

Orthogonal iteration

iteration:

input: $\mathbf{Q}_0 \in \mathbb{R}^{n \times k}$ with orthonormal columns

$$\tilde{\mathbf{X}}_{\ell+1} := \mathbf{A}\mathbf{Q}_\ell$$

$$[\mathbf{Q}_{\ell+1}, \mathbf{R}_{\ell+1}] := \text{qr}(\tilde{\mathbf{X}}_{\ell+1})$$

Thm.: Let $\hat{\mathbf{Q}}_\ell \in \mathbb{R}^{n \times (n-k)}$ be selected such that $\mathbf{Q} = (\mathbf{Q}_\ell, \hat{\mathbf{Q}}_\ell) \in \mathbb{R}^{n \times n}$ is an orthogonal matrix. Then the matrix

$$\mathbf{Q}^\top \mathbf{A} \mathbf{Q} = \begin{pmatrix} \mathbf{Q}_\ell^\top \mathbf{A} \mathbf{Q}_\ell & \mathbf{Q}_\ell^\top \mathbf{A} \hat{\mathbf{Q}}_\ell \\ \hat{\mathbf{Q}}_\ell^\top \mathbf{A} \mathbf{Q}_\ell & \hat{\mathbf{Q}}_\ell^\top \mathbf{A} \hat{\mathbf{Q}}_\ell \end{pmatrix} \quad \text{satisfies the bound} \quad \|\hat{\mathbf{Q}}_\ell^\top \mathbf{A} \mathbf{Q}_\ell\|_2 \leq C\delta^\ell,$$

where $\delta < 1$, if the EV of \mathbf{A} satisfy

$$|\lambda_1| \geq |\lambda_2| \geq \cdots |\lambda_k| > |\lambda_{k+1}| \geq |\lambda_{k+2}| \geq \cdots \geq |\lambda_n|.$$

Orthogonal iteration with $Q_0 = e_1$:

$$A = \begin{pmatrix} 3.5488 & 15.593 & 8.5775 & -4.0123 \\ \mathbf{2.3595} & 24.524 & 14.596 & -5.8157 \\ \mathbf{0.0899} & 27.599 & 21.438 & -5.8415 \\ \mathbf{1.9227} & 55.667 & 39.717 & -10.558 \end{pmatrix}$$

$$Q_1^\top A Q_1 = \begin{pmatrix} 23.88 & 7.4511 & -18.7348 & 48.5777 \\ \mathbf{21.81} & 13.8262 & -19.4457 & 53.4859 \\ \mathbf{0.05} & 0.9388 & 0.7410 & 1.1440 \\ \mathbf{-0.38} & -0.6794 & 0.0344 & 0.5050 \end{pmatrix}$$

$$Q_5^\top A Q_5 = \begin{pmatrix} 2.9995_{+01} & -3.2632_{+01} & -7.2229_{+01} & -4.5374_{+00} \\ \mathbf{2.0674}_{-02} & 5.9635_{+00} & 1.8037_{+00} & 2.9067_{-01} \\ \mathbf{6.7202}_{-06} & 1.6060_{-02} & 2.0253_{+00} & 1.2553_{-01} \\ \mathbf{-9.5156}_{-07} & -1.4322_{-03} & -1.5312_{-01} & 9.6943_{-01} \end{pmatrix}$$

$$Q_{10}^\top A Q_{10} = \begin{pmatrix} 2.9966_{+01} & -3.2941_{+01} & -7.0716_{+01} & 1.4757_{+01} \\ \mathbf{6.6205}_{-06} & 5.9990_{+00} & 1.7890_{+00} & -5.5720_{-01} \\ \mathbf{8.9954}_{-12} & 6.7581_{-05} & 2.0006_{+00} & -2.7247_{-01} \\ \mathbf{3.6005}_{-14} & 1.6941_{-07} & 4.2406_{-03} & 9.8680_{-01} \end{pmatrix}$$

$$Q_{15}^\top A Q_{15} = \begin{pmatrix} 2.9966_{+01} & -3.2942_{+01} & -7.0655_{+01} & -1.5044_{+01} \\ \mathbf{2.1287}_{-09} & 5.9990_{+00} & 1.7866_{+00} & 5.6447_{-01} \\ \mathbf{1.1896}_{-17} & 2.7806_{-07} & 1.9994_{+00} & 2.7658_{-01} \\ \mathbf{-1.4008}_{-21} & -2.0498_{-11} & -1.2463_{-04} & 9.8791_{-01} \end{pmatrix}$$

orthogonal iteration with $Q_0 = [e_1, e_2]$:

$$A = \begin{pmatrix} 3.5488 & 15.593 & 8.5775 & -4.0123 \\ 2.3595 & 24.524 & 14.596 & -5.8157 \\ \mathbf{0.0899} & \mathbf{27.599} & 21.438 & -5.8415 \\ \mathbf{1.9227} & \mathbf{55.667} & 39.717 & -10.558 \end{pmatrix}$$

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