problem sheet 9

discussion: week of Monday, 6.12.2021

9.1. (**to be uploaded in TUWEL**) The function $f(x) = \sin x$ is to be approximated by a polynomial of the form $\pi(x) = a_1 x + a_3 x^3$. To this end, the coefficients a_1, a_3 are determined using the least squares method by minimizing $\sum_{j=0}^{m}(\pi(x_j) - f(x_j))^2$, where $x_0, \dots, x_m$ are given points. Set up the least squares problem for a_1 and a_3. Write a program that computes the coefficients a_1, a_3 for the following 9 choices of knots x_j: the x_j are N randomly chosen points in the interval $[-1/N, 1/N]$ for $N = 2^n$, $n = 2, \dots, 10$. Do the values a_1, a_3 converge to a limit as $N \to \infty$? Which limit do you expect?

9.2. The least squares method can also be used to fit the parameters of certain nonlinear problems. How would you determine the parameters C, k to fit given data (t_i, y_i), $i = 1, \dots, N$, to the law $y(t) = Ce^{-kt}$? How do you proceed to determine C, α for the law $y(t) = Ct^{\alpha}$?

9.3. Let $\mathbf{Q}$ be an orthogonal matrix. Show:

a) $\mathbf{x}^\top \mathbf{y} = ((\mathbf{Q})\mathbf{x})^\top (\mathbf{Q}\mathbf{y})$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$.

b) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m > n$ and its QR-factorization $\mathbf{A} = \mathbf{QR}$. Show: If $\mathbf{A}$ has full rank (i.e., $\operatorname{rank}(\mathbf{A}) = n$), then the diagonal entries of $\mathbf{R}$ are non-zero. Show that the first n columns of $\mathbf{Q}$ form an orthogonal basis of the range of $\mathbf{A}$.

9.4. Let $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ be the SVD of $\mathbf{A} \in \mathbb{R}^{m \times n}$. Let $\sigma_1, \dots, \sigma_{\min\{m,n\}}$ be the singular values of $\mathbf{A}$ and assume $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > \sigma_{r+1} = \dots = \sigma_{\min\{m,n\}} = 0$.

a) Show: the columns of $\mathbf{U}(:, [1:r])$ are an ONB of the range $\operatorname{Im} \mathbf{A}$ of $\mathbf{A}$.

b) Show: The columns of $\mathbf{V}(:, [r+1:n])$ are an ONB of $\operatorname{Ker} \mathbf{A}$.

c) Show: r is the rank of $\mathbf{A}$.