problem sheet 4

discussion: week of Monday, 28.10.2019

4.1. (**to be uploaded on TUWEL**) Write a program that realizes the composite trapezoidal rule for integration on $[a,b]$. The rule is based on a subdivision of $[a,b]$ into N subintervals of length $h=(b-a)/N$. Input are a function handle for f, N, and a, b.

Consider, for $[a,b]=[-1,1]$ the 5 integrands

$$f_1(x)=x^2, \qquad f_2(x)=|x|, \qquad f_3(x)=\begin{cases}\frac{1}{2}e^x & x<1/3\\ e^x & x\geq 1/3\end{cases}, \qquad f_4(x)=\sin(\pi x), \qquad f_5(x)=\sin(4\pi x),$$

Plot in `loglog`-scale the quadrature error versus h for $h=(b-a)2^{-i}$, $i=1,2,\ldots,20$. What do you observe? Explain your observations for the functions f_1, f_2, f_3.

4.2. Explain the convergence behavior in Exercise 4.1 for the integrand f_4. You may use Euler's formula $e^{\mathbf{i}z}=\cos z+\mathbf{i}\sin z$ in the form $\sin z=\frac{1}{2\mathbf{i}}(e^{\mathbf{i}z}-e^{-\mathbf{i}z})$ with the imaginary unit $\mathbf{i}$ and the geometric series

$$\sum_{j=0}^{N-1} q^j=\frac{q^N-1}{q-1}, \qquad q\neq 1.$$

*H*int: what is is the exact value of the integral?

4.3.
- **a)** Show that the weights of the Newton-Cotes formulas satisfy $\sum_{i=0}^n w_i=1(=\text{length of the interval }[0,1])$. (*Hint:* apply the quadrature formula to a suitable function f.)
- **b)** Show that both the closed and open Newton-Cotes quadrature formulas $\widehat{Q}_n^{cNC}$, $\widehat{Q}_n^{oNC}$ are exact for $f\in\mathcal{P}_n$.
- **c)** Show the symmetry property $w_{n-i}=w_i$, $i=0,\ldots,n$. (*Hint:* use the symmetry of the points, i.e., $x_j=1-x_{n-j}$.
- **d)** Let $n=2m$ be even. Consider the function $f=(x-1/2)^{n+1}$, which is antisymmetric with respect to $1/2$. Show: $\int_0^1 f(x)\,dx=0=\widehat{Q}_n^{cNC}(f)=\widehat{Q}_n^{oNC}(f)$. Conclude that the quadrature formulas $\widehat{Q}_n^{cNC}$ and $\widehat{Q}_n^{oNC}$ are exact for polynomials of degree $n+1$. In particular, the midpoint rule is exact for polynomials in $\mathcal{P}_1$, and the Simpson rule is exact for polynomials in $\mathcal{P}_3$.

4.4. Let C, $\alpha>0$ and consider the function $h\mapsto f(h)=Ch^\alpha$. Why is this function a straight line in a `loglog`-plot? What is its slope? Other popular plotting schemes are, `semilogx` and `semilogy`. Which one would you use to plot functions of the form $N\mapsto Ce^{-bN}$? How would you proceed if you suspect that a function $h\mapsto f(h)$ has the form $f(h)=Ce^{-b/h}$? What if you suspect $f(h)=Ce^{-b/h^2}$?