problem sheet 5

discussion: week of Monday, 8.11.2021

5.1. We wish to show that the extrapolation of the composite trapezoidal rule is the composite Simpson rule. To that end, let $T(h)$ be the composite trapezoidal rule with step size $h = (b-a)/N$ and $S(h)$ be the composite Simpson rule with step size $h = (b-a)/N$. Use Romberg extrapolation with step sizes $h_i = (b-a)2^{-i}$, $i = 0, 1, \dots,$

a) Extrapolation of the composite trapezoidal rule (with step size h) has in column $m = 0$ the values $T(h_i)$. Show: in column $m = 1$ of the Neville scheme are the values

$$N_i := T(h_{i+1}) + \frac{1}{3}(T(h_{i+1}) - T(h_i))$$

b) Show: $N_i = S(h_i)$.

5.2. **(to be uploaded in TUWEL)** Write a program with signature $I = \texttt{adapt}(f, a, b, \tau, h_{min})$ that realizes an adaptive quadrature for $\int_a^b f(x)\,dx$. The quadrature should be based on the Simpson rule. τ is the desired (absolute) accuracy and h_{min} the minimal interval length. To estimate the accuracy, compare the value of the Simpson rule $S_{\{a,b\}}(f)$ for the integration on $[a, b]$ with the value $S_{\{a,m\}}(f) + S_{\{m,b\}}(f)$ with $m = (a+b)/2$. Use your algorithm for the integration of the function

$$\begin{cases} \frac{1}{2}e^x & x < 1/3 \\ e^x & x \geq 1/3 \end{cases}$$

over $[0, 1]$. Use $\tau = h_{min} = 2^{-j}$, $j = 0, \dots, 10$. Plot the error versus τ. What convergence do you observe? Why was $h_{min} = \tau$ chosen?

5.3. Develop an adaptive algorithm for the integration of functions over the rectangle $[a_x, b_x] \times [a_y, b_y]$. Base your algorithm on the midpoint rule, i.e., $Q_{[a,b]\times[c,d]}(f) = (b-a)(d-c)f((a+b)/2, (c+d)/2)$. *Hint:* adapt the ideas of the 1d-adaptive algorithm of Problem 5.2. Test your adaptive algorithm for the integration over $[0, 1]^2$ of the following functions:

$$f_1(x, y) = x^2 \quad \text{and} \quad f_2(x, y) = \begin{cases} 0 & x < y \\ 1 & x \geq y \end{cases}$$

Use the tolerances $\tau = 2^{-i}$, $i = 0, \dots, 15$, and make a convergence plot (quadrature error versus tolerance) in `loglog` scale.

5.4. (sinc quadrature) For certain integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ the simple trapezoidal rule works astonishingly well: Define the quadrature rule

$$Q^N(f) := h \sum_{i=-N}^{N} f(x_i), \qquad h := \frac{1}{\sqrt{N}}, \qquad x_i := ih.$$

Apply this rule to the integrand $f(x) = e^{-x^2} \sin^2(x)$. Plot the error versus N in a suitable scale.