problem sheet 10

discussion: Tuesday, 15.12.20

10.1. (connection of 1. and 2. Piola-Kirchhoff tensor) Let $W : \mathbb{S}(3) \to \mathbb{R}$ be a (smooth) function that is defined on the vector space of the symmetric 3×3 matrices. Define on the space of matrices (with determinant > 0) the function $\widehat{W}(F) := W(F^\top F)$. Show:

$$\frac{\partial \widehat{W}(F)}{\partial F} = 2F \frac{\partial W(C)}{\partial C}, \qquad C = F^\top F.$$

Hint: Recall that for a scalar function W that is defined on the set of matrices, the derivative $\frac{\partial W}{\partial C}$ given by $W(C+\Delta) = W(C) + \frac{\partial W}{\partial C} : \Delta + \cdots$ and that we agreed that $\frac{\partial W(C)}{\partial C}$ is symmetric.

10.2. (Neo-Hookean materials) The deformation energy is

$$W(C) = \frac{1}{2}\mu\left[\operatorname{tr}(C - I) + \frac{2}{\beta}\left\{(\det C)^{-\beta/2} - 1\right\}\right], \qquad \beta > 0.$$

Show that for *small* strains $E = \frac{1}{2}(C - I)$, one obtains the classical Hooke law. What are the corresponding Lamé parameters λ and μ? *Hint:* You may use that E is symmetric and the representation of $\det(\lambda - A) = \lambda^3 - i_1(A)\lambda^2 + i_2(A)\lambda + i_3(A)$ with $i_1(A) = \operatorname{tr} A$, $i_2(A) = \frac{1}{2}(\operatorname{tr} A)^2 - \frac{1}{2}\operatorname{tr} A^2$, $i_3(A) = \det A$ to show $\det C = 1 + 2\operatorname{tr} E + 2(\operatorname{tr} E)^2 - 2E : E + \cdots$.

10.3. In elasticity theory the energy functional $A \mapsto W(A)$ is defined on (a subset of) the set of matrices. Convexity (of functionals, sets) is an important tool to show existence of minimizers. Show the following negative result: The set $M := \{A \in \mathbb{R}^{d\times d} \mid \det A > 0\}$ is *not* convex.

10.4. (incompressible materials) Consider the energy minimization

$$J(u) := \int_\Omega \mu\varepsilon(u) : \varepsilon(u) + \frac{\lambda}{2}(\operatorname{tr}\varepsilon(u))^2\, dx - \int_\Omega f\cdot u - \int_{\Gamma_N} g\cdot u, \qquad u \in (H^1_D(\Omega))^3 = \{u \in (H^1(\Omega))^3 \mid u|_{\Gamma_D} = 0\}.$$

Incompressible materials are characterized by $\lambda \to \infty$. For fixed μ, f, g, let u_λ the minimizer of J.

a) Show that $\|u_\lambda\|_{H^1(\Omega)}$ is bounded uniformly in λ (for $\lambda \to \infty$). Therefore, the is at least a subsequence with a weak limit in $(H^1_D(\Omega))^3$. (In fact, even the full sequence converges in $(H^1_D)^3$.)

b) Show that u_λ also solves the following saddle point problem: Find $u_\lambda \in (H^1_D(\Omega))^3$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} 2\mu(\varepsilon(u_\lambda), \varepsilon(v))_{L^2(\Omega)} + (\nabla\cdot v, p)_{L^2(\Omega)} &= l(v) := (f, v)_{L^2(\Omega)} + (g, v)_{L^2(\Gamma_N)} \qquad \forall v \in (H^1_D(\Omega))^3 \\ (\nabla\cdot u_\lambda, q)_{L^2(\Omega)} - \frac{1}{\lambda}(p, q)_{L^2(\Omega)} &= 0 \qquad \forall q \in L^2(\Omega). \end{aligned}$$

How was p defined? Here, $(\varepsilon(u), \varepsilon(v))_{L^2(\Omega)} = \int_\Omega \varepsilon(u) : \varepsilon(v)$. *Remark:* If $\Gamma_D = \partial\Omega$, then one can/should actually additionally require $\int_\Omega p = 0$ and replace $L^2(\Omega)$ with the factor space $L^2(\Omega)\setminus \mathbb{R}$.

10.5. For special geometries ("plates", "shells") special equations of elasticity are commonly used. The key idea to derive these equations to assume that the displacement has a certain form in the transverse direction.

Let $\Omega = \omega \times (-t/2, t/2)$, where ω denotes the mid surface and $t > 0$ the thickness of the plate.

Make the following assumptions ("assumptions of Reissner and Mindlin")

(i) No surface forces are applied but only volume forces f that are of the form $f(x, y, z) = f(x, y)(0, 0, 1)^\top$.

(ii) The stress σ_{33} satisfies $\sigma_{33} = 0$ in Ω

(iii) The displacements have the form

$$u_i(x,y,z) = -z\theta_i(x,y), \qquad i \in \{1,2\}, \qquad u_3(x,y,z) = w(x,y)$$

Show: If one inserts the ansatz (i) and (iii) into the minimization problem of linear elastiicty, then one obtains a minimization problem for (θ_1, θ_2, w) in which the functional Π is minimized:

$$\begin{aligned} \Pi(\theta, w) &:= \frac{t^3}{12} a(\theta,\theta) + \frac{\mu t}{2} \int_\omega |\nabla w - \theta|^2 \, dx_1 \, dx_2 - t \int_\omega f w \, dx_1 \, dx_2 \\ a(\theta, \psi) &:= \mu \int_\omega \varepsilon(\theta) : \varepsilon(\psi) + \widetilde{\lambda} \operatorname{div} \theta \operatorname{div} \psi \, dx_1 \, dx_2, \end{aligned}$$

where $\theta = (\theta_1, \theta_2)$ and the 2×2-matrix $\varepsilon(\theta)$ is given by $(\varepsilon(\theta))_{ij} = \frac{1}{2}(\frac{\partial \theta_i}{\partial x_j} + \frac{\partial \theta_j}{\partial x_i})$. What is the parameter $\widetilde{\lambda}$?

Remarks:

(1) The condition $\sigma_{33} = 0$ is not used in your derivation. In the classical Reissner-Mindlin plate theory, it is accounted for and leads to $\widetilde{\lambda} = \frac{\lambda}{2} \frac{\mu}{\lambda + 2\mu}$ (cf. [Chap. VI, Sec. 5, Braess: FEM]).

(2) a good scaling for the load f is $f = t^2 \widetilde{f}$ for some function $\widetilde{f}$ that is $O(1)$. (In fact, the plate cannot sustain a load that is larger as it would break for small thickness t.) By dividing Π by t^{-3} (which doesn't change the minimizer!) one see that for small t, the method enforces the constraint $\theta = \nabla w$.

(3) The constraint $\theta = \nabla w$ can also be used directly, which is known as "Kirchhoff" or "Kirchhoff-Love" assumption. Physically, one then assumes that the lines that are orthogonal to the mid plane ω prior to the deformation are also orthogonal to the mid plane after the deformation. This leads to the ansatz

$$\theta_i(x,y) = \frac{\partial w}{\partial x_i}(x,y), \qquad u_i(x,y) = z \frac{\partial w}{\partial x_i}, \qquad i = 1, 2;$$

inserting this ansatz into the minimization problem of linear elasticity leads to an equation of fourth order.