problem sheet 6

discussion: Tuesday, 17.11.20

6.1. Let $\widehat{\sigma} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ be a function with the property

$$\widehat{\sigma}(\partial_t Q Q^\top + QAQ^\top) = Q\widehat{\sigma}(A)Q^\top$$

for all smooth matrix-valued functions $Q : \mathbb{R} \to O(d)$, where $O(d)$ denote the set of orthogonal $d \times d$-matrices with determinant 1. Show:

$$\widehat{\sigma}(A) = \widehat{\sigma}(\frac{1}{2}(A + A^\top)).$$

Hint: Let W be a skew-symmetric matrix and $Q(t) := e^{-tW}$. Show: $Q(t) \in O(d)$ for every $t \in \mathbb{R}$. Next, show that $\widehat{\sigma}(-W + A) = \widehat{\sigma}(A)$. Select W.

6.2. (Hagen-Poiseuille) Consider a viscous, incompressible, homogeneous flow in a pipe

$$\Omega = \{x \in \mathbb{R}^3 \,|\, 0 < x_1 < L, \quad x_2^2 + x_3^2 < R^2\}$$

of length L and radius R. The flow is driven by a pressure difference.

a) Solve the stationary Navier-Stokes equations

$$\nabla \cdot v = 0, \qquad -\eta\Delta v + (v \cdot \nabla)v + \nabla p = 0$$

assuming that v has the form $v = (v_1(r), 0, 0)$ with $r = \sqrt{x_2^2 + x_3^2}$. Assume furthermore that the pressure at $x_1 = 0$ is p_1 und at $x_1 = L$ is p_2. Assume $v(R) = 0$.

b) Compute the flow rate of the pipe.

6.3. (Potential flows)

a) Let v be a stationary, incompressible, irrotational flow (in 2D or 3D). Let ρ be constant. Show that v solves the Euler equations with $p = -\frac{\rho}{2}|v|^2$ and $f = 0$.

b) Consider now the 2D flow on $\mathbb{R}^2 \setminus \{(0,0)\}$ with velocity field

$$v(x_1, x_2) = \frac{1}{x_1^2 + x_2^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

Show that v satisfies the assumptions of a) (i.e., stationary, incompressible, irrotational).

c) Show that the velocity field of b) is not a potential flow.

6.4. Consider a stationary flow between two infinitely long, coaxial cylinders that rotate (slowly). The geometry is:

$$\Omega = \{(x, y, z) \,|\, R_1 < r < R_2, \quad z \in \mathbb{R}\}, \qquad r^2 = x^2 + y^2$$

The homogeneous fluid is assumed to be desribed by the incompressible Navier-Stokes equations with no-slip boundary conditions. Determine an axisymmetric stationary solution assuming $f = 0$. That is: the flow is only in the (x, y)-plane and axially symmetric there. In particular, you may assume that v and p do not depend on z. The outer cylinder has angular velocity ω_2 and the inner one angular velocity ω_1.

Hint: The ODE $y'' + \frac{1}{r}y' - \frac{1}{r^2}y = 0$ has solutions of the form $y(r) = r^\alpha$.