problem sheet 3

discussion: week of Monday, 25.10.2021

Hint: one way to evaluate interpolating polynomials is to reuse the Neville scheme that you have programmed already in Problem 2.2a. This could also be used to compute the Lagrange interpolation polynomials ℓ_i.

3.1. Consider the function $f(x) = (4-x^2)^{-1}$. Estimate $\min_{q\in\mathcal{P}_n} \|f-q\|_{\infty,[-1,1]}$ by considering the Taylor polynomial of f about a suitable point. Plot semilogarithmically (`semilogy`) the error $\|f - I_n^{Cheb} f\|_{\infty,[-1,1]}$ versus $n \in \{1,\dots,20\}$, where $I_n^{Cheb} f$ is the Chebyshev interpolant of degree n. Approximate the error $\|f - I_n^{Cheby} f\|_{\infty,[-1,1]}$ by taking the maximal interpolation error in 100 uniformly distributed points in the interval $[-1,1]$. Include in your graph also the error $\|f - T_{2n} f\|_{\infty,[-1,1]}$ where $T_{2n} f$ is the Taylor polynomial of f about $x = 0$ of degree $2n$. Compare the approximation quality of the Taylor polynomial with that of the Chebyshev interpolant.

3.2. For $n \in \{10, 20, 40\}$ consider interpolation on the interval $[-5,5]$ in $n+1$ points. Compare the uniformly distributed points $x_i^{unif} := 5(-1 + 2\frac{i}{n})$, $i = 0,\dots,n$ and the Chebyshev points $x_i^{Cheb} = 5\cos(\frac{i+0.5}{n+1}\pi)$, $i = 0,\dots,n$.

a) For the function $f(x) = (1+x^2)^{-1}$, plot the two interpolating polynomials on $[-5,5]$.

b) **(code to be uploaded in TUWEL)** Investigate numerically the Lebesgue constant

$$\Lambda_n := \max_{x\in[-5,5]} \sum_{i=0}^{n} |\ell_i(x)|,$$

for both the uniform interpolation point distribution and the Chebyshev points. To that end, plot Λ_n versus n in semilogarithmic scale (`semilogy`).

c) For the uniform point distribution, one can in fact show that $\Lambda_n \approx Ce^{bn}$ for some C, $b > 0$. Determine C and b from your data as follows by taking the logarithm $\log \Lambda_n \approx \log C + bn$ and fit your data for the values for $n \in \{20, 40\}$. *Hint:* you can let `polyfit` do the work for you to compute $\log C$ and b or you solve a 2×2 system.

3.3. (lazy change of basis for polynomials) Let two bases $\{p_0(x),\dots,p_n(x)\}$ and $\{q_0(x),\dots,q_n(x)\}$ of the space $\mathcal{P}_n$ of polynommials of degree n be given. Let interpolation points x_i, $i = 0,\dots,n$, be given. Define the matrices $\mathbf{G}$ and $\mathbf{H}$ by

$$\mathbf{G}_{ij} = p_j(x_i), \qquad \mathbf{H}_{ij} = q_j(x_i).$$

Show: the matrix $\mathbf{G}^{-1}\mathbf{H}$ realizes the change of basis, i.e.,

$$\mathbf{c} = \mathbf{G}^{-1}\mathbf{H}\mathbf{d}$$

implies $\sum_i \mathbf{c}_i p_i = \sum_i \mathbf{d}_i q_i$.