0 Introduction

Literature:

1. Giaquita & Martinazzi, [GM12]
2. Gilbarg-Trudinger, [GT77]
3. Beck, [Bec16]
4. Jost, [Jos02]

1 some existence theory: Hilbert space techniques

1.1 Lax-Milgram for the scalar case

For a bounded Lipschitz domain, we consider the scalar model problem

$$-D_\alpha A^{\alpha\beta} D_\beta u = f \quad \text{in } \Omega \subset \mathbb{R}^d, \qquad\qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where the matrix $(A^{\alpha\beta})_{\alpha,\beta=1}^d \in L^\infty(\Omega)$ is uniformly SPD, i.e., there exists $\lambda > 0$ such that

$$A^{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d, \quad x \in \Omega \tag{1.2}$$

Example 1.1 $A = \mathrm{I}$ *corresponds to the operator* Δ. ∎

The weak formulation is: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega A^{\alpha\beta}(x) D_\beta u D_\alpha v = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega) \tag{1.3}$$

The bilinear form $a(\cdot\,\cdot)$ is continuous and coercive on $H_0^1(\Omega)$ so that the Lax-Milgram Lemma is applicable:

Lemma 1.2 *For every* $f \in H^{-1}(\Omega) := (H_0^1(\Omega))'$ *there exists a unique* u *satisfying (1.3), and there is* $C > 0$ *(depending only on* λ, $\|A\|_{L^\infty}$, *and* Ω*) such that*

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{1.4}$$

The problem (1.3) can be formulated in operator notation: associated with the bilinear form a is a bounded linear operator $\mathbf{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ given by

$$\langle \mathbf{A}u, v\rangle_{H^{-1}\times H_0^1} = a(u, v) \tag{1.5}$$

and (1.3) is equivalent to finding $u \in H_0^1$ such that

$$\mathbf{A}u = \langle f, \cdot\rangle_{H^{-1}\times H_0^1} \tag{1.6}$$

1.2 Gårding inequality

Elliptic PDEs often contain “lower order terms”. For regularity purposes or even existence theory, they usually play a minor role. We illustrate this here for existence theory.

Definition 1.3 (coercive, Gårding inequality) *A bounded linear operator* $\mathbf{A} : \mathbf{X} \to \mathbf{X}'$ *is* coercive *if there exists* $\alpha > 0$ *such that*

$$\forall u \in \mathbf{X}: \quad \alpha\|u\|_{\mathbf{X}}^2 \leq \langle \mathbf{A}u, u\rangle$$

A bounded linear operator $\mathbf{A} : \mathbf{X} \to \mathbf{X}'$ *is said to satisfy a* Gårding inequality *if there is a compact operator* $\mathbf{K} : \mathbf{X} \to \mathbf{X}'$ *such that* $\mathbf{A} + \mathbf{K}$ *is coercive, i.e., there is* $\alpha > 0$ *such that*

$$\langle \mathbf{A}u, u\rangle \geq \alpha\|u\|_{\mathbf{X}}^2 - \langle \mathbf{K}u, u\rangle. \tag{1.7}$$

Example 1.4 *Let the matrix $A^{\alpha\beta}$ be as in Sec. 1.1. Consider, for $k > 0$,*

$$-D_\alpha A^{\alpha\beta} D_\beta u - k^2 u = f \quad \textit{in } \Omega, \qquad u = 0 \quad \textit{on } \partial\Omega.$$

With the bilinear form $a(\cdot,\cdot)$ of (1.3), the weak form is: find $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega): \quad \widetilde{a}(u,v) := a(u,v) - (k^2 u, v)_{L^2(\Omega)} = \langle f, v\rangle \tag{1.8}$$

With the operator $\mathbf{A}$ of (1.5) and the bounded linear operator $\mathbf{C} : L^2(\Omega) \to L^2(\Omega)$ given by $(\mathbf{C}u, v)_{L^2(\Omega)} = (k^2 u, v)_{L^2(\Omega)}$, problem (1.8) reads

$$(\mathbf{A} - \mathbf{C})u = f$$

Select $\mathbf{K} = \mathbf{C}$. In view of the compact embedding $H^1 \subset L^2$, the operator $\mathbf{A} - \mathbf{C} : H_0^1 \to H^{-1}$ satisfies a Gårding inequality.
In terms of bilinear forms, the Gårding inequality satisfied by $\widetilde{a}(\cdot,\cdot)$ takes the form

$$\forall u \in H^1: \quad \widetilde{a}(u,u) \geq c\|u\|_{H^1}^2 - C\|u\|_{L^2}^2$$

for some C, $c > 0$. This is typical of Gårding inequalities. In the context of variationally posed elliptic problems with bilinear form $b(\cdot,\cdot)$, the Gårding inequality typically takes the form

$$\forall u \in H^s: \quad b(u,u) \geq c\|u\|_{H^s}^2 - C\|u\|_{H^t}^2 \tag{1.9}$$

where $t < s$. ■

In the context of existence theory, the importance of the Gårding inequality arises from the fact that the Fredholm Alternative applies:

Lemma 1.5 (Fredholm alternative) *Let $\mathbf{X}$ be a Hilbert space.*[1] *Let $\mathbf{A} : \mathbf{X} \to \mathbf{X}'$ satisfy a Gårding inequality. Then: If $\mathbf{A}$ is injective then it is a bijection with bounded inverse.*

Proof: Let $\mathbf{K}$ be the compact operator such that $\mathbf{A} + \mathbf{K}$ is coercive. Then $\mathbf{A} + \mathbf{K} : \mathbf{X} \to \mathbf{X}'$ is boundedly invertible. Upon writing $\mathbf{A} = \mathbf{A} + \mathbf{K} - \mathbf{K}$ we see that

$$\mathbf{A}u = f \quad \iff \quad (\mathrm{I} - (\mathbf{A} + \mathbf{K})^{-1}\mathbf{K})u = (\mathbf{A} + \mathbf{K})^{-1} f$$

Since $\mathbf{K}$ is compact and $(\mathbf{A} + \mathbf{K})^{-1}$ bounded, linear, the operator $(\mathbf{A} + \mathbf{K}^{-1})\mathbf{K}$ is compact and the assertion follows from the (Fredholm) theory for compact operators. (cf. FANA). □

Corollary 1.6 *Let the continuous bilinear form b satisfy the Gårding inequality (1.9). Assume injectivity, i.e., $b(u,\cdot) = 0$ implies $u = 0$. Then, there exists $C > 0$ such that for every $f \in (H^s)'$, the problem*

$$\forall v \in H^s: \quad b(u,v) = \langle f, v\rangle$$

has a unique solution u and a priori the bound

$$\|u\|_{H^s} \leq C\|f\|_{(H^s)'}$$

holds.

[1] reflexive would suffice

1.3 Solvability theory for elliptic systems

For $\Omega \subset \mathbb{R}^d$, we consider systems of the form

$$-D_\alpha \left(A_{ij}^{\alpha\beta}(x) D_\beta u^j \right) = f_i - D_\alpha F_i^\alpha, \quad i = 1, \ldots, m \tag{1.10}$$

again with the understanding that summation takes place over repeated indices. Here, α, $\beta \in \{1, \ldots, d\}$ and i, $j \in \{1, \ldots, m\}$.
There are several possible generalizations of the concept of coercivity from the scalar case to the vectorial one.

Definition 1.7 (Legendre and Legendre-Hadamard conditions) *The tensor* $(A_{ij}^{\alpha\beta}(x))_{i,j=1,\ldots,m}^{\alpha,\beta=1,\ldots,d}$ *satisfies the*

(i) very strong ellipticity/superstrong ellipticity/Legendre condition *at x if for some $\lambda > 0$*

$$\forall \xi \in \mathbb{R}^{m \times d}: \quad A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq \lambda |\xi|^2 \tag{L}$$

(ii) the strong ellipticity/Legendre-Hadamard condition *at x, if for some $\lambda > 0$*

$$\forall \xi \in \mathbb{R}^d \quad \forall \eta \in \mathbb{R}^n: \quad A_{ij}^{\alpha\beta} \xi_\alpha \xi_\beta \eta^i \eta^j \geq \lambda |\xi|^2 |\eta|^2 \tag{LH}$$

Remark 1.8
- *Whereas the Legendre condition (L) requires positive definiteness of $A_{ij}^{\alpha\beta}$ on the set of all matrices, the Legendre-Hadamard condition (LH) is much weaker as it requires it on the set of rank-1 matrices $\xi \otimes \eta$ for $\xi \in \mathbb{R}^d$, $\eta \in \mathbb{R}^m$. In particular, (L) implies (LH) by setting $\xi_\alpha^i = \xi_\alpha \eta^i$*
- *(LH) implies (L) for $m = 1 = d$.*
- *In general, (LH) does* not *imply (L); see [GM12, Ex. 3.38].* ∎

Example 1.9 (linear elasticity) *In linear elasticity with Lamé-parameters λ, $\mu > 0$, one seeks the displacement field $\mathbf{u} \in \mathbb{R}^d$ such that*

$$-\mu \Delta \mathbf{u} - (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) = \mathbf{f}.$$

We have

$$A_{ij}^{\alpha\beta} = \mu \delta_{ij} \delta_{\alpha\beta} + (\lambda + \mu) \delta_{i\alpha} \delta_{j\beta}$$

so that

$$A_{ij}^{\alpha\beta} \xi_\alpha \xi_\beta \eta^i \eta^j = \mu |\xi|^2 |\eta|^2 + (\lambda + \mu)(\xi \cdot \eta)^2 \geq \mu |\xi|^2 |\eta|^2.$$

∎

The importance of the condition (LH) stems from the fact that the corresponding bilinear form is coercive.

Theorem 1.10 *Let $A_{ij}^{\alpha\beta} \in C^0(\overline{\Omega})$ satisfy (LH) for some $\lambda > 0$ independent of $x \in \overline{\Omega}$. Define the bilinear form*

$$b(u, v) := \int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha v^i \tag{1.11}$$

Then:

(i) If $(A_{ij}^{\alpha\beta})$ *is* constant *on* Ω*, then there exists* $C > 0$ *such that*

$$\forall u \in H_0^1(\Omega;\mathbb{R}^m): \quad b(u,u) \geq C\|u\|^2_{H^1(\Omega)} \tag{1.12}$$

(ii) there exist C*,* $c > 0$ *such that*

$$\forall u \in H_0^1(\Omega;\mathbb{R}^m): \quad b(u,u) \geq c\|u\|^2_{H^1(\Omega)} - C\|u\|^2_{L^2(\Omega)} \tag{1.13}$$

Proof: (see [GM12, Thm. 3.42] for details) We tacitly extend $u \in H^1(\Omega)$ by zero to $\mathbb{R}^d$.
Proof of (i): We employ Fourier techniques. Set

$$\widehat{u}(\xi) := \int_{\mathbb{R}^d} u(y)e^{-2\pi \mathbf{i}\xi\cdot y}\,dy$$

and note

$$\widehat{D_\alpha u}(\xi) = 2\pi\mathbf{i}\xi_\alpha\widehat{u}(\xi), \qquad (\widehat{u},\widehat{v})_{L^2} = (u,v)_{L^2}.$$

We also note that $b(u,u)$ is real. Therefore,

$$\begin{aligned}
b(u,u) &= \operatorname{Re}\left(A_{ij}^{\alpha\beta}\int_{\mathbb{R}^d}\widehat{D_\beta u^j}\,\overline{\widehat{D_\alpha u^i}}\right) = \operatorname{Re}(2\pi)^2 A_{ij}^{\alpha\beta}\int_{\mathbb{R}^d}\xi_\beta\widehat{u}^j\xi_\alpha\overline{\widehat{u}^i} \\
&= (2\pi)^2 A_{ij}^{\alpha\beta}\int_{\mathbb{R}^d}\xi_\alpha\xi_\beta\left(\operatorname{Re}\widehat{u}^j\operatorname{Re}\widehat{u}^i + \operatorname{Im}\widehat{u}^j\operatorname{Im}\widehat{u}^i\right) \\
&\overset{(LH)}{\geq} (2\pi)^2\lambda\int_{\mathbb{R}^d}|\xi|^2|\widehat{u}|^2\,d\xi = (2\pi)^2\lambda\delta^{\alpha\beta}\delta_{ij}\int_{\mathbb{R}^d}\xi_\alpha\xi_\beta\widehat{u}^j\overline{\widehat{u}^i}\,d\xi \\
&= \lambda\delta^{\alpha\beta}\delta_{ij}\int_{\mathbb{R}^d}D_\alpha u^j\overline{D_\beta u^i}\,d\xi = \lambda\int_{\mathbb{R}^d}|Du|^2\,dx = \lambda|u|^2_{H^1(\Omega)}.
\end{aligned}$$

Proof of (ii): Define the modulus of continuity finis 1.DS

$$\omega(r) := \max_{i,j,\alpha,\beta,x,y}|A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(y)|.$$

Step 1: Let $u \in H_0^1(B_r(x_0))$. Then

$$\begin{aligned}
b(u,u) &= A_{ij}^{\alpha\beta}(x_0)\int_\Omega D_\beta u^j D_\alpha u^i + \int_\Omega\left[A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(x_0)\right]D_\beta u^j D_\alpha u^i \\
&\overset{(i)}{\geq} \lambda|u|^1_{H^1(\Omega)} - \omega(r)|u|^2_{H^1(\Omega)}.
\end{aligned}$$

Step 2: Since $A_{ij}^{\alpha\beta}$ is uniformly continuous on $\overline{\Omega}$, we can find $r > 0$ such that $\omega(r) < \lambda$ (e.g., $\leq \lambda/2$). Cover Ω by finitely many balls $B_r(x_k)$ and pick a partition of unity $\{\varphi_k^2\}$ subordinate to that covering. Then, using $u \equiv \sum_k \varphi_k^2 u$ on Ω we compute

$$b(u,u) = \int_\Omega A_{ij}^{\alpha\beta}(x)\sum_k\varphi_k^2 D_\beta u^j D_\alpha u^i$$

The key observation is that

$$\varphi_k^2 D_\beta u^j D_\alpha u^i = D_\beta(\varphi_k u^j)D_\alpha(\varphi_k u^i) + \text{terms that are products of a) } u^j \text{ and } D_\alpha u^i \text{ or b) } D_\beta u^j \text{ and } u^i \text{ or c) } u^i \text{ and } u^j$$

Therefore,

$$
\begin{aligned}
b(u,u) &\overset{\text{Step 1}}{\geq} (\lambda - \omega(r)) \sum_k |\varphi_k u|^2_{H^1(\Omega)} - C\|u\|_{L^2}\|u\|_{H^1} \\
&\geq (\lambda - \omega(r)) \sum_k \int_\Omega \varphi_k^2 |Du|^2 \, dx - C'\|u\|_{L^2}\|u\|_{H^1} \\
&= (\lambda - \omega(r)) \int_\Omega |Du|^2 \, dx - C'\|u\|_{L^2}\|u\|_{H^1}.
\end{aligned}
$$

The proof is complete. □

Remark 1.11 *The use of Fourier techniques is very common for PDEs with* constant *coefficients as it turns partial differential equations into algebraic ones.* ■

2 Caccioppoli inequality and difference quotient techniques

2.1 Caccioppoli inequality/Reverse Poincaré inequality

The Caccioppoli/reverse Poincaré inequality is a key property of elliptic equations. It allows one to estimate higher derivatives by lower ones (at the expense of slightly increasing the integration domain).

Theorem 2.1 *Let $u \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of*

$$-D_\alpha A_{ij}^{\alpha\beta} D_\beta u^j = f_i - D_\alpha F_i^\alpha$$

with f, $F^\alpha \in L^2(\Omega)$. Assume that one of the following three conditions is satisfied:

(i) $A_{ij}^{\alpha\beta} \in L^\infty$ satisfies (L)

(ii) $A_{ij}^{\alpha\beta} = const$ satisfies (LH)

(iii) $A_{ij}^{\alpha\beta} \in C(\overline{\Omega})$ satisfies (LH)

Then, for any ball $B_R(x_0) \subset \Omega$, any $0 < \rho < R$, and any $\xi \in \mathbb{R}^m$ the following Caccioppoli/reverse Poincaré inequality holds:

$$\int_{B_\rho(x_0)} |Du|^2 \leq C \left[(R-\rho)^{-2} \int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\xi|^2 + R^2 \int_{B_R(x_0)} |f|^2 + \int_{B_R(x_0)} |F|^2 \right] \tag{2.1}$$

In cases (i), (ii), the constant C depends on λ, $\Lambda := \|A\|_{L^\infty}$; in case (iii), it depends additionally on R and the modulus of continuity of A.

Proof: see [GM12, Thm. 4.4]. We illustrate the main point for the scalar problem

$$-\Delta u = f - D_\alpha F^\alpha.$$

Step 1: Let $f \equiv 0$. Let $\eta \in C_0^\infty(B_R(x_0))$ with $\eta \equiv 1$ on $B_\rho(x_0)$ and $\|\nabla \eta\|_{L^\infty} \leq 2(R-\rho)^{-1}$. The test function $v = \eta^2(u-\xi)$ yields

$$\int_{B_R} \nabla u \cdot \nabla(\eta^2(u-\xi)) = \int_{B_R} F \cdot \nabla(\eta^2(u-\xi)).$$

Therefore,

$$\begin{aligned}
\|\eta \nabla u\|_{L^2}^2 &= \int_{B_R} \eta^2 |\nabla u|^2 = \int_{B_R} \eta^2 \nabla u \cdot \nabla u = \int_\Omega \nabla u \cdot \big(\nabla(\eta^2(u-\xi)) - \nabla(\eta^2)(u-\xi)\big) \\
&= \int_\Omega F \cdot \nabla(\eta^2(u-\xi)) - (\nabla \eta^2) \cdot (u-\xi) \nabla u \\
&= \int_\Omega F \cdot 2\eta \nabla \eta (u-\xi) + F \cdot \eta^2 \nabla u - 2\eta \nabla \eta \cdot \nabla u (u-\xi) \\
&\leq \|F\|_{L^2(B_R)} \frac{4}{R-\rho} \|u-\xi\|_{L^2(B_R \setminus B_\rho)} + \|F\eta\|_{L^2} \|\eta \nabla u\|_{L^2} + \frac{2}{R-\rho} \|\eta \nabla u\|_{L^2} \|u-\xi\|_{L^2(B_R \setminus B_\rho)}
\end{aligned}$$

The proof is complete with Young's inequality that allows one to move $\|\eta\nabla u\|_{L^2}$ to the left-hand side.

Step 2 (reduction to Step 1): Let $f \not\equiv 0$. Define

$$\widetilde{F}^1 := \int_{-\infty}^{x_1} f(t, x_2, \ldots, x_d)\chi_{B_R(x_0)}(t, x_2, \ldots, x_d)\, dt.$$

Then, $f = D_1\widetilde{F}^1$ and we may apply Step 1 with F replaced with $(\widetilde{F}^1, 0, \ldots, 0)^\top + F$. In view of the 1D estimate $\int_0^{2R}(\int_0^x g(t)\, dt)^2 \le \frac{(2R)^2}{2}\int_0^{2R} g^2$ we estimate

$$\|\widetilde{F}^1\|^2_{L^2(B_R)} \le CR^2\|f\|^2_{L^2(B_R)}.$$

The proof is complete. □

2.2 Interior estimates via difference quotients

A classical technique to obtain regularity estimates for PDEs is the difference quotient method.

2.2.1 preliminaries

We start with Jensen's inequality:

Lemma 2.2 (Jensen's inequality) *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex and μ a probability measure on Ω, i.e., $\int_\Omega d\mu = 1$. Then for $f : \Omega \to \mathbb{R}$ there holds*

$$\varphi\left(\int_\Omega f\, d\mu\right) \le \int_\Omega \varphi \circ f\, d\mu. \tag{2.2}$$

In particular, for an interval $[a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ we have for convex functions φ

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)\, dx\right) \le \frac{1}{(b-a)}\int_a^b \varphi(f(x))\, dx.$$

Proof: see literature. □

For $h \in \mathbb{R}$ and $s \in \{1, \ldots, d\}$ we define with the sth unit vector $e_s \in \mathbb{R}^d$

$$(\tau_{h,s}u)(x) := \frac{u(x + he_s) - u(x)}{h}. \tag{2.3}$$

We observe for u, v compactly supported in Ω and h sufficiently small (by a change of variables)

$$\int_\Omega (\tau_{h,s}u)v = -\int_\Omega u(\tau_{-h,s}v) \tag{2.4}$$

Lemma 2.3 *Let $1 < p < \infty$ and $\Omega_0 \subset\subset \Omega$.*

(i) If $u \in W^{1,p}(\Omega)$ then $\|\tau_{h,s}u\|_{L^p(\Omega_0)} \le \|D_s u\|_{L^p(\Omega)}$ if $h < \frac{1}{2}\operatorname{dist}(\Omega_0, \partial\Omega)$

(ii) If $u \in L^p(\Omega)$ and $\|\tau_{h,s}u\|_{L^p(\Omega_0)} \leq L$ for all $0 < h < \operatorname{dist}(\Omega_0, \partial\Omega)$ and all $s \in \{1, \dots, d\}$, then $u \in W^{1,p}(\Omega_0)$ and $\|Du\|_{L^p(\Omega_0)} \leq L$. Furthermore, $\tau_{h,s}u \to D_s u$ in $L^p(\Omega_1)$ for $\Omega_1 \subset\subset \Omega_0$.

Proof: *ad (i):* By density, we restrict to $u \in C^\infty(\Omega)$, [MS64]. By writing $u(x + he_s) - u(x) = \int_0^h D_s u(x + te_s)\, dt$ we get $\tau_{h,s}u(x) = \frac{1}{h}\int_0^h D_s u(x + te_s)\, dt$ and thus

$$\begin{aligned}
\|\tau_{h,s}u\|^p_{L^p(\Omega_0)} &= \int_{\Omega_0} \left(\frac{1}{h}\int_0^h D_s u(x + te_s)\, dt\right)^p dx \\
&\overset{\text{Jensen}}{\leq} \int_{\Omega_0} h^{-1} \int_0^h |D_s u(x + te_s)|^p\, dt \overset{\text{Fubini}}{\leq} h^{-1} \int_{t=0}^h \int_{x \in \Omega_0} |D_s u(x + te_s)|^p\, dx\, dt \\
&\leq \|D_s u\|^p_{L^p(\Omega)}.
\end{aligned}$$

ad (ii): Observe that $L^p(\Omega)$ is reflexive for $p \in (1, \infty)$. By Banach-Alaoglu there exists a subsequence $\tau_{h_n,s}u$ and a $g \in L^p(\Omega_0)$ such that

$$\tau_{h_n,s}u \xrightarrow{L^p(\Omega_0)} g$$

By (sequential weak lower semicontinuity of the norm[1]), we also have

$$\|g\|_{L^p(\Omega_0)} \leq \liminf_{h_n \to 0} \|\tau_{h_n,s}u\|_{L^p(\Omega_0)} \leq L.$$

Claim: $g = D_s u$. To see this, let $\varphi \in C_0^\infty(\Omega_0)$. Then, by the definition of distributional derivative

$$\int_{\Omega_0} g\varphi = \lim_{h_n \to 0} \int_{\Omega_0} \tau_{h_n,s}u\varphi = \lim_{h_n \to 0} -\int_{\Omega_0} u\tau_{-h_n,s}\varphi = -\int_{\Omega_0} uD_s\varphi,$$

so that g is indeed the weak derivative of u. Hence, $D_s u \in L^p(\Omega_0)$ and $\|D_s u\|_{L^p(\Omega_0)} \leq L$. Furthermore, by uniqueness of the limit, $\tau_{h,s}u \rightharpoonup D_s u$ in $L^p(\Omega_0)$ as $h \to 0$.
To see $\lim_{h\to 0} \tau_{h,s}u = D_s u$ in $L^p(\Omega_1)$, we let $w \in C^\infty(\Omega_0)$. By density, we can make $\|u - w\|_{W^{1,p}(\Omega_0)}$ small (Meyers-Serrin [MS64]). Then

$$\tau_{h,s}u - D_s u = \underbrace{\tau_{h,s}(u - w)}_{\|\cdot\|_{L^p(\Omega_1)} \leq \|D_s(u-w)\|_{L^p(\Omega_0)} \to 0 \text{by (i)}} + \underbrace{\tau_{h,s}w - D_s w}_{\to 0 \text{ as } h \to 0 \text{ by smoothness of } w} + \underbrace{D_s(w - u)}_{\to 0 \text{ by Meyers-Serrin}}$$

□

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2.2.2 Interior regularity

Theorem 2.4 *Let $u \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of*

$$-D_\alpha A_{ij}^{\alpha\beta} D_\beta u^j = f_i - D_\alpha F_i^\alpha$$

with $f_i \in L^2(\Omega)$ and $F_i^\alpha \in H^1(\Omega)$. Let $A_{ij}^{\alpha\beta} \in \operatorname{Lip}(\Omega)$ satisfy (LH). Then $u \in H^2_{loc}(\Omega; \mathbb{R}^m)$ and for each $\Omega_0 \subset\subset \Omega$ there is $C > 0$ such that

$$\|u\|_{H^2(\Omega_0)} \leq C\left[\|f\|_{L^2(\Omega)} + \|DF\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right].$$

[1]here: for $v \in L^{p'}$ we have $\langle g, v\rangle = \lim_n \langle \tau_{h_n,s}u, v\rangle \leq \liminf_n \|\tau_{h_n,s}u\|_{L^p}\|v\|_{L^{p'}}$

Proof: See [GM12, Thm. 4.9] for details. Here, we restrict to the scalar case $-\Delta u = f$. Let $\eta \in C_0^\infty(\Omega)$ with $\eta \equiv 1$ on Ω_0. Then, with the test function $v = \tau_{-h,s}(\eta^2 \tau_{h,s} u) \in H_0^1(\Omega)$ we have

$$lhs := \int_\Omega \nabla u \cdot \nabla \tau_{-h,s}(\eta^2 \tau_{h,s} u) = \int_\Omega f \tau_{-h,s}(\eta^2 \tau_{h,s} u) =: rhs$$

We estimate lhs and rhs separately for h sufficiently small:

$$\begin{aligned} lhs &= \int_\Omega \nabla \tau_{h,s} u \cdot \nabla(\eta^2 \tau_{h,s} u) = \int_\Omega \eta^2 |\nabla \tau_{h,s} u|^2 + \int_\Omega \nabla \tau_{h,s} u \cdot ((\nabla \eta^2) \tau_{h,s} u) \\ &\geq \|\eta \nabla \tau_{h,s} u\|^2_{L^2(\Omega)} - 2\|\eta \nabla \tau_{h,s} u\|_{L^2(\Omega)} \|(\nabla \eta)\tau_{h,s} u\|_{L^2(\Omega)}, \\ rhs &\overset{L.\ 2.3}{\leq} \|f\|_{L^2(\Omega)} \|\nabla(\eta^2 \tau_{h,s} u)\|_{L^2(\Omega)} \overset{\eta \leq 1}{\leq} \|f\|_{L^2(\Omega)} \left(\|\eta \nabla \tau_{h,s} u\|_{L^2(\Omega)} + \|(\nabla \eta)\tau_{h,s} u\|_{L^2(\Omega)} \right) \end{aligned}$$

Combining these two estimates with appropriate use of Young's inequality yields

$$\|\eta \nabla \tau_{h,s} u\|^2_{L^2(\Omega)} \leq C \left(\|f\|^2_{L^2(\Omega)} + \|(\nabla \eta)\tau_{h,s} u\|^2_{L^2(\Omega)} \right)$$

Select $\widetilde{\Omega}_0$ with $\Omega_0 \subset\subset \widetilde{\Omega}_0 \subset\subset \Omega$. For h sufficiently small, we obtain from Theorem 2.1

$$\|\eta \tau_{h,s} \nabla u\|^2_{L^2(\Omega)} \leq C \left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\widetilde{\Omega}_0)} \right) \tag{2.5}$$

Finally, the uniform-in-h control of difference quotients in (2.5) allows us to appeal to Lemma 2.3 to get in view of $\eta \equiv 1$ on Ω_0

$$\|D_s \nabla u\|_{L^2(\Omega_0)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\widetilde{\Omega}_0)} \right) \tag{2.6}$$

Remark (motivation for the choice of test function): For $-\Delta u = f$ we consider the equations satisfied by u and by $u(\cdot + h e_s)$:

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v &= \int_\Omega f v \quad \forall v \in H_0^1(\Omega) \\ \int_\Omega \nabla u(\cdot + h e_s) \cdot \nabla v &= \int_\Omega f(\cdot + h e_s) v \quad \forall v \text{ with suitable support} \\ \hline \int_\Omega \nabla \tau_{h,s} u \cdot \nabla v &= \int_\Omega \tau_{h,s} f v = -\int_\Omega f \tau_{-h,s} v \end{aligned}$$

The left-hand side suggests to select for v a localized version of $\tau_{h,s} u$, i.e., $\eta^2 \tau_{h,s} u$. □

By differentiating the equation, one can iterate the argument and obtain higher regularity:

Theorem 2.5 *Assume the hypotheses of Theorem 2.4. Assume additionally*

(i) $A_{ij}^{\alpha\beta} \in C^{k,1}(\Omega)$

(ii) $f_i \in H^k(\Omega)$

(iii) $F_i^\alpha \in H^{k+1}(\Omega)$

Then, $u \in H^{k+2}_{loc}(\Omega)$ *and for every* $\Omega_0 \subset\subset \Omega$ *there is* $C > 0$ *such that*

$$\|u\|_{H^{k+2}(\Omega_0)} \leq C \left[\|u\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)} + \|DF\|_{H^k(\Omega)} \right]. \tag{2.7}$$

Proof: see [GM12, Thm. 4.11]. □

Corollary 2.6 *Assume the hypotheses of Theorem 2.4. If* $A_{ij}^{\alpha\beta} \in C^\infty(\Omega)$, $f_i \in C^\infty(\Omega)$, $F_i^\alpha \in C^\infty(\Omega)$, *then* $u \in C^\infty(\Omega)$.

Exercise 2.7 *The iterative application of Theorem 2.4 allows for some quantitative bounds for the constant* C *in (2.7). Show for the case* $-\Delta u = 0$ *on* B_R *that*

$$\|D^k u\|_{L^2(B_\rho)} \leq C_k (R-\rho)^{-k} \|u\|_{L^2(B_R)},$$

where C_k *depends only on* k. ∎

Exercise 2.8 *Consider* u *with* $-\Delta u = 0$ *on* $B_R(x_0)$. *Denote by* u_R *the average of* u *on* B_R. *Show:*

$$\sup_{B_{R/2}} |Du|^2 \leq C R^{-(d+2)} \|u - u_R\|^2_{L^2(B_R)} \leq C R^{-d} \|Du\|^2_{L^2(B_R)}.$$

Hint: *Consider the scaled function* $\hat{u}(x) := u(x_0 + xR)$ *on* $B_1(0)$, *which satisfies* $-\Delta \hat{u} = 0$. *Use Sobolev embedding theorem* $H^k \subset L^\infty$ *for* $k > d/2$. *Obtain the powers of* R *by scaling.* ∎

Remark 2.9 *Theorems 2.4, 2.5 only consider equations with principal term. However, regularity for equations with lower order terms can be obtained by moving the lower order terms to the right-hand side. For example, for the equation* $-\Delta u + \mathbf{b} \cdot \nabla u + cu = f$, *one write*

$$-\Delta u = \widetilde{f} := f - \mathbf{b} \cdot \nabla u - cu$$

and applies the above theorems with $\widetilde{f}$. ∎

2.2.3 Boundary regularity

The method of difference quotients is able to get regularity *up to the boundary* provided suitable boundary conditions are provided. A typical result is:

Theorem 2.10 *Assume the hypotheses of Theorem 2.5. Assume additionally* $u \in H_0^1(\Omega)$ *and* $\partial\Omega \in C^{k+2}$. *Then:*

$$\|u\|_{H^{k+2}(\Omega)} \leq C \left[\|f\|_{H^k(\Omega)} + \|DF\|_{H^k(\Omega)} \right].$$

Proof: see [GM12, Thm. 4.14]. The general technique is:

(I) localize and reduce the problem to that on a half-space problem by locally flattening the boundary

(II) control the tangential derivatives by the method of difference quotients

(III) control the normal derivatives by using the equation.

ad (I): Let $F : \widehat{\omega} \to \omega$ be a C^1-diffeomorphism. Define

$$\widehat{u} := u \circ F, \qquad \widehat{A} := u \circ F, \qquad \widehat{f} := f \circ F, \qquad J := |\det(DF)|.$$

Note that

$$(\nabla \widehat{u})^\top = ((\nabla u)^\top \circ F) DF. \tag{2.8}$$

Assume that u satisfies (in strong form) $-\nabla \cdot (A\nabla u) = f$ in ω. One could simply use the chain rule to determine the equation satisfied by $\widehat{u}$. It is easier to do this using the weak formulation:

$$\underbrace{\int_\omega (\nabla v)^\top A(x) \nabla u}_{\int_{\widehat{\omega}} (\nabla \widehat{v})^\top (DF)^{-1} \widehat{A} (DF)^{-T} \nabla \widehat{u} J} = \underbrace{\int_\omega f v \quad \forall v \in H^1_0(\omega)}_{\int_{\widehat{\omega}} \widehat{f} \widehat{v} J}$$

so that, in strong form, the transformed equation is

$$-\nabla \cdot \Big((J (DF)^{-1} \widehat{A} (DF)^{-T}) \nabla \widehat{u} \Big) = J \widehat{f} \quad \text{in } \widehat{\omega},$$

which is of a similar structure as the original equation.
ad (II): We introduce half-balls $B_r^+ := \{x = (x', x_d) \,|\, |x| < r,\, x_d > 0\}$. To simplify the presentation, consider u satisfying

$$-\Delta u = 0 \quad \text{in } B_R^+, \qquad\qquad u = 0 \quad \text{on } x_d = 0. \tag{2.9}$$

The weak form is

$$\int_{B_R^+} \nabla u \cdot \nabla v = \int_{B_R^+} f v \quad \forall v \in H^1_0(B_R^+).$$

For $s \in \{1 \ldots, d-1\}$, consider the test function $v = \tau_{-h,s}(\eta^2 \tau_{h,s})$ and note $v \in H^1_0(B_R^+)$ for h sufficiently small. Exactly as in the proof of Theorem 2.4, one obtains for $0 < r < R$ (and a constant depending on r, R)

$$\|D_s \nabla u\|_{L^2(B_r^+)} \leq C \left[\|f\|_{L^2(B_R^+)} + \|u\|_{L^2(B_R^+)} \right]. \tag{2.10}$$

ad (III): The bounds (2.10) lead to (local) control of all second derivatives of u with the exception of $D_d^2 u$. From the differential equation $-\Delta u = f$, we get

$$-D_d^2 u = f + \sum_{i=1}^{d-1} D_i^2 u,$$

so that

$$\|D_d^2 u\|_{L^2(B_r^+)} \leq \|f\|_{L^2(B_r^+)} + \sum_{i=1}^{d-1} \|D_i^2 u\|_{L^2(B_r^+)} \overset{(2.10)}{\leq} C \left[\|f\|_{L^2(Bk_R)} + \|u\|_{L^2(B_R^+)} \right].$$

⟦ note that we know already that the second derivatives of u exist almost everywhere so that the equation really holds pointwise almost everywhere ⟧ □

finis 3.DS

Remark 2.11 *(i) Thm. 2.10 also holds for Neumann boundary conditions.*

(ii) For scalar equations with constant coefficients, there is an alternative way to show regularity up to the boundary using reflection principles. Consider the Dirichlet case of (2.9). Extend u to the ball B_R by antisymmetric/odd extension: $u(x', x_d) := -u(x', -x_d)$ for

$x_d < 0$ and extend $f(x', x_d) := -f(x', -x_d)$. It can be checked that the extended function is in $H^1(B_R)$ and that it is a weak solution of $-\Delta u = f$ in B_R.

⟦ split the test function $v = \mathcal{A}v + \mathcal{S}v$ into the antisymmetric part $\mathcal{A}(x', x_d) = \frac{1}{2}(v(x', x_d) - v(x', -x_d))$ and the symmetric part $\mathcal{S}(x', x_d) = \frac{1}{2}(v(x', x_d) + v(x', -x_d))$ and compute for $v \in H_0^1(B_R)$, using $\mathcal{A}v \in H_0^1(B_R^+)$ denoting, for clarification, by $\widetilde{u}$ the extended function

$$\int_{B_R} \nabla\widetilde{u} \cdot \nabla v = \underbrace{\int_{B_R} \nabla\widetilde{u} \cdot \nabla \mathcal{A}v}_{=2\int_{B_R^+} \nabla u \cdot \nabla \mathcal{A}v = 2\int_{B_R^+} f\mathcal{A}v = \int_{B_R} f\mathcal{A}v} + \int_{B_R} \nabla\widetilde{u} \cdot \nabla \mathcal{S}v$$

$$= \int_{B_R} f\mathcal{A}v + \underbrace{\int_{B_R} \nabla_{x'}\widetilde{u} \cdot \nabla_{x'} \mathcal{S}v}_{=0} + \underbrace{\int_{B_R} D_d u D_d \mathcal{S}v}_{=0}$$

$$= \int_{B_R} fv$$

⟧

(iii) the reflection principle also works for Neumann boundary conditions using an even/symmetric extension.

■

2.3 difference quotient techniques for Lipschitz domains

The difference quotient technique employed above uses tangential difference quotients to properly make use of the boundary conditions. It appears not be to be suitable for Lipschitz domains: the flattening of the boundary leads to a PDE with L^∞ coefficients and the difference quotient technique requires some differentiability of the coefficients. For the case of pure Dirichlet or Neumann boundary conditions, [Sav98] showed that the difference quotient technique may nevertheless be used for both linear and non-linear (specifically: "monotone") operators.

Example 2.12 *Consider the Dirichlet problem*

$$-\Delta u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{2.11}$$

with weak formulation $a(u, v) := \int_\Omega \nabla u \cdot \nabla v = \ell(v) := \int_\Omega fv$. The weak formulation is the Euler-Lagrange equation of the minimization problem: Find $u \in H_0^1(\Omega)$ such that

$$J(u) := \frac{1}{2}a(u, u) - \ell(u) \tag{2.12}$$

is minimized over $H_0^1(\Omega)$. ⟦ compute the Gâteaux derivative of J by computing derivative of $t \mapsto J(u + tv)$ and evaluate at $t = 0$ ⟧ At the minimizer u, J is quadratic:

$$J(v) - J(u) = \frac{1}{2}a(v, v) - \ell(v) - \frac{1}{2}a(u, u) + \ell(u) = \frac{1}{2}a(v - u, v + u) - \ell(v - u)$$
$$= \frac{1}{2}a(v - u, v - u) + a(v - u, u) - \ell(v - u) = \frac{1}{2}a(v - u, v - u)$$

[[*this property is used, e.g., in the FEM where $v = u_h$ is can be used to compute energy norm errors*]] ■

The idea of the technique that we will present consists in choosing v as a translate of the exact solution u and then estimate $J(v) - J(u)$. In this way, difference quotients can be estimated.

2.3.1 monotone operators

The method of [Sav98] is able to deal with certain non-linear problems such as monotone operators, which generalize the quadratic behavior of the functional J at a minimizer in Example 2.12.
An (not necessarily linear) operator $\mathcal{A} : X \to X'$ is called *monotone/strongly montone* if

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle \geq 0 \quad \forall u, v \in X \tag{2.13}$$

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle \geq c\|u - v\|_X^2 \quad \forall u, v \in X. \tag{2.14}$$

Remark 2.13 *Monotone operators arise naturally as the (Gâteaux)-derivatives of convex functionals. Indeed, let $J : X \to \mathbb{R}$ be differentiable. Consider $t \mapsto \varphi(t) := J(u + t(v - u))$ and note*

$$\varphi'(t) = \langle J'(u + t(v - u)), v - u\rangle.$$

- *Let J be convex. Then φ is convex and therefore its derivative is monotone increasing. From $\varphi'(1) \geq \varphi'(0)$ we obtain $\langle J'(u) - J'(v), u - v\rangle \geq 0$.*
- *Let J' be monotone. Then for $s < t$ we have*

$$\begin{aligned}\varphi'(t) - \varphi'(s) &= \langle J'(u + t(v - u)) - J'(u + s(v - u)), v - u\rangle \\ &= \frac{1}{t - s}\langle J'(u + t(v - u)) - J'(u + s(v - u)), u + t(v - u) - (u + s(v - u))\rangle \geq 0\end{aligned}$$

  so that the function φ' is monotone increasing. Hence, φ is convex and thus also J.

Note that the functional J of Example 2.12 is such a convex functional with even strongly monotone derivative. A good reference for convex operators is [Zei85]. ■

Minimizers u of J satisfy[2]

$$J'(u) = 0. \tag{2.15}$$

Lemma 2.14 *Let $J' = \mathcal{A}$ be p-coercive, w.r.t. a seminorm $|\cdot|_*$ for a given $p \in [2, \infty)$, i.e.,*

$$\exists \alpha > 0 : \quad \langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle \geq \alpha|u - v|_*^p \qquad \forall u, v \in X.$$

Then: If u realizes a minimum of J, then

$$\frac{\alpha}{p}|u - v|_*^p \leq J(v) - J(u) \qquad \forall v \in X.$$

[2] in case of minimization over convex sets K, one only has the variational inequality $\langle J'(u), v - u\rangle \geq 0$ for all $v \in K$

Proof: Consider the function $\varphi(t) := J(u + t(v-u))$ with derivative $\varphi'(t) = \langle \mathcal{A}(u + t(v - u)), v - u\rangle$. Compute

$$\begin{aligned} J(v) - J(u) - \underbrace{\langle \mathcal{A}(u), v-u\rangle}_{=0 \text{ by } (2.15)} &= \varphi(1) - \varphi(0) - \varphi'(0) = \int_{t=0}^1 \varphi'(t) - \varphi'(0)\, dt \\ &= \int_{t=0}^1 \langle \mathcal{A}(u + t(v-u)) - \mathcal{A}(u), t(v-u)\rangle \frac{dt}{t} \\ &\geq \alpha \int_{t=0}^1 t^p |v-u|_*^p \frac{dt}{t} = \frac{\alpha}{p}|v-u|_*^p. \end{aligned}$$

□

Remark 2.15 *The restriction $p \geq 2$ comes from the purely practical observation that the φ' would be nowhere differentiable anymore since $\varphi'(t+h) - \varphi'(t) = \langle \mathcal{A}(u + t(v-u) + h(v-u)) - \mathcal{A}(u + t(v-u)), v-u\rangle \geq \frac{\alpha}{p}(h|v-u|_*)^p h^{-1}$. However, convex functions are a.e. twice differentiable.* ■

prominent examples include

- $J(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega fu$, which leads to the Poisson problem (e.g., with Dirichlet conditions if $X = H^1_0(\Omega)$)
- Let $p \in (1, \infty)$. Define on $X = W^{1,p}_0(\Omega)$ the functional $J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \langle f, u\rangle$, which leads to the "$p$-Laplacian". In strong form

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = f.$$

2.3.2 difference quotient techniques

<u>idea:</u> estimate $|u - u_h|_*$ where

$$u_h(x) := u(x+h) \tag{2.16}$$

for vectors h in the form $|u - u_h|_* = O(h^\alpha)$ for some $\alpha > 0$. This is a measure of the smoothness of u. A technical issue is that only certain directions h are allowed.

Example 2.16 *For $s \in (0,1)$ and a function v defined on $\mathbb{R}^d$, its Slobodecki (semi-)norm $|\cdot|_{H^s(\mathbb{R}^d)}$ is given by*

$$|v|^2_{H^s(\mathbb{R}^d)} = \int_{x\in\mathbb{R}^d}\int_{y\in\mathbb{R}^d} \frac{|v(x)-v(y)|^2}{|x-y|^{2s+d}}\, dy\, dx = \int_{x\in\mathbb{R}^d}\int_{h\in\mathbb{R}^d} \frac{|v(x+h)-v(x)|^2}{|h|^{2s+d}}\, dh\, dx, \tag{2.17}$$

which can be interpreted as a measure of how fast $v - v_h$ tends to zero as $h \to 0$. ■

To fix ideas, we consider the simplest case of the linear Dirichlet problem (2.11) on a Lipschitz domain Ω. We may apply Lemma 2.14 with $|\cdot|_* = \|\nabla \cdot\|_{L^2(\Omega)}$, $X = H^1_0(\Omega)$, and $J(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega fu$. Theorem 2.17 applies to more general situations such as the p-Laplacian mentioned above.
We require some notation:

- $\widetilde{u}$ is the extension by zero outside Ω

- given $h \in \mathbb{R}^d$, we set $u_h(x) := \widetilde{u}(x+h)$
- given $x_0 \in \mathbb{R}^d$, $\rho > 0$, we fix a cut-off function $\varphi \in C_0^\infty(B_{2\rho}(x_0))$ with $\varphi \equiv 1$ on $B_\rho(x_0)$ and $0 \le \varphi \le 1$.
- for $h \in \mathbb{R}^d$, we define

$$T_h v := \varphi v_h + (1-\varphi) v$$

Theorem 2.17 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $f \in L^2(\Omega)$. Then the solution u (implicitly extended by zero outside Ω) of (2.11) satisfies*

$$|u(\cdot+h) - u(\cdot)|_{H^1(\Omega)} \le C|h|^{1/2}\|f\|_{L^2(\Omega)} \qquad \forall h \in \mathbb{R}^d: \quad |h| \le 1 \tag{2.18}$$

and in particular for every $0 < t < 1/2$ there is $C > 0$ such that

$$|\nabla u|_{H^t(\mathbb{R}^d)} \le C\|f\|_{L^2(\Omega)}. \tag{2.19}$$

Proof: Before proving the theorem, let us mention that this characterization (2.18) implies that u is in the Besov space $B^{3/2}_{2,\infty}(\Omega)$. This in itself would already imply $u \in H^{3/2-\varepsilon}(\Omega)$ for every $\varepsilon > 0$ so that (2.19) could be argued to be a consequence of (2.18). We also mention that the regularity requirement $f \in L^2(\Omega)$ is not the minimal one for this regularity assertion for u.

0. step: We have the *a priori* estimate

$$\|u\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

1. step: away from $\partial\Omega$, the solution u is in H^2 by "standard" difference quotient arguments.
2. step: We note that it suffices to consider small h since for $|h| > c > 0$ bounded away from 0, we have by the triangle inequality $\|u_h - u\|_{L^2(\Omega)} \le 2\|u\|_{L^2(\Omega)} \le 2Cc^{-1}|h|\|f\|_{L^2(\Omega)}$.
3. step: Fix $x_0 \in \partial\Omega$ and consider a ball $B_{3\rho}(x_0)$. Define the set of *admissible directions* D by

$$D := \{h \in \mathbb{R}^d \,|\, T_h u \in H_0^1(\Omega)\} \cap B_\rho(0).$$

Our goal will be to bound for $h \in D$ the expression

$$\frac{1}{2}|T_h u - u|^2_{H^1(\Omega)} \overset{\text{Lemma 2.14}}{\le} J(T_h u) - J(u) = \frac{1}{2}\left(a(T_h u, T_h u) - a(u,u)\right) - \int_\Omega f(T_h u - u)$$

We note that we can simplify

$$T_h u - u = \varphi(u - u_h).$$

3. step (estimate of $\int_\Omega f(T_h u - u)$): using $|h| \le \rho$ and Lemma 2.3, we calculate

$$|\int_\Omega f(T_h u - u)| = |\int_\Omega f\varphi(u_h - u)| \le \|f\|_{L^2(B_{2\rho}(x_0))}\|u - u_h\|_{L^2(B_{2\rho}(x_0))} \le |h|\|f\|_{L^2(B_{2\rho}(x_0))}\|\nabla u\|_{L^2(B_{3\rho}(x_0))}$$

4. step (estimate of $a(T_h u, T_h u) - a(u,u)$): ⟦ ignoring the effect of φ and considering the full space, the difference $a(u_h, u_h) - a(u,u) = 0$ by translation invariance. Hence, there is a chance that this difference is small ⟧ finis 4.DS

$$\begin{aligned}
\int_\Omega |\nabla T_h u|^2 - |\nabla u|^2 &= \int_\Omega |\nabla\varphi u_h + \varphi\nabla u_h - \nabla\varphi u + (1-\varphi)\nabla u|^2 - |\nabla u|^2 \\
&= \int_\Omega |\nabla\varphi(u_h - u) + T_h\nabla u|^2 - |\nabla u|^2 \\
&= \int_\Omega |\nabla\varphi(u_h - u)|^2 + 2(T_h\nabla u)\cdot\nabla\varphi(u_h - u) + \int_\Omega |T_h\nabla u|^2 - |\nabla u|^2 \\
&=: T_1 + T_2
\end{aligned}$$

For the first term, we estimate with the support properties of φ and Lemma 2.3, which provides $\|u_h - u\|_{L^2(B_{2\rho}(x_0))} \le |h|\|\nabla u\|_{L^2(B_{3\rho}(x_0))}$,

$$T_1 \le C|h|^2\|\nabla u\|^2_{L^2(B_{3\rho}(x_0))} + C\|T_h\nabla u\|_{L^2(B_{2\rho}(x_0))}|h|\|\nabla u\|_{L^2(B_{3\rho}(x_0))} \le C|h|\|\nabla u\|^2_{L^2(B_{3\rho}(x_0))}.$$

For the second term, we use $0 \le \varphi \le 1$ to simplify[3] and estimate $|T_h\nabla u|^2 \le \varphi|\nabla u_h|^2 + (1-\varphi)|\nabla u|^2$ so that

$$\begin{aligned} T_2 &= \int_\Omega |T_h\nabla u|^2 - |\nabla u|^2 \le \int_\Omega \varphi(|\nabla u_h|^2 - |\nabla u|^2) \\ &\overset{\text{change of variables}}{=} \int_{B_{3\rho}(x_0)} (\varphi(x-h) - \varphi(x))|\nabla u|^2 \overset{\varphi \text{ smooth}}{\le} C|h|\|\nabla u\|^2_{L^2(B_{3\rho}(x_0))}. \end{aligned}$$

5. step (removing the restriction to $h \in D$): The fact that Ω is a bounded Lischitz domain implies that there is a cone $\mathcal{C} = \{h \in \mathbb{R}^d \,|\, |h \cdot \mathbf{n}| \ge \delta|h|\} \cap B_R(0)$ (for some unit vector $\mathbf{n}$ and δ, $R > 0$) such that $\mathcal{C} \subset D$ for every $x \in B_{2\rho}(x_0)$. ⟦ draw picture: for $T_h u \in H^1_0(\Omega)$ it suffices that $u(x+h) = 0$ for $x \in \partial\Omega \cap B_{2\rho}(x_0)$, and this is ensured if $x + \mathcal{C} \subset \Omega^c$ for every $x \in \partial\Omega \cap B_{2\rho}(x_0)$. Existence of $\mathcal{C}$ now practically follows from the definition of Lipschitz boundary ⟧
From

$$\frac{(h + c_D|h|\mathbf{n}) \cdot \mathbf{n}}{|h + c_D|h|\mathbf{n}\,|} = \frac{1 + h/(|h|c_D)}{|\mathbf{n} + h/(|h|c_D)|} \to 1 \quad \text{for } c_D \to \infty$$

we conclude that there exists $c_D > 0$ (depending only on δ) such that for arbitrary $h \in \mathbb{R}^d$ we have $\widetilde{h} := h + c_D\mathbf{n}|h|$ is in the infinite cone $\mathcal{C}^\infty = \{h \in \mathbb{R}^d \,|\, |h \cdot \mathbf{n}| \ge \delta|h|\}$. Hence, for $|h| \le R/(1 + c_D)$ we have that $\widetilde{h} \in \mathcal{C} \subset D$.
For a v defined on $\mathbb{R}^d$ we write for arbitrary h with $|h| \le R/(1 + c_D)$ for $\widetilde{h} = h + c_D\mathbf{n}|h|$

$$v(x) - v_h(x) = v(x) - v(x+h) = v(x) - v(x + \widetilde{h}) + v((x+h) + c_D|h|\mathbf{n}) - v(x+h).$$

Integrating over $B_{\rho'}(x_0)$ and changing variables yields

$$\|v - v_h\|^2_{L^2(B_{\rho'}(x_0))} \le 2\|v - v_{h+c_D|h|\mathbf{n}}\|^2_{L^2(B_{\rho'}(x_0))} + 2\|v - v_{c_D|h|\mathbf{n}}\|^2_{L^2(B_{\rho'}(x_0+h))}$$

Selecting $\rho' = \rho/2$ and ensuring furthermore $|h| \le \rho/2$, we arrive at

$$\|v - v_h\|^2_{L^2(B_{\rho'}(x_0))} \le 2\|v - v_{h+c_D|h|\mathbf{n}}\|^2_{L^2(B_{\rho'}(x_0))} + 2\|v - v_{c_D|h|\mathbf{n}}\|^2_{L^2(B_\rho(x_0))}.$$

We observe that both $\widetilde{h}$ and $c_D|h|\mathbf{n}$ are in $\mathcal{C} \subset D$. Hence, applying this estimate to $v = \nabla u$ we get in view of Steps 3–5 (and using $\varphi \equiv 1$ on $B_{\rho/2}(x_0)$):

$$\|\nabla u - \nabla u_h\|^2_{L^2(B_{\rho/2}(x_0))} \le C|h|\left[\|f\|^2_{L^2(B_{3\rho}(x_0))} + \|\nabla u\|^2_{L^2(B_{3\rho}(x_0))}\right]. \tag{2.20}$$

6. step: A covering argument finally extends the estimate (2.20) to any point $x_0 \in \partial\Omega$ for a fixed ρ that solely depends on Ω. This shows (2.18).

[3] $|T_h\nabla u|^2 = \varphi^2|\nabla u_h|^2 + 2\varphi(1-\varphi)\nabla u_h \cdot \nabla u + (1-\varphi)^2|\nabla u|^2 = \varphi|\nabla u_h|^2 + (1-\varphi)|\nabla u|^2 - \varphi(1-\varphi)|\nabla u_h - \nabla u|^2 \le \varphi|\nabla u_h|^2 + (1-\varphi)|\nabla u|^2$

7. step (Proof of (2.19)): We start by observing that the above arguments can be generalized to assert (2.18) with Ω on the left-hand side replaced with $\mathbb{R}^d$, i.e.,

$$\|\nabla u - \nabla u_h\|_{L^2(\mathbb{R}^d)} \leq C|h|^{1/2}\|f\|_{L^2(\Omega)} \qquad \forall |h| \leq 1. \tag{2.21}$$

To obtain (2.19) from this, we use (2.17) to estimate

$$\begin{aligned}
|\nabla\widetilde{u}|^2_{H^t(\mathbb{R}^d)} &= \int_{x\in\mathbb{R}^d}\int_{h\in\mathbb{R}^d} \frac{|\nabla\widetilde{u}(x+h) - \widetilde{\nabla} u(x)|^2}{|h|^{2t+d}}\,dh\,dx \\
&\leq \int_{x\in\mathbb{R}^d}\int_{|h|\leq 1} \frac{|\nabla\widetilde{u}(x+h) - \widetilde{\nabla} u(x)|^2}{|h|^{2t+d}}\,dh\,dx + 2\|\nabla\widetilde{u}\|^2_{L^2(\mathbb{R}^d)}|\mathbb{S}^{d-1}|\int_{r=1}^{\infty} r^{-2t-1}\,dr \\
&\leq C\|f\|^2_{L^2(\Omega)}\int_{|h|\leq 1}|h|^{1-(2t+d)}\,dh + C\frac{1}{2t}\|\nabla\widetilde{u}\|^2_{L^2(\Omega)} \\
&\leq C\left[\|f\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right].
\end{aligned}$$

□

Remark 2.18 *Theorem 2.17 works similarly for Neumann boundary conditions and other elliptic equations that p-coercive in the sense of Lemma 2.14.* ■

2.4 Hole filling technique of Widman

Consider

$$-D_\alpha A^{\alpha\beta}_{ij} D_\beta u^j = 0 \quad \text{in } \Omega. \tag{2.22}$$

and assume the Caccioppoli inequality

$$\forall \rho < R: \qquad \int_{B_\rho(x_0)} |Du|^2 \leq \frac{C_C}{(R-\rho)^2}\int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\xi|^2. \tag{2.23}$$

For *smooth* $A^{\alpha\beta}_{ij}$, Theorem 2.5 gives smoothness of u and Du so that $\int_{B_\rho(x_0)}|Du|^2 \leq c\rho^d$ can be shown. An alternative technique (which gives less decay as $\rho \to 0$ but requires less regularity of $A^{\alpha\beta}_{ij}$) is the *hole filling technique of Widman*, which merely needs $A^{\alpha\beta}_{ij} \in L^\infty$.
For a set A denote by

$$u_A := \frac{1}{|A|}\int_A u =: \fint_A u \tag{2.24}$$

Selecting $\xi = u_{B_R\setminus B_{R/2}}$ we get from the Poincaré inequality with the Poincaré constant C_P for the annulus $B_1 \setminus B_{1/2}$

$$\int_{B_{R/2}} |Du|^2 \leq \frac{4C_C}{R^2}\int_{B_R\setminus B_{R/2}} |u-\xi|^2 \leq \underbrace{\frac{4C_C}{R^2}C_P^2R^2}_{=:C'}\int_{B_R\setminus B_{R/2}}|Du|^2 =: C'\int_{B_R\setminus B_{R/2}}|Du|^2$$

We now "fill the hole" in the annulus $B_R \setminus B_{R/2}$:

$$(1+C')\int_{B_{R/2}}|Du|^2 \leq C'\int_{B_R}|Du|^2$$

and conclude

$$\int_{B_{R/2}} |Du|^2 \leq \underbrace{\frac{C'}{1+C'}}_{<1!} \int_{B_R} |Du|^2$$

Since C' is independent of R, we can iterate this estimate to get for all $k \in \mathbb{N}$

$$\int_{B_{2^{-k}R/2}} |Du|^2 \leq \left(\frac{C'}{1+C'}\right)^k \int_{B_R} |Du|^2$$

This implies

Lemma 2.19 (hole filling of Widman) *Assume u satisfies (2.23). Then there are α, $C > 0$ such that for all $0 < \rho < R$*

$$\int_{B_\rho(x_0)} |Du|^2 \leq C(\rho/R)^\alpha \int_{B_R(x_0)} |Du|^2. \tag{2.25}$$

Proof: Choose k such that $2^{-k}R = \rho$ ⟦ to be precise: select $k = \lfloor \log_2 R/\rho \rfloor$ and note that this implies $B_\rho \subset B_{2^{-k}R}$ as well as $k \leq \log_2(\rho/R) \leq k+1$ ⟧

$$\begin{aligned}
\int_{B_\rho(x_0)} |Du|^2 \leq \int_{B_{2^{-k}R}} |Du|^2 &\leq \left(\frac{C'}{1+C'}\right)^k \int_{B_R} |Du|^2 \leq \left(\frac{C'}{1+C'}\right)^{\log_2(R/\rho)-1} \int_{B_R} |Du|^2 \\
&= \frac{1+C'}{C'} \exp\left(\frac{\ln(R/\rho)}{\ln 2 \ln(C'/(1+C'))}\right) \int_{B_R} |Du|^2 \\
&= \frac{1+C'}{C'} (\rho/R)^\alpha \int_{B_R} |Du|^2
\end{aligned}$$

with

$$\alpha = \frac{1}{\ln 2 \ln((1+C')/C')}$$

□

We conclude that we control

$$\sup_{0<\rho<R} \rho^{-\alpha} \int_{B_\rho(x_0)} |Du|^2.$$

This observation motivates the upcoming definition of the Morrey and Campanato spaces. finis 5.DS