vector iteration

iteration:

repeat$\{$

$$x_{\ell+1} := \frac{\mathbf{A}\mathbf{x}_\ell}{\|\mathbf{A}\mathbf{x}_\ell\|_2}$$

$$\widetilde{\lambda}_{\ell+1} = \mathbf{x}_{\ell+1}^\top \mathbf{A}\mathbf{x}_{l+1}$$

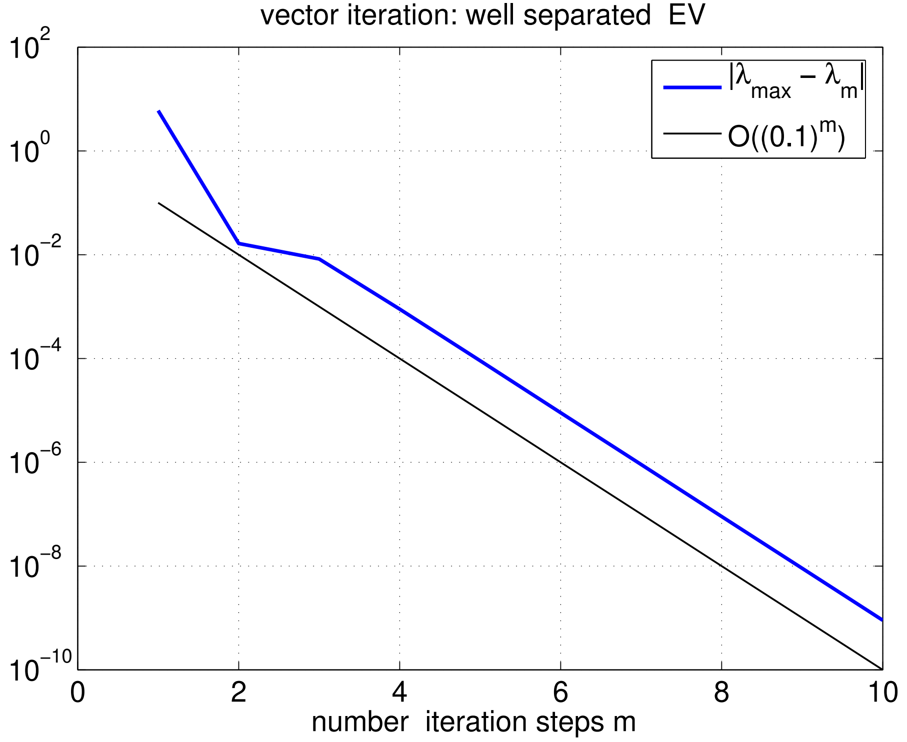
$$\ell := \ell + 1$$

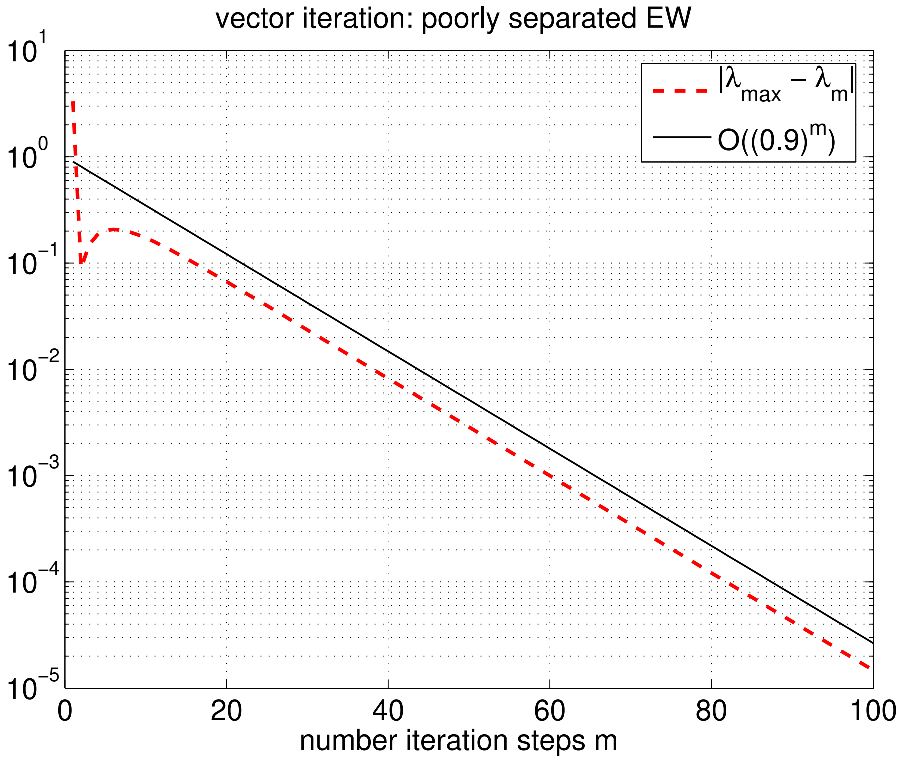
$\}$**until** sufficiently accurate

Example 1: well separated and poorly separated EV; $x_0 = (1,1,1)^\top$

$$\mathbf{A}_1 = \begin{pmatrix} 10 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_1 = 10, \quad \lambda_2 = 1, \quad \lambda_3 = 0, \qquad \left|\frac{\lambda_2}{\lambda_1}\right| = 0.1$$

$$\mathbf{A}_2 = \begin{pmatrix} 10 & 1 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_1 = 10, \quad \lambda_2 = 9, \quad \lambda_3 = 0, \qquad \left|\frac{\lambda_2}{\lambda_1}\right| = 0.9$$





Example 2: EV not separated (in absolute value); $\mathbf{x}_0 = (1, 1, 1)^\top$

$$\mathbf{A}_3 = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 0.1 \end{pmatrix},$$

$$c = \cos(\pi/3), \quad s = \sin(\pi/3),$$

$$\lambda = 0.5 \pm 0.5\sqrt{3}\,\mathbf{i}, \quad \lambda = 0.1$$

iteration step ℓ	$\widetilde{\lambda}_\ell$
1	0.36666666666667
2	0.49800995024876
3	0.49998000099995
4	0.49999980000010
5	0.49999999800000
6	0.49999999998000
7	0.49999999999980
8	0.50000000000000
9	0.50000000000000
10	0.50000000000000
11	0.50000000000000

of course, convergence to complex EV cannot be expected for real matrices and real starting values....

Example 2: EV not separated (in absolute value);

$\mathbf{x}_0 = (1 + 7\mathbf{i}, 2 + 11\mathbf{i}, 3 + 17\mathbf{i})^\top$

$$\mathbf{A}_3 = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 0.1 \end{pmatrix},$$

$c = \cos(\pi/3), \quad s = \sin(\pi/3)$

$$\lambda_{1,2} = 0.5 \pm 0.5\sqrt{3}\,\mathbf{i} \approx 0.5 \pm 0.866\mathbf{i}$$

$\lambda_3 = 0.1$

ℓ	$\widetilde{\lambda}_\ell$
1	0.247991543340381 + 0.010985523092403i
2	0.493302618271716 + 0.029195147896992i
3	0.499931897311201 + 0.029687244244732i
4	0.499999318858303 + 0.029692248996665i
5	0.499999993188572 + 0.029692299052706i
6	0.499999999931886 + 0.029692299553267i
7	0.499999999999319 + 0.029692299558273i
8	0.499999999999993 + 0.029692299558323i
9	0.500000000000000 + 0.029692299558324i
10	0.500000000000000 + 0.029692299558324i
11	0.500000000000000 + 0.029692299558324i

still no convergence... solution: use inverse iteration with shift ($\rightarrow$ later)

Inverse iteration and Rayleigh quotient iteration

Inverse iteration with shift λ:

$$\widetilde{\lambda}_\ell = \mathbf{x}_\ell^\top \mathbf{A}\mathbf{x}_\ell$$

$$\widetilde{\mathbf{x}}_{\ell+1} := (\mathbf{A} - \lambda)^{-1}\mathbf{x}_\ell$$

$$\mathbf{x}_{\ell+1} := \frac{\widetilde{\mathbf{x}}_{\ell+1}}{\|\widetilde{\mathbf{x}}_{\ell+1}\|_2}$$

Rayleigh quotient method

$$\widetilde{\lambda}_\ell = \mathbf{x}_\ell^\top \mathbf{A}\mathbf{x}_\ell$$

$$\widetilde{\mathbf{x}}_{\ell+1} := (\mathbf{A} - \widetilde{\lambda}_\ell)^{-1}\mathbf{x}_\ell$$

$$\mathbf{x}_{\ell+1} := \frac{\widetilde{\mathbf{x}}_{\ell+1}}{\|\widetilde{\mathbf{x}}_{\ell+1}\|_2}$$

Example

$$\mathbf{A} = \begin{pmatrix} 10 & 1 & 0 \\ 1 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_1 \approx 10.6180, \quad \lambda_2 \approx 8.3820, \quad \lambda_3 = 0,$$

$$\mathbf{x}_0 = (1, -1, 0)^\top \rightsquigarrow \text{Rayleigh quotient iteration converges to } \lambda_2$$