Norms

Definition 1. [Norm] A mapping $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a norm, if:

(i) $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$ (positivity)

(ii) $\|x\| = 0 \iff x = 0$ (definiteness)

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$ (homogeneity)

(iv) $\|x + y\| \leq \|x\| + \|y\|$ for all x, $y \in \mathbb{R}^n$. (triangle inequality)

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examples:

1. die ℓ_1-norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$
2. die ℓ_∞-norm: $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$
3. die ℓ_p-norm $(1 \leq p < \infty)$: $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$

matrix norms

Definition 2. [Matrixnorm] Let $\|\cdot\|_{\mathbb{R}^n}$ and $\|\cdot\|_{\mathbb{R}^m}$ be norms on $\mathbb{R}^n$, $\mathbb{R}^m$. These two norms induce on the (vector) space of the $m \times n$ matrices the (induced) matrix norm by

$$\|A\| := \sup_{0 \neq x \in \mathbb{R}^n} \frac{\|Ax\|_{\mathbb{R}^m}}{\|x\|_{\mathbb{R}^n}}.$$

properties:

1. norm: $A \mapsto \|A\|$ is a norm
2. submultiplicativity: $\|AB\| \leq \|A\| \, \|B\|$ for matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times \mu}$
3. $\|I\| = 1$ ($I =$ identity matrix)
4. $\|A^{-1}\| \geq \|A\|^{-1}$ (if A is regular/invertible)