problem sheet 11

discussion: Tuesday, 22.12.20

11.1. (Homogenization in 1D) Let $\Omega = (0,1)$ and $f \in L^2(\Omega)$. Let $A \in C^1(\mathbb{R})$ be positive on $\mathbb{R}$ and additionally 1-periodic. Consider

$$-(A(x/\varepsilon)u')' = f \qquad \text{on } \Omega, \qquad u(0) = u(1) = 0.$$

a) What is the homogenized equation?

b) Determine the "first order approximation" $u_0(x) + \varepsilon u_1(x, x/\varepsilon)$ for the specific case

$$f(x) \equiv 1, \qquad A(x) = \frac{1}{2 + \cos(2\pi x)}$$

c) The volume and boundary residual can be expected to of size $O(\varepsilon^\alpha)$. Which value of α do you expect in the setting of b)? What would be the "correct" norm to measure the volume residual?

11.2. Let $A \in C^1(\mathbb{R}^d)$ be pointwise symmetric positive definite. Let A be $Y := [0,1)^d$-periodic.

The goal is to show: the homogenized matrix A^0 is SPD.

To that end, let the functions χ_j of the lecture be defined by

$$-\nabla \cdot (A \nabla \chi_j) = -\sum_i \partial_{y_i} A_{ij}, \qquad \chi_j \text{ is } Y\text{-periodic}$$

and let e_j be the j-th unit vector. Define

$$w_j(y) := y_j - \chi_j(y)$$

a) Show:

$$A^0_{ij} = \int_Y (A(y)\nabla w_i) \cdot \nabla w_j \, dy \tag{1}$$

(i.e., A^0 is symmetric). To that end, show:

$$\begin{aligned}
\int_Y (A\nabla \chi_j) \cdot \nabla v \, dy &= \int_Y (Ae_j) \cdot \nabla v \, dy \qquad \forall v \in H^1_{per}(Y) \\
\int_Y (A\nabla w_j) \cdot \nabla \chi_i \, dy &= 0 \\
A^0_{ij} &= \int_Y (A\nabla w_j) \cdot \nabla y_i \, dy
\end{aligned}$$

b) Use that A is positive definite and the representation (1) to show for some c:

$$(A^0\xi) \cdot \xi \geq c\|\xi\|_2^2 \qquad \forall \xi \in \mathbb{R}^d.$$