problem sheet 8

discussion: Tuesday, 1.12.20

8.1. Consider the singularly perturbed problem

$$\varepsilon y'' + 2y' + y = e^x, \qquad x \in (0,1), \qquad y(0) = y(1) = 0 \tag{1}$$

for small $\varepsilon > 0$.

a) Formulate the equations for the "outer expansion"

b) Formulate the equations for the "inner expansion" on the interval $(0,\varepsilon)$. For that, use that the right-hand side $f(x) = e^x$ can be expressed on $(0,\varepsilon)$ in the new variable ξ as $f(x) = f(\varepsilon\xi) = f(0) + \varepsilon\xi f'(0) + \cdots$.

c) Determine the terms y_0, y_1 of the outer expansion and the terms Y_0, Y_1 of the inner expansion using "matching". Determine the uniform approximation. (It suffices to write it down—you need not prove that it is a sensible approximation.)

8.2. Consider the singularly perturbed problem

$$-\varepsilon^2 y'' + y = f(x), \qquad x \in (0,1), \qquad y(0) = y(1) = 0$$

a) Formulate the equations for the outer expansion. What boundary conditions can you impose?

b) Boundary layers appear at both endpoints $x = 0$ and $x = 1$. Consider the solution behavior at the endpoint $x = 0$. For the inner expansion make (for the left endpoint $x = 0$) the scaling ansatz $x = \varepsilon^\alpha \xi$ with $\alpha > 0$ and determine α.

c) Construct as many terms of the outer and the inner expansion so that you obtain an approximation $\widetilde{y}$ whose residual (both the volume residual $-\varepsilon^2 \widetilde{y}'' + \widetilde{y}$ and the boundary residual $|\widetilde{y}(0)| + |\widetilde{y}(1)|$) is of order $O(\varepsilon^2)$.

d) The Lax-Milgram Lemma show for the solution y that

$$\varepsilon^2 \|y'\|^2_{L^2(0,1)} + \|y\|^2_{L^2(0,1)} \le \|f\|^2_{L^2(0,1)}.$$

Using this, show that the uniform approximation constructed by you is indeed a sensible approximation to the exact solution.

8.3. (Problems with "turning points")

a) Consider for small $\varepsilon > 0$ the problem

$$-\varepsilon y'' + xy' + y = 1, \qquad x \in (-1,1), \qquad y(-1) = y(1) = 0.$$

Formulate the equations for the functions y_i of the "outer expansion" $\sum_i \varepsilon^i y_i$. Compute the y_i.
Hint: 1) The ansatz $y(x) = x^\alpha$ yields the fundamental system for the reduced equation. 2) Argue why you cannot impose boundary conditions at $x = -1$ or $x = 1$ for the functions y_i.
Where do you expect boundary layers?

b) Consider for small $\varepsilon > 0$ the problem

$$-\varepsilon y'' - xy' + y = 1, \qquad x \in (-1,1), \qquad y(-1) = y(1) = 0.$$

Construct a *piecewise* smooth y_0 that is (presumably) a good approximation to the solution. Where do you expect "boundary" layers? What is the length scale on which you expect these layers to arise?

8.4. The KdV equation (2) has an infinite number of conserved quantities. Show:

$$m_1(u) := \int_{-\infty}^{\infty} u(x,t)\,dx \qquad \text{(the "volume" of the wave)}$$
$$m_2(u) := \int_{-\infty}^{\infty} u^2(x,t)\,dx$$

are conserved[1]. *Hint:* 1) You may assume that u (and all required derivatives) decay sufficiently rapidly as $x \to \pm\infty$. 2) Try to write down an equation for u or u^2 of the form $D(u)_t + F(u)_x = 0$ where $D(u) = u$ in the first case and $D(u) = u^2$ in the second case. To determine F in the second case, it is useful to note $(2uu_{xx})_x - ((u_x)^2)_x = 2uu_{xxx}$.

8.5. (optional) The KdV-equation

$$u_t + uu_x + u_{xxx} = 0 \tag{2}$$

has, for every $c > 0$ a "soliton" solution

$$u(x,t) = 3c\frac{1}{\cosh^2(\frac{\sqrt{c}}{2}(x - ct))}$$

as can be checked by elementary calculations. In order to "derive" the soliton solution, make the ansatz

$$u(x,t) = U(x - ct), \qquad \lim_{\xi\to\pm\infty} U(\xi) = 0.$$

a) Check that U has to satisfy the equation

$$U'' + \left[\frac{1}{2}U^2 - cU - A\right] = 0 \tag{3}$$

for an integration constant A. Select $A = 0$.

b) Rewrite this second order equation as a first order system. Show that it has two equilibria: a stable center one and an unstable saddle point. ((4) below is connected to a Ljapunov function).

c) Sketch the phase portrait (e.g., with MAPLE, *phaseportrait*). Why can the sought solution U only be the homoclinic orbit that connects the saddle point at $(0, 0)$ with itself?

d) Conclude from (3) (with $A = 0$) that

$$\frac{1}{2}(U')^2 + \frac{1}{6}U^3 - \frac{c}{2}U^2 = E, \tag{4}$$

for some new integration constant E. Reason that $E = 0$, since U corresponds to a homoclinic orbit.

Remark: The substitution $U = v^2$ leads to a separable system

$$(v')^2 + \frac{1}{12}v^4 - \frac{c}{4}v^2 = 0,$$

that can be solved explicitly using cosh.

[1] a less "obvious" conserved quantity is $m_3(u) = \int_{-\infty}^{\infty} \frac{1}{6}u^3 + \frac{1}{2}uu_{xx}\,dx$