ASC Report No. 5/2018

On commuting p-version projection-based interpolation on tetrahedra (extended version)

J.M. Melenk, C. Rojik

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-00-1

Most recent ASC Reports

4/2018	*F. Achleitner, A. Arnold, and B. Signorello* On optimal decay estimates for ODEs and PDEs with modal decomposition
3/2018	*L. Brugnano, G. Gurioli, F. Iavernaro, E.B. Weinmüller* Line integral solution of Hamiltonian systems with holonomic constraints
2/2018	*W. Auzinger, J. Burkotová , I. Rachůnková , V. Wenin* Shooting methods for state-dependent impulsive boundary value problems, with applications
1/2018	*W. Auzinger, H. Hofstätter, O. Koch, M. Quell and M. Thalhammer* A posteriori error estimation for Magnus-type integrators
33/2017	*M.M. Tung and E.B. Weinmüller* Metamaterial Acoustics on the (2+1)D Schwarzschild Plane
32/2017	*J.M. Gambi, M.L. Garcia del Pinto, J. Schwindl, E.B. Weinmüller* Post-Newtonian equations of motion for LEO debris objects and space-based Acquisition, Pointing and Tracking laser systems
31/2017	*J. Burkotova. I. Rachunkova, E.B. Weinmüller* On singular BVPs with unsmooth data: Analysis of the linear case with variable coefficient matrix
30/2017	*J. Burkotova. I. Rachunkova, E.B. Weinmüller* On singular BVPs with unsmooth data. Part 2: Convergence of the collocation schemes
29/2017	*G. Hrkac, C.-M. Pfeiler, D. Praetorius, M. Ruggeri, A. Segatti, and B. Stiftner* Convergent tangent plane integrators for the simulation of chiral magnetic skyrmion dynamics
28/2017	*G. Kitzler and J. Schöberl* A polynomial spectral method for the spatially homogeneous Boltzmann equation in 3 dimensions

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-00-1

On commuting p-version projection-based interpolation on tetrahedra (extended version)

J.M. Melenk* C. Rojik*

February 1, 2018

Abstract

On the reference tetrahedron $\widehat{K}$, we define three projection-based interpolation operators on $H^2(\widehat{K})$, $\mathbf{H}^1(\widehat{K}, \mathbf{curl})$, and $\mathbf{H}^1(\widehat{K}, \mathrm{div})$. These operators are projections onto space of polynomials, they have the commuting diagram property and feature the optimal convergence rate as the polynomial degree increases in $H^{1-s}(\widehat{K})$, $\mathbf{H}^{-s}(\widehat{K}, \mathbf{curl})$, $\mathbf{H}^{-s}(\widehat{K}, \mathrm{div})$ for $0 \leq s \leq 1$.

1 Introduction

Operators that approximate a given function by a (piecewise) polynomial are fundamental tools in numerical analysis. The case of scalar functions is rather well-understood and many such approximation operators exist both for fixed order approximation where accuracy is achieved by refining the mesh, the so-called h-version, and the p-version, where accuracy is obtained by increasing the polynomial degree p; for the p-version in an H^1-conforming setting we refer to [3, 5, 29] and references therein. For the approximation of vector-valued functions, specifically, the approximation in the spaces $H(\mathbf{curl})$ and $H(\mathrm{div})$, the situation is less developed since the approximation operators are typically required to satisfy, in addition to having certain approximation properties, also the requirement to be projections and to have a commuting diagram property. While various operators with all these desirable properties have been developed for the h-version, optimal results in the p-version are missing in the literature. The present paper is devoted to the analysis of a p-version projection-based interpolation operator that has the optimal polynomial approximation properties under suitable regularity assumptions.

High order polynomial projection-based interpolation operators with the projection and commuting diagram properties have been developed by L. Demkowicz and several coworkers, [11, 16–18]; a very nice and comprehensive presentation of these results can be found in [15], which will also be the basis for the present work. The projection-based interpolation operators presented in [15] are a) projections, b) have the commuting diagram property, and c) admit element-by-element construction. The last point means that the operators are defined elementwise by specifying them on the reference element and that the appropriate interelement continuity is ensured by defining the interpolant in terms of pertinent traces: for scalar functions, the projection-based interpolant interpolates in the vertices and its restriction to an edge or a face is completely determined by the restriction of the function to that edge or face; for the $H(\mathbf{curl})$-conforming interpolant, its tangential component on an edge or face is completely determined by the tangential trace of the function on that edge or face; for the $H(\mathrm{div})$-conforming interpolant, the normal component on a face is fully dictated by the normal component of the function on that face. Such a construction is only possible under additional regularity assumptions beyond the minimal one (which would be H^1, $H(\mathbf{curl})$ or $H(\mathrm{div})$). Indeed, in 3D, the construction described in [15] requires the regularity H^{1+s} with $s > 1/2$ for scalar functions, $H^s(\mathbf{curl})$ with $s > 1/2$ and $H^s(\mathrm{div})$ with $s > 0$ for the vectorial ones. Under these regularity assumptions, it is shown in [15, Thm. 5.3] that the projection-based interpolation operator has, up to logarithmic factors, the optimal algebraic convergence properties (as $p \to \infty$), for function with finite Sobolev regularity as measured by s. In this note, we remove the logarithmic factors, i.e., show optimal rates of convergence, under the more stringent regularity assumption $s \geq 1$ (cf. Theorem 2.8 for the case of tetrahedra and Theorem 2.11 for the case of triangles).

*(melenk@tuwien.ac.at, claudio.rojik@tuwien.ac.at), Institut für Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstrasse 8-10, A-1040 Wien, Austria.

The projection-based interpolation operator analyzed in the present work is of the type studied in [15]. Correspondingly, many tools used in [15] are also used here, most notably, the polynomial lifting operators developed for tetrahedra in [19–21] and for the simpler case of triangles in [2]; we mention in passing that suitable polynomial lifting operators are also available for the case of the cube [12]. Another tool that [15] uses are right inverses of the gradient, curl and div operators ("Poincaré maps"). Here, we use a more recent and powerful variant, namely, the regularized right inverses of [13]. This breakthrough paper [13] allows for stable decompositions of functions in $H(\mathbf{curl})$ and $H(\mathrm{div})$ with appropriate mapping properties in scales of Sobolev spaces and is an essential component in the analysis of the p-version in $H(\mathbf{curl})$, [6,8,23]. The distingushing technical difference between [15] and the present work, which is responsible for the removal of the logarithmic factor, is the treatment of the non-local norms on the boundary. Non-local norms on the boundary are written in [15] (following [17]) as a sum of contributions over the boundary parts (that is, faces in 3D and edges in 2D); in finite-dimensional spaces of piecewise polynomials, this localization procedure is possible at the price of logarithmic factors. Instead of localizing a non-local norm, the approach taken here is to realize the non-local norm by interpolating between two norms related to integer order Sobolev norms, which both can be localized, i.e., written as sums of contributions over boundary parts. In turn, this requires to analyze the error of the projection-based interpolation in two norms instead of a single one. The estimate in the stronger norm is obtained by a best approximation argument as done in [15], the estimate in the weaker norm is obtained by a duality argument.
The gradient operator ∇ for scalar functions u and the divergence operator div for $\mathbb{R}^d$-valued functions $\mathbf{u}$ are defined in the usual way: $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_d} u)^\top$ and $\operatorname{div} \mathbf{u} = \sum_{i=1}^d \partial_{x_i} \mathbf{u}_i$. For $d = 3$ and $\mathbb{R}^3$-valued functions $\mathbf{u}$ the $\mathbf{curl}$-operator is defined as $\mathbf{curl}\,\mathbf{u} := (\partial_{x_2}\mathbf{u}_3 - \partial_{x_3}\mathbf{u}_2, -(\partial_{x_1}\mathbf{u}_3 - \partial_{x_3}\mathbf{u}_1), \partial_{x_1}\mathbf{u}_2 - \partial_{x_2}\mathbf{u}_1)^\top$. For $d = 2$ we distinguish between the scalar-valued and vector-valued curl operator: for a scalar function u, we defined $\mathbf{curl}\, u := (\partial_{x_2} u, -\partial_{x_1} u)^\top$ and for an $\mathbb{R}^2$-valued function $\mathbf{u}$ we set $\operatorname{curl} \mathbf{u} := \partial_{x_1}\mathbf{u}_2 - \partial_{x_2}\mathbf{u}_1$.
For Lipschitz domains $\omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) and scalar functions, we employ the usual Sobolev spaces $H^s(\omega)$, $s \geq 0$, as defined, e.g., in [1]. For $s > 0$ the space $\widetilde{H}^{-s}(\omega) := (H^s(\omega))'$ is the dual space of $H^s(\omega)$ characterized by the norm

$$\|u\|_{\widetilde{H}^{-s}(\omega)} := \sup_{v \in H^s(\omega)} \frac{(u, v)_{L^2(\omega)}}{\|v\|_{H^s(\omega)}}, \tag{1.1}$$

where $(\cdot, \cdot)_{L^2(\omega)}$ denotes the (extended) L^2-scalar product. Vector-valued analogs $\mathbf{H}^s(\omega)$ are defined to be elements of $H^s(\omega)$ componentwise and also the dual norm $\|\cdot\|_{\widetilde{\mathbf{H}}^{-s}(\omega)}$ is defined analogously to (1.1). For $s \geq 0$ and $d = 3$, we set $\mathbf{H}^s(\omega, \mathbf{curl}) = \{\mathbf{u} \in \mathbf{H}^s(\omega) \mid \mathbf{curl}\,\mathbf{u} \in \mathbf{H}^s(\omega)\}$ and $\mathbf{H}^s(\omega, \operatorname{div}) = \{\mathbf{u} \in \mathbf{H}^s(\omega) \mid \operatorname{div} \mathbf{u} \in H^s(\omega)\}$; for $d = 2$ we have $\mathbf{H}^s(\omega, \operatorname{curl}) = \{\mathbf{u} \in \mathbf{H}^s(\omega) \mid \operatorname{curl} \mathbf{u} \in H^s(\omega)\}$. For $s \geq 0$, we define

$$\|\mathbf{u}\|^2_{\widetilde{\mathbf{H}}^{-s}(\omega,\mathbf{curl})} := \|\mathbf{u}\|^2_{\widetilde{\mathbf{H}}^{-s}(\omega)} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{\widetilde{\mathbf{H}}^{-s}(\omega)}$$

and analogously the norms $\|\mathbf{u}\|^2_{\widetilde{\mathbf{H}}^{-s}(\omega,\operatorname{div})}$ and $\|\mathbf{u}\|^2_{\widetilde{\mathbf{H}}^{-s}(\omega,\operatorname{curl})}$. The space $H^{1/2}(\partial\omega)$ will be understood as the trace space of $H^1(\omega)$ and $H^{-1/2}(\partial\omega)$ denotes its dual. The spaces $\mathbf{H}_0(\omega, \mathbf{curl})$ and $\mathbf{H}_0(\omega, \operatorname{div})$ are the subspaces of $\mathbf{H}(\omega, \mathbf{curl})$ and $\mathbf{H}(\omega, \operatorname{div})$ with vanishing tangential or normal trace, defined as the closure of $(C_0^\infty(\omega))^d$ under the norms $\|\cdot\|_{\mathbf{H}(\omega,\mathbf{curl})}$ and $\|\cdot\|_{\mathbf{H}(\omega,\operatorname{div})}$.

2 Projection based interpolation

$\widehat{K} \subset \mathbb{R}^3$ denotes the reference tetrahedron, which is taken to be the regular tetrahedron, i.e., its 4 faces are equilateral triangles. The sets $\mathcal{F}(\widehat{K})$, $\mathcal{E}(\widehat{K})$ and $\mathcal{V}(\widehat{K})$ denote the sets of faces, edges and vertices of $\widehat{K}$, respectively. In the two-dimensional space, we use the notation $\widehat{f}$ for the reference triangle, which is taken to be the equilateral triangle with interior angles $\pi/3$, and $\mathcal{E}(\widehat{f})$ and $\mathcal{V}(\widehat{f})$ for the set of edges and vertices of $\widehat{f}$. We also need the tangential trace and tangential component operators: For a sufficiently smooth function $\mathbf{u}$ on $\widehat{K}$ we set $\Pi_\tau \mathbf{u} := \mathbf{n} \times (\mathbf{u}|_{\partial\widehat{K}} \times \mathbf{n})$ and $\gamma_\tau \mathbf{u} := \mathbf{u}|_{\partial\widehat{K}} \times \mathbf{n}$, where $\mathbf{n}$ denotes the outer normal vector of $\widehat{K}$. For a face $f \in \mathcal{F}(\widehat{K})$ we will write $\Pi_{\tau,f}$ for the (in-plane) tangential trace on ∂f, i.e., with the in-plane exterior normal $\mathbf{n}_{\partial f}$ and sufficiently smooth tangential fields $\mathbf{u}$ we set $\Pi_{\tau,f}\mathbf{u} = \mathbf{u} - \mathbf{n}_{\partial f}(\mathbf{n}_{\partial f} \cdot \mathbf{u})$. For sufficiently smooth $\mathbf{u}$, we have for each edge $e \in \mathcal{E}(\widehat{K})$ $\Pi_{\tau,f}\mathbf{u}|_e = \mathbf{u} \cdot \mathbf{t}_e$. Here, $\mathbf{t}_e$ is the tangential vector of the edge e; its orientation is assumed to be fixed.
We have the integration by parts formula

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{v})_{L^2(\widehat{K})} = (\mathbf{curl}\,\mathbf{v}, \mathbf{u})_{L^2(\widehat{K})} - \langle \Pi_\tau \mathbf{u}, \gamma_\tau \mathbf{v} \rangle_{L^2(\partial\widehat{K})} \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\widehat{K}), \tag{2.1}$$

which actually extends to $\mathbf{u}$, $\mathbf{v} \in \mathbf{H}(\widehat{K}, \mathbf{curl})$, [25, Thm. 3.29]. In 2D, we have the integration by parts formula (Stokes formula)

$$\int_{\widehat{f}} \mathbf{curl}\, v \cdot \mathbf{F} = \int_{\widehat{f}} v \operatorname{curl} \mathbf{F} - \int_{\partial \widehat{f}} v \mathbf{F} \cdot \mathbf{t} \tag{2.2}$$

where the piecewise constant tangential vector $\mathbf{t}$ is oriented such that $\widehat{f}$ is "on the left".
For each face $f \in \mathcal{F}(\widehat{K})$ and $s \geq 0$ we define the Sobolev spaces $H^s(f)$, $\widetilde{H}^{-s}(f)$ as well as $\mathbf{H}^s_T(f, \operatorname{curl})$ and $\widetilde{\mathbf{H}}^{-s}_T(f, \operatorname{curl})$ by identifying the face f with a subset of $\mathbb{R}^2$ via an affine congruence map. The subscript T indicates that *tangential* fields are considered. Also the spaces $H^s(e)$ and $\widetilde{H}^{-s}(e)$ on an edge $e \in \mathcal{E}(\widehat{K})$ are defined by such an identification.

2.0.1 Spaces on the reference element

On $\widehat{K}$ we introduce the classical Nédélec type I and Raviart-Thomas elements of degree $p \geq 0$ (see, e.g., [25]):

$$\mathcal{P}_p(\widehat{K}) := \operatorname{span}\{x^\alpha \,|\, |\alpha| \leq p\}, \tag{2.3}$$

$$\boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\widehat{K}) := \{\mathbf{p}(\mathbf{x}) + \mathbf{x} \times \mathbf{q}(\mathbf{x}) \,|\, \mathbf{p}, \mathbf{q} \in (\mathcal{P}_p(\widehat{K}))^3\}, \tag{2.4}$$

$$\mathbf{RT}_p(\widehat{K}) := \{\mathbf{p}(\mathbf{x}) + q(\mathbf{x})\mathbf{x} \,|\, \mathbf{p} \in (\mathcal{P}_p(\widehat{K}))^3, q \in \mathcal{P}_p(\widehat{K})\}. \tag{2.5}$$

Recall the exact sequences on the continuous level

$$\mathbb{R} \xrightarrow{\mathrm{id}} H^2(\widehat{K}) \xrightarrow{\nabla} \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \xrightarrow{\mathbf{curl}} \mathbf{H}^1(\widehat{K}, \operatorname{div}) \xrightarrow{\operatorname{div}} H^1(\widehat{K}) \xrightarrow{0} \{0\} \tag{2.6}$$

and on the discrete level

$$\mathbb{R} \xrightarrow{\mathrm{id}} \mathcal{P}_{p+1}(\widehat{K}) \xrightarrow{\nabla} \boldsymbol{\mathcal{N}}^I_p(\widehat{K}) \xrightarrow{\mathbf{curl}} \mathbf{RT}_p(\widehat{K}) \xrightarrow{\operatorname{div}} \mathcal{P}_p(\widehat{K}) \xrightarrow{0} \{0\}. \tag{2.7}$$

Using the notation

$$W_{p+1}(\widehat{K}) := \mathcal{P}_{p+1}(\widehat{K}), \qquad \mathbf{Q}_p(\widehat{K}) := \boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\widehat{K}), \qquad \mathbf{V}_p(\widehat{K}) := \mathbf{RT}_p(\widehat{K}),$$

we present here projection operators $\widehat{\Pi}^{\mathrm{grad},3d}_{p+1}$, $\widehat{\Pi}^{\mathrm{curl},3d}_p$, $\widehat{\Pi}^{\mathrm{div},3d}_p$, $\widehat{\Pi}^{L^2}_p$ that enjoy the commuting diagram property

$$\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\mathrm{id}} & H^2(\widehat{K}) & \xrightarrow{\nabla} & \mathbf{H}^1(\widehat{K}, \mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \mathbf{H}^1(\widehat{K}, \operatorname{div}) & \xrightarrow{\operatorname{div}} & H^1(\widehat{K}) & \xrightarrow{0} & \{0\} \\
 & & \big\downarrow \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} & & \big\downarrow \widehat{\Pi}^{\mathrm{curl},3d}_p & & \big\downarrow \widehat{\Pi}^{\mathrm{div},3d}_p & & \big\downarrow \widehat{\Pi}^{L^2}_p & & \\
\mathbb{R} & \xrightarrow{\mathrm{id}} & W_{p+1}(\widehat{K}) & \xrightarrow{\nabla} & \mathbf{Q}_p(\widehat{K}) & \xrightarrow{\mathbf{curl}} & \mathbf{V}_p(\widehat{K}) & \xrightarrow{\operatorname{div}} & W_p(\widehat{K}) & \xrightarrow{0} & \{0\}
\end{array} \tag{2.8}$$

In the two-dimensional setting, the Nédélec type I elements are defined by

$$\mathbf{Q}_p(\widehat{f}) := \boldsymbol{\mathcal{N}}^I_p(\widehat{f}) := \{\mathbf{p}(\mathbf{x}) + q(\mathbf{x})(y, -x)^T \,|\, \mathbf{p} \in (\mathcal{P}_p(\widehat{f}))^2, q \in \widetilde{\mathcal{P}}_p(\widehat{f})\},$$

where $\widetilde{\mathcal{P}}_p(\widehat{f})$ denotes the homogeneous polynomials of degree p. Here we have shorter exact sequences of the forms

$$\mathbb{R} \xrightarrow{\mathrm{id}} H^{3/2}(\widehat{f}) \xrightarrow{\nabla} \mathbf{H}^{1/2}(\widehat{f}, \operatorname{curl}) \xrightarrow{\operatorname{curl}} H^{1/2}(\widehat{f}) \xrightarrow{0} \{0\} \tag{2.9}$$

on the continuous level and

$$\mathbb{R} \xrightarrow{\mathrm{id}} \mathcal{P}_{p+1}(\widehat{f}) \xrightarrow{\nabla} \boldsymbol{\mathcal{N}}^I_p(\widehat{f}) \xrightarrow{\operatorname{curl}} \mathcal{P}_p(\widehat{f}) \xrightarrow{0} \{0\} \tag{2.10}$$

one the discrete level. We then define projection operators $\widehat{\Pi}^{\mathrm{grad},2d}_{p+1}$, $\widehat{\Pi}^{\mathrm{curl},2d}_p$, $\widehat{\Pi}^{L^2}_p$ which satisfy the commuting diagram property

$$\begin{array}{ccccccccc}
\mathbb{R} & \xrightarrow{\mathrm{id}} & H^{3/2}(\widehat{f}) & \xrightarrow{\nabla} & \mathbf{H}^{1/2}(\widehat{f}, \operatorname{curl}) & \xrightarrow{\operatorname{curl}} & H^{1/2}(\widehat{f}) & \xrightarrow{0} & \{0\} \\
 & & \big\downarrow \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} & & \big\downarrow \widehat{\Pi}^{\mathrm{curl},2d}_p & & \big\downarrow \widehat{\Pi}^{L^2}_p & & \\
\mathbb{R} & \xrightarrow{\mathrm{id}} & W_{p+1}(\widehat{f}) & \xrightarrow{\nabla} & \mathbf{Q}_p(\widehat{f}) & \xrightarrow{\operatorname{curl}} & W_p(\widehat{f}) & \xrightarrow{0} & \{0\}
\end{array} \tag{2.11}$$

2.0.2 Trace spaces on the boundary

We will also need the traces of the spaces $W_{p+1}(\widehat{K})$, $\mathbf{Q}_p(\widehat{K})$ and $\mathbf{V}_p(\widehat{K})$ on various parts of the boundary. For faces $f \in \mathcal{F}(\widehat{K})$ the corresponding spaces are defined by trace operations:

$$W_{p+1}(f) := W_{p+1}(\widehat{K}))|_f, \quad \mathbf{Q}_p(f) := (\Pi_\tau \mathbf{Q}_p(\widehat{K}))|_f, \quad V_p(f) := \mathbf{V}_p(\widehat{K}) \cdot \mathbf{n}_f,$$

where Π_τ is the tangential component and $\mathbf{n}_f$ the normal vector of f. These traces are well-known objects: Identifying a face f with the reference triangle $\widehat{f}$ via the affine element map, the space $W_{p+1}(f)$ coincides with the space $\mathcal{P}_{p+1}(\mathbb{R}^2)$ of bivariate polynomials of (total) degree $p+1$; the space $\mathbf{Q}_p(f)$ turns out to be $\boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(f)$, the type-I Nédélec element on triangles; and $V_p(f)$ is the space $\mathcal{P}_p(\mathbb{R}^2)$. Lowering the dimension even further, we introduce for each edge $e \in \mathcal{E}(\widehat{K})$ the spaces

$$W_{p+1}(e) := W_{p+1}(\widehat{K}))|_e, \quad Q_p(e) := \mathbf{Q}_p(\widehat{K}) \cdot \mathbf{t}_e,$$

where $\mathbf{t}_e$ is the tangential vector of the edge e. Similar to the case of the faces, the space $W_{p+1}(e)$ can be identified with the univariate polynomials of degree $p+1$ and $Q_p(e)$ with the univariate polynomimals of degree p.
We also need subspaces of functions vanishing on the boundary in the appropriate sense. We set

$$\mathring{W}_{p+1}(\widehat{K}) := W_{p+1}(\widehat{K}) \cap H^1_0(\widehat{K}), \mathring{\mathbf{Q}}_p(\widehat{K}) := \{\mathbf{u} \in \mathbf{Q}_p(\widehat{K}) \,|\, \Pi_\tau \mathbf{u} = 0\}, \mathring{\mathbf{V}}_p(\widehat{K}) := \{\mathbf{u} \in \mathbf{V}_p(\widehat{K}) \,|\, \mathbf{n} \cdot \mathbf{u} = 0\}.$$

We also need $W^{aver}_p(\widehat{K}) := \{u \in W_{p+1}(\widehat{K}) \,|\, \int_{\widehat{K}} u = 0\}$. Corresponding spaces on lower-dimensional manifolds are defined as follows:

$$\begin{aligned} \mathring{W}_{p+1}(f) &:= W_{p+1}(f) \cap H^1_0(f), \mathring{\mathbf{Q}}_p(f) := \{\mathbf{u} \in \mathbf{Q}_p(f) \,|\, \Pi_{\tau,f}\mathbf{u} = 0\}, \\ \mathring{V}_p(f) &:= \{u \in V_p(f) \,|\, \int_f u = 0\}. \end{aligned}$$

Finally, we set for edges $e \in \mathcal{E}(\widehat{K})$

$$\mathring{W}_{p+1}(e) := W_{p+1}(e) \cap H^1_0(e), \qquad \mathring{Q}_p(e) := \{u \in Q_p(e) \,|\, \int_e u = 0\}.$$

By e.g., [15] or [23] (actually, [23] uses the tangential trace operator γ_τ instead of Π_τ in the definition of the spaces $\mathbf{Q}_p(f)$ and correspondingly identifies the space $\mathbf{Q}_p(f)$ with a Raviart-Thomas space instead of a Nédélec space) we have the following diagrams for faces $f \in \mathcal{F}(\widehat{K})$ and edges $e \in \mathcal{E}(\widehat{K})$

$$\begin{array}{ccccccccccc} \{0\} & \xrightarrow{\mathrm{Id}} & \mathring{W}_{p+1}(\widehat{K}) & \xrightarrow{\nabla} & \mathring{\mathbf{Q}}_p(\widehat{K}) & \xrightarrow{\mathbf{curl}} & \mathring{\mathbf{V}}_p(\widehat{K}) & \xrightarrow{\mathrm{div}} & W^{aver}_p(\widehat{K}) & \xrightarrow{0} & \{0\} \\ \{0\} & \xrightarrow{\mathrm{Id}} & \mathring{W}_{p+1}(f) & \xrightarrow{\nabla_f} & \mathring{\mathbf{Q}}_p(f) & \xrightarrow{\mathrm{curl}_f} & \mathring{V}_p(f) & \xrightarrow{0} & \{0\} & & \\ \{0\} & \xrightarrow{\mathrm{Id}} & \mathring{W}_{p+1}(e) & \xrightarrow{\nabla_e} & \mathring{Q}_p(e) & \xrightarrow{0} & \{0\} & & & & \end{array} \tag{2.12}$$

In this diagram (and in what follows), the operators ∇_f, ∇_e represent surface gradients on a face f and tangential differentiation on an edge e, respectively. The operator curl_f is the surface curl on face f.
In two dimensions, we set

$$\mathring{W}_{p+1}(\widehat{f}) := W_{p+1}(\widehat{f}) \cap H^1_0(\widehat{f}), \quad \mathring{\mathbf{Q}}_p(\widehat{f}) := \{\mathbf{u} \in \mathbf{Q}_p(\widehat{f}) \,|\, \mathbf{u} \cdot \mathbf{t}_e = 0 \,\forall e \in \mathcal{E}(\widehat{f})\}.$$

One again looks at shortened sequences, namely,

$$\begin{array}{ccccccccc} \{0\} & \xrightarrow{\mathrm{Id}} & \mathring{W}_{p+1}(\widehat{f}) & \xrightarrow{\nabla} & \mathring{\mathbf{Q}}_p(\widehat{f}) & \xrightarrow{\mathrm{curl}} & \mathring{V}_p(\widehat{f}) & \xrightarrow{0} & \{0\} \\ \{0\} & \xrightarrow{\mathrm{Id}} & \mathring{W}_{p+1}(e) & \xrightarrow{\nabla_e} & \mathring{Q}_p(e) & \xrightarrow{0} & \{0\} & & \end{array} \tag{2.13}$$

2.1 Definition of the operators $\widehat{\Pi}^{\mathrm{grad},3d}_{p+1}$, $\widehat{\Pi}^{\mathrm{curl},3d}_{p}$, $\widehat{\Pi}^{\mathrm{div},3d}_{p}$

The construction is similar to that in [15,17]. The difference is that all inner products are integer order inner products.

2.1.1 The operators in 3D

Definition 2.1 ($\widehat{\Pi}_{p+1}^{\text{grad},3d}$)**.** *The operator* $\widehat{\Pi}_{p+1}^{\text{grad},3d} : H^2(\widehat{K}) \to W_{p+1}(\widehat{K})$ *is defined by*

$$(\nabla(u - \widehat{\Pi}_{p+1}^{\text{grad},3d} u), \nabla v)_{L^2(\widehat{K})} = 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{K}), \tag{2.14a}$$

$$(\nabla_f(u - \widehat{\Pi}_{p+1}^{\text{grad},3d} u), \nabla_f v)_{L^2(f)} = 0 \quad \forall v \in \mathring{W}_{p+1}(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{2.14b}$$

$$(\nabla_e(u - \widehat{\Pi}_{p+1}^{\text{grad},3d} u), \nabla_e v)_{L^2(e)} = 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall e \in \mathcal{E}(\widehat{K}), \tag{2.14c}$$

$$u(V) - \widehat{\Pi}_{p+1}^{\text{grad},3d} u(V) = 0 \quad \forall V \in \mathcal{V}(\widehat{K}). \tag{2.14d}$$

Definition 2.2 ($\widehat{\Pi}_{p}^{\text{curl},3d}$)**.** *The operator* $\widehat{\Pi}_{p}^{\text{curl},3d} : \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \to \mathbf{Q}_p(\widehat{K})$ *is defined by the following conditions*

$$(\mathbf{curl}(\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},3d}\mathbf{u}), \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}), \tag{2.15a}$$

$$((\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},3d}\mathbf{u}), \nabla v)_{L^2(\widehat{K})} = 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{K}), \tag{2.15b}$$

$$(\text{curl}_f\, \Pi_\tau(\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},3d}\mathbf{u}), \text{curl}_f\, \mathbf{v})_{L^2(f)} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{2.15c}$$

$$(\Pi_\tau(\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},3d}\mathbf{u}), \nabla_f v)_{L^2(f)} = 0 \quad \forall v \in \mathring{W}_{p+1}(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{2.15d}$$

$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},3d}\mathbf{u}), \nabla_e v)_{L^2(e)} = 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall e \in \mathcal{E}(\widehat{K}), \tag{2.15e}$$

$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},3d}\mathbf{u}), 1)_{L^2(e)} = 0 \quad \forall e \in \mathcal{E}(\widehat{K}). \tag{2.15f}$$

Definition 2.3 ($\widehat{\Pi}_{p}^{\text{div},3d}$)**.** *The operator* $\widehat{\Pi}_{p}^{\text{div},3d} : \mathbf{H}^{1/2}(\widehat{K}, \text{div}) \to \mathbf{V}_p(\widehat{K})$ *is defined by the following conditions:*

$$(\text{div}(\mathbf{u} - \widehat{\Pi}_{p}^{\text{div},3d}\mathbf{u}), \text{div}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}), \tag{2.16a}$$

$$((\mathbf{u} - \widehat{\Pi}_{p}^{\text{div},3d}\mathbf{u}), \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}), \tag{2.16b}$$

$$(\mathbf{n}_f \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\text{div},3d}\mathbf{u}), v)_{L^2(f)} = 0 \quad \forall v \in \mathring{V}_p(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{2.16c}$$

$$(\mathbf{n}_f \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\text{div},3d}\mathbf{u}), 1)_{L^2(f)} = 0 \quad \forall f \in \mathcal{F}(\widehat{K}). \tag{2.16d}$$

Definition 2.4 ($\widehat{\Pi}_{p}^{L^2}$)**.** *The operator* $\widehat{\Pi}_{p}^{L^2} : L^2(\widehat{K}) \to W_p(\widehat{K})$ *is defined by the conditions*

$$(u - \widehat{\Pi}_{p}^{L^2} u, v)_{L^2(\widehat{K})} = 0 \qquad \forall v \in W_p(\widehat{K}). \tag{2.17}$$

2.1.2 The operators in 2D

We define the projection operators following the lines of Section 2.1.1. The operators are then well-defined by the following equations, which can be shown by checking the numbers of conditions the same way as in Section 2.1.1.

Definition 2.5 ($\widehat{\Pi}_{p+1}^{\text{grad},2d}$)**.** *The operator* $\widehat{\Pi}_{p+1}^{\text{grad},2d} : H^{3/2}(\widehat{f}) \to W_{p+1}(\widehat{f})$ *is defined by*

$$(\nabla(u - \widehat{\Pi}_{p+1}^{\text{grad},2d} u), \nabla v)_{L^2(\widehat{f})} = 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{f}), \tag{2.18a}$$

$$(\nabla_e(u - \widehat{\Pi}_{p+1}^{\text{grad},2d} u), \nabla_e v)_{L^2(e)} = 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall e \in \mathcal{E}(\widehat{f}), \tag{2.18b}$$

$$u(V) - \widehat{\Pi}_{p+1}^{\text{grad},2d} u(V) = 0 \quad \forall V \in \mathcal{V}(\widehat{f}). \tag{2.18c}$$

Definition 2.6 ($\widehat{\Pi}_{p}^{\text{curl},2d}$)**.** *The operator* $\widehat{\Pi}_{p}^{\text{curl},2d} : \mathbf{H}^{1/2}(\widehat{f}, \text{curl}) \to \mathbf{Q}_p(\widehat{f})$ *is defined by*

$$(\text{curl}(\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},2d}\mathbf{u}), \text{curl}\,\mathbf{v})_{L^2(\widehat{f})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{f}), \tag{2.19a}$$

$$((\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},2d}\mathbf{u}), \nabla v)_{L^2(\widehat{f})} = 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{f}), \tag{2.19b}$$

$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},2d}\mathbf{u}), \nabla_e v)_{L^2(e)} = 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall e \in \mathcal{E}(\widehat{f}), \tag{2.19c}$$

$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\text{curl},2d}\mathbf{u}), 1)_{L^2(e)} = 0 \quad \forall e \in \mathcal{E}(\widehat{f}). \tag{2.19d}$$

Definition 2.7 ($\widehat{\Pi}_p^{L^2}$)**.** *The operator* $\widehat{\Pi}_p^{L^2} : L^2(\widehat{f}) \to W_p(\widehat{f})$ *is defined by*

$$(u - \widehat{\Pi}_p^{L^2} u, v)_{L^2(\widehat{f})} = 0 \qquad \forall v \in W_p(\widehat{f}). \tag{2.20}$$

It is worth pointing out that, up to identifying a face $f \in \mathcal{F}(\widehat{K})$ with the reference triangle $\widehat{f}$, the 2D operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$, $\widehat{\Pi}_p^{\mathrm{curl},2d}$ coincide with the restrictions to the face f of $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, $\widehat{\Pi}_p^{\mathrm{curl},3d}$.

2.2 Main results

We can now formulate the main theorems. The proofs are postponed to the later sections.

Theorem 2.8 (Projection-based interpolation in 3D)**.** *There are constants C_s and $C_{s,k}$ (depending only on s and k) such that:*

(i) The operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, $\widehat{\Pi}_p^{\mathrm{curl},3d}$, $\widehat{\Pi}_p^{\mathrm{div},3d}$, $\widehat{\Pi}_p^{L^2}$ are well-defined, projections, and the diagram (2.8) commutes.

(ii) For all $\varphi \in H^2(\widehat{K})$ there holds

$$\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} \varphi\|_{H^{1-s}(\widehat{K})} \le C_s p^{-(1+s)} \inf_{v \in W_{p+1}(\widehat{K})} \|\varphi - v\|_{H^2(\widehat{K})}, \qquad s \in [0,1].$$

(iii) For all $\mathbf{u} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$ there holds

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d} \mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \le C_s p^{-(1+s)} \inf_{\mathbf{v} \in \mathbf{Q}_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}, \qquad s \in [0,1].$$

(iv) For all $k \ge 1$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ with $\mathbf{curl}\, \mathbf{u} \in \mathbf{V}_p(\widehat{K}) = \mathbf{RT}_p(\widehat{K}) \supset (\mathcal{P}_p(\widehat{K}))^3$ there holds

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d} \mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \le C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}, \qquad s \in [0,1]. \tag{2.21}$$

If $p \ge k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

(v) For all $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$ there holds

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d} \mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \le C_s p^{-(1/2+s)} \inf_{\mathbf{v} \in \boldsymbol{V}_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}, \qquad s \in [0,1].$$

(vi) For all $k \ge 1$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ with $\mathrm{div}\, \mathbf{u} \in \mathcal{P}_p(\widehat{K})$ there holds

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d} \mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \le C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}, \qquad s \in [0,1]. \tag{2.22}$$

If $p \ge k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

Proof. Statement (i) asserts that the pertinent traces are well-defined and in L^2-based spaces. That $\mathbf{H}^1(\widehat{K}, \mathbf{curl})$-functions have L^2-traces on the edges is shown with the arguments given at the beginning of Lemma 4.11. That $\mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$-functions have normal traces in L^2 on the faces is shown in Lemma 5.16. The commuting diagram property follows by arguments very similar to those given in [15]; details can be found in Section B. For (ii) see Theorem 5.10. Item (iii) is shown Theorem 5.14 and (iv) in Lemma 5.15. Statement (v) is given in Theorem 5.20, and statement (vi) is shown in Lemma 5.21. □

The projection property of the operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, $\widehat{\Pi}_p^{\mathrm{curl},3d}$, $\widehat{\Pi}_p^{\mathrm{div},3d}$ together with the best approximation property of Lemma 4.1 implies:

Corollary 2.9. *For $k \ge 1$ and $s \in [0,1]$ there are constants $C_{s,k}$ depending only on k, s such that*

$$\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} \varphi\|_{H^{1-s}(\widehat{K})} \le C_{s,k} p^{-(k+s)} \|\varphi\|_{H^{k+1}(\widehat{K})}, \tag{2.23}$$

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d} \mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \le C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathbf{curl})} \tag{2.24}$$

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d} \mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \le C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathrm{div})}. \tag{2.25}$$

Proof. The estimate (2.23) follows directly from Theorem 2.8, (ii) and the best approximation result Lemma 4.1. For the proof of the estimate (2.24) we use Theorem 2.8, (iii) and Lemma 4.1 in the following way: With Lemma 5.5, we write $\mathbf{u} = \nabla\varphi + \mathbf{z}$ with $\|\varphi\|_{H^{k+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathbf{curl})}$ and $\|\mathbf{z}\|_{\mathbf{H}^{k+1}(\widehat{K})} \lesssim \|\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$. From Theorem 2.8, (iii) and Lemma 4.1 we infer

$$
\begin{aligned}
\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}\|_{\mathbf{H}^{-s}(\widehat{K},\mathbf{curl})} &\lesssim p^{-(1+s)} \inf_{v\in W_p(\widehat{K}),\mathbf{q}\in\mathbf{Q}_p(\widehat{K})} \|\nabla\varphi + \mathbf{z} - (\nabla v + \mathbf{q})\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \\
&\lesssim p^{-(1+s)} \left[\inf_{v\in W_p(\widehat{K})} \|\varphi - v\|_{H^2(\widehat{K})} + \inf_{\mathbf{q}\in\mathbf{Q}_p(\widehat{K})} \|\mathbf{z}-\mathbf{q}\|_{\mathbf{H}^2(\widehat{K})} \right] \\
&\overset{\text{Lem. 4.1}}{\lesssim} p^{-(1+s)-(k+1-2)} \left[\|\varphi\|_{H^{k+1}(\widehat{K})} + \|\mathbf{z}\|_{\mathbf{H}^{k+1}(\widehat{K})} \right] \lesssim p^{-(s+k)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathbf{curl})}.
\end{aligned}
$$

The bound (2.25) is shown in a similar way, using, for $\mathbf{u} \in \mathbf{H}^k(\widehat{K}, \mathrm{div})$ the decomposition $\mathbf{u} = \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z}$ with $\|\boldsymbol{\varphi}\|_{\mathbf{H}^{k+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathrm{div})}$ and $\|\mathbf{z}\|_{\mathbf{H}^{k+1}(\widehat{K})} \lesssim \|\operatorname{div}\mathbf{u}\|_{H^k(\widehat{K})}$ given by Lemma 5.6 and arguing with Theorem 2.8, (v) and Lemma 4.1, thus:

$$
\begin{aligned}
\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\|_{\mathbf{H}^{-s}(\widehat{K},\mathrm{div})} &\lesssim p^{-(1/2+s)} \inf_{\mathbf{q}\in\mathbf{Q}_p(\widehat{K}),\mathbf{v}\in\mathbf{V}_p(\widehat{K})} \|\mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z} - (\mathbf{curl}\,\mathbf{q} + \mathbf{v})\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})} \\
&\lesssim p^{-(1/2+s)} \left[\inf_{\mathbf{q}\in\mathbf{Q}_p(\widehat{K})} \|\boldsymbol{\varphi} - \mathbf{q}\|_{\mathbf{H}^{3/2}(\widehat{K})} + \inf_{\mathbf{v}\in\mathbf{V}_p(\widehat{K})} \|\mathbf{z}-\mathbf{v}\|_{\mathbf{H}^{3/2}(\widehat{K})} \right] \\
&\overset{\text{Lem. 4.1}}{\lesssim} p^{-(s+k)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathrm{div})}.
\end{aligned}
$$

□

Remark 2.10. *The operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, $\widehat{\Pi}_p^{\mathrm{curl},3d}$, $\widehat{\Pi}_p^{\mathrm{div},3d}$, $\widehat{\Pi}_p^{L^2}$ admit element-by-element constructions as in Definition C.1. The global operators $\Pi_{p+1}^{\mathrm{grad}}$, Π_p^{curl}, Π_p^{div}, $\Pi_p^{L^2}$ obtained from the operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, $\widehat{\Pi}_p^{\mathrm{curl},3d}$, $\widehat{\Pi}_p^{\mathrm{div},3d}$, $\widehat{\Pi}_p^{L^2}$ by an element-by-element construction are also linear projection operators with the commuting diagram property*

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\mathrm{id}} & H^2(\Omega) & \xrightarrow{\nabla} & \mathbf{H}^1(\Omega,\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \mathbf{H}^1(\Omega,\mathrm{div}) & \xrightarrow{\mathrm{div}} & H^1(\Omega) & \xrightarrow{0} & \{0\} \\
 & & \downarrow \Pi_{p+1}^{\mathrm{grad}} & & \downarrow \Pi_p^{\mathrm{curl}} & & \downarrow \Pi_p^{\mathrm{div}} & & \downarrow \Pi_p^{L^2} & & \\
\mathbb{R} & \xrightarrow{\mathrm{id}} & W_{p+1}(\mathcal{T}) & \xrightarrow{\nabla} & \mathbf{Q}_p(\mathcal{T}) & \xrightarrow{\mathbf{curl}} & \mathbf{V}_p(\mathcal{T}) & \xrightarrow{\mathrm{div}} & W_p(\mathcal{T}) & \xrightarrow{0} & \{0\}
\end{array} . \tag{2.26}
$$

This is a direct consequence of Theorem 2.8, (i) and the fact that the operators are constructed element by element. ■

Theorem 2.11 (Projection-based interpolation in 2D)**.** *There are constants $C_{s,k}$ depending only on s, k such that the following holds:*

(i) The operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$, $\widehat{\Pi}_p^{\mathrm{curl},2d}$, $\widehat{\Pi}_p^{L^2}$ are well-defined, projections, and the diagram (2.11) commutes.

(ii) For all $\varphi \in H^{3/2}(\widehat{f})$ there holds

$$
\begin{aligned}
\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d}\varphi\|_{H^{1-s}(\widehat{f})} &\le C_{s,k} p^{-(1/2+s)} \inf_{v\in W_{p+1}(\widehat{f})} \|\varphi - v\|_{H^{3/2}(\widehat{f})}, & s\in[0,1] \\
\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d}\varphi\|_{\widetilde{H}^{1-s}(\widehat{f})} &\le C_{s,k} p^{-(1/2+s)} \inf_{v\in W_{p+1}(\widehat{f})} \|\varphi - v\|_{H^{3/2}(\widehat{f})}, & s\in[1,3).
\end{aligned}
$$

(iii) For all $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \mathrm{curl})$ there holds

$$
\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \le C_{s,k} p^{-(1/2+s)} \inf_{\mathbf{v}\in\mathbf{Q}_p(\widehat{f})} \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}, \qquad s\in[0,3).
$$

(iv) For all $k \ge 1$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{f})$ with $\operatorname{curl}\mathbf{u} \in \mathcal{P}_p(\widehat{f})$ there holds

$$
\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \le C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}, \qquad s\in[0,3). \tag{2.27}
$$

If $p \ge k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{f})}$.

Proof. The proof of (i) follows by arguments very similar to those given in [15]; details can be found in Section B. Item (ii) is shown in Theorem 4.8 and item (iii) in Lemma 4.13. For statement (iv), see Lemma 4.14. □

The following corollary is the two-dimensional analog of Corollary 2.9:

Corollary 2.12. *For $k \geq 1$,*

$$\|\varphi - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1}\varphi\|_{H^{1-s}(\widehat{f})} \leq C_{s,k} p^{-(k+s)} \|\varphi\|_{H^{k+1}(\widehat{f})}, \qquad s \in [0,1], \tag{2.28}$$

$$\|\varphi - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1}\varphi\|_{\widetilde{H}^{1-s}(\widehat{f})} \leq C_{s,k} p^{-(k+s)} \|\varphi\|_{H^{k+1}(\widehat{f})}, \qquad s \in [1,3), \tag{2.29}$$

$$\|\mathbf{u} - \widehat{\Pi}^{\mathrm{curl},2d}_{p}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \leq C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^{k}(\widehat{f},\mathrm{curl})}, \qquad s \in [0,3). \tag{2.30}$$

Proof. The proof follows as in Corollary 2.9, relying on Lemma 4.5 for the proof of (2.30). □

3 Stability of the projection operators in one space dimension

In the one-dimensional space, the following result holds true.

Lemma 3.1. *Let $\widehat{e} = (-1,1)$. Let $\widehat{\Pi}^{\mathrm{grad},1d}_p : H^1(\widehat{e}) \to \mathcal{P}_p$ be defined by*

$$\begin{aligned} ((u - \widehat{\Pi}^{\mathrm{grad},1d}_p u)', v')_{L^2(\widehat{e})} &= 0 \qquad \forall v \in \mathcal{P}_p \cap H^1_0(\widehat{e}), \\ u(\pm 1) &= (\widehat{\Pi}^{\mathrm{grad},1d}_p u)(\pm 1). \end{aligned} \tag{3.1}$$

Then for every $s \geq 0$ there is C_s such that

$$\|u - \widehat{\Pi}^{\mathrm{grad},1d}_p u\|_{H^{1-s}(\widehat{e})} \leq C_s p^{-s} \inf_{v \in \mathcal{P}_p(\widehat{e})} \|u - v\|_{H^1(\widehat{e})}, \qquad \text{if } s \in [0,1] \tag{3.2a}$$

$$\|u - \widehat{\Pi}^{\mathrm{grad},1d}_p u\|_{\widetilde{H}^{1-s}(\widehat{e})} \leq C_s p^{-s} \inf_{v \in \mathcal{P}_p(\widehat{e})} \|u - v\|_{H^1(\widehat{e})}, \qquad \text{if } s \geq 1. \tag{3.2b}$$

Proof. The case $s = 0$ in (3.2a) reflects the well-known best approximation property of $\widehat{\Pi}^{\mathrm{grad},1d}_p$. For $s \geq 1$, one proceeds by a standard duality argument. We set $\widetilde{e} := u - \widehat{\Pi}^{\mathrm{grad},1d}_p u$ and $t = -(1-s) \geq 0$. We need an estimate for

$$\|\widetilde{e}\|_{\widetilde{H}^{-t}(\widehat{e})} = \sup_{v \in H^t(\widehat{e})} \frac{(\widetilde{e}, v)_{L^2(\widehat{e})}}{\|v\|_{H^t(\widehat{e})}}.$$

For every $v \in H^t(\widehat{e})$, there exists a unique solution $z \in H^{t+2}(\widehat{e}) \cap H^1_0(\widehat{e})$ of the problem

$$-z'' = v \text{ on } \widehat{e}, \qquad z = 0 \text{ on } \partial\widehat{e}$$

satisfying $\|z\|_{H^{t+2}(\widehat{e})} \lesssim \|v\|_{H^t(\widehat{e})}$. Thus, we obtain using integration by parts, the orthogonality condition (3.1) and the estimate for $s = 0$

$$\begin{aligned} |(\widetilde{e}, v)_{L^2(\widehat{e})}| &= |(\widetilde{e}', z')_{L^2(\widehat{e})}| \leq \|\widetilde{e}'\|_{L^2(\widehat{e})} \inf_{\pi \in \mathcal{P}_p \cap H^1_0(\widehat{e})} \|z' - \pi'\|_{L^2(\widehat{e})} \\ &\lesssim \|\widetilde{e}'\|_{L^2(\widehat{e})} p^{-(t+1)} \|z\|_{H^{t+2}(\widehat{e})} \overset{(3.2a) \text{ with } s=0}{\lesssim} p^{-(t+1)} \inf_{v \in \mathcal{P}_p(\widehat{e})} \|u - v\|_{H^1(\widehat{e})} \|v\|_{H^t(\widehat{e})}, \end{aligned}$$

which implies (3.2b) for $s \geq 1$. Noting that $\widetilde{H}^0(\widehat{e}) = L^2(\widehat{e}) = H^0(\widehat{e})$, the remaining cases $s \in (0,1)$ follow by interpolation. □

4 Stability of the projection operators in two space dimensions

4.1 Preliminaries

We recall the following unconstrained approximation results:

Lemma 4.1. *Let K be the reference tetrahedron $\widehat{K}$ or the reference triangle $\widehat{f}$. Fix $0 \le r$ and $d \in \mathbb{N}$. Then there are approximation operators $J_p : \mathbf{H}^r(K) \to (\mathcal{P}_p)^d$ such that*

$$\|\mathbf{u} - J_p\mathbf{u}\|_{\mathbf{H}^s(K)} \le C(p+1)^{-(r-s)}\|\mathbf{u}\|_{\mathbf{H}^r(K)}, \qquad \forall p \in \mathbb{N}_0, \quad 0 \le s \le r.$$

Proof. The scalar case $d = 1$ is well-known, a proof can be found, e.g., in [24, Thm. 5.1]. The case $d > 1$ follows from a componentwise application of the case $d = 1$. $\square$

Lemma 4.2 ([15])**.** *Let $P^{\mathrm{grad},2d}u \in W_{p+1}(\widehat{f})$ be defined by the conditions*

$$(\nabla(u - P^{\mathrm{grad},2d}u), \nabla v)_{L^2(\widehat{f})} = 0 \qquad \forall v \in W_{p+1}(\widehat{f}), \tag{4.1a}$$
$$(u - P^{\mathrm{grad},2d}u, 1)_{L^2(\widehat{f})} = 0. \tag{4.1b}$$

Then, for $r > 1$, there holds $\|u - P^{\mathrm{grad},2d}u\|_{H^1(\widehat{f})} \le C_r p^{-(r-1)}\|u\|_{H^r(\widehat{f})}$.

Lemma 4.3 ([15])**.** *Let $P^{\mathrm{curl},2d}\mathbf{u} \in \mathbf{Q}_p(\widehat{f})$ be defined by the conditions*

$$(\operatorname{curl}(\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}), \operatorname{curl}\mathbf{v})_{L^2(\widehat{f})} = 0 \qquad \forall \mathbf{v} \in \mathbf{Q}_p(\widehat{f}), \tag{4.2a}$$
$$(\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}, \nabla v)_{L^2(\widehat{f})} = 0 \qquad \forall v \in W_{p+1}(\widehat{f}). \tag{4.2b}$$

Then, for $r > 0$ there holds $\|\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} \le Cp^{-r}\|\mathbf{u}\|_{\mathbf{H}^r(\widehat{f},\mathrm{curl})}$.

The next lemma provides right inverses for the differential operators ∇ and curl;

Lemma 4.4 ([13], [6, Sec. 2.3])**.** *Let $B \subset \widehat{f}$ be a ball. Let $\theta \in C_0^\infty(B)$ with $\int_B \theta = 1$. Define the operators*

$$R^{\mathrm{grad}}\mathbf{u}(\mathbf{x}) := \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^1 \mathbf{u}(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))\,dt \cdot (\mathbf{x}-\mathbf{a})\,d\mathbf{a},$$
$$\mathbf{R}^{\mathrm{curl}}u(\mathbf{x}) := \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^1 tu(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))\,dt \begin{pmatrix} -(\mathbf{x}_2 - \mathbf{a}_2) \\ \mathbf{x}_1 - \mathbf{a}_1 \end{pmatrix} d\mathbf{a}.$$

Then:

(i) For $u \in L^2(\widehat{f})$, there holds $\operatorname{curl}\mathbf{R}^{\mathrm{curl}}u = u$.

(ii) For $\mathbf{u}$ with $\mathbf{curl}\,\mathbf{u} = 0$, there holds $\nabla R^{\mathrm{grad}}\mathbf{u} = \mathbf{u}$.

(iii) If $\mathbf{u} \in \mathbf{Q}_p(\widehat{f})$, then $R^{\mathrm{grad}}\mathbf{u} \in W_{p+1}(\widehat{f})$.

(iv) If $u \in V_p(\widehat{f})$, then $\mathbf{R}^{\mathrm{curl}}u \in \mathbf{Q}_p(\widehat{f})$.

(v) For every $k \ge 0$, the operators R^{grad} and $\mathbf{R}^{\mathrm{curl}}$ are bounded linear operators $\mathbf{H}^k(\widehat{f}) \to H^{k+1}(\widehat{f})$ and $H^k(\widehat{f}) \to \mathbf{H}^{k+1}(\widehat{f})$, respectively.

Lemma 4.4 can now be used to construct regular Helmholtz-like decompositions.

Lemma 4.5. *Let $s \ge 0$. Then each $\mathbf{u} \in \mathbf{H}^s(\widehat{f}, \mathrm{curl})$ can be written as $\mathbf{u} = \nabla\varphi + \mathbf{z}$ with $\varphi \in H^{s+1}(\widehat{f})$, $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{f})$.*

Proof. With the aid of the operators $\mathbf{R}^{\mathrm{curl}}$, R^{grad} of Lemma 4.4, we write $\mathbf{u} = \nabla R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})) + \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})$. The mapping properties of $\mathbf{R}^{\mathrm{curl}}$ and R^{grad} of Lemma 4.4 then imply the result. $\square$

Lemma 4.6 (discrete Friedrichs inequality in 2D)**.** *There exists $C > 0$ independent of p and $\mathbf{u}$ such that*

$$\|\mathbf{u}\|_{L^2(\widehat{f})} \le C\|\operatorname{curl}\mathbf{u}\|_{L^2(\widehat{f})} \tag{4.3}$$

in the following two cases:

(i) $\mathbf{u} \in \mathbf{Q}_p(\widehat{f})$ satisfies $(\mathbf{u}, \nabla v)_{L^2(\widehat{f})} = 0$ for all $v \in W_{p+1}(\widehat{f})$.

(ii) $\mathbf{u} \in \mathring{\mathbf{Q}}_p(\widehat{f})$ *satisfies* $(\mathbf{u}, \nabla v)_{L^2(\widehat{f})} = 0$ *for all* $v \in \mathring{W}_{p+1}(\widehat{f})$.

Proof. Statement (i) is proved in [17, Lemma 6] or [15, Lemma 4.1]. Statement (ii) is shown with similar techniques. Let R^{grad} and $\mathbf{R}^{\text{curl}}$ be the operators of Lemma 4.4. We decompose $\mathbf{u} \in \mathring{\mathbf{Q}}_p(\widehat{f})$ as

$$\mathbf{u} = \nabla\psi + \mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u}), \qquad \psi := R^{\text{grad}}(\mathbf{u} - \mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u})).$$

Since $\mathbf{u} \in \mathbf{Q}(\widehat{f})$ we have $\psi \in W_{p+1}(\widehat{f})$. The property $\mathbf{u} \in \mathring{\mathbf{Q}}_p(\widehat{f})$ implies with the tangential vector $\mathbf{t}$ on the boundary $\partial\widehat{f}$

$$\mathbf{t} \cdot \nabla\psi = -\mathbf{t} \cdot \mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u}).$$

Since ψ is continuous at the vertices of $\widehat{f}$, we infer

$$\begin{aligned} |\psi|_{H^{1/2}(\partial\widehat{f})} &\lesssim |\psi|_{H^1(\partial\widehat{f})} = \|\mathbf{t}\cdot\mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u})\|_{L^2(\partial\widehat{f})} \le |\mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u})|_{L^2(\partial\widehat{f})} \\ &\lesssim \|\mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u})\|_{H^{1/2}(\partial\widehat{f})} \lesssim \|\mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u})\|_{H^1(\widehat{f})} \lesssim \|\operatorname{curl}\mathbf{u}\|_{L^2(\widehat{f})}. \end{aligned}$$

Next, we decompose $\psi = \psi_0 + \mathcal{L}(\psi|_{\partial\widehat{f}})$, where $\mathcal{L} : H^{1/2}(\partial\widehat{f}) \to H^1(\widehat{f})$ is the lifting operator of [4]. Since $\mathcal{L}$ produces a polynomial and $\psi \in W_{p+1}(\widehat{f})$, we get that $\psi_0 \in \mathring{W}_{p+1}(\widehat{f})$ and estimate

$$\begin{aligned} \|\mathbf{u}\|^2_{L^2(\widehat{f})} &= (\mathbf{u}, \nabla\psi_0 + \nabla\mathcal{L}(\psi|_{\partial\widehat{f}}) + \mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u}))_{L^2(\widehat{f})} = (\mathbf{u}, \nabla\mathcal{L}(\psi|_{\partial\widehat{f}}) + \mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u}))_{L^2(\widehat{f})} \\ &\le \|\mathbf{u}\|_{L^2(\widehat{f})} \left\{ \|\nabla\mathcal{L}(\psi|_{\partial\widehat{f}})\|_{L^2(\widehat{f})} + \|\mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u})\|_{L^2(\widehat{f})} \right\} \\ &\le \|\mathbf{u}\|_{L^2(\widehat{f})} \left\{ \|\psi\|_{H^{1/2}(\partial\widehat{f})} + \|\mathbf{R}^{\text{curl}}(\operatorname{curl}\mathbf{u})\|_{L^2(\widehat{f})} \right\} \lesssim \|\mathbf{u}\|_{L^2(\widehat{f})} \|\operatorname{curl}\mathbf{u}\|_{L^2(\widehat{f})}. \end{aligned}$$

□

Recall that the reference triangle $\widehat{f}$ is the equilateral triangle with interior angles $\pi/3$. Thus we have the following well-known shift theorem for the Laplacian.

Lemma 4.7. *For every $s \in [0, 2)$ there is $C_s > 0$ such that the following shift theorems are true:*

(i) For every $v \in H^s(\widehat{f})$ the solution z of the problem

$$-\Delta z = v \text{ on } \widehat{f}, \quad z = 0 \text{ on } \partial\widehat{f},$$

satisfies $z \in H^{s+2}(\widehat{f}) \cap H^1_0(\widehat{f})$ with the estimate $\|z\|_{H^{s+2}(\widehat{f})} \le C_s \|v\|_{H^s(\widehat{f})}$.

(ii) For every $v \in H^s(\widehat{f})$ and data $g \in L^2(\partial\widehat{f})$ with $g|_e \in H^{s+1/2}(e)$, $e \in \mathcal{E}(\widehat{f})$ that satisfies additionally the compatibility condition $\int_{\widehat{f}} v + \int_{\partial\widehat{f}} g = 0$, the solution z of the problem

$$-\Delta z = v \text{ on } \widehat{f}, \quad \partial_n z = g \text{ on } \partial\widehat{f}, \qquad \int_{\widehat{f}} z = 0,$$

satisfies $z \in H^{s+2}(\widehat{f})$ together with $\|z\|_{H^{s+2}(\widehat{f})} \le C_s \left[\|v\|_{H^s(\widehat{f})} + \sum_{e\in\mathcal{E}(\widehat{f})} \|g\|_{H^{s+1/2}(e)}\right]$.

Proof. 1. step: It follows from [14, 22] that both regularity assertions are satisfied for the case of homogeneous Dirichlet and Neumann conditions (i.e., $g = 0$). The key observation is that the leading corner singularities for both the homogeneous Dirichlet and Neumann problem are in $H^{4-\varepsilon}$ for every $\varepsilon > 0$, since they are of the form $O(r^3 \log r)$, where r measures the distance from the vertex (with which the singularity function is associated).
2. step: For the case of inhomogeneous Neumann conditions $g \neq 0$, one constructs a vector field $\boldsymbol{\sigma} \in \mathbf{H}^{s+1}(\widehat{f})$ such that $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ on $\partial\widehat{f}$. It is easy to construct such a vector field away from the vertices, and near the vertices, an affine coordinate change reduces the construction to one in a quarter plane, where each component of $\boldsymbol{\sigma}$ can be constructed separately by lifting from one of the coordinate axes. Next, one solves the two problems

$$\begin{aligned} -\Delta z_0 &= v + \operatorname{div}\boldsymbol{\sigma} \quad \text{in } \widehat{f}, \qquad \partial_n z_0 = 0 \quad \text{on } \partial\widehat{f}, \\ -\Delta \widetilde{z}_0 &= \operatorname{curl}\boldsymbol{\sigma} \quad \text{in } \widehat{f}, \qquad \widetilde{z}_0 = 0 \quad \text{on } \partial\widehat{f}. \end{aligned}$$

From step 1, one has that z_0, $\widetilde{z}_0 \in H^{s+2}(\widehat{f})$. It remains to see that $\nabla z = \boldsymbol{\sigma} + \operatorname{\mathbf{curl}}\widetilde{z}_0 + \nabla z_0$. This follows from the observation that the difference $\boldsymbol{\delta} := \nabla z - (\boldsymbol{\sigma} + \operatorname{\mathbf{curl}}\widetilde{z}_0 + \nabla z_0)$ satisfies $\operatorname{div}\boldsymbol{\delta} = 0 = \operatorname{curl}\boldsymbol{\delta}$ as well as $\boldsymbol{\delta} \cdot \mathbf{n} = 0$ on $\partial\widehat{f}$. □

4.2 Stability of the operator $\widehat{\Pi}^{\mathrm{grad},2d}_{p+1}$

Theorem 4.8. *For every $s \in [0,3)$ there is C_s such that*

$$\|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{H^{1-s}(\widehat{f})} \le C_s p^{-(1/2+s)} \inf_{v \in \mathcal{P}_p(\widehat{f})} \|u - v\|_{H^{3/2}(\widehat{f})} \qquad \text{if } s \in [0,1], \tag{4.4a}$$

$$\|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{\widetilde{H}^{1-s}(\widehat{f})} \le C_s p^{-(1/2+s)} \inf_{v \in \mathcal{P}_p(\widehat{f})} \|u - v\|_{H^{3/2}(\widehat{f})} \qquad \text{if } s \in [1,3). \tag{4.4b}$$

Proof. The first observation is that it suffices to show the estimates (4.4a), (4.4b) for the special case $v = 0$ in the infimum by the projection property of $\widehat{\Pi}^{\mathrm{grad},2d}_{p+1}$. We will therefore show in a first step (4.4a) for the case $s = 0$. In a second step, we show (4.4b) for the cases $s \in [1,3)$. The remaining cases $s \in (0,1)$ are obtained by interpolating between the case $s = 0$ and the case $s = 1$ (for which (4.4a) and (4.4b) coincide).
We note that the trace theorem gives $u \in H^1(e)$ for each edge $e \in \mathcal{E}(\widehat{f})$ with $\|u\|_{H^1(e)} \lesssim \|u\|_{H^{3/2}(\widehat{f})}$. By Lemma 3.1, we have for every edge $e \in \mathcal{E}(\widehat{f})$

$$\|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{H^{1-s}(e)} \le C p^{-s} \|u\|_{H^{3/2}(\widehat{f})}, \qquad s \in [0,1]. \tag{4.5}$$

Since $\widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u$ is piecewise polynomial and continuous on $\partial \widehat{f}$, we infer in particular for $s = 0$ and $s = 1$ the bounds

$$\|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{H^{1-s}(\partial \widehat{f})} \le C p^{-s} \|u\|_{H^{3/2}(\widehat{f})}, \tag{4.6}$$

and then, by interpolation, also for the intermediate $s \in (0,1)$. Next, we show (4.4a) for $s = 0$. In view of the existence of a polynomial preserving lifting of [4] that is continuous $H^{1/2}(\partial \widehat{f}) \to H^1(\widehat{f})$ and the fact that $P^{\mathrm{grad},2d} u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u$ is discrete harmonic, i.e.,

$$(\nabla (P^{\mathrm{grad},2d} u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u), \nabla v)_{L^2(\widehat{f})} = 0 \qquad \forall v \in \mathring{W}_{p+1}(\widehat{f}), \tag{4.7}$$

we infer from Lemma 4.2 and (4.6) for the seminorm $|\cdot|_{H^1(\widehat{f})}$

$$\begin{aligned} |u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u|_{H^1(\widehat{f})} &\le |u - P^{\mathrm{grad},2d} u|_{H^1(\widehat{f})} + |P^{\mathrm{grad},2d} u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u|_{H^1(\widehat{f})} \\ &\lesssim p^{-1/2} \|u\|_{H^{3/2}(\widehat{f})} + \|P^{\mathrm{grad},2d} u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{H^{1/2}(\partial \widehat{f})} \\ &\lesssim p^{-1/2} \|u\|_{H^{3/2}(\widehat{f})} + \|u - P^{\mathrm{grad},2d} u\|_{H^1(\widehat{f})} \lesssim p^{-1/2} \|u\|_{H^{3/2}(\widehat{f})}, \end{aligned} \tag{4.8}$$

which is (4.4a) for $s = 0$. We next show the estimate (4.4b) for $s \in [1,3)$ by a duality argument. Let $\widetilde{e} = u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u$, and set $t = -(1-s)$. To estimate

$$\|\widetilde{e}\|_{\widetilde{H}^{-t}(\widehat{f})} = \sup_{v \in H^t(\widehat{f})} \frac{(\widetilde{e}, v)_{L^2(\widehat{f})}}{\|v\|_{H^t(\widehat{f})}} \tag{4.9}$$

let $v \in H^t(\widehat{f})$ and $z \in H^{t+2}(\widehat{f}) \cap H^1_0(\widehat{f})$ solve (cf. Lemma 4.7)

$$-\Delta z = v \quad \text{in } \widehat{f}, \qquad z|_{\partial \widehat{f}} = 0.$$

Note the *a priori* estimate $\|z\|_{H^{t+2}(\widehat{f})} \le C \|v\|_{H^t(\widehat{f})}$. Then, integration by parts yields

$$(\widetilde{e}, v)_{L^2(\widehat{f})} = \int_{\widehat{f}} \nabla \widetilde{e} \cdot \nabla z - \int_{\partial \widehat{f}} \partial_n z \widetilde{e}. \tag{4.10}$$

For the first term in (4.10) we get by the orthogonality properties satisfied by $\widetilde{e}$, Lemma 4.1 and (4.8)

$$\begin{aligned} \left| \int_{\widehat{f}} \nabla z \cdot \nabla \widetilde{e} \right| &\le \inf_{\pi \in \mathcal{P}_p \cap H^1_0(\widehat{f})} \|z - \pi\|_{H^1(\widehat{f})} \|\nabla \widetilde{e}\|_{L^2(\widehat{f})} \lesssim p^{-(t+1)} \|z\|_{H^{t+2}(\widehat{f})} \|\nabla \widetilde{e}\|_{L^2(\widehat{f})} \\ &\lesssim p^{-(t+1)} \|\nabla \widetilde{e}\|_{L^2(\widehat{f})} \|v\|_{H^t(\widehat{f})} \overset{\text{(4.4a) with } s=0}{\lesssim} p^{-(1/2+s)} \|u\|_{H^{3/2}(\widehat{f})} \|v\|_{H^t(\widehat{f})}. \end{aligned} \tag{4.11}$$

For the second term in (4.10) we use Lemma 3.1 to obtain on each edge $e \in \mathcal{E}(\widehat{f})$

$$|(\partial_n z, \widetilde{e})_{L^2(e)}| \lesssim \|\widetilde{e}\|_{\widetilde{H}^{-(t+1/2)}(e)} \|\partial_n z\|_{H^{t+1/2}(e)} \lesssim p^{-(3/2+t)} \|u\|_{H^1(e)} \|z\|_{H^{t+2}(\widehat{f})} \lesssim p^{-(1/2+s)} \|u\|_{H^{3/2}(\widehat{f})} \|v\|_{H^t(\widehat{f})}. \tag{4.12}$$

Inserting (4.11) and (4.12) in (4.9) yields (4.4b) for $s \in [1,3)$. The estimate (4.4a) for $s \in (0,1)$ now follows by interpolation between $s = 0$ and $s = 1$. □

4.3 Stability of the operator $\widehat{\Pi}_p^{\mathrm{curl},2d}$

The following lemmata present the duality arguments that are needed later on to estimate negative Sobolev norms.

Lemma 4.9. *Let* $\mathbf{E} \in \mathbf{H}(\widehat{f}, \mathrm{curl})$ *satisfy the orthogonality conditions*

$$(\mathrm{curl}\,\mathbf{E}, \mathrm{curl}\,\mathbf{v})_{L^2(\widehat{f})} = 0 \qquad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{f}), \tag{4.13a}$$

$$(\mathbf{E}, \nabla\varphi)_{L^2(\widehat{f})} = 0 \qquad \forall \varphi \in \mathring{W}_{p+1}(\widehat{f}), \tag{4.13b}$$

$$(\mathbf{E}\cdot\mathbf{t}_e, \nabla_e\varphi)_{L^2(e)} = 0 \qquad \forall \varphi \in \mathring{W}_{p+1}(e), \quad \forall e \in \mathcal{E}(\widehat{f}), \tag{4.13c}$$

$$(\mathbf{E}\cdot\mathbf{t}_e, 1)_{L^2(e)} = 0 \qquad \forall e \in \mathcal{E}(\widehat{f}). \tag{4.13d}$$

Then, for $s \in [0, 3)$*, there holds* $\|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f})} \le Cp^{-s}\|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}$.

Proof. 1. step: We may restrict to the case $s \ge 1$ as the case $s = 0$ is trivial and the remaining cases $s \in [0, 1]$ follow then by interpolation.
2. step: Any $\mathbf{v} \in \mathbf{H}^s(\widehat{f})$ can be decomposed as

$$\mathbf{v} = \nabla\varphi + \mathbf{curl}\,z, \tag{4.14}$$

where $\varphi, z \in H^{s+1}(\widehat{f})$ are determined by the following equations:

$$-\Delta\varphi = -\operatorname{div}\mathbf{v}, \qquad \varphi = 0 \quad \text{on } \partial\widehat{f}, \tag{4.15a}$$

$$-\Delta z = \mathrm{curl}\,\mathbf{v}, \qquad \partial_n z = -\mathbf{t}\cdot\mathbf{curl}\,z = -\mathbf{t}\cdot(\mathbf{v} - \nabla\varphi) \quad \text{on } \partial\widehat{f}, \qquad \int_{\widehat{f}} z = 0. \tag{4.15b}$$

Here, $\mathbf{t}$ denotes the unit tangent vector on $\partial\widehat{f}$ oriented such that $\widehat{f}$ is "on the left". We note that (4.15b) is a Neumann problem; integration by parts shows that the solvability condition is satisfied. We have by Lemma 4.7 the *a priori* estimates

$$\|\varphi\|_{H^{s+1}(\widehat{f})} \lesssim \|\operatorname{div}\mathbf{v}\|_{H^{s-1}(\widehat{f})}, \qquad \|z\|_{H^{s+1}(\widehat{f})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})}. \tag{4.16}$$

Together with integration by parts (cf. (2.2)) we compute

$$(\mathbf{E}, \mathbf{v})_{L^2(\widehat{f})} = (\mathbf{E}, \nabla\varphi)_{L^2(\widehat{f})} + (\mathbf{E}, \mathbf{curl}\,z)_{L^2(\widehat{f})} = (\mathbf{E}, \nabla\varphi)_{L^2(\widehat{f})} + (\mathrm{curl}\,\mathbf{E}, z)_{L^2(\widehat{f})} - \int_{\partial\widehat{f}} z\mathbf{E}\cdot\mathbf{t}. \tag{4.17}$$

and estimate each of the three terms separately.
3. step: Using the orthogonalities satisfied by $\mathbf{E}$ and $\varphi \in H_0^1(\widehat{f}) \cap H^{s+1}(\widehat{f})$ we obtain for the first term in (4.17)

$$(\mathbf{E}, \nabla\varphi)_{L^2(\widehat{f})} = \inf_{w \in \mathring{W}_{p+1}(\widehat{f})} (\mathbf{E}, \nabla(\varphi - w))_{L^2(\widehat{f})} \lesssim p^{-s}\|\operatorname{div}\mathbf{v}\|_{H^{s-1}(\widehat{f})}\|\mathbf{E}\|_{L^2(\widehat{f})} \lesssim p^{-s}\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})}\|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}.$$

4. step: The term $(z, \mathbf{E}\cdot\mathbf{t})_{L^2(\partial\widehat{f})}$ in (4.17) can be treated using the orthogonalities satisfied by $\mathbf{E}$: Using that $z \in H^{s+1}(\widehat{f})$ so that $z \in C(\partial\widehat{f})$ and $z \in H^{s+1/2}(e)$ for each edge $e \in \mathcal{E}(\widehat{f})$ and the orthogonality properties (4.13c) and (4.13d), we get

$$\begin{aligned}\left|\int_{\partial\widehat{f}} \mathbf{E}\cdot\mathbf{t}z\right| &= \inf_{w \in W_p(\partial\widehat{f})} \left|\int_{\partial\widehat{f}} \mathbf{E}\cdot\mathbf{t}(z - w)\right| \lesssim \|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})} \inf_{w \in W_p(\partial\widehat{f})} \|z - w\|_{H^{1/2}(\partial\widehat{f})} \\ &\lesssim p^{-s}\|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})}\|z\|_{H^{s+1}(\widehat{f})} \lesssim p^{-s}\|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})},\end{aligned}$$

where, in the final step, we used the continuity of the tangential trace map: $\|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}$. (cf., e.g., [15, (eq. (154)]).
5. step: For the first term in (4.17) , we introduce an auxiliary function $\mathbf{z}$ with the following key properties:

$$\mathrm{curl}\,\mathbf{z} = z, \quad \mathbf{z}\cdot\mathbf{t} = 0$$

Such a function can be obtained as $\mathbf{z} = \mathbf{curl}\,\widetilde{z}$, where $\widetilde{z}$ solves the following Neumann problem (note that $\int_{\widehat{f}} z = 0$, so the solvability condition is satisfied)

$$-\Delta \widetilde{z} = z \quad \text{in } \widehat{f}, \qquad \partial_n \widetilde{z} = 0 \quad \text{on } \partial \widehat{f}.$$

We obtain

$$\begin{aligned}(\operatorname{curl} \mathbf{E}, z)_{L^2(\widehat{f})} &= (\operatorname{curl} \mathbf{E}, \operatorname{curl} \mathbf{z})_{L^2(\widehat{f})} \overset{(4.13a)}{=} \inf_{\mathbf{w} \in \mathring{\mathbf{Q}}_p(\widehat{f})} (\operatorname{curl} \mathbf{E}, \operatorname{curl}(\mathbf{z} - \mathbf{w}))_{L^2(\widehat{f})} \\ &\lesssim p^{-s} \| \operatorname{curl} \mathbf{E}\|_{L^2(\widehat{f})} \|\mathbf{z}\|_{\mathbf{H}^s(\widehat{f},\operatorname{curl})} \overset{\text{Lem. 4.3}}{\lesssim} p^{-s} \|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\operatorname{curl})} \|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})}. \end{aligned}$$

□

Lemma 4.10. *Let $\mathbf{E} \in \mathbf{H}(\widehat{f}, \operatorname{curl})$ satisfy (4.13a), (4.13d). Then, for $s \in [0, 3)$, there holds $\| \operatorname{curl} \mathbf{E}\|_{\widetilde{H}^{-s}(\widehat{f})} \leq C_s p^{-s} \| \operatorname{curl} \mathbf{E}\|_{L^2(\widehat{f})}$.*

Proof. As in the proof of Lemma 4.9, we restrict to $s \geq 1$ and argue by interpolation for $s \in [0, 1]$. Let $v \in H^s(\widehat{f})$ and $\overline{v} := (\int_{\widehat{f}} v)/|\widehat{f}| \in \mathbb{R}$ be its average. Integration by parts yields

$$(\operatorname{curl} \mathbf{E}, v)_{L^2(\widehat{f})} = (\operatorname{curl} \mathbf{E}, v - \overline{v})_{L^2(\widehat{f})} + \overline{v}(\mathbf{E} \cdot \mathbf{t}, 1)_{L^2(\partial \widehat{f})} \overset{(4.13d)}{=} (\operatorname{curl} \mathbf{E}, v - \overline{v})_{L^2(\widehat{f})}.$$

Next, we define the auxiliary function $\varphi \in H^{s+1}(\widehat{f})$ as the solution of

$$-\Delta \varphi = v - \overline{v} \quad \text{in } \widehat{f}, \qquad \partial_n \varphi = 0 \quad \text{on } \partial \widehat{f}$$

and set $\mathbf{v} := \mathbf{curl}\,\varphi$. (Lemma 4.7 is applicable since $s+1 < 4$; for $s < 2$ Lemma 4.7 even asserts $\varphi \in H^{s+2}(\widehat{f})$.) We note $\operatorname{curl} \mathbf{v} = -\Delta \varphi = v - \overline{v}$ in $\widehat{f}$ and $\mathbf{t} \cdot \mathbf{v} = -\partial_n \varphi = 0$ on $\partial \widehat{f}$ so that integration by parts gives

$$\begin{aligned}(\operatorname{curl} \mathbf{E}, v - \overline{v})_{L^2(\widehat{f})} &= (\operatorname{curl} \mathbf{E}, \operatorname{curl} \mathbf{v})_{L^2(\widehat{f})} \overset{(4.13a)}{=} \inf_{\mathbf{w} \in \mathring{\mathbf{Q}}_p(\widehat{f})} (\operatorname{curl} \mathbf{E}, \operatorname{curl}(\mathbf{v} - \mathbf{w}))_{L^2(\widehat{f})} \\ &\overset{\text{Lem. 4.3}}{\lesssim} p^{-s} \| \operatorname{curl} \mathbf{E}\|_{L^2(\widehat{f})} \|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f},\operatorname{curl})} \lesssim p^{-s} \| \operatorname{curl} \mathbf{E}\|_{L^2(\widehat{f})} \|v\|_{H^s(\widehat{f})}. \end{aligned}$$

□

After the next lemma about approximation on edges $e \in \mathcal{E}(\widehat{f})$, we can prove the stability results in 2D as stated in Theorem 2.11.

Lemma 4.11. *For each edge $e \in \mathcal{E}(\widehat{f})$ we have for $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \operatorname{curl})$ and $s \geq 0$*

$$\|(\mathbf{u} - \widehat{\Pi}_p^{\operatorname{curl},2d} \mathbf{u}) \cdot \mathbf{t}_e\|_{\widetilde{H}^{-s}(e)} \leq C_s p^{-s} \inf_{v \in \mathcal{P}_p(e)} \|\mathbf{u} \cdot \mathbf{t}_e - v\|_{L^2(e)}. \tag{4.18}$$

Proof. Note that $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \operatorname{curl})$ ensures that $\mathbf{u} \cdot \mathbf{t}_e \in L^2(e)$ since $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \operatorname{curl})$ can be decomposed as $\mathbf{u} = \nabla \varphi + \mathbf{z}$ with $\varphi \in H^{3/2}(\widehat{f})$, $\mathbf{z} \in \mathbf{H}^{3/2}(\widehat{f})$.
We recall that on edges, the operator $\widehat{\Pi}_p^{\operatorname{curl},2d}$ is simply the L^2-projection. Thus, (4.18) holds for $s = 0$. For $s > 0$, (4.18) is shown by a standard duality argument. Let $\widetilde{e} := \left(\mathbf{u} - \widehat{\Pi}_p^{\operatorname{curl},2d} \mathbf{u}\right) \cdot \mathbf{t}_e$ be the error and $v \in H^1(e)$. Note that a function $w \in \mathcal{P}_p(\mathbb{R})$ can be decomposed into $w(x) = \overline{w} + \left(\int_0^x w - \overline{w}\right)'$, where $\overline{w}$ denotes the average of w on e. Hence, $(\widetilde{e}, w)_{L^2(e)}$ by (2.15e) and (2.15f), and we obtain

$$(\widetilde{e}, v)_{L^2(e)} = \inf_{w \in \mathcal{P}_p} (\widetilde{e}, v - w)_{L^2(e)} \leq \|\widetilde{e}\|_{L^2(e)} \inf_{w \in \mathcal{P}_p} \|v - w\|_{L^2(e)} \lesssim p^{-1} \|\widetilde{e}\|_{L^2(e)} \|v\|_{H^1(e)}.$$

□

Remark 4.12. *For $w \in L^2(\partial \widehat{f})$, the estimate $\|w\|_{H^{-1/2}(\partial \widehat{f})} \lesssim \sum_{e \in \mathcal{E}(\widehat{f})} \|w\|_{\widetilde{H}^{-1/2}(e)}$ holds.* ■

Lemma 4.13. *For $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \operatorname{curl})$ there holds*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\operatorname{curl},2d} \mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\operatorname{curl})} \leq C_s p^{-(1/2+s)} \inf_{\mathbf{v} \in \mathbf{Q}_p(\widehat{f})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{f},\operatorname{curl})}, \qquad s \in [0, 3).$$

Proof. By the projection property of $\widehat{\Pi}_p^{\mathrm{curl},2d}$, it suffices to show the bound with $\mathbf{v} = 0$ in the infimum.

1. step: We recall the existence of a lifting from the boundary: As discussed in [15, Sec. 4.2] (which relies on [2]) there is a polynomial-preserving lifting $\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d} : H^{-1/2}(\partial\widehat{f}) \to \mathbf{H}(\widehat{f}, \mathrm{curl})$ that is uniformly (in p) bounded.

2. step: Let $P^{\mathrm{curl},2d}\mathbf{u}$ be the polynomial best approximation of Lemma 4.3. Following the procedure suggested in [15], we define

$$\mathbf{E} := P^{\mathrm{curl},2d}\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}.$$

Note that $\mathbf{E} \in \mathbf{Q}_p(\widehat{f})$ and that $\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t}) \in \mathring{\mathbf{Q}}_p(\widehat{f})$. We get from the orthogonalities (2.15c) and (4.2)

$$\left(\mathrm{curl}(\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})), \mathrm{curl}\,\mathbf{E}\right)_{L^2(\widehat{f})} = 0. \tag{4.19}$$

Hence,

$$\begin{aligned}\|\mathrm{curl}\,\mathbf{E}\|^2_{L^2(\widehat{f})} &= \Big(\mathrm{curl}(\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})), \mathrm{curl}\,\mathbf{E}\Big)_{L^2(\widehat{f})} + \Big(\mathrm{curl}\,\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t}), \mathrm{curl}\,\mathbf{E}\Big)_{L^2(\widehat{f})} \\ &\overset{(4.19)}{\leq} \|\mathrm{curl}\,\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{L^2(\widehat{f})}\|\mathrm{curl}\,\mathbf{E}\|_{L^2(\widehat{f})},\end{aligned}$$

from which we obtain with the stability properties of the lifting operator $\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}$

$$\|\mathrm{curl}\,\mathbf{E}\|_{L^2(\widehat{f})} \lesssim \|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})}. \tag{4.20}$$

3. step: The discrete Friedrichs inequality of Lemma 4.6, (ii) then gives also

$$\begin{aligned}\|\mathbf{E}\|_{L^2(\widehat{f})} &\leq \|\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{L^2(\widehat{f})} + \|\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{L^2(\widehat{f})} \qquad (4.21)\\ &\lesssim \|\mathrm{curl}(\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t}))\|_{L^2(\widehat{f})} + \|\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{L^2(\widehat{f})} \\ &\lesssim \|\mathrm{curl}\,\mathbf{E}\|_{L^2(\widehat{f})} + \|\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{H(\widehat{f},\mathrm{curl})} \overset{(4.20)}{\lesssim} \|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})}. \qquad (4.22)\end{aligned}$$

4. step: With the triangle inequality and the approximation property of Lemma 4.3, we arrive at

$$\begin{aligned}\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} &\lesssim \|\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} + \|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} \\ &\overset{(4.20),(4.22)}{\leq} \|\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} + \|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})} \\ &\lesssim \|\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} + \|((\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u})\cdot\mathbf{t})\|_{H^{-1/2}(\partial\widehat{f})} \\ &\overset{\text{Lemma 4.3, Lemma 4.11}}{\lesssim} p^{-1/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}. \qquad (4.23)\end{aligned}$$

5. step: From Lemma 4.9 and Lemma 4.10 together with interpolation, it follows immediately

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \lesssim p^{-s}\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} \overset{(4.23)}{\lesssim} p^{-(1/2+s)}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}. \qquad \square$$

In the case of discrete curl, we get the following result.

Lemma 4.14. *For all $k \geq 1$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{f})$ with $\mathrm{curl}\,\mathbf{u} \in \mathcal{P}_p(\widehat{f})$ there holds*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \leq C_{s,k}p^{-(k+s)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}, \qquad s \in [0,3). \tag{4.24}$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{f})}$.

Proof. We employ the regularized right inverses of the operators ∇ and **curl** and proceed as in [23, Lemma 5.8]. We write, using the decomposition of Lemma 4.5,

$$\mathbf{u} = \nabla R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{u}) + \mathbf{R}^{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{u} =: \nabla\varphi + \mathbf{v}$$

with $\varphi \in H^{k+1}(\widehat{f})$ and $\mathbf{v} \in \mathbf{H}^k(\widehat{f})$ together with

$$\|\varphi\|_{H^{k+1}(\widehat{f})} + \|\mathbf{v}\|_{\mathbf{H}^k(\widehat{f})} \leq C\left(\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})} + \|\mathrm{curl}\,\mathbf{u}\|_{H^{k-1}(\widehat{f})}\right) \leq C\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}. \tag{4.25}$$

The assumption $\operatorname{curl} \mathbf{u} \in \mathcal{P}_p(\widehat{f})$ and Lemma 4.4, (iv) imply $\mathbf{v} = \mathbf{R}^{\mathrm{curl}} \operatorname{curl} \mathbf{u} \in \mathbf{Q}_p(\widehat{f})$; furthermore, since $\widehat{\Pi}_p^{\mathrm{curl},2d}$ is a projection, we conclude $\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{v} = 0$. Thus, together with the commuting diagram property $\nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},2d} = \widehat{\Pi}_p^{\mathrm{curl},2d}\nabla$ and the bound (4.4) we get

$$\begin{aligned}
\|(I - \widehat{\Pi}_p^{\mathrm{curl},2d})\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} &= \|(I - \widehat{\Pi}_p^{\mathrm{curl},2d})\nabla\varphi + \underbrace{(I - \widehat{\Pi}_p^{\mathrm{curl},2d})\mathbf{v}}_{=0}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \\
&= \|\nabla(I - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d})\varphi\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f})} \lesssim p^{-(k+s)}\|\varphi\|_{H^{k+1}(\widehat{f})}.
\end{aligned}$$

The proof of (4.24) is complete in view of (4.25). Replacing $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}$ with $|\mathbf{u}|_{\mathbf{H}^k(\widehat{f})}$ follows from the observation that the projector $\widehat{\Pi}_p^{\mathrm{curl},2d}$ reproduces polynomials of degree p. ◻

5 Stability of the projection operators in three space dimensions

5.1 Preliminaries

For the approximation properties of $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, we need the following approximation results.

Lemma 5.1 ([15]). *Let $P^{\mathrm{grad},3d}u \in W_{p+1}(\widehat{K})$ be defined by the conditions*

$$(\nabla(u - P^{\mathrm{grad},3d}u), \nabla v)_{L^2(\widehat{K})} = 0 \qquad \forall v \in W_{p+1}(\widehat{K}), \tag{5.1a}$$

$$(u - P^{\mathrm{grad},3d}u, 1)_{L^2(\widehat{K})} = 0. \tag{5.1b}$$

Then, for $r > 1$, there holds $\|u - P^{\mathrm{grad},3d}u\|_{H^1(\widehat{K})} \le C_r p^{-(r-1)}\|u\|_{H^r(\widehat{K})}$.

Lemma 5.2 ([15,17]). *Let $P^{\mathrm{curl},3d}\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$ be defined by the conditions*

$$(\mathbf{curl}(\mathbf{u} - P^{\mathrm{curl},3d}\mathbf{u}), \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathbf{Q}_p(\widehat{K}), \tag{5.2a}$$

$$(\mathbf{u} - P^{\mathrm{curl},3d}\mathbf{u}, \nabla v)_{L^2(\widehat{K})} = 0 \qquad \forall v \in W_{p+1}(\widehat{K}). \tag{5.2b}$$

Then, for $r > 0$, there holds $\|\mathbf{u} - P^{\mathrm{curl},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \le C_r p^{-r}\|\mathbf{u}\|_{\mathbf{H}^r(\widehat{K},\mathbf{curl})}$.

Lemma 5.3 ([15, Thm. 5.2]). *Let $P^{\mathrm{div},3d}\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ be defined by the conditions*

$$(\operatorname{div}(\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u}), \operatorname{div}\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_p(\widehat{K}), \tag{5.3a}$$

$$(\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u}, \operatorname{div}\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathbf{Q}_p(\widehat{K}). \tag{5.3b}$$

Then, for $r > 0$, there holds $\|\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \le C_r p^{-r}\|\mathbf{u}\|_{\mathbf{H}^r(\widehat{K},\mathrm{div})}$.

In the next lemma, right inverses for the differential operators are defined and some properties are stated.

Lemma 5.4 ([13], see also [23, Sec. 2]). *Let $B \subset \widehat{K}$ be a ball. Let $\theta \in C_0^\infty(B)$ with $\int_B \theta = 1$. Define the operators*

$$\begin{aligned}
R^{\mathrm{grad}}\mathbf{u}(\mathbf{x}) &:= \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^1 \mathbf{u}(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))\,dt \cdot (\mathbf{x} - \mathbf{a})\,d\mathbf{a}, \\
\mathbf{R}^{\mathrm{curl}}\mathbf{u}(\mathbf{x}) &:= \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^1 t\mathbf{u}(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))\,dt \times (\mathbf{x} - \mathbf{a})\,d\mathbf{a}, \\
\mathbf{R}^{\mathrm{div}}u(\mathbf{x}) &:= \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^1 t^2 u(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))\,dt(\mathbf{x} - \mathbf{a})\,d\mathbf{a}.
\end{aligned}$$

Then:

(i) For $\mathbf{u}$ with $\operatorname{div}\mathbf{u} = 0$, there holds $\mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}\mathbf{u} = \mathbf{u}$.

(ii) For $\mathbf{u}$ with $\mathbf{curl}\,\mathbf{u} = 0$, there holds $\nabla R^{\mathrm{grad}}\mathbf{u} = \mathbf{u}$.

(iii) *For* $u \in L^2(\widehat{K})$*, there holds* $\operatorname{div} \mathbf{R}^{\mathrm{div}} u = u$.

(iv) *If* $\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$*, then* $R^{\mathrm{grad}}\mathbf{u} \in W_{p+1}(\widehat{K})$.

(v) *If* $\mathbf{u} \in \mathbf{V}_p(\widehat{K})$*, then* $\mathbf{R}^{\mathrm{curl}}\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$.

(vi) *If* $u \in W_p(\widehat{K})$*, then* $\mathbf{R}^{\mathrm{div}} u \in \mathbf{V}_p(\widehat{K})$.

(vii) *For every* $k \geq 0$*, the operators* R^{grad}, $\mathbf{R}^{\mathrm{curl}}$ *and* $\mathbf{R}^{\mathrm{div}}$ *are bounded linear operators* $\mathbf{H}^k(\widehat{K}) \to H^{k+1}(\widehat{K})$, $\mathbf{H}^k(\widehat{K}) \to \mathbf{H}^{k+1}(\widehat{K})$ *and* $H^k(\widehat{K}) \to \mathbf{H}^{k+1}(\widehat{K})$*, respectively.*

The right inverses can now be used to construct regular Helmholtz-like decompositions of functions in $\mathbf{H}^s(\widehat{K}, \mathbf{curl})$ and $\mathbf{H}^s(\widehat{K}, \operatorname{div})$.

Lemma 5.5. *Let* $s \geq 0$*. Then each* $\mathbf{u} \in \mathbf{H}^s(\widehat{K}, \mathbf{curl})$ *can be written as* $\mathbf{u} = \nabla\varphi + \mathbf{z}$ *with* $\varphi \in H^{s+1}(\widehat{K})$, $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{K})$ *satisfying* $\|\varphi\|_{H^{s+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s(\widehat{K},\mathbf{curl})}$ *and* $\|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^s(\widehat{K})}$.

Proof. With the aid of the operators $\mathbf{R}^{\mathrm{curl}}$, R^{grad} of Lemma 5.4, we write $\mathbf{u} = \nabla R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})) + \mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})$. The mapping properties of $\mathbf{R}^{\mathrm{curl}}$ and R^{grad} of Lemma 5.4 then imply the result. For the desired estimates, we use the stability properties of the operators $\mathbf{R}^{\mathrm{curl}}$ and R^{grad} to get

$$
\begin{aligned}
\|\varphi\|^2_{H^{s+1}(\widehat{K})} &\lesssim \|\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})\|^2_{\mathbf{H}^s(\widehat{K})} \lesssim \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K})} + \|\mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})\|^2_{\mathbf{H}^{s+1}(\widehat{K})} \\
&\lesssim \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K})} + \|\mathbf{curl}\,\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K})} = \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K},\mathbf{curl})} \\
\|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} &= \|\mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^s(\widehat{K})}. \qquad \square
\end{aligned}
$$

Lemma 5.6. *Let* $s \geq 0$*. Then each* $\mathbf{u} \in \mathbf{H}^s(\widehat{K}, \operatorname{div})$ *can be written as* $\mathbf{u} = \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z}$ *with* $\boldsymbol{\varphi} \in \mathbf{H}^{s+1}(\widehat{K})$, $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{K})$ *satisfying* $\|\boldsymbol{\varphi}\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s(\widehat{K},\operatorname{div})}$ *and* $\|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\operatorname{div}\mathbf{u}\|_{H^s(\widehat{K})}$.

Proof. Using the operators $\mathbf{R}^{\mathrm{curl}}$ and $\mathbf{R}^{\mathrm{div}}$ of Lemma 5.4, we write $\mathbf{u} = \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\mathbf{u} - \mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})) + \mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})$. The mapping properties of $\mathbf{R}^{\mathrm{curl}}$ and $\mathbf{R}^{\mathrm{div}}$ of Lemma 5.4 then imply the result. For the desired estimates, we use the stability properties of $\mathbf{R}^{\mathrm{curl}}$ and $\mathbf{R}^{\mathrm{div}}$ and get

$$
\begin{aligned}
\|\boldsymbol{\varphi}\|^2_{\mathbf{H}^{s+1}(\widehat{K})} &\lesssim \|\mathbf{u} - \mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})\|^2_{\mathbf{H}^s(\widehat{K})} \lesssim \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K})} + \|\mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})\|^2_{H^{s+1}(\widehat{K})} \\
&\lesssim \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K})} + \|\operatorname{div}\mathbf{u}\|^2_{H^s(\widehat{K})} = \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K},\operatorname{div})}, \\
\|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} &= \|\mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\operatorname{div}\mathbf{u}\|_{H^s(\widehat{K})}. \qquad \square
\end{aligned}
$$

We now state the Friedrichs inequalities for the operators **curl** and div.

Lemma 5.7 (discrete Friedrichs inequality for $\mathbf{H}(\mathbf{curl})$ in 3D, [15, Lemma 5.1])**.** *There exists* $C > 0$ *independent of* p *and* $\mathbf{u}$ *such that*

$$\|\mathbf{u}\|_{L^2(\widehat{K})} \leq C\|\mathbf{curl}\,\mathbf{u}\|_{L^2(\widehat{K})} \tag{5.4}$$

in the following two cases:

(i) $\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$ *satisfies* $(\mathbf{u}, \nabla v)_{L^2(\widehat{K})} = 0$ *for all* $v \in W_{p+1}(\widehat{K})$,

(ii) $\mathbf{u} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K}) := \{\mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}) \colon (\mathbf{v}, \nabla\psi)_{L^2(\widehat{K})} = 0 \quad \forall \psi \in \mathring{W}_{p+1}(\widehat{K})\}$.

Lemma 5.8 (discrete Friedrichs inequality for $\mathbf{H}(\operatorname{div})$)**.** *There exists* $C > 0$ *independent of* p *and* $\mathbf{u}$ *such that*

$$\|\mathbf{u}\|_{L^2(\widehat{K})} \leq C\|\operatorname{div}\mathbf{u}\|_{L^2(\widehat{K})} \tag{5.5}$$

in the following two cases:

(i) $\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ *satisfies* $(\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0$ *for all* $\mathbf{v} \in \mathbf{Q}_p(\widehat{K})$,

(ii) $\mathbf{u} \in \mathring{\mathbf{V}}_p(\widehat{K})$ *satisfies* $(\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0$ *for all* $\mathbf{v} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$.

Proof. The statement (i) is taken from [15, Lemma 5.2]. It is also shown in [15, Lemma 5.2] that the Friedrichs inequality (5.5) holds for all $\mathbf{u}$ satisfying

$$\mathbf{u} \in \mathring{\mathbf{V}}_p(\widehat{K}) \text{ satisfies } (\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \text{ for all } \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}). \tag{5.6}$$

To see that the condition (ii) in Lemma 5.8 suffices, assume that $\mathbf{u}$ satisfies the condition (ii) in Lemma 5.8 and write $\mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K})$ as $\mathbf{v} = \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v} + (\mathbf{v} - \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v})$, where $\Pi_{\nabla \mathring{W}_{p+1}}$ denotes the L^2-projection on $\nabla \mathring{W}_{p+1}(\widehat{K}) \subset \mathring{\mathbf{Q}}_p(\widehat{K})$. Then observe that $\mathbf{v} - \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$ so that

$$(\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = \underbrace{(\mathbf{u}, \mathbf{curl}(\Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v}))_{L^2(\widehat{K})}}_{=0} + \underbrace{(\mathbf{u}, \mathbf{curl}(\mathbf{v} - \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v}))_{L^2(\widehat{K})}}_{=0 \text{ since } \mathbf{v} - \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v} \in \mathring{\mathbf{Q}}_{p,\perp}} = 0;$$

hence, $\mathbf{u}$ satisfies in fact (5.6). Thus, it satisfies the Friedrichs inequality (5.5). ☐

Remark 5.9. *The arguments of the proof of Lemma 5.8 also show that we have the equivalence of (5.7) and (5.8):*

$$\mathbf{u} \in \mathring{\mathbf{V}}_p(\widehat{K}) \textit{ satisfies } (\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}) \qquad \Longleftrightarrow \tag{5.7}$$

$$\mathbf{u} \in \mathring{\mathbf{V}}_p(\widehat{K}) \textit{ satisfies } (\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K}). \tag{5.8}$$

5.2 Stability of the operator $\widehat{\Pi}^{\mathrm{grad},3d}_{p+1}$

The three-dimensional analog of Theorem 4.8 is:

Theorem 5.10. *For every $s \in [0,1]$ there is $C_s > 0$ such that*

$$\|u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u\|_{H^{1-s}(\widehat{K})} \le C_s p^{-(1+s)} \inf_{v \in W_{p+1}(\widehat{K})} \|u - v\|_{H^2(\widehat{K})}. \tag{5.9}$$

Proof. The proof proceeds along the same lines as the 2D case. First, we observe from the projection property of $\widehat{\Pi}^{\mathrm{grad},3d}_{p+1}$ that it suffices to show (5.9) with $v = 0$ in the infimum. Next, from the trace theorem, we have $u|_f \in H^{3/2}(f)$ for every face $f \in \mathcal{F}(\widehat{K})$. From Theorem 4.8 we get, for every face $f \in \mathcal{F}(\widehat{K})$ and $s \in [0,1]$

$$\|u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u\|_{H^{1-s}(f)} \le C p^{-(1/2+s)} \|u\|_{H^2(\widehat{K})}. \tag{5.10}$$

Since $u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u \in C(\partial\widehat{K})$, we conclude

$$\|u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u\|_{H^{1-s}(\partial\widehat{K})} \le C p^{-(1/2+s)} \|u\|_{H^2(\widehat{K})} \tag{5.11}$$

for $s \in \{0,1\}$ and then, by interpolation for all $s \in [0,1]$. Next, we show (5.9) for $s = 0$. As in the 2D case, we get from Lemma 5.1, the estimate (5.11), the existence of a polynomial preserving lifting (cf. [26]) and the fact that $P^{\mathrm{grad},3d} u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u$ is discrete harmonic the bound

$$\begin{aligned} |u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u|_{H^1(\widehat{K})} &\le |u - P^{\mathrm{grad},3d} u|_{H^1(\widehat{K})} + |P^{\mathrm{grad},3d} u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u|_{H^1(\widehat{K})} \\ &\lesssim p^{-1} \|u\|_{H^2(\widehat{K})} + \|P^{\mathrm{grad},3d} u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u\|_{H^{1/2}(\partial\widehat{K})} \\ &\lesssim p^{-1} \|u\|_{H^2(\widehat{K})} + \|u - P^{\mathrm{grad},3d} u\|_{H^1(\widehat{K})} \lesssim p^{-1} \|u\|_{H^2(\widehat{K})}. \end{aligned} \tag{5.12}$$

To get the L^2-estimate, we proceed by a duality argument: Let $z \in H^2(\widehat{K}) \cap H^1_0(\widehat{K})$ by given by

$$-\Delta z = \widetilde{e} := u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u \quad \text{on } \widehat{K}, \qquad z|_{\partial\widehat{K}} = 0.$$

Integration by parts gives

$$\|\widetilde{e}\|^2_{L^2(\widehat{K})} = \int_{\widehat{K}} \nabla z \cdot \nabla \widetilde{e} - \int_{\partial\widehat{K}} \partial_n z \widetilde{e}. \tag{5.13}$$

For the first term in (5.13) we can use the orthogonality properties satisfied by $\widetilde{e}$ and (5.12) to get

$$|(\nabla z, \nabla \widetilde{e})_{L^2(\widehat{K})}| \leq \inf_{\pi \in \mathring{W}_{p+1}(\widehat{K})} \|z - \pi\|_{H^1(\widehat{K})} \|\nabla \widetilde{e}\|_{L^2(\widehat{K})} \lesssim p^{-1} \|\widetilde{e}\|_{L^2(\widehat{K})} \|\nabla \widetilde{e}\|_{L^2(\widehat{K})}. \tag{5.14}$$

For the second term in (5.13), we use Theorem 4.8 to obtain on each face $f \in \mathcal{F}(\widehat{K})$

$$|(\partial_n z, \widetilde{e})_{L^2(f)}| \leq \|\partial_n z\|_{H^{1/2}(f)} \|\widetilde{e}\|_{\widetilde{H}^{-1/2}(f)} \lesssim p^{-2} \|\partial_n z\|_{H^{1/2}(f)} \|u\|_{H^{3/2}(f)} \lesssim p^{-2} \|\widetilde{e}\|_{L^2(\widehat{K})} \|u\|_{H^2(\widehat{K})}. \tag{5.15}$$

Inserting (5.14), (5.15) in (5.13) gives the desired estimate for $s = 1$. Interpolation gives the intermediate values $s \in (0, 1)$. ☐

5.3 Stability of the operator $\widehat{\Pi}_p^{\mathrm{curl},3d}$

As in the proof of Lemma 4.13, a key ingredient is the existence of a polynomial preserving lifting operator from the boundary to the element with the appropriate mapping properties and an additional orthogonality property. For $\mathbf{H}(\widehat{K}, \mathbf{curl})$, a lifting operator has been constructed in [20]. We formulate a simplified version of their results and also explicitly modify that lifting to ensure a convenient orthogonality property.

Lemma 5.11. *Introduce on the trace space $\Pi_\tau \mathbf{H}(\widehat{K}, \mathbf{curl})$ the norm*

$$\|\mathbf{z}\|_{\mathbf{X}^{-1/2}} := \inf\{\|\mathbf{v}\|_{\mathbf{H}(\widehat{K}, \mathbf{curl})} \,|\, \Pi_\tau \mathbf{v} = \mathbf{z}\}. \tag{5.16}$$

There exists a lifting operator $\boldsymbol{\mathcal{L}}^{\mathrm{curl},3d} : \Pi_\tau \mathbf{H}(\widehat{K}, \mathbf{curl}) \to \mathbf{H}(\widehat{K}, \mathbf{curl})$ with the following properties:

(i) $\boldsymbol{\mathcal{L}}^{\mathrm{curl},3d}\mathbf{z} \in \mathbf{Q}_p(\widehat{K})$ *if* $\mathbf{z} \in \Pi_\tau \mathbf{H}(\widehat{K}, \mathbf{curl})$ *satisfies* $\mathbf{z}|_f \in \mathbf{Q}_p(f)$ *for all* $f \in \mathcal{F}(\widehat{K})$.

(ii) There holds $\|\boldsymbol{\mathcal{L}}^{\mathrm{curl},3d}\mathbf{z}\|_{\mathbf{H}(\widehat{K}, \mathbf{curl})} \leq C \|\mathbf{z}\|_{\mathbf{X}^{-1/2}}$.

(iii) There holds the orthogonality $(\boldsymbol{\mathcal{L}}^{\mathrm{curl},3d}\mathbf{z}, \nabla v)_{L^2(\widehat{K})} = 0$ *for all* $v \in \mathring{W}_{p+1}(\widehat{K})$.

(iv) Let $\mathbf{T} := \Pi_\tau \mathbf{H}^2(\widehat{K})$. *A function* $\mathbf{z} \in \mathbf{T}$ *is in* $L^2(\partial\widehat{K})$ *and facewise in* $\mathbf{H}^{3/2}$. *Its surface curl,* $\mathrm{curl}_{\partial\widehat{K}}\, \mathbf{z}$, *is an* $L^2(\widehat{K})$*-function, which coincides with the facewise curl* $\mathrm{curl}_f\, \mathbf{z}$. *Furthermore, there holds*

$$\|\mathbf{z}\|_{\mathbf{X}^{-1/2}} \leq C \sum_{f \in \mathcal{F}(\widehat{K})} \left[\|\mathbf{z}\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} + \|\,\mathrm{curl}_f\, \mathbf{z}\|_{\widetilde{H}^{-1/2}(f)} \right].$$

Here, we recall that $\widetilde{\mathbf{H}}_T^{-1/2}(f)$ *is the dual space of the space* $\mathbf{H}_T^{1/2}(f)$ *of tangential fields.*

Proof. The lifting operator $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ constructed in [20] has the desired polynomial preserving property (i) and continuity property (ii), [20, Thm. 7.2]. Our goal is to define the desired lifting operator by $\boldsymbol{\mathcal{L}}^{\mathrm{curl},3d}\mathbf{z} := \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z} - \mathbf{w}_0$, where $\mathbf{w}_0$ is defined by the following saddle point problem: Find $\mathbf{w}_0 \in \mathring{\mathbf{Q}}_p(\widehat{K})$ and $\varphi \in \mathring{W}_{p+1}(\widehat{K})$ such that

$$(\mathbf{curl}\, \mathbf{w}_0, \mathbf{curl}\, \mathbf{q})_{L^2(\widehat{K})} + (\mathbf{q}, \nabla\varphi)_{L^2(\widehat{K})} = (\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}), \mathbf{curl}\, \mathbf{q})_{L^2(\widehat{K})} \qquad \forall \mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K}) \tag{5.17a}$$

$$(\mathbf{w}_0, \nabla\mu)_{L^2(\widehat{K})} = (\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}, \nabla\mu)_{L^2(\widehat{K})} \qquad \forall \mu \in \mathring{W}_{p+1}(\widehat{K}). \tag{5.17b}$$

Problem (5.17) is uniquely solvable: Define the bilinear forms $a(\mathbf{w}, \mathbf{q}) := (\mathbf{curl}\, \mathbf{w}, \mathbf{curl}\, \mathbf{q})_{L^2(\widehat{K})}$ and $b(\mathbf{w}, \varphi) := (\mathbf{w}, \nabla\varphi)_{L^2(\widehat{K})}$ for $\mathbf{w}, \mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K})$ and $\varphi \in \mathring{W}_{p+1}(\widehat{K})$. Coercivity of a on the kernel of b with $\ker b = \{\mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K})\colon (\mathbf{q}, \nabla\mu)_{L^2(\widehat{K})} = 0\, \forall \mu \in \mathring{W}_{p+1}\} = \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$, follows from the Friedrichs inequality (Lemma 5.7) by

$$a(\mathbf{v}, \mathbf{v}) = \|\,\mathbf{curl}\, \mathbf{v}\|^2_{L^2(\widehat{K})} \geq \frac{1}{2C^2}\|\mathbf{v}\|^2_{L^2(\widehat{K})} + \frac{1}{2}\|\,\mathbf{curl}\, \mathbf{v}\|^2_{L^2(\widehat{K})} \geq \min\{\frac{1}{2C^2}, \frac{1}{2}\}\|\mathbf{v}\|^2_{\mathbf{H}(\widehat{K}, \mathbf{curl})}$$

for all $\mathbf{v} \in \ker b$. Next, we show the inf-sup condition

$$\inf_{\varphi \in \mathring{W}_{p+1}(\widehat{K})} \sup_{\mathbf{w} \in \mathring{\mathbf{Q}}_p(\widehat{K})} \frac{b(\mathbf{w}, \varphi)}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K}, \mathbf{curl})} \|\varphi\|_{H^1(\widehat{K})}}.$$

Given $\varphi \in \mathring{W}_{p+1}(\widehat{K})$, choose $\mathbf{w} = \nabla\varphi \in \mathring{\mathbf{Q}}_p(\widehat{K})$. Hence,

$$\frac{b(\mathbf{w},\varphi)}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\|\varphi\|_{H^1(\widehat{K})}} = \frac{\|\nabla\varphi\|^2_{L^2(\widehat{K})}}{\|\nabla\varphi\|_{L^2(\widehat{K})}\|\varphi\|_{H^1(\widehat{K})}} \geq C$$

by Poincaré's inequality. Thus, the saddle point problem (5.17) has a unique solution $(\mathbf{w}_0,\varphi) \in \mathring{\mathbf{Q}}_p(\widehat{K}) \times \mathring{W}_{p+1}(\widehat{K})$. In fact, taking $\mathbf{q} = \nabla\varphi$ in (5.17a) reveals $\varphi = 0$. The lifting operator $\boldsymbol{\mathcal{L}}^{\text{curl},3d}$ now obviously satisfies (i) and (iii) by construction. For (ii) note that the solution $\mathbf{w}_0$ satisfies the estimate $\|\mathbf{w}_0\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|f\|+\|g\|$, where $f(\mathbf{v}) = (\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\text{curl}}\mathbf{z}),\mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}$, $g(v) = (\boldsymbol{\mathcal{E}}^{\text{curl}}\mathbf{z},\nabla v)_{L^2(\widehat{K})}$, and $\|\cdot\|$ denotes the operator norm. Thus,

$$\|f\| = \sup_{\|\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\leq 1} |(\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\text{curl}}\mathbf{z}),\mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}| \leq \|\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\text{curl}}\mathbf{z})\|_{L^2(\widehat{K})} \lesssim \|\mathbf{z}\|_{\mathbf{X}^{-1/2}}.$$

The estimate $\|g\| \lesssim \|\mathbf{z}\|_{\mathbf{X}^{-1/2}}$ is shown in a similar way. Hence, (ii) follows from

$$\|\boldsymbol{\mathcal{L}}^{\text{curl},3d}\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \leq \|\boldsymbol{\mathcal{E}}^{\text{curl}}\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\mathbf{w}_0\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|\mathbf{z}\|_{\mathbf{X}^{-1/2}}.$$

We now show (iv), proceeding several steps.
1. step: Clearly, $\mathbf{z}$ is in $L^2(\partial\widehat{K})$ and facewise in $\mathbf{H}^{3/2}_T$. The surface curl, $\text{curl}_{\partial\widehat{K}}\,\mathbf{z}$, of $\mathbf{z} \in \mathbf{T}$ is defined by $\mathbf{n}\cdot\mathbf{curl}\,\widetilde{\mathbf{z}} \in H^{-1/2}(\partial\widehat{K})$ for any lifting $\widetilde{\mathbf{z}} \in \mathbf{H}(\widehat{K},\mathbf{curl})$ of $\mathbf{z}$. This definition is indeed independent of the lifting since the difference $\boldsymbol{\delta}$ of two liftings is in $\mathbf{H}_0(\widehat{K},\mathbf{curl})$ and by the deRham diagram (see, e.g., [25, eqn. (3.60)]) we then have $\mathbf{curl}\,\boldsymbol{\delta} \in \mathbf{H}_0(\widehat{K},\text{div})$. Furthermore, since an $\mathbf{H}^2$-lifting of $\mathbf{z}$ exists, $\text{curl}_{\partial\widehat{K}}\,\mathbf{z} \in H^{-1/2}(\partial\widehat{K})$ is facewise in $\mathbf{H}^{1/2}_T$ and coincides facewise with $\text{curl}_f\,\mathbf{z}$.
2. step: We construct a particular lifting $\mathbf{Z} \in \mathbf{H}(\widehat{K},\mathbf{curl})$ of $\mathbf{z} \in \mathbf{X}^{-1/2}$ and will use $\|\mathbf{z}\|_{\mathbf{X}^{-1/2}} \leq \|\mathbf{Z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}$. This lifting $\mathbf{Z}$ is taken to be the solution of the following (constrained) minimization problem:

$$\text{Minimize } \|\mathbf{curl}\,\mathbf{Y}\|_{L^2(\widehat{K})} \text{ under the constraints } \Pi_\tau\mathbf{Y} = \mathbf{g} \text{ and } (\mathbf{Y},\nabla\varphi)_{L^2(\widehat{K})} = 0 \text{ for all } \varphi \in H^1_0(\widehat{K}). \quad (5.18)$$

This minimization problem can be solved with the method of Lagrange multipliers as was done in (5.17). Without repeating the arguments, one obtains, in strong form, the problem: Find $(\mathbf{Z},\varphi) \in \mathbf{H}(\widehat{K},\mathbf{curl})\times H^1_0(\widehat{K})$ such that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{Z} + \nabla\varphi = 0 \quad \text{in } \widehat{K}, \qquad \Pi_\tau\mathbf{Z} = \mathbf{z}.$$

As was observed above, the Lagrange multiplier φ in fact vanishes so that we conclude that the minimizer $\mathbf{Z}$ solves

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{Z} = 0, \qquad \text{div}\,\mathbf{Z} = 0, \qquad \Pi_\tau\mathbf{Z} = \mathbf{g}.$$

3. step: We bound $\mathbf{w} := \mathbf{curl}\,\mathbf{Z}$. We have

$$\mathbf{curl}\,\mathbf{w} = 0, \qquad \text{div}\,\mathbf{w} = 0, \qquad \mathbf{n}\cdot\mathbf{w} = \text{curl}_{\partial\widehat{K}}\,\mathbf{z}. \quad (5.19)$$

From $\mathbf{curl}\,\mathbf{w} = 0$, we get that $\mathbf{w}$ is a gradient: $\mathbf{w} = \nabla\psi$. The second and third conditions in (5.19) show

$$-\Delta\psi = 0 \qquad \partial_n\psi = \mathbf{n}\cdot\mathbf{w} = \text{curl}_{\partial\widehat{K}}\,\mathbf{z}.$$

Noting that the integrability condition is satisfied since $(\mathbf{n}\cdot\mathbf{w},1)_{L^2(\partial\widehat{K})} = (\text{div}\,\mathbf{w},1)_{L^2(\widehat{K})} = 0$, we conclude by standard *a priori* estimates for the Laplace problem

$$\|\mathbf{curl}\,\mathbf{Z}\|_{L^2(\widehat{K})} = \|\mathbf{w}\|_{L^2(\widehat{K})} = \|\nabla\psi\|_{L^2(\widehat{K})} \lesssim \|\text{curl}_{\partial\widehat{K}}\,\mathbf{g}\|_{H^{-1/2}(\partial\widehat{K})}. \quad (5.20)$$

4. step: To bound $\mathbf{Z}$, we write it with the operators $\mathbf{R}^{\text{curl}}$ and R^{grad} of Lemma 5.4 as

$$\mathbf{Z} = \nabla\phi + \widetilde{\mathbf{z}}, \qquad \widetilde{\mathbf{z}} := \mathbf{R}^{\text{curl}}(\mathbf{curl}\,\mathbf{Z}), \qquad \phi := R^{\text{grad}}(\mathbf{Z} - \mathbf{R}^{\text{curl}}(\mathbf{curl}\,\widetilde{\mathbf{z}})), \quad (5.21)$$

$$\text{with } \|\widetilde{\mathbf{z}}\|_{H^1(\widehat{K})} \lesssim \|\mathbf{curl}\,\mathbf{Z}\|_{L^2(\widehat{K})} \lesssim \|\text{curl}_{\partial\widehat{K}}\,\mathbf{z}\|_{H^{-1/2}(\partial\widehat{K})}. \quad (5.22)$$

For the control of ϕ, proceed by an integration by parts argument. Noting that $\text{div}\,\mathbf{Z} = 0$, we have

$$\nabla\phi + \widetilde{\mathbf{z}} = \mathbf{Z} = \mathbf{curl}\,\mathbf{R}^{\text{curl}}(\mathbf{Z}) = \mathbf{curl}\,\mathbf{R}^{\text{curl}}(\nabla\phi) + \mathbf{curl}\,\mathbf{R}^{\text{curl}}(\widetilde{\mathbf{z}}).$$

With the integration by parts formula (2.1) (which is actually valid for functions in $\mathbf{H}(\widehat{K}, \mathbf{curl})$ as shown in [25, Thm. 3.29]) we get

$$(\mathbf{curl}\,\mathbf{Z}, \mathbf{v})_{L^2(\widehat{K})} \overset{(2.1)}{=} (\mathbf{Z}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} - \langle \mathbf{z}, \gamma_\tau \mathbf{v} \rangle_{\partial \widehat{K}}.$$

Selecting $\mathbf{v} = \mathbf{R}^{\mathrm{curl}}(\nabla\phi) \in \mathbf{H}^1(\widehat{K})$

$$(\mathbf{curl}\,\mathbf{Z}, \mathbf{R}^{\mathrm{curl}}(\nabla\phi))_{L^2(\widehat{K})} = (\nabla\phi + \widetilde{\mathbf{z}}, \nabla\phi + \widetilde{\mathbf{z}} - \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\widetilde{\mathbf{z}}))_{L^2(\widehat{K})} + \langle \mathbf{z}, \gamma_\tau \mathbf{R}^{\mathrm{curl}}(\nabla\phi) \rangle_{\partial\widehat{K}}.$$

In view of the mapping property $\mathbf{R}^{\mathrm{curl}} : \mathbf{L}^2(\widehat{K}) \to \mathbf{H}^1(\widehat{K})$

$$\begin{aligned} \|\nabla\phi\|^2_{L^2(\widehat{K})} &\lesssim \|\,\mathbf{curl}\,\mathbf{Z}\|_{L^2(\widehat{K})} \|\nabla\phi\|_{L^2(\widehat{K})} + \|\widetilde{\mathbf{z}}\|_{L^2(\widehat{K})} \|\widetilde{\mathbf{z}} - \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}\widetilde{\mathbf{z}}\|_{L^2(\widehat{K})} \\ &\quad + \|\widetilde{\mathbf{z}} - \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\widetilde{\mathbf{z}})\|_{L^2(\widehat{K})} \|\nabla\phi\|_{L^2(\widehat{K})} + \|\widetilde{\mathbf{z}}\|_{L^2(\widehat{K})} \|\nabla\phi\|_{L^2(\widehat{K})} + \left|\langle \mathbf{z}, \gamma_\tau \mathbf{R}^{\mathrm{curl}}(\nabla\phi)\rangle_{\partial\widehat{K}}\right|. \end{aligned} \tag{5.23}$$

Combining (5.21), (5.22), (5.23) shows

$$\|\mathbf{Z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|\widetilde{\mathbf{z}}\|_{L^2(\widehat{K})} + \|\nabla\phi\|_{L^2(\widehat{K})} + \|\,\mathbf{curl}\,\mathbf{Z}\|_{L^2(\widehat{K})} \lesssim \sup_{\mathbf{v}\in\mathbf{H}^1(\widehat{K})} \frac{\langle \mathbf{z}, \gamma_\tau \mathbf{v}\rangle_{\partial\widehat{K}}}{\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}} + \|\,\mathrm{curl}_{\partial\widehat{K}}\,\mathbf{z}\|_{H^{-1/2}(\partial\widehat{K})}. \tag{5.24}$$

5. step: Since $\mathbf{z}$ and $\mathrm{curl}_{\partial\widehat{K}}\,\mathbf{z}$ are actually L^2-functions, the norm $\|\cdot\|_{\mathbf{X}^{-1/2}}$ can be estimated in a localized fashion: The continuity of the inclusions $H^{1/2}(\partial\widehat{K}) \subset \prod_{f\in\mathcal{F}(\widehat{K})} H^{1/2}(f)$ and $\gamma_\tau \mathbf{H}^1(\widehat{K}) \subset \prod_{f\in\mathcal{F}(\widehat{K})} \mathbf{H}^{1/2}(f)$ implies

$$\|\,\mathrm{curl}_{\partial\widehat{K}}\,\mathbf{z}\|_{H^{-1/2}(\partial\widehat{K})} \lesssim \sum_{f\in\mathcal{F}(\widehat{K})} \|\,\mathrm{curl}_f\,\mathbf{z}\|_{\widetilde{H}^{-1/2}(f)}, \qquad \sup_{\mathbf{v}\in\mathbf{H}^1(\widehat{K})} \frac{\langle \mathbf{z}, \gamma_\tau \mathbf{v}\rangle_{\partial\widehat{K}}}{\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}} \lesssim \sum_{f\in\mathcal{F}(\widehat{K})} \|\mathbf{z}\|_{\widetilde{\mathbf{H}}^{-1/2}(f)}. \tag{5.25}$$

We finally obtain the desired estimate

$$\|\mathbf{z}\|_{\mathbf{X}^{-1/2}} \lesssim \|\mathbf{Z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \overset{(5.24),\,(5.25)}{\lesssim} \sum_{f\in\mathcal{F}(\widehat{K})} \|\mathbf{z}\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} + \|\,\mathrm{curl}_f\,\mathbf{z}\|_{\widetilde{H}^{-1/2}(f)}.$$

This concludes the proof. We mention that an alternative proof of the assertion (iv) could be based on the intrinsic characterization of the trace spaces of $\mathbf{H}(\widehat{K}, \mathbf{curl})$ given in [9, 10]. ∎

Theorem 5.12. *There exists $C > 0$ independent of p such that for all $\mathbf{u} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \le Cp^{-1} \inf_{\mathbf{v}\in\mathbf{Q}_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}. \tag{5.26}$$

Proof. 1. step: Since $\widehat{\Pi}_p^{\mathrm{curl},3d}$ is projection operator, it suffices to show the bound with $\mathbf{v} = 0$ in the infimum.
2. step: Write, with the operators R^{grad}, $\mathbf{R}^{\mathrm{curl}}$ of Lemma 5.4, the function $\mathbf{u} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$ as $\mathbf{u} = \nabla\varphi + \mathbf{v}$ with $\varphi \in H^2(\widehat{K})$ and $\mathbf{v} \in \mathbf{H}^2(\widehat{K})$. We have $\|\varphi\|_{H^2(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}$ and $\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})} \lesssim \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^1(\widehat{K})}$. From the commuting diagram property, we readily get

$$\|\nabla\varphi - \widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} = \|\nabla(\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi)\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} = |\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi|_{H^1(\widehat{K})} \overset{\text{Thm. 5.10}}{\lesssim} p^{-1}\|\varphi\|_{H^2(\widehat{K})}.$$

3. step: We claim

$$\|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\mathbf{X}^{-1/2}} \le Cp^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})}. \tag{5.27}$$

To see this, we note $\mathbf{v} \in \mathbf{H}^2(\widehat{K})$ and estimate with Lemma 5.11

$$\|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\mathbf{X}^{-1/2}} \lesssim \sum_{f\in\mathcal{F}(\widehat{K})} \|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} + \|\,\mathrm{curl}_f(\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v}))\|_{\widetilde{H}^{-1/2}(f)}.$$

We consider each face $f \in \mathcal{F}(\widehat{K})$ separately. Lemmas 4.9, 4.10, 4.13 imply with the aid of the continuity of the trace $\Pi_\tau : \mathbf{H}^2(\widehat{K}) \to \mathbf{H}^{3/2}_T(f) \subset \mathbf{H}^{1/2}(f, \operatorname{curl})$

$$
\begin{aligned}
\|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v})\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} &\lesssim p^{-1/2}\|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v})\|_{\mathbf{H}(f,\operatorname{curl})} \\
&\lesssim p^{-1/2-1/2}(\|\Pi_\tau \mathbf{v}\|_{\mathbf{H}^{1/2}(f,\operatorname{curl})} + \|\Pi_\tau \mathbf{v}\|_{\mathbf{H}^1(f)}) \lesssim p^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})}, \\
\|\operatorname{curl}(\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v}))\|_{\widetilde{H}^{-1/2}(f)} &\lesssim p^{-1/2}\|\operatorname{curl}(\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v}))\|_{L^2(f)} \\
&\lesssim p^{-1/2-1/2}(\|\Pi_\tau \mathbf{v}\|_{\mathbf{H}^{1/2}(f,\operatorname{curl})} + \|\Pi_\tau \mathbf{v}\|_{\mathbf{H}^1(f)}) \lesssim p^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})}.
\end{aligned}
$$

4. step: Since $\mathbf{v} \in \mathbf{H}^2(\widehat{K})$, the approximation $P^{\operatorname{curl},3d}\mathbf{v} \in \mathbf{Q}_p(\widehat{K})$ given by Lemma 5.2 satisfies

$$
\|\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \leq Cp^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})}. \tag{5.28}
$$

We note

$$
\begin{aligned}
\|\mathbf{v} - \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} &\leq \|\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \\
&\leq p^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})} + \|\widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}.
\end{aligned}
$$

For the term $\|\widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}$, we introduce the abbreviation $\mathbf{E} := \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v} \in \mathbf{Q}_p(\widehat{K})$ and observe that the orthogonality conditions (2.15a), (2.15b) satisfied by $\widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v}$ and the conditions (5.2a), (5.2b) satisfied by $P^{\operatorname{curl},3d}\mathbf{v}$, lead to two orthogonalities:

$$
(\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{w} \in \mathring{\mathbf{Q}}_p(\widehat{K}), \qquad (\mathbf{E}, \nabla w)_{L^2(\widehat{K})} = 0 \quad \forall w \in \mathring{W}_{p+1}(\widehat{K}). \tag{5.29}
$$

By Lemma 5.11, the orthogonality condition

$$
(\boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}\Pi_\tau \mathbf{E}, \nabla w)_{L^2(\widehat{K})} = 0 \quad \forall w \in \mathring{W}_{p+1}(\widehat{K})
$$

holds. Hence, the discrete Friedrichs inequality of Lemma 5.7 is applicable to $\mathbf{E} - \boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}\mathbf{E}$, and we get

$$
\begin{aligned}
\|\mathbf{E}\|_{L^2(\widehat{K})} &\leq \|\boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}\Pi_\tau \mathbf{E}\|_{L^2(\widehat{K})} + \|\mathbf{E} - \boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}\Pi_\tau \mathbf{E}\|_{L^2(\widehat{K})} \\
&\lesssim \|\boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}\Pi_\tau \mathbf{E}\|_{L^2(\widehat{K})} + \|\mathbf{curl}(\mathbf{E} - \boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}\Pi_\tau \mathbf{E})\|_{L^2(\widehat{K})} \lesssim \|\boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}\Pi_\tau \mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\mathbf{curl}\,\mathbf{E}\|_{L^2(\widehat{K})} \\
&\lesssim \|\Pi_\tau \mathbf{E}\|_{\mathbf{X}^{-1/2}} + \|\mathbf{curl}\,\mathbf{E}\|_{L^2(\widehat{K})}.
\end{aligned} \tag{5.30}
$$

Using again the lifting $\boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}$ of Lemma 5.11 and the first orthogonality of (5.29), we get

$$
\|\mathbf{curl}\,\mathbf{E}\|_{L^2(\widehat{K})} \leq \|\mathbf{curl}\,\boldsymbol{\mathcal{L}}^{\operatorname{curl},3d}\Pi_\tau \mathbf{E}\|_{L^2(\widehat{K})} \lesssim \|\Pi_\tau \mathbf{E}\|_{\mathbf{X}^{-1/2}}. \tag{5.31}
$$

We conclude the proof by observing

$$
\begin{aligned}
\|\mathbf{v} - \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} &\leq \|\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \\
&\overset{(5.30),(5.31)}{\lesssim} \|\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\Pi_\tau \mathbf{E}\|_{\mathbf{X}^{-1/2}} \\
&\lesssim \|\mathbf{v} - P^{\operatorname{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{v})\|_{\mathbf{X}^{-1/2}} \overset{(5.28),(5.27)}{\lesssim} p^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})}.
\end{aligned}
$$
□

For negative norm estimates $\|\mathbf{u} - \widehat{\Pi}_p^{\operatorname{curl},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})}$ with $s \geq 0$ we need Helmholtz decompositions:

Lemma 5.13 (Helmholtz decomposition)**.** *A function $\mathbf{v} \in \mathbf{H}^1(\widehat{K})$ can be written as*

$$
\mathbf{v} = \nabla\varphi_0 + \mathbf{curl}\,\mathbf{curl}\,\mathbf{z}_0, \tag{5.32}
$$

$$
\mathbf{v} = \nabla\varphi_1 + \mathbf{curl}\,\mathbf{z}_1, \tag{5.33}
$$

where $\varphi_0 \in H^2(\widehat{K}) \cap H^1_0(\widehat{K})$ and $\mathbf{z}_0 \in \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ and where $\varphi_1 \in H^2(\widehat{K})$ and $\mathbf{z}_1 \in \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ together with the estimates

$$
\begin{aligned}
\|\varphi_0\|_{H^2(\widehat{K})} + \|\mathbf{z}_0\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} &\leq C\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}, \\
\|\varphi_1\|_{H^2(\widehat{K})} + \|\mathbf{z}_1\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} &\leq C\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.
\end{aligned}
$$

Proof. Before proving these decompositions, we recall the continuous embeddings

$$\mathbf{H}_0(\widehat{K},\mathbf{curl})\cap\mathbf{H}(\widehat{K},\operatorname{div})\subset\mathbf{H}^1(\widehat{K})\quad\text{ and }\quad\mathbf{H}(\widehat{K},\mathbf{curl})\cap\mathbf{H}_0(\widehat{K},\operatorname{div})\subset\mathbf{H}^1(\widehat{K}),\tag{5.34}$$

which hinge on the convexity of $\widehat{K}$ (see [7, 28] and the discussion in [25, Rem. 3.48]).
We construct the decomposition (5.33): We define $\varphi_1\in H^1(\widehat{K})$ as the solution of

$$-\Delta\varphi_1=-\operatorname{div}\mathbf{v}\quad\text{ in }\widehat{K},\qquad\partial_n\varphi_1=\mathbf{n}\cdot\mathbf{v}\quad\text{ on }\partial\widehat{K}.$$

The contribution $\mathbf{z}_1$ is defined by the saddle point problem: Find $(\mathbf{z}_1,\psi)\in\mathbf{H}_0(\widehat{K},\mathbf{curl})\times H^1_0(\widehat{K})$ such that

$$\begin{aligned}(\mathbf{curl}\,\mathbf{z}_1,\mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})}-(\nabla\psi,\mathbf{w})_{L^2(\widehat{K})}&=(\mathbf{curl}\,\mathbf{v},\mathbf{w})_{L^2(\widehat{K})}\qquad\forall\mathbf{w}\in\mathbf{H}_0(\widehat{K},\mathbf{curl}),\\ (\mathbf{z}_1,\nabla q)_{L^2(\widehat{K})}&=0\qquad\forall q\in H^1_0(\widehat{K}).\end{aligned}$$

This problem is uniquely solvable, we have $\psi=0$ (since $\operatorname{div}\mathbf{curl}\,\mathbf{v}=0$) and the *a priori* estimate

$$\|\mathbf{z}_1\|_{\mathbf{H}(\mathbf{curl},\widehat{K})}\lesssim\|\,\mathbf{curl}\,\mathbf{v}\|_{L^2(\widehat{K})}\lesssim\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.$$

(In the proof of Lemma 5.11, we considered a similar problem in a discrete setting; here, the appeal to the discrete Friedrichs inequality of Lemma 5.7 needs to replaced with that to the continuous one, [25, Cor. 3.51]) From $\operatorname{div}\mathbf{z}_1=0$ and (5.34), we furthermore infer $\|\mathbf{z}_1\|_{\mathbf{H}^1(\widehat{K})}\lesssim\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}$. The representation (5.33) is obtained from the observation that the difference $\boldsymbol{\delta}:=\mathbf{v}-\nabla\varphi_1-\mathbf{curl}\,\mathbf{z}_1$ satisfies, by construction, $\operatorname{div}\boldsymbol{\delta}=0$, $\mathbf{curl}\,\boldsymbol{\delta}=0$, $\mathbf{n}\cdot\boldsymbol{\delta}=(\mathbf{n}\cdot\mathbf{v}-\partial_n\varphi_1)-\mathbf{n}\cdot\mathbf{curl}\,\mathbf{z}_1=0-\operatorname{curl}_{\partial\widehat{K}}\Pi_\tau\mathbf{z}_1=0-0=0$ so that again (5.34) (specifically, in the form [28, Thm. 4.1]) implies $\boldsymbol{\delta}=0$. Finally, from $\mathbf{v}\in\mathbf{H}^1(\widehat{K})$, $\varphi_1\in H^2(\widehat{K})$ and the representation (5.33), we infer $\mathbf{curl}\,\mathbf{z}_1\in\mathbf{H}^1(\widehat{K})$.
We construct the decomposition (5.32): We define $\varphi_0\in H^1_0(\widehat{K})$ as the solution of

$$-\Delta\varphi_0=-\operatorname{div}\mathbf{v}\quad\text{ in }\widehat{K},\qquad\varphi_0=0\quad\text{ on }\partial\widehat{K}.$$

Next, we define $(\mathbf{z}_0,\psi)\in\mathbf{H}_0(\widehat{K},\mathbf{curl})\times H^1_0(\widehat{K})$ as the solution of the saddle point problem

$$\begin{aligned}(\mathbf{curl}\,\mathbf{z}_0,\mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})}-(\nabla\psi,\mathbf{w})_{L^2(\widehat{K})}&=(\mathbf{v}-\nabla\varphi_0,\mathbf{w})_{L^2(\widehat{K})}\qquad\forall\mathbf{w}\in\mathbf{H}_0(\widehat{K},\mathbf{curl}),\\ (\mathbf{z}_0,\nabla q)_{L^2(\widehat{K})}&=0\qquad\forall q\in H^1_0(\widehat{K}).\end{aligned}$$

Again, this problem is uniquely solvable and, in fact $\psi=0$ (since $\operatorname{div}(\mathbf{v}-\nabla\varphi_0)=0$). We have $\|\mathbf{z}_0\|_{\mathbf{H}(\mathbf{curl},\widehat{K})}\lesssim\|\mathbf{v}-\nabla\varphi_0\|_{L^2(\widehat{K})}\lesssim\|\mathbf{v}\|_{L^2(\widehat{K})}$. Since $\operatorname{div}\mathbf{z}_0=0$, we get from (5.34) that $\|\mathbf{z}_0\|_{\mathbf{H}^1(\widehat{K})}\lesssim\|\mathbf{v}\|_{L^2(\widehat{K})}$. Finally, an integration by parts reveals

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{z}_0=\mathbf{v}-\nabla\varphi_0,$$

which is representation (5.32). ☐

We control the approximation error in negative Sobolev norms.

Theorem 5.14. *For $s\in[0,1]$ and all $\mathbf{u}\in\mathbf{H}^1(\widehat{K},\mathbf{curl})$ there holds the estimate*

$$\|\mathbf{u}-\widehat{\Pi}^{\mathrm{curl},3d}_p\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})}\le C_s p^{-(1+s)}\inf_{\mathbf{v}\in\mathbf{Q}_p}\|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}.$$

Proof. By the familiar argument that $\widehat{\Pi}^{\mathrm{curl},3d}_p$ is a projection, we may restrict the proof to the case $\mathbf{v}=0$ in the infimum. The case $s=0$ is covered by Theorem 5.12. In the remainder the proof, we will show the case $s=1$ as the case $s\in(0,1)$ then follows by interpolation.
We write $\mathbf{E}:=\mathbf{u}-\widehat{\Pi}^{\mathrm{curl},3d}_p\mathbf{u}$ for simplicity. By definition we have

$$\|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-1}(\widehat{K},\mathbf{curl})}\sim\|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-1}(\widehat{K})}+\|\,\mathbf{curl}\,\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-1}(\widehat{K})}=\sup_{\mathbf{v}\in\mathbf{H}^1(\widehat{K})}\frac{(\mathbf{E},\mathbf{v})_{L^2(\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}}+\sup_{\mathbf{v}\in\mathbf{H}^1(\widehat{K})}\frac{(\mathbf{curl}\,\mathbf{E},\mathbf{v})_{L^2(\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}}\tag{5.35}$$

We start with estimating the first supremum in (5.35). According to Lemma 5.13, any $\mathbf{v}\in\mathbf{H}^1(\widehat{K})$ can be decomposed as

$$\mathbf{v}=\nabla\varphi+\mathbf{curl}\,\mathbf{curl}\,\mathbf{z}$$

with $\varphi \in H^2(\widehat{K}) \cap H^1_0(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$. We also observe $\mathbf{curl}\,\mathbf{z} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$, thus by Lemma 5.5 we can further decompose $\mathbf{curl}\,\mathbf{z}$ as

$$\mathbf{curl}\,\mathbf{z} = \nabla\varphi_2 + \mathbf{z}_2 \tag{5.36}$$

with $\varphi_2 \in H^2(\widehat{K})$ and $\mathbf{z}_2 \in \mathbf{H}^2(\widehat{K})$. We estimate each term in the decomposition $(\mathbf{E}, \mathbf{v})_{L^2(\widehat{K})} = (\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})} + (\mathbf{E}, \mathbf{curl}\,\mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})}$ separately. Using the orthogonality condition (2.15b) and Theorem 5.12, we get

$$\begin{aligned}(\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})} &= \inf_{w \in \mathring{W}_{p+1}(\widehat{K})} (\mathbf{E}, \nabla(\varphi - w))_{L^2(\widehat{K})} \lesssim p^{-1} \|\varphi\|_{H^2(\widehat{K})} \|\mathbf{E}\|_{L^2(\widehat{K})} \\ &\lesssim p^{-1} \|\mathbf{v}\|_{H^1(\widehat{K})} \|\mathbf{E}\|_{\mathbf{H}(\widehat{K}, \mathbf{curl})} \lesssim p^{-2} \|\mathbf{v}\|_{H^1(\widehat{K})} \|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})}.\end{aligned} \tag{5.37}$$

Integration by parts and (5.36) give

$$\begin{aligned}(\mathbf{E}, \mathbf{curl}\,\mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} &= (\mathbf{E}, \mathbf{curl}\,\mathbf{z}_2)_{L^2(\widehat{K})} = (\mathbf{curl}\,\mathbf{E}, \mathbf{z}_2)_{L^2(\widehat{K})} + (\Pi_\tau \mathbf{E}, \gamma_\tau \mathbf{z}_2)_{L^2(\partial\widehat{K})} \\ &= (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} - (\mathbf{curl}\,\mathbf{E}, \nabla\varphi_2)_{L^2(\widehat{K})} + (\Pi_\tau \mathbf{E}, \gamma_\tau \mathbf{z}_2)_{L^2(\partial\widehat{K})} \\ &= (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} - (\mathrm{curl}_{\partial\widehat{K}}\, \Pi_\tau \mathbf{E}, \varphi_2)_{L^2(\widehat{K})} + (\Pi_\tau \mathbf{E}, \gamma_\tau \mathbf{z}_2)_{L^2(\partial\widehat{K})}.\end{aligned} \tag{5.38}$$

We estimate these three terms separately. For the first term in (5.38), we use the orthogonality (2.15a) and Theorem 5.12 to get

$$\begin{aligned}(\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} &= \inf_{\mathbf{w} \in \mathring{\mathbf{Q}}_p(\widehat{K})} (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}(\mathbf{z} - \mathbf{w}))_{L^2(\widehat{K})} \lesssim p^{-1} \|\,\mathbf{curl}\,\mathbf{E}\|_{L^2(\widehat{K})} \|\mathbf{z}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})} \\ &\lesssim p^{-1} \|\mathbf{v}\|_{H^1(\widehat{K})} \|\mathbf{E}\|_{\mathbf{H}(\widehat{K}, \mathbf{curl})} \lesssim p^{-2} \|\mathbf{v}\|_{H^1(\widehat{K})} \|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})}.\end{aligned} \tag{5.39}$$

For the second term in (5.38), we note that $\Pi_\tau \mathbf{E}$ is sufficiently regular on $\partial\widehat{K}$ to split the integral over $\partial\widehat{K}$ into a sum of face contributions. We get for each face contribution, using Lemmata 4.10 and 4.13,

$$\begin{aligned}\left|(\mathrm{curl}_f\, \Pi_\tau \mathbf{E}, \varphi_2)_{L^2(f)}\right| &\overset{\text{Lem. 4.10}}{\lesssim} p^{-3/2} \|\,\mathrm{curl}_f\, \Pi_\tau \mathbf{E}\|_{L^2(f)} \|\varphi_2\|_{H^{3/2}(f)} \\ &\overset{\text{Lem. 4.13}}{\lesssim} p^{-2} \|\Pi_\tau \mathbf{u}\|_{\mathbf{H}^{1/2}(\mathrm{curl}, f)} \|\varphi_2\|_{H^2(\widehat{K})} \lesssim p^{-2} \|\mathbf{u}\|_{\mathbf{H}^1(\mathbf{curl}, \widehat{K})} \|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.\end{aligned} \tag{5.40}$$

Finally, for the third term in (5.38) we infer with Lemmata 4.9, 4.13

$$\begin{aligned}(\Pi_\tau \mathbf{E}, \gamma_\tau \mathbf{z}_2)_{L^2(f)} &\overset{\text{Lem. 4.9}}{\lesssim} p^{-3/2} \|\Pi_\tau \mathbf{E}\|_{\mathbf{H}(f, \mathrm{curl})} \|\gamma_\tau \mathbf{z}_2\|_{\mathbf{H}^{3/2}(f)}, \\ &\overset{\text{Lem. 4.13}}{\lesssim} p^{-2} \|\Pi_\tau \mathbf{u}\|_{\mathbf{H}^{1/2}(f, \mathrm{curl})} \|\mathbf{z}_2\|_{\mathbf{H}^2(\widehat{K})} \lesssim p^{-2} \|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})} \|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.\end{aligned} \tag{5.41}$$

Adding (5.40) and (5.41) over all faces and taking note of (5.39) shows that we estimate the first supremum (5.35) in the desired fashion.

We turn to estimating the second supremum in (5.35). We start with decomposing $\mathbf{v} \in \mathbf{H}^1(\widehat{K})$ as

$$\mathbf{v} = \nabla\varphi + \mathbf{curl}\,\mathbf{z}$$

with $\varphi \in H^2(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ according to Lemma 5.13. Thus we have to control $(\mathbf{curl}\,\mathbf{E}, \mathbf{v})_{L^2(\widehat{K})} = (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} + (\mathbf{curl}\,\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})}$. Using the orthogonality condition (2.15a) and Theorem 5.12, the first term is estimated by

$$\begin{aligned}(\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} &= \inf_{\mathbf{w} \in \mathring{\mathbf{Q}}_p(\widehat{K})} (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}(\mathbf{z} - \mathbf{w}))_{L^2(\widehat{K})} \lesssim p^{-1} \|\mathbf{E}\|_{\mathbf{H}(\widehat{K}, \mathbf{curl})} \|\mathbf{z}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})} \\ &\lesssim p^{-2} \|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})} \|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})}.\end{aligned}$$

Concerning the second term, an integration by parts yields in view of $\mathrm{curl}_f\, \Pi_\tau \mathbf{E} = \mathbf{n} \cdot \mathbf{curl}\,\mathbf{E}$

$$(\mathbf{curl}\,\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})} = \sum_{f \in \mathcal{F}(\widehat{K})} (\mathrm{curl}_f\, \Pi_\tau \mathbf{E}, \varphi)_{L^2(f)},$$

where the decomposition into face contributions is again permitted by the regularity of $\mathbf{E}$ and φ. We obtain

$$(\mathrm{curl}_f\, \Pi_\tau \mathbf{E}, \varphi)_{L^2(f)} \lesssim p^{-3/2} \|\Pi_\tau \mathbf{E}\|_{\mathbf{H}(f, \mathrm{curl})} \|\varphi\|_{H^{3/2}(f)} \lesssim p^{-2} \|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})} \|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}$$

by Lemma 4.10 and Lemma 4.13, which finishes the proof. ∎

For functions $\mathbf{u}$ with discrete **curl**, we have the following result.

Lemma 5.15. *For all $k \geq 1$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ with $\mathbf{curl}\,\mathbf{u} \in \mathbf{V}_p(\widehat{K}) \supset (\mathcal{P}_p(\widehat{K}))^3$ there holds*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\text{curl},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \leq C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}, \qquad s \in [0,1]. \tag{5.42}$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

Proof. We employ the regularized right inverses of the operators ∇ and **curl** and proceed as in Lemma 4.14. We write, using the decomposition of Lemma 5.5, $\mathbf{u} = \nabla R^{\text{grad}}(\mathbf{u} - \mathbf{R}^{\text{curl}}\,\mathbf{curl}\,\mathbf{u}) + \mathbf{R}^{\text{curl}}\,\mathbf{curl}\,\mathbf{u} =: \nabla\varphi + \mathbf{v}$ with $\varphi \in H^{k+1}(\widehat{K})$ and $\mathbf{v} \in \mathbf{H}^k(\widehat{K})$ together with

$$\|\varphi\|_{H^{k+1}(\widehat{K})} + \|\mathbf{v}\|_{\mathbf{H}^k(\widehat{K})} \leq C\left(\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})} + \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^{k-1}(\widehat{K})}\right) \leq C\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}. \tag{5.43}$$

The assumption $\mathbf{curl}\,\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ and Lemma 5.4, (v) imply $\mathbf{v} = \mathbf{R}^{\text{curl}}\,\mathbf{curl}\,\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$; furthermore, since $\widehat{\Pi}_p^{\text{curl},3d}$ is a projection, we conclude $\mathbf{v} - \widehat{\Pi}_p^{\text{curl},3d}\mathbf{v} = 0$. Thus, together with the commuting diagram property $\nabla\widehat{\Pi}_{p+1}^{\text{grad},3d} = \widehat{\Pi}_p^{\text{curl},3d}\nabla$ we get

$$\begin{aligned}\|(I - \widehat{\Pi}_p^{\text{curl},3d})\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} &= \|(I - \widehat{\Pi}_p^{\text{curl},3d})\nabla\varphi + \underbrace{(I - \widehat{\Pi}_p^{\text{curl},3d})\mathbf{v}}_{=0}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \\ &= \|\nabla(I - \widehat{\Pi}_{p+1}^{\text{grad},3d})\varphi\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K})} \lesssim p^{-(k+s)}\|\varphi\|_{H^{k+1}(\widehat{K})}.\end{aligned}$$

The proof of (5.42) is complete in view of (5.43). Replacing $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ with $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$ follows from the observation that the projector $\widehat{\Pi}_p^{\text{curl},3d}$ reproduces polynomials of degree p. ∎

5.4 Stability of the operator $\widehat{\Pi}_p^{\text{div},3d}$

Similar to Lemma 4.11, we state the following result:

Lemma 5.16. *For each face $f \in \mathcal{F}(\widehat{K})$ we have for $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \text{div})$ and every $s \geq 0$*

$$\|(\mathbf{u} - \widehat{\Pi}_p^{\text{div},3d}\mathbf{u}) \cdot \mathbf{n}_f\|_{\widetilde{H}^{-s}(f)} \leq C_s p^{-s} \inf_{v \in V_p(f)} \|\mathbf{u} \cdot \mathbf{n}_f - v\|_{L^2(f)}. \tag{5.44}$$

Proof. We first show that, for $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \text{div})$ the normal trace $\mathbf{n}_f \cdot \mathbf{u} \in L^2(f)$ for each face f. To that end, one writes with the aid of Lemma 5.6 $\mathbf{u} = \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z}$ with $\boldsymbol{\varphi}$, $\mathbf{z} \in \mathbf{H}^{3/2}(\widehat{K})$. We have $\mathbf{n}_f \cdot \mathbf{z} \in \mathbf{H}^1(f)$. Noting $\boldsymbol{\varphi}|_f \in \mathbf{H}^1(f)$ and $(\mathbf{n}_f \cdot \mathbf{curl}\,\boldsymbol{\varphi})|_f = \text{curl}_f(\Pi_\tau\boldsymbol{\varphi})|_f$, we conclude that $(\mathbf{n}_f \cdot \mathbf{curl}\,\boldsymbol{\varphi})|_f \in L^2(f)$.
Note that (2.16c) and (2.16d) imply that on faces, the operator $\widehat{\Pi}_p^{\text{div},3d}$ is the L^2-projection onto $V_p(f)$. Thus, (5.44) holds for $s = 0$. The case $s > 0$ follows by a standard duality argument. To that end define $\tilde{e} := \left(\mathbf{u} - \widehat{\Pi}_p^{\text{div},3d}\mathbf{u}\right) \cdot \mathbf{n}_f$ and let $v \in H^s(f)$. Note that $w \in \mathcal{P}_p(\mathbb{R}^2)$ can be written as $w = \overline{w} + (w - \overline{w})$, where $\overline{w}$ denotes the average of w on f. Since $w - \overline{w} \in \mathring{V}_p(f)$, (2.16c) and (2.16d) imply $(\tilde{e}, w)_{L^2(f)} = 0$. Thus we have

$$(\tilde{e}, v)_{L^2(f)} = \inf_{w \in \mathcal{P}_p} (\tilde{e}, v - w)_{L^2(f)} \leq \|\tilde{e}\|_{L^2(f)} \inf_{w \in \mathcal{P}_p} \|v - w\|_{L^2(f)} \lesssim p^{-s}\|\tilde{e}\|_{L^2(f)}\|v\|_{H^s(f)}.$$

∎

Remark 5.17. *Note that for $u \in L^2(\partial\widehat{K})$, we have*

$$\|u\|_{H^{-1/2}(\partial\widehat{K})} \leq \sum_{f \in \mathcal{F}(\widehat{K})} \|u\|_{\widetilde{H}^{-1/2}(f)}. \tag{5.45}$$

As in the analysis of the operators in the previous sections, the existence of a polynomial preserving lifting operator from the boundary $\partial\widehat{K}$ to $\widehat{K}$ with appropriate properties will play an important role. Such a lifting operator has been constructed in [21]. We modify this lifting slightly to explicitly ensure an additional orthogonality property.

Lemma 5.18. *There exists a lifting operator $\boldsymbol{\mathcal{L}}^{\text{div},3d}$ with the following properties:*

(i) $\boldsymbol{\mathcal{L}}^{\mathrm{div},3d} z \in \mathbf{V}_p(\widehat{K})$ *if* $z|_f \in V_p(f)$ *for all faces* $f \in \mathcal{F}(\widehat{K})$*.*

(ii) There holds the extension property $(\boldsymbol{\mathcal{L}}^{\mathrm{div},3d} z \cdot \mathbf{n}_f)|_f = z$*.*

(iii) There holds $\|\boldsymbol{\mathcal{L}}^{\mathrm{div},3d} z\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \le C\|z\|_{\widetilde{H}^{-1/2}(\partial\widehat{K})}$*.*

(iv) There holds the orthogonality $(\boldsymbol{\mathcal{L}}^{\mathrm{div},3d} z, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0$ *for all* $\mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K})$*.*

Proof. Recall the space $\mathring{\mathbf{Q}}_{p,\perp}(\widehat{K}) = \{\mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K})\colon (\mathbf{q}, \nabla\psi)_{L^2(\widehat{K})} = 0\,\forall\psi \in \mathring{W}_{p+1}(\widehat{K})\}$ defined in Lemma 5.7. Let $z \in \widetilde{H}^{-1/2}(\partial\widehat{K})$ be a function with the property $z|_f \in V_p(f)$ for all faces $f \in \mathcal{F}(\widehat{K})$. The goal is to define the lifting operator by $\boldsymbol{\mathcal{L}}^{\mathrm{div},3d} z := \boldsymbol{\mathcal{E}}^{\mathrm{div}} z - \mathbf{w}_0$, where $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$ denotes the lifting operator from [21], and where $\mathbf{w}_0$ is defined by the following saddle point problem: Find $\mathbf{w}_0 \in \mathring{\mathbf{V}}_p(\widehat{K})$ and $\boldsymbol{\varphi} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$ such that

$$(\operatorname{div}\mathbf{w}_0, \operatorname{div}\mathbf{v})_{L^2(\widehat{K})} + (\mathbf{v}, \mathbf{curl}\,\boldsymbol{\varphi})_{L^2(\widehat{K})} = (\operatorname{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z), \operatorname{div}\mathbf{v})_{L^2(\widehat{K})} \qquad \forall \mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}) \tag{5.46a}$$

$$(\mathbf{w}_0, \mathbf{curl}\,\boldsymbol{\mu})_{L^2(\widehat{K})} = (\boldsymbol{\mathcal{E}}^{\mathrm{div}} z, \mathbf{curl}\,\boldsymbol{\mu})_{L^2(\widehat{K})} \qquad \forall \boldsymbol{\mu} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K}). \tag{5.46b}$$

Unique solvability of Problem (5.46) is seen as follows: Define the bilinear forms $a(\mathbf{w}, \mathbf{q}) := (\operatorname{div}\mathbf{w}, \operatorname{div}\mathbf{q})_{L^2(\widehat{K})}$ and $b(\mathbf{w}, \boldsymbol{\varphi}) := (\mathbf{w}, \mathbf{curl}\,\boldsymbol{\varphi})_{L^2(\widehat{K})}$ for $\mathbf{w}, \mathbf{q} \in \mathring{\mathbf{V}}_p(\widehat{K})$ and $\boldsymbol{\varphi} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$. Coercivity of a on the kernel of b, $\ker b = \{\mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}) : (\mathbf{v}, \mathbf{curl}\,\boldsymbol{\mu})_{L^2(\widehat{K})} = 0\,\forall\boldsymbol{\mu} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})\}$, follows from the Friedrichs inequality for the divergence operator (cf. Lemma 5.8). That is,

$$a(\mathbf{v}, \mathbf{v}) = \|\operatorname{div}\mathbf{v}\|^2_{L^2(\widehat{K})} \ge \frac{1}{2C^2}\|\mathbf{v}\|^2_{L^2(\widehat{K})} + \frac{1}{2}\|\operatorname{div}\mathbf{v}\|^2_{L^2(\widehat{K})} \ge \min\{\frac{1}{2C^2}, \frac{1}{2}\}\|\mathbf{v}\|^2_{\mathbf{H}(\widehat{K},\mathrm{div})} \qquad \forall \mathbf{v} \in \ker b.$$

Next, the inf-sup condition for b follows easily by considering, for given $\boldsymbol{\varphi} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$, the function $\mathbf{w} = \mathbf{curl}\,\boldsymbol{\varphi} \in \mathring{\mathbf{V}}_p(\widehat{K})$ in $b(\mathbf{w}, \boldsymbol{\varphi})$ and using the Friedrichs inequality for the $\mathbf{curl}$ (Lemma 5.7). That is,

$$\frac{b(\mathbf{w}, \boldsymbol{\varphi})}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}} = \frac{\|\mathbf{curl}\,\boldsymbol{\varphi}\|^2_{L^2(\widehat{K})}}{\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{L^2(\widehat{K})}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}} \overset{\text{Lem. 5.7}}{\ge} C.$$

Thus, the saddle point problem (5.46) has a unique solution $(\mathbf{w}_0, \boldsymbol{\varphi}) \in \mathring{\mathbf{V}}_p(\widehat{K}) \times \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$. In fact, selecting $\mathbf{v} = \mathbf{curl}\,\boldsymbol{\varphi}$ in (5.46a) shows $\boldsymbol{\varphi} = 0$. The lifting operator $\boldsymbol{\mathcal{L}}^{\mathrm{div},3d}$ now obviously satisfies (i), (ii) and (iv) by construction, cf. [21, Theorem 7.1] for the properties of the operator $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$. For (iii) note that the solution $\mathbf{w}_0$ satisfies the estimate $\|\mathbf{w}_0\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \lesssim \|f\| + \|g\|$, where $f(\mathbf{v}) = (\operatorname{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z), \operatorname{div}\mathbf{v})_{L^2(\widehat{K})}$, $g(v) = (\boldsymbol{\mathcal{E}}^{\mathrm{div}} z, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}$, and $\|\cdot\|$ denotes the operator norm. Thus,

$$\|f\| = \sup_{\|\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \le 1} |(\operatorname{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z), \operatorname{div}\mathbf{v})_{L^2(\widehat{K})}| \le \|\operatorname{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z)\|_{L^2(\widehat{K})} \lesssim \|z\|_{\widetilde{H}^{-1/2}(\partial\widehat{K})}.$$

The estimate $\|g\| \lesssim \|z\|_{\widetilde{H}^{-1/2}(\partial\widehat{K})}$ is shown in a similar way. Hence, (iii) follows from

$$\|\boldsymbol{\mathcal{L}}^{\mathrm{div},3d} z\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \le \|\boldsymbol{\mathcal{E}}^{\mathrm{div}} z\|_{\mathbf{H}(\widehat{K},\mathrm{div})} + \|\mathbf{w}_0\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \lesssim \|z\|_{\widetilde{H}^{-1/2}(\partial\widehat{K})}.$$

□

Theorem 5.19. *There exists* $C > 0$ *independent of* p *such that for all* $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \le Cp^{-1/2} \inf_{\mathbf{v} \in \mathbf{V}_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}. \tag{5.47}$$

Proof. 1. step: By the projection property of $\widehat{\Pi}_p^{\mathrm{div},3d}$, it suffices to show (5.47) for $\mathbf{v} = 0$.
2. step: As shown in Lemma 5.16, $\mathbf{u} \cdot \mathbf{n}_f \in L^2(f)$ on each face $f \in \mathcal{F}(\widehat{K})$. Thus we get from Lemma 5.16

$$\|(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}) \cdot \mathbf{n}_f\|_{\widetilde{H}^{-1/2}(f)} \lesssim p^{-1/2}\|\mathbf{u} \cdot \mathbf{n}_f\|_{L^2(f)} \lesssim p^{-1/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}. \tag{5.48}$$

3. step: The volume error $\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}$ is estimated using the approximation $P^{\mathrm{div},3d}\mathbf{u}$ of Lemma 5.3. We abbreviate $\mathbf{E} := \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ and note that, since $\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}$ satisfies the orthogonality conditions

(2.16a) and (2.16b), and $P^{\mathrm{div},3d}\mathbf{u}$ satisfies the conditions (5.3a) and (5.3b), we have the two orthogonality conditions

$$(\operatorname{div}\mathbf{E},\operatorname{div}\mathbf{v})_{L^2(\widehat{K})}=0\qquad\forall\mathbf{v}\in\mathring{\mathbf{V}}_p(\widehat{K}),\qquad(\mathbf{E},\mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}=0\qquad\forall\mathbf{v}\in\mathring{\mathbf{Q}}_p(\widehat{K}).\tag{5.49}$$

By Lemma 5.18, the orthogonality condition

$$(\boldsymbol{\mathcal{L}}^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n}),\mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}=0\qquad\forall\mathbf{v}\in\mathring{\mathbf{Q}}_p(\widehat{K})$$

holds; hence the discrete Friedrichs inequality (Lemma 5.8, (ii)) can be applied to $\mathbf{E}-\boldsymbol{\mathcal{L}}^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n})\in\mathring{\mathbf{V}}_p(\widehat{K})$. Thus, we obtain

$$\begin{aligned}\|\mathbf{E}\|_{L^2(\widehat{K})}&\le\|\boldsymbol{\mathcal{L}}^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n})\|_{L^2(\widehat{K})}+\|\mathbf{E}-\boldsymbol{\mathcal{L}}^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n})\|_{L^2(\widehat{K})}\\&\lesssim\|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})}+\|\operatorname{div}(\mathbf{E}-\boldsymbol{\mathcal{L}}^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n}))\|_{L^2(\widehat{K})}\\&\lesssim\|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})}+\|\operatorname{div}\mathbf{E}\|_{L^2(\widehat{K})}.\end{aligned}\tag{5.50}$$

4. step: Using the first part of (5.49), we get

$$\|\operatorname{div}\mathbf{E}\|^2_{L^2(\widehat{K})}=(\operatorname{div}\mathbf{E},\operatorname{div}\boldsymbol{\mathcal{L}}^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n}))_{L^2(\widehat{K})}\le\|\operatorname{div}\mathbf{E}\|_{L^2(\widehat{K})}\|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})}.\tag{5.51}$$

Combining (5.50), (5.51) we arrive at

$$\|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}\lesssim\|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})}.\tag{5.52}$$

5. step: With the triangle inequality and the continuity of the normal trace operator

$$\begin{aligned}\|\mathbf{u}-\widehat{\Pi}^{\mathrm{div},3d}_p\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}&\le\|\mathbf{u}-P^{\mathrm{div},3\mathrm{d}}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}+\|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}\\&\overset{(5.52)}{\lesssim}\|\mathbf{u}-P^{\mathrm{div},3\mathrm{d}}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}+\|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})}\\&\lesssim\|\mathbf{u}-P^{\mathrm{div},3\mathrm{d}}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}+\sum_{f\in\mathcal{F}(\widehat{K})}\|(\mathbf{u}-\widehat{\Pi}^{\mathrm{div},3\mathrm{d}}_p\mathbf{u})\cdot\mathbf{n}_f\|_{\widetilde{H}^{-1/2}(f)}\\&\overset{(5.48),\text{Lem. }5.3}{\lesssim}p^{-1/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})},\end{aligned}$$

□

Considering the approximation error in negative Sobolev norms is the next step.

Theorem 5.20. *For $s\in[0,1]$ and for all $\mathbf{u}\in\mathbf{H}^{1/2}(\widehat{K},\operatorname{div})$ there holds the estimate*

$$\|\mathbf{u}-\widehat{\Pi}^{\mathrm{div},3d}_p\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})}\le C_sp^{-1/2-s}\inf_{\mathbf{v}\in\mathbf{V}_p(\widehat{K})}\|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}.$$

Proof. In view of the projection property of $\widehat{\Pi}^{\mathrm{div},3d}_p$, we restrict to showing the estimate with $\mathbf{v}=0$. The case $s=0$ is shown in Theorem 5.19. We will therefore merely focus on the case $s=1$ as the cases $s\in(0,1)$ follow by interpolation.
We write $\mathbf{E}:=\mathbf{u}-\widehat{\Pi}^{\mathrm{div},3d}_p\mathbf{u}$ for simplicity. By definition we have

$$\|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-1}(\widehat{K},\mathrm{div})}\sim\|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-1}(\widehat{K})}+\|\operatorname{div}\mathbf{E}\|_{\widetilde{H}^{-1}(\widehat{K})}=\sup_{\mathbf{v}\in\mathbf{H}^1(\widehat{K})}\frac{(\mathbf{E},\mathbf{v})_{L^2(\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}}+\sup_{v\in H^1(\widehat{K})}\frac{(\operatorname{div}\mathbf{E},v)_{L^2(\widehat{K})}}{\|v\|_{H^1(\widehat{K})}}.\tag{5.53}$$

We start with estimating the first supremum in (5.53). We decompose $\mathbf{v}\in\mathbf{H}^1(\widehat{K})$ as

$$\mathbf{v}=\nabla\varphi+\mathbf{curl}\,\mathbf{z}$$

with $\varphi\in H^2(\widehat{K})$ and $\mathbf{z}\in\mathbf{H}^1(\widehat{K},\mathbf{curl})\cap\mathbf{H}_0(\widehat{K},\mathbf{curl})$ according to Lemma 5.13 and have to bound the two terms in $(\mathbf{E},\mathbf{v})_{L^2(\widehat{K})}=(\mathbf{E},\mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})}+(\mathbf{E},\nabla\varphi)_{L^2(\widehat{K})}$. For the first term, by Theorem 5.19, the estimate

$$\begin{aligned}(\mathbf{E},\mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})}&=\inf_{\mathbf{w}\in\mathring{\mathbf{Q}}_p(\widehat{K})}(\mathbf{E},\mathbf{curl}(\mathbf{z}-\mathbf{w}))_{L^2(\widehat{K})}\lesssim p^{-1}\|\mathbf{E}\|_{L^2(\widehat{K})}\|\mathbf{z}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}\\&\lesssim p^{-3/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}\end{aligned}$$

holds. For the second term, we employ integration by parts to get

$$(\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})} = -(\operatorname{div}\mathbf{E}, \varphi)_{L^2(\widehat{K})} + \sum_{f\in\mathcal{F}(\widehat{K})} (\mathbf{E}\cdot\mathbf{n}, \varphi)_{L^2(f)} \tag{5.54}$$

Denote by $\overline{\varphi} := (\int_{\widehat{K}} \varphi)/|\widehat{K}|$ the average of φ. Now the integration by parts formula gives

$$(\operatorname{div}\mathbf{E}, \varphi)_{L^2(\widehat{K})} = (\operatorname{div}\mathbf{E}, \varphi - \overline{\varphi})_{L^2(\widehat{K})} + \overline{\varphi}(\mathbf{E}\cdot\mathbf{n}, 1)_{L^2(\partial\widehat{K})} \overset{(2.16d)}{=} (\operatorname{div}\mathbf{E}, \varphi - \overline{\varphi})_{L^2(\widehat{K})}. \tag{5.55}$$

We then define the auxiliary function ψ by

$$\Delta\psi = \varphi - \overline{\varphi}, \qquad \partial_n\psi = 0 \text{ on } \partial\widehat{K}$$

and set $\mathbf{\Phi} := \nabla\psi$. Since $\operatorname{div}\mathbf{\Phi} = \Delta\psi = \varphi - \overline{\varphi}$, we get

$$\left|(\operatorname{div}\mathbf{E}, \varphi - \overline{\varphi})_{L^2(\widehat{K})}\right| = \left|(\operatorname{div}\mathbf{E}, \operatorname{div}\mathbf{\Phi})_{L^2(\widehat{K})}\right| \overset{(2.16a)}{=} \left|\inf_{\mathbf{w}\in\mathring{\mathbf{V}}_p(\widehat{K})} (\operatorname{div}\mathbf{E}, \operatorname{div}(\mathbf{\Phi} - \mathbf{w}))_{L^2(\widehat{K})}\right| \tag{5.56}$$

$$\lesssim p^{-1}\|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\operatorname{div})}\|\mathbf{\Phi}\|_{H^1(\widehat{K},\operatorname{div})} \lesssim p^{-3/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\operatorname{div})}\|\varphi\|_{H^1(\widehat{K})} \tag{5.57}$$

$$\lesssim p^{-3/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\operatorname{div})}\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}. \tag{5.58}$$

Thus, only estimates for the boundary terms in (5.54) are missing. The orthogonality properties (2.16c) and (2.16d) as well as Lemma 5.16 lead to

$$\begin{aligned}(\mathbf{E}\cdot\mathbf{n}, \varphi)_{L^2(f)} &= \inf_{w\in V_p(f)} (\mathbf{E}\cdot\mathbf{n}, \varphi - w)_{L^2(f)} \lesssim p^{-1}\|\mathbf{E}\cdot\mathbf{n}\|_{\widetilde{H}^{-1/2}(f)}\|\varphi\|_{H^{3/2}(f)} \\ &\overset{\text{Lem. 5.16}}{\lesssim} p^{-3/2}\|\mathbf{u}\cdot\mathbf{n}\|_{L^2(f)}\|\varphi\|_{H^2(\widehat{K})} \lesssim p^{-3/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\operatorname{div})}\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.\end{aligned}$$

Thus, we have estimated the first term of (5.53).
We now handle the second supremum in (5.53). Such estimates have already been derived in (5.55) and (5.56); we merely have to note that the function φ in these lines satisfied $\varphi \in H^2(\widehat{K})$, but $H^1(\widehat{K})$-regularity is indeed sufficient as is visible in (5.57). □

If we assume discrete divergence, we get a result similar to Lemma 5.15.

Lemma 5.21. *For all $k \geq 1$, all $s \in [0,1]$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ with $\operatorname{div}\mathbf{u} \in \mathcal{P}_p(\widehat{K})$ there holds*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\operatorname{div},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\operatorname{div})} \leq C_{s,k}p^{-(k+s)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}. \tag{5.59}$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

Proof. We write, using the decomposition of Lemma 5.6, $\mathbf{u} = \mathbf{curl}\,\mathbf{R}^{\text{curl}}(\mathbf{u} - \mathbf{R}^{\text{div}}\operatorname{div}\mathbf{u}) + \mathbf{R}^{\text{div}}\operatorname{div}\mathbf{u} =: \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z}$ with $\boldsymbol{\varphi} \in \mathbf{H}^{k+1}(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^k(\widehat{K})$ together with

$$\|\boldsymbol{\varphi}\|_{\mathbf{H}^{k+1}(\widehat{K})} + \|\mathbf{z}\|_{\mathbf{H}^k(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})} + \|\operatorname{div}\mathbf{u}\|_{H^{k-1}(\widehat{K})} \leq C\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}. \tag{5.60}$$

The assumption $\operatorname{div}\mathbf{u} \in \mathcal{P}_p(\widehat{K})$ and Lemma 5.4, (vi) imply $\mathbf{z} = \mathbf{R}^{\text{div}}\operatorname{div}\mathbf{u} \in \mathbf{V}_p(\widehat{K})$; furthermore, since $\widehat{\Pi}_p^{\operatorname{div},3d}$ is a projection, we conclude $\mathbf{z} - \widehat{\Pi}_p^{\operatorname{div},3d}\mathbf{z} = 0$. Thus, we get from the commuting diagram and Corollary 2.9

$$\begin{aligned}\|(I - \widehat{\Pi}_p^{\operatorname{div},3d})\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\operatorname{div})} &= \|(I - \widehat{\Pi}_p^{\operatorname{div},3d})\,\mathbf{curl}\,\boldsymbol{\varphi} + \underbrace{(I - \widehat{\Pi}_p^{\operatorname{div},3d})\mathbf{z}}_{=0}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\operatorname{div})} \\ &\lesssim \|(I - \widehat{\Pi}_p^{\text{curl},3d})\boldsymbol{\varphi}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \lesssim p^{-(k+s)}\|\boldsymbol{\varphi}\|_{\mathbf{H}^k(\widehat{K},\mathbf{curl})} \lesssim p^{-(k+s)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}.\end{aligned}$$

Replacing $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ with $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$ follows from the observation that the projector $\widehat{\Pi}_p^{\operatorname{div},3d}$ reproduces polynomials of degree p. □

A Equivalence of $\mathbf{X}^{-1/2}$ and $\|\cdot\|_{H^{-1/2}} + \|\operatorname{\mathbf{curl}} \cdot\|_{H^{-1/2}}$

Lemma A.1. *Let $\widehat{K}$ be the reference tetrahedron. Then, for $\mathbf{g} \in \mathbf{X}^{-1/2}$ (defined in Lemma 5.11) we have*

$$\|\mathbf{g}\|_{\mathbf{X}^{-1/2}} \sim \left[\|\mathbf{g}\|_{\mathbf{H}_T^{-1/2}(\partial\widehat{K})} + \|\operatorname{curl}_{\partial\widehat{K}} \mathbf{g}\|_{H^{-1/2}(\partial\widehat{K})}\right], \qquad \|\mathbf{v}\|_{\mathbf{H}_T^{-1/2}(\partial\widehat{K})} := \sup_{\mathbf{u}\in\mathbf{H}^1(\widehat{K})} \frac{\langle \mathbf{v}, \gamma_\tau \mathbf{u}\rangle_{\partial\widehat{K}}}{\|\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}}, \tag{A.1}$$

with constant depending solely on $\widehat{K}$. Here, $\langle\cdot,\cdot\rangle_{\partial\widehat{K}}$ denotes a duality pairing introduced in the proof below. The surface curl, $\operatorname{curl}_{\partial\widehat{K}} \mathbf{g}$, is defined as $\mathbf{n}\cdot\operatorname{\mathbf{curl}}\mathbf{z}$ for any lifting $\mathbf{z} \in \mathbf{H}(\widehat{K},\mathbf{curl})$ of $\mathbf{g} \in \mathbf{X}^{-1/2}$.

Proof. The workhorse is the integration by parts formula

$$\langle \Pi_\tau \mathbf{u}, \gamma_\tau \mathbf{v}\rangle_{\partial\widehat{K}} = (\operatorname{\mathbf{curl}}\mathbf{v}, \mathbf{u})_{L^2(\widehat{K})} - (\operatorname{\mathbf{curl}}\mathbf{u}, \mathbf{v})_{L^2(\widehat{K})} \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}(\widehat{K},\mathbf{curl}), \tag{A.2}$$

which also defines the duality pairing. To give a few more details, one defines the range $\mathbf{Y}^{-1/2} := \gamma_\tau \mathbf{H}(\widehat{K},\mathbf{curl})$ endowed with the quotient norm $\|\mathbf{g}\|_{\mathbf{Y}^{-1/2}} := \inf\{\|\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \,|\, \gamma_\tau \mathbf{v} = \mathbf{g}\}$. By [25, Thm. 3.31], the trace operator Π_τ maps into $\left(\mathbf{Y}^{-1/2}\right)'$ via (A.2), which therefore defines $\langle\cdot,\cdot\rangle_{\partial\widehat{K}}$ on $\mathbf{X}^{-1/2} \times \mathbf{Y}^{-1/2}$.
Proof of the bound $\|\mathbf{g}\|_{\mathbf{X}^{-1/2}} \gtrsim \|\operatorname{curl}_{\partial\widehat{K}} \mathbf{g}\|_{H^{-1/2}(\partial\widehat{K})} + \|\mathbf{g}\|_{\mathbf{H}_T^{-1/2}(\partial\widehat{K})}$:
Let $\mathbf{z} \in \mathbf{H}(\widehat{K},\mathbf{curl})$ and set $\mathbf{g} := \Pi_\tau \mathbf{z}$. Then (A.2) yields $\|\mathbf{g}\|_{\mathbf{H}_T^{-1/2}(\partial\widehat{K})} \lesssim \|\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}$. To control $\operatorname{curl}_{\partial\widehat{K}} \mathbf{g}$, we first note $\operatorname{div}\operatorname{\mathbf{curl}}\mathbf{z} = 0$ so that $\mathbf{n}\cdot\operatorname{\mathbf{curl}}\mathbf{z} \in H^{-1/2}(\partial\widehat{K})$ is well-defined and is taken as the definition of $\operatorname{curl}_{\partial\widehat{K}} \mathbf{g}$. Indeed, this definition is independent of the lifting $\mathbf{z}$: The difference $\boldsymbol{\delta} := \mathbf{z}_1 - \mathbf{z}_2 \in \mathbf{H}_0(\widehat{K},\mathbf{curl})$ of two liftings of $\mathbf{g}$ satisfies $\operatorname{\mathbf{curl}}\boldsymbol{\delta} \in \mathbf{H}_0(\widehat{K},\operatorname{div})$ by the deRham diagram property (see, e.g., [25, eqn. (3.60)]). Next, we estimate for arbitrary $\varphi \in H^1(\widehat{K})$

$$|\langle \operatorname{curl}_{\partial\widehat{K}} \mathbf{g}, \varphi\rangle_{\partial\widehat{K}}| \stackrel{\text{by def.}}{=} |(\mathbf{n}\cdot\operatorname{\mathbf{curl}}\mathbf{z}, \varphi)_{L^2(\partial\widehat{K})}| = |(\operatorname{\mathbf{curl}}\mathbf{z}, \nabla\varphi)_{L^2(\widehat{K})}| \leq \|\mathbf{z}\|_{\mathbf{H}(\mathbf{curl},\widehat{K})} \|\nabla\varphi\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}.$$

Proof of the bound $\|\mathbf{g}\|_{\mathbf{X}^{-1/2}} \lesssim \|\operatorname{curl}_{\partial\widehat{K}} \mathbf{g}\|_{H^{-1/2}(\partial\widehat{K})} + \|\mathbf{g}\|_{\mathbf{H}_T^{-1/2}(\partial\widehat{K})}$:
Since the norm $\|\cdot\|_{\mathbf{X}^{-1/2}}$ is defined by the minimum norm extension, we merely need to construct a lifting $\mathbf{Z} \in \mathbf{H}(\widehat{K},\mathbf{curl})$ with a good bound on $\mathbf{Z}$. We define $\mathbf{Z}$ as the solution of the following (constrained) minimization problem:

$$\text{Minimize } \|\operatorname{\mathbf{curl}}\mathbf{Y}\|_{L^2(\widehat{K})} \text{ under the constraints } \Pi_\tau \mathbf{Y} = \mathbf{g} \text{ and } (\mathbf{Y}, \nabla\varphi)_{L^2(\widehat{K})} = 0 \text{ for all } \varphi \in H_0^1(\widehat{K}). \tag{A.3}$$

This minimization problem can be solved with the method of Lagrange multipliers as discussed in [15, Sec. 4.4] (in the discrete setting) and the proof of Lemma 5.11. One obtains, in strong form, the problem: Find $(\mathbf{Z}, \varphi) \in \mathbf{H}(\widehat{K},\mathbf{curl}) \times H_0^1(\widehat{K})$ such that

$$\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}}\mathbf{Z} + \nabla\varphi = 0 \quad \text{in } \widehat{K}, \qquad \Pi_\tau \mathbf{Z} = \mathbf{g}.$$

It can be checked (this is observed, for example, in [15, Sec. 4.4] and also the case in the proof of Lemma 5.11) that the Lagrange multiplier φ vanishes. Therefore, $\mathbf{Z}$ solves

$$\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}}\mathbf{Z} = 0, \qquad \operatorname{div}\mathbf{Z} = 0, \qquad \Pi_\tau \mathbf{Z} = \mathbf{g}.$$

Let us focus on $\mathbf{w} := \operatorname{\mathbf{curl}}\mathbf{Z}$. We have

$$\operatorname{\mathbf{curl}}\mathbf{w} = 0, \qquad \operatorname{div}\mathbf{w} = 0, \qquad \mathbf{n}\cdot\mathbf{w} = \operatorname{curl}_{\partial\widehat{K}} \mathbf{g}.$$

From $\operatorname{\mathbf{curl}}\mathbf{w} = 0$, we get that $\mathbf{w}$ is a gradient: $\mathbf{w} = \nabla\psi$. The second and third conditions show

$$-\Delta\psi = 0 \qquad \partial_n\psi = \mathbf{n}\cdot\mathbf{w} = \operatorname{curl}_{\partial\widehat{K}} \mathbf{g}.$$

Noting that the integrability condition is satisfied since $(\mathbf{n}\cdot\mathbf{w}, 1)_{L^2(\partial\widehat{K})} = (\operatorname{div}\mathbf{w}, 1)_{L^2(\widehat{K})} = 0$, we conclude by standard *a priori* estimates for the Laplace problem

$$\|\mathbf{w}\|_{L^2(\widehat{K})} = \|\nabla\psi\|_{L^2(\widehat{K})} \lesssim \|\operatorname{curl}_{\partial\widehat{K}} \mathbf{g}\|_{H^{-1/2}(\partial\widehat{K})}.$$

Hence, $\|\operatorname{\mathbf{curl}}\mathbf{Z}\|_{L^2(\widehat{K})} \lesssim \|\operatorname{curl}_{\partial\widehat{K}}\mathbf{g}\|_{H^{-1/2}(\partial\widehat{K})}$. To get more information about $\mathbf{Z}$, we write it as

$$\mathbf{Z} = \nabla\phi + \mathbf{z}, \qquad \mathbf{z} := \mathbf{R}^{\mathrm{curl}}(\operatorname{\mathbf{curl}}\mathbf{Z}), \qquad \phi = R^{\mathrm{grad}}(\mathbf{Z} - \mathbf{R}^{\mathrm{curl}}(\operatorname{\mathbf{curl}}\mathbf{z})) \tag{A.4}$$

with, by Lemma 5.4,

$$\|\mathbf{z}\|_{H^1(\widehat{K})} \lesssim \|\operatorname{\mathbf{curl}}\mathbf{Z}\|_{L^2(\widehat{K})} \lesssim \|\operatorname{curl}_{\partial\widehat{K}}\mathbf{g}\|_{H^{-1/2}(\partial\widehat{K})}. \tag{A.5}$$

For the control of ϕ, proceed by an integration by parts argument. Noting that $\operatorname{div}\mathbf{Z} = 0$, we have

$$\nabla\phi + \mathbf{z} = \mathbf{Z} = \operatorname{\mathbf{curl}}\mathbf{R}^{\mathrm{curl}}(\mathbf{Z}) = \operatorname{\mathbf{curl}}\mathbf{R}^{\mathrm{curl}}(\nabla\phi) + \operatorname{\mathbf{curl}}\mathbf{R}^{\mathrm{curl}}(\mathbf{z}).$$

Next, we employ the integration by parts formula (A.2)

$$(\operatorname{\mathbf{curl}}\mathbf{Z}, \mathbf{v})_{L^2(\widehat{K})} \stackrel{(A.2)}{=} (\mathbf{Z}, \operatorname{\mathbf{curl}}\mathbf{v})_{L^2(\widehat{K})} - \langle \mathbf{g}, \gamma_\tau \mathbf{v}\rangle_{\partial\widehat{K}}.$$

Selecting $\mathbf{v} = \mathbf{R}^{\mathrm{curl}}(\nabla\phi) \in \mathbf{H}^1(\widehat{K})$

$$(\operatorname{\mathbf{curl}}\mathbf{Z}, \mathbf{R}^{\mathrm{curl}}(\nabla\phi))_{L^2(\widehat{K})} = (\nabla\phi + \mathbf{z}, \nabla\phi + \mathbf{z} - \operatorname{\mathbf{curl}}\mathbf{R}^{\mathrm{curl}}(\mathbf{z}))_{L^2(\widehat{K})} + \langle \mathbf{g}, \gamma_\tau \mathbf{R}^{\mathrm{curl}}(\nabla\phi)\rangle_{\partial\widehat{K}}.$$

In view of the mapping property $\mathbf{R}^{\mathrm{curl}} : \mathbf{L}^2(\widehat{K}) \to \mathbf{H}^1(\widehat{K})$

$$\begin{aligned}\|\nabla\phi\|^2_{L^2(\widehat{K})} \lesssim{}& \|\operatorname{\mathbf{curl}}\mathbf{Z}\|_{L^2(\widehat{K})}\|\nabla\phi\|_{L^2(\widehat{K})} + \|\mathbf{z}\|_{L^2(\widehat{K})}\|\mathbf{z} - \operatorname{\mathbf{curl}}\mathbf{R}^{\mathrm{curl}}\mathbf{z}\|_{L^2(\widehat{K})} \\ &+ \|\mathbf{z} - \operatorname{\mathbf{curl}}\mathbf{R}^{\mathrm{curl}}(\mathbf{z})\|_{L^2(\widehat{K})}\|\nabla\phi\|_{L^2(\widehat{K})} + \|\mathbf{z}\|_{L^2(\widehat{K})}\|\nabla\phi\|_{L^2(\widehat{K})} + \|\mathbf{g}\|_{\mathbf{H}_T^{-1/2}(\partial\widehat{K})}\|\nabla\phi\|_{L^2(\widehat{K})}.\end{aligned} \tag{A.6}$$

Combining (A.4), (A.5), (A.6), we infer $\|\mathbf{Z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|\mathbf{g}\|_{\mathbf{H}_T^{-1/2}(\partial\widehat{K})} + \|\operatorname{curl}_{\partial\widehat{K}}\mathbf{g}\|_{H^{-1/2}(\partial\widehat{K})}$, which concludes the proof. □

Remark A.2. *We include an alternative proof of Lemma 5.11, (iv), which is based on the intrinsic characterization of the trace norm* $\|\cdot\|_{\mathbf{X}^{-1/2}}$*:*
The norm $\|\mathbf{z}\|_{\mathbf{X}^{-1/2}}$ *can be estimated using the characterization of the trace spaces given in [9,10]. Specifically, using the notation of [9,10], one has by [9, Thm. 4.6] that the mapping* $\Pi_\tau : \mathbf{H}(\widehat{K}, \mathbf{curl}) \to \mathbf{H}_\perp^{-1/2}(\partial\widehat{K}, \mathbf{curl})$ *is linear, continuous, and surjective, where the associated norm is*

$$\|\mathbf{z}\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl},\partial\widehat{K})} = \|\mathbf{z}\|^2_{\mathbf{H}_\perp^{-1/2}(\partial\widehat{K})} + \|\operatorname{curl}\mathbf{z}\|^2_{H_\perp^{-1/2}(\partial\widehat{K})}.$$

Here, the norm $\|\cdot\|_{\mathbf{H}_\perp^{-1/2}(\partial\widehat{K})}$ *is the dual norm (with pivot space* $L^2_t(\partial\widehat{K})$*) of* $\mathbf{H}_\perp^{1/2}(\partial\widehat{K})$*; analogously the norm* $\|\cdot\|_{H_\perp^{-1/2}(\partial\widehat{K})}$ *is the dual norm (with pivot space* $L^2_t(\partial\widehat{K})$*) of* $H_\perp^{1/2}(\partial\widehat{K})$*. The precise characterization of these two latter spaces in [10] gives the continuous embeddings* $\mathbf{H}_\perp^{1/2}(\partial\widehat{K}) \subset \prod_{f\in\mathcal{F}(\widehat{K})}\mathbf{H}^{1/2}(f)$ *and* $H_\perp^{1/2}(\partial\widehat{K}) \subset \prod_{f\in\mathcal{F}(\widehat{K})} H^{1/2}(f)$*. In turn, this implies the estimates*

$$\|z\|_{H_\perp^{-1/2}(\partial\widehat{K})} \leq C \sum_{f\in\mathcal{F}(\widehat{K})} \|z\|_{\widetilde{H}^{-1/2}(f)}, \qquad \|\mathbf{z}\|_{\mathbf{H}_\perp^{-1/2}(\partial\widehat{K})} \leq C \sum_{f\in\mathcal{F}(\widehat{K})} \|\mathbf{z}\|_{\widetilde{\mathbf{H}}^{-1/2}(f)}.$$

It remains to see that $\operatorname{curl}\mathbf{z}$ *in the above formula can be interpreted facewise. This is the case because* $\mathbf{z} \in \mathbf{T}$ *is facewise sufficiently smooth (it is in* $\mathbf{H}^{3/2}(f)$*) and satisfies appropriate continuity conditions across the edges of* $\widehat{K}$ *(by the assumption that* $\mathbf{T} = \Pi_\tau \mathbf{H}^2(\widehat{K})$*).* ■

B Well-definedness of the projection operators and commuting diagram property

Lemma B.1. *The operator* $\widehat{\Pi}^{\mathrm{grad},3d}_{p+1}$ *is well-defined.*

Proof. One needs to check that the traces of $u \in H^2(\widehat{K})$ on the edges are in H^1. This follows from the trace theorem: a two-fold trace estimate (from $\widehat{K}$ to the faces and then from the faces to the edges) shows for an edge e that the trace operator maps $H^{2+\varepsilon}(\widehat{K}) \to H^{1+\varepsilon}(e)$ for sufficiently small $\varepsilon > 0$ and $\varepsilon < 0$. The mapping property $H^2(\widehat{K}) \to H^1(e)$ then follows by interpolation.
We check the number of conditions in (2.14):

$$\dim W_{p+1}(\widehat{K}) = \frac{1}{6}(p+4)(p+3)(p+2),$$
$$\text{number of conditions} = \frac{1}{6}p(p-1)(p-2) + 4\frac{(p-1)p}{2} + 6p + 4 = \dim W_{p+1}(\widehat{K}).$$

Hence, the defining equations (2.14) represent a square linear system. For $u = 0$ (2.14d) shows $\widehat{\Pi}_{p+1}^{\text{grad},3d}u(V) = 0$ for all vertices $V \in \mathcal{V}(\widehat{K})$. The conditions (2.14c) then imply that $\widehat{\Pi}_{p+1}^{\text{grad},3d}u = 0$ on all edges of $\widehat{K}$; next (2.14b) leads to $\widehat{\Pi}_{p+1}^{\text{grad},3d}u = 0$ vanishing on all faces of $\widehat{K}$ and finally next (2.14a) shows $\widehat{\Pi}_{p+1}^{\text{grad},3d}u = 0$. Thus, $\widehat{\Pi}_{p+1}^{\text{grad},3d}$ is well-defined. ☐

Lemma B.2. *The operator $\widehat{\Pi}_p^{\text{curl},3d}$ is well-defined.*

Proof. First, one needs to check that for a $\mathbf{u} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$ the face traces $(\Pi_\tau \mathbf{u})|_f$ and edge traces $\mathbf{t}_e \cdot \mathbf{u}$ are in L^2. The trace theorem gives, for each face f, $\Pi_\tau \mathbf{u} \in \mathbf{H}^{1/2}(f, \text{curl}_f)$. The argument at the outset of the proof of Lemma 4.11 then shows that $\mathbf{t}_e \cdot \mathbf{u} \in L^2(e)$.
We check the number of conditions in (2.15). With the notation

$$\ker \mathbf{curl} = \{\mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K}) : \mathbf{curl}\, \mathbf{q} = \mathbf{0}\},$$

we have

$$\dim \mathring{\mathbf{Q}}_p(\widehat{K}) = \dim \mathbf{curl}\, \mathring{\mathbf{Q}}_p(\widehat{K}) + \dim \ker \mathbf{curl} = \dim \mathbf{curl}\, \mathring{\mathbf{Q}}_p(\widehat{K}) + \dim \nabla \mathring{W}_{p+1}(\widehat{K})$$

in view of the exactness of the sequence (2.12). Hence,

$$\text{the number of conditions in (2.15a), (2.15b) } = \dim \mathring{\mathbf{Q}}_p(\widehat{K}).$$

Analogously, we argue with the exactness of the second sequence in (2.12) that

$$\text{the number of conditions in (2.15c), (2.15d) } = \dim \mathring{\mathbf{Q}}_p(f), \qquad \forall \text{ faces } f \in \mathcal{F}(\widehat{K}).$$

Finally, we check

$$\text{the number of conditions in (2.15e)} = p - 1, \qquad \forall \text{ edges } e \in \mathcal{E}(\widehat{K}),$$
$$\text{the number of conditions in (2.15f)} = 6.$$

In total, the number of conditions in (2.15) coincides with $\dim \mathbf{Q}_p$. We conclude that (2.15) represents a square system of equations. As in the case of Lemma B.1, see that $\mathbf{u} = 0$ implies $\widehat{\Pi}_p^{\text{curl},3d}\mathbf{u} = 0$ in the following way: (2.15e), (2.15f) imply that the tangential component of $\widehat{\Pi}_p^{\text{curl},3d}\mathbf{u}$ vanishes on all edges of $\widehat{K}$. From that, (2.15c), (2.15d) together with the exact sequence property (2.13) gives that the tangential component $\Pi_\tau \widehat{\Pi}_p^{\text{curl},3d}\mathbf{u}$ vanishes on all faces of $\widehat{K}$. Finally, (2.15a), (2.15b) together with again the exact sequence property (2.12) yields $\widehat{\Pi}_p^{\text{curl},3d}\mathbf{u} = 0$. ☐

Lemma B.3. *The operator $\widehat{\Pi}_p^{\text{div},3d}$ is well-defined.*

Proof. We first show that, for $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \text{div})$ the normal trace $\mathbf{n}_f \cdot \mathbf{u} \in L^2(f)$ for each face f. To that end, one write with the aid of Lemma 5.6 $\mathbf{u} = \mathbf{curl}\, \boldsymbol{\varphi} + \mathbf{z}$ with $\boldsymbol{\varphi}$, $\mathbf{z} \in \mathbf{H}^{3/2}(\widehat{K})$. We have $\mathbf{n}_f \cdot \mathbf{z} \in \mathbf{H}^1(f)$. Noting $\boldsymbol{\varphi}|_f \in \mathbf{H}^1(f)$ and $(\mathbf{n}_f \cdot \mathbf{curl}\, \boldsymbol{\varphi})|_f = \text{curl}_f(\Pi_\tau \boldsymbol{\varphi})|_f$, we conclude that $(\mathbf{n}_f \cdot \mathbf{curl}\, \boldsymbol{\varphi})|_f \in L^2(f)$.
We check the number of conditions in (2.16). In view of the exactness of the sequence in (2.12) we get, using the notation

$$\ker \text{div} = \{\mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}) : \text{div}\, \mathbf{v} = 0\},$$

the equality

$$\dim \mathring{\mathbf{V}}_p(K) = \dim \operatorname{div} \mathring{\mathbf{V}}_p + \dim \ker \operatorname{div} = \dim \operatorname{div} \mathring{\mathbf{V}}_p + \dim \mathbf{curl}\, \mathring{\mathbf{Q}}_p$$

so that

$$\text{number of conditions in (2.16a), (2.16b) } = \dim \mathring{\mathbf{V}}_p(K).$$

Furthermore, we have

$$\text{number of conditions in (2.16c), (2.16d) } = 4 \dim W_p(f)$$

and

$$\dim \mathring{\mathbf{V}}_p(K) + 4 \dim W_p(f) = \frac{1}{2}(p+2)(p+1)p + 4\frac{(p+1)(p+2)}{2} = \dim \mathbf{V}_p$$

We check that $\mathbf{u} = 0$ implies $\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} = 0$: Conditions (2.16c), (2.16d) produce $\mathbf{n}_f \cdot \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} = 0$ for all faces $f \in \mathcal{F}(\widehat{K})$. The exact sequence property (2.12) and conditions (2.16a), (2.16b) then imply $\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} = 0$. ☐

Theorem B.4. *The diagrams* (2.8) *and* (2.11) *commute.*

Proof. Proof of $\nabla \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} = \widehat{\Pi}_p^{\mathrm{curl},3d}\nabla$*:* Let $\mathbf{u} = \nabla\varphi$ for some $\varphi \in H^2(\widehat{K})$. We first claim that

$$\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi = \nabla\varphi_p \quad \text{for some } \varphi_p \in W_{p+1}(\widehat{K}). \tag{B.1}$$

For each edge e with endpoints V_1, V_2, we compute $\int_e \mathbf{u}\cdot\mathbf{t}_e = \varphi(V_1) - \varphi(V_2)$, so that we get from (2.15f) for each face f (and orienting the tangential vectors of the edges $e \in \mathcal{E}(f)$ so that f is always "on the left")

$$\int_{\partial f} \Pi_{\tau,f}\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} = \sum_{e\subset\partial f}\int_e \mathbf{u}\cdot\mathbf{t}_e = 0. \tag{B.2}$$

We conclude with integration by parts in view of $\operatorname{curl}_f \Pi_\tau \mathbf{u} = \operatorname{curl}_f \Pi_\tau \nabla\varphi = 0$

$$\int_f \operatorname{curl}_f \Pi_\tau \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} = \int_{\partial f} \Pi_{\tau,f}\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} \overset{(B.2)}{=} 0. \tag{B.3}$$

Furthermore, the exact sequence property (2.12) gives us $\operatorname{curl}_f \mathring{\mathbf{Q}}_p(f) = \mathring{V}_p(f)$ so that (2.15c) gives

$$\operatorname{curl}_f \Pi_\tau \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} = \text{const}\,. \tag{B.4}$$

(B.3) and (B.4) together imply $\operatorname{curl}_f \Pi_\tau \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} = 0$ so that on each face $(\Pi_\tau \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u})|_f$ is a gradient of a polynomial: $(\Pi_\tau \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u})|_f = \nabla\varphi_{p,f}$ for some $\varphi_{p,f} \in W_{p+1}(f)$ for each face $f \in \mathcal{F}(\widehat{K})$.
We claim that this piecewise polynomial can be chosen to be continuous on $\partial\widehat{K}$. Fix a vertex $V \in \mathcal{V}(\widehat{K})$. By fixing the constant of the polynomials $\varphi_{p,f}$ we may assume that $\varphi_{p,f}(V) = 0$ for each face f that has V as a vertex. From (2.15e), (2.15f) we conclude that $\varphi_{p,f}$ is continuous across all edges e that have V as an endpoint. Hence, the piecewise polynomial φ_p given by $\varphi_p|_f = \varphi_{p,f}$ is continuous in all vertices of $\widehat{K}$. We conclude that φ_p is continuous on $\partial\widehat{K}$. This continuous, piecewise polynomial φ_p has, by [19,26], a polynomial lifting to $\widehat{K}$ (again denoted $\varphi_p \in W_{p+1}(\widehat{K})$). We note

$$\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} - \nabla\varphi_p \in \mathring{\mathbf{Q}}_p(\widehat{K})$$

so that (2.15a) with test function $\mathbf{v} = \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} - \nabla\varphi_p \in \mathring{\mathbf{Q}}_p(\widehat{K})$ implies

$$\mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} = 0. \tag{B.5}$$

Since the second line of (2.8) expresses an exact sequency property, we conclude that (B.1) holds.
We now show that $\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi = \nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi$. From (B.1) we get $\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi = \nabla\varphi_p$ for some $\varphi_p \in W_{p+1}(\widehat{K})$. We fix the constant in the function φ_p by stipulating $\varphi_p(V) = \varphi(V)$ for one selected vertex $V \in \mathcal{V}(\widehat{K})$. From (2.15f), we then get $\varphi(V') = \varphi_p(V')$ for all vertices $V' \in \mathcal{V}(\widehat{K})$. Next, (2.15e) and (2.14c) imply $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi = \varphi_p$ on all edges $e \in \mathcal{E}(\widehat{K})$. Comparing (2.15d) and (2.14b) reveals $\nabla_f \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi = \Pi_\tau\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi$ on each face $f \in \mathcal{F}(\widehat{K})$. Finally, comparing (2.15b) with (2.14a) shows $\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi = \nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi$.

Proof of $\mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d} = \widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}$: First, we show

$$\operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = 0. \tag{B.6}$$

To see this, we note from the second line of (2.8) that $\operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} \in W_p(\widehat{K})$. Additionally,

$$\int_{\partial\widehat{K}} \mathbf{n}\cdot\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} \overset{(2.16\mathrm{d})}{=} \int_{\partial\widehat{K}} \mathbf{n}\cdot\mathbf{curl}\,\mathbf{u} = \int_{\widehat{K}} \operatorname{div}\mathbf{curl}\,\mathbf{u} = 0. \tag{B.7}$$

Finally, the exact sequence property of the first line of (2.12) informs us that $\operatorname{div} \mathring{\mathbf{V}}_p(\widehat{K}) \to W_p^{aver}(\widehat{K})$ is surjective. Hence, we get from (2.16a) that $\operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = 0$, i.e., indeed the claim (B.6) holds. Next, (B.6) and the exact sequence property of (2.8) imply

$$\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = \mathbf{curl}\,\mathbf{u}_p \tag{B.8}$$

for some $\mathbf{u}_p \in \mathbf{Q}_p(\widehat{K})$.
We next claim $\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}$. To that end, we check that $\mathbf{v} := \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ satisfies the equations (2.16) for $\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u}$. That is, we check:

$$(\operatorname{div}(\mathbf{curl}\,\mathbf{u} - \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), \operatorname{div}\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}), \tag{B.9a}$$

$$(\mathbf{curl}\,\mathbf{u} - \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}), \tag{B.9b}$$

$$(\mathbf{n}_f\cdot(\mathbf{curl}\,\mathbf{u} - \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), v)_{L^2(f)} = 0 \quad \forall v \in \mathring{V}_p(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{B.9c}$$

$$(\mathbf{n}_f\cdot(\mathbf{curl}\,\mathbf{u} - \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), 1)_{L^2(f)} = 0 \quad \forall f \in \mathcal{F}(\widehat{K}). \tag{B.9d}$$

(B.9a) is obviously satisfied and (B.9b) is a rephrasing of (2.15a). Noting $\mathbf{n}_f\cdot\mathbf{curl} = \operatorname{curl}_f \Pi_\tau$, we rephrase (B.9c) as

$$(\operatorname{curl}_f \Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), v)_{L^2(f)} = 0 \quad \forall v \in \mathring{V}_p(f). \tag{B.10}$$

In view of the exact sequence property of (2.12), the space $\mathring{V}_p(f)$ is the image of $\operatorname{curl}_f \mathring{\mathbf{Q}}_p(f)$ so that (2.15c) implies (B.10). Finally, for (B.9d) we perform an integration by parts to get

$$(\operatorname{curl}_f \Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), 1)_{L^2(f)} = \sum_{e\subset\partial f} (\Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), \mathbf{t}_e)_{L^2(e)} \overset{(2.15\mathrm{f})}{=} 0.$$

Proof of $\operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d} = \widehat{\Pi}_p^{L^2} \operatorname{div}$*:* Again, this follows from the exact sequence property (2.12). We check that $v := \operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}$ satisfies (2.17). To that end, we note

$$(\operatorname{div}\mathbf{u} - v, 1)_{L^2(\widehat{K})} = (\operatorname{div}\mathbf{u} - \operatorname{div}\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}, 1)_{L^2(\widehat{K})} = \int_{\partial\widehat{K}} \mathbf{n}\cdot(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}) \overset{(2.16\mathrm{d})}{=} 0. \tag{B.11}$$

Furthermore, from the exact sequence property (2.12), we have that every $w \in W_p^{aver}(\widehat{K})$ has the form $w = \operatorname{div}\mathbf{w}$ for some $\mathbf{w} \in \mathring{\mathbf{V}}_p(\widehat{K})$. We therefore conclude for every $w \in W_p^{aver}(\widehat{K})$

$$(\operatorname{div}\mathbf{u} - v, w)_{L^2(\widehat{K})} = (\operatorname{div}\mathbf{u} - v, \operatorname{div}\mathbf{w})_{L^2(\widehat{K})} \overset{(2.16\mathrm{a})}{=} 0.$$

This concludes the proof of the commutativity of (2.8) in the three-dimensional setting. The commuting digram (2.11) in 2D is shown by very similar arguments. □

C Meshes and spaces

The classical example of curl-conforming and div-conforming FE spaces are the (type I) Nédélec [27] and Raviart-Thomas elements. These spaces are based on a regular, shape-regular triangulation $\mathcal{T}$ of $\Omega \subset \mathbb{R}^3$. That is, $\mathcal{T}$ satisfies:

(i) The (open) elements $K \in \mathcal{T}$ cover Ω, i.e., $\overline{\Omega} = \cup_{K\in\mathcal{T}}\overline{K}$.

(ii) Associated with each element K is the *element map*, a C^1-diffeomorphism $F_K : \overline{\widehat{K}} \to \overline{K}$. The set $\widehat{K}$ is the *reference tetrahedron*.

(iii) Denoting $h_K = \operatorname{diam} K$, there holds, with some *shape-regularity constant* γ,

$$h_K^{-1} \|F_K'\|_{L^\infty(\widehat{K})} + h_K \|(F_K')^{-1}\|_{L^\infty(\widehat{K})} \le \gamma.$$

(iv) The elements $K \in \mathcal{T}_h$ cover Ω. Their intersection is only empty, a vertex, an edge, a face, or they coincide (here, vertices, edges, and faces are the images of the corresponding entities on the reference tetrahedron $\widehat{K}$). The parametrization of common edges or faces are compatible. That is, if two elements K, K' share an edge (i.e., $F_K(e) = F_K(e')$ for edges e, e' of $\widehat{K}$) or a face (i.e., $F_K(f) = F_K(f')$ for faces f, f' of $\widehat{K}$), then $F_K^{-1} \circ F_{K'} : f' \to f$ is an affine isomorphism.

The global finite element spaces $S_{p+1}(\mathcal{T})$, $\boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\mathcal{T})$, $\mathbf{RT}_p(\mathcal{T})$ on Ω are defined as in [25, (3.76), (3.77)] by transforming covariantly $\boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\widehat{K})$ and $\mathbf{RT}_p(\widehat{K})$ with the aid of the Piola transform:

$$S_{p+1}(\mathcal{T}) := \{u \in H^1(\Omega) \,|\, u|_K \circ F_K \in \mathcal{P}_{p+1}(\widehat{K})\}, \tag{C.1a}$$

$$\boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\mathcal{T}) := \{\mathbf{u} \in \mathbf{H}(\Omega, \mathbf{curl}) \,|\, (F_K')^T \mathbf{u}|_K \circ F_K \in \boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\widehat{K})\}, \tag{C.1b}$$

$$\mathbf{RT}_p(\mathcal{T}) := \{\mathbf{u} \in \mathbf{H}(\Omega, \operatorname{div}) \,|\, (\det F_K')(F_K')^{-1} \mathbf{u}|_K \circ F_K \in \mathbf{RT}_p(\widehat{K})\}, \tag{C.1c}$$

We restrict our attention to approximation operators that are constructed element-by-element.

Definition C.1 (element-by-element construction)**.** *An operator* $\widehat{\Pi}^{\mathrm{grad}} : H^2(\widehat{K}) \to \mathcal{P}_{p+1}$ *is said to admit* element-by-element construction *if the operator* $\Pi^{\mathrm{grad}} : H^1(\Omega) \cap \prod_{K \in \mathcal{T}} H^2(K)$ *defined elementwise by* $(\Pi^{\mathrm{grad}} u)|_K := (\widehat{\Pi}^{\mathrm{grad}}(u \circ F_K)) \circ F_K^{-1}$ *maps into the conforming subspace* $S^{p+1}(\mathcal{T}) \subset H^1(\Omega)$.
An operator $\widehat{\Pi}^{\mathrm{curl}} : \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \to \boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\widehat{K})$ *is said to admit* element-by-element construction *if the operator* $\Pi^{\mathrm{curl}} : \mathbf{H}(\Omega, \mathbf{curl}) \cap \prod_{K \in \mathcal{T}} \mathbf{H}^1(K, \mathbf{curl})$ *defined elementwise by* $(\Pi^{\mathrm{curl}} u)|_K := (F_K')^{-T}(\widehat{\Pi}^{\mathrm{curl}}((F_K')^T \mathbf{u} \circ F_K)) \circ F_K^{-1}$ *maps into the conforming subspace* $\boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\mathcal{T}) \subset \mathbf{H}(\Omega, \mathbf{curl})$.
An operator $\widehat{\Pi}^{\mathrm{div}} : \mathbf{H}^1(\widehat{K}, \operatorname{div}) \to \mathbf{RT}_p(\widehat{K})$ *is said to admit* element-by-element construction *if the operator* $\Pi^{\mathrm{div}} : \mathbf{H}(\Omega, \operatorname{div}) \cap \prod_{K \in \mathcal{T}} \mathbf{H}^1(K, \operatorname{div})$ *defined elementwise by*

$$(\Pi^{\mathrm{div}} u)|_K := (\det(F_K'))^{-1} F_K' (\widehat{\Pi}^{\mathrm{div}} (\det F_K')(F_K')^{-1} \mathbf{u} \circ F_K)) \circ F_K^{-1}$$

maps into the conforming subspace $\mathbf{RT}_p(\mathcal{T}) \subset \mathbf{H}(\Omega, \operatorname{div})$.

Acknowledgement

JMM is grateful to his colleague Joachim Schöberl (TU Wien) for inspiring discussions on the topic of the paper and, in particular, for pointing out the arguments of Theorem 4.8.

References

[1] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] Mark Ainsworth and Leszek Demkowicz. Explicit polynomial preserving trace liftings on a triangle. *Math. Nachr.*, 282(5):640–658, 2009.

[3] T. Apel and J.M. Melenk. Interpolation and quasi-interpolation in h- and hp-version finite element spaces. In E. Stein, R. de Borst, and T.J.R. Hughes, editors, *Encyclopedia of Computational Mechanics*. 2017.

[4] I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta. Efficient preconditioning for the p version finite element method in two dimensions. *SIAM J. Numer. Anal.*, 28(3):624–661, 1991.

[5] I. Babuška and M. Suri. The p and h-p versions of the finite element method, basic principles and properties. *SIAM review*, 36(4):578–632, 1994.

[6] Alexei Bespalov and Norbert Heuer. Optimal error estimation for H(curl)-conforming p-interpolation in two dimensions. *SIAM J. Numer. Anal.*, 47(5):3977–3989, 2009.

[7] M. Sh. Birman and M. Z. Solomyak. L_2-theory of the Maxwell operator in arbitrary domains. *Uspekhi Mat. Nauk*, 42(6(258)):61–76, 247, 1987.

[8] D. Boffi, M. Costabel, M. Dauge, L. Demkowicz, and R. Hiptmair. Discrete compactness for the p-version of discrete differential forms. *SIAM J. Numer. Anal.*, 49(1):135–158, 2011.

[9] A. Buffa and P. Ciarlet, Jr. On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Math. Methods Appl. Sci.*, 24(1):31–48, 2001.

[10] A. Buffa and P. Ciarlet, Jr. On traces for functional spaces related to Maxwell's equations. Part I: An integration by parts formula in Lipschitz polyhedra. *Math. Meth. Appl. Sci.*, 21:9–30, 2001.

[11] W. Cao and L. Demkowicz. Optimal error estimate of a projection based interpolation for the p-version approximation in three dimensions. *Comput. Math. Appl.*, 50(3-4):359–366, 2005.

[12] M. Costabel, M. Dauge, and L. Demkowicz. Polynomial extension operators for H^1, $H(\mathrm{curl})$ and $H(\mathrm{div})$-spaces on a cube. *Math. Comp.*, 77(264):1967–1999, 2008.

[13] Martin Costabel and Alan McIntosh. On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Math. Z.*, 265(2):297–320, 2010.

[14] M. Dauge. *Elliptic boundary value problems on corner domains*, volume 1341 of *Lecture Notes in Mathematics*. Springer Verlag, 1988.

[15] L. Demkowicz. Polynomial exact sequences and projection-based interpolation with applications to Maxwell's equations. In D. Boffi, F. Brezzi, L. Demkowicz, L.F. Durán, R. Falk, and M. Fortin, editors, *Mixed Finite Elements, Compatibility Conditions, and Applications*, volume 1939 of *Lectures Notes in Mathematics*. Springer Verlag, 2008.

[16] L. Demkowicz and I. Babuška. p interpolation error estimates for edge finite elements of variable order in two dimensions. *SIAM J. Numer. Anal.*, 41(4):1195–1208, 2003.

[17] L. Demkowicz and A. Buffa. H^1, $H(\mathrm{curl})$ and $H(\mathrm{div})$-conforming projection-based interpolation in three dimensions. Quasi-optimal p-interpolation estimates. *Comput. Methods Appl. Mech. Engrg.*, 194(2-5):267–296, 2005.

[18] L. Demkowicz, P. Monk, L. Vardapetyan, and W. Rachowicz. de Rham diagram for *hp* finite element spaces. *Comput. Math. Appl.*, 39(7-8):29–38, 2000.

[19] Leszek Demkowicz, Jayadeep Gopalakrishnan, and Joachim Schöberl. Polynomial extension operators. I. *SIAM J. Numer. Anal.*, 46(6):3006–3031, 2008.

[20] Leszek Demkowicz, Jayadeep Gopalakrishnan, and Joachim Schöberl. Polynomial extension operators. II. *SIAM J. Numer. Anal.*, 47(5):3293–3324, 2009.

[21] Leszek Demkowicz, Jayadeep Gopalakrishnan, and Joachim Schöberl. Polynomial extension operators. Part III. *Math. Comp.*, 81(279):1289–1326, 2012.

[22] P.G. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston, 1985.

[23] R. Hiptmair. Discrete compactness for p-version of tetrahedral edge elements. Technical report, 2008. arxiv 0901.0761.

[24] J.M. Melenk. *hp*-interpolation of nonsmooth functions and an application to *hp*-a posteriori error estimation. *SIAM J. Numer. Anal.*, 43(1):127–155, 2005.

[25] Peter Monk. *Finite element methods for Maxwell's equations*. Oxford University Press, New York, 2003.

[26] R. Muñoz-Sola. Polynomial liftings on a tetrahedron and applications to the *hp*-version of the finite element method in three dimensions. *SIAM J. Numer. Anal.*, 34(1):282–314, 1997.

[27] J.-C. Nédélec. Mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 35(3):315–341, 1980.

[28] Jukka Saranen. On an inequality of Friedrichs. *Math. Scand.*, 51(2):310–322 (1983), 1982.

[29] Ch. Schwab. *p- and hp-finite element methods.* The Clarendon Press Oxford University Press, New York, 1998. Theory and applications in solid and fluid mechanics.