QR-Algorithm with and without shift

QR-iteration w/o shift

$$\begin{aligned} A &=: QR \\ A &:= RQ \end{aligned}$$

QR-iteration with single shift

$$\begin{aligned} \sigma &:= A_{nn} \qquad \% \text{ choose shift} \\ A - \sigma I &=: QR \\ A &:= RQ + \sigma I \end{aligned}$$

check whether A decouples.

If yes: use QR-Alg. with shift for upper block

we expect:

1. linear convergence for QR-algorithm without shift
   (more precisely: if the EV satisfy $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$).

2. rapid (asymptotically quadratic) convergence for QR-algorith. with shift

Example

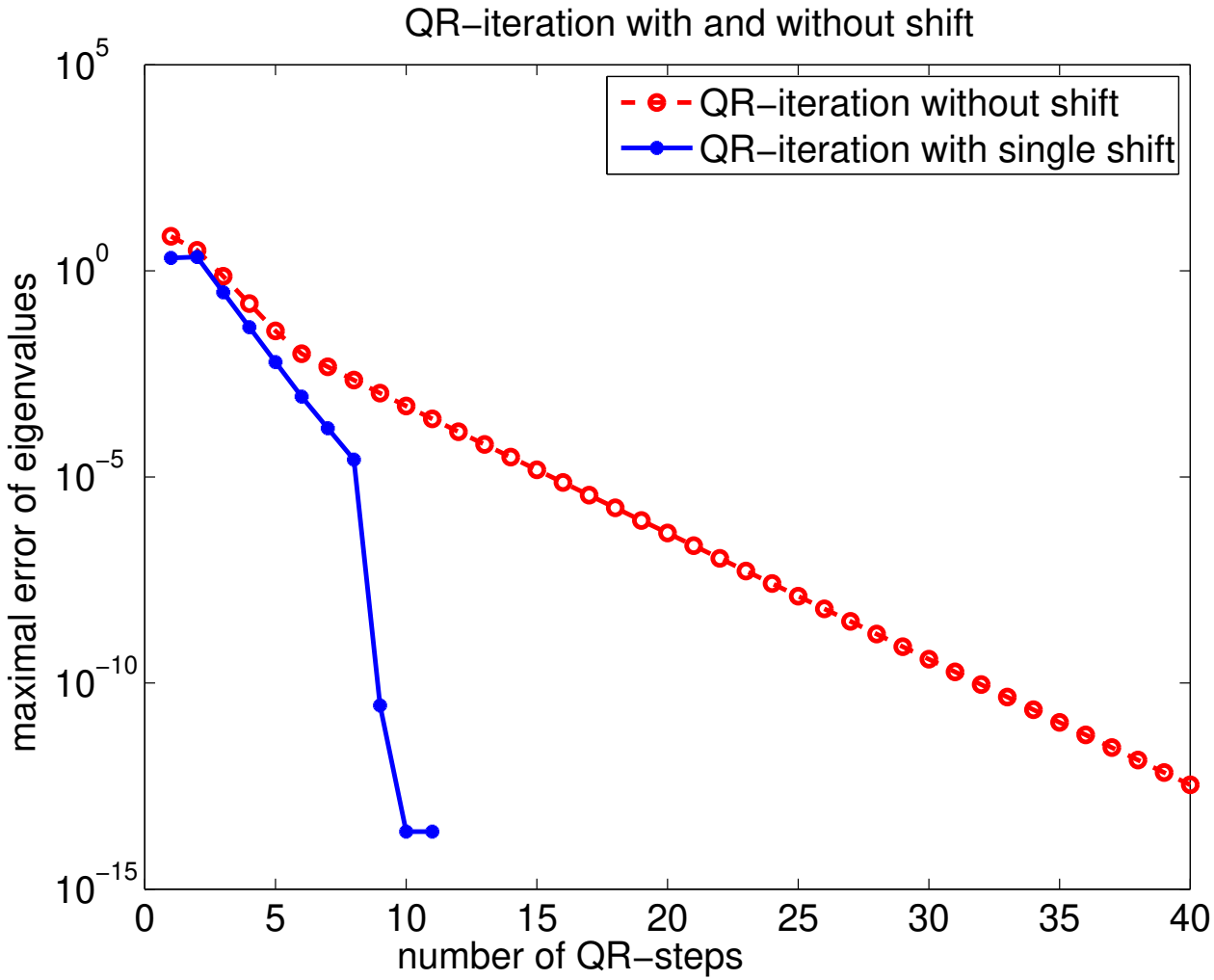
initial matrix $$A = \begin{pmatrix} 3.5488 & 15.593 & 8.5775 & -4.0123 \\ 2.3595 & 24.524 & 14.596 & -5.8157 \\ 0.0899 & 27.599 & 21.438 & -5.8415 \\ 1.9227 & 55.667 & 39.717 & -10.558 \end{pmatrix},$$

Hessenberg form $$Q^\top A Q = H = \begin{pmatrix} 3.5488 & -9.8025 & 4.1046 & -14.8282 \\ -3.0450 & 36.5229 & -8.3593 & 55.8301 \\ 0 & 25.8343 & -2.5900 & 41.15901 \\ 0 & 0 & -0.5372 & 1.4710 \end{pmatrix}$$

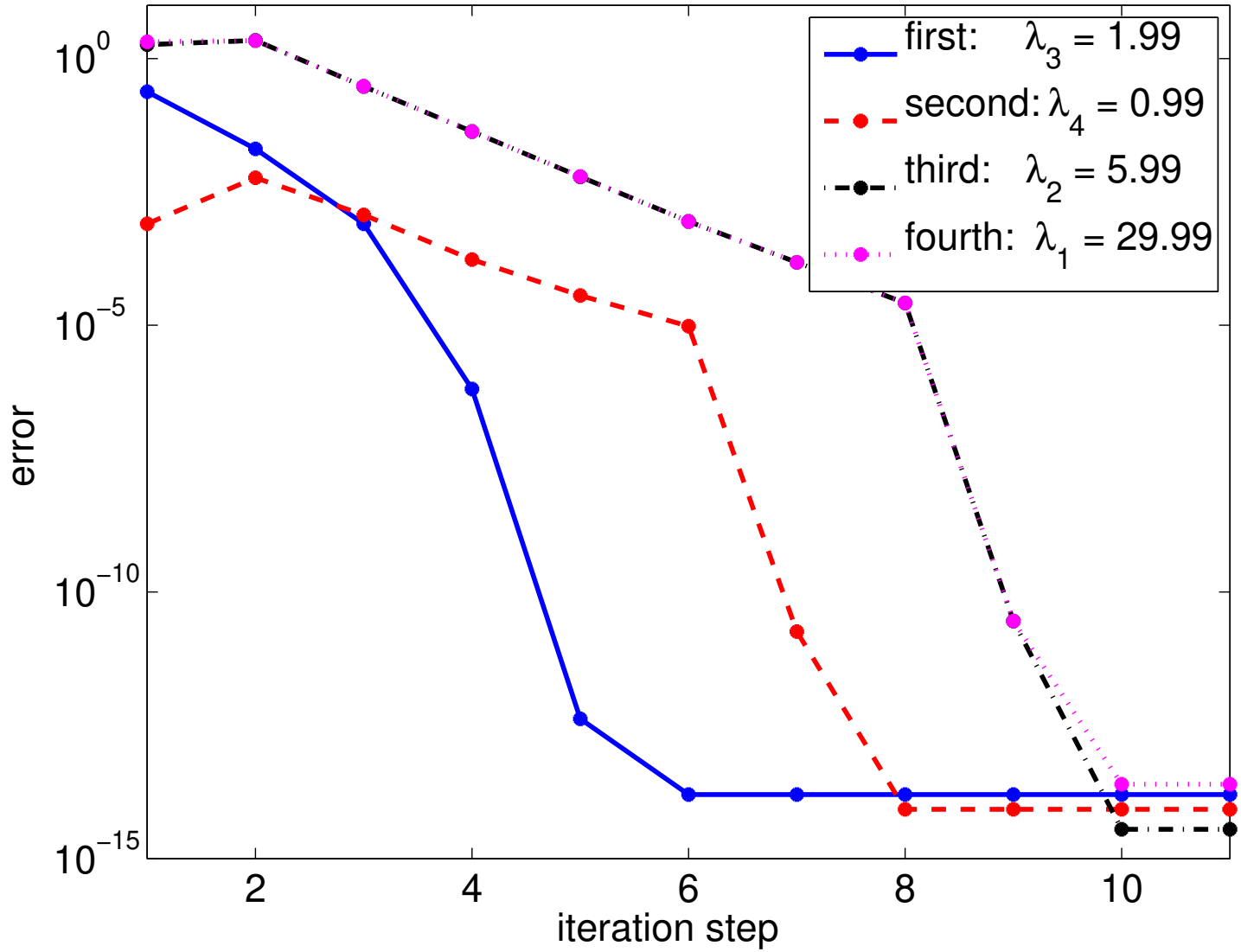
eigenvalues (MATLAB):

29.96642406212449
5.99902388398071
1.99941121279189
0.98794084110292

comparison QR with and without shift



convergence history of all eigenvalue for QR with shift



asymptot. quadr. convergence of one EV and simultaneously linear convergence of the other EV

QR-Iteration without shift

$$A_1 = \begin{pmatrix} 23.9 & -6.6 & 39.2 & 34.5 \\ \mathbf{-21.8} & 12.8 & -42.1 & -38.6 \\ 0 & \mathbf{1.0} & 0.63 & -0.6 \\ 0 & 0 & \mathbf{0.72} & 1.6 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 29.99 & 32.66 & 71.6 & -10.6 \\ \mathbf{-2.1_{-2}} & 6.0 & 1.7 & -0.44 \\ 0 & \mathbf{1.5_{-2}} & 2.02 & -0.21 \\ 0 & 0 & \mathbf{6.4_{-2}} & 0.97 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} 29.97 & 32.9 & 70.68 & 14.9 \\ \mathbf{-6.6_{-6}} & 6.0 & 1.79 & 0.56 \\ 0 & \mathbf{6.2_{-5}} & 2.0 & 0.27 \\ 0 & 0 & \mathbf{-1.8_{-3}} & 0.99 \end{pmatrix}$$

$$A_{15} = \begin{pmatrix} 29.9 & 32.9 & 70.65 & -15.05 \\ \mathbf{-2.1_{-9}} & 6.0 & 1.79 & -0.56 \\ 0 & \mathbf{2.6_{-7}} & 2.0 & -0.28 \\ 0 & 0 & \mathbf{5.3_{-5}} & 0.99 \end{pmatrix}$$

$$A_{20} = \begin{pmatrix} 29.97 & 32.9 & 70.65 & 15.1 \\ \mathbf{-6.8_{-13}} & 6.0 & 1.79 & 0.56 \\ 0 & \mathbf{1.1_{-9}} & 2.0 & 0.28 \\ 0 & 0 & \mathbf{-1.6_{-6}} & 0.99 \end{pmatrix}$$

QR-iteration with shift and deflation

$$A_1 = \begin{pmatrix} 32.0 & 1.66 & -28.9 & 49.9 \\ -23.4 & 4.17 & 23.6 & -42.1 \\ 0 & -0.65 & 0.99 & 0.03 \\ 0 & 0 & \mathbf{-0.33} & 1.76 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 32.1 & 30.8 & 8.1 & -71.9 \\ -1.8 & 3.8 & -0.05 & 3.2 \\ 0 & 0.14 & 0.98 & 0.085 \\ 0 & 0 & \mathbf{-0.16} & 2.02 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 30.3 & 32.7 & -4.17 & 72.15 \\ -0.2 & 5.70 & 0.08 & 1.21 \\ 0 & -4.0_{-2} & 0.99 & -0.29 \\ 0 & 0 & \mathbf{2.98_{-3}} & 2.0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 30.1 & 32.9 & -4.06 & -72.12 \\ -0.03 & 6.0 & 0.089 & -1.78 \\ 0 & 1.01_{-2} & 0.99 & 0.27 \\ 0 & 0 & \mathbf{-2.3_{-6}} & 2.0 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 30.0 & 32.9 & -4.14 & 72.12 \\ -4.5_{-3} & 6.0 & 0.07 & 1.86 \\ 0 & -2.56_{-3} & 0.99 & -0.28 \\ 0 & 0 & \mathbf{1.5_{-12}} & 2.0 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} \mathbf{30.0} & \mathbf{32.9} & \mathbf{-4.12} & -72.12 \\ \mathbf{-6.4_{-4}} & \mathbf{6.0} & \mathbf{0.07} & -1.87 \\ \mathbf{0} & \mathbf{6.5_{-4}} & \mathbf{0.99} & 0.28 \\ 0 & 0 & \mathbf{0.0} & 2.0 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 30.0 & 32.9 & 4.12 \\ -1.1_{-5} & 6.0 & -0.07 \\ 0 & \mathbf{1.2_{-9}} & 0.99 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} \mathbf{30.0} & \mathbf{32.9} & -4.12 \\ \mathbf{-2.0_{-5}} & \mathbf{6.0} & 0.073 \\ 0 & \mathbf{0.0} & 0.99 \end{pmatrix}$$

$$A_9 = \begin{pmatrix} 30.0 & -32.9 \\ \mathbf{2.1_{-11}} & 6.0 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} \mathbf{30.0} & 32.9 \\ \mathbf{-2.5_{-23}} & 6.0 \end{pmatrix}$$

obere Dreiecksgestalt R gerundet:

$$R = \begin{pmatrix} 30.0 & 32.94 & -4.12 & -72.1 \\ 0.0 & 6.0 & 0.073 & -1.87 \\ 0 & 0.0 & 0.99 & 0.28 \\ 0 & 0 & 0.0 & 2.0 \end{pmatrix}$$

comparison: (maximal error of EV):

without shift: $\approx 10^{-5}$ after 20 steps

Single Shift: $\approx 10^{-15}$ after 10 steps