problem sheet 3

discussion: Tuesday, 27.10.20

3.1. Consider Burgers' equation.

a) Consider the initial condition

$$u_0(x) = \begin{cases} 2, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$

What is the solution[1]? Sketch the charakteristics and the shock curves.

b) Consider the initial condition

$$u_0(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$

Sketch the solution (at least for small times t).

3.2. Consider the following scalar conservation law with a term of order zero:

$$\partial_t u + \partial_x(f(u)) = g(u) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+;$$

here, f, g are smooth and bounded. What is the Rankine-Hugoniot condition?

3.3. Consider the piecewise constant function

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st. \end{cases} \tag{1}$$

a) Show: For such a function to be an entropy solution of the hyperbolic conservation law $\partial_t u + \partial_x(f(u)) = 0$, it has to satisfy the Rankine-Hugoniot condition and additionally

$$s(\eta(u_r) - \eta(u_l)) \geq \psi(u_r) - \psi(u_l)$$

for all entropy-entropy flux pairs (η, ψ).

b) Show for Burgers' equation (i.e., $f(u) = \frac{1}{2}u^2$) and $u_l > u_r$ that the function given by (1) is an entropy solution, if s is given by the Rankine-Hugoniot condition.

c) Show for Burgers' equation and $u_l < u_r$, that the function given by (1) is a weak solution but not an entropy solution (*hint:* $\eta(u) = u^2$).

3.4. Consider Burgers' equation.

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = 0 \quad \text{auf } \mathbb{R} \times \mathbb{R}^+, \qquad u(\cdot, 0) = u_0(\cdot).$$

a) Compute the entropy flux ψ for the entropy $\eta(u) = \frac{1}{2}u^2$.

b) Let u_0 be bounded and have compact support. Assume additionally that the entropy solution is bounded and satisfies (as in the lecture) the condition $u \in C^0([0,\infty); L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$. Show that

$$\|u(\cdot,t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}.$$

Hint: Proceed somewhat formally as in the lecture by taking the test function $\psi = \chi_K$ as in the proof of Theorem 1.18. A full argument would work with a smoothed verion of χ_K and a limiting argument.
Remark: in general, it *not* true that $\|u(\cdot,t) - v(\cdot,t)\|_{L^2(\mathbb{R})} \leq \|u(\cdot,0) - v(\cdot,0)\|_{L^2(\mathbb{R})}$ for entropy solutions u, v.

[1] of course, the entropy solution is meant, which consists of shocks

3.5. Theorem 1.19 of the lecture stated a monotonicity property for entropy solutions: $u_0 \le v_0$ implies $u(\cdot,t) \le v(\cdot,t)$. Show that this statement follows with the techniques of the proof of Theorem 1.18. *Hints:*

1. Set $C_0^1(\mathbb{R}\times\mathbb{R}^+) := \{v \in C^1(\mathbb{R}\times\mathbb{R}^+) \mid \operatorname{supp} v \subset \mathbb{R}\times\mathbb{R}^+ \text{ is compact }\}$. Recall from the proof of Theorem 1.18 that for all $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$ with $\varphi \ge 0$ one has

$$\int_{\mathbb{R}\times\mathbb{R}^+} |u(x,t) - v(x,t)|\varphi_t(x,t) + \operatorname{sign}(u(x,t) - v(x,t))(f(u(x,t)) - f(v(x,t))\varphi_x(x,t)\,dx\,dt \ge 0$$

(Note that $\varphi(\cdot,0) = 0$ is assumed so that no integral overe $\mathbb{R}\times\{0\}$ appears.)

2. Check that 2 entropy solutions u, v satisfy

$$0 = \int_{\mathbb{R}\times\mathbb{R}^+} (u-v)\varphi_t + (f(u) - f(v))\varphi_x \qquad \forall \varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+).$$

3. Show that for the function $\Phi(u) := |u| + u$ and a suitable (lipschitz continuous) function $(u,v) \mapsto \Psi(u,v)$ one has

$$0 \le \int_{\mathbb{R}\times\mathbb{R}^+} \Phi(u-v)\varphi_t + \Psi(u,v)\varphi_x \quad \varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+), \quad \varphi \ge 0.$$

4. Show for $0 < t_1 < t$

$$\int_{\mathbb{R}} \Phi(u-v)(\cdot,t) \le \int_{\mathbb{R}} \Phi(u-v)(\cdot,t_1)$$

by taking a suitable test function. As in the proof of Theorem 1.18, you may proceed formally by taking the test function as a characteristic function of a suitable set.

5. Take the limit $t_1 \to 0$ and infer $(u-v)(x,t) \le 0$ for all $x \in \mathbb{R}$ (up to a set of measure zero, of course).