problem sheet 8

discussion: week of Monday, 29.11.2021

8.1. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. For $n\times n$ matrices $\mathbf{A}\in\mathbb{R}^{n\times n}$ define a norm $\|\mathbf{A}\|$ by

$$\max_{0\neq\mathbf{x}\in\mathbb{R}^n}\frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}.$$

a) Show that for arbitrary matrices $\mathbf{A}$, $\mathbf{B}\in\mathbb{R}^{n\times n}$ one has $\|\mathbf{A}\mathbf{B}\|\leq\|\mathbf{A}\|\|\mathbf{B}\|$.

b) Show for the norm $\|\mathbf{x}\|_\infty=\max_{i=1,\dots,n}|x_i|$ that $\|\mathbf{A}\|_\infty\leq\max_i\sum_{j=1}^n|a_{ij}|$. *Remark:* In fact, there holds $\|\mathbf{A}\|_\infty=\max_i\sum_{j=1}^n|a_{ij}|$.

c) Show for the norm $\|\mathbf{x}\|_1=\sum_{i=1}^n|x_i|$ that $\|\mathbf{A}\|_1\leq\max_j\sum_{i=1}^n|a_{ij}|$. *Remark:* In fact, there holds $\|\mathbf{A}\|_1=\max_j\sum_{i=1}^n|a_{ij}|$.

8.2. Consider the matrix

$$\mathbf{A}=\begin{pmatrix}10^{-6} & 1\\ 1 & 0\end{pmatrix}$$

a) Compute by hand the LU-factorization of $\mathbf{A}$. Calculate $\mathbf{A}^{-1}$, $\mathbf{L}^{-1}$, $\mathbf{U}^{-1}$ and compute the three condition numbers $\kappa_\infty(\mathbf{A})=\|\mathbf{A}\|_\infty\|\mathbf{A}^{-1}\|_\infty$, $\kappa_\infty(\mathbf{L})=\|\mathbf{L}\|_\infty\|\mathbf{L}^{-1}\|_\infty$, $\kappa_\infty(\mathbf{U})=\|\mathbf{U}\|_\infty\|\mathbf{U}^{-1}\|_\infty$. Here, $\|\cdot\|_\infty$ is the row-sum norm of Problem 8.1.

b) Repeat the calculation of a) for the matrix $\widetilde{\mathbf{A}}$ that is obtained from $\mathbf{A}$ by interchanging the two rows. What do you observe?

8.3. Let $A\in\mathbb{R}^{n\times n}$ be a tridiagonal matrix of the form

$$A=\begin{pmatrix}d_1 & e_1 & & & \\ c_2 & d_2 & e_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e_{n-1}\\ & & & c_n & d_n\end{pmatrix}$$

Assume that A has an LU-factorization.

a) Show: the factors L and U have the form

$$L=\begin{pmatrix}1 & & & & \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \ddots & \ddots & \\ & & & l_n & 1\end{pmatrix},\qquad U=\begin{pmatrix}u_1 & f_1 & & & \\ & u_2 & f_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & f_{n-1}\\ & & & & u_n\end{pmatrix}$$

Hint: do the induction step of Thm. 4.17 of the notes.

b) Formulate an algorithm that computes the l_i and the u_i for $i=2,\dots,n$

c) (**to be uploaded on TUWEL**) realize your algorithm of b) in matlab/python. Input are the vectors $\mathbf{d}$, $\mathbf{e}$, $\mathbf{c}$ (i.e., the diagonals of A), output are the vectors $\mathbf{l}$, $\mathbf{u}$, and $\mathbf{f}$ (i.e., the diagonals of L and U).

8.4. The lengths of the 3 edges (meeting at a corner) and the circumferences (orthogonal to the first and second edge) of a brick are measured. The measured values are:

edge 1: $26mm$, edge 2: $38mm$, edge 3: $55mm$

circumference $\perp$ edge 1: $188mm$, circumference $\perp$ edge 2: $163mm$.

Determine the edge lengths using the method of least squares.