\delta \to 0$ for almost every $t \in (0,T)$. Choosing $\phi(t,x) = \varphi_{\delta n}(t)h(x)$, with $h \in S$, by (7.27) we obtain

$$\begin{aligned} &\frac{2\rho}{\delta} \int_{t_0}^{t_0+\delta} \int_\Omega \exp\Big(-\frac{t}{\varepsilon_n}\Big) \ddot{u}^{\varepsilon_n}(t) \cdot h(x)\,dx\,dt \\ &\quad + \int_{t_0}^{T} \int_\Omega \exp\Big(-\frac{t}{\varepsilon_n}\Big) \varphi_{\delta n}(t) \mathbb{C} e^{\varepsilon_n}(t) : Eh(x)\,dx\,dt = 0. \end{aligned}$$

Letting $\delta \to 0$, (7.28) and the dominated convergence theorem yield

$$\rho \int_\Omega \ddot{u}^{\varepsilon_n}(t_0) \cdot h(x)\,dx + \frac{1}{\varepsilon_n^2} \int_{t_0}^{T} \int_\Omega \exp\Big(\frac{t_0 - t}{\varepsilon_n}\Big)(t-t_0) \mathbb{C} e^{\varepsilon_n}(t) : Eh(x)\,dx\,dt = 0.$$

By (5.42), there holds

$$\begin{aligned} &\Big| \frac{1}{\varepsilon_n} \int_{t_0}^{T} \int_\Omega \exp\Big(\frac{t_0 - t}{\varepsilon_n}\Big)(t-t_0) \mathbb{C} e^{\varepsilon_n}(t) : Eh(x)\,dx\,dt \Big| \\ &\quad \leq \frac{C}{\varepsilon_n} \|e^{\varepsilon_n}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}))} \|Eh\|_{L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})} \Big(\int_{t_0}^{T} \exp\Big(\frac{2(t_0-t)}{\varepsilon_n}\Big)(t-t_0)^2\,dt \Big)^{\frac{1}{2}} \\ &\quad \leq C\sqrt{\varepsilon_n} \|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)} \Big(\int_0^{\frac{(T-t_0)}{\varepsilon_n}} t^2 \exp(-2t)\,dt \Big)^{\frac{1}{2}} \leq C \|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)}, \end{aligned}$$

where in the last inequality we used the fact that for t large there holds $t^2 \exp(-2t) \leq 1$.

Thus,

$$\rho \Big| \int_\Omega \ddot{u}^{\varepsilon_n}(t_0) h(x)\,dx \Big| \leq C \|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)},$$

where the constant C is independent of ε_n and t_0. In particular, we obtain the uniform estimate

$$\rho \|\ddot{u}^{\varepsilon_n}\|_{L^\infty(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))} \leq C. \tag{7.29}$$

By combining (5.42), (7.3), and (7.29), we deduce that

$$\|\dot{u}^{\varepsilon_n}\|_{W^{1,2}(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))} \leq C.$$

In particular, as $W^{1,2}(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))$ embeds into $C_w([0,T];W^{-1,2}(\Omega;\mathbb{R}^3))$, up to the extraction of a (non–relabeled) subsequence we obtain claim (7.26), which in turn yields $\dot{u}(0) = u^1$.

As pointed out in [36, Remark 3.2], arguing as in [7] one obtains that (c3) holds with an equality. The additional regularity in time of the solution follows by adapting the argument in [36, Proof of Theorem 3.1]. The thesis follows now by the uniqueness of solutions for the dynamic plasticity problem (see Theorem 2.2).

We point out that the assertion of Theorem 2.3 still holds if we generalize the minimum problem (2.14) by imposing ε-dependent initial data satisfying suitable compatibility assumptions. To be precise, for every ε, define the set

$$\begin{aligned}\mathcal{V}_\varepsilon := \{&(u,e,p) \in W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^1(0,T;BD(\Omega))\\ &\times L^2((0,T)\times\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \times BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)) :\\ &(u(t),e(t),p(t)) \in \mathscr{A}(w(t)) \text{ for a.e. } t\in[0,T],\\ &Eu(t) = e(t)+p(t) \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \text{ for every } t\in[0,T],\\ &u(0)=u^0_\varepsilon,\ \dot{u}(0)=u^1_\varepsilon,\ e(0)=e^0_\varepsilon,\ p(0)=p^0_\varepsilon\},\end{aligned}$$

with $(u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon) \in \mathscr{A}(w(0))$, and $u^1_\varepsilon \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that $u^1_\varepsilon = \dot{w}(0)$ on Γ_0. Assuming that the initial data are *well prepared*, namely,

$$\begin{aligned}&u^0_\varepsilon \rightharpoonup^* u^0 \quad \text{weakly* in } BD(\Omega),\\ &e^0_\varepsilon \rightharpoonup e^0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}),\\ &p^0_\varepsilon \rightharpoonup^* p^0 \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D),\\ &u^1_\varepsilon \to u^1 \quad \text{strongly in } W^{-1,2}(\Omega;\mathbb{R}^3),\end{aligned}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to0}\Big[\int_\Omega Q(e^0_\varepsilon)\,dx + \frac{\rho}{2}\int_\Omega |u^1_\varepsilon|^2\,dx - \rho\int_\Omega u^1_\varepsilon\cdot\dot{w}(0)\,dx\Big]\\ &\quad= \int_\Omega Q(e^0)\,dx + \frac{\rho}{2}\int_\Omega |u^1|^2\,dx - \rho\int_\Omega u^1\cdot\dot{w}(0)\,dx,\end{aligned}$$

one can again prove there exists a sequence of triples $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$, with $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \subset \mathcal{V}_\varepsilon$ for every ε such that

$$I_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) = \min_{(v,f,q)\in\mathcal{V}_\varepsilon} I_\varepsilon(v,f,q),$$

such that $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$ converges to the solution (u,e,p) of dynamic perfect plasticity, namely (c1), (c2), and (c3), in the sense of Theorem 2.3.

Appendix A. This appendix is devoted to the proof of Proposition 6.7. We start with a somewhat technical lemma.

LEMMA A.1. *Let* $\mu \in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$, *and let* $\varphi \in C^\infty_c(0,T)$. *Then,*

$$\hat{D}_{\mathcal{H}}(\varphi;\mu;0,T) = -\int_0^T \dot{\varphi}(t) D_{\mathcal{H}}(\mu;0,t)\,dt.$$

Proof. In view of [21, Theorem 4.5], there holds (see also [21, Theorem 2.15])

$$\hat{D}_{\mathcal{H}}(\varphi;\mu;0,T) = -PMS\int_0^T D_{\mathcal{H}}(\mu;0,t)\,d\varphi$$
$$= -RS\int_0^T D_{\mathcal{H}}(\mu;0,t)\,d\varphi = -\int_0^T D_{\mathcal{H}}(\mu;0,t)\dot{\varphi}(t)\,dt,$$

where $PMS\int$ and $RS\int$ denote the Pollard–Moore–Stieltjes and the Riemann–Stieltjes integrals, respectively (see [21, section 4]), and where the last equality is due to the regularity of φ and to classical properties of the Riemann–Stieltjes integral. ∎

We are now in a position to prove Proposition 6.7.

Proof of Proposition 6.7. We argue as in [38, Proposition 4.1] by comparing the energy associated to $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ with that of a rescaled triple $(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon)$. Consider an increasing diffeomorphism

$$\beta_\delta : [0,T] \to [0,T]$$

such that $\beta_\delta \in C^2([0,T])$, $\beta_\delta(0)=0$, $\beta_\delta(T)=T$, and $\dot{\beta}_\delta(0)=1$. We set

$$\tilde{u}^\varepsilon(s) := u^\varepsilon(\beta_\delta^{-1}(s)) - w(\beta_\delta^{-1}(s)) + w(s), \quad \tilde{p}^\varepsilon(s) := p^\varepsilon(\beta_\delta^{-1}(s)),$$

for every $s \in [0,T]$, and

$$\tilde{e}^\varepsilon(s) := e^\varepsilon(\beta_\delta^{-1}(s)) - Ew(\beta_\delta^{-1}(s)) + Ew(s)$$

for every $s \in [0,T]$. It is easy to check that $(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) \in \mathcal{V}$. Hence, by the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ there holds

$$G_\varepsilon(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) - G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) \geq 0. \tag{A.1}$$

Using the definition of $(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon)$, we can rewrite its associated energy as

$$\begin{aligned}G_\varepsilon(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) &= \int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot{\beta}_\delta(t)\int_\Omega Q\big(e^\varepsilon(t) - Ew(t) + Ew(\beta_\delta(t))\big)\,dx\,dt\\
&+ \frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot{\beta}_\delta(t)\Big(\int_\Omega\Big|\frac{\ddot{u}^\varepsilon(t)}{(\dot{\beta}_\delta(t))^2} - \frac{\dot{u}^\varepsilon(t)\ddot{\beta}_\delta(t)}{(\dot{\beta}_\delta(t))^3} - \frac{\ddot{w}(t)}{(\dot{\beta}_\delta(t))^2}\\
&+ \frac{\dot{w}(t)\ddot{\beta}_\delta(t)}{(\dot{\beta}_\delta(t))^3} + \ddot{w}(\beta_\delta(t))\Big|^2\,dx\Big)\,dt + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);\tilde{p}^\varepsilon;0,T).\end{aligned}$$

Along the lines of [38, Proposition 4.1], we fix $\varphi \in C_c^\infty(0,T)$. Let $\delta \in (0,1)$ be such that $\varepsilon\delta\dot{\varphi}(t) < \exp(-t/\varepsilon)$ for every $t \in [0,T]$, and define β_δ as the solution to

$$\exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big) - \exp\Big(-\frac{t}{\varepsilon}\Big) = \delta\varphi(t). \tag{A.2}$$

We immediately see that $\beta_\delta(0)=0$ and $\beta_\delta(T)=T$. In addition, deriving (A.2) with respect to time, we have

$$\dot{\beta}_\delta(t) = \exp\Big(\frac{\beta_\delta(t)}{\varepsilon}\Big)\Big(\exp\Big(-\frac{t}{\varepsilon}\Big) - \varepsilon\delta\dot{\varphi}(t)\Big) \tag{A.3}$$

for every $t \in [0,T]$, yielding $\dot{\beta}_\delta(t) > 0$ for every $t \in (0,T)$ and $\dot{\beta}_\delta(0)=1$. As already observed in [38, Proposition 4.1],

$$\beta_\delta(t) = t - \varepsilon\delta\varphi(t)\exp\Big(\frac{t}{\varepsilon}\Big) + \mathrm{O}(\delta^2). \tag{A.4}$$

In addition, by (A.2) and (A.3),

$$\dot\beta_\delta(t) = 1 - \delta(\varphi(t) + \varepsilon\dot\varphi(t)) \exp\Big(\frac{t}{\varepsilon}\Big) + \mathrm{O}(\delta^2), \tag{A.5}$$

and by performing a further derivation in time of (A.3),

$$\ddot\beta_\delta(t) = -\delta\Big(\frac{\varphi(t)}{\varepsilon} + 2\dot\varphi(t) + \varepsilon\ddot\varphi(t)\Big) \exp\Big(\frac{t}{\varepsilon}\Big) + \mathrm{O}(\delta^2). \tag{A.6}$$

Let us first observe that

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega Q\big(e^\varepsilon(t) - Ew(t) + Ew(\beta_\delta(t))\big)\,dx\,dt\\
&\quad - \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\Big\}\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^T \Big(\exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t) - \exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\
&\quad + \int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega Q\big(Ew(t) - Ew(\beta_\delta(t))\big)\,dx\,dt\\
&\quad - \int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega \mathbb{C}e^\varepsilon(t) : \Big(Ew(t) - Ew(\beta_\delta(t))\Big)\,dx\,dt\Big\}.
\end{aligned} \tag{A.7}$$

In view of (A.2) and (A.5), and by the dominated convergence theorem, the first term in the right-hand side of (A.7) becomes

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \Big(\exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t) - \exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \Big(\Big(\delta\varphi(t) + \exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\dot\beta_\delta(t) - \exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\
&= -\varepsilon\int_0^T \dot\varphi(t)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt.
\end{aligned} \tag{A.8}$$

By the regularity of w and by (A.4) there holds

$$|Ew(t) - Ew(\beta_\delta(t))| = \Big|\int_t^{\beta_\delta(t)} E\dot w(\xi)\,d\xi\Big| \le \sqrt{\delta}\|w\|_{W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}.$$

Hence, by (A.2) and (A.5) one obtains

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega Q\big(Ew(t) - Ew(\beta_\delta(t))\big)\,dx\,dt = 0. \tag{A.9}$$

Finally, by (A.2), (A.5), and the mean value theorem we get

(A.10)
$$
\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega \mathbb{C}e^\varepsilon(t):\Big(Ew(t)-Ew(\beta_\delta(t))\Big)\,dx\,dt\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \Big(\exp\Big(-\frac{t}{\varepsilon}\Big)+\delta\varepsilon\dot\varphi(t)\Big)\int_\Omega \mathbb{C}e^\varepsilon(t):\Big(\int_t^{\beta_\delta(t)} E\dot w(\xi)\,d\xi\Big)\,dx\,dt\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \Big(\exp\Big(-\frac{t}{\varepsilon}\Big)+\delta\varepsilon\dot\varphi(t)\Big)\int_t^{\beta_\delta(t)}\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(\xi)\,dx\,d\xi\,dt\\
&=-\varepsilon\lim_{\delta\to 0}\int_0^T\varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(\xi^t)\,dx\,dt=-\varepsilon\int_0^T\varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(t)\,dx\,dt,
\end{aligned}
$$

where, in the second-to-last line, for every $t\in[0,T]$, ξ^t is an intermediate value between t and $\beta_\delta(t)$. By combining (A.7)–(A.10) we obtain

$$
\begin{aligned}
\text{(A.11)}\quad &\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega Q\big(e^\varepsilon(\beta_\delta(t))-Ew(t)+Ew(\beta_\delta(t))\big)\,dx\,dt\\
&\quad-\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\Big\}\\
&=-\varepsilon\int_0^T\dot\varphi(t)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt+\varepsilon\int_0^T\varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(t)\,dx\,dt.
\end{aligned}
$$

We proceed by performing the analogous computation for the inertial term. We seek to estimate

(A.12)
$$
\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\Big(\int_\Omega\Big|\frac{\ddot u^\varepsilon(t)}{(\dot\beta_\delta(t))^2}-\frac{\dot u^\varepsilon(t)\ddot\beta_\delta(t)}{(\dot\beta_\delta(t))^3}-\frac{\ddot w(t)}{(\dot\beta_\delta(t))^2}\\
&\quad+\frac{\dot w(t)\ddot\beta_\delta(t)}{(\dot\beta_\delta(t))^3}+\ddot w(\beta_\delta(t))\Big|^2\,dx\Big)\,dt-\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\Big\}.
\end{aligned}
$$

By (A.2) and (A.5) we have

$$
\begin{aligned}
\text{(A.13)}\quad &\lim_{\delta\to 0}\frac{1}{\delta}\frac{\varepsilon^2\rho}{2}\int_0^T\Big(\frac{1}{(\dot\beta_\delta(t))^3}\exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)-\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\\
&=\frac{3\varepsilon^3\rho}{2}\int_0^T\dot\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt+2\varepsilon^2\rho\int_0^T\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt.
\end{aligned}
$$

By (A.2), (A.5), and (A.6) there holds

(A.14)
$$
\lim_{\delta\to 0}\frac{1}{\delta}\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\int_\Omega\Big[\frac{(\ddot\beta_\delta(t))^2}{(\dot\beta_\delta(t))^5}(|\dot u^\varepsilon(t)|^2+|\dot w(t)|^2-2\dot u^\varepsilon(t)\cdot\dot w(t))\Big]\,dx\,dt=0,
$$

as well as

$$
\begin{aligned}
\text{(A.15)}\qquad & \lim_{\delta\to 0}\frac{1}{\delta}\varepsilon^2\rho\int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\int_\Omega \frac{\ddot\beta_\delta(t)}{(\dot\beta_\delta(t))^4}\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt\\
&= -\varepsilon^3\rho\int_0^T \ddot\varphi(t)\ddot u^\varepsilon(t)\cdot(\dot w(t)\\
&- \dot u^\varepsilon(t))\,dx\,dt - \varepsilon\rho\int_0^T(\varphi(t)+2\varepsilon\dot\varphi(t))\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt.
\end{aligned}
$$

To estimate the remaining term, we observe that by (A.5) and in view of the regularity of the boundary datum,

$$
\begin{aligned}
&-\frac{\ddot w(t)}{(\dot\beta_\delta(t))^2}+\ddot w(\beta_\delta(t)) = -\frac{\ddot w(t)}{(\dot\beta_\delta(t))^2}(1-(\dot\beta_\delta(t))^2)+\int_t^{\beta_\delta(t)}\ddot w(\xi)\,d\xi\\
&= -\frac{2\delta\ddot w(t)}{(\dot\beta_\delta(t))^2}(\varphi(t)+\varepsilon\dot\varphi(t))\exp\Big(\frac{t}{\varepsilon}\Big)+\int_t^{\beta_\delta(t)}\ddot w(\xi)\,d\xi+\mathrm{O}(\delta^2).
\end{aligned}
$$

By the regularity of w, by (A.4), and by Lebesgue's theorem,

$$
\begin{aligned}
&\lim_{\delta\to 0}\Big\|\frac{1}{\delta}\int_t^{\beta_\delta(t)}\ddot w(\xi)\,d\xi+\varepsilon\ddot w(t)\varphi(t)\exp\Big(\frac{t}{\varepsilon}\Big)\Big\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\\
&= \lim_{\delta\to 0}\Big\|\frac{1}{\delta}\int_t^{\beta_\delta(t)}(\ddot w(\xi)-\ddot w(t))\,d\xi\Big\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\\
&\le \lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\int_{t-\delta\varepsilon\|\varphi\|_{L^\infty(0,T)}\exp(T/\varepsilon)}^{t+\delta\varepsilon\|\varphi\|_{L^\infty(0,T)}\exp(T/\varepsilon)}\int_\Omega|\ddot w(\xi)-\ddot w(t)|^2\,dx\,d\xi\,dt = 0.
\end{aligned}
$$

Therefore, by (A.5) and (A.6),

$$
\begin{aligned}
\text{(A.16)}\qquad & \lim_{\delta\to 0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega\Big[\Big|-\frac{\ddot w(t)}{(\dot\beta_\delta(t))^2}+\ddot w(\beta_\delta(t))\Big|^2\\
&\quad +2\Big(-\frac{\ddot w(t)}{(\dot\beta_\delta(t))^2}+\ddot w(\beta_\delta(t))\Big)\cdot\frac{(\dot w(t)-\dot u^\varepsilon(t))\ddot\beta_\delta(t)}{(\dot\beta_\delta(t))^3}\Big]\,dx\,dt\Big\} = 0,
\end{aligned}
$$

and

(A.17)

$$
\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\varepsilon^2\rho\int_0^T\exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega\frac{\ddot u^\varepsilon(t)}{(\dot\beta_\delta(t))^2}\cdot\Big(-\frac{\ddot w(t)}{(\dot\beta_\delta(t))^2}+\ddot w(\beta_\delta(t))\Big)\,dx\,dt\\
&= -2\varepsilon^2\rho\int_0^T\int_\Omega \ddot u^\varepsilon(t)\cdot\ddot w(t)(\varphi(t)+\varepsilon\dot\varphi(t))\,dx\,dt\\
&\quad -\varepsilon^3\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\ddot w(t)\varphi(t)\,dx\,dt.
\end{aligned}
$$

By combining (A.12)–(A.17), we obtain

$$\begin{aligned}
\text{(A.18)}\quad & \lim_{\delta\to 0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{\beta_\delta(t)}{\varepsilon}\Big)\dot\beta_\delta(t)\int_\Omega\Big|\frac{\ddot u^\varepsilon(t)}{(\dot\beta_\delta(t))^2}-\frac{\dot u^\varepsilon(t)\ddot\beta_\delta(t)}{(\dot\beta_\delta(t))^3}-\frac{\ddot w(t)}{(\dot\beta_\delta(t))^2}\\
&+\frac{\dot w(t)\ddot\beta_\delta(t)}{(\dot\beta_\delta(t))^3}+\ddot w(\beta_\delta(t))\Big|^2\,dx\,dt-\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\Big\}\\
&=\frac{3\varepsilon^3\rho}{2}\int_0^T\dot\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt+2\varepsilon^2\rho\int_0^T\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\\
&-\varepsilon^3\rho\int_0^T\int_\Omega\ddot\varphi(t)\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt\\
&-\varepsilon\rho\int_0^T\int_\Omega(\varphi(t)+2\varepsilon\dot\varphi(t))\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt\\
&-2\varepsilon^2\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\ddot w(t)(\varphi(t)+\varepsilon\dot\varphi(t))\,dx\,dt\\
&-\varepsilon^3\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\ddot w(t)\varphi(t)\,dx\,dt.
\end{aligned}$$

To complete the proof of the ε-energy inequality it remains to estimate from above the quantity

$$\text{(A.19)}\qquad \limsup_{\delta\to 0}\frac{1}{\delta}\big(D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);\tilde p^\varepsilon;0,T)-D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,T)\big).$$

To this end, fix $\lambda>0$, and let $0\le t_0<t_1<\cdots<t_m\le T$ be such that

$$D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);\tilde p^\varepsilon;0,T)\le\sum_{i=1}^m\exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal H(\tilde p^\varepsilon(t_i)-\tilde p^\varepsilon(r_{i-1}))+\lambda.$$

For $i=0,\dots,m$, let $s_i\in[0,T]$ be such that $\beta_\delta(s_i)=t_i$. By the properties of β_δ, it follows that $0\le s_0<s_1<\cdots<s_n\le T$. In view of (A.2), we have

$$\begin{aligned}
&\sum_{i=1}^m\exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal H(\tilde p^\varepsilon(t_i)-\tilde p^\varepsilon(r_{i-1}))=\sum_{i=1}^m\exp\Big(-\frac{\beta_\delta(s_i)}{\varepsilon}\Big)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))\\
&=\sum_{i=1}^m\exp\Big(-\frac{s_i}{\varepsilon}\Big)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))\\
&+\sum_{i=1}^m\Big(\exp\Big(-\frac{\beta_\delta(s_i)}{\varepsilon}\Big)-\exp\Big(-\frac{s_i}{\varepsilon}\Big)\Big)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))\\
&\le D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,T)+\delta\sum_{i=1}^m\varphi(s_i)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1})).
\end{aligned}$$

By considering finer and finer refinements of $\{t_0,\dots,t_m\}$, in view of the definition of $\hat D_{\mathcal H}$, and by the arbitrariness of λ we conclude that

$$D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);\tilde p^\varepsilon;0,T)\le D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,T)+\delta\hat D_{\mathcal H}(\varphi;p^\varepsilon;0,T).$$

Thus, we can bound (A.19) from above as

(A.20)

$$\limsup_{\delta\to 0}\frac{1}{\delta}\big(D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);\tilde p^\varepsilon;0,T)-D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,T)\big)\le\hat D_{\mathcal H}(\varphi;p^\varepsilon;0,T),$$

where $\hat{D}_{\mathcal{H}}$ is the quantity defined in (2.8). Combining (A.1), (A.11), (A.18), (A.20), and Lemma A.1 we finally obtain the inequality

$$
\begin{aligned}
0 &\le \limsup_{\delta\to 0} \frac{1}{\varepsilon\delta}\left(G_\varepsilon(\tilde{u}^\varepsilon,\tilde{e}^\varepsilon,\tilde{p}^\varepsilon) - G(u^\varepsilon,e^\varepsilon,p^\varepsilon)\right)\\
&\le -\int_0^T \dot{\varphi}(t)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt - \int_0^T \varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot{w}(t)\,dx\,dt\\
&\quad + \frac{3\varepsilon^2\rho}{2}\int_0^T \dot{\varphi}(t)\int_\Omega |\ddot{u}^\varepsilon(t)|^2\,dx\,dt + 2\varepsilon\rho\int_0^T \varphi(t)\int_\Omega |\ddot{u}^\varepsilon(t)|^2\,dx\,dt\\
&\quad - \varepsilon^2\rho\int_0^T\int_\Omega \ddot{\varphi}(t)\ddot{u}^\varepsilon(t)\cdot(\dot{w}(t)-\dot{u}^\varepsilon(t))\,dx\,dt\\
&\quad - \rho\int_0^T\int_\Omega (\varphi(t)+2\varepsilon\dot{\varphi}(t))\ddot{u}^\varepsilon(t)\cdot(\dot{w}(t)-\dot{u}^\varepsilon(t))\,dx\,dt\\
&\quad - 2\varepsilon\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon(t)\cdot\ddot{w}(t)(\varphi(t)+\varepsilon\dot{\varphi}(t))\,dx\,dt\\
&\quad - \varepsilon^2\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon(t)\cdot\ddot{w}(t)\varphi(t)\,dx\,dt\\
&\quad - \int_0^T D_{\mathcal{H}}(p^\varepsilon;0,t)\dot{\varphi}(t)\,dt
\end{aligned}
\tag{A.21}
$$

for every $\varphi \in C_c^\infty(0,T)$. The weak energy equality (6.18) follows now by performing an integration by parts. ∎

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Dynamic perfect plasticity and damage in viscoelastic solids

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Abstract

Abstract. In this paper we analyze an isothermal and isotropic model for viscoelastic media combining linearized perfect plasticity (allowing for concentration of plastic strain and development of shear bands) and damage effects in a dynamic setting. The interplay between the viscoelastic rheology with inertia, elasto-plasticity, and unidirectional rate-dependent incomplete damage affecting both the elastic and viscous response, as well as the plastic yield stress, is rigorously characterized by showing existence of weak solutions to the constitutive and balance equations of the model. The analysis relies on the notions of plastic-strain measures and bounded-deformation displacements, on sophisticated time-regularity estimates to establish a duality between acceleration and velocity of the elastic displacement, on the theory of rate-independent processes for the energy conservation in the dynamical-plastic part, and on the proof of the strong convergence of the elastic strains. Existence of a suitably defined weak solutions is proved rather constructively by using a staggered two-step time discretization scheme.

Keywords: Perfect plasticity, inertia, cohesive damage, Kelvin-Voigt viscoelastic rheology, functions of bounded deformation, staggered time discretisation, weak solution.

AMS Subj. Classificaiton: 35Q74, 37N15, 74C05, 74R05.

1 Introduction

Plasticity and damage are inelastic phenomena providing the macroscopical evidence of defect formation and evolution at the atomistic level. Plasticity results from the accumulation of slip defects (dislocations), which determine the behavior of a body to change from elastic and reversible to plastic and irreversible, once the magnitude of the stress reaches a certain threshold and a plastic flow develops. Damage evolution originates from the formation of cracks and voids in the microstructure of the material.

The mathematical modeling of inelastic phenomena is a very active research area, at the triple point between mathematics, physics, and materials science. A vast literature concerning damage in viscoelastic materials, both in the quasistatic and the dynamical setting is currently available. We refer, e.g., to [39, 41, 46, 51, 53] and the references therein for an overview of the main results.

The interplay between plasticity and damage has been already extensively investigated, prominently in the quasistatic framework. The interaction between damage and strain gradient plasticity is addressed in [19] whereas a perfect-plastic model has been proposed in [1], where the one-dimensional response is also studied. Existence results in general dimensions have been obtained in [18, 20], see also [21] for some recent associated lower semicontinuity results. The coupling between damage and rate-independent small-strain plasticity with hardening is the subject of [10, 44, 49]. Quasistatic perfect plasticity and damage with healing are analyzed in [48]. The identification of fracture models as limits of damage coupled with plasticity has also been considered [24, 25].

The analysis of dynamic perfect plasticity without damage has been initiated in [5]. A derivation of the equations via vanishing hardening, and vanishing viscoplasticity has been performed in [15, 16]. A generalization via the so-called cap-model approximation has been obtained in [6]. An approximation of the

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equations of dynamic plasticity relying on the minimization of a parameter-dependent functional defined on trajectories is the subject of [26], whereas an alternative approach based on hyperbolic conservation laws has been proposed in [7]. Dimension reduction for dynamic perfectly plastic plates has been carried on in [40]. Convergence of dynamic models to quasistatic ones has been analyzed in [23, 43].

To our best knowledge, the combination of perfect plasticity, damage, and inertia has been so far tackled in the engineering and geophysical literature (see, e.g., [27,32,52]), whilst a mathematical counterpart to the applicative analysis is still missing. The focus of this paper is to provide a rigorous analysis of an isothermal and isotropic model for viscoelastic media combining both small-strain perfect plasticity and damage effects in a dynamic setting.

More specifically, our main result (Theorem 2.2) shows existence of suitably weak solutions to the following system of equations and differential inclusions, complemented by suitable boundary conditions and initial data

$$\rho\ddot{u} - \operatorname{div}\sigma = f, \quad \sigma := \mathbb{C}(\alpha)e_{\rm el} + \mathbb{D}(\alpha)\dot{e}_{\rm el}, \;\; e_{\rm el} = e(u) - \pi, \tag{1a}$$

$$\sigma_{\rm YLD}(\alpha)\mathrm{Dir}(\dot{\pi}) \ni \operatorname{dev}\sigma, \tag{1b}$$

$$\partial\zeta(\dot{\alpha}) + \frac{1}{2}\mathbb{C}'(\alpha)e_{\rm el} : e_{\rm el} \ni \phi'(\alpha) + \operatorname{div}(\kappa|\nabla\alpha|^{p-2}\nabla\alpha), \tag{1c}$$

where u, π, and α denote the displacement, the plastic strain, and the damage variable, respectively, $\mathbb{C}(\cdot)$, $\mathbb{D}(\cdot)$, and $\sigma_{\rm YLD}(\cdot)$ are the damage-dependent elasticity tensor, viscosity tensor, and yield surface, and $e(u) = (\nabla u + \nabla u^\top)$ is the linearized strain. The notation Dir stands for the set-valued "direction" (see Subsection 2.5), dev σ identifies the deviatoric part of the stress σ, namely $\operatorname{dev}\sigma := \sigma - \operatorname{tr}(\sigma)\mathrm{Id}/d$, ζ is the local potential of dissipative damage-driving force (see (7)), constraining the damage process to be unidirectional (no healing). Finally ϕ is the energy associated to the creation of microvoids or microcracks during the damaging process, κ is the length scale of the damage profile, and ρ the mass density. We refer to Section 2 for the precise setting of the problem, the definition of weak solution to (1a)–(1c), and the statement of Theorem 2.2.

The analysis of model (1a) presents several technical challenges. Perfect plasticity allows for plastic strain concentrations along the (possibly infinitesimally thin) slip-bands and calls for weak formulations in the spaces of bounded Radon measures for plastic strains and bounded-deformation (BD) for displacements. This requires a delicate notion of stress-strain duality (see Subsection 4.1). Considering inertia and the related kinetic energy renders the analysis quite delicate because of the interaction of possible elastic waves with nonlinearly responding slip bands, as pointed out already in [8]. Various natural extensions such as allowing healing instead of unidirectional damage, or mutually independent damage in the viscous and the elastic response (in contrast to (22b) below), or different damage behaviors in relation to compression/tension mode leading to a non-quadratic stored energy, or an enhancement by heat generation/transfer with some thermal coupling to the mechanical part, seem difficult and remain currently open.

The proof strategy relies on a staggered discretization scheme, in which at each time-step we first identify the damage variable as a solution to the damage evolution equation, and we then determine the plastic strain and elastic displacements as minimizer of a damage-dependent energy inequality (see Section 4). A standard test of (1a)–(1c) leads to the proof of a first a-priori estimate in Proposition 5.6. In order to ensure the strong convergence of the time-discrete elastic strains $e_{\rm el}$, needed for the limit passage in the damage flow rule, a further higher order test is performed in Proposition 5.7. The convergence of the elastic strains is then achieved by means of a delicate limsup estimate (see Proposition 6.2). Due to the failure of energy conservation under basic data qualification, the flow rule is only recovered, in the limit, in the form of an energy inequality (see Remark 2.9).

A motivation for tackling the simultaneous occurrence of dynamical perfect plasticity and damaging is the mathematical modeling of cataclasite zones in geophysics. During fast slips, lithospheric faults in elastic rocks tend to emit elastic (seismic) waves, which in turn determine the occurrence of (tectonic) earthquakes, and the local arising of cataclasis. This latter phenomenon consists in a gradual fracturing of mineral grains into core zones of lithospheric faults, which tend to arrange themselves into slip bands, sliding plastically on each other without further fracturing of the material. On the one hand, cataclasite core zone are often very narrow (sometimes centimeters wide) in comparison with the surrounding compact rocks (which typically extend for many kilometers), and can be hence modeled for rather small time scales (minutes of ongoing earthquakes or years between them, rather than millions of years) via small-strain perfect (no-gradient) plasticity. On the other hand the partially damaged area surrounding the thin cataclasite core can by relatively wide, and thus calls for a modeling via gradient-damage theories (see [45, 47]).

The novelty of our contribution is threefold. First, we extend the mathematical modeling of damage-evolution effects to an inelastic setting. Second, we characterize the interaction between damage onset and plastic slips formation in the framework of perfect plasticity, with no gradient regularization and in the absence of hardening. Third, we complement the study of dynamic perfect plasticity, by keeping track of the effects of damage both on the plastic yield surface, and on the viscoelastic behavior of the material.

The paper is organized as follows: In Section 2, we introduce some basic notation and modeling assumptions, and we state our main existence result. Section 3 highlights the formal strategy that will be employed afterward for the proof of Theorem 2.2, whereas Section 4 focuses on the formulation of our staggered two-step discretization scheme. In Section 5 we establish some a-priori energy estimates. Finally Section 6 is devoted to the proof of the main result.

2 Setting of the problem and statement of the main result

We devote this section to specify the mathematical setting of the model, and to present our main result. We first introduce some basic notation and assumptions, and we recall some notions from measure theory.

In what follows, let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded open set with C^2 boundary. In our model, the domain Ω represents the reference configuration of a linearly viscoelastic, perfectly plastic Kelvin-Voigt body subject to a possible damage in its elastic as well as in its viscous and plastic response.

We assume that the boundary $\partial\Omega =: \Gamma$ is partitioned into the union of two disjoint sets $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$. In particular, we require $\Gamma_{\rm D}$ to be a connected open subset of Γ (in the relative topology of Γ) such that $\partial_\Gamma \Gamma_{\rm D}$ is a connected, $(d-2)$-dimensional, C^2 manifold, whereas $\Gamma_{\rm N}$ is defined as $\Gamma_{\rm N} := \Gamma \setminus \Gamma_{\rm D}$.

For any map $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ we will denote by $\dot{f}$ its time derivative, and by ∇f its spatial gradient. We will adopt the notation $\mathbb{R}^{d\times d}$ to indicate the set of $d \times d$ matrices. Given $M, N \in \mathbb{R}^{d\times d}$, their scalar product will be denoted by $M : N := \mathrm{tr}(M^\top N)$ where tr is the trace operator, and the superscript stands for transposition. We will write $\mathrm{dev}\, M$ to identify the deviatoric part of M, namely $\mathrm{dev}\, M := M - \mathrm{tr}\,(M)\mathrm{Id}/d$, where Id is the identity matrix. The symbols $\mathbb{R}^{d\times d}_{\rm sym}$ and $\mathbb{R}^{d\times d}_{\rm dev}$ will represent the set of symmetric $d \times d$ matrices, and that of symmetric matrices having null trace, respectively.

2.1 Function spaces, measures and functions with bounded deformation

We use the standard notation L^p, $W^{k,p}$, and $L^p(0,T;X)$ or $W^{1,p}(0,T;X)$ for Lebesgue, Sobolev, and Bochner or Bochner-Sobolev spaces. By $C_{\rm w}(0,T;X)$ we denote the space of weakly continuous mappings with value in the Banach space X. We also use the shorthand convention $H^k := W^{k,2}$.

Given a Borel set $B \subset \mathbb{R}^d$ the symbol $\mathcal{M}_b(B;\mathbb{R}^m)$ denotes the space of bounded Borel measures on B with values in $\mathbb{R}^m$ $(m \in \mathbb{N})$. When $m = 1$ we will simply write $\mathcal{M}_b(B)$. We will endow $\mathcal{M}_b(B;\mathbb{R}^m)$ with the norm $\|\mu\|_{\mathcal{M}_b(B;\mathbb{R}^m)} := |\mu|(B)$, where $|\mu| \in \mathcal{M}_b(B)$ is the total variation of the measure μ.

If the relative topology of B is locally compact, by the Riesz representation Theorem the space $\mathcal{M}_b(B;\mathbb{R}^m)$ can be identified with the dual of $C_0(B;\mathbb{R}^m)$, which is the space of continuous functions $\varphi : B \to \mathbb{R}^m$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $\mathcal{M}_b(B;\mathbb{R}^m)$ is defined using this duality.

The space $BD(\Omega;\mathbb{R}^d)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(\Omega;\mathbb{R}^d)$ whose symmetric gradient

$$e(u) := \frac{\nabla u + (\nabla u)^\top}{2}$$

(defined in the sense of distributions) belongs to $\mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. It is easy to see that $BD(\Omega;\mathbb{R}^d)$ is a Banach space when endowed with the norm

$$\|u\|_{L^1(\Omega;\mathbb{R}^d)} + \|e(u)\|_{\mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}.$$

A sequence $\{u^k\}$ is said to converge to u weakly* in $BD(\Omega;\mathbb{R}^d)$ if $u^k \to u$ weakly in $L^1(\Omega;\mathbb{R}^d)$ and $e(u^k) \to e(u)$ weakly* in $\mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Every bounded sequence in $BD(\Omega;\mathbb{R}^d)$ has a weakly* converging subsequence. In our setting, since Ω is bounded and has C^2 boundary, $BD(\Omega;\mathbb{R}^d)$ can be embedded into $L^{d/(d-1)}(\Omega;\mathbb{R}^d)$ and every function $u \in BD(\Omega;\mathbb{R}^d)$ has a trace, still denoted by u, which belongs to $L^1(\Gamma;\mathbb{R}^d)$. For every nonempty subset γ of Γ_{D} which is open in the relative topology of Γ_{D} , there exists a constant $C > 0$, depending on Ω and γ, such that the following Korn inequality holds true

$$\|u\|_{L^1(\Omega;\mathbb{R}^d)} \leq C\|u\|_{L^1(\gamma;\mathbb{R}^d)} + C\|e(u)\|_{\mathcal{M}_b(U;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \tag{2}$$

(see [50, Chapter II, Proposition 2.4 and Remark 2.5]). For the general properties of the space $BD(\Omega;\mathbb{R}^d)$ we refer to [50].

2.2 State of the system and admissible displacements and strains

At each time $t \in [0,T]$, the viscoelastic perfectly-plastic behavior of the body is described by three basic state variables: the displacement $u(t) : \Omega \to \mathbb{R}^d$, the plastic strain $\pi(t) : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{dev}}$, and the damage variable $\alpha(t) : \Omega \to [0,1]$. In particular, we adopt the convention (used in mathematics, in contrast to the opposite convention used in engineering and geophysics) that $\alpha = 1$ corresponds to the undamaged elastic material, whereas $\alpha = 0$ describes the situation in which the material is totally damaged. The abstract state q will be here given by the triple $q = (u, \pi, \alpha)$.

On Γ_{D} we prescribe a boundary datum $u_{\mathrm{D}} \in H^{1/2}(\Gamma_{\mathrm{D}};\mathbb{R}^d)$, later being considered to be time dependent. With a slight abuse of notation we also denote by u_{D} a $H^1(\Omega;\mathbb{R}^d)$-extension of the boundary condition to the set Ω.

The *set of admissible displacements and strains* for the boundary datum u_{D} is given by

$$\begin{aligned}\mathscr{A}(u_{\mathrm{D}}) :=\Big\{&(u, e_{\mathrm{el}}, \pi) \in \big(BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega)\big) \times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}) :\\ &e(u) = e_{\mathrm{el}} + \pi \text{ in } \Omega, \quad \pi = (u_{\mathrm{D}} - u) \odot \nu_\Gamma \mathcal{H}^{d-1} \text{ on } \Gamma_{\mathrm{D}}\Big\},\end{aligned} \tag{3}$$

where $\odot$ stands for the symmetrized tensor product, namely

$$a \odot b := (a \otimes b + b \otimes a)/2 \quad \forall\, a, b \in \mathbb{R}^d,$$

ν_Γ is the outer unit normal to Γ, and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. Note that the kinematic relation $e(u) = e_{\mathrm{el}} + \pi$ in $\mathscr{A}(u_{\mathrm{D}})$ is classic in linearized elastic theories and it is usually referred to as additive strain decomposition.

We point out that the constraint

$$\pi = (u_{\mathrm{D}} - u) \odot \nu_\Gamma \mathcal{H}^{d-1} \text{ on } \Gamma_{\mathrm{D}} \tag{4}$$

is a relaxed formulation of the boundary condition $u = u_{\mathrm{D}}$ on Γ_{D}; see also [42]. As remarked in [22], the mechanical meaning of (4) is that whenever the boundary datum is not attained a plastic slip develops, whose amount is directly proportional to the difference between the displacement u and the boundary condition u_{D}.

2.3 Stored energy

Let $\mathcal{L}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ denote the space of linear symmetric (self-adjoint) operators $\mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, being understood as 4th-order symmetric tensors.

We assume the elastic tensor $\mathbb{C} : \mathbb{R} \to \mathcal{L}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ to be continuously differentiable, and nondecreasing in the sense of the Löwner ordering, i.e. the ordering of $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ with respect to the cone of positive semidefinite matrices. Additionally, we require $\mathbb{C}(\alpha)$ to be positive semi-definite for every $\alpha \in \mathbb{R}$. Note that, in view of the pointwise semi-definiteness of $\mathbb{C}$, the possibility of having complete damage in the elastic part is also encoded in the model. We additionally assume that $\mathbb{C}(\alpha) = \mathbb{C}(0)$ for every $\alpha < 0$, and that $\mathbb{C}'(0) = 0$. This corresponds to the situation in which the damage is cohesive.

The *stored energy* of the model will be given by

$$\mathscr{E}(q) = \mathscr{E}(u, \pi, \alpha) = \int_\Omega \Big(\frac{1}{2}\mathbb{C}(\alpha)e_{\mathrm{el}} : e_{\mathrm{el}} - \phi(\alpha) + \frac{\kappa}{p}|\nabla\alpha|^p\Big)\,\mathrm{d}x \quad \text{with} \quad e_{\mathrm{el}} = e(u) - \pi, \tag{5}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ stands for the specific *energy of damage*, motivated by extra energy of microvoids or microcracks created by degradation of the material during the damaging process, whereas κ represents a length scale for the damage profile. When $\phi'(\alpha) > 0$, the damage evolution is an activated processes, even if there is no activation threshold in the dissipation potential, as indeed considered in (7) below.

For the sake of allowing full generality to the choice of initial conditions, we will assume that $\mathrm{dev}\mathbb{C}e = \mathbb{C}\mathrm{dev}\, e$. Note that this is the case for isotropic materials.

2.4 Other ingredients: dissipation and kinetic energy

For the sake of notational simplicity, we consider isotropic materials as far as plastification is concerned.

Let the *yield stress* σ_{YLD} as a function of damage $\sigma_{\mathrm{YLD}} : [0, 1] \to (0, +\infty)$ be continuously differentiable and non-decreasing. For every $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}}; \mathbb{R}^{d\times d}_{\mathrm{dev}})$ let $\mathrm{d}\pi/\mathrm{d}|\pi|$ be the Radon-Nikodým derivative of π with respect to its total variation $|\pi|$. Assuming that $\alpha : [0, T] \times \Omega \to [0, 1]$ is continuous, we consider the positively one-homogeneous function $M \mapsto \sigma_{\mathrm{YLD}}(\alpha)|M|$ for every $M \in \mathbb{R}^{d\times d}$, and, according to the theory of convex functions of measures [34], we introduce the functional

$$\mathcal{R}(\alpha, \pi) := \int_{\Omega\cup\Gamma_{\mathrm{D}}} \sigma_{\mathrm{YLD}}(\alpha)\frac{\mathrm{d}\pi}{\mathrm{d}|\pi|}\,\mathrm{d}|\pi|$$

for every $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}}; \mathbb{R}^{d\times d}_{\mathrm{dev}})$.

In what follows, we will refer to $\mathcal{R}$ as to the *damage-dependent plastic dissipation potential*. Note that, by Reshetnyak's lower semicontinuity theorem (see [2, Theorem 2.38]), the functional $\mathcal{R}$ is lower-semicontinuous in its second variable with respect to the weak* convergence in $\mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}}; \mathbb{R}^{d\times d}_{\mathrm{dev}})$.

For α continuous and such that $\dot\alpha \le 0$ in $[0, T] \times \Omega$, we define the *α-weighted $\mathcal{R}$-dissipation* of a map $t \mapsto \pi(t)$ in the interval $[s_1, s_2]$ as

$$D_{\mathcal{R}}(\alpha; \pi; s_1, s_2) := \sup\Bigg\{\sum_{j=1}^{n}\mathcal{R}\big(\alpha(t_j), \pi(t_j) - \pi(t_{j-1})\big) :\ s_1 \le t_0 < t_1 < \cdots < t_n \le s_2,\ n \in \mathbb{N}\Bigg\}. \tag{6}$$

We will work under the assumption that the damage is unidirectional, i.e. $\dot\alpha \le 0$. Constraining the rate rather than the state itself, this constraint is to be incorporated into the dissipation potential. For a (small) damage-viscosity parameter $\eta > 0$, we define the local potential of dissipative damage-driving force as

$$\zeta(\dot\alpha) := \begin{cases} \dfrac{1}{2}\eta\dot\alpha^2 & \text{if } \dot\alpha \le 0, \\ +\infty. & \text{otherwise} \end{cases} \tag{7}$$

Let the viscoelastic tensor $\mathbb{D} : \mathbb{R} \to \mathcal{L}_{\rm sym}(\mathbb{R}^{d\times d}_{\rm sym})$ be given and define the overall potential of dissipative forces

$$
\begin{aligned}
\mathscr{R}(q;\dot q) &= \mathscr{R}(\alpha;\dot u,\dot\pi,\dot\alpha)\\
&= \int_\Omega \Big(\frac12\mathbb{D}(\alpha)\dot e_{\rm el} : \dot e_{\rm el} + \zeta(\dot\alpha)\Big)\,{\rm d}x + \int_{\Omega\cup\Gamma_{\rm D}} \sigma_{\rm YLD}(\alpha)\frac{{\rm d}\dot\pi}{{\rm d}|\dot\pi|}\,{\rm d}|\dot\pi| \quad \text{where } \dot e_{\rm el} = e(\dot u) - \dot\pi.
\end{aligned} \tag{8}
$$

Let $\rho \in L^\infty(\Omega)$, with $\rho > 0$ almost everywhere in Ω denote the mass density. We will additionally consider the *kinetic energy* given by

$$
\mathscr{T}(\dot u) = \int_\Omega \frac12 \rho|\dot u|^2\,{\rm d}x. \tag{9}
$$

2.5 Governing equations by Hamilton variational principle

We formulate the model via *Hamilton's variational principle* generalized for dissipative systems [9]. This prescribes that, among all admissible motions $q = q(t)$ on a fixed time interval $[0, T]$ given the initial and final states $q(0)$ and $q(T)$, the actual motion is a stationary point of the *action*

$$
\int_0^T \mathscr{L}\big(t, q, \dot q\big)\,{\rm d}t \tag{10}
$$

where $\dot q = \frac{\partial}{\partial t}q$ and the *Lagrangean* $\mathscr{L}(t, q, \dot q)$ is defined as

$$
\mathscr{L}\big(t,q,\dot q\big) := \mathscr{T}\big(\dot q\big) - \mathscr{E}(q) + \langle F(t), q\rangle \quad \text{with} \quad F = F_0(t) - \partial_{\dot q}\mathscr{R}(q,\dot q)\,. \tag{11}
$$

This corresponds to the sum of external time-dependent loading and the (negative) nonconservative force assumed for a moment fixed. In addition to $\mathscr{E}$, $\mathscr{R}$, and $\mathscr{T}$ from Sections 2.3 and 2.4, we define the outer loading F_0 as $\langle F_0(t), q\rangle = \int_\Omega f\cdot u\,{\rm d}x$, where f is a time-dependent external body load.

The corresponding Euler-Lagrange equations read

$$
\partial_u\mathscr{L}\big(t,q,\dot q\big) - \frac{\rm d}{{\rm d}t}\partial_{\dot q}\mathscr{L}\big(t,q,\dot q\big) = 0. \tag{12}
$$

This gives the abstract 2nd-order evolution equation

$$
\partial^2\mathscr{T}\,\ddot q + \partial_{\dot q}\mathscr{R}(q,\dot q) + \mathscr{E}'(q) = F_0(t) \tag{13}
$$

where ∂ indicates the (partial) Gâteaux differential. Let us now rewrite the abstract relation (13) in terms of our specific choices (5), (7)-(9). We have

the following equation/inclusion on $[0, T] \times \Omega$:

$$
\rho\ddot u - {\rm div}\,\sigma = f, \quad \sigma := \mathbb{C}(\alpha)e_{\rm el} + \mathbb{D}(\alpha)\dot e_{\rm el}, \;\; e_{\rm el} = e(u) - \pi, \tag{14a}
$$

$$
\sigma_{\rm YLD}(\alpha){\rm Dir}(\dot\pi) \ni {\rm dev}\,\sigma, \tag{14b}
$$

$$
\partial\zeta(\dot\alpha) + \frac12\mathbb{C}'(\alpha)e_{\rm el} : e_{\rm el} \ni \phi'(\alpha) + {\rm div}\,(\kappa|\nabla\alpha|^{p-2}\nabla\alpha), \tag{14c}
$$

complemented by the boundary conditions

$$
\sigma\nu_\Gamma = 0 \quad \text{on } [0,T]\times\Gamma_{\rm N}, \quad u = u_{\rm D} \quad \text{on } [0,T]\times\Gamma_{\rm D}, \quad \nabla\alpha\cdot\nu_\Gamma = 0 \quad \text{on } [0,T]\times\Gamma. \tag{15}
$$

The notation ${\rm Dir} : \mathbb{R}^{d\times d}_{\rm dev} \rightrightarrows \mathbb{R}^{d\times d}_{\rm dev}$ in (14b) means the set-valued "direction" mapping defined by ${\rm Dir}(\dot\pi) := [\partial|\cdot|](\dot\pi)$. In particular

$$
{\rm Dir}(\dot\pi) = \begin{cases} \dot\pi/|\dot\pi| & \text{if } \dot\pi \neq 0 \\ \{d \in \mathbb{R}^{d\times d}_{\rm dev} \;:\; |d| \le 1\} & \text{if } \dot\pi = 0 \end{cases}
$$

Relations (14a), (14b), and (14c) correspond to the equilibrium equation and constitutive relation, the plastic flow rule, and the evolution law for damage, respectively.

The above boundary-value problem is complemented with initial conditions as follows ,

$$u(0) = u_0, \quad \pi(0) = \pi_0, \quad \alpha(0) = \alpha_0, \quad \dot{u}(0) = v_0. \tag{16}$$

We point out that the monotonicity of $\mathbb{C}$, combined with the unidirectionality ($\dot{\alpha} \leq 0$) of damage implies that

$$\dot{\alpha}\mathbb{C}'(\alpha)e : e \leq 0 \quad \text{for every } e \in \mathbb{R}^{d\times d}, \tag{17}$$

namely $\dot{\alpha}\mathbb{C}'(\alpha)$ is negative semi-definite. By the monotonicity of σ_{YLD}, the unidirectionality of damage also yields that

$$\dot{\alpha}\sigma'_{\text{YLD}}(\alpha) \leq 0. \tag{18}$$

The *energetics* of the model (14)-(15), obtained by standard tests of (14) successively against $\dot{u}$, $\dot{\pi}$, and $\dot{\alpha}$, is formally encoded by the following energy equality

$$\begin{aligned}
&\underbrace{\int_\Omega \frac{\rho}{2}|\dot{u}(t)|^2 \,\mathrm{d}x}_{\substack{\text{kinetic energy}\\ \text{at time } t}} + \underbrace{\int_\Omega \frac{1}{2}\mathbb{C}(\alpha(t))e_{\text{el}}(t) : e_{\text{el}}(t) - \phi(\alpha(t)) + \frac{\kappa}{p}|\nabla\alpha(t)|^p \,\mathrm{d}x}_{\text{stored energy at time } t} \\
&\quad + \underbrace{\int_0^t\int_\Omega \eta\dot{\alpha}^2 + \mathbb{D}(\alpha)\dot{e}_{\text{el}} : \dot{e}_{\text{el}} \,\mathrm{d}x\,\mathrm{d}s + \sigma_{\text{YLD}}(\alpha)|\dot{\pi}| \,\mathrm{d}x\,\mathrm{d}s}_{\text{dissipation on } [0,t]} \\
&= \underbrace{\int_\Omega \frac{\rho}{2}|v_0|^2 \,\mathrm{d}x}_{\substack{\text{kinetic energy}\\ \text{at time } 0}} + \underbrace{\int_\Omega \frac{1}{2}\mathbb{C}(\alpha_0)e_{\text{el}}(0) : e_{\text{el}}(0) - \phi(\alpha_0) + \frac{\kappa}{p}|\nabla\alpha_0|^p \,\mathrm{d}x}_{\text{stored energy at time } 0} \\
&\quad + \underbrace{\int_0^t\int_\Omega f\cdot\dot{u} \,\mathrm{d}x\,\mathrm{d}s}_{\substack{\text{energy of}\\ \text{external bulk load}}} + \underbrace{\int_0^t\int_{\Gamma_{\mathrm{D}}} \sigma\nu_\Gamma\cdot\dot{u}_{\mathrm{D}} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s}_{\substack{\text{energy of}\\ \text{boundary condition}}}
\end{aligned} \tag{19}$$

where the last term has to be interpreted in the sense of (40) below. A rigorous derivation of the energy equality above will be presented in Subsection 3.1.

2.6 Statement of the main result

Let $p > d$ be given and assume that the data of the problem satisfy the following conditions:

$$u_0 \in L^2(\Omega;\mathbb{R}^d) \cap BD(\Omega;\mathbb{R}^d), \;\; v_0 \in H^1(\Omega;\mathbb{R}^d),$$

$$\pi_0 \in \mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\text{dev}}), \;\; \dot{\pi}_0 \in L^2(\Omega;\mathbb{R}^{d\times d}_{\text{dev}}), \tag{20a}$$

$$(u_0, e(u_0)-\pi_0, \pi_0) \in \mathscr{A}(u_{\mathrm{D}}(0)), \;\; (v_0, e(v_0)-\dot{\pi}_0, \dot{\pi}_0) \in \mathscr{A}(\dot{u}_{\mathrm{D}}(0)), \tag{20b}$$

$$\alpha_0 \in W^{1,p}(\Omega), \quad 0 \leq \alpha_0 \leq 1,$$

$$\sigma_{\text{YLD}}(\alpha_0)\mathrm{Dir}(\dot{\pi}_0) \ni \mathrm{dev}\,(\mathbb{C}(\alpha_0)(e(u_0){-}\pi_0) + \mathbb{D}(\alpha_0)(e(v_0){-}\dot{\pi}_0)), \tag{20c}$$

$$f \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \quad u_{\mathrm{D}} \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^1(0,T;H^1(\Omega;\mathbb{R}^d)). \tag{20d}$$

The regularity requirements in (20) for v_0 and $\dot{\pi}_0$ and the compatibility condition in (20c) are needed in order to make some higher-order estimate rigorous, see Subsection 3.2.

We now introduce the notion of weak solution to (14)–(16).

Definition 2.1 (Weak solution to (14)–(16)). A quadruple

$$
\begin{aligned}
u &\in L^\infty(0,T;BD(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))\\
e_{\rm el} &\in H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})),\\
\pi &\in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})),\\
\alpha &\in \big(H^1(0,T;L^2(\Omega)) \cap C_{\rm w}(0,T;W^{1,p}(\Omega)\big)
\end{aligned}
$$

is a *weak solution* to (14)–(16) if it satisfies (16), and the following conditions are fulfilled:

(C1) $(u(t),e_{\rm el}(t),\pi(t)) \in \mathscr{A}(u_{\rm D}(t))$ for every $t\in[0,T]$ (see (3));

(C2) The equilibrium equation (14a) holds almost everywhere in $\Omega\times(0,T)$;

(C3) The quadruple $(u,e_{\rm el},\pi,\alpha)$ satisfies the energy inequality

$$
\begin{aligned}
&\int_\Omega \frac{\rho}{2}|\dot u(T)|^2\,\mathrm{d}x + \int_0^T\!\!\int_\Omega \rho\dot u\cdot\ddot u_{\rm D}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_\Omega \Big(\frac12\mathbb{C}(\alpha(T))e_{\rm el}(T):e_{\rm el}(T) - \phi(\alpha(T)) + \frac{\kappa}{p}|\nabla\alpha(T)|^p\Big)\,\mathrm{d}x\\
&\quad+ D_{\mathcal{R}}(\alpha;\pi;0,T) + \int_0^T\!\!\int_\Omega \Big(\mathbb{D}(\alpha)\dot e_{\rm el}:\dot e_{\rm el} + \eta\dot\alpha^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_\Omega \frac{\rho}{2}v_0^2\,\mathrm{d}x + \int_\Omega\Big(\frac12\mathbb{C}(\alpha_0)(e(u_0)-\pi_0):(e(u_0)-\pi_0) - \phi(\alpha_0) + \frac{\kappa}{p}|\nabla\alpha_0|^p\Big)\,\mathrm{d}x\\
&\quad+\int_\Omega \rho\dot u(T)\cdot\dot u_{\rm D}(T)\,\mathrm{d}x + \int_\Omega \rho v_0\cdot\dot u_{\rm D}(0)\,\mathrm{d}x\\
&\quad+\int_0^T\!\!\int_\Omega\Big(\mathbb{C}(\alpha)e_{\rm el}:e(\dot u_{\rm D}) + \mathbb{D}(\alpha)\dot e_{\rm el}:e(\dot u_{\rm D}) + f\cdot(\dot u-\dot u_{\rm D})\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

(C4) The quadruple $(u,e_{\rm el},\pi,\alpha)$ satisfies the damage inequality

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \phi'(\alpha)\varphi - \kappa|\nabla\alpha|^{p-2}\nabla\alpha\cdot\nabla\varphi - \frac12(\varphi-\dot\alpha)\mathbb{C}'(\alpha)e_{\rm el}:e_{\rm el} - \eta\dot\alpha\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\le \int_\Omega \phi(\alpha(T)) - \phi(\alpha_0) - \frac{\kappa}{p}|\nabla\alpha(T)|^p + \frac{\kappa}{p}|\nabla\alpha_0|^p\,\mathrm{d}x - \int_0^T\!\!\int_\Omega \eta\dot\alpha^2\,\mathrm{d}x\,\mathrm{d}t, \qquad (21)
\end{aligned}
$$

for all $\varphi\in W^{1,p}(\Omega)$ with $\varphi(x)\le 0$ for a.e. $x\in\Omega$.

The main result of the paper consists in showing existence of weak solutions to (14)–(16). Let us summarize the assumption on the data of the model:

$$\mathbb{C}:\mathbb{R}\to\mathcal{L}_{\rm sym}(\mathbb{R}^{d\times d}_{\rm sym}) \text{ continously differentiable, positive semidefinite, nondecreasing,} \qquad (22a)$$
$$\mathbb{D}(\cdot) = \mathbb{D}_0 + \chi\mathbb{C}(\cdot), \mathbb{D}_0 \text{ positive definite, } \ \chi\ge 0, \qquad (22b)$$
$$\phi:\mathbb{R}\to\mathbb{R} \text{ continuously differentiable, nondecreasing,} \qquad (22c)$$
$$\sigma_{\rm YLD}:\mathbb{R}\to\mathbb{R} \text{ continuously differentiable, positive, and nondecreasing,,} \qquad (22d)$$
$$\mathbb{C}'(0) = 0, \quad \phi'(0)\ge 0, \qquad (22e)$$
$$\eta\in L^\infty(\Omega), \ \ \eta\ge\eta_0>0 \text{ a.e.,} \qquad (22f)$$
$$\kappa\in L^\infty(\Omega), \ \ \kappa\ge\kappa_0>0 \text{ a.e.,} \qquad (22g)$$

$$\rho \in L^\infty(\Omega), \quad \rho \geq \rho_0 > 0 \ \text{a.e.} \tag{22h}$$

where $\chi > 0$ is a constant denoting a relaxation time. The structural assumption (22b) is instrumental in making our existence theory amenable. It arises naturally by assuming $\mathbb{C}(\cdot)$ and $\mathbb{D}(\cdot)$ to be pure second-order polynomials of the damage variable α, namely $\mathbb{C}(\alpha) = \alpha^2\mathbb{C}_2$ (recall (22e)) and $\mathbb{D}(\alpha) = \mathbb{D}_0 + \alpha^2\mathbb{D}_2$. By assuming the two tensors $\mathbb{C}_2$ and $\mathbb{D}_2$ to be spherical, namely $\mathbb{C}_2 = c_2 I_4$ and $\mathbb{D}_2 = d_2 I_4$ for some $c_2,\ d_2 > 0$ where I_4 is the identity 4-tensor, one can define $\chi = d_2/c_2$ in order to get (22b). Assumption (22e) ensures that α stays non-negative during the evolution even if the constraint $\alpha \geq 0$ is not explicitly included in the problem, see Remark 2.4 below.

Theorem 2.2 (Existence)**.** *Under assumptions* (20) *on initial conditions and loading and* (22) *on data there exists a weak solution to* (14)–(16) *in the sense of Definition 2.1. Moreover, this solution has the additional regularity* $(u, e_{\rm el}, \pi) \in W^{1,\infty}(0,T; BD(\Omega;\mathbb{R}^d)) \times W^{1,\infty}(0,T; L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})) \times W^{1,\infty}(0,T; \mathcal{M}_b(\Omega\cup \Gamma_{\rm D}; \mathbb{R}^{d\times d}_{\rm dev}))$.

The proof of Theorem 2.2 is postponed to Section 6, where we present a conceptually implementable, numerically stable, and convergent numerical algorithm. Instead, we conclude this section with some final remarks.

Remark 2.3 (Body and surface loads)**.** As pointed out in [6, Introduction], for quasistatic evolution in perfect plasticity one has to impose a compatibility condition between body and surface loads, namely a *safe load* to ensure that the body is not in a free flow. In the dynamic case, under the assumption of null surface loads, this condition can be weaken for what concerns body loads; see, *e.g.*, [36].

Remark 2.4 (Cohesive damage assumption)**.** We will not include in the model reaction forces to the constraint $0 \leq \alpha \leq 1$. This would be encoded by rewriting (14c) as

$$\partial\zeta(\dot\alpha) + \frac{1}{2}\mathbb{C}'(\alpha)e_{\rm el} : e_{\rm el} + p_{\rm R} \ni \phi'(\alpha) + {\rm div}\,(\kappa|\nabla\alpha|^{p-2}\nabla\alpha) \quad \text{where} \quad p_{\rm R} \in N_{[0,1]}(\alpha)\,;$$

here $N_{[0,1]}(\cdot)$ denote the normal cone and $p_{\rm R}$ is a "reaction pressure" to the constraints $0 \leq \alpha \leq 1$. We point out that the presence of this additional term in the damage flow rule would cause a loss of regularity for the damage variable. In order to avoid such problem we will rather enforce the constraint $0 \leq \alpha \leq 1$ by exploiting the irreversibility of damage, and by restricting our analysis to the situation in which the damage is cohesive.

Remark 2.5 (Regularity of Γ)**.** We remark that the C^2-regularity of Γ is needed in order to apply [37, Proposition 2.5], and define a duality between stresses and plastic strains. For $d = 2$, owing to the results in [30], it is also possible to analyze the setting in which Γ is Lipschitz. The same strategy can not be applied for $d = 3$, for it would require ${\rm div}\,\sigma \in L^3(\Omega)$, whereas here we can only achieve ${\rm div}\,\sigma \in L^2(\Omega)$.

Remark 2.6 (The role of the term $\eta\dot\alpha$)**.** The term $\eta\dot\alpha$ in (14c) guarantees strong convergence of the damage-interpolants in the time-discretization scheme to the limit damage variable. This, in turn, is a key point to ensure strong convergence of the elastic stresses, which is fundamental for the proof of the damage inequality in condition (C4). From a modeling point of view, this might be interpreted as some additional dissipation related with the speed of the damaging process contributing to the heat production, possibly leading to an increase of temperature. The microscopical idea behind it is that faster mechanical processes cause higher heat production and therefore higher dissipation.

Remark 2.7 (Phase-field fracture)**.** Our cohesive damage with $\mathbb{C}'(0) = 0$ has the drawback that, while α approaches zero, the driving force needed for its evolution rises to infinity. This model is anyhow used in the phase-field approximation of fracture.

$$\mathscr{E}(u,\alpha) := \int_\Omega \frac{(\varepsilon/\varepsilon_0)^2{+}\alpha^2}{2}\mathbb{C}_1 e(u){:}e(u) + \underbrace{G_\mathrm{c}\Big(\frac{1}{2\varepsilon}(1{-}\alpha)^2 + \frac{\varepsilon}{2}|\nabla\alpha|^2\Big)}_{\text{crack surface density}}\,\mathrm{d}x \tag{23}$$

with with G_c denoting the energy of fracture and with ε controlling a "characteristic" width of the phase-field fracture zone(s). The physical dimension of ε_0 as well as of ε is m (meters) while the physical dimension of G_c is J/m^2. In the model (5), it means $\mathbb{C}(\alpha) = (\varepsilon^2/\varepsilon_0^2{+}\alpha^2)\mathbb{C}_1$ and $\phi(\alpha) = -G_\mathrm{c}(1{-}\alpha)^2/(2\varepsilon)$ while $\kappa = \varepsilon$ and $p = 2$. This is known as the so-called *Ambrosio-Tortorelli functional*. Its motivation came from the static case, where this approximation was proposed by Ambrosio and Tortorelli [3, 4] and the asymptotic analysis for $\varepsilon \to 0$ was rigorously proved first for the scalar-valued case. The generalization for the vectorial case is due to Focardi [28]. Later, it was extended to the evolution situation, namely for a rate-independent cohesive damage, in [33], see also [11, 12, 14, 38, 41] where inertial forces are incorporated in the description. Note however that plasticity was not involved in all these references. Some modifications have been addressed in [13], see also [46] for various other models, and [17, 29, 31, 35] for the linearized and cohesive-fracture settings.

Remark 2.8 (Ductile damage/fracture)**.** A combination of damage/fracture with plasticity is sometimes denoted by the adjective "ductile", in contrast to "brittle", if plasticity is not considered. There are various scenarios of combination of plastification processes with damage, that can model various phenomena in fracture mechanics. Here, we address the case of damage-dependent elastic response and the yield stress.

Remark 2.9 (Influence of damage on the energy equality)**.** We point out that, in the absence of damage, energy conservation could be recovered. Indeed, it would be possible to prove the energy equality, which would then ensure the validity of the flow rule (14b) as well. A detailed analysis of an analogous albeit quasistationary case has been performed in [22, Section 6] in the quasistatic framework. An adaptation of the argument yields the analogous statements in the dynamic setting.

3 Some formal calculus first

We first highlight a formal strategy that will lead to the proof of Theorem 2.2, avoiding (later necessary) technicalities. In particular, we first derive the energetics of the model by performing some standard tests of (14) against the time derivatives $(\dot u, \dot\pi, \dot\alpha)$. Further a-priori estimates will be obtained by performing a test of the same equations against higher-order time-derivatives of the maps. Eventually, a direct strong-convergence argument will be presented.

All the arguments will be eventually made rigorous in Sections 5–6 by means of a time-discretization procedure, combined with a passage to the limit as the time-step vanishes. The estimates described in Subsections 3.2–3.3 will be essential to pass to the limit in the time-discrete damage equation.

3.1 Energetics of the model and first estimates

A formal test of the equations/inclusion (14) successively against $\dot u$, $\dot\pi$, and $\dot\alpha$ yields

$$\int_\Omega \Big(\rho\ddot u(t)\cdot\dot u(t) + \sigma(t) : e(\dot u(t))\Big)\,\mathrm{d}x = \int_\Omega f(t)\cdot\dot u(t)\,\mathrm{d}x + \int_\Gamma \sigma(t)\nu_\Gamma\cdot\dot u_\mathrm{D}(t)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{24a}$$

$$\int_\Omega \operatorname{dev}\sigma(t) : \dot\pi(t)\,\mathrm{d}x = \int_\Omega \sigma_{\mathrm{YLD}}(\alpha(t))|\dot\pi(t)|\,\mathrm{d}x, \tag{24b}$$

$$\int_\Omega \eta\dot\alpha(t)^2\,\mathrm{d}x = \int_\Omega \Big(\phi'(\alpha(t))\dot\alpha(t) - \frac{1}{2}\mathbb{C}'(\alpha(t))\dot\alpha(t)e_{\mathrm{el}}(t) : e_{\mathrm{el}}(t) - \kappa|\nabla\alpha(t)|^{p-2}\nabla\alpha(t)\cdot\nabla\dot\alpha(t)\Big)\,\mathrm{d}x. \tag{24c}$$

Integrating (24a) in time, by (16), (24b), and by the definition of e_{el}, we obtain

$$\begin{aligned}\int_\Omega \Big(\frac{\rho}{2}|\dot u(t)|^2 + \frac{1}{2}\mathbb{C}(\alpha(t))e_{\mathrm{el}}(t) : e_{\mathrm{el}}(t)\Big)\,\mathrm{d}x - \int_0^t\!\!\int_\Omega \frac{1}{2}\mathbb{C}'(\alpha)\dot\alpha e_{\mathrm{el}} : e_{\mathrm{el}}\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t\!\!\int_\Omega \mathbb{D}(\alpha)\dot e_{\mathrm{el}} : \dot e_{\mathrm{el}}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega \sigma_{\mathrm{YLD}}(\alpha)|\dot\pi|\,\mathrm{d}x\,\mathrm{d}s \\ = \int_\Omega \Big(\frac{\rho}{2}|v_0|^2 + \frac{1}{2}\mathbb{C}(\alpha_0)e_{\mathrm{el}}(0) : e_{\mathrm{el}}(0)\Big)\,\mathrm{d}x + \int_0^t\!\!\int_\Omega f\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_{\Gamma_{\mathrm{D}}} \sigma\nu_\Gamma\cdot\dot u_{\mathrm{D}}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s.\end{aligned} \tag{25}$$

In view of (15) and (16), an integration in time of (24c) yields

$$\int_0^t\!\!\int_\Omega \Big(\eta\dot\alpha^2 + \frac{1}{2}\dot\alpha\mathbb{C}'(\alpha)e_{\mathrm{el}} : e_{\mathrm{el}}\Big)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \Big(\frac{\kappa}{p}|\nabla\alpha(t)|^p - \phi(\alpha(t))\Big)\,\mathrm{d}x = \int_\Omega \Big(\frac{\kappa}{p}|\nabla\alpha_0|^p - \phi(\alpha_0)\Big)\,\mathrm{d}x. \tag{26}$$

Thus, summing (25) and (26), by (15) we deduce the energy equality (19).

To see the energy-based estimates from (19), here we should use the Gronwall inequality for the term $f\cdot\dot u$ benefitting from having the kinetic energy on the left-hand side, and the by-part integration of the Dirichlet loading term. We stress that the last term in (25) can be rigorously defined as in (40). This way, we can see the estimates

$$u \in L^\infty(0,T;BD(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{27a}$$

$$e_{\mathrm{el}} \in H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{27b}$$

$$\pi \in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})), \tag{27c}$$

$$\alpha \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)). \tag{27d}$$

Unfortunately, these estimates do not suffice for the convergence analysis as the time step goes to 0. In particular, in relation (35) later on one needs to handle the term $\rho\ddot u_k\cdot\dot u$, which is still not integrable under (27a).

3.2 Higher-order tests

In this subsection we perform an extension of the regularity estimate in Subsection 3.1, relying on the unidirectionality of the damage evolution, on the fact that $\sigma_{\mathrm{YLD}}(\cdot)$ is nondecreasing, and on the monotonicity of $\mathbb{C}(\cdot)$ with respect to the Löwner ordering. We introduce the abbreviation

$$w := u + \chi\dot u, \quad \varepsilon_{\mathrm{el}} := e_{\mathrm{el}} + \chi\dot e_{\mathrm{el}}, \quad \text{and} \quad \varpi = \pi + \chi\dot\pi, \tag{28}$$

and observe that, $\ddot u = (\dot w - \dot u)/\chi$. Hence, the equilibrium equation rewrites as

$$\rho\frac{\dot w}{\chi} - \operatorname{div}\sigma = f + \rho\frac{\dot u}{\chi}. \tag{29}$$

We first argue by testing the plastic flow rule (14b) against $\dot{\varpi}$. We use the (here formal) calculus

$$\sigma_{\text{YLD}}(\alpha)\mathrm{Dir}(\dot{\pi})\colon \ddot{\pi} = \frac{\partial}{\partial t}\Big(\sigma_{\text{YLD}}(\alpha)|\dot{\pi}|\Big) - \dot{\alpha}\sigma'_{\text{YLD}}(\alpha)|\dot{\pi}| \geq \frac{\partial}{\partial t}\Big(\sigma_{\text{YLD}}(\alpha)|\dot{\pi}|\Big) \tag{30}$$

because $\dot{\alpha}\sigma'_{\text{YLD}}(\alpha)|\dot{\pi}| \leq 0$ when assuming $\sigma_{\text{YLD}}(\cdot)$ nondecreasing and using $\dot{\alpha} \leq 0$, cf. (18). This formally yields

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \sigma_{\text{YLD}}(\alpha)|\dot{\pi}|\,\mathrm{d}x\,\mathrm{d}t + \chi\int_\Omega \Big(\sigma_{\text{YLD}}(\alpha(t))|\dot{\pi}(t)| - \chi\sigma_{\text{YLD}}(\alpha_0)|\dot{\pi}(0)|\Big)\,\mathrm{d}x\\
&\quad= \int_0^T\!\!\int_\Omega \sigma_{\text{YLD}}(\alpha)|\dot{\pi}|\,\mathrm{d}x\,\mathrm{d}t + \chi\int_\Omega \sigma_{\text{YLD}}(\alpha(t))|\dot{\pi}(t)|\,\mathrm{d}x - \chi\int_\Omega \sigma_{\text{YLD}}(\alpha(0))|\dot{\pi}(0)|\,\mathrm{d}x\\
&\qquad\qquad \leq \int_0^T\!\!\int_\Omega \sigma\colon\dot{\varpi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{31}$$

Analogously, testing (29) against $\dot{w}$ and integrating in time, by (15) we deduce

$$\int_0^T\!\!\int_\Omega \Big(\frac{\rho}{\chi}|\dot{w}|^2 + \sigma : e(\dot{w})\Big)\,\mathrm{d}x\mathrm{d}t = \int_0^T\!\!\int_\Omega \Big(f\cdot\dot{w} + \frac{\rho}{\chi}\dot{u}\cdot\dot{w}\Big)\,\mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_{\Gamma_{\text{D}}} \sigma\nu_\Gamma\cdot(\dot{u}_{\text{D}} + \chi\ddot{u}_{\text{D}})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t. \tag{32}$$

By the definition of the tensor $\mathbb{D}$ (see Subsection 2.3), and by (17), we infer that

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \sigma : e(\dot{w})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega \Big(\mathbb{C}(\alpha)\varepsilon_{\text{el}} : \dot{\varepsilon}_{\text{el}} + \mathbb{D}_0\dot{e}_{\text{el}} : \dot{\varepsilon}_{\text{el}} + \sigma : \dot{\varpi}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\geq \int_\Omega \frac{1}{2}\mathbb{C}(\alpha(t))\varepsilon_{\text{el}}(t) : \varepsilon_{\text{el}}(t)\,\mathrm{d}x + \int_0^T\!\!\int_\Omega \mathbb{D}_0\dot{e}_{\text{el}} : \dot{e}_{\text{el}}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \frac{1}{2}\mathbb{C}(\alpha_0)\varepsilon_{\text{el}}(0) : \varepsilon_{\text{el}}(0)\,\mathrm{d}x\\
&\quad + \frac{\chi}{2}\int_\Omega \mathbb{D}_0\dot{e}_{\text{el}}(t) : \dot{e}_{\text{el}}(t)\,\mathrm{d}x - \frac{\chi}{2}\int_\Omega \mathbb{D}_0\dot{e}_{\text{el}}(0) : \dot{e}_{\text{el}}(0)\,\mathrm{d}x + \int_0^T\!\!\int_\Omega \sigma : \dot{\varpi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{33}$$

Thus, by combining (31), with (32) and (33), we obtain the inequality

$$\begin{aligned}
&\frac{1}{\chi}\int_0^T\!\!\int_\Omega \rho|\dot{w}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega \mathbb{C}(\alpha(t))\varepsilon_{\text{el}}(t) : \varepsilon_{\text{el}}(t)\,\mathrm{d}x\\
&\quad + \int_0^T\!\!\int_\Omega \mathbb{D}_0\dot{e}_{\text{el}} : \dot{e}_{\text{el}}\,\mathrm{d}x\,\mathrm{d}t + \frac{\chi}{2}\int_\Omega \mathbb{D}_0\dot{e}_{\text{el}}(t) : \dot{e}_{\text{el}}(t)\,\mathrm{d}x + \int_0^T\!\!\int_\Omega \sigma_{\text{YLD}}(\alpha)|\dot{\pi}|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \chi\int_\Omega \sigma_{\text{YLD}}(\alpha(t))|\dot{\pi}(t)|\,\mathrm{d}x \leq \frac{1}{2}\int_\Omega \mathbb{C}(\alpha_0)\varepsilon_{\text{el}}(0) : \varepsilon_{\text{el}}(0)\,\mathrm{d}x\\
&\quad + \frac{\chi}{2}\int_\Omega \mathbb{D}_0\dot{e}_{\text{el}}(0) : \dot{e}_{\text{el}}(0)\,\mathrm{d}x + \chi\int_\Omega \sigma_{\text{YLD}}(\alpha_0)|\dot{\pi}(0)|\,\mathrm{d}x\\
&\quad + \int_0^T\!\!\int_\Omega f\cdot\dot{w}\,\mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_{\Gamma_{\text{D}}} \sigma\nu_\Gamma\cdot(\dot{u}_{\text{D}} + \chi\ddot{u}_{\text{D}})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t + \frac{1}{\chi}\int_0^T\!\!\int_\Omega \rho\dot{u}\cdot\dot{w}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Let us note that we can use (27a) in order to control $\dot{u}$ in the last term above. As for initial data, we need here that $\dot{e}_{\text{el}}(0) \in L^2(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})$ and $\dot{\pi}(0) \in L^2(\Omega;\mathbb{R}^{d\times d}_{\text{dev}})$, which follows under the provisions of (20). Eventually, by (19), and (28) this yields the following additional regularity for the displacement, and for the elastic and plastic strains

$$u \in W^{1,\infty}(0,T;BD(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{34a}$$

$$e_{\text{el}} \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{d\times d})), \tag{34b}$$

$$\pi \in W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\text{D}};\mathbb{R}^{d\times d}_{\text{dev}})). \tag{34c}$$

3.3 One more estimate for the strong convergence of $e_{\rm el}$'s

The strong convergence of the elastic strains $e_{\rm el}$ is needed for the limit passage in the damage flow rule. The failure of energy conservation (see Remark 2.9) prevents the usual "limsup-strategy", but one can estimate directly the difference between the (presently still unspecified) approximate solution (u_k,π_k) and its limit (u,π) punctually as:

$$
\begin{aligned}
&\int_Q \mathbb{D}(\alpha_k)(\dot e_{{\rm el},k}-\dot e_{\rm el})\colon(\dot e_{{\rm el},k}-\dot e_{\rm el})\,{\rm d}x{\rm d}t\\
&\qquad\qquad+\int_\Omega\frac12\mathbb{C}(\alpha_k(T))(e_{{\rm el},k}(T)-e_{\rm el}(T))\colon(e_{{\rm el},k}(T)-e_{\rm el}(T))\,{\rm d}x\\
&\le\int_Q\int_\Omega\big(\mathbb{D}(\alpha_k)(\dot e_{{\rm el},k}-\dot e_{\rm el})+\mathbb{C}(\alpha_k)(e_{{\rm el},k}-e_{\rm el})\big)\colon(\dot e_{{\rm el},k}-\dot e_{\rm el})\,{\rm d}x{\rm d}t\\
&\le\int_Q\bigg((f-\rho\ddot u_k)\cdot(\dot u_k-\dot u)-\big(\mathbb{D}(\alpha_k)\dot e_{\rm el}+\mathbb{C}(\alpha_k)e_{\rm el}\big)\colon(\dot e_{{\rm el},k}-\dot e_{\rm el})\\
&\qquad\qquad+\sigma_{\rm YLD}(\alpha_k)(|\dot\pi|-|\dot\pi_k|)\bigg)\,{\rm d}x{\rm d}t. \qquad (35)
\end{aligned}
$$

The first inequality in (35) is due to the monotonicity of $\mathbb{C}(\cdot)$ with respect to the Löwner ordering so that, due to (17), it holds

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac12\mathbb{C}(\alpha_k)(e_{{\rm el},k}-e_{\rm el}){:}(e_{{\rm el},k}-e_{\rm el})\right)\\
&=\frac12\dot\alpha_k\mathbb{C}'(\alpha_k)(e_{{\rm el},k}-e_{\rm el}){:}(e_{{\rm el},k}-e_{\rm el})+\mathbb{C}(\alpha_k)\frac{\partial}{\partial t}\left(\frac12(e_{{\rm el},k}-e_{\rm el}){:}(e_{{\rm el},k}-e_{\rm el})\right)\\
&\le\mathbb{C}(\alpha_k)(e_{{\rm el},k}-e_{\rm el}){:}(\dot e_{{\rm el},k}-\dot e_{\rm el})\,. \qquad (36)
\end{aligned}
$$

while the second step in (35) is due to the inequality ${\rm dev}\,\sigma\colon(\dot\pi_k-\dot\pi)\ge\sigma_{\rm YLD}(\alpha_k)(|\dot\pi_k|-|\dot\pi|)$, following from the plastic flow rule $\sigma_{\rm YLD}(\alpha_k){\rm Dir}(\dot\pi_k)\ni{\rm dev}\,\sigma$, with σ from (14a).

By using weak* upper semicontinuity and the uniform convergence $\alpha_k\to\alpha$ one checks that the limit superior of the right-hand side in (35) can be estimated from above by zero (so that, in fact, the limit does exist and equals to zero). We refer to Proposition 6.2 for the rigorous implementation of the above argument.

4 Staggered two-step time-discretization scheme

This section is devoted to the solution of a discrete counterpart of the system of equations (14)–(16), and to the proof of a-priori estimates for the associated piecewise constant, piecewise affine, and piecewise quadratic in-time interpolants.

Fix $n\in\mathbb{N}$, set $\tau:=T/n$, and consider the equidistant time partition of the interval $[0,T]$ with step τ. We define the discrete body-forces by setting $f_\tau^k:=\int_{(k-1)\tau}^{k\tau}f(t)\,{\rm d}t$ for all $k\in\{1,\dots,T/\tau\}$. We consider the following time-discretization scheme:

$$\rho\delta^2u_\tau^k-{\rm div}\big(\mathbb{C}(\alpha_\tau^{k-1})e_{{\rm el},\tau}^k+\mathbb{D}(\alpha_\tau^{k-1})\delta e_{{\rm el},\tau}^k\big)=f_\tau^k, \qquad (37a)$$

$$\sigma_{\rm YLD}(\alpha_\tau^{k-1}){\rm Dir}(\delta\pi_\tau^k)\ni{\rm dev}\big(\mathbb{C}(\alpha_\tau^{k-1})e_{{\rm el},\tau}^k+\mathbb{D}(\alpha_\tau^{k-1})\delta e_{{\rm el},\tau}^k\big), \qquad (37b)$$

$$\partial\zeta(\delta\alpha_\tau^k)+\frac12\mathbb{C}^\circ(\alpha_\tau^k,\alpha_\tau^{k-1})e_{{\rm el},\tau}^k:e_{{\rm el},\tau}^k\ni\phi^\circ(\alpha_\tau^k,\alpha_\tau^{k-1})+{\rm div}\,(\kappa|\nabla\alpha_\tau^k|^{p-2}\nabla\alpha_\tau^k), \qquad (37c)$$

to be complemented with the boundary conditions

$$\Big(\mathbb{C}(\alpha_\tau^{k-1})e_{{\rm el},\tau}^k+\mathbb{D}(\alpha_\tau^{k-1})\delta e_{{\rm el},\tau}^k\Big)\,\nu_\Gamma=0\quad\text{on }\Gamma_{\rm N} \qquad (38a)$$

$$\kappa|\nabla\alpha_\tau^k|^{p-2}\nabla\alpha_\tau^k\cdot\nu_\Gamma = 0 \quad \text{on } \Gamma. \tag{38b}$$

Here, δ and δ^2 denote the first and second order finite-difference operator, that is

$$\delta u_\tau^k := \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \quad \text{and} \quad \delta^2 u_\tau^k := \delta\big[\delta u_\tau^k\big] = \frac{u_\tau^k - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2},$$

and where the tensor $\mathbb{C}^\circ(\alpha,\tilde\alpha)$ and the scalar $\phi^\circ(\alpha,\tilde\alpha)$ are defined for all $\alpha,\ \tilde\alpha\in\mathbb{R}$ as

$$\mathbb{C}^\circ(\alpha,\tilde\alpha) := \begin{cases} \dfrac{\mathbb{C}(\alpha)-\mathbb{C}(\tilde\alpha)}{\alpha-\tilde\alpha} & \text{if } \alpha\neq\tilde\alpha \\ \mathbb{C}'(\alpha) = \mathbb{C}'(\tilde\alpha) & \text{if } \alpha=\tilde\alpha, \end{cases}$$

$$\phi^\circ(\alpha,\tilde\alpha) := \begin{cases} \dfrac{\phi(\alpha)-\phi(\tilde\alpha)}{\alpha-\tilde\alpha} & \text{if } \alpha\neq\tilde\alpha\,, \\ \phi'(\alpha) = \phi'(\tilde\alpha) & \text{if } \alpha=\tilde\alpha\,. \end{cases}$$

Let us note that, if $\phi(\cdot)$ is affine, then simply $\phi^\circ(\alpha_\tau^k,\alpha_\tau^{k-1}) = \phi'$. Similarly, if $\mathbb{C}(\cdot)$ were affine, then $\mathbb{C}^\circ(\alpha_\tau^k,\alpha_\tau^{k-1}) = \mathbb{C}'$. We point out that this case would be in conflict with (22e) unless $\mathbb{C}$ would be independent of damage.

4.1 Weak solutions to the time-discretization scheme

In order to define a notion of weak solutions to (37b), we need to preliminary introduce a duality between stresses and plastic strains. We work along the footsteps of [37] and [22, Subsection 2.3]. We first define the set

$$\Sigma(\Omega) := \big\{\sigma\in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}):\ \operatorname{dev}\sigma\in L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}) \text{ and } \operatorname{div}\sigma\in L^2(\Omega;\mathbb{R}^d)\big\}. \tag{39}$$

By [37, Proposition 2.5 and Corollary 2.6], for every $\sigma\in\Sigma(\Omega)$ there holds

$$\sigma\in L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}).$$

In addition, we can introduce the trace $[\sigma\nu_\Gamma]\in H^{-1/2}(\Gamma;\mathbb{R}^d)$ (see e.g. [50, Theorem 1.2, Chapter I]) as

$$\langle[\sigma\nu_\Gamma],\,\psi\rangle_\Gamma := \int_\Omega \operatorname{div}\sigma\cdot\psi\,\mathrm{d}x + \int_\Omega \sigma : e(\psi)\,\mathrm{d}x \tag{40}$$

for every $\psi\in H^1(\Omega;\mathbb{R}^d)$. Defining the normal and the tangential part of $[\sigma\nu_\Gamma]$ as

$$[\sigma\nu_\Gamma]_\nu := ([\sigma\nu_\Gamma]\cdot\nu_\Gamma)\nu_\Gamma \quad \text{and} \quad [\sigma\nu_\Gamma]_\nu^\perp := [\sigma\nu_\Gamma] - ([\sigma\nu_\Gamma]\cdot\nu_\Gamma)\nu_\Gamma,$$

by [37, Lemma 2.4] we have that $[\sigma\nu_\Gamma]_\nu^\perp\in L^\infty(\Gamma;\mathbb{R}^d)$, and

$$\|[\sigma\nu_\Gamma]_\nu^\perp\|_{L^\infty(\Gamma;\mathbb{R}^d)} \le \frac{1}{\sqrt{2}}\|\operatorname{dev}\sigma\|_{L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}.$$

Let $\sigma\in\Sigma(\Omega)$ and let $u\in BD(\Omega;\mathbb{R}^d)\cap L^2(\Omega;\mathbb{R}^d)$, with $\operatorname{div}u\in L^2(\Omega)$. We define the distribution $[\operatorname{dev}\sigma:\operatorname{dev}e(u)]$ on Ω as

$$\langle[\operatorname{dev}\sigma:\operatorname{dev}e(u)],\,\varphi\rangle := -\int_\Omega \varphi\operatorname{div}\sigma\cdot u\,\mathrm{d}x - \frac{1}{d}\int_\Omega \varphi\operatorname{tr}\sigma\cdot\operatorname{div}u\,\mathrm{d}x - \int_\Omega \sigma:(u\odot\nabla\varphi)\,\mathrm{d}x \tag{41}$$

for every $\varphi\in C_c^\infty(\Omega)$. By [37, Theorem 3.2] it follows that $[\operatorname{dev}\sigma:\operatorname{dev}e(u)]$ is a bounded Radon measure on Ω, whose variation satisfies

$$|[\operatorname{dev}\sigma:\operatorname{dev}e(u)]| \le \|\operatorname{dev}\sigma\|_{L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}|\operatorname{dev}e(u)| \quad \text{in } \Omega.$$

Let $\Pi_{\Gamma_{\rm D}}(\Omega)$ be the set of admissible plastic strains, namely the set of maps $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\rm D}; \mathbb{R}^{d\times d}_{\rm dev})$ such that there exist $u \in BD(\Omega; \mathbb{R}^d) \cap L^2(\Omega; \mathbb{R}^d)$, $e \in L^2(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$, and $w \in W^{1,2}(\Omega; \mathbb{R}^d)$ with $(u, e, \pi) \in \mathscr{A}(w)$. Note that the additive decomposition $e(u) = e + \pi$ implies that $\operatorname{div} u \in L^2(\Omega)$.

It is possible to define a duality between elements of $\Sigma(\Omega)$ and $\Pi_{\Gamma_{\rm D}}(\Omega)$. To be precise, given $\pi \in \Pi_{\Gamma_{\rm D}}(\Omega)$, and $\sigma \in \Sigma(\Omega)$, we fix (u, e, w) such that $(u, e, \pi) \in \mathscr{A}(w)$, with $u \in L^2(\Omega; \mathbb{R}^d)$, and we define the measure $[\operatorname{dev}\sigma : \pi] \in \mathcal{M}_b(\Omega \cup \Gamma_{\rm D}; \mathbb{R}^{d\times d}_{\rm dev})$ as

$$[\operatorname{dev}\sigma : \pi] := \begin{cases} [\operatorname{dev}\sigma : \operatorname{dev} e(u)] - \operatorname{dev}\sigma : \operatorname{dev} e & \text{in } \Omega \\ [\sigma\nu_\Gamma]^\perp_\nu \cdot (w - u)\mathcal{H}^{d-1} & \text{on } \Gamma_{\rm D}, \end{cases}$$

so that

$$\int_{\Omega\cup\Gamma_{\rm D}} \varphi \,\mathrm{d}[\operatorname{dev}\sigma : \pi] = \int_\Omega \varphi \,\mathrm{d}[\operatorname{dev}\sigma : \operatorname{dev} e(u)] - \int_\Omega \varphi \operatorname{dev}\sigma : \operatorname{dev} e \,\mathrm{d}x + \int_{\Gamma_{\rm D}} \varphi [\sigma\nu]^\perp_\nu \cdot (w - u)\,\mathrm{d}\mathcal{H}^{d-1}$$

for every $\varphi \in C(\bar{\Omega})$. Arguing as in [22, Section 2], one can prove that the definition of $[\operatorname{dev}\sigma : \pi]$ is independent of the choice of (u, e, w), and that if $\operatorname{dev}\sigma \in C(\bar{\Omega}; \mathbb{R}^{d\times d}_{\rm dev})$ and $\varphi \in C(\bar{\Omega})$, then

$$\int_{\Omega\cup\Gamma_{\rm D}} \varphi \,\mathrm{d}[\operatorname{dev}\sigma : \pi] = \int_{\Omega\cup\Gamma_{\rm D}} \varphi \operatorname{dev}\sigma : \,\mathrm{d}\pi.$$

We are now in a position to state the definition of weak solutions to the time-discretization scheme.

Definition 4.1 (Weak discrete solutions)**.** For every $k \in \{1, \dots, T/\tau\}$, a quadruple $(u^k_\tau, e^k_{{\rm el},\tau}, \pi^k_\tau, \alpha^k_\tau)$ is a weak solution to (37) if $(u^k_\tau, e^k_\tau, \pi^k_\tau) \in \mathscr{A}(u^k_{{\rm D},\tau})$, $\alpha^k_\tau \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies $0 \le \alpha^k_\tau \le 1$, the quadruple solves (37c) and (38), property (37a) holds almost everywhere, and the following discrete flow-rule is fulfilled

$$[\operatorname{dev}\sigma^k_\tau : \delta\pi^k_\tau](\Omega \cup \Gamma_{\rm D}) = \mathcal{R}(\alpha^{k-1}_\tau, \delta\pi^k_\tau), \quad \text{with} \quad \sigma^k_\tau := \mathbb{C}(\alpha^{k-1}_\tau) e^k_{{\rm el},\tau} + \mathbb{D}(\alpha^{k-1}_\tau)\delta e^k_{{\rm el},\tau}. \tag{42}$$

Remark 4.2 (The discrete flow-rule)**.** A crucial difference with respect to the results in [6, Proposition 3.3] is the fact that condition (42) is much weaker than the differential inclusion (37b). This is due to a key peculiarity of our model, for we consider a viscous contribution involving only the elastic strain, but we still allow for perfect plasticity. In fact, in our setting (37b) is only formal, as for every τ and k, the map $\delta\pi^k_\tau$ is a bounded Radon measure. In particular the quantity $\sigma_{\rm YLD}(\alpha^{k-1}_\tau)\mathrm{Dir}(\delta\pi^k_\tau)$ does not have a pointwise meaning. As customary in the setting of perfect plasticity, the discrete flow-rule is thus only recovered in a weaker form.

4.2 Existence of weak solutions

Let us start by specifying the discretization of the boundary Dirichlet data as system

$$u^0_{{\rm D},\tau} := u_{\rm D}(0), \quad u^{-1}_{{\rm D},\tau} := u_{\rm D}(0) - \tau\dot{u}_{\rm D}(0), \quad u^k_{{\rm D},\tau} := u_{\rm D}(k\tau) \quad \text{for every } k \in \{1, \dots, T/\tau\}.$$

As for initial data, we recall (20) and prescribe

$$u^0_\tau := u_0, \;\; \pi^0_\tau := \pi_0, \;\; \alpha^0_\tau := \alpha_0, \;\; e^0_{{\rm el},\tau} = e(u_0) - \pi_0.$$

In order to reproduce the higher-order tests of Subsection 3.2 at the discrete level we need to specify additionally the following

$$u^{-1}_\tau := u_0 - \tau v_0, \;\; \pi^{-1}_\tau := \pi_0 - \tau\dot{\pi}_0, \;\; \alpha^{-1}_\tau := \alpha^0_\tau, \;\; e^{-1}_{{\rm el},\tau} = e(u_0) - \tau(e(v_0) - \dot{\pi}_0).$$

In particular, the last condition in (20c) ensures that the discrete flow rule (37b) holds at level $k = 0$ as well.

In order to check for the solvability of the discrete system (37) we proceed in two steps. For given $\alpha_\tau^{k-1} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $0 \le \alpha_\tau^{k-1} \le 1$ we look for the triple $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ given by the unique solution to the minimum problem

$$\min\Bigg\{ \int_\Omega \Big(\frac{1}{2}\mathbb{C}(\alpha_\tau^{k-1})e : e + \frac{1}{2\tau}\mathbb{D}(\alpha_\tau^{k-1})(e - e_{\mathrm{el},\tau}^{k-1}) : (e - e_{\mathrm{el},\tau}^{k-1}) - f_\tau^k \cdot u\Big)\,\mathrm{d}x$$
$$+ \frac{\rho}{2\tau^2}\|u - 2u_\tau^{k-1} + u_\tau^{k-2}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \mathcal{R}(\alpha_\tau^{k-1}, \pi - \pi_\tau^{k-1}) \ : \ (u, e, \pi) \in \mathscr{A}(u_{\mathrm{D},\tau}^k)\Bigg\}. \tag{43}$$

where $\mathscr{A}(\cdot)$ is defined in (3). The existence and uniqueness of solutions to (43) is ensured by Lemma 4.3 below.

Once $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ are found, we determine α_τ^k by solving

$$\min\Bigg\{ \int_\Omega \Bigg(\tau\zeta\Big(\frac{\alpha - \alpha_\tau^{k-1}}{\tau}\Big) + \frac{\kappa}{p}|\nabla\alpha|^p$$
$$+ \int_0^{\alpha(x)} \frac{1}{2}\mathbb{C}^\circ(s, \alpha_\tau^{k-1}(x))\, e_{\mathrm{el},\tau}^k(x) : e_{\mathrm{el},\tau}^k(x) - \phi^\circ(s, \alpha_\tau^{k-1}(x))\,\mathrm{d}s\Bigg)\,\mathrm{d}x \ : \tag{44}$$
$$\alpha \in W^{1,p}(\Omega),\ 0 \le \alpha \le 1\Bigg\} \tag{45}$$

in Lemma 4.5 below.

Lemma 4.3 (Existence of time-discrete displacements and strains)**.** *Let $\alpha_\tau^{k-1} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $0 \le \alpha_\tau^{k-1} \le 1$, be given. Then, there exists a unique triple $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k) \in \mathscr{A}(u_{\mathrm{D},\tau}^k)$ solving* (43).

Proof. The result follows by compactness, lower-semicontinuity, and by Korn's inequality (2). The uniqueness of the solution is a consequence of the strict convexity of the functional, and the fact that $\mathscr{A}(u_{\mathrm{D},\tau}^k)$ is affine. $\square$

Minimizers of (43) satisfy the following first order optimality conditions.

Proposition 4.4 (Time-discrete Euler-Lagrange equations for displacement and strains)**.** *Let $\alpha_\tau^{k-1} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a solution to* (37c) *satisfying $0 \le \alpha_\tau^{k-1} \le 1$. Let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of* (43)*. Then, $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ solves* (37a) *and* (42)*, $\operatorname{div}\sigma_\tau^k \in L^2(\Omega;\mathbb{R}^d)$, and $[\sigma_\tau^k \nu_\Gamma] = 0$ on Γ_{N}.*

Proof. We omit the proof of (37a), as it follows closely the argument in [6, Proposition 3.3]. The proof of (42) is postponed to Corollary 5.3. $\square$

We conclude this subsection by showing existence of solutions to (37c).

Lemma 4.5 (Existence of admissible time-discrete damage variables)**.** *Let $k \in \{1, \dots, T/\tau\}$, and assume that $\alpha_\tau^{k-1} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $0 \le \alpha_\tau^{k-1} \le 1$, is given. Let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of* (43)*. Then there exists $\alpha_\tau^k \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ solving* (37c)*, and satisfying $0 \le \alpha_\tau^k \le 1$.*

Proof. We preliminary observe that α_τ^k solves (37c) if and only if it minimizes the functional in (45). The existence of a minimizer $\alpha_\tau^k \in W^{1,p}(\Omega)$ follows by the continuity of $\phi(\cdot)$ and $\mathbb{C}(\cdot)$, by lower-semicontinuity, and by the Dominated Convergence Theorem. The fact that $\alpha_\tau^k(x) \leq 1$ for every $x \in \Omega$ is a consequence of the assumption that $0 \leq \alpha_\tau^{k-1} \leq 1$ in Ω, and of the constraint $\alpha_\tau^k \leq \alpha_\tau^{k-1}$. The constraint $0 \leq \alpha_\tau^k$ instead is satisfied due to the assumptions on $\mathbb{C}(\cdot)$ and $\phi(\cdot)$ (see Subsections 2.3 and 2.5), and owing to a truncation argument. □

5 A-priori energy estimates

In order to pass to the limit in the discrete scheme as the fineness τ of the partition goes to 0 we establish a few a-priori estimates on time interpolants between the quadruple identified via the time-discretization scheme of Section 4. We first rewrite [22, Proposition 2.2] in our framework.

Lemma 5.1 (Integration by parts)**.** *Let $\sigma \in \Sigma(\Omega)$, $u_{\mathrm{D}} \in H^1(\Omega;\mathbb{R}^d)$, and $(u, e_{\mathrm{el}}, \pi) \in \mathscr{A}(u_{\mathrm{D}})$ with $\mathscr{A}(\cdot)$ from* (3)*, with $u \in L^2(\Omega;\mathbb{R}^d)$. Assume that $[\sigma\nu_\Gamma] = 0$ on Γ_{N}. Then*

$$[\operatorname{dev}\sigma : \pi](\Omega \cup \Gamma_{\mathrm{D}}) + \int_\Omega \sigma : (e_{\mathrm{el}} - e(u_{\mathrm{D}}))\,\mathrm{d}x = -\int_\Omega \operatorname{div}\sigma \cdot (u - u_{\mathrm{D}})\,\mathrm{d}x.$$

Note that the above lemma serves as definition of $[\operatorname{dev}\sigma : \pi](\Omega \cup \Gamma_{\mathrm{D}})$, which is a priori not defined for $\operatorname{dev}\sigma \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ and $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$.

We are now in a position of providing, in the following lemmas and corollary, further optimality conditions for triples $(u_\tau^k, e_\tau^k, \pi_\tau^k)$ solving (43).

Lemma 5.2 (Discrete Euler-Lagrange equations for the plastic strain)**.** *Let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of* (43)*, and let σ_τ^k be the quantity defined in* (42)*. Then, there holds*

$$\mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k) - [\operatorname{dev}\sigma_\tau^k : \pi](\Omega \cup \Gamma_{\mathrm{D}}) \geq 0 \tag{46}$$

for every $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that there exist $u \in BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega;\mathbb{R}^d)$, and $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ with $(u, e, \pi) \in \mathscr{A}(0)$.

Proof. Considering variations of the form $(u_\tau^k, e_\tau^k, \pi_\tau^k) + \lambda(u, e, \pi)$ for $\lambda \geq 0$ and $(u, e, \pi) \in \mathscr{A}(0)$ in (43), by the convexity of $\mathcal{R}$ in its second variable we obtain

$$\frac{1}{\lambda}\big(\mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \lambda\pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k)\big) \leq \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k),$$

which yields

$$\int_\Omega \sigma_\tau^k : e\,\mathrm{d}x + \int_\Omega \rho\delta^2 u_\tau^k \cdot u\,\mathrm{d}x + \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k) - \int_\Omega f_\tau^k \cdot u\,\mathrm{d}x \geq 0, \tag{47}$$

for every $u \in BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega;\mathbb{R}^d)$, $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, and $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that $(u, e, \pi) \in \mathscr{A}(0)$. In view of Lemma 5.1, and by (37a) the previous inequality implies (46). □

Corollary 5.3 (Discrete flow-rule)**.** *Let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of* (43)*, let α_τ^k be the solution to* (37c) *provided by Lemma 4.5, and let σ_τ^k be the quantity defined in* (42)*. Then, $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k, \alpha_\tau^k)$ solve the discrete flow-rule* (42)*.*

Proof. The assert follows by choosing $\pi = \tau\delta\pi_\tau^k$, and $\pi = -\tau\delta\pi_\tau^k$ in (46). □

Lemma 5.4. *For $k \in \{1, \dots T/\tau\}$, let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of* (43)*, and let σ_τ^k be the quantity defined in* (42)*. Then, there holds*

$$\begin{aligned}\mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \pi) + \mathcal{R}(\alpha_\tau^{k-2}, \tau\delta\pi_\tau^{k-1} - \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k)\\ - \mathcal{R}(\alpha_\tau^{k-2}, \tau\delta\pi_\tau^{k-1}) - \tau[\mathrm{dev}\,\delta\sigma_\tau^k : \pi](\Omega \cup \Gamma_{\mathrm{D}}) \geq 0\end{aligned}$$

for every $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}}; \mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that there exist $u \in BD(\Omega; \mathbb{R}^d) \cap L^2(\Omega; \mathbb{R}^d)$, and $e \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ with $(u, e, \pi) \in \mathscr{A}(0)$.

Proof. Considering variations of the form $(u_\tau^{k-1}, e_\tau^{k-1}, \pi_\tau^{k-1}) - \lambda(u, e, \pi)$ for $\lambda \geq 0$ and $(u, e, \pi) \in \mathscr{A}(0)$ in (43) at level $i - 1$, the convexity of $\mathcal{R}$ in its second variable yields

$$\mathcal{R}(\alpha_\tau^{k-2}, \tau\delta\pi_\tau^{k-1} - \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^{k-1}) - \int_\Omega \Big(\sigma_\tau^{k-1} : e + \rho\delta^2 u_\tau^{k-1} \cdot u - f_\tau^{k-1} \cdot u\Big)\,\mathrm{d}x \geq 0, \quad (48)$$

for every $u \in BD(\Omega; \mathbb{R}^d) \cap L^2(\Omega; \mathbb{R}^d)$, $e \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, and $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}}; \mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that $(u, e, \pi) \in \mathscr{A}(0)$. The assert follows by summing (47) and (48), and by applying Lemma 5.1, and (37a). □

Let now $\underline{u}_\tau$, and $\bar{u}_\tau$ be the backward- and forward- piecewise constant in-time interpolants associated to the maps u_τ^k, namely

$$\underline{u}_\tau(0) := u_0, \quad \underline{u}_\tau(t) := u_\tau^{k-1} \quad \text{for every } t \in [(k-1)\tau, k\tau),\ k \in \{1, \dots, T/\tau\}, \quad (49)$$

and

$$\bar{u}_\tau(0) := u_0, \quad \bar{u}_\tau(t) := u_\tau^k \quad \text{for every } t \in ((k-1)\tau, k\tau],\ k \in \{1, \dots, T/\tau\}. \quad (50)$$

Denote by u_τ the associated piecewise affine in-time interpolant, that is

$$u_\tau(0) := u_0, \quad u_\tau(t) := \frac{(t-(k-1)\tau)}{\tau} u_\tau^k + \Big(1 - \frac{(t-(k-1)\tau)}{\tau}\Big) u_\tau^{k-1}, \quad (51)$$

for every $t \in ((k-1)\tau, k\tau]$, $k \in \{1, \dots, T/\tau\}$, and let finally $\widetilde{u}_\tau$ be the piecewise quadratic interpolant satisfying $\widetilde{u}(k\tau) = u_\tau^k$, and

$$\ddot{\widetilde{u}}_\tau(t) = \delta^2 u_\tau^k \quad \text{for every } t \in ((k-1)\tau, k\tau],\ k \in \{1, \dots, T/\tau\}.$$

Let $\underline{\alpha}_\tau, \bar{\pi}_\tau, \bar{e}_{\mathrm{el},\tau}, \bar{\alpha}_\tau, \pi_\tau, e_\tau$, and α_τ be defined analogously. The following proposition provides a first uniform estimate for the above quantities.

Proposition 5.5 (Discrete energy inequality)**.** *Under assumptions* (20)*, the following energy inequality holds true*

$$\begin{aligned}&\int_\Omega \frac{\rho}{2}|\dot{u}_\tau(T)|^2\,\mathrm{d}x + \frac{\tau}{2}\int_0^T\int_\Omega \rho|\ddot{\widetilde{u}}_\tau|^2\,\mathrm{d}x\,\mathrm{d}s + \int_\tau^T\int_\Omega \rho\dot{u}_\tau(\cdot - \tau)\cdot\ddot{\widetilde{u}}_{\mathrm{D},\tau}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + D_{\mathcal{R}}(\alpha_\tau; \pi_\tau; 0, T) + \int_\Omega \Big(\frac{1}{2}\mathbb{C}(\underline{\alpha}_\tau(T))\bar{e}_{\mathrm{el},\tau}(T) : \bar{e}_{\mathrm{el},\tau}(T) - \phi(\alpha_\tau(T))\,\mathrm{d}x + \frac{\kappa}{p}|\nabla\alpha_\tau(T)|^p\Big)\,\mathrm{d}x\\
&\quad + \int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\mathrm{el},\tau} : \dot{e}_{\mathrm{el},\tau}\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \eta\dot{\alpha}_\tau(T)^2\,\mathrm{d}x\\
&\leq \int_\Omega \Big(\frac{\rho}{2}v_0^2 + \rho\dot{u}_\tau(T)\cdot\dot{u}_{\mathrm{D},\tau}(t) + \rho v_0\cdot\delta u^1_{\mathrm{D},\tau}\Big)\,\mathrm{d}x\\
&\quad + \int_\Omega \frac{1}{2}\mathbb{C}(\alpha_0)(e(u_0) - \pi_0) : (e(u_0) - \pi_0) - \phi(\alpha_0)\,\mathrm{d}x + \frac{\kappa}{p}|\nabla\alpha_0|^p\,\mathrm{d}x\\
&\quad + \int_0^T\int_\Omega \Big(\mathbb{C}(\underline{\alpha}_\tau)\bar{e}_{\mathrm{el},\tau} : e(\dot{u}_{\mathrm{D},\tau}) + \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\mathrm{el},\tau} : e(\dot{u}_{\mathrm{D},\tau}) + \bar{f}_\tau\cdot(\dot{u}_\tau - \dot{u}_{\mathrm{D},\tau})\Big)\,\mathrm{d}x\,\mathrm{d}s. \quad (52)\end{aligned}$$

Proof. Fix $k \in \{1, \ldots, T/\tau\}$. Testing (37c) against $\delta\alpha_\tau^k$, we deduce the equality

$$\int_\Omega \eta|\delta\alpha_\tau^k|^2 \,\mathrm{d}x + \int_\Omega \frac{1}{2}\mathbb{C}^\circ(\alpha_\tau^k, \alpha_\tau^{k-1})\delta\alpha_\tau^k e_{\mathrm{el},\tau}^k : e_{\mathrm{el},\tau}^k \,\mathrm{d}x \\ + \int_\Omega \Big(-\phi^\circ(\alpha_\tau^k, \alpha_\tau^{k-1})\delta\alpha_\tau^k + |\nabla\alpha_\tau^k|^{p-2}\nabla\alpha_\tau^k \cdot \nabla(\delta\alpha_\tau^k)\Big) \,\mathrm{d}x = 0. \tag{53}$$

Taking $\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k$ as test function in (37a), we have

$$\int_\Omega \rho\delta^2 u_\tau^k \cdot (\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k) \,\mathrm{d}x - \int_\Omega \operatorname{div}\sigma_\tau^k \cdot (\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k) \,\mathrm{d}x = \int_\Omega f_\tau^k \cdot (\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k) \,\mathrm{d}x,$$

which by Lemma 5.1, and by the fact that $[\sigma_\tau^k \nu_\Gamma] = 0$ on Γ_{N} (see Proposition 4.4), yields

$$\int_\Omega \rho\delta^2 u_\tau^k \cdot (\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k) \,\mathrm{d}x + [\operatorname{dev}\sigma_\tau^k : \delta\pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}}) + \int_\Omega \sigma_\tau^k : (\delta e_{\mathrm{el},\tau}^k - e(\delta u_{\mathrm{D},\tau}^k)) \,\mathrm{d}x \\ = \int_\Omega f_\tau^k \cdot (\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k) \,\mathrm{d}x.$$

In view of Corollary 5.3, we obtain

$$\int_\Omega \rho\delta^2 u_\tau^k \cdot (\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k) \,\mathrm{d}x + \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi_\tau^k) + \int_\Omega \sigma_\tau^k : (\delta e_{\mathrm{el},\tau}^k - e(\delta u_{\mathrm{D},\tau}^k)) \,\mathrm{d}x \\ = \int_\Omega f_\tau^k \cdot (\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k) \,\mathrm{d}x. \tag{54}$$

For $n \in \{1, \ldots, T/\tau\}$, a discrete integration by parts in time yields

$$\tau \sum_{k=1}^n \rho\delta^2 u_\tau^k \cdot \delta u_\tau^k = \sum_{k=1}^n \rho\big((\delta u_\tau^k)^2 - \delta u_\tau^k \cdot \delta u_\tau^{k-1}\big) = \frac{1}{2}\rho(\delta u_\tau^n)^2 - \frac{1}{2}\rho v_0^2 + \frac{\tau^2}{2}\sum_{k=1}^n \rho(\delta^2 u_\tau^k)^2 \tag{55}$$

a.e. on Ω. Analogously, we deduce that

$$-\tau \sum_{k=1}^n \rho\delta^2 u_\tau^k \cdot \delta u_{\mathrm{D},\tau}^k = \tau \sum_{k=1}^n \rho\delta u_\tau^{k-1} \cdot \delta^2 u_{\mathrm{D},\tau}^k - \rho\delta u_\tau^n \cdot \delta u_{\mathrm{D},\tau}^n - \rho v_0 \cdot \delta u_{\mathrm{D},\tau}^0 \tag{56}$$

a.e. on Ω. Additionally, by the monotonicity of $\mathbb{C}$ in the Löwner ordering, and (22b), we have

$$\tau \sum_{k=1}^n \int_\Omega \sigma_\tau^k : \delta e_{\mathrm{el},\tau}^k \,\mathrm{d}x = \tau \sum_{k=1}^n \int_\Omega \mathbb{C}(\alpha_\tau^{k-1}) e_{\mathrm{el},\tau}^k : \delta e_{\mathrm{el},\tau}^k \,\mathrm{d}x + \tau \sum_{k=1}^n \int_\Omega \mathbb{D}(\alpha_\tau^{k-1})\delta e_{\mathrm{el},\tau}^k : \delta e_{\mathrm{el},\tau}^k \,\mathrm{d}x \\ \geq \int_\Omega \frac{1}{2}\mathbb{C}(\alpha_\tau^n) e_{\mathrm{el},\tau}^n : e_{\mathrm{el},\tau}^n \,\mathrm{d}x - \int_\Omega \frac{1}{2}\mathbb{C}(\alpha_0)(e(u_0) - \pi_0) : (e(u_0) - \pi_0) \,\mathrm{d}x \\ - \tau \sum_{k=1}^n \int_\Omega \frac{1}{2}\delta[\mathbb{C}(\alpha_\tau^k)] e_{\mathrm{el},\tau}^k : e_{\mathrm{el},\tau}^k \,\mathrm{d}x + \tau \sum_{k=1}^n \int_\Omega \mathbb{D}(\alpha_\tau^k)\delta e_{\mathrm{el},\tau}^k : \delta e_{\mathrm{el},\tau}^k \,\mathrm{d}x, \tag{57}$$

and

$$\frac{\tau}{2}\sum_{k=1}^n \int_\Omega \underbrace{\big(\mathbb{C}^\circ(\alpha_\tau^k, \alpha_\tau^{k-1})\delta\alpha_\tau^k - \delta[\mathbb{C}(\alpha_\tau^k)]\big)}_{=0} e_{\mathrm{el},\tau}^k : e_{\mathrm{el},\tau}^k \,\mathrm{d}x = 0. \tag{58}$$

Thus, multiplying (53) and (54) by τ, and summing for $k = 1, \dots, T/\tau$, in view of (55), (56), (57), and (58) we deduce

$$\int_\Omega \frac{\rho}{2}|\dot{u}_\tau(T)|^2 \,\mathrm{d}x + \frac{\tau}{2}\int_0^T\int_\Omega \rho|\ddot{\widetilde{u}}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\tau^T\int_\Omega \rho\dot{u}_\tau(\cdot-\tau)\cdot\ddot{\widetilde{u}}_{\mathrm{D},\tau}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\tau\sum_{k=1}^{T/\tau}\mathcal{R}(\alpha_\tau^{k-1},\delta\pi_\tau^k) + \frac{1}{2}\int_\Omega \mathbb{C}(\overline{\alpha}_\tau(T))\bar{e}_{\mathrm{el},\tau}(T) : \bar{e}_{\mathrm{el},\tau}(T)\,\mathrm{d}x$$
$$+\int_0^T\int_\Omega \mathbb{D}(\overline{\alpha}_\tau)\dot{e}_{\mathrm{el},\tau} : \dot{e}_{\mathrm{el},\tau}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega\Big(\eta\dot{\alpha}_\tau(T)^2 - \phi(\alpha_\tau(T)) + \frac{\kappa}{p}|\nabla\alpha_\tau(T)|^p\Big)\,\mathrm{d}x$$
$$\leq \int_\Omega\Big(\frac{\rho}{2}v_0^2 + \rho\dot{u}_\tau(T)\cdot\dot{u}_{\mathrm{D},\tau}(T) + \rho v_0\cdot\delta u^0_{\mathrm{D},\tau}\Big)\,\mathrm{d}x$$
$$+\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\alpha_0)(e(u_0)-\pi_0) : (e(u_0)-\pi_0) - \phi(\alpha_0) + \frac{\kappa}{p}|\nabla\alpha_0|^p\Big)\,\mathrm{d}x$$
$$+\int_0^T\int_\Omega \mathbb{C}(\underline{\alpha}_\tau)\bar{e}_{\mathrm{el},\tau} : e(\dot{u}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\mathrm{el},\tau} : e(\dot{u}_{\mathrm{D},\tau}) + \bar{f}_\tau\cdot(\dot{u}_\tau - \dot{u}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s. \tag{59}$$

Additionally, recalling definition (6), and observing that π_τ jumps exactly only in the points τk, $k \in \{1, \dots, T/\tau\}$, by the monotonicity of the maps α_τ (see Subsection 2.4), we have

$$D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T) = \tau\sum_{k=1}^{T/\tau}\mathcal{R}(\alpha_\tau^{k-1},\delta\pi_\tau^k). \tag{60}$$

This concludes the proof of the energy inequality (52) and of the proposition. □

Owing to the previous discrete energy inequality, we are now in a position to deduce some first a-priori estimates for the piecewise affine interpolants.

Proposition 5.6 (A-priori estimates)**.** *Under assumptions* (20)*, for τ small enough there exists a constant C, dependent only on the initial conditions, on f, and on u_{D}, such that*

$$\|\alpha_\tau\|_{H^1(0,T;L^2(\Omega))} + \|\alpha_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|e_{\mathrm{el},\tau}\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))}$$
$$+\|u_\tau\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} + \|u_\tau\|_{BV(0,T;BD(\Omega;\mathbb{R}^d))} + \|\pi_\tau\|_{BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}))}$$
$$+\|\underline{\alpha}_\tau\|_{L^\infty((0,T)\times\Omega)} + \|\bar{\alpha}_\tau\|_{L^\infty((0,T)\times\Omega)} + \|\bar{e}_{\mathrm{el},\tau}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \leq C. \tag{61}$$

Proof. The assert follows by Proposition 5.5, by the regularity of the applied force f and of the boundary datum u_{D}, and by applying Hölder's and discrete Gronwall's inequalities, for τ small enough. □

We proceed by performing at the discrete level the higher-order test with the strategy formally sketched in Subsection 3.2.

Proposition 5.7 (Second a-priori estimates)**.** *Under assumptions* (20)*, for τ small enough there exists a constant C, dependent only on the initial conditions, on f, and on u_{D}, such that*

$$\|\widetilde{u}_\tau\|_{H^2(0,T;L^2(\Omega;\mathbb{R}^d))} + \|u_\tau\|_{W^{1,\infty}(0,T;BD(\Omega;\mathbb{R}^d))}$$
$$+\|\pi_\tau\|_{W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}))} + \|e_{\mathrm{el},\tau}\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \leq C.$$

Proof. Fix $k \in \{1, \dots, T/\tau\}$, and consider the map $w_\tau^k := u_\tau^k + \chi \delta u_\tau^k$, where $\chi > 0$ is the constant introduced in Subsection 2.3. Since $\delta^2 u_\tau^k = (\delta w_\tau^k - \delta u_\tau^k)/\chi$, equation (37a) rewrites as

$$\rho\Big(\frac{\delta w_\tau^k}{\chi}\Big) - \operatorname{div} \sigma_\tau^k = f_\tau^k + \rho\Big(\frac{\delta u_\tau^k}{\chi}\Big). \tag{62}$$

Now, testing (62) against $\delta w_\tau^k - (\delta u_{\mathrm{D},\tau}^k + \chi \delta^2 u_{\mathrm{D},\tau}^k)$, by Lemma 5.1 we deduce the estimate

$$\begin{aligned}
&\frac{1}{\chi}\int_\Omega \rho |\delta w_k^\tau|^2 \,\mathrm{d}x + [\operatorname{dev} \sigma_\tau^k : (\delta \pi_\tau^k + \chi \delta^2 \pi_\tau^k)](\Omega \cup \Gamma_{\mathrm{D}}) \\
&\qquad + \int_\Omega \sigma_\tau^k : (\delta e_{\mathrm{el},\tau}^k + \chi \delta^2 e_{\mathrm{el},\tau}^k - e(\delta u_{\mathrm{D},\tau}^k) - \chi e(\delta^2 u_{\mathrm{D},\tau}^k)) \,\mathrm{d}x \\
&\quad = \int_\Omega f_\tau^k \cdot (\delta w_\tau^k - \delta u_{\mathrm{D},\tau}^k - \chi \delta^2 u_{\mathrm{D},\tau}^k) \,\mathrm{d}x + \frac{1}{\chi}\int_\Omega \rho \delta u_\tau^k \cdot (\delta w_\tau^k - \delta u_{\mathrm{D},\tau}^k - \chi \delta^2 u_{\mathrm{D},\tau}^k) \,\mathrm{d}x \\
&\qquad + \frac{1}{\chi}\int_\Omega \rho \delta w_\tau^k \cdot (\delta u_{\mathrm{D},\tau}^k + \chi \delta^2 u_{\mathrm{D},\tau}^k) \,\mathrm{d}x.
\end{aligned} \tag{63}$$

In view of Lemma 5.1 we have

$$[\operatorname{dev} \sigma_\tau^k : (\delta \pi_\tau^k + \chi \delta^2 \pi_\tau^k)](\Omega \cup \Gamma_{\mathrm{D}}) = [\operatorname{dev} \sigma_\tau^k : \delta \pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}}) + \chi [\operatorname{dev} \sigma_\tau^k : \delta^2 \pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}}).$$

Now, Corollary 5.3 yields

$$[\operatorname{dev} \sigma_\tau^k : \delta \pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}}) = \mathcal{R}(\alpha_\tau^{k-1}, \delta \pi_\tau^k), \tag{64}$$

whereas Lemma 5.4 entails

$$\begin{aligned}
\chi[\operatorname{dev} \sigma_\tau^k : \delta^2 \pi_\tau^k)](\Omega \cup \Gamma_{\mathrm{D}}) &= \chi \delta \{[\operatorname{dev} \sigma_\tau^k : \delta \pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}})\} - \chi [\operatorname{dev} \delta \sigma_\tau^k : \delta \pi_\tau^{k-1}](\Omega \cup \Gamma_{\mathrm{D}}) \\
&\geq \chi \delta \{[\operatorname{dev} \sigma_\tau^k : \delta \pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}})\} + \mathcal{R}(\alpha_\tau^{k-1}, \delta \pi_\tau^k) + \mathcal{R}(\alpha_\tau^{k-2}, \delta \pi_\tau^{k-1}) - \mathcal{R}(\alpha_\tau^{k-1}, \delta \pi_\tau^k + \delta \pi_\tau^{k-1}) \\
&\geq \chi \delta \{[\operatorname{dev} \sigma_\tau^k : \delta \pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}})\} + \mathcal{R}(\alpha_\tau^{k-1}, \delta \pi_\tau^k) + \mathcal{R}(\alpha_\tau^{k-1}, \delta \pi_\tau^{k-1}) - \mathcal{R}(\alpha_\tau^{k-1}, \delta \pi_\tau^k + \delta \pi_\tau^{k-1}) \\
&\geq \chi \delta \{[\operatorname{dev} \sigma_\tau^k : \delta \pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}})\},
\end{aligned} \tag{65}$$

where the second-to-last step follows by the fact that σ_{YLD} is nondecreasing (see Subsection 2.4), and the last step is a consequence of the triangle inequality. By combining (63), (64), and (65), we obtain

$$\begin{aligned}
&\frac{1}{\chi}\int_\Omega \rho |\delta w_\tau^k|^2 \,\mathrm{d}x + \chi \delta \{[\operatorname{dev} \sigma_\tau^k : \delta \pi_\tau^k](\Omega \cup \Gamma_{\mathrm{D}})\} + \mathcal{R}(\alpha_\tau^{k-1}, \delta \pi_\tau^k) \\
&\qquad + \int_\Omega \sigma_\tau^k : (\delta e_{\mathrm{el},\tau}^k + \chi \delta^2 e_{\mathrm{el},\tau}^k - e(\delta u_{\mathrm{D},\tau}^k) - \chi e(\delta^2 u_{\mathrm{D},\tau}^k)) \,\mathrm{d}x \\
&\quad \leq \int_\Omega f_\tau^k \cdot (\delta w_\tau^k - \delta u_{\mathrm{D},\tau}^k - \chi \delta^2 u_{\mathrm{D},\tau}^k) \,\mathrm{d}x + \frac{1}{\chi}\int_\Omega \rho \delta u_\tau^k \cdot (\delta w_\tau^k - \delta u_{\mathrm{D},\tau}^k - \chi \delta^2 u_{\mathrm{D},\tau}^k) \,\mathrm{d}x \\
&\qquad + \frac{1}{\chi}\int_\Omega \rho \delta w_\tau^k \cdot (\delta u_{\mathrm{D},\tau}^k + \chi \delta^2 u_{\mathrm{D},\tau}^k) \,\mathrm{d}x.
\end{aligned} \tag{66}$$

Multiplying (66) by τ, summing for $k = 1, \dots, n$, with $n \in \{1, \dots, T/\tau\}$, and using again (64) with $k = n$, we infer that

$$\begin{aligned}
&\frac{\tau}{\chi}\sum_{k=1}^n \int_\Omega \rho |\delta w_\tau^k|^2 \,\mathrm{d}x + \chi \mathcal{R}(\alpha_\tau^{n-1}, \delta \pi_\tau^n) - \chi \mathcal{R}(\alpha_\tau^{-1}, \delta \pi_\tau^0) \\
&\quad + \tau \sum_{k=1}^n \mathcal{R}(\alpha_\tau^{k-1}, \delta \pi_\tau^k) + \tau \sum_{k=1}^n \int_\Omega \sigma_\tau^k : (\delta e_{\mathrm{el},\tau}^k + \chi \delta^2 e_{\mathrm{el},\tau}^k) \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le \tau\sum_{k=1}^n\int_\Omega f_\tau^k\cdot(\delta w_\tau^k-\delta u_{\mathrm{D},\tau}^k-\chi\delta^2 u_{\mathrm{D},\tau}^k)\,\mathrm{d}x+\frac{\tau}{\chi}\sum_{k=1}^n\int_\Omega\rho\delta u_\tau^k\cdot(\delta w_\tau^k-\delta u_{\mathrm{D},\tau}^k-\chi\delta^2 u_{\mathrm{D},\tau}^k)\,\mathrm{d}x\\
&\quad+\frac{\tau}{\chi}\sum_{k=1}^n\int_\Omega\rho\delta w_\tau^k\cdot(\delta u_{\mathrm{D},\tau}^k+\chi\delta^2 u_{\mathrm{D},\tau}^k)\,\mathrm{d}x+\tau\sum_{k=1}^n\int_\Omega\sigma_\tau^k:(e(\delta u_{\mathrm{D},\tau}^k)+\chi e(\delta^2 u_{\mathrm{D},\tau}^k))\,\mathrm{d}x. \qquad (67)
\end{aligned}
$$

By (22b),

$$
\begin{aligned}
\tau\sum_{k=1}^n\int_\Omega\sigma_\tau^k:(\delta e_{\mathrm{el},\tau}^k+\chi\delta^2 e_{\mathrm{el},\tau}^k)\,\mathrm{d}x&=\tau\sum_{k=1}^n\int_\Omega\mathbb{C}(\alpha_\tau^{k-1})(e_{\mathrm{el},\tau}^k+\chi\delta e_{\mathrm{el},\tau}^k):(\delta e_{\mathrm{el},\tau}^k+\chi\delta^2 e_{\mathrm{el},\tau}^k)\,\mathrm{d}x\\
&\quad+\tau\sum_{k=1}^n\int_\Omega\mathbb{D}_0\delta e_{\mathrm{el},\tau}^k:(\delta e_{\mathrm{el},\tau}^k+\chi\delta^2 e_{\mathrm{el},\tau}^k)\,\mathrm{d}x.
\end{aligned}
$$

Thus, arguing as in (57), we have

$$
\begin{aligned}
\tau\sum_{k=1}^n\int_\Omega\sigma_\tau^k:(\delta e_{\mathrm{el},\tau}^k+\chi\delta^2 e_{\mathrm{el},\tau}^k)\,\mathrm{d}x&\ge\int_\Omega\frac12\mathbb{C}(\alpha_\tau^n)(e_{\mathrm{el},\tau}^n+\chi\delta e_{\mathrm{el},\tau}^n):(e_{\mathrm{el},\tau}^n+\chi\delta e_{\mathrm{el},\tau}^n)\,\mathrm{d}x\\
&\quad-\int_\Omega\frac12\mathbb{C}(\alpha_0)(e(u_0)-\pi_0+\chi(e(v_0)-\dot\pi_0)):(e(u_0)-\pi_0+\chi(e(v_0)-\dot\pi_0)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big(\tau\sum_{k=1}^n\mathbb{D}_0\delta e_{\mathrm{el},\tau}^k:\delta e_{\mathrm{el},\tau}^k+\frac{\chi}{2}\mathbb{D}_0\delta e_{\mathrm{el},\tau}^n:\delta e_{\mathrm{el},\tau}^n\,\mathrm{d}x-\frac{\chi}{2}\mathbb{D}_0 e(v_0):e(v_0)\Big)\,\mathrm{d}x. \qquad (68)
\end{aligned}
$$

Eventually, by (67) and (68), and by recalling (60), for every $t=k\tau$,

$$
\begin{aligned}
&\frac{1}{\chi}\int_0^t\!\!\int_\Omega\rho|\dot w_\tau|^2\,\mathrm{d}x\,\mathrm{d}s+\chi\mathcal{R}(\bar\alpha_\tau(t),\dot\pi_\tau(t))+D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,t)+\frac{\chi}{2}\int_\Omega\mathbb{D}_0\dot e_{\mathrm{el},\tau}(t):\dot e_{\mathrm{el},\tau}(t)\,\mathrm{d}x\\
&\qquad+\int_\Omega\mathbb{C}(\underline{\alpha}(t))(\bar e_{\mathrm{el},\tau}(t)+\chi\dot e_{\mathrm{el},\tau}(t)):(\bar e_{\mathrm{el},\tau}(t)+\chi\dot e_{\mathrm{el},\tau}(t))\,\mathrm{d}x+\int_0^t\!\!\int_\Omega\mathbb{D}_0\dot e_{\mathrm{el},\tau}:\dot e_{\mathrm{el},\tau}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\le\int_\Omega\Big(\frac12\mathbb{C}(\alpha_0)(e(u_0){-}\pi_0{+}\chi e(v_0){-}\chi\dot\pi_0){:}(e(u_0){-}\pi_0{+}\chi e(v_0){-}\chi\dot\pi_0)+\frac{\chi}{2}\mathbb{D}_0 e(v_0){:}e(v_0)\Big)\mathrm{d}x\\
&\qquad+\int_0^t\!\!\int_\Omega\bar f_\tau\cdot(\dot w_\tau-\dot u_{\mathrm{D},\tau}-\chi\ddot{\tilde u}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s+\frac{1}{\chi}\int_0^t\!\!\int_\Omega\dot u_\tau\cdot(\dot w_\tau-\dot u_{\mathrm{D},\tau}-\chi\ddot{\tilde u}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+\frac{1}{\chi}\int_0^t\!\!\int_\Omega\dot w_\tau\cdot(\dot u_{\mathrm{D},\tau}+\chi\ddot{\tilde u}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s+\chi\mathcal{R}(\alpha_0,\dot\pi_0)\\
&\qquad+\int_0^t\!\!\int_\Omega\big(\mathbb{C}(\underline{\alpha}_\tau)\bar e_{\mathrm{el},\tau}+\mathbb{D}(\underline{\alpha}_\tau)\dot e_{\mathrm{el},\tau}\big):(e(\dot u_{\mathrm{D},\tau})+\chi e(\ddot{\tilde u}_{\mathrm{D},\tau})))\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

The assert follows by Hölder's inequality, Proposition 5.6, and the assumptions on σ_{YLD} (see Subsection 2.4). □

6 Convergence and proof of Theorem 2.2

Proposition 6.1 (Compactness)**.** *Under the assumptions of Theorem 2.2, there exist α, e_{el}, π, and u such that $(u(t),e_{\mathrm{el}}(t),\pi(t))\in\mathscr{A}(u_{\mathrm{D}}(t))$ for every $t\in[0,T]$ (see* (3)*), the initial conditions* (16) *are satisfied, and up to the extraction of a (non-relabeled) subsequence, there holds*

$$
\alpha_\tau\to\alpha\quad\textit{weakly* in }\ H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;W^{1,p}(\Omega)), \qquad (69a)
$$

$$e_{\mathrm{el},\tau} \to e_{\mathrm{el}} \quad \textit{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{69b}$$

$$\pi_\tau \to \pi \quad \textit{weakly* in } W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})), \tag{69c}$$

$$u_\tau \to u \quad \textit{weakly* in } W^{1,\infty}(0,T;BD(\Omega;\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{69d}$$

$$\underline{\alpha}_\tau \to \alpha \;\textit{ and }\; \bar{\alpha}_\tau \to \alpha \quad \textit{weakly* in } L^\infty((0,T)\times\Omega), \tag{69e}$$

$$\bar{e}_{\mathrm{el},\tau} \to e_{\mathrm{el}} \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{69f}$$

$$\widetilde{u}_\tau \to u \quad \textit{weakly in } H^2(0,T;L^2(\Omega;\mathbb{R}^d)). \tag{69g}$$

Proof. Properties (69a)–(69d) are a consequence of Propositions 5.6 and 5.7. The admissibility condition (C1) (see Definition 2.1) follows by the same argument as in [22, Lemma 2.1]. Additionally, by Proposition 5.6 there holds

$$u_\tau \to u \quad \text{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{70}$$

and there exist $\breve{\alpha},\hat{\alpha}\in L^\infty((0,T)\times\Omega)$, and $\hat{e}\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ such that

$$\bar{\alpha}_\tau \to \breve{\alpha} \;\text{ and }\; \underline{\alpha}_\tau \to \hat{\alpha} \quad \text{weakly* in } L^\infty((0,T)\times\Omega)$$

and

$$\bar{e}_{\mathrm{el},\tau} \to \hat{e} \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})).$$

Additionally by Proposition 5.7 there exists a map $\hat{u}\in H^2(0,T;L^2(\Omega;\mathbb{R}^d))$ such that, up to the extraction of a (non-relabeled) subsequence,

$$\widetilde{u}_\tau \to \hat{u} \quad \text{weakly in } H^2(0,T;L^2(\Omega;\mathbb{R}^d)). \tag{71}$$

By the compact embeddings of $W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))$ and $H^2(0,T;L^2(\Omega;\mathbb{R}^d))$ into $C_{\mathrm{w}}(0,T;L^2(\Omega;\mathbb{R}^d))$, we deduce

$$u_\tau(t) \to u(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^d), \tag{72}$$

and

$$\widetilde{u}_\tau(t) \to \hat{u}(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^d), \tag{73}$$

for every $t\in[0,T]$. To complete the proof of (69) , it remains to show that $\breve{\alpha}=\hat{\alpha}=\alpha$, $\hat{e}=e_{\mathrm{el}}$, and $\hat{u}=u$.

We proceed by showing this last equality; the proof of the other two identities is analogous. Fix $k\in\{1,\dots,T/\tau\}$, and $t\in((k-1)\tau,k\tau]$. Then, using the fact that

$$\dot{\widetilde{u}}_\tau(t)=\frac{(t-(k-1)\tau)}{\tau}\delta u^k_\tau+\Big(1-\frac{(t-(k-1)\tau)}{\tau}\Big)\delta u^{k-1}_\tau$$

for every $t\in((k-1)\tau,k\tau]$, we have

$$\int_0^T \|\dot{\widetilde{u}}_\tau(t)-\dot{u}_\tau(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}\mathrm{d}t=\sum_{k=1}^N\int_{(k-1)\tau}^{k\tau}\|\dot{\widetilde{u}}_\tau(t)-\dot{u}_\tau(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}\mathrm{d}t$$
$$=\tau^2\sum_{k=1}^N\int_{(k-1)\tau}^{k\tau}(1-\overline{\alpha}_\tau(t))^2\,\mathrm{d}t\left\|\frac{\dot{u}_\tau-\dot{u}_\tau(\cdot-\tau)}{\tau}\right\|^2=\frac{\tau^2}{3}\sum_{k=1}^N\tau\|\delta^2u_k\|^2_{L^2(\Omega;\mathbb{R}^d)}\le C\tau^2, \tag{74}$$

where the last inequality follows by Proposition 5.7. The assert follows then by combining (72), (73), and (74). □

Proposition 6.2 (Strong convergence of the elastic strains)**.** *Let $e_{\rm el}$ be the map identified in Proposition 6.1. Under the assumptions of Theorem 2.2, there holds*

$$e_{\mathrm{el},\tau} \to e_{\rm el} \quad \textit{strongly in } H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})), \tag{75}$$

and

$$\bar{e}_{\mathrm{el},\tau}(t) \to e_{\rm el}(t) \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})) \quad \textit{for a.e. } t\in[0,T]. \tag{76}$$

Proof. For $k \in \{1,\dots,T/\tau\}$, denote by $\delta e_{\rm el}(k\tau)$ the quantity

$$\delta e_{\rm el}(k\tau) := \frac{e_{\rm el}(k\tau) - e_{\rm el}((k-1)\tau)}{\tau},$$

and by $\bar{e}^\tau_{\rm el}$, $e^\tau_{\rm el}$ the forward-piecewise constant and the affine interpolants between the values $\{e(k\tau)\}_{k=1,\dots,T/\tau}$ (see (49) and (51)). Let $\delta u(k\tau)$, $\delta\pi(k\tau)$, $\bar{u}^\tau$, u^τ, $\bar{\pi}^\tau$, and π^τ be defined analogously. Note that here we cannot directly use the values at time t, for this would prevent relation (81) to hold. Here, the pointwise value of π is simply that of its right-continuous representative.

Fix $k \in \{1,\dots,T/\tau\}$. We proceed by testing the time-discrete equilibrium equation (37a) by $\delta u^k_\tau - \delta u(k\tau)$. On the one hand, by Lemma 5.1, we have

$$\begin{aligned}\int_\Omega \rho\delta^2 u^k_\tau \cdot (\delta u^k_\tau - \delta u(k\tau))\,\mathrm{d}x + [\operatorname{dev}\sigma^k_\tau : (\delta\pi^k_\tau - \delta\pi(k\tau))](\Omega\cup\Gamma_{\rm D})&\\ + \int_\Omega \sigma^k_\tau : (\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau))\,\mathrm{d}x - \int_\Omega f^k_\tau\cdot(\delta u^k_\tau - \delta u(k\tau))\,\mathrm{d}x = 0.&\end{aligned} \tag{77}$$

On the other hand, Lemma 5.2 yields

$$[\operatorname{dev}\sigma^k_\tau : (\delta\pi^k_\tau - \delta\pi(k\tau))](\Omega\cup\Gamma_{\rm D}) \ge \mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau) - \mathcal{R}(\alpha^{k-1}_\tau,\delta\pi(k\tau)). \tag{78}$$

By combining (77) and (78), we obtain

$$\int_\Omega \sigma^k_\tau : (\delta e^k_{\mathrm{el},\tau} - \delta e(k\tau))\,\mathrm{d}x \le \mathcal{R}(\alpha^{k-1}_\tau,\delta\pi(k\tau)) - \mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau) + \int_\Omega (f^k_\tau - \rho\delta^2 u^k_\tau)\cdot(\delta u^k_\tau - \delta u(k\tau))\,\mathrm{d}x. \tag{79}$$

In view of the definition of σ_k there holds

$$\begin{aligned}\int_\Omega \sigma^k_\tau : (\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau))\,\mathrm{d}x &= \int_\Omega \mathbb{C}(\alpha^{k-1}_\tau)(e^k_{\mathrm{el},\tau} - e_{\rm el}(k\tau)) : (\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau))\,\mathrm{d}x\\ &+ \int_\Omega \mathbb{D}(\alpha^{k-1}_\tau)(\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau)):(\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau)) + \mathbb{D}(\alpha^{k-1}_\tau)\delta e_{\rm el}(k\tau):(\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau))\,\mathrm{d}x\\ &+ \int_\Omega \mathbb{C}(\alpha(k\tau))e_{\rm el}(k\tau) : (\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau))\,\mathrm{d}x\\ &- \int_\Omega (\mathbb{C}(\alpha(k\tau)) - \mathbb{C}(\alpha^{k-1}_\tau))e_{\rm el}(k\tau) : (\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau))\,\mathrm{d}x.\end{aligned} \tag{80}$$

Let now $n \in \{1,\dots,T/\tau\}$. By the monotonicity of $\mathbb{C}$ in the Löwner order, arguing as in the proof of (57), we deduce

$$\begin{aligned}\tau\sum_{k=1}^n \int_\Omega \mathbb{C}(\alpha^{k-1}_\tau)(e^k_{\mathrm{el},\tau} - e_{\rm el}(k\tau)) : (\delta e^k_{\mathrm{el},\tau} - \delta e_{\rm el}(k\tau))\,\mathrm{d}x&\\ \ge \int_\Omega \frac{1}{2}\mathbb{C}(\alpha^n_\tau)(e^n_{\mathrm{el},\tau} - e_{\rm el}(n\tau)) : (e^n_{\mathrm{el},\tau} - \delta e_{\rm el}(n\tau))\,\mathrm{d}x.&\end{aligned} \tag{81}$$

Multiplying (79) by τ, and summing for $k = 1, \dots, T/\tau$, in view of (80) and (81), we obtain the estimate

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\mathbb{C}(\underline{\alpha}_\tau(T))(\bar{e}_{\mathrm{el},\tau}(T) - \bar{e}_{\mathrm{el}}^\tau(T)) : (\bar{e}_{\mathrm{el},\tau}(T) - \bar{e}_{\mathrm{el}}^\tau(T))\,\mathrm{d}x \\
&\quad + \int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)(\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}^\tau) : (\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \le \tau\sum_{k=1}^{T/\tau} \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi(k\tau)) - D_{\mathcal{R}}(\alpha_\tau; \pi_\tau; 0, T) + \int_0^T\int_\Omega (\bar{f}_\tau - \ddot{\tilde{u}}_\tau)\cdot(\dot{u}_\tau - \dot{u}^\tau)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\mathrm{el}}^\tau : (\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s - \int_0^T\int_\Omega \mathbb{C}(\overline{\alpha}^\tau)\bar{e}_{\mathrm{el}}^\tau : (\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T\int_\Omega \big(\mathbb{C}(\overline{\alpha}^\tau) - \mathbb{C}(\underline{\alpha}_\tau)\big)\bar{e}_{\mathrm{el}}^\tau : (\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}^\tau)\big)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

By Proposition 6.1 we infer that

$$
\begin{aligned}
&\limsup_{\tau\to 0}\int_0^T\int_\Omega \mathbb{D}(\overline{\alpha}_\tau)(\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}) : (\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}})\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \limsup_{\tau\to 0}\Big\{ \int_0^T \mathcal{R}(\underline{\alpha}_\tau, \dot{\pi}^\tau)\,\mathrm{d}s - D_{\mathcal{R}}(\alpha_\tau; \pi_\tau; 0, T) + \int_0^T\int_\Omega (\bar{f}_\tau - \ddot{\tilde{u}}_\tau)\cdot(\dot{u}_\tau - \dot{u}^\tau)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad - \int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\mathrm{el}}^\tau : (\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s - \int_0^T\int_\Omega \mathbb{C}(\underline{\alpha}^\tau)\bar{e}_{\mathrm{el}}^\tau : (\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_0^T\int_\Omega \big(\mathbb{C}(\bar{\alpha}^\tau) - \mathbb{C}(\underline{\alpha}_\tau)\big)\bar{e}_{\mathrm{el}}^\tau : (\dot{e}_{\mathrm{el},\tau} - \dot{e}_{\mathrm{el}}^\tau)\big)\,\mathrm{d}x\,\mathrm{d}s\Big\}.
\end{aligned}
$$

Since $u \in H^2(0, T; L^2(\Omega; \mathbb{R}^d))$ and $e_{\mathrm{el}} \in W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$, it follows that

$$
u^\tau \to u \quad \text{strongly in } L^2((0,T)\times\Omega; \mathbb{R}^d), \tag{82}
$$

and

$$
\bar{e}_{\mathrm{el}}^\tau \to \bar{e}_{\mathrm{el}} \quad \text{strongly in } L^2((0,T)\times\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}). \tag{83}
$$

Additionally, by the definition of the affine interpolants,

$$
\dot{e}_{\mathrm{el}}^\tau \to \dot{e}_{\mathrm{el}} \quad \text{strongly in } L^2((0,T)\times\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{84}
$$

$$
\dot{\pi}^\tau \to \dot{\pi} \quad \text{strongly in } L^1(0,T; \mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}}; \mathbb{R}^{d\times d}_{\mathrm{dev}})). \tag{85}
$$

By (69a) and by the Aubin-Lions Lemma, up to the extraction of a (non-relabeled) subsequence,

$$
\alpha_\tau \to \alpha \quad \text{strongly in } C([0,T]\times\bar{\Omega}). \tag{86}
$$

Since $\alpha \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; W^{1,p}(\Omega))$,

$$
\bar{\alpha}^\tau,\ \underline{\alpha}^\tau \to \alpha \quad \text{strongly in } L^2((0,T)\times\Omega). \tag{87}
$$

Thus, by the Dominated Convergence Theorem, we deduce that

$$
\mathbb{C}(\bar{\alpha}^\tau)\bar{e}_{\mathrm{el}}^\tau \to \mathbb{C}(\alpha)e_{\mathrm{el}} \quad \text{strongly in } L^2((0,T)\times\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{88}
$$

$$
\big(\mathbb{C}(\bar{\alpha}^\tau) - \mathbb{C}(\underline{\alpha}_\tau)\big)\bar{e}_{\mathrm{el}}^\tau \to 0 \quad \text{strongly in } L^2((0,T)\times\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{89}
$$

$$\mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\rm el}^\tau \to \mathbb{D}(\alpha)\dot{e}_{\rm el} \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^{d\times d}_{\rm sym}). \tag{90}$$

Finally, by the assumptions on f, we have

$$\bar{f}_\tau \to f \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^d). \tag{91}$$

By combining (82)–(91) we conclude that

$$\begin{aligned}\limsup_{\tau\to 0}&\int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)(\dot{e}_{{\rm el},\tau}-\dot{e}_{\rm el}):(\dot{e}_{{\rm el},\tau}-\dot{e}_{\rm el})\,\mathrm{d}x\,\mathrm{d}s\\ &\le \limsup_{\tau\to 0}\int_0^T \mathcal{R}(\bar{\alpha}_\tau,\dot{\pi}^\tau)\,\mathrm{d}s - \liminf_{\tau\to 0} D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T).\end{aligned} \tag{92}$$

Arguing as in [22, Theorem 7.1], since $\pi \in W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))$ we deduce the uniform bound

$$\int_0^T \|\dot{\pi}^\tau\|_{\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})} = \tau\sum_{k=1}^{T/\tau}\|\delta\pi(k\tau)\|_{\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})} \le \int_0^T \|\dot{\pi}\|_{\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})}\,\mathrm{d}s \le C. \tag{93}$$

Hence, by (93) and by the continuity and monotonicity of $\sigma_{\rm YLD}(\cdot)$ (see Subsection 2.4), there holds

$$\begin{aligned}&\limsup_{\tau\to 0}\int_0^T \mathcal{R}(\bar{\alpha}_\tau,\dot{\pi}^\tau)\,\mathrm{d}s \le \limsup_{\tau\to 0}\int_0^T \mathcal{R}(\alpha_\tau,\dot{\pi}^\tau)\,\mathrm{d}s\\ &\le \limsup_{\tau\to 0}\Big\{\int_0^T \mathcal{R}(\alpha,\dot{\pi}^\tau)\,\mathrm{d}s + \Big|\int_0^T(\mathcal{R}(\alpha_\tau,\dot{\pi}^\tau)-\mathcal{R}(\alpha,\dot{\pi}^\tau))\,\mathrm{d}s\Big|\Big\}\\ &\le \limsup_{\tau\to 0}\int_0^T \mathcal{R}(\alpha,\dot{\pi}^\tau)\,\mathrm{d}s + C\limsup_{\tau\to 0}\|\sigma_{\rm YLD}(\alpha_\tau)-\sigma_{\rm YLD}(\alpha)\|_{L^\infty((0,T)\times\Omega)} = \int_0^T \mathcal{R}(\alpha,\dot{\pi})\,\mathrm{d}s,\end{aligned} \tag{94}$$

where the last step follows by (85).

To complete the proof of (75), it remains to show that

$$D_{\mathcal{R}}(\alpha;\pi;0,T) \le \liminf_{\tau\to 0} D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T). \tag{95}$$

Let $0 < t_0 < t_1 < \cdots < t_n \le T$. By the definition of $D_{\mathcal{R}}$, we have

$$\begin{aligned}D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T) &\ge \sum_{j=1}^{T/\tau}\mathcal{R}(\alpha_\tau(t_j),\pi_\tau(t_j)-\pi_\tau(t_{j-1})) \ge \sum_{j=1}^{T/\tau}\mathcal{R}(\alpha(t_j),\pi_\tau(t_j)-\pi_\tau(t_{j-1}))\\ &\quad - \tau\sum_{j=1}^{T/\tau}\|\sigma_{\rm YLD}(\alpha_\tau(t_j))-\sigma_{\rm YLD}(\alpha(t_j))\|_{L^\infty(\Omega)}\|\dot{\pi}_\tau\|_{L^\infty(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))}.\end{aligned}$$

Now, by (69c) and (86),

$$\lim_{\tau\to 0}\tau\sum_{j=1}^{T/\tau}\|\sigma_{\rm YLD}(\alpha_\tau(t_j))-\sigma_{\rm YLD}(\alpha(t_j))\|_{L^\infty(\Omega)}\|\dot{\pi}_\tau\|_{L^\infty(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))} = 0.$$

Thus, by (69c),

$$\liminf_{\tau\to 0} D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T) \ge \liminf_{\tau\to 0}\sum_{j=1}^{T/\tau}\mathcal{R}(\alpha(t_j),\pi_\tau(t_j)-\pi_\tau(t_{j-1})) \ge \sum_{j=1}^{T/\tau}\mathcal{R}(\alpha(t_j),\pi(t_j)-\pi(t_{j-1})).$$

By the arbitrariness of the partition, we deduce (95), which in turn yields (75).

Property (76) follows arguing exactly as in the proof of (74). □

Let us now conclude the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $(u, e_{\rm el}, \pi, \alpha)$ be the limit quadruple identified in Proposition 6.1. By Proposition 6.1 we already know that condition (C1) in Definition 2.1 is fulfilled. For convenience of the reader we subdivide the proof of the remaining conditions into three steps.
Step 1: We first show that the limit quadruple satisfies the equilibrium equation (14a). In view of (37a) we have

$$\rho\ddot{\tilde{u}}_\tau - \operatorname{div}\left(\mathbb{C}(\underline{\alpha}_\tau)\bar{e}_{{\rm el},\tau} + \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{{\rm el},\tau}\right) = \bar{f}_\tau$$

for a.e. $x \in \Omega$ and $t \in [0,T]$, and for all $\tau > 0$. In particular, for all $\varphi \in C_c^\infty(0,T; C_c^\infty(\Omega))$ there holds

$$\int_0^T\int_\Omega \rho\ddot{\tilde{u}}_\tau \cdot \varphi\, + (\mathbb{C}(\underline{\alpha}_\tau)\bar{e}_{{\rm el},\tau} + \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{{\rm el},\tau}) : e(\varphi)\,{\rm d}x\,{\rm d}t = \int_0^T\int_\Omega \bar{f}_\tau \cdot \varphi\,{\rm d}x\,{\rm d}t.$$

By (69e-g) and (91), we infer that

$$\int_0^T\int_\Omega \rho\ddot{u} \cdot \varphi + (\mathbb{C}(\alpha)e_{\rm el} + \mathbb{D}(\alpha)\dot{e}_{\rm el}) : e(\varphi)\,{\rm d}x\,{\rm d}t = \int_0^T\int_\Omega f \cdot \varphi\,{\rm d}x\,{\rm d}t,$$

which in turn yields (14a) for a.e. $x \in \Omega$, and $t \in [0,T]$. In particular, (69g) guarantees that $u(0) = u^0$, and $\dot{u}(0) = v_0$.

Step 2: The limit energy inequality is a direct consequence of (52), Propositions 6.1 and 6.2, and (95).
Step 3: We now pass to the limit in the discrete damage law. In view of (37c), for every $k \in \{1,\ldots,T/\tau\}$ we deduce the inequality

$$\int_\Omega \Big(\phi^\circ(\alpha_\tau^k, \alpha_\tau^{k-1}) + \operatorname{div}\left(|\nabla\alpha_\tau^k|^{p-2}\nabla\alpha_\tau^k\right) - \frac{1}{2}\mathbb{C}^\circ(\alpha_\tau^k, \alpha_\tau^{k-1})e_{{\rm el},\tau}^k : e_{{\rm el},\tau}^k - \eta\delta\alpha_\tau^k\Big)(\varphi - \delta\alpha_\tau^k)\,{\rm d}x = 0$$

for all $\varphi \in W^{1,p}(\Omega)$ such that $\varphi(x) \le 0$ for a.e. $x \in \Omega$. Thus, summing in k, we conclude that

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(\phi^\circ(\bar{\alpha}_\tau, \underline{\alpha}_\tau)\varphi - |\nabla\bar{\alpha}_\tau|^{p-2}\nabla\bar{\alpha}_\tau \cdot \nabla\varphi - \frac{1}{2}\mathbb{C}^\circ(\bar{\alpha}_\tau, \underline{\alpha}_\tau)\bar{e}_{{\rm el},\tau} : \bar{e}_{{\rm el},\tau}\varphi - \eta\dot{\alpha}_\tau\varphi\Big)\,{\rm d}x\,{\rm d}t \\
&\quad\le \int_\Omega \Big(\phi(\alpha_\tau(T)) - \phi(\alpha_0) - \frac{\kappa}{p}|\nabla\bar{\alpha}_\tau(T)|^p + \frac{\kappa}{p}|\nabla\alpha_0|^p\Big)\,{\rm d}x \\
&\qquad - \int_0^T\int_\Omega \frac{1}{2}\Big(\mathbb{C}^\circ(\bar{\alpha}_\tau, \underline{\alpha}_\tau)\bar{e}_{{\rm el},\tau} : \bar{e}_{{\rm el},\tau}\Big)\dot{\alpha}_\tau\,{\rm d}x\,{\rm d}t - \int_0^T\int_\Omega \eta\dot{\alpha}_\tau^2\,{\rm d}x\,{\rm d}t\,.
\end{aligned}$$

Condition (C4) in Definition 2.1 follows then in view of Propositions 6.1 and 6.2. □

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