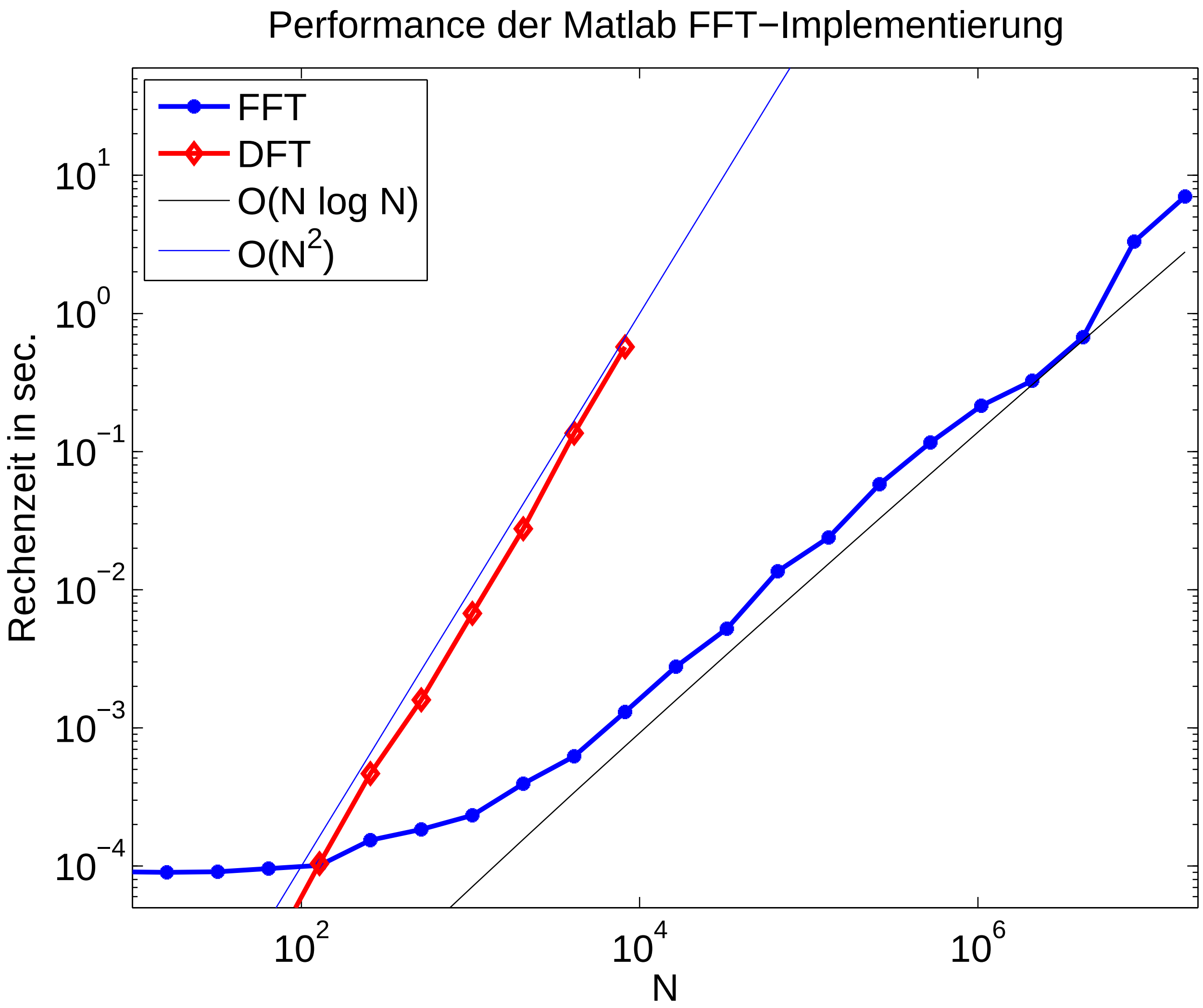
FFT



Performance der Matlab FFT−Implementierung
FFT
DFT
O(N log N)
O(N^2)
Rechenzeit in sec.
10^1
10^0
10^−1
10^−2
10^−3
10^−4
10^2
10^4
10^6
N

DFT for periodic sequences $\mathfrak{f} = (\mathfrak{f}_j)_{j\in\mathbb{Z}}$

- periodic sequences: $\mathfrak{f} \in \mathbb{C}^n_{per}$, if $\mathfrak{f}_{k+n} = \mathfrak{f}_k \qquad \forall k \in \mathbb{Z}$

- DFT: $\widehat{\mathfrak{f}} := \mathcal{F}_n(\mathfrak{f}) := (\sum_{j=0}^{n-1} \omega_n^{jk}\mathfrak{f}_j)_{k\in\mathbb{Z}}, \qquad \omega_n := e^{-2\pi\mathbf{i}/\mathbf{n}}$.

- convolution: The convolution $\mathfrak{f} * \mathfrak{g} \in \mathbb{C}^n_{per}$ of $\mathfrak{f}$, $\mathfrak{g} \in \mathbb{C}^n_{per}$ is defined by

$$(\mathfrak{f} * \mathfrak{g})_k := \sum_{j=0}^{n-1} \mathfrak{f}_{k-j}\mathfrak{g}_j, \qquad k \in \mathbb{Z}$$

- pointwise multiplication: For $\mathfrak{f}$, $\mathfrak{g} \in \mathbb{C}^n_{per}$ the product $\mathfrak{f} \cdot \mathfrak{g} \in \mathbb{C}^n_{per}$ is defined by

$$(\mathfrak{f} \cdot \mathfrak{g})_k := \mathfrak{f}_k \cdot \mathfrak{g}_k, \qquad k \in \mathbb{Z}$$

- inner product on $\mathbb{C}^n_{per}$:

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathbb{C}^n_{per}} := \sum_{j=0}^{n-1} \mathfrak{f}_j \overline{\mathfrak{g}_j}$$

properties of the DFT for periodic sequences

- $\mathcal{F}_n : \mathbb{C}^n_{per} \to \mathbb{C}^n_{per}$ is linear.

- $\mathcal{F}_n^{-1} : \mathbb{C}^n_{per} \to \mathbb{C}^n_{per}$ is given by

$$\mathcal{F}_n^{-1}(\mathfrak{f}) = \left(\frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{-jk} \mathfrak{f}_j \right)_{k \in \mathbb{Z}}$$

- Parseval: $\langle \widehat{\mathfrak{f}}, \widehat{\mathfrak{g}} \rangle_{\mathbb{C}^n_{per}} = n \langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathbb{C}^n_{per}}$

- convolution theorem: $\widehat{\mathfrak{f} * \mathfrak{g}} := \mathcal{F}_n(\mathfrak{f} * \mathfrak{g}) = \widehat{\mathfrak{f}} \cdot \widehat{\mathfrak{g}}$, i.e., $(\widehat{\mathfrak{f} * \mathfrak{g}})_k = \widehat{\mathfrak{f}}_k \widehat{\mathfrak{g}}_k$ for all $k \in \mathbb{Z}$.

- modulation $\leftrightarrow$ translation:

$$\mathcal{F}_n((\mathfrak{f}_{k+m})_{k \in \mathbb{Z}}) = (\widehat{\mathfrak{f}}_k \omega_n^{-mk})_{k \in \mathbb{Z}} \qquad \text{translation} \to \text{modulation}$$

$$\mathcal{F}_n\left((\mathfrak{f}_k \omega_n^{km})_{k \in \mathbb{Z}} \right) = (\widehat{\mathfrak{f}}_{k+m})_{k \in \mathbb{Z}} \qquad \text{modulation} \to \text{translation}$$