ASC Report No. 13/2017

Tensor FEM for spectral fractional diffusion

L. Banjai, J.M. Melenk, R.H. Nochetto, E. Otarola, A. Salgado, and C. Schwab

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-00-1

Most recent ASC Reports

12/2017 *S. Börm, C. Börst, J.M. Melenk*
An analysis of a butterfly algorithm

11/2017 *X. Chen and A. Jüngel*
A note on the uniqueness of weak solutions to a class of cross-diffusion styles

10/2017 *A. Gerstenmayer and A. Jüngel*
Analysis of a degenerate parabolic cross-diffusion system for ion transport

09/2017 *M. Löhndorf and J.M. Melenk*
On thin plate spline interpolation

08/2017 *A. Sokolova and H. Woracek*
Termination in Convex Sets of Distrubtions

07/2017 *R. Pruckner*
Density of the spectrum of Jacobi matrices with power asymptotics

06/2017 *T. Führer and D. Praetorius*
A linear Uzawa-type solver for nonlinear transmission problems

05/2017 *A. Jüngel and S. Schuchnigg*
A discrete Bakry-Emery method and its application to the porous-medium equation

04/2017 *M. Bulicek, M. Pokorny, and N. Zamponi*
Existence analysis for incompressible fluid model of electrically charged chemically reacting and heat conducting mixtures

03/2017 *M. Faustmann, J.M. Melenk*
Local convergence of the boundary element method on polyhedral domains

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-00-1

TENSOR FEM FOR SPECTRAL FRACTIONAL DIFFUSION*

LEHEL BANJAI[†], JENS M. MELENK[‡], RICARDO H. NOCHETTO[§], ENRIQUE OTÁROLA[¶], ABNER J. SALGADO[‖], AND CHRISTOPH SCHWAB[**]

Abstract. We design and analyze several Finite Element Methods (FEMs) applied to the Caffarelli-Silvestre extension that localizes the fractional powers of symmetric, coercive, linear elliptic operators in bounded domains with Dirichlet boundary conditions. We consider open, bounded, polytopal but not necessarily convex domains $\Omega \subset \mathbb{R}^d$ with $d = 1, 2$. For the solution to the extension problem, we establish analytic regularity with respect to the extended variable $y \in (0, \infty)$. We prove that the solution belongs to countably normed, power–exponentially weighted Bochner spaces of analytic functions with respect to y, taking values in corner-weighted Kondat'ev type Sobolev spaces in Ω. In $\Omega \subset \mathbb{R}^2$, we discretize with continuous, piecewise linear, Lagrangian FEM (P_1-FEM) with mesh refinement near corners, and prove that first order convergence rate is attained for compatible data $f \in \mathbb{H}^{1-s}(\Omega)$.

We also prove that tensorization of a P_1-FEM in Ω with a suitable *hp*-FEM in the extended variable achieves log-linear complexity with respect to $\mathcal{N}_\Omega$, the number of degrees of freedom in the domain Ω. In addition, we propose a novel, *sparse tensor product FEM* based on a multilevel P_1-FEM in Ω and on a P_1-FEM on radical–geometric meshes in the extended variable. We prove that this approach also achieves log-linear complexity with respect to $\mathcal{N}_\Omega$. Finally, under the stronger assumption that the data be analytic in $\overline{\Omega}$, and without compatibility at $\partial\Omega$, we establish *exponential rates of convergence of hp-FEM for spectral, fractional diffusion operators* in energy norm. This is achieved by a combined tensor product *hp*-FEM for the Caffarelli-Silvestre extension in the truncated cylinder $\Omega \times (0, \mathcal{Y})$ with anisotropic geometric meshes that are refined towards $\partial\Omega$. We also report numerical experiments for model problems which confirm the theoretical results. We indicate several extensions and generalizations of the proposed methods to other problem classes and to other boundary conditions on $\partial\Omega$.

Key words. Fractional diffusion, nonlocal operators, weighted Sobolev spaces, regularity estimates, finite elements, anisotropic *hp*–refinement, corner refinement, sparse grids, exponential convergence.

AMS subject classifications. 26A33, 65N12, 65N30.

1. Introduction.

We are interested in the design and analysis of a variety of efficient numerical techniques to solve problems involving certain fractional powers of the linear, elliptic, self-adjoint, second order, differential operator $\mathcal{L}w = -\mathrm{div}(A\nabla w)+cw$, supplemented with homogeneous Dirichlet boundary conditions. The coefficient $A \in L^\infty(\Omega, \mathsf{GL}(\mathbb{R}^d))$ is symmetric and uniformly positive definite and $0 \leq c \in L^\infty(\Omega, \mathbb{R})$

*The results in this paper were obtained when the authors met at the MFO Oberwolfach during the WS 1711 in March 2017. The research of RHN was supported in part by NSF grants DMS-1109325 and DMS-1411808. The research of EO was supported in part by CONICYT through project FONDECYT 3160201. The research of AJS was supported in part by NSF grant DMS-1418784. The research of JMM was supported by the Austrian Science Fund (FWF) project F 65.

[†]Maxwell Institute for Mathematical Sciences, School of Mathematical & Computer Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK (l.banjai@hw.ac.uk).

[‡]Institut für Analysis und Scientific Computing, Technische Universität Wien, A-1040 Vienna, Austria (melenk@tuwien.ac.at).

[§]Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA (rhn@math.umd.edu).

[¶]Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile (enrique.otarola@usm.cl).

[‖]Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA (asalgad1@utk.edu).

[**]Seminar for Applied Mathematics, ETH Zürich, ETH Zentrum, HG G57.1, CH8092 Zürich, Switzerland (christoph.schwab@sam.math.ethz.ch).

(additional regularity requirements will be imposed in the course of our convergence rate analysis ahead). We denote by Ω a bounded domain of $\mathbb{R}^d$ $(d = 1, 2)$, with Lipschitz boundary $\partial\Omega$ and further properties imposed as required: the FEM convergence theory in Section 5 will focus on polygonal domains $\Omega \subset \mathbb{R}^2$, the hp-FEM results in Section 7 require analytic $\partial\Omega$.

The Dirichlet problem for the fractional Laplacian is as follows: Given a function f and $s \in (0, 1)$, we seek u such that

$$\mathcal{L}^s u = f \quad \text{in } \Omega \,. \tag{1.1}$$

An essential difficulty in the analysis of (1.1) and in the design of efficient numerical methods for this problem is that $\mathcal{L}^s$ is a nonlocal operator [13, 14, 15, 17, 32]. In the case of the Dirichlet Laplacian $\mathcal{L} = -\Delta$, Caffarelli and Silvestre in [15] localize it by using a nonuniformly elliptic PDE posed in one more spatial dimension. They showed that any power $s \in (0, 1)$ of the fractional Laplacian in $\mathbb{R}^d$ can be realized as the Dirichlet-to-Neumann map of an extension to the upper half-space $\mathbb{R}^{d+1}_+$. This result was extended by Cabré and Tan [14] and by Stinga and Torrea [55] to bounded domains Ω and more general operators, thereby obtaining an extension posed on the semi–infinite cylinder $\mathcal{C} := \Omega \times (0, \infty)$; we also refer to [17]. This extension is the following local boundary value problem

$$\begin{cases} \mathfrak{L}\mathscr{U} = -\mathrm{div}\,(y^\alpha \boldsymbol{A}\nabla\mathscr{U}) + cy^\alpha\mathscr{U} = 0 & \text{in } \mathcal{C}, \\ \mathscr{U} = 0 & \text{on } \partial_L\mathcal{C}, \\ \partial_{\nu^\alpha}\mathscr{U} = d_s f & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.2}$$

where $\boldsymbol{A} = \mathrm{diag}\{A, 1\} \in L^\infty(\bar{\mathcal{C}}, \mathsf{GL}(\mathbb{R}^{d+1}))$, $\partial_L\mathcal{C} := \partial\Omega \times (0, \infty)$ signifies the lateral boundary of $\mathcal{C}$, $d_s := 2^{1-2s}\Gamma(1-s)/\Gamma(s)$ is a positive normalization constant and the parameter α is defined as $\alpha = 1 - 2s \in (-1, 1)$ [15, 55]. The so–called conormal exterior derivative of $\mathscr{U}$ at $\Omega \times \{0\}$ is

$$\partial_{\nu^\alpha}\mathscr{U} = -\lim_{y\to 0^+} y^\alpha \mathscr{U}_y. \tag{1.3}$$

We shall refer to y as the *extended variable* and to the dimension $d+1$ in $\mathbb{R}^{d+1}_+$ the *extended dimension* of problem (1.2). Throughout the text, points $x \in \mathcal{C}$ will be written as $x = (x', y)$ with $x' \in \Omega$ and $y > 0$. The limit in (1.3) must be understood in the distributional sense [14, 15, 55]. With the extension $\mathscr{U}$ at hand, the fractional powers of $\mathcal{L}$ in (1.1) and the Dirichlet-to-Neumann operator of problem (1.2) are related by

$$d_s\mathcal{L}^s u = \partial_{\nu^\alpha}\mathscr{U} \quad \text{in } \Omega \,. \tag{1.4}$$

In [41] the extension problem (1.2) was first used as a way to obtain a numerical technique to approximate the solution to (1.1). A piecewise linear finite element method (P_1-FEM) was proposed and analyzed. In this work, we extend the results of [41] in several directions:

a) In Theorem 5.9, we generalize the error analysis of [41], based on the localization of $\mathcal{L}^s$ given by (1.2), to nonconvex polygonal domains $\Omega \subset \mathbb{R}^2$, under the requirement of Lipschitz regularity in Ω for A and c, and for $f \in \mathbb{H}^{1-s}(\Omega)$ in (2.2) ahead.

b) In Theorem 4.7 we prove, again under Lipschitz regularity in Ω for A and c, *weighted H^2 (with respect to the extended variable y) regularity estimates* for the solution $\mathscr{U}$ of (1.2). We use these to propose a novel, sparse tensor product P_1-FEM in $\mathcal{C}$ which is realized by invoking (in parallel) $\mathcal{O}(\log \mathcal{N}_\Omega)$ many instances of anisotropic tensor product P_1-FEM in $\mathcal{C}$. We prove, in Theorem 5.12, that, when the base of the cylinder $\mathcal{C}$ is a polygonal domain $\Omega \subset \mathbb{R}^2$, this approach yields a method with $\mathcal{O}(\mathcal{N}_\Omega \log \mathcal{N}_\Omega)$ degrees of freedom realizing the (optimal) asymptotic convergence rate of $\mathcal{N}_\Omega^{-1/2}$.

c) We show, in Theorem 5.14, that a full tensor product approach of an *hp*-FEM in the extended variable y with P_1-FEM in Ω yields the same rate. To achieve this, we establish *weighted analytic regularity of* $\mathscr{U}$ *with respect to the extended variable* y, in terms of countably normed weighted Bochner-Sobolev spaces. This extends, in the case $d = 2$, recent work [33] to a general diffusion operator $\mathcal{L}$ in (1.1) and to nonconvex, polygonal domains, under the requirement of Lipschitz regularity in Ω for A and c.

d) We propose in Section 6 a novel *diagonalization technique* which decouples the degrees of freedom introduced by a Galerkin (semi-)discretization in the extended variable. It reduces the y-semidiscrete Caffarelli-Stinga extension to the solution of independent, singularly perturbed *second order* reaction-diffusion equations in Ω. This decoupling allows us to establish exponential convergence for *analytic data* f *without boundary compatibility* as discussed in the following item e). The diagonalization also permits to block-diagonalize the stiffness matrix of the fully discrete problem with corresponding befits for the solver complexity of the linear system of equations.

e) We establish an *exponential convergence* rate (7.8) of a local *hp*-FEM for the fractional differential operator $\mathfrak{L}$ in (1.2). This requires, however, the data A, c and f to be *analytic* in $\bar{\Omega}$ and the boundary $\partial\Omega$ to be analytic as well. For brevity of exposition, we detail the mathematical argument in intervals $\Omega \subset \mathbb{R}^1$ and in bounded domains $\Omega \subset \mathbb{R}^2$ with analytic boundary $\partial\Omega$, and for constant coefficients A and c, and only outline the necessary extensions, with references, for polygons $\Omega \subset \mathbb{R}^2$; see Theorems 7.3, 7.7 and Remark 7.8.

f) We present numerical experiments in each of the previous cases which illustrate our results, and indicate their sharpness.

g) We indicate how the presently developed discretizations and error bounds extend in several directions, in particular to three dimensional polyhedral domains Ω, to Neumann or mixed Dirichlet-Neumann boundary conditions on $\partial\Omega$, etc.

To close the introduction, we comment on other numerical approaches to fractional PDEs. In addition to [41], numerical schemes that deal with spectral fractional powers of elliptic operators have been proposed in [33] and [11]. The very recent work [33] adopts the same Galerkin framework as [41] and the present article and, independently, proposes to use high order discretizations in the extended variable to exploit analyticity. The starting point of [11] is the so-called Balakrishnan formula, a contour integral representation of the inverse $\mathcal{L}^{-s}$. Upon discretizing the integral by a suitable quadrature formula, the numerical scheme of [11] results in a collection of (decoupled) singularly perturbed reaction diffusion problems in Ω. This connects [11] with our approach in Section 7. However, the decoupled reaction diffusion problems in Ω which arise in our approach result from a Galerkin discretization in the extended variable. For the integral definition of the fractional Laplacian in several dimensions we mention, in particular, the analysis of [2, 21]. We refer the reader to [10] for a detailed account of all the approaches mentioned above.

2. Notation and preliminaries. We adopt the notation of [41, 45]: For $\mathcal{Y} > 0$ the truncated cylinder with base Ω and height $\mathcal{Y}$ is $\mathcal{C}_\mathcal{Y} = \Omega \times (0, \mathcal{Y})$, its lateral boundary is $\partial_L \mathcal{C}_\mathcal{Y} = \partial\Omega \times (0, \mathcal{Y})$. If $x \in \mathcal{C}$ we set $x = (x', y)$ with $x' \in \Omega$ and $y \in (0, \infty)$. By $a \lesssim b$ we mean $a \le Cb$, with a constant C that neither depends on a, b or the discretization parameters. The notation $a \sim b$ signifies $a \lesssim b \lesssim a$. The value of C might change at each occurrence.

2.1. Fractional powers of elliptic operators. To define $\mathcal{L}^s$, as in [41], we invoke spectral theory [9]. The operator $\mathcal{L}$ induces an inner product $a_\Omega(\cdot,\cdot)$ on $H_0^1(\Omega)$

$$a_\Omega(w, v) = \int_\Omega (A\nabla w \cdot \nabla v + cwv) \, \mathrm{d}x', \tag{2.1}$$

and $\mathcal{L}$ is an isomorphism $H_0^1(\Omega) \to H^{-1}(\Omega)$ given by $u \mapsto a_\Omega(u, \cdot)$. The eigenvalue problem: Find $(\lambda, \phi) \in \mathbb{R} \times H_0^1(\Omega) \setminus \{0\}$ such that

$$a_\Omega(\phi, v) = \lambda(\phi, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

has a countable collection of solutions $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \times H_0^1(\Omega)$, with the real eigenvalues enumerated in increasing order, counting multiplicities, and such that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $(H_0^1(\Omega), a_\Omega(\cdot,\cdot))$. In terms of these eigenpairs, we introduce, for $s \ge 0$, the spaces

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^\infty w_k \varphi_k : \|w\|^2_{\mathbb{H}^s(\Omega)} = \sum_{k=1}^\infty \lambda_k^s w_k^2 < \infty \right\}. \tag{2.2}$$

We denote by $\mathbb{H}^{-s}(\Omega)$ the dual space of $\mathbb{H}^s(\Omega)$. The duality pairing between $\mathbb{H}^s(\Omega)$ and $\mathbb{H}^{-s}(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$. Through this duality pairing, we identify elements of $f \in \mathbb{H}^{-s}(\Omega)$ with sequences $(f_k)_k$ with $\sum_k f_k^2 \lambda_k^{-2s} = \|f\|^2_{\mathbb{H}^{-s}(\Omega)}$, which allows us to extend the definition of the norm in (2.2) to $s < 0$. We have the isometries $\|w\|^2_{L^2(\Omega)} = \|w\|^2_{\mathbb{H}^0}$ and $a_\Omega(w, w) = \|w\|^2_{\mathbb{H}^1}$; by (real) interpolation between $L^2(\Omega)$ and $H_0^1(\Omega)$, we infer for $s \in (0,1)$ that $\mathbb{H}^s(\Omega) = [L^2(\Omega), H_0^1(\Omega)]_s$.

For functions $w = \sum_k w_k \varphi_k \in \mathbb{H}^1(\Omega)$, the operator $\mathcal{L} : \mathbb{H}^1(\Omega) \to \mathbb{H}^{-1}(\Omega)$ takes the form $\mathcal{L}w = \sum_k \lambda_k w_k \varphi_k$. For $s \in (0,1)$ and $w = \sum_k w_k \varphi_k \in \mathbb{H}^s(\Omega)$, the operator $\mathcal{L}^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$ is defined by

$$\mathcal{L}^s w = \sum_{k=1}^\infty \lambda_k^s w_k \varphi_k. \tag{2.3}$$

2.2. The extension property. Both extensions, the one by Caffarelli–Silvestre for $\Omega = \mathbb{R}^d$ [15] and that of Cabré–Tan [14] and Stinga–Torrea for Ω bounded and general elliptic operators [55] require us to deal with the *nonuniformly* (but local) linear, second order elliptic equation (1.2). Here, Lebesgue and Sobolev spaces with the weight y^α for $\alpha \in (-1, 1)$ [12, 14, 15, 17] naturally arise. If $D \subset \mathbb{R}^{d+1}$, we define $L^2(y^\alpha, D)$ as the Lebesgue space for the measure $|y|^\alpha \, \mathrm{d}x$. We also define the weighted Sobolev space

$$H^1(y^\alpha, D) = \left\{ w \in L^2(y^\alpha, D) : |\nabla w| \in L^2(y^\alpha, D) \right\},$$

where ∇w is the distributional gradient of w. We equip $H^1(y^\alpha, D)$ with the norm

$$\|w\|_{H^1(y^\alpha,D)} = \left(\|w\|^2_{L^2(y^\alpha,D)} + \|\nabla w\|^2_{L^2(y^\alpha,D)} \right)^{\frac{1}{2}}. \tag{2.4}$$

In view of the fact that $\alpha \in (-1,1)$, the weight y^α belongs to the Muckenhoupt class $A_2(\mathbb{R}^{d+1})$ [23, 24, 27, 39, 56]. This, in particular, implies that $H^1(y^\alpha, D)$ with norm (2.4) is Hilbert and $C^\infty(D) \cap H^1(y^\alpha, D)$ is dense in $H^1(y^\alpha, D)$ (cf. [56, Proposition 2.1.2, Corollary 2.1.6], [31] and [27, Theorem 1]).

To analyze problem (1.2) we define the weighted Sobolev space

$$\mathring{H}^1(y^\alpha, \mathcal{C}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}) : w = 0 \text{ on } \partial_L \mathcal{C} \right\}. \tag{2.5}$$

As [41, inequality (2.21)] shows, the following *weighted Poincaré inequality* holds:

$$\|w\|_{L^2(y^\alpha,\mathcal{C})} \lesssim \|\nabla w\|_{L^2(y^\alpha,\mathcal{C})} \quad \forall w \in \mathring{H}^1(y^\alpha, \mathcal{C}). \tag{2.6}$$

Consequently, the seminorm on $\mathring{H}^1(y^\alpha, \mathcal{C})$ is equivalent to (2.4). For $w \in H^1(y^\alpha, \mathcal{C})$, $\operatorname{tr}_\Omega w$ denotes its trace onto $\Omega \times \{0\}$ which satisfies (see [41, Proposition 2.5])

$$\operatorname{tr}_\Omega \mathring{H}^1(y^\alpha, \mathcal{C}) = \mathbb{H}^s(\Omega), \qquad \|\operatorname{tr}_\Omega w\|_{\mathbb{H}^s(\Omega)} \le C_{\operatorname{tr}_\Omega} \|w\|_{\mathring{H}^1(y^\alpha,\mathcal{C})}. \tag{2.7}$$

Define the bilinear form $a_\mathcal{C} : \mathring{H}^1(y^\alpha, \mathcal{C}) \times \mathring{H}^1(y^\alpha, \mathcal{C}) \to \mathbb{R}$ by

$$a_\mathcal{C}(v, w) = \int_\mathcal{C} y^\alpha (\boldsymbol{A} \nabla v \cdot \nabla w + cvw) \, \mathrm{d}x' \, \mathrm{d}y, \tag{2.8}$$

and note that it is continuous and, owing to (2.6), it is also coercive. Consequently, it induces an inner product on $\mathring{H}^1(y^\alpha, \mathcal{C})$ and the *energy norm* $\|\cdot\|_\mathcal{C}$:

$$\|v\|_\mathcal{C}^2 := a_\mathcal{C}(v, v) \sim \|\nabla v\|^2_{L^2(y^\alpha,\mathcal{C})}. \tag{2.9}$$

Occasionally, we will restrict the integration to the truncated cylinder $\mathcal{C}_\mathcal{Y}$. The corresponding bilinear form and norm are denoted by

$$a_{\mathcal{C}_\mathcal{Y}}(v, w) := \int_{\mathcal{C}_\mathcal{Y}} y^\alpha (\boldsymbol{A} \nabla v \cdot \nabla w + cvw) \, \mathrm{d}x' \, \mathrm{d}y, \qquad \|v\|^2_{\mathcal{C}_\mathcal{Y}} = a_{\mathcal{C}_\mathcal{Y}}(v, v). \tag{2.10}$$

With these definitions at hand, the weak formulation of (1.2) reads: Find $\mathscr{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$ such that

$$a_\mathcal{C}(\mathscr{U}, v) = d_s \langle f, \operatorname{tr}_\Omega v \rangle \qquad \forall v \in \mathring{H}^1(y^\alpha, \mathcal{C}). \tag{2.11}$$

The fundamental result of Caffarelli and Silvestre [15] then reads as follows (see also [14, Proposition 2.2] and [55, Theorem 1.1] for bounded domains and for general elliptic operators): given $f \in \mathbb{H}^{-s}(\Omega)$, let $u \in \mathbb{H}^s(\Omega)$ solve (1.1). If $\mathscr{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$ solves (2.11), then $u = \operatorname{tr}_\Omega \mathscr{U}$ and

$$d_s \mathcal{L}^s u = \partial_{\nu^\alpha} \mathscr{U} \quad \text{in } \Omega. \tag{2.12}$$

3. A first order FEM for fractional diffusion. The first work that, in a numerical setting, exploits the identity (2.12) for the design and analysis of a finite element approximation of solutions to (1.1) is [41]; see also [45]. Let us briefly review the main results of [41].

First, [41] truncates $\mathcal{C}$ to $\mathcal{C}_\mathcal{Y}$ and places homogeneous Dirichlet boundary conditions on $y = \mathcal{Y}$, thus obtaining an approximation $\mathcal{U}$ (which, by slight abuse of notation, is understood to coincide with its extension by zero from $\mathcal{C}_\mathcal{Y}$ to $\mathcal{C}$). The error committed in this approximation is exponentially small: There holds (see [41, Theorem 3.5])

$$\|\nabla(\mathscr{U} - \mathcal{U})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4}\|f\|_{\mathbb{H}^{-s}(\Omega)},$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $\mathcal{L}$.

Second, [41] develops a regularity theory for $\mathscr{U}$ in weighted Sobolev spaces; see Theorem 4.7 below for a generalization. These results reveal that the second order regularity of $\mathscr{U}$ in the extended direction is lost as $y \downarrow 0$. Thus, *graded* meshes in the extended variable y play a fundamental role. In the notation of the present work, with a mesh $\mathcal{T}$ on Ω and a mesh $\mathcal{G}^M$ on $(0,\mathcal{Y})$ that is graded towards $y = 0$, the truncated cylinder $\mathcal{C}_\mathcal{Y}$ is partitioned by tensor product elements $K \times I$ with $K \in \mathcal{T}$ and $I \in \mathcal{G}^M$. On this mesh, the tensor product space $\mathbb{V}^{1,1}_{h,M}(\mathcal{T},\mathcal{G}^M)$ of piecewise bilinears in $\Omega \times (0,\mathcal{Y})$ (see (5.1) for the precise definition) is used in a Galerkin method. The Galerkin approximation $\mathscr{U}_{h,M} \in \mathbb{V}^{1,1}_{h,M}(\mathcal{T},\mathcal{G}^M)$ of $\mathcal{U}$ satisfies a best approximation property à la Céa. From there, upon studying piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces [41, 42] error estimates were obtained under the assumption that $f \in \mathbb{H}^{1-s}(\Omega)$ and that Ω is convex (see [41, Theorem 5.4] and [41, Corollary 7.11]):

Theorem 3.1 (a priori error estimate). *Let $\mathcal{G}^M$ be suitably graded towards $y = 0$ and $\mathbb{V}^{1,1}_{h,M}$ be constructed with tensor product elements and $\mathscr{U}_{h,M} \in \mathbb{V}^{1,1}_{h,M}$ denote the Galerkin approximation to $\mathcal{U}$. Then, for suitable truncation parameter $\mathcal{Y} \sim \log \mathcal{N}_{\Omega,\mathcal{Y}}$ we have, with the total number of unkowns $\mathcal{N}_{\Omega,\mathcal{Y}} := \#\mathcal{T}\#\mathcal{G}^M$*

$$\begin{aligned}\|u - \operatorname{tr}_\Omega \mathscr{U}_{h,M}\|_{\mathbb{H}^s(\Omega)} &\lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \\ &\lesssim |\log \mathcal{N}_{\Omega,\mathcal{Y}}|^s(\mathcal{N}_{\Omega,\mathcal{Y}})^{-1/(d+1)}\|f\|_{\mathbb{H}^{1-s}(\Omega)}.\end{aligned}$$

Remark 3.2 (complexity). *Up to logarithmic factors, Theorem 3.1 yields rates of convergence of $(\mathcal{N}_{\Omega,\mathcal{Y}})^{-1/(d+1)}$. In terms of error versus work, this P_1-FEM is sub-optimal as a method to compute in Ω. In this paper we propose and study P_1-FE methods in Ω that afford an error decay $(\mathcal{N}_{\Omega,\mathcal{Y}})^{-1/d}$ (up to possibly logarithmic terms).* ∎

4. Analytic regularity.

We obtain regularity results for the solution of (1.2) that will underlie the analysis of the various FEMs in Section 5 and 7. We begin by recalling that if $u = \sum_{k=1}^\infty u_k\varphi_k$ solves (1.1), then the unique solution $\mathscr{U}$ of problem (1.2) admits the representation [41, formula (2.24)]

$$\mathscr{U}(x',y) = \sum_{k=1}^{\infty} u_k\varphi_k(x')\psi_k(y), \quad u_k := \lambda_k^{-s}f_k. \tag{4.1}$$

We also recall that $\{\lambda_k,\varphi_k\}_{k\in\mathbb{N}}$ is the set of eigenpairs of the elliptic operator $\mathcal{L}$, supplemented with homogeneous Dirichlet boundary conditions. The functions ψ_k solve

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_k(y) + \dfrac{\alpha}{y}\dfrac{\mathrm{d}}{\mathrm{d}y}\psi_k(y) - \lambda_k\psi_k(y) = 0, & y \in (0,\infty), \\ \psi_k(0) = 1, & \lim\limits_{y\to\infty}\psi_k(y) = 0. \end{cases} \tag{4.2}$$

Thus, if $s = \frac{1}{2}$, we have $\psi_k(y) = \exp(-\sqrt{\lambda_k} y)$ [14, Lemma 2.10]; more generally, if $s \in (0,1) \setminus \{\frac{1}{2}\}$, then [17, Proposition 2.1]

$$\psi_k(y) = c_s(\sqrt{\lambda_k} y)^s K_s(\sqrt{\lambda_k} y),$$

where $c_s = 2^{1-s}/\Gamma(s)$ and K_s denotes the modified Bessel function of the second kind. We refer the reader to [1, Chapter 9.6] for a comprehensive treatment of the Bessel function K_s and recall the following properties.

LEMMA 4.1 (properties of K_ν). *The modified Bessel function of the second kind K_ν satisfies:*

(i) For $\nu > -1$ and $z > 0$, $K_\nu(z)$ is real and positive [1, Chapter 9.6].
(ii) For $\nu \in \mathbb{R}$, $K_\nu(z) = K_{-\nu}(z)$ [1, Chapter 9.6].
(iii) For $\nu > 0$, [1, estimate (9.6.9)]

$$\lim_{z \downarrow 0} \frac{K_\nu(z)}{\frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}} = 1. \tag{4.3}$$

(iv) For $\ell \in \mathbb{N}$, [1, formula (9.6.28)]

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^\ell (z^\nu K_\nu(z)) = (-1)^\ell z^{\nu-\ell} K_{\nu-\ell}(z). \tag{4.4}$$

(v) For $z > 0$, $z^{\min\{\nu,1/2\}} e^z K_\nu(z)$ is a decreasing function [38, Theorem 5].
(vi) For $\nu > 0$, [1, estimate (9.7.2)]

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \qquad z \to \infty, \quad |\arg z| \le 3\pi/2 - \delta, \quad \delta > 0.$$

REMARK 4.2 (consistency for $s = \frac{1}{2}$). *A basic computation allows us to conclude that $c_{\frac{1}{2}} = \sqrt{\frac{2}{\pi}}$. On the other hand, formulas (9.2.10) and (9.6.10) in [1] yield $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$. We thus have arrived at*

$$\lim_{s \to \frac{1}{2}} \psi_k(y) = \exp(-\sqrt{\lambda_k} y) \quad \forall y > 0$$

for all $y > 0$. ■

We now analyze the regularity properties of $\mathscr{U}$ when $s \in (0,1)$. On the basis of the representation formula (4.1) we see that it is essential to derive regularity estimates for the solution ψ_k of problem (4.2). To accomplish this task, we define the function $\psi(z) = c_s z^s K_s(z)$ and notice that

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi(z) - \psi(z) + \frac{\alpha}{z}\frac{\mathrm{d}}{\mathrm{d}z}\psi(z) = 0, \quad z \in (0,\infty), \quad \psi(0) = 1, \quad \lim_{z \to \infty} \psi(z) = 0. \tag{4.5}$$

This, for any $\ell \in \mathbb{N}_0$, allows us to obtain that

$$\begin{aligned} \frac{\mathrm{d}^{\ell+2}}{\mathrm{d}z^{\ell+2}}\psi(z) &= \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\psi(z) - \alpha \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\left(z^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\psi(z)\right) \\ &= \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\psi(z) - \alpha \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\mathrm{d}^j}{\mathrm{d}z^j}(z^{-1}) \frac{\mathrm{d}^{\ell-j}}{\mathrm{d}z^{\ell-j}}\psi'(z). \end{aligned}$$

We thus have arrived at the bound

$$\left|\frac{\mathrm{d}^{\ell+2}}{\mathrm{d}z^{\ell+2}}\psi(z)\right| \le \left|\frac{\mathrm{d}^{\ell}}{\mathrm{d}z^{\ell}}\psi(z)\right| + |\alpha| \sum_{j=0}^{\ell} \frac{\ell!}{(\ell-j)!} z^{-(1+j)} \left|\frac{\mathrm{d}^{\ell+1-j}}{\mathrm{d}z^{\ell+1-j}}\psi(z)\right|, \tag{4.6}$$

which is essential to derive the following asymptotic result.

Lemma 4.3 (behavior of ψ near $z=0$). *Let ψ solve* (4.5)*, $z \in (0,1)$ and $\ell \in \mathbb{N}$. Then there is a constant C independent of z, ℓ and s such that*

$$\left|\frac{\mathrm{d}^{\ell}}{\mathrm{d}z^{\ell}}\psi(z)\right| \le C d_s \ell! z^{2s-\ell}, \tag{4.7}$$

where, as before, $d_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$.

Proof. We proceed by induction. Let us first assume that $\ell = 1$. The differentiation formula (4.4) with $\ell = 1$ yields that

$$\psi'(z) = c_s(z^s K_s(z))' = -c_s z^s K_{s-1}(z) = -c_s z^s K_{1-s}(z), \tag{4.8}$$

where we used Lemma 4.1 (ii). The asymptotic formula (4.3) shows that there is $\tilde{C}$ independent of s such that, for every $z \in (0,1)$, we have

$$\left| \frac{K_{1-s}(z)}{\frac{1}{2}\Gamma(1-s)(\frac{1}{2}z)^{-(1-s)}} - 1 \right| \le \tilde{C}.$$

Set $C = \tilde{C} + 1$ to arrive at the fact that we have, for all $z \in (0,1)$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\psi(z)\right| \le \left|\frac{K_{1-s}(z)}{\frac{1}{2}\Gamma(1-s)(\frac{1}{2}z)^{-(1-s)}}\right| \left(\frac{1}{2}\Gamma(1-s)\left(\frac{1}{2}z\right)^{-(1-s)}\right) c_s z^s \le C d_s z^{2s-1},$$

which is (4.7) for $\ell = 1$.

We now assume that (4.7) holds for every $j \le \ell + 1$. This, on the basis of the bound (4.6), implies that

$$\begin{aligned}\left|\frac{\mathrm{d}^{\ell+2}}{\mathrm{d}z^{\ell+2}}\psi(z)\right| &\le C d_s \ell! z^{2s-\ell} + C d_s z^{2s-\ell-2} \sum_{j=0}^{\ell} \frac{\ell!}{(\ell-j)!}(\ell+1-j)! \\ &\le C d_s \ell! z^{2s-\ell-2}\left[1 + \sum_{i=1}^{\ell+1} i\right] = C d_s \ell! z^{2s-\ell-2}\left[1 + \frac{1}{2}(\ell+1)(\ell+2)\right],\end{aligned}$$

because $z \in (0,1)$. Therefore

$$\left|\frac{\mathrm{d}^{\ell+2}}{\mathrm{d}z^{\ell+2}}\psi(z)\right| \le C d_s (\ell+2)! z^{2s-\ell-2},$$

as we intended to show. ∎

We now analyze the behavior of ψ for large values of z. In particular, we will show that ψ and all its derivatives decay exponentially with respect to z.

Lemma 4.4 (behavior of ψ for z large). *Let ψ solve* (4.5)*, $z \ge 1$, $\ell \in \mathbb{N}_0$ and $\epsilon \in (0,1)$. Then there is a constant $C_{\epsilon,s}$ that is independent of z and ℓ such that*

$$\left|\frac{\mathrm{d}^{\ell}}{\mathrm{d}z^{\ell}}\psi(z)\right| \le C_{\epsilon,s} \ell! \epsilon^{-\ell} z^{s-\ell-\frac{1}{2}} e^{-(1-\epsilon)z}, \tag{4.9}$$

where $C_{\epsilon,s}$ blows up when $\epsilon \uparrow 1$.

Proof. The proof is a consequence of Cauchy's integral formula for derivatives [3, 18] and Lemma 4.1 (vi). Let $\epsilon \in (0,1)$ and $B_\sigma(\zeta)$ denote the ball with center ζ and radius σ. For a fixed $z \geq 1$, we thus have that

$$\left|\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\psi(z)\right| = \left|\frac{\ell!}{2\pi i}\int_{\zeta\in\partial B_{\epsilon z}(z)} \frac{\psi(\zeta)}{(\zeta - z)^{\ell+1}}\,\mathrm{d}\zeta\right| \leq \ell!\epsilon^{-\ell}z^{-\ell} \max_{\zeta\in\partial B_{\epsilon z}(z)} |\psi(\zeta)|,$$

where $\ell \in \mathbb{N}_0$. We now recall that $\psi(z) = c_s z^s K_s(z)$ and invoke Lemma 4.1 (vi) to conclude that

$$\left|\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\psi(z)\right| \leq C_{\epsilon,s} c_s \ell! \epsilon^{-\ell} z^{s-\ell-\frac{1}{2}} e^{-(1-\epsilon)z},$$

with $C_{\epsilon,s} = C\max\{(1+\epsilon)^{s-\frac{1}{2}}, (1-\epsilon)^{s-\frac{1}{2}}\}$ and C such that $K_s(z) \leq Cz^{-\frac{1}{2}}e^{-z}$ for $z \geq 1$. Notice that $C_{\epsilon,s}$ can be bounded independently of $s \in (0,1)$ and that blows up when $\epsilon \uparrow 1$. This concludes the proof. □

REMARK 4.5 (Cauchy's integral formula). *The technique used in the proof of Lemma 4.4 that is based on the Cauchy's integral formula can also be applied to analyze the behavior of ψ near $z = 0$. However, the obtained estimate with such a technique is not quite as sharp as* (4.7) *since it includes the term $\epsilon^{-\ell}$ with $\epsilon \in (0,1)$, as it appears in the estimate* (4.9). ■

To analyze global regularity properties of the α–harmonic extension $\mathscr{U}$, we define the weight

$$\omega_{\beta,\gamma}(y) = y^\beta e^{\gamma y}, \qquad 0 \leq \gamma < 2\sqrt{\lambda_1}, \tag{4.10}$$

with a parameter $\beta \in \mathbb{R}$ that will be specified later, and we recall that the parameter $\lambda_1 > 0$ is the smallest eigenvalue of $\mathcal{L}$. With the weight (4.10) at hand, we define the weighted norm

$$\|v\|_{L^2(\omega_{\beta,\gamma},\mathcal{C})} := \left(\int_0^\infty \int_\Omega \omega_{\beta,\gamma}(y)|v(x',y)|^2\,\mathrm{d}x'\,\mathrm{d}y\right)^{\frac{1}{2}}. \tag{4.11}$$

We now proceed to study how certain weighted integrals of the derivatives of ψ behave. To do so, we define, for β, $\delta \in \mathbb{R}$, $\ell \in \mathbb{N}$, and $\lambda > 0$

$$\Phi(\delta,\gamma,\lambda) = \int_0^\infty z^\delta e^{\gamma z/\sqrt{\lambda}}\,|\psi(z)|^2\,\mathrm{d}z \tag{4.12}$$

and

$$\Psi_\ell(\beta,\gamma,\lambda) = \int_0^\infty z^{\beta+2\ell} e^{\gamma z/\sqrt{\lambda}} \left|\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\psi(z)\right|^2 \mathrm{d}z; \tag{4.13}$$

γ is such that (4.10) holds. Let us now bound the integrals $\Phi(\delta,\gamma,\lambda)$ and $\Psi_\ell(\beta,\gamma,\lambda)$.

LEMMA 4.6 (bounds on Φ and Ψ_ℓ). *Let $\delta > -1$, $\beta > -1-4s$, $\ell \in \mathbb{N}$ and let γ be such that $0 \leq \gamma < 2\sqrt{\lambda_1}$. If $\lambda \geq \lambda_1$, then we have that*

$$\Phi(\delta,\gamma,\lambda) \lesssim 1, \tag{4.14}$$

where the hidden constant is independent of λ*. In addition, there exists* $\kappa > 1$ *such that we have the following bound*

$$\Psi_\ell(\beta,\gamma,\lambda) \lesssim \kappa^{2\ell}(\ell!)^2, \tag{4.15}$$

where the hidden constant is independent of ℓ *and* λ*.*

Proof. We derive (4.15). As a first step, we write $\Psi_\ell = \Psi_\ell(\beta,\gamma,\lambda)$ as follows:

$$\Psi_\ell = \int_0^1 z^{\beta+2\ell} e^{\frac{\gamma z}{\sqrt{\lambda}}} \left| \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\psi(z) \right|^2 \mathrm{d}z + \int_1^\infty z^{\beta+2\ell} e^{\frac{\gamma z}{\sqrt{\lambda}}} \left| \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\psi(z) \right|^2 \mathrm{d}z = \mathrm{I} + \mathrm{II}, \tag{4.16}$$

and estimate each term separately.

We start by bounding I. To accomplish this task we notice that, since $0 \leq \gamma < 2\sqrt{\lambda_1}$ and $\lambda \geq \lambda_1$ we have that

$$\sup_{z\in(0,1)} e^{\frac{\gamma z}{\sqrt{\lambda}}} < \sup_{z\in(0,1)} e^{2z} \leq e^2.$$

Consequently, an application of the results of Lemma 4.3 yields

$$\mathrm{I} = \int_0^1 z^{\beta+2\ell} e^{\frac{\gamma z}{\sqrt{\lambda}}} \left| \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\psi(z) \right|^2 \mathrm{d}z \lesssim d_s^2(\ell!)^2 \int_0^1 z^{\beta+2\ell+2(2s-\ell)}\,\mathrm{d}z \lesssim d_s^2(\ell!)^2,$$

where last integral converges because $\beta > -1-4s$. Notice that the hidden constant blows up when $\beta \downarrow -1-4s$.

We now estimate the term II in (4.16). To do this we utilize the estimate (4.9) of Lemma 4.4 as follows:

$$\mathrm{II} \leq C_\epsilon^2 c_s^2 (\ell!)^2 \epsilon^{-2\ell} \int_1^\infty z^{\beta+2\ell} z^{2s-2\ell-1} e^{\frac{\gamma z}{\sqrt{\lambda}}} e^{-2(1-\epsilon)z}\,\mathrm{d}z.$$

Define

$$\hat{\gamma} := \sup_{\lambda\geq\lambda_1} \left(\frac{\gamma}{\sqrt{\lambda}} - 2(1-\epsilon) \right).$$

Notice that, since $0 \leq \frac{\gamma}{\sqrt{\lambda_1}} < 2$ by (4.10), the parameter $\epsilon \in (0,1)$ can be selected such that $\hat{\gamma} < 0$. Consequently

$$\mathrm{II} \lesssim C_\epsilon^2 c_s^2 (\ell!)^2 \epsilon^{-2\ell} \int_1^\infty z^{\beta+2s-1} e^{\hat{\gamma} z}\,\mathrm{d}z \lesssim C_\epsilon^2 c_s^2 (\ell!)^2 \epsilon^{-2\ell}.$$

Replacing the estimates for the terms I and II into (4.16) and considering $\kappa = \epsilon^{-1} > 1$ we arrive at the desired estimate (4.15). To obtain the estimate (4.14) we decompose Φ as in (4.16) and use that, as estimate (4.3) shows, ψ is bounded as $z \downarrow 0^+$ and decays exponentially to zero as $z \uparrow \infty$; see Lemma 4.1 (v) and (vi). For brevity, we skip details. ∎

Now, on the basis of Lemma 4.6, we provide global regularity results for the α-harmonic extension $\mathscr{U}$ in weighted Sobolev spaces.

Theorem 4.7 (global regularity of $\mathscr{U}$). *Let* $\mathscr{U} \in \mathring{H}^1(y^\alpha,\mathcal{C})$ *solve* (1.2) *with* $s \in (0,1)$*. Let* $0 \leq \tilde{\nu} < s$ *and* $0 \leq \nu < 1+s$*. Then there exists* $\kappa > 1$ *such that the* *following holds for all $\ell \in \mathbb{N}_0$ with the weight $w_{\beta,\gamma}$ given by (4.10):*

$$\|\partial_y^{\ell+1}\mathscr{U}\|_{L^2(\omega_{\alpha+2\ell-2\tilde{\nu},\gamma},\mathcal{C})} \lesssim \kappa^{\ell+1}(\ell+1)!\|f\|_{\mathbb{H}^{-s+\tilde{\nu}}(\Omega)}, \tag{4.17}$$

$$\|\nabla_{x'}\partial_y^{\ell+1}\mathscr{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} \lesssim \kappa^{\ell+1}(\ell+1)!\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, \tag{4.18}$$

$$\|\mathcal{L}_{x'}\partial_y^{\ell+1}\mathscr{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} \lesssim \kappa^{\ell+1}(\ell+1)!\|f\|_{\mathbb{H}^{1-s+\nu}(\Omega)}. \tag{4.19}$$

In all these inequalities, the implied constants are independent of ℓ, $\mathscr{U}$ and f. In addition, if $0 \le \nu' < 1-s$ then

$$\|\mathcal{L}_{x'}\mathscr{U}\|_{L^2(\omega_{\alpha-2\nu'},\gamma,\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1-s+\nu'}(\Omega)}, \tag{4.20}$$

$$\|\nabla_{x'}\mathscr{U}\|_{L^2(\omega_{\alpha-2\nu'},\gamma,\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{-s+\nu'}(\Omega)}, \tag{4.21}$$

$$\|\mathscr{U}\|_{L^2(\omega_{\alpha-2\nu'},\gamma,\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{-1-s+\nu'}(\Omega)}, \tag{4.22}$$

where the constant implied in $\lesssim$ is independent of $\mathscr{U}$ and f.

Proof. We follow [41, Theorem 2.7] and thus invoke the representation formula (4.1) to arrive at

$$\|\partial_y^{\ell+1}\mathscr{U}\|^2_{L^2(\omega_{\alpha+2\ell-2\sigma,\gamma},\mathcal{C})} = \sum_{k=1}^{\infty} f_k^2\lambda_k^{-2s}\int_0^\infty y^{\alpha+2\ell-2\sigma}e^{\gamma y}\left|\frac{\mathrm{d}^{\ell+1}}{\mathrm{d}y^{\ell+1}}\psi_k(y)\right|^2 \mathrm{d}y\ .$$

We introduce the change of variable $z = \sqrt{\lambda_k}y$ and recall that $\psi(z) = c_s z^s K_s(z)$ and $\psi_k(y) = \psi(\sqrt{\lambda_k}y)$ as well as the definition of Ψ_ℓ given as in (4.13), to obtain that

$$\begin{aligned}\|\partial_y^{\ell+1}\mathscr{U}\|^2_{L^2(\omega_{\alpha+2\ell-2\sigma,\gamma},\mathcal{C})} &= \sum_{k=1}^{\infty} f_k^2\lambda_k^{-2s+(\ell+1)-\left(\frac{\alpha+2\ell-2\sigma}{2}\right)-\frac{1}{2}}\Psi_{\ell+1}(\alpha-2\sigma-2,\gamma,\lambda_k)\\ &\lesssim (\ell+1)!^2\kappa^{2(\ell+1)}\sum_{k=1}^{\infty} f_k^2\lambda_k^{\sigma-s} = (\ell+1)!^2\kappa^{2(\ell+1)}\|f\|^2_{\mathbb{H}^{-s+\sigma}(\Omega)},\end{aligned}$$

where the last inequality follows from the estimate (4.15) with $\beta = \alpha-2\sigma-2 = 1-2s-2\sigma-2 > -1-4s$.

We now derive (4.19); the proof of the estimate (4.18) follows by using similar arguments. As before, we arrive at

$$\begin{aligned}&\|\mathcal{L}_{x'}\partial_y^{\ell+1}\mathscr{U}\|^2_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})}\\ &= \sum_{k=1}^{\infty} f_k^2\lambda_k^{2(1-s)}\int_0^\infty y^{\alpha+2(\ell+1)-2\nu}e^{\gamma y}\left|\frac{\mathrm{d}^{\ell+1}}{\mathrm{d}y^{\ell+1}}\psi_k(y)\right|^2 \mathrm{d}y\\ &= \sum_{k=1}^{\infty} f_k^2\lambda_k^{2(1-s)+(\ell+1)-\left(\frac{\alpha+2(\ell+1)-2\nu}{2}\right)-\frac{1}{2}}\Psi_{\ell+1}(\alpha-2\nu,\gamma,\lambda_k),\end{aligned}$$

where we applied again the change of variable $z = \sqrt{\lambda_k}y$ and used the definition of Ψ_ℓ given by (4.13). We now notice that $\alpha-2\nu > 1-2s-4-2s = -1-4s$. Thus an application of the estimate (4.15) with $\beta = \alpha-2\nu$ reveals that

$$\|\mathcal{L}_{x'}\partial_y^{\ell+1}\mathscr{U}\|^2_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2\|f\|^2_{\mathbb{H}^{1-s+\nu}(\Omega)}.$$

This yields (4.19).

The proofs of (4.20), (4.21), (4.22) rely on similar arguments using that $\nu' < 1-s$ implies $\delta := \alpha-2\nu' = 1-2s-2\nu' > 1-2s-2(1-s) = -1$, and thus, as a consequence of (4.14), that $\Phi(\delta,\gamma,\lambda) \lesssim 1$. This concludes the proof. ∎

5. h-FE discretization in Ω. We now begin with the discretization of (2.11). The structure of this section is as follows: in Section 5.1, we introduce the FE approximation in Ω and fix notation on Finite Element spaces. Section 5.2 introduces the FE discretization in $\mathcal{C}$ in abstract form. Section 5.3 next addresses a basic decomposition of the FE discretization error which decomposes the FE discretization error into two parts: a semidiscretization error with respect to $x' \in \Omega$, and a corresponding error with respect to $y \in (0, \mathcal{Y})$, where $0 < \mathcal{Y} < \infty$ denotes a truncation parameter of the cylinder $(0, \infty)$. Section 5.4 then addresses two first order tensor product FEMs in $\mathcal{C}$. The first one, as in [41], is a full tensor product FEM and for it we show the first order rate of convergence in Ω, but at superlinear complexity in terms of the number $\mathcal{N}_\Omega$ of degrees of freedom in Ω. To reduce the complexity, we propose the second, novel approach: by sparse tensor product P_1 discretization of the extended problem in $\mathcal{C}$, we show the same convergence rate, but with (essentially) linear complexity in terms of $\mathcal{N}_\Omega$ requiring only marginally more regularity of the data f in Ω. Section 5.5 addresses the use of an hp-FEM in the extended variable y, combined with a P_1-FEM in Ω.

5.1. Notation and FE spaces. For a truncation parameter $\mathcal{Y} > 0$ (which is fixed, and which will be selected ahead), we denote by $\mathcal{G}^M$ a generic partition of $[0, \mathcal{Y}]$ into M intervals. In particular, the following two types of partitions, that are refined towards $y = 0$, will be essential for our purposes:

- *Graded* meshes $\mathcal{G}^k_{gr,\eta}$. Here k indicates the mesh size near $y = 1$ and η characterizes the mesh grading towards $y = 0$; see Section 5.4.2 for details.
- *Geometric* meshes $\mathcal{G}^M_{geo,\sigma}$. This mesh has M elements and $\sigma \in (0, 1)$ is the subdivision ratio; see Section 5.5.1 for details.

Given a mesh $\mathcal{G}^M = \{I_m\}_{m=1}^M$ in $[0, \mathcal{Y}]$, where $I_m = [y_{m-1}, y_m]$, $y_0 = 0$ and $y_M = \mathcal{Y}$, we associate to $\mathcal{G}^M$ a *polynomial degree distribution* $\boldsymbol{r} = (r_1, r_2, \ldots, r_M) \in \mathbb{N}^M$. With these ingredients at hand we define the finite element space

$$S^{\boldsymbol{r}}((0, \mathcal{Y}), \mathcal{G}^M) = \left\{ v_M \in C[0, \mathcal{Y}] : v_M|_{I_m} \in \mathbb{P}_{r_m}(I_m), I_m \in \mathcal{G}^M, m = 1, \ldots, M \right\}.$$

We also define the subspace of $S^{\boldsymbol{r}}((0, \mathcal{Y}), \mathcal{G}^M)$ of functions that vanish at $y = \mathcal{Y}$:

$$S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0, \mathcal{Y}), \mathcal{G}^M) = \left\{ v_M \in S^{\boldsymbol{r}}((0, \mathcal{Y}), \mathcal{G}^M) : v_M(\mathcal{Y}) = 0 \right\}.$$

In the particular case that $r_i = r$ for $i = 1, \ldots, M$, we write $S^r((0, \mathcal{Y}), \mathcal{G}^M)$ or $S^r_{\{\mathcal{Y}\}}((0, \mathcal{Y}), \mathcal{G}^M)$ as appropriate. In Ω, we consider Lagrangian FEM of polynomial degree $q \geq 1$ based on shape-regular, simplicial triangulations denoted by $\mathcal{T}$. Denote by $h(\mathcal{T}) = \max\{\text{diam}(K) : K \in \mathcal{T}\}$ the mesh width of $\mathcal{T}$. We thus introduce

$$S^q_0(\Omega, \mathcal{T}) = \left\{ v_h \in C(\bar{\Omega}) : v_h|_K \in \mathbb{P}_q(K) \quad \forall K \in \mathcal{T},\ v_h|_{\partial\Omega} = 0 \right\}.$$

In what follows we will also consider nested sequences $\{\mathcal{T}^\ell\}_{\ell \geq 0}$ of triangulations of Ω that are generated by bisection–tree refinement of a coarse, regular initial triangulation $\mathcal{T}^0$ of Ω. We denote by $h_\ell = \max\{\text{diam}(K) : K \in \mathcal{T}^\ell\}$ the mesh width of $\mathcal{T}^\ell$.

By $\Pi^q_{x'} : H^1_0(\Omega) \to S^q_0(\Omega, \mathcal{T})$, we denote a FE quasi–interpolation operator defined on $L^2(\Omega)$ that, when restricted to $H^1_0(\Omega)$, preserves homogeneous Dirichlet boundary conditions. We assume that $\Pi^q_{x'}$ has optimal asymptotic approximation properties in $L^2(\Omega)$ and $H^1(\Omega)$ on regular, locally refined, and nested bisection–tree mesh sequences $\{\mathcal{T}^\ell\}_{\ell \geq 0}$ in Ω. In addition, we assume that $\Pi^q_{x'}$ is concurrently stable in $L^2(\Omega)$ and

$H^1(\Omega)$. In the particular case that $q \le 12$ we will set $\Pi^q_{x'}$ to be the $L^2(\Omega)$ projection onto $S^q_0(\Omega, \mathcal{T})$. We refer, in particular, to [25] for a verification of the requisite stability and approximation properties over nested bisection–tree meshes.

We define the finite–dimensional *tensor product space*

$$\mathbb{V}^{q,\boldsymbol{r}}_{h,M}(\mathcal{T}, \mathcal{G}^M) := S^q_0(\Omega, \mathcal{T}) \otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M) \subset \mathring{H}^1(y^\alpha, \mathcal{C}) \ , \tag{5.1}$$

and write $\mathbb{V}_{h,M}$ if the arguments are clear from the context. In the ensuing error analysis, we also require semidiscretizations which are based on the following (infinite–dimensional) Hilbertian tensor product spaces

$$\begin{aligned} \mathbb{V}^q_h(\mathcal{C}_\mathcal{Y}) &:= S^q_0(\Omega, \mathcal{T}_h) \otimes \mathring{H}^1(y^\alpha, (0,\mathcal{Y})) \subset \mathring{H}^1(y^\alpha, \mathcal{C}) \ , \\ \mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_\mathcal{Y}) &:= H^1_0(\Omega) \otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M) \subset \mathring{H}^1(y^\alpha, \mathcal{C}) \ . \end{aligned} \tag{5.2}$$

Both of them are closed subspaces of $\mathring{H}^1(y^\alpha, \mathcal{C})$, so that Galerkin projections with respect to the inner product given by the bilinear form $a_{\mathcal{C}_\mathcal{Y}}$ in (2.10) are well defined. We denote these projections by G^q_h and $G^{\boldsymbol{r}}_M$, respectively. To the space $\mathbb{V}^{q,\boldsymbol{r}}_{h,M}(\mathcal{T}, \mathcal{G}^M)$, defined in (5.1), we can also associate a Galerkin projection with respect to $a_{\mathcal{C}_\mathcal{Y}}$. We remark that this projector is the composition of the semidiscrete projections:

$$G^{q,\boldsymbol{r}}_{h,M} = G^q_h \circ G^{\boldsymbol{r}}_M = G^{\boldsymbol{r}}_M \circ G^q_h : \ \mathring{H}^1(y^\alpha, \mathcal{C}) \to \mathbb{V}^{q,\boldsymbol{r}}_{h,M}(\mathcal{T}, \mathcal{G}^M) \ . \tag{5.3}$$

5.2. FE discretization and quasioptimality. The *FE approximation* $\mathscr{U}_{h,M}$ is defined as $\mathscr{U}_{h,M} = G^{q,\boldsymbol{r}}_{h,M} \mathscr{U} \in \mathbb{V}_{h,M}$, i.e., it satisfies

$$a_{\mathcal{C}_\mathcal{Y}}(\mathscr{U}_{h,M}, \phi) = d_s \langle f, \operatorname{tr}_\Omega \phi \rangle \quad \forall \phi \in \mathbb{V}_{h,M} \ . \tag{5.4}$$

Coercivity of $a_{\mathcal{C}_\mathcal{Y}}$ immediately implies existence and uniqueness of $\mathscr{U}_{h,M}$. In addition, Galerkin orthogonality gives quasioptimality of $\mathscr{U}_{h,M}$. More precisely, as in [41, Section 4], we have the following result.

Lemma 5.1 (Céa and truncation). *Let* $\mathscr{U}$ *be the solution to problem* (2.11) *and let* $\mathscr{U}_{h,M} = G^{q,\boldsymbol{r}}_{h,M} \mathscr{U}$ *its finite element approximation that solves* (5.4). *Then we have*

$$\begin{aligned} \|\nabla(\mathscr{U} - \mathscr{U}_{h,M})\|_{L^2(y^\alpha, \mathcal{C})} &\lesssim \min_{v_{h,M} \in \mathbb{V}_{h,M}} \|\nabla(\mathscr{U} - v_{h,M})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} \\ &\quad + \|\nabla \mathscr{U}\|_{L^2(y^\alpha, \mathcal{C} \setminus \mathcal{C}_\mathcal{Y})} \ , \end{aligned} \tag{5.5}$$

where the hidden constant does not depend on $\mathbb{V}_{h,M}$.

As already noted in [41, Prop. 3.1], the second term on the right hand side of (5.5) is exponentially small in $\mathcal{Y}$. More precisely, using (4.17) and (4.21) we get, with the selection $\gamma < 2\sqrt{\lambda_1}$, that

$$\|\nabla \mathscr{U}\|_{L^2(y^\alpha, \mathcal{C} \setminus \mathcal{C}_\mathcal{Y})} \lesssim \exp(-\gamma \mathcal{Y}/2) \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{5.6}$$

5.3. FE error splitting. As (5.6) shows, the second term on the right hand side of of (5.5) decays exponentially in $\mathcal{Y}$. Thus, we now concentrate on estimating the first one.

As in [41, 33], we separate the errors incurred by discretizations with respect to x' and y as follows.

Lemma 5.2 (dimensional error splitting). *Let* $\mathscr{U}$ *be the solution to problem* (2.11) *and let* $\mathscr{U}_{h,M}$ *denote its approximation defined as the solution to* (5.4). *Assume* *that on the sequence* $\{\mathcal{T}^\ell\}_{\ell\geq 1}$ *of regular, simplicial triangulations of* Ω *the quasi-interpolation operator* $\Pi^q_{x'}$ *is concurrently uniformly stable on* $L^2(\Omega)$ *and* $H^1(\Omega)$*. Let* $\pi^{\boldsymbol{r}}_y : H^1(y^\alpha,(0,\mathcal{Y})) \to S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M)$ *be a linear projector. Then*

$$\min_{v_{h,M}\in\mathbb{V}_{h,M}} \|\nabla(\mathscr{U}-v_{h,M})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \|\nabla(\mathscr{U}-\Pi^q_{x'}\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + \|\nabla(\mathscr{U}-\pi^{\boldsymbol{r}}_y\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}\ , \tag{5.7}$$

where the hidden constant does not depend on the dimension of $\mathbb{V}_{h,M}$*.*

Proof. The desired estimate follows from the tensor-product structure of the finite element space defined in (5.1) and the triangle inequality, upon choosing in (5.7) the function $v_{h,M} := \Pi^q_{x'}\otimes\pi^{\boldsymbol{r}}_y\mathscr{U}$. □

5.4. h-FE error analysis. In the present subsection we analyze convergence rates and complexity for two particular instances of the FE-space $\mathbb{V}^{q,\boldsymbol{r}}_{h,M}(\mathcal{T},\mathcal{G}^M)$:

(a) The case when $\boldsymbol{r}=(1,1,\dots,1)$ on a graded mesh $\mathcal{G}^M$ and $q=1$. A particular instance of this was first introduced in [41]; see Section 3. Generalizing the results of [41, 33], we allow $\Omega\subset\mathbb{R}^2$ to be a polygon with finitely many straight sides and corners $\{\boldsymbol{c}\}$. This will mandate the use of a sequence of nested triangulations $\{\mathcal{T}^\ell\}_{\ell\geq1}$ of the domain Ω with, in general, local refinement towards the corners $\boldsymbol{c}\in\partial\Omega$.

(b) The case $\boldsymbol{r}=(1,1,\dots,1)$ on a nested sequence $\{\mathcal{G}^{\ell'}\}_{\ell'\geq1}$ of graded meshes in $(0,\mathcal{Y})$. At the same time, we also consider multilevel approximations in Ω on a sequence $\{\mathcal{T}^\ell\}_{\ell\geq1}$ of nested triangulations with appropriate corner refinement in Ω, a particular instance being the so-called bisection–tree refinements.

In all cases, we bound the first term on the right hand side of (5.5).

5.4.1. P_1-FEM in Ω with mesh refinement at $\boldsymbol{c}$. In a bounded polygon $\Omega\subset\mathbb{R}^2$ with straight sides and corners $\boldsymbol{c}$ we consider the Dirichlet problem

$$\mathcal{L}w = g \text{ in } \Omega\ ,\quad w=0 \text{ on } \partial\Omega\ , \tag{5.8}$$

for $g\in H^{-1}(\Omega)$. It is immediate that problem (5.8) has a unique solution $w\in H^1_0(\Omega)$. However, in general the solution w does not belong to $H^2(\Omega)$. Under additional regularity assumptions on A and c, it rather belongs to weighted Sobolev spaces of Kondrat'ev type in Ω which we now define.

For a finite set $\{\boldsymbol{c}\}$ of corners of Ω and $x\in\Omega$ we define $\Phi(x)=\prod_{\boldsymbol{c}}|x-\boldsymbol{c}|$. To follow standard notation, for $0\leq\beta\in\mathbb{R}$, we set $L^2_\beta(\Omega)=L^2(\Phi^{2\beta},\Omega)$. We also define the space $H^2_\beta(\Omega)$ as the closure of $H^2(\Omega)\cap H^1_0(\Omega)$ with respect to the norm

$$\|w\|_{H^2_\beta(\Omega)} = \|w\|_{H^1(\Omega)} + \|D^2w\|_{L^2(\Phi^{2\beta},\Omega)}\ . \tag{5.9}$$

With this setting at hand, we present the following result on regularity shift in weighted Sobolev spaces for the solution of problem (5.8).

Proposition 5.3 (weighted regularity estimate). *Let* $A\in W^{1,\infty}(\Omega,\mathsf{GL}(\mathbb{R}^2))$ *be uniformly positive definite,* $c\in W^{1,\infty}(\Omega,\mathbb{R})$ *and* $g\in L^2_\beta(\Omega)$*. Then, for every polygon* $\Omega\subset\mathbb{R}^2$*, there exists* $\beta\geq0$ *such that the solution* w *of* (5.8) *belongs to* $H^2_\beta(\Omega)$ *and*

$$\|w\|_{H^2_\beta(\Omega)} \lesssim \|\mathcal{L}w\|_{L^2_\beta(\Omega)} = \|g\|_{L^2_\beta(\Omega)}\ , \tag{5.10}$$

where the hidden constant is independent of g*.*

Proof. This is result is a particular case of [8, Theorem 1.1]. It suffices to set, in the notation of this reference, $m = 1$, $b_j = 0$, and $\beta = 1 - a$. ☐

REMARK 5.4 (Laplacian). *In the special case that $\mathcal{L} = -\Delta$, i.e., when (5.8) corresponds to the Dirichlet Poisson problem in a polygon Ω, the parameter β must satisfy $\beta > 1 - \min_{\boldsymbol{c}} \pi/\omega_{\boldsymbol{c}}$, where $0 < \omega_{\boldsymbol{c}} < 2\pi$ is the interior opening angle of Ω at the vertex $\boldsymbol{c}$. If Ω is convex, the choice $\beta = 0$ is admissible, and then (5.10) reduces to the classical regularity shift for the Dirichlet problem of the Poisson equation in convex domains. We refer the reader to the discussion in [8, equations (2) and (3)] for more details.* ∎

Proposition 5.3 and the regularity of $\mathscr{U}$ given in Theorem 4.7 imply the following regularity result for $\mathscr{U}$ in weighted norms in Ω.

PROPOSITION 5.5 (global regularity of $\mathscr{U}$: weighted estimates in Ω). *Let $\mathscr{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$ solve (1.2) with $s \in (0,1)$. Let $0 \le \nu' < 1 - s$. Assume that $\Omega \subset \mathbb{R}^2$ is a polygon and that A and c satisfy the assumptions of Proposition 5.3. Then there exists $\beta \ge 0$, which depends only on Ω, A, and c, such that*

$$\|\mathscr{U}\|_{L^2(\omega_{\alpha-2\nu',\gamma},(0,\infty);H^2_\beta(\Omega))} \lesssim \|f\|_{\mathbb{H}^{1-s+\nu'}(\Omega)}, \tag{5.11}$$

where the weight $\omega_{\beta,\gamma}$ is defined as in (4.10). In addition, for $\ell \in \mathbb{N}_0$, and $0 \le \tilde{\nu} < 1 + s$, there exists $\kappa > 1$ such that

$$\|\partial_y^{\ell+1}\mathscr{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\tilde{\nu},\gamma},(0,\infty);H^2_\beta(\Omega))} \lesssim \kappa^{\ell+1}(\ell+1)!\|f\|_{\mathbb{H}^{1-s+\tilde{\nu}}(\Omega)} \, . \tag{5.12}$$

In both estimates, the hidden constants are independent of $\mathscr{U}$ and f.

Proof. The proof for (5.12) follows from (4.17) and that of (5.11) from (4.22) by using the the weighted regularity shift (5.10). In fact, for a fixed $y > 0$ and $m \in \mathbb{N}_0$, set $w = \partial_y^m \mathscr{U}(\cdot, y)$ in (5.8). Notice that $g = \partial_y^m \mathcal{L}_{x'} \mathscr{U}(\cdot, y)$. Since $\beta \ge 0$ we have that $g \in L^2_\beta(\Omega)$ and estimate (5.10) holds. Square it and multiply it by either $\omega_{\alpha-2\nu',\gamma}$ if $m = 0$, or $\omega_{\alpha+2m-2\nu,\gamma}$ when $m \ge 1$. Integration with respect to y over $(0,\infty)$ allows us then to conclude. ☐

The previous regularity result will be the basis for the analysis of a P_1-FEM on properly refined meshes in Ω and it will allow us to recover the full first order convergence rate; see Theorem 5.9 below.

To accomplish this task, we associate with $H^2_\beta(\Omega)$ a sequence $\{\mathcal{T}^\ell_\beta\}_{\ell \ge 0}$ of bisection–tree meshes in Ω which, as constructed in [26], are properly refined towards the corners $\{\boldsymbol{c}\}$ of Ω. Bisection–tree meshes are uniformly shape regular (see, e.g., [43, Lemma 1]) and, as shown in [25], the L^2-projections $\Pi^\ell_\beta := \Pi^1_{x'} : L^2(\Omega) \to S^1_0(\Omega, \mathcal{T}^\ell_\beta)$ are uniformly stable in $L^2(\Omega)$ and also in $H^1(\Omega)$. In addition, they satisfy optimal asymptotic error bounds, i.e., for every $\ell \ge 0$ and every $w \in H^1_0(\Omega)$ we have

$$N_\ell \|w - \Pi^\ell_\beta w\|^2_{L^2(\Omega)} \lesssim \|w\|^2_{H^1(\Omega)}, \tag{5.13}$$

where $N_\ell = \dim S^1_0(\Omega, \mathcal{T}^\ell_\beta) = \mathcal{O}(h_\ell^{-2})$. In addition, for every $w \in H^2_\beta(\Omega)$, there holds

$$N_\ell \|w - \Pi^\ell_\beta w\|^2_{L^2(\Omega)} + \|\nabla_{x'}(w - \Pi^\ell_\beta w)\|^2_{L^2(\Omega)} \lesssim N_\ell^{-1} \|w\|^2_{H^2_\beta(\Omega)} \, . \tag{5.14}$$

In view of the embedding $H^2_\beta(\Omega) \hookrightarrow C^0(\bar{\Omega})$, the nodal interpolant is well–defined and [40, Section 5] shows that (5.14) holds for such an interpolant. We now use that Π^ℓ_β reproduces the discrete space $S^1_0(\Omega, \mathcal{T}^\ell_\beta)$ and, owing to [25], that it is bounded uniformly with respect to ℓ concurrently in $L^2(\Omega)$ and in $H^1(\Omega)$ to conclude (5.14).

REMARK 5.6 (other quasi-interpolants). *The L^2-projection in the previous argument can also be replaced with Scott-Zhang type quasi-interpolants that are projections* *onto $S_0^1(\Omega, \mathcal{T}_\beta^\ell)$ and have suitable local stability properties in both L^2 and H^1. Such operators are constructed, e.g., in [6, Lemma 4] by dropping in the classical Scott-Zhang operator [54] the degrees of freedom associated with nodes on $\partial\Omega$ and noting that the remaing operator is well-defined and (locally) stable in $L^2(\Omega)$.* ∎

5.4.2. Linear interpolant π_η^1 on radical-geometric meshes in $[0,\mathscr{Y}]$. To approximate the solution $\mathscr{U}$ with respect to the extended variable y, we shall use a continuous, piecewise linear interpolant on suitably refined meshes $\mathcal{G}^k_{gr,\eta}$ in $[0,\mathscr{Y}]$. The mesh is radical on $[0,1]$ and geometric on $[1,\mathscr{Y}]$, and the parameter k indicates the mesh size near the point 1. Specifically, for $\mathscr{Y} > 1$, $\eta > 0$ and $k = 1/N$ for an integer $N \in \mathbb{N}$, the mesh $\mathcal{G}^k_{gr,\eta}$ is given by

$$\mathcal{G}^k_{gr,\eta} := \{I_i \,|\, i = 1, \dots, N\} \cup \{J_j \,|\, j = 1, \dots, N'\}, \tag{5.15a}$$
$$I_i = \big[((i-1)k)^\eta, (ik)^\eta\big], \qquad i = 1, \dots, N, \tag{5.15b}$$
$$J_j = \big[\exp((j-1)k), \exp(jk)\big], \qquad j = 1, \dots, N'-1 := \lfloor N \log \mathscr{Y} \rfloor - 1, \tag{5.15c}$$
$$J_{N'} = \big[\exp((N'-1)k), \mathscr{Y}\big]. \tag{5.15d}$$

Given η and $\mathscr{Y}$, we denote by $\pi_\eta^1 : C((0,\mathscr{Y}]) \to S^1((0,\mathscr{Y}), \mathcal{G}^k_{gr,\eta})$ the piecewise linear interpolation operator over all the elements of the mesh $\mathcal{G}^k_{gr,\eta}$ with the exception of the first one, i.e., I_1. On that element, π_η^1 corresponds to the linear interpolant in the midpoint of I_1 and the right endpoint of I_1. The operator

$$\pi^1_{\eta,\{\mathscr{Y}\}} : C((0,\mathscr{Y}]) \to S^1_{\{\mathscr{Y}\}}\left((0,\mathscr{Y}), \mathcal{G}^k_{gr,\eta}\right)$$

is obtained from π_η^1 by subtracting a linear function on the element abutting at $\mathscr{Y}$ so as to satisfy $(\pi^1_{\eta,\{\mathscr{Y}\}}u)(\mathscr{Y}) = 0$. These operators naturally extend to Hilbert space valued functions. The approximation properties of these operators are as follows.

Lemma 5.7 (interpolation error estimates). *Let X be a Hilbert space, $\alpha \in (-1,1)$, $\theta \in (0,1]$, and $0 \le \gamma' < \gamma$. Let the mesh grading parameter η that defines the mesh $\mathcal{G}^k_{gr,\eta}$ satisfy $\eta\theta \ge 1$. In this setting the following assertions hold.*

(i) The number of elements in $\mathcal{G}^k_{gr,\eta}$ is bounded by $k^{-1}(1 + \log \mathscr{Y})$.

(ii) For every $u \in C((0,\mathscr{Y}]; X)$ with $u' \in L^2(\omega_{\alpha+2(1-\theta),\gamma}, (0,\mathscr{Y}); X)$ we have

$$\|u - \pi_\eta^1 u\|_{L^2(\omega_{\alpha,\gamma'},(0,\mathscr{Y});X)} \lesssim k \|u'\|_{L^2(\omega_{\alpha+2(1-\theta),\gamma},(0,\mathscr{Y});X)}, \tag{5.16}$$
$$\|u - \pi^1_{\eta,\{\mathscr{Y}\}} u\|_{L^2(\omega_{\alpha,\gamma'},(0,\mathscr{Y});X)} \lesssim k \|u'\|_{L^2(\omega_{\alpha+2(1-\theta),\gamma},(0,\mathscr{Y});X)} + \sqrt{\mathscr{Y}k}\mathscr{Y}^\alpha \exp(\mathscr{Y}\gamma'/2)\|u(\mathscr{Y})\|_X. \tag{5.17}$$

Furthermore, under the assumption that $\lim_{y\to\infty} u(y) = 0$ in X and the constraint

$$\mathscr{Y}^{1/2+\alpha} \exp(-\mathscr{Y}\gamma/2) \le k^{1/2} \tag{5.18}$$

the following estimate holds:

$$\|u - \pi^1_{\eta,\{\mathscr{Y}\}} u\|_{L^2(y^\alpha,(0,\mathscr{Y});X)} \lesssim k \|u'\|_{L^2(\omega_{\alpha+2(1-\theta),\gamma},(0,\mathscr{Y});X)}. \tag{5.19}$$

(iii) For $u \in C((0,\mathscr{Y}]; X)$ with $u'' \in L^2(\omega_{\alpha+2(1-\theta),\gamma}, (0,\mathscr{Y}); X)$ and $j \in \{0,1\}$

$$\|(u - \pi_\eta^1 u)^{(j)}\|_{L^2(\omega_{\alpha,\gamma'},(0,\mathscr{Y});X)} \lesssim k^{2-j} \|u''\|_{L^2(\omega_{\alpha+2(1-\theta),\gamma}(0,\mathscr{Y});X)}, \tag{5.20}$$
$$\|(u - \pi^1_{\eta,\{\mathscr{Y}\}} u)^{(j)}\|_{L^2(\omega_{\alpha,\gamma'},(0,\mathscr{Y});X)} \lesssim k^{2-j} \|u''\|_{L^2(\omega_{\alpha+2(1-\theta),\gamma}(0,\mathscr{Y});X)} + (\mathscr{Y}k)^{1/2-j}\mathscr{Y}^\alpha \exp(\mathscr{Y}\gamma'/2)\|u(\mathscr{Y})\|_X. \tag{5.21}$$

Furthermore, under the assumption that, for $j \in \{0,1\}$, $\lim_{y\to\infty} u^{(j)}(y) = 0$ *in* X*, and the constraint*

$$\mathcal{Y}^{1/2+2\alpha} \exp(-\mathcal{Y}\gamma/2) \le k^2 \tag{5.22}$$

the following estimate holds for $j \in \{0,1\}$*:*

$$\|(u - \pi^1_{\eta,\{\mathcal{Y}\}} u)^{(j)}\|_{L^2(y^\alpha,(0,\mathcal{Y});X)} \lesssim k^{2-j} \|u''\|_{L^2(\omega_{\alpha+2(1-\theta),\gamma},(0,\mathcal{Y});X)}. \tag{5.23}$$

Proof. We present the details for the proof of (ii), as that of (iii) is similar. The technique used to obtain interpolation error estimates on the radical mesh on $[0,1]$ is well-established; see, for instance, [51, Example 3.47]. We introduce the mesh points $y_i := (ik)^\eta$, $i = 0,\dots,N$ so that $I_i = [y_{i-1}, y_i]$.

For the first element $I_1 = [y_0, y_1] = [0, k^\eta]$, we invoke the estimate (A.3) with the choice $\delta = 1-\theta \in [0,1)$ and a scaling argument to conclude that

$$\|u - \pi^1_\eta u\|^2_{L^2(y^\alpha,I_1;X)} \lesssim k_1^{2\theta} \|u'\|^2_{L^2(y^{\alpha+2(1-\theta)},I_1;X)}, \tag{5.24}$$

where $k_1 = |I_1|$; we recall that $\theta \in (0,1]$.

Over the remaining elements I_i, $i = 2,\dots,N$, of $[0,1]$, we use that $k_i \lesssim k y_{i-1}^{(\eta-1)/\eta}$, where $k_i = |I_i| = y_i - y_{i-1}$ and η defines the radical mesh on $[0,1]$ as in (5.15b). We thus recall the standard interpolation estimate

$$\|u - \pi^1_\eta u\|^2_{L^2(I_i)} \lesssim k_i^2 \|u'\|^2_{L^2(I_i)}$$

and obtain, upon using that $\max_{y\in I_i} y^\alpha \lesssim \min_{y \in I_i} y^\alpha$ and tensorization with X, the bound

$$\begin{aligned}\|u - \pi^1_\eta u\|^2_{L^2(y^\alpha,I_i;X)} &\lesssim k_i^2 \|u'\|^2_{L^2(y^\alpha,I_i;X)} \lesssim k^2 y_{i-1}^{2(\eta-1)/\eta} \|u'\|^2_{L^2(y^\alpha,I_i;X)} \\ &\lesssim k^2 \|u'\|^2_{L^2(y^{\alpha+2(\eta-1)/\eta},I_i;X)} \lesssim k^2 \|u'\|^2_{L^2(y^{\alpha+2(1-\theta)},I_i;X)}\,. \end{aligned} \tag{5.25}$$

The last relation holds because $\eta\theta \ge 1$.

For the elements beyond $y = 1$, we begin by setting, for $j = 1,\dots,N'$, $J_j := [\widetilde{y}_{j-1}, \widetilde{y}_j] = \big[\exp((j-1)k, \exp(jk)\big]$. Let us now notice that, since $k \le 1$,

$$|J_j| = \exp((j-1)k)(1-e^k) \sim \widetilde{y}_{j-1} k, \qquad j = 1,\dots,N'-1\,. \tag{5.26}$$

Using that the weight functions $\omega_{\alpha,\gamma'}$ and $\omega_{\alpha,\gamma}$, defined as in (4.10), are slowly varying over the intervals J_j, i.e.,

$$\max_{y\in J_j} \omega_{\alpha,\gamma'}(x) \lesssim \min_{y\in J_j} \omega_{\alpha,\gamma'}(x) \quad \text{and} \quad \max_{y\in J_j} \omega_{\alpha,\gamma}(x) \lesssim \min_{y\in J_j} \omega_{\alpha,\gamma}(x), \tag{5.27}$$

we obtain

$$\begin{aligned}\sum_j \|u - \pi^1_\eta u\|^2_{L^2(\omega_{\alpha,\gamma'},J_j;X)} &\lesssim \sum_j |J_j|^2 \|u'\|^2_{L^2(\omega_{\alpha,\gamma'},J_j;X)} \\ &\lesssim k^2 \sum_j \widetilde{y}_{j-1}^2 e^{-(\gamma-\gamma')\widetilde{y}_{j-1}} \|u'\|^2_{L^2(\omega_{\alpha,\gamma},J_j;X)},\end{aligned}$$

where in the last step we used (5.26). Using now that $\widetilde{y}_{j-1}^2 e^{-(\gamma-\gamma')\widetilde{y}_{j-1}} \lesssim 1$ and (5.27), again, we finally arrive at

$$\sum_j \|u - \pi^1_\eta u\|^2_{L^2(\omega_{\alpha,\gamma'},J_j;X)} \lesssim k^2 \sum_j \|u'\|^2_{L^2(\omega_{\alpha,\gamma},J_j;X)}. \tag{5.28}$$

Combining (5.24), (5.25), and (5.28) finishes the proof of the approximation properties of π^1_η. The correction on the last element to obtain (5.17) for the operator $\pi^1_{\eta,\{\mathscr{Y}\}}$ is straight forward in view of (5.26). The estimate (5.19) follows from (5.17) by controlling $\|u(\mathscr{Y})\|_X$ with the aid of Lemma A.2.

The proof of (iii) follows along similar lines. ◻

It is worth stressing that the choices $k = 2^{-\ell}$ lead to nested meshes.

COROLLARY 5.8 (nested meshes). *For every fixed $\eta \geq 0$, $\mathscr{Y} \geq 1$ and for $k_\ell = 2^{-\ell}$, the sequence $\{\mathcal{G}^{k_\ell}_{gr,\eta}\}_{\ell=0}^\infty$ of graded meshes in $(0,\mathscr{Y})$ is nested and has $\mathcal{O}(2^\ell(1+\log\mathscr{Y}))$ elements.*

Proof. For fixed $\mathscr{Y} > 0$, it follows directly from the definition of the mesh points (5.15), in terms of k, that the meshes are nested. ◻

5.4.3. Tensor P_1-FEM in $\mathcal{C}$ with corner mesh refinement in Ω. We now provide a convergence estimate in refined meshes over, not necessarily convex, polygons.

THEOREM 5.9 (error estimates). *Let $u \in \mathbb{H}^s(\Omega)$ and $\mathscr{U} \in \mathring{H}^1(y^\alpha,\mathcal{C})$ solve (1.1) and (1.2), respectively, with $f \in \mathbb{H}^{1-s}(\Omega)$ and $\Omega \subset \mathbb{R}^2$ a bounded polygon with straight sides and (a finite set of) corners $\{\boldsymbol{c}\}$. Let $\beta \geq 0$ be such that (5.10) holds and let $\{\mathcal{T}^\ell_\beta\}_\ell$ be a sequence of graded meshes that satisfy (5.13) and (5.14). Let $\mathcal{G}^k_{gr,\eta}$ be the graded–exponential mesh of (5.15) with η chosen to satisfy $\eta s > 1$, $k = 1/N$ with $N \in \mathbb{N}$ chosen so that $2h_\ell^{-1} \geq 1/k \geq h_\ell^{-1}$, and with the cut-off $\mathscr{Y} > 0$ chosen as*

$$\mathscr{Y} \sim |\log h_\ell| \,. \tag{5.29}$$

Denote by $\mathscr{U}_{h_\ell,M}$ the solution of (5.4) over the space $\mathbb{V}^{1,1}_{h,M}(\mathcal{T}^\ell_\beta,\mathcal{G}^k_{gr,\eta})$. In this setting we have the following error estimate

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{h,M}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h_\ell \|f\|_{\mathbb{H}^{1-s}(\Omega)}. \tag{5.30}$$

In addition, the total number of degrees of freedom behaves like

$$\mathcal{N}_{\Omega,\mathscr{Y}} := \dim \mathbb{V}^{1,1}_{h,M}(\mathcal{T}^\ell_\beta,\mathcal{G}^k_{gr,\eta}) = \mathcal{O}(h_\ell^{-3}\log|\log h_\ell|) = \mathcal{O}(\mathcal{N}_\Omega^{1+1/2}\log\log\mathcal{N}_\Omega)\,, \tag{5.31}$$

where $\mathcal{N}_\Omega = \#\mathcal{T}^\ell_\beta$.

Before proving Theorem 5.9, we note a corollary that follows from a simple interpolation argument.

COROLLARY 5.10 (reduced regularity). *Assume that the meshes are constructed as in Theorem 5.9 and that $f \in \mathbb{H}^{-s+\sigma}(\Omega)$, with $\sigma \in [0,1]$. Then we have*

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{h,M}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h_\ell^\sigma \|f\|_{\mathbb{H}^{\sigma-s}(\Omega)}, \tag{5.32}$$

where the hidden constant also depends on σ.

The proof of Theorem 5.9 follows similar arguments to [41] and [33, Section 4.1] and uses the stability and approximation properties (5.14) of Π^β_ℓ. For completeness we provide the details.

Proof of Theorem 5.9: For the given choice of k, η and $\mathscr{Y}$, we denote by $\pi^{1,\ell}_{\eta,\{\mathscr{Y}\}}$ the nodal interpolation operator on the mesh (5.15), which we analyzed in Lemma 5.7.

By Lemmas 5.1 and 5.2, and by the choice (5.29) (recall (5.6)) it suffices to bound

$$\|\nabla(\mathscr{U} - \pi^{1,\ell}_{\eta,\{\mathscr{Y}\}}\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathscr{Y})} + \|\nabla(\mathscr{U} - \Pi^\ell_\beta\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathscr{Y})} =: I + II\,.$$

Recalling that $\nabla = (\nabla_{x'}, \partial_y)$ we split the first term I into

$$I \lesssim \|\partial_y(\mathscr{U} - \pi^{1,\ell}_{\eta,\{\mathcal{Y}\}}\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + \|\nabla_{x'}(\mathscr{U} - \pi^{1,\ell}_{\eta,\{\mathcal{Y}\}}\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} =: I_a + I_b \ .$$

In view of (5.29), we immediately obtain that the conditions (5.18) and (5.22) of Lemma 5.7 are satisfied. We can thus, since $\eta s > 1$, bound the term I_a using Lemma 5.7, item (iii), with $j = 1$ and $X = L^2(\Omega)$ and the term I_b using Lemma 5.7, item (ii) with $X = H^1_0(\Omega)$. We have thus arrived at

$$I \lesssim I_a + I_b \lesssim h_\ell \|f\|_{\mathbb{H}^0(\Omega)},$$

where we have also used the regularity estimates of Theorem 4.7.

We apply the same splitting to the term II to arrive at

$$II \lesssim \|\partial_y(\mathscr{U} - \Pi^\ell_\beta\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + \|\nabla_{x'}(\mathscr{U} - \Pi^\ell_\beta\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} =: II_a + II_b \ .$$

Since $N_\ell = \mathcal{O}(h_\ell^{-2})$ we have, from (5.13), that

$$II_a = \|\partial_y\mathscr{U} - \Pi^\ell_\beta(\partial_y\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim h_\ell\|\partial_y\mathscr{U}\|_{L^2(\omega_{\alpha,0},(0,\mathcal{Y});H^1(\Omega))} \ .$$

To bound II_b we use (5.14) and obtain

$$II_b \lesssim h_\ell\|\mathscr{U}\|_{L^2(\omega_{\alpha,0},(0,\mathcal{Y});H^2_\beta(\Omega))} \ .$$

Using the regularity estimate (5.11) with $\nu' = 0$ we conclude the proof of (5.30).

To obtain (5.31), we first note that by Lemma 5.7 item (i), the number of elements in $\mathcal{G}^k_{gr,\eta}$ with $h_\ell = 2^{-\ell}$ and with the choice $\mathcal{Y} \simeq |\log h_\ell| \simeq \ell$ is $\mathcal{O}(2^\ell \log \ell)$. We finally observe that the total number of degrees of freedom in the tensor product space is the product of the dimensions of the component spaces, i.e., $\mathcal{O}(h_\ell^{-2}h_\ell^{-1}\log|\log h_\ell|)$. $\square$

5.4.4. Sparse grid P_1-FEM with corner mesh refinement. The convergence order (5.30) is optimal, however, the complexity of the method implied by (5.31) is superlinear with respect to the number of degrees of freedom in Ω, $\mathcal{N}_\Omega$.

To reduce the complexity to nearly linear, in what follows we develop a sparse tensor product approach. It is based on the subspace hierarchies

$$\{S^1_0(\Omega,\mathcal{T}^\ell_\beta)\}_{\ell\geq 1} \ , \quad \{S^1_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^{2^{-\ell'}}_{gr,\eta})\}_{\ell'\geq 1},$$

where $\{\mathcal{T}^\ell_\beta\}_{\ell\geq 1}$ is the nested sequence of bisection–tree meshes in Ω which are β-graded toward the corners $\{\boldsymbol{c}\}$ in such a way that first-order convergence in $h_\ell = \mathcal{O}(2^{-\ell})$ is achieved; the sequence $\{\mathcal{G}^{2^{-\ell'}}_{gr,\eta}\}_{\ell'\geq 1}$ consists of nested graded meshes on $[0,\mathcal{Y}]$ that achieve, for functions belonging to weighted H^2-spaces in $(0,\mathcal{Y})$, as introduced in Theorem 4.7, first order convergence (cf. the precise statements in Lemma 5.7 and in Corollary 5.8).

For $\ell, \ell' \geq 0$, we denote by

$$\Pi^\ell_\beta : L^2(\Omega) \to S^1_0(\Omega,\mathcal{T}^\ell_\beta) \text{ and } \pi^{1,\ell'}_{\eta,\{\mathcal{Y}\}} : C((0,\mathcal{Y}]) \to S^1_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^{2^{-\ell'}}_{gr,\eta})$$

the corresponding (quasi)interpolatory projections introduced in Section 5.1. Define in addition $\Pi^{-1}_\beta := 0$ and $\pi^{1,-1}_{\eta,\{\mathcal{Y}\}} := 0$. Then, for $L \in \mathbb{N}_0$, we define the *sparse tensor product space* as

$$\hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y}) = \sum_{\ell,\ell'\geq 0,\ell+\ell'\leq L} S^1_0(\Omega,\mathcal{T}^\ell_\beta) \otimes S^1_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^{2^{-\ell'}}_{gr,\eta}) \ . \tag{5.33}$$

We immediately comment that the sum in (5.33) is not direct, and by zero extension we, evidently, have $\hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y}) \subset \mathring{H}^1(y^\alpha, \mathcal{C})$.

We define the approximation $\hat{\mathscr{U}}_L \in \hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y})$ as the solution to (5.4) with $\hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y})$ taking the role of $\mathbb{V}_{h,M}$ there.

REMARK 5.11 (implementation). *The computation of the sparse tensor FE approximation $\hat{\mathscr{U}}_L \in \hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y})$ by directly evaluating* (5.4) *would require an explicit representation of the sparse tensor product subspace $\hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y})$ and therefore, in particular, an explicit basis for the "increment spaces" in* (5.33)*, i.e., for the complements of $S^1_0(\Omega, \mathcal{T}^{\ell-1}_\beta)$ in $S^1_0(\Omega, \mathcal{T}^{\ell}_\beta)$ and the complements of $S^1_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^{2^{-(\ell'-1)}}_{gr,\eta})$ in $S^1_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^{2^{-\ell'}}_{gr,\eta})$. Construction of bases for the increment spaces is possible, based on ideas from multiresolution analyses. We opt, instead, to compute $\hat{\mathscr{U}}_L \in \hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y})$ from the so-called combination formula (see, e.g., [29, Section 4.2, Equation (4.6)]). It is based on anisotropic $\mathring{H}^1(y^\alpha, \mathcal{C})$-Galerkin projections*

$$G^{1,1}_{\ell,\ell'} := G^1_{h_\ell} \circ G^1_{2^{-\ell'}} : \mathring{H}^1(y^\alpha; \mathcal{C}) \to \mathbb{V}^{1,1}_{h,M}(\mathcal{T}_\ell, \mathcal{G}^{2^{-\ell'}}_{gr,\eta}) \, , \tag{5.34}$$

with the semidiscrete projections in (5.3)*. The projectors $G^{1,1}_{\ell,\ell'}$ in* (5.34) *can be realized with standard FE bases in Ω and in $(0,\mathcal{Y})$. The combination formula then takes the following form: denoting by $\mathscr{U}_{\ell,\ell'} := G^{1,1}_{\ell,\ell'}\mathscr{U}$, there holds, with the understanding that $\mathscr{U}_{-1,j} = 0$ for $j \in \mathbb{N}_0$,*

$$\hat{\mathscr{U}}_L \quad = \quad \sum_{\ell=0}^{L} (\mathscr{U}_{\ell,L-\ell} - \mathscr{U}_{\ell-1,L-\ell}) \, .$$

The convergence of our sparse grids scheme is the content of the next result.

THEOREM 5.12 (convergence for sparse grids). *Let $\beta \geq 0$ be such that* (5.10) *holds. Let $1 < \nu < 1 + s$. Let $\eta(\nu - 1) \geq 1$. Select $\mathcal{Y} \sim |\log h_L|$ with a sufficiently large implied constant. Let $f \in \mathbb{H}^{-s+\nu}(\Omega)$. Then the sparse tensor product space $\hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y})$ of* (5.33) *and the corresponding Galerkin approximation $\hat{\mathscr{U}}_L \in \hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y})$ to $\mathscr{U}$ satisfy*

$$\|\nabla(\mathscr{U} - \hat{\mathscr{U}}_L)\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h_L |\log h_L| \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, \tag{5.35}$$

$$\dim \hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y}) \lesssim \mathcal{N}_\Omega \log\log \mathcal{N}_\Omega. \tag{5.36}$$

Proof. We begin by proving (5.36). From the condition $\mathcal{Y} \sim |\log h_L| \sim L$, we have, by Lemma 5.7, item (i), that $\#(\mathcal{G}^{2^{-\ell'}}_{gr,\eta}) \lesssim 2^{\ell'} |\log h_L| \sim 2^{\ell'} \log L$. Consequently,

$$\dim \hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y}) \lesssim \sum_{\ell,\ell' \geq 0, \ell+\ell' \leq L} 2^{2\ell+\ell'} |\log L| \lesssim 2^{2L} \log L \sim \mathcal{N}_\Omega \log\log \mathcal{N}_\Omega \, , \tag{5.37}$$

where we have also used that $N_\ell = \dim(S^1_0(\Omega, \mathcal{T}^\ell_\beta)) \sim 2^{2\ell}$.

We now study the error of our method. From Lemma 5.1 and (5.6) it suffices to study the best approximation error in $\mathbb{V}^{1,1}_L(\mathcal{C}_\mathcal{Y})$. To do so, we introduce the *sparse tensor product interpolation* projector

$$\hat{\Pi}^L_\mathcal{Y} : C((0,\mathcal{Y}]; L^2(\Omega)) \to \hat{\mathbb{V}}^{1,1}_L(\mathcal{C}_\mathcal{Y})$$

which is defined by

$$\hat{\Pi}^L_{\mathscr{Y}} w := \sum_{\ell,\ell'\geq 0, \ell+\ell'\leq L} (\Pi^\ell_\beta - \Pi^{\ell-1}_\beta) \otimes (\pi^{1,\ell'}_{\eta,\{\mathscr{Y}\}} - \pi^{1,\ell'-1}_{\eta,\{\mathscr{Y}\}}) w \ . \tag{5.38}$$

We can now, as in the proof of Theorem 5.9, split the error into

$$\begin{aligned} \min_{\hat{v}_L \in \hat{\mathbb{V}}_L} \|\nabla(\mathscr{U} - \hat{v}_L)\|^2_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} &\lesssim \|\partial_y(\mathscr{U} - \hat{\Pi}^L_{\mathscr{Y}}\mathscr{U})\|^2_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \\ &\quad + \|\nabla_{x'}(\mathscr{U} - \hat{\Pi}^L_{\mathscr{Y}}\mathscr{U})\|^2_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} =: I + II \ . \end{aligned} \tag{5.39}$$

Each one of these terms can now be bounded in the usual sparse grid fashion, *provided* that $\mathscr{U}$ has so-called mixed regularity. To do this we introduce the operators

$$Q^\ell_\beta := \Pi^\ell_\beta - \Pi^{\ell-1}_\beta \ , \quad q^{1,\ell'}_\eta := \pi^{1,\ell'}_{\eta,\{\mathscr{Y}\}} - \pi^{1,\ell'-1}_{\eta,\{\mathscr{Y}\}} \ .$$

Let us bound term I in (5.39). From the estimate (4.18) of Theorem 4.7 we infer

$$\|\partial^2_y \mathscr{U}\|_{L^2(\omega_{\alpha+2(2-\nu),\gamma},(0,\infty);H^1(\Omega))} \lesssim \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, \qquad 0 \leq \nu < 1+s. \tag{5.40}$$

Of interest to us is the case $1 < \nu < 1+s < 2$. Then, with the mesh grading parameter η satisfying $\eta(-1+\nu) \geq 1$ and upon assuming that $\mathscr{Y} \geq CL$ for $C > 0$ sufficiently large so that the condition (5.22) is satisfied we estimate

$$\begin{aligned} I &\leq \sum_{\ell+\ell'>L} \|\partial_y(Q^\ell_\beta \otimes q^{1,\ell'}_y \mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \\ &\leq \sum_{\ell+\ell'>L} \|\partial_y[((I_{x'} \otimes q^{1,\ell'}_y) \circ (Q^\ell_\beta \otimes I_y)\mathscr{U}]\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \\ &\lesssim \sum_{\ell+\ell'>L} 2^{-\ell} \|\partial_y[(I_{x'} \otimes q^{1,\ell'}_y)\mathscr{U}]\|_{L^2(y^\alpha,(0,\mathscr{Y});H^1(\Omega))}, \end{aligned}$$

where, in the last step, we used the approximation property (5.13). We now apply the estimate (5.23) with $j = 0$, $\theta = \nu - 1$ and $X = H^1(\Omega)$, to arrive at

$$I \lesssim \sum_{\ell+\ell'>L} 2^{-\ell-\ell'} \|\partial^2_y \mathscr{U}\|_{L^2(\omega_{\alpha+2(2-\nu),\gamma},(0,\mathscr{Y});H^1(\Omega))} \lesssim L 2^{-L} \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)} \ ,$$

where in the last step we have used the regularity estimate (5.40).

Let us now bound, using similar arguments, the term II in (5.39). From (5.11) and (5.12) we obtain, for $1 \leq \nu < 2 - s$, the regularity estimate

$$\|\partial_y \mathscr{U}\|_{L^2(\omega_{\alpha+2(2-\nu),\gamma},(0,\infty);H^2_\beta(\Omega))} \lesssim \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}. \tag{5.41}$$

Hence, for $\eta(-1+\nu) \geq 1$, and again under the condition that $\mathscr{Y} \geq CL$ so that (5.18) is satisfied, we can estimate

$$\begin{aligned} II &\leq \sum_{\ell+\ell'>L} \|\nabla_{x'}(Q^\ell_\beta \otimes q^{1,\ell'}_y \mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \\ &\leq \sum_{\ell+\ell'>L} \|\nabla_{x'}[((I_{x'} \otimes q^{1,\ell'}_y) \circ (Q^\ell_\beta \otimes I_y)\mathscr{U}]\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \\ &\lesssim \sum_{\ell+\ell'>L} 2^{-\ell} \|(I_{x'} \otimes q^{1,\ell'}_y)\mathscr{U}\|_{L^2(y^\alpha,(0,\mathscr{Y});H^2_\beta(\Omega))} \end{aligned}$$

where in the last step we used the approximation properties of Π_β^ℓ, as stated in (5.14). The approximation properties of $\pi^{1,\ell'}_{\eta,\{\mathscr{Y}\}}$ given in (5.19) with the regularity estimate of (5.41) allow us to conclude that

$$II \lesssim \sum_{\ell+\ell'>L} 2^{-\ell-\ell'} \|\partial_y \mathscr{U}\|_{L^2(\omega_{\alpha+2(2-\nu),\gamma},(0,\mathscr{Y});H^2_\beta(\Omega))} \lesssim L2^{-L}\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)} \ .$$

Collecting the bounds obtained for I and II yields the result. □

Theorem 5.12 shows that it is possible to obtain near optimal order convergence for fractional diffusion in Ω, by using only P_1-FEM in both Ω and the extended dimension. An alternative approach is based on exploiting *analytic regularity* of the solution of the extended problem. In this case, *exponentially convergent hp-FEM with respect to the extended variable y* will achieve near optimal order for conforming P_1-FEM in Ω, as observed recently in [33], and, as we show (by a different argument) in Section 5.5, see Theorem 5.14.

5.5. *hp*-FEM in $(0,\infty)$ and P_1-FEM in Ω. The discretizations in the preceding Sections 5.4.4 and 5.4.3 were of first order in x' and y. We showed that full tensor product FEM allows to achieve first order convergence in Ω at the expense of superlinear complexity (5.31). Here, we address the use of the so-called *hp*-FEM in $(0,\mathscr{Y})$; the analytic regularity estimates derived in Section 4 allow us to prove *exponential convergence estimates* for corresponding high-order discretizations in $(0,\mathscr{Y})$. We consider two situations:

a) The case where $\boldsymbol{r}$ is a so-called linear degree vector in $(0,\mathscr{Y})$, which will imply exponential convergence with respect to y (cf. Lemma 6.2 below). If fixed order FEM on a sequence $\{\mathcal{T}^\ell_\beta\}_{\ell\geq 0}$ of regular, simplicial corner-refined meshes in Ω are used, near optimal, algebraic convergence rates (with respect to the number $\mathcal{N}_\Omega$ of degrees of freedom in Ω) result for the solution of (1.1) in Ω (Theorem 5.14). We mention [30] where, in a structurally similar context, analyticity in the extended variable is also exploited by an *hp*-FEM.
b) The case where $\boldsymbol{r}$ is a linear degree vector in $(0,\mathscr{Y})$, and where we use the *hp*-FEM in Ω; in this case, and under the additional assumption (7.1) of analyticity on the data c, f, A, *exponential convergence in terms of the number $\mathcal{N}_{\Omega,\mathscr{Y}}$ of degrees of freedom in $\mathcal{C}_\mathscr{Y}$* can be achieved. We confine the exposition to $\Omega = (0,1)$ and to $\Omega \subset \mathbb{R}^2$ with analytic boundary. This will be the content of Section 7.

5.5.1. A univariate *hp*-interpolation operator. We present here the construction of a univariate interpolation operator that leads to exponential convergence for analytic functions that may have a singularity at $y = 0$. The construction is essentially taken from the work by Babuška and collaborators, [28, 7] and discussed in the literature on *hp*-FEM (see, e.g., [51, Sec. 4.4.1], [5, Thm. 8] and also [33]).

To make matters precise, we consider geometric meshes $\mathcal{G}^M_{geo,\sigma}$ on $[0,\mathscr{Y}]$ with M elements and grading factor $\sigma \in (0,1)$: $\{I_i \,|\, i = 1,\dots,M\}$ with $I_1 = [0,\mathscr{Y}\sigma^{M-1}]$ and $I_i = [\mathscr{Y}\sigma^{M-i+1},\mathscr{Y}\sigma^{M-i}]$ for $i = 2,\dots,M$. On such meshes, we consider a *linear degree vector* $\boldsymbol{r}$ with slope $\mathfrak{s}$ given by

$$r_i := \max\{1, \lceil \mathfrak{s} i\rceil\} , \quad i = 1,2,...,M \ . \tag{5.42}$$

We denote by $\widehat{K} = (-1,1)$ the reference interval. We will require a base interpolation operator $\widehat{\Pi}_r : H^1(\widehat{K}) \to \mathbb{P}_r(\widehat{K})$ that allows for exponential convergence in r for analytic functions with the following two properties:

1. $(\widehat{\Pi}_r \widehat{u})(\pm 1) = \widehat{u}(\pm 1)$ for all $\widehat{u} \in H^1(\widehat{K})$.
2. For every $K_u > 0$ there exist $C = C(K_u)$, $b = b(K_u) > 0$ such that if, for all $\ell \in \mathbb{N}_0$, we have $\|\widehat{u}^{(\ell)}\|_{L^2(\widehat{K})} \le C_u K_u^{\ell+1}(\ell+1)!$ then

$$\|\widehat{u} - \widehat{\Pi}_r \widehat{u}\|_{H^1(\widehat{K})} \lesssim C_u e^{-br} \qquad \forall r \in \mathbb{N}.$$

Classical examples of such operators include the Gauss-Lobatto interpolation operator and the "Babuška-Szabó operator" Π_r^{BS} as described, e.g., in the survey [5, Example 13] or in [51, Theorem 3.14].

With the aid of $\widehat{\Pi}_r$ we introduce the operators $\pi_y^{\boldsymbol{r}}$ and $\pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}}$ on an arbitrary mesh $\mathcal{G}^M$ on $[0,\mathcal{Y}]$ with M elements and polynomial degree distribution $\boldsymbol{r} \in \mathbb{N}^M$ in an element-by-element fashion in the usual way below. However, for $\pi_y^{\boldsymbol{r}}$ we modify the approximation on the first element $I_1 = [0, y_1]$ by interpolating in the points $y_1/2$ and y_1 instead of the endpoints. The operator $\pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}}$ is obtained by a further modification that enforces $\pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}}(\mathcal{Y}) = 0$. Specifically, with $F_{I_i} : \widehat{K} \to I_i$ denoting the affine, orientation-preserving element maps for element $I_i \in \mathcal{G}^M$ we have

$$\begin{aligned}
((\pi_y^{\boldsymbol{r}} u)|_{I_1} \circ F_{I_1})(\xi) &= 2(u \circ F_{I_1})(1)(\xi - 1/2) + 2(u \circ F_{I_1})(1/2)(1-\xi),\\
((\pi_y^{\boldsymbol{r}} u)|_{I_i} \circ F_{I_i})(\xi) &= \widehat{\Pi}_{\boldsymbol{r}_m}(u \circ F_{I_i}), \qquad i = 2, \dots, M,\\
(\pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}} u)|_{I_i} &= (\pi_y^{\boldsymbol{r}} u)|_{I_i}, \qquad i = 1, \dots, M-1,\\
((\pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}} u)|_{I_M} \circ F_{I_M})(\xi) &= ((\pi_y^{\boldsymbol{r}} u)|_{I_M} \circ F_{I_M})(\xi) - (u \circ F_{I_M})(1)(\xi+1)/2\,.
\end{aligned}$$

The definition of $\pi_y^{\boldsymbol{r}}$, $\pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}}$ is naturally extended for functions $u \in C^0((0,\mathcal{Y}];X)$, where X denotes a Hilbert space. We will apply these operators to functions from the following two classes of analytic functions of the extended variable y:

$$\begin{aligned}
\mathcal{B}^1_{\beta,\gamma}(C_u, K_u; X) := \big\{ & u \in C^\infty((0,\infty);X) \,:\, \|u\|_{L^2(\omega_{\alpha,\gamma},(0,\infty);X)} < C_u,\\
& \|u^{(\ell+1)}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\beta,\gamma},(0,\infty);X)} < C_u K_u^{\ell+1}(\ell+1)! \quad \forall \ell \in \mathbb{N}_0 \big\}
\end{aligned} \tag{5.43}$$

and

$$\begin{aligned}
\mathcal{B}^2_{\beta,\gamma}(C_u, K_u, X) := \big\{ & u \in C^\infty((0,\infty);X) \,:\\
& \|u\|_{L^2(\omega_{\alpha,\gamma},(0,\infty);X)} + \|u'\|_{L^2(\omega_{\alpha,\gamma},(0,\infty);X)} \le C_u,\\
& \|u^{(\ell+2)}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\beta,\gamma},(0,\infty);X)} \le C_u K_u^{\ell+2}(\ell+2)! \quad \forall \ell \in \mathbb{N}_0 \big\}.
\end{aligned} \tag{5.44}$$

We recall that the weight $\omega_{\beta,\gamma}$ is defined as in (4.10). In the case that $X = \mathbb{R}$, we omit the tag X in (5.43), (5.44).

The approximation properties of the operators $\pi_y^{\boldsymbol{r}}$ and $\pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}}$ are given below.

LEMMA 5.13 (exponential interpolation error estimates). *Let $\beta \in (0,1]$, $\gamma > 0$, C_u, $K_u \ge 0$. Let $\sigma \in (0,1)$. Then there exists a slope $\mathfrak{s}_{min} > 0$ for the degree vector such that on the geometric mesh $\mathcal{G}^M_{geo,\sigma}$ the following estimates hold for any polynomial degree distribution $\boldsymbol{r}$ with $\boldsymbol{r}_i \ge 1 + \mathfrak{s}_{min}(i-1)$:*

(i) If $u \in \mathcal{B}^1_{\beta,\gamma}(C_u, K_u; X)$ and $\sigma^M \mathcal{Y} \le 1$, then

$$\|u - \pi_y^{\boldsymbol{r}} u\|_{L^2(\omega_{\alpha,\gamma},(0,\mathcal{Y});X)} \lesssim C_u \mathcal{Y}^\beta e^{-bM}, \tag{5.45}$$

$$\|u - \pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}} u\|_{L^2(\omega_{\alpha,\gamma},(0,\mathcal{Y});X)} \lesssim C_u \left(\mathcal{Y}^\beta e^{-bM} + \mathcal{Y}^{-1/2+\beta} e^{-\gamma\mathcal{Y}/2} \right). \tag{5.46}$$

(ii) If $u \in \mathcal{B}^2_{\beta,\gamma}(C_u, K_u; X)$ and $\sigma^M \mathcal{Y} \le 1$, then

$$\|(u - \pi_y^{\boldsymbol{r}} u)'\|_{L^2(\omega_{\alpha,\gamma},(0,\mathcal{Y});X)} \lesssim C_u \mathcal{Y}^\beta e^{-bM}, \tag{5.47}$$

$$\|(u - \pi_{y,\{\mathcal{Y}\}}^{\boldsymbol{r}} u)'\|_{L^2(\omega_{\alpha,\gamma},(0,\mathcal{Y});X)} \lesssim C_u \left(\mathcal{Y}^\beta e^{-bM} + \mathcal{Y}^{-3/2+\beta} e^{-\gamma\mathcal{Y}/2}\right). \tag{5.48}$$

In all the estimates, the hidden constant and b depend only on β, γ, α, σ, and K_u.

Proof. See Appendix A. ∎

5.5.2. hp-discretization in y and P_1 FEM in Ω. With the hp-approximation operator $\pi_y^{\boldsymbol{r}}$ of the previous section at hand, we can analyze the properties of the space $\mathbb{V}^{1,\boldsymbol{r}}_{h,M}(\mathcal{T}^\ell_\beta, \mathcal{G}^M_{geo,\sigma})$. The following result generalizes [33] in that we allow for a general elliptic operator $\mathcal{L}$ and in that the appropriate mesh grading in Ω is included to compensate for the lack of a full elliptic shift theorem.

Theorem 5.14 (error estimates). *Let $u \in \mathbb{H}^s(\Omega)$ and $\mathscr{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$ solve (1.1) and (1.2), respectively, with $f \in \mathbb{H}^{1-s}(\Omega)$ and $\Omega \subset \mathbb{R}^2$ a bounded polygon with straight sides and (a finite set of) corners $\{\boldsymbol{c}\}$. Let $\beta \ge 0$ be such that (5.10) holds and let $\{\mathcal{T}^\ell_\beta\}_\ell$ be a sequence of graded meshes that satisfy (5.13) and (5.14). Let $\mathcal{G}^M_{geo,\sigma}$ be a geometric mesh on $(0,\mathcal{Y})$ with $\mathcal{Y} \sim |\log h_\ell|$ with a sufficiently large constant. Let $\mathscr{U}_{h_\ell,M}$ be the solution of (5.4) over the space $\mathbb{V}^{1,\boldsymbol{r}}_{h,M}(\mathcal{T}^\ell_\beta, \mathcal{G}^M_{geo,\sigma})$. Then there exists a minimal slope $\mathfrak{s}_{min}$ independent of h_ℓ and f such that for linear degree vectors $\boldsymbol{r}$ with slope $\mathfrak{s} \ge \mathfrak{s}_{min}$ there holds*

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{h,M}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{h_\ell,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h_\ell \|f\|_{\mathbb{H}^{1-s}(\Omega)}. \tag{5.49}$$

In addition, the total number of degrees of freedom behaves like

$$\dim \mathbb{V}^{1,\boldsymbol{r}}_{h,M}(\mathcal{T}^\ell_\beta, \mathcal{G}^M_{geo,\sigma}) \sim \mathcal{N}_{\Omega,\mathcal{Y}} \sim M^2 h_\ell^{-2} \sim h_\ell^{-2}(\log h_\ell)^2 \sim \mathcal{N}_\Omega \log \mathcal{N}_\Omega,$$

where $\mathcal{N}_\Omega = \#\mathcal{T}^\ell_\beta$. More generally, if $f \in \mathbb{H}^{\sigma-s}(\Omega)$ for $\sigma \in [0,1]$, then the bound (5.49) takes the form

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{h,M}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{h_\ell,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h_\ell^\sigma \|f\|_{\mathbb{H}^{\sigma-s}(\Omega)}.$$

Proof. The starting point is again the error decomposition (5.7). The univariate hp-interpolation operator $\pi_y^{\boldsymbol{r}}$ constructed in Section 5.5.1 makes the semidiscretization error $\mathscr{U} - \pi_y^{\boldsymbol{r}}\mathscr{U}$ in y exponentially small in M (see Lemma 6.2 below for details). In turn, the assumption $M \sim |\log h_\ell|$ implies any desired algebraic convergence in h_ℓ by suitably selecting the implied constant. On the other hand, the error $\mathscr{U} - \Pi^q_{x'}\mathscr{U}$ in (5.7) is controlled as in the proof of Theorem 5.9.

Finally, the estimate for $f \in \mathbb{H}^{\sigma-s}(\Omega)$ follows by interpolation. ∎

6. Diagonalization: semidiscretization in y. We now explore the possibilities offered by a semidiscretization in y. We will observe, among other things, that this leads to a sequence of decoupled singularly perturbed, *linear second order* elliptic problems in Ω.

For an arbitrary mesh $\mathcal{G}^M$ on $[0,\mathcal{Y}]$ and for a polynomial degree distribution $\boldsymbol{r}$, we consider the following y-semidiscrete problem: Find $\mathscr{U}_M \in \mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_\mathcal{Y})$ such that

$$a_\mathcal{C}(\mathscr{U}_M, \phi) = d_s\langle f, \operatorname{tr}_\Omega \phi\rangle \qquad \forall \phi \in \mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_\mathcal{Y}), \tag{6.1}$$

where $\mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_\mathcal{Y})$ is defined as in (5.2) and is a closed subspace of $\mathring{H}^1(y^\alpha,\mathcal{C})$. In what follows we obtain an explicit formula for $\mathscr{U}_M$. To accomplish this, we consider the following eigenvalue problem: Find $(v,\mu) \in S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M)\setminus\{0\}\times\mathbb{R}$ such that

$$\mu\int_0^{\mathcal{Y}} y^\alpha v'(y)w'(y)\,\mathrm{d}y = \int_0^{\mathcal{Y}} y^\alpha v(y)w(y)\,\mathrm{d}y \qquad \forall w\in S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M), \tag{6.2}$$

where $S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M)$ is defined as in Section 5.1. All eigenvalues μ are positive, and the space $S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M)$ has an eigenbasis $(v_i)_{i=1}^{\mathcal{M}}$, with $\mathcal{M}:=\dim S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M)$, such that, for $i,\, j\in\{1,\dots,\mathcal{M}\}$,

$$\int_0^{\mathcal{Y}} y^\alpha v_i'(y)v_j'(y)\,\mathrm{d}y = \delta_{i,j}, \qquad \int_0^{\mathcal{Y}} y^\alpha v_i(y)v_j(y)\,\mathrm{d}y = \mu_i\delta_{i,j}. \tag{6.3}$$

We now write $\mathscr{U}_M(x',y) := \sum_{j=1}^{\mathcal{M}} U_j(x')v_j(y)$ and consider $\phi(x',y)=V(x')v_i(y)$, with $V\in H^1_0(\Omega)$ as a test function, in (6.1). This yields the following system of *decoupled* problems for $i=1,\dots,\mathcal{M}$: Find $U_i\in H^1_0(\Omega)$ such that

$$a_{\mu_i,\Omega}(U_i,V) = d_s v_i(0)\langle f,V\rangle \qquad \forall V\in H^1_0(\Omega), \tag{6.4}$$

where

$$a_{\mu_i,\Omega}(U,V) := \mu_i a_\Omega(U,V) + \int_\Omega UV\,\mathrm{d}x',$$

and a_Ω is introduced in (2.1). An important observation is that, for functions of the form $Z(x',y)=\sum_{i=1}^{\mathcal{M}} V_i(x')v_i(y)$ with $V_i\in H^1_0(\Omega)$, we have the equality

$$a_{\mathcal{C}}(Z,Z) = a_{\mathcal{C}_\mathcal{Y}}(Z,Z) = \sum_{i=1}^{\mathcal{M}} \|V_i\|^2_{\mu_i,\Omega}, \qquad \|V\|^2_{\mu_i,\Omega} := a_{\mu_i,\Omega}(V,V). \tag{6.5}$$

To obtain a fully discrete scheme, select a mesh $\mathcal{T}$ on Ω and the corresponding space $S^q_0(\Omega,\mathcal{T})$ and let $\Pi_i : H^1_0(\Omega)\to S^q_0(\Omega,\mathcal{T})$ be the Ritz projectors for the bilinear forms $a_{\mu_i,\Omega}$:

$$a_{\mu_i,\Omega}(u-\Pi_i u, v) = 0 \qquad \forall v\in S^q_0(\Omega,\mathcal{T}). \tag{6.6}$$

With this notation at hand, we can formulate an explicit representation of the Galerkin approximation $\mathscr{U}_{h,M}\in S^q_0(\Omega,\mathcal{T})\otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}(\mathcal{G}^M)$ to $\mathscr{U}$ as well as an error representation.

Lemma 6.1 (error representation). *Let $(\mu_i,v_i)_{i=1}^{\mathcal{M}}$ be the eigenpairs given by* (6.2), (6.3). *Let $U_i\in H^1_0(\Omega)$ be the solution to* (6.4) *and $\Pi_i : H^1_0(\Omega)\to S^q_0(\Omega,\mathcal{T})$ given as in* (6.6). *Let $\mathscr{U}_M$ be the solution to the semidiscrete problem* (6.1). *Then the Galerkin approximation $\mathscr{U}_{h,M}\in S^q_0(\Omega,\mathcal{T})\otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}(\mathcal{G}^M)$ to $\mathscr{U}$ satisfies*

$$\mathscr{U}_{h,M}(x',y) = \sum_{i=1}^{\mathcal{M}} \Pi_i U_i(x')v_i(y), \tag{6.7}$$

$$a_{\mathcal{C}}(\mathscr{U}_M-\mathscr{U}_{h,M},\mathscr{U}_M-\mathscr{U}_{h,M}) = \sum_{i=1}^{\mathcal{M}} \|U_i-\Pi_i U_i\|^2_{\mu_i,\Omega}. \tag{6.8}$$

Proof. Expression (6.7) follows from (6.4) and (6.6), whereas (6.8) is a consequence of (6.5). ∎

We next show that the semidiscretization error $\mathscr{U} - \mathscr{U}_M$ can be made exponentially small on geometric meshes $\mathcal{G}^M_{geo,\sigma}$.

Lemma 6.2 (exponential convergence). *Let $f \in \mathbb{H}^{-s+\nu}(\Omega)$ for $\nu \in (0,s)$. Let $c_1 M \le \mathscr{Y} \le c_2 M$. Consider the geometric mesh $\mathcal{G}^M_{geo,\sigma}$ on $(0,\mathscr{Y})$. Then there exist C, $\mathfrak{s}_{min}$, $b > 0$ (depending solely on s, $\mathcal{L}$, c_1, c_2, σ, ν) such that for any linear degree $\boldsymbol{r}$ with slope $\mathfrak{s} \ge \mathfrak{s}_{min}$ there holds*

$$\|\nabla(\mathscr{U} - \mathscr{U}_M)\|_{L^2(y^\alpha,\mathcal{C})} \le Ce^{-bM}\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}. \tag{6.9}$$

Proof. We begin the proof by invoking Galerkin orthogonality to arrive at

$$\begin{aligned}\|\mathscr{U} - \mathscr{U}_M\|^2_{\mathcal{C}} &\le \|\mathscr{U} - \pi^{\boldsymbol{r}}_{y,\{\mathscr{Y}\}}\mathscr{U}\|^2_{\mathcal{C}} \\ &\lesssim \|\mathscr{U} - \pi^{\boldsymbol{r}}_{y,\{\mathscr{Y}\}}\mathscr{U}\|^2_{\mathcal{C}_{\mathscr{Y}}} + \|\nabla\mathscr{U}\|^2_{L^2(y^\alpha,\mathcal{C}\setminus\mathcal{C}_{\mathscr{Y}})},\end{aligned}$$

where $\|\cdot\|_{\mathcal{C}}$ and $\|\cdot\|_{\mathcal{C}_{\mathscr{Y}}}$ are defined by (2.9) and (2.10), respectively. Since (5.6) shows that $\|\nabla\mathscr{U}\|_{L^2(y^\alpha;\mathcal{C}\setminus\mathcal{C}_{\mathscr{Y}})}$ is exponentially small in $\mathscr{Y}$ we thus focus on the interpolation error term. To control such a term we first observe that, in view of the definitions of the spaces $\mathcal{B}^j_{\beta,\gamma}$, $j \in \{0,1\}$, given by (5.43), (5.44), the regularity estimates (4.17) and (4.18) of Theorem 4.7, imply that $\mathscr{U}$ viewed as a function in $C^\infty((0,\infty),L^2(\Omega)) \cap C^\infty((0,\infty),H^1_0(\Omega))$ satisfies for $\nu \in (0,s)$ and $K > \kappa$ (with κ as in Theorem 4.7)

$$\mathscr{U} \in \mathcal{B}^1_{\nu,\gamma}(C\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, K; H^1_0(\Omega)) \cap \mathcal{B}^2_{\nu,\gamma}(C\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, K; L^2(\Omega)). \tag{6.10}$$

From Lemma 5.13 together with the fact that $\mathscr{Y} \sim M$ we conclude that

$$\|\nabla_{x'}(\mathscr{U} - \pi^{\boldsymbol{r}}_{y,\{\mathscr{Y}\}}\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \le Ce^{-bM}\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, \tag{6.11}$$

$$\|\partial_y(\mathscr{U} - \pi^{\boldsymbol{r}}_{y,\{\mathscr{Y}\}}\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \le Ce^{-bM}\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, \tag{6.12}$$

with $b > 0$ slightly smaller than that in (5.46) and (5.48). This implies the desired estimate (6.9) and concludes the proof. ∎

Finally, for the geometric mesh $\mathcal{G}^M_{geo,\sigma}$ with the linear degree vector $\boldsymbol{r}$ and truncation parameter $\mathscr{Y} \sim M$, we have the following estimates for the eigenvalues μ_i of problem (6.2) and for the point values $v_i(0)$ in (6.4).

Lemma 6.3 (properties of the eigenpairs). *Let $\mathcal{G}^M_{geo,\sigma}$ be a geometric mesh on $(0,\mathscr{Y})$ and $\boldsymbol{r}$ a linear degree vector with slope $\mathfrak{s}$. If $c_i M \le \mathscr{Y} \le c_2 M$, then there are constants C, b depending only on σ such that for the eigenpairs $(\mu_i, v_i)_{i=1}^{\mathcal{M}}$ given by* (6.2), (6.3) *we have that:*

$$\|v_i\|_{L^\infty(0,\mathscr{Y})} \le CM^{(1-\alpha)/2}, \qquad C^{-1}\mathfrak{s}^{-2}M^{-1}\sigma^M \le \mu_i \le CM^2.$$

Proof. The results follow from Lemmas B.1, B.2, and B.3. ∎

The previously described approach that perform a semidiscretization in y leads to structural insight into the regularity properties of the solution $\mathscr{U}$: it shows that, up to an exponentially small, in $\mathscr{Y}$, error introduced by cutting off at $\mathscr{Y}$, the solution $\mathscr{U}$ can be expressed in terms of solutions of singularly perturbed reaction–diffusion type problems. (A similar structural property for $\mathscr{U}(\cdot,0)$ can also be seen from the Balakrishnan formula, e.g., [11, Equation (4)]). In what follows we will exploit this to design appropriate approximation spaces in the x'-variable. Nevertheless, the diagonalization (6.1)–(6.4) has more far-reaching ramifications:

- The diagonalization technique can be exploited numerically as it is not restricted to the semi-discrete case. It holds for arbitrary, closed tensor product approximation spaces $\mathbb{W}\otimes\mathbb{Q}$, where $\mathbb{W} \subset H^1_0(\Omega)$ and $\mathbb{Q} \subset H^1_{\{\mathcal{Y}\}}(y^\alpha,(0,\mathcal{Y}))$. It completely decouples the solution of the full Galerkin problem, based on $\mathbb{W}\otimes\mathbb{Q}$, into the (parallel) solution of $\dim\mathbb{Q}$ problems of size $\dim\mathbb{W}$. The numerical experiments in Section 8 exploit this observation; see Remark 8.1 below.
- The observation (6.5) allows one to gauge the impact of solving approximately the $\dim\mathbb{Q}$ problems that are of (singularly perturbed) reaction–diffusion type. For convex domains Ω and spaces $\mathbb{W}$ based on piecewise linears on quasi-uniform meshes, robust, (with respect to the singular perturbation parameter), multigrid methods are available (see, e.g., [44]).
- The diagonalization technique (6.2)–(6.4) also suggests another numerical technique: approximate each solution U_i from a different (closed) space $W_i \subset H^1_0(\Omega)$. This leads to the approximation of $\mathscr{U}$ in the space $\sum_{i=1}^{\mathcal{M}} v^i(y)W_i$. The resulting Galerkin approximation still satisfies (6.7) and (6.8). This approach produces approximation spaces in $\Omega\times(0,\mathcal{Y})$ that do not have tensor product structure but still provides exponential convergence. As in the sparse grids case of Section 5.4.4 this approach allows for reducing the number of degrees of freedom without sacrificing much accuracy; specifically, the exponent $1/4$ in the exponential convergence bound (7.8) that we obtain in the next section could be reduced to $1/3$ if Ω is an interval and the exponent $1/5$ in (7.13) could be reduced to $1/4$ if $\Omega \subset \mathbb{R}^2$ has an analytic boundary, albeit at the expense of breaking the tensor product structure of the discretization.

7. *hp*-FE discretization in Ω. Up to this point, we have exploited the analytic regularity of the solution $\mathscr{U}$ in the extended variable y in order to recover (up to logarithmic terms) optimal complexity of a P_1-FEM, for (1.1) posed in the polygon $\Omega \subset \mathbb{R}^2$, by full tensorization of a *hp*-FEM with respect to y with the P_1-FEM in Ω

As a final goal, in this section we employ, *in addition*, an *hp*-FEM in Ω to obtain an *exponentially convergent, local FEM* for the fractional diffusion problem (1.1). Naturally, stronger regularity assumptions on the data f, A and c will be required: in addition to the previously made assumptions on these data, we assume in Section 7.1

$$c, f \in \mathcal{A}(\overline{\Omega},\mathbb{R})\ , \quad A \in \mathcal{A}(\overline{\Omega},\mathsf{GL}(\mathbb{R}^d))\ . \tag{7.1}$$

Here, $\mathcal{A}(\overline{\Omega},G)$ denotes the set of functions which are analytic in $\overline{\Omega}$ and take values in the group G.

7.1. Tensorized *hp*-FEM in $\Omega\times(0,\mathcal{Y})$. The choice of the meshes $\mathcal{G}^M$ and $\mathcal{T}$ as well as the degree vector $\boldsymbol{r}$ and the polynomial degree q were not specified in Section 6. Mesh design principles for problems as (6.4) are available in the literature. For meshes, in an h-version context, we mention the so–called Shishkin meshes and refer to [46] for an in-depth discussion of numerical methods for singular perturbation problems. Here, we focus on the *hp*-version. Appropriate mesh design principles ensuring robust exponential convergence of *hp*-FEM have been developed in [52, 53, 34, 36, 35]. In these references, linear second order elliptic singular perturbations with a single length scale and exponential boundary layers were considered. As is revealed by the diagonalization (6.4), the y-semidiscrete solution (6.1) contains $\mathcal{M}$

separate length scales μ_i, $i = 1, ..., \mathcal{M}$. These need to be resolved *simultaneously* by the x'-discretization space. To this end, based on [52, 53, 34, 36, 35], we employ a mesh that is geometrically refined towards $\partial\Omega$ such that the smallest length scale $\mu_{\mathcal{M}}$ is resolved. We illustrate the key points in the following Sections 7.1.1 and 7.1.2 in dimension $d = 1$, and in dimension $d = 2$ for smooth boundaries.

7.1.1. Exponential convergence of *hp*-FEM in one dimension. To gain insight into how to discretize the family of problems (6.4), we first consider the following reaction-diffusion problem in $\Omega = (0, 2)$: given $f \in \mathcal{A}(\overline{\Omega}; \mathbb{R})$ and a parameter $0 < \varepsilon \leq 1$, find $u_\varepsilon \in H_0^1(\Omega)$ such that

$$-\varepsilon^2 u_\varepsilon'' + u_\varepsilon = f \quad \text{on } \Omega, \qquad u_\varepsilon(0) = u_\varepsilon(2) = 0 \,. \tag{7.2}$$

For (7.2), *hp*-Galerkin FEM afford *robust exponential convergence*. The following result is a particular instance of [34, Proposition 20].

Proposition 7.1 (exponential convergence). *Let $\Omega = (0, 2)$. Let $\mathcal{T}^{1D,L}_{geo,\sigma}$ be a mesh on Ω that is geometrically refined towards $\partial\Omega = \{0, 2\}$ with L layers and grading factor $\sigma \in (0, 1)$:*

$$\mathcal{T}^{1D,L}_{geo,\sigma} := \{(0, \sigma^L), (2 - \sigma^L, 2)\} \cup \{(\sigma^{L-i+1}, \sigma^{L-i}), (2 - \sigma^{L-i}, 2 - \sigma^{L-i+1})\}_{i=1}^{L}. \tag{7.3}$$

Select L such that $\sigma^L \leq \varepsilon \leq 1$. Let f satisfy the analytic regularity estimates

$$\|f^{(\ell)}\|_{L^2(\Omega)} \leq C_f K_f^\ell \ell! \qquad \forall \ell \in \mathbb{N}_0, \tag{7.4}$$

for some constants C_f, $K_f > 0$ that depend on f. Then there exist constants C, $b > 0$ independent of $\varepsilon \in (0, 1]$ such that for the Galerkin approximation $u_\varepsilon^{q,L} \in S_0^q(\Omega, \mathcal{T}^{1D,L}_{geo,\sigma})$ of the solution u_ε of (7.2) one has exponential convergence in the energy norm, given by $\|w\|^2_{\varepsilon^2,\Omega} := \varepsilon^2 \|w'\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}$, i.e.

$$\|u_\varepsilon - u_\varepsilon^{q,L}\|_{\varepsilon^2,\Omega} \lesssim C_f e^{-bq}.$$

Here the hidden constant and the constant b are independent of ε, but depend on σ and K_f. Furthermore, $L = \mathcal{O}(1 + |\log \varepsilon|)$ so that $\dim S_0^q(\Omega, \mathcal{T}^{1D,L}_{geo,\sigma}) = \mathcal{O}(q^2(1 + |\log \varepsilon|))$.

Remark 7.2 (exponential convergence). *The discretization described in Proposition 7.1 and its properties warrant the following comments.*

- *The case $\epsilon \geq 1$: Although Proposition 7.1 restricts to $\varepsilon \in (0, 1]$, one can check that for $\varepsilon \geq 1$, the mesh degenerates into a fixed mesh with three points $\{0, 1, 2\}$ and the corresponding approximation result reads*

$$\|u_\varepsilon - u_\varepsilon^{q,L}\|_{\varepsilon^2,\Omega} \lesssim (1 + \varepsilon) C_f e^{-bq} \,. \tag{7.5}$$

- *Different length scales: Proposition 7.1 gives* robust exponential convergence *and* does not require explicit knowledge of the singular perturbation parameter ε, *but only a lower bound for it. This is crucial for the presently considered fractional diffusion problem, where the decoupled problems* (6.4) *depend on several length scales given by λ_i (which, in turn, depend on the discretization in the extended variable $y \in (0, \mathcal{Y})$). Applying a tensor product hp-FE space directly (i.e., without explicit diagonalization* (6.1)–(6.4)*) to the extended problem* (1.2) *based on the tensor product of the hp-FE space $S_0^q(\Omega, \mathcal{T}^{1D,L}_{geo,\sigma})$ and on the hp-FE space $S^r_{\{\mathcal{Y}\}}((0, \mathcal{Y}), \mathcal{G}^M_\sigma)$ obviates the numerical solution of the generalized eigenproblem* (6.2)*. It requires, however, the hp-space $S_0^q(\Omega, \mathcal{T}^{1D,L}_{geo,\sigma})$ to* concurrently *approximate the solutions of* all singularly perturbed problems (6.4) *in Ω with exponential convergence rates.*

- *Different meshes: If an eigenbasis* $(v_i)_{i=1}^{\mathcal{M}}$ *satisfying* (6.3) *is available, then for each of the decoupled singularly perturbed problems in* Ω*, a geometric boundary layer mesh is not mandatory to achieve robust exponential convergence. A coarser mesh, tailored to the specific length scale* μ_i *in the* i*-th equation of* (6.4)*, will then suffice; we refer to [52, 51] for details.* ∎

Lemma 6.3 asserts that the reaction-diffusion problems (6.4) are singularly perturbed with length scale μ_i ranging from $\mathcal{O}(M^{-1}\sigma^M)$ to $\mathcal{O}(M^2)$. Proposition 7.1 implies exponential convergence rates under the analyticity assumption (7.1). In the next result, we combine these two observations to obtain an exponentially convergent hp-FEM for the fractional diffusion problem in Ω.

Theorem 7.3 (exponential convergence). *Let* $u \in \mathbb{H}^s(\Omega)$ *and* $\mathscr{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$ *solve* (1.1) *and* (1.2)*, respectively, with* $\Omega = (0,2)$*,* $A = I$*,* $c = 0$ *and* f *satisfying* (7.1)*. Given fixed constants* c_1*,* $c_2 > 0$*, let* $\mathcal{G}^M_{geo,\sigma}$ *be a geometric mesh on* $[0, \mathcal{Y}]$ *with grading factor* $\sigma \in (0,1)$ *and such that* $c_1 M \le \mathcal{Y} \le c_2 M$*. Let* $\boldsymbol{r}$*, on* $\mathcal{G}^M_{geo,\sigma}$*, be the linear degree vector with slope* $\mathfrak{s}$*. Let* $\mathcal{T}^{1D,L}_{geo,\sigma}$ *be a geometric mesh in* Ω *as described in Proposition 7.1 with an integer* L *such that*

$$\sigma^{2L} \le \mathcal{Y}(\mathfrak{s}M)^{-2}\sigma^M \, . \tag{7.6}$$

Then, there are constants b*,* $\mathfrak{s}_{min} > 0$ *independent of* M *and* $\mathcal{Y}$ *such that for* $\mathfrak{s} \ge \mathfrak{s}_{min}$ *the Galerkin approximation* $\mathscr{U}_{q,\boldsymbol{r}} \in S^q_0(\Omega, \mathcal{T}^{1D,L}_{geo,\sigma}) \otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M_{geo,\sigma})$ *to* $\mathscr{U}$ *satisfies*

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{q,\boldsymbol{r}}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{q,\boldsymbol{r}})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim \left(M^2 e^{-bq} + e^{-bM}\right), \tag{7.7}$$

where the hidden constant is independent of M *and* $\mathcal{Y}$*. In addition, as* $M \to \infty$*, with* L *and* M *related by* (7.6)*, we have that, uniformly in* $q \in \mathbb{N}$*, the total number of degrees of freedom behaves like*

$$\mathcal{N}_{\Omega,\mathcal{Y}} := \dim S^q_0(\Omega, \mathcal{T}^{1D,L}_{geo,\sigma}) \otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M_{\sigma}) = \mathcal{O}(qM^3).$$

Choosing, in particular, $q \sim M$ *yields a convergence rate bound in terms of the total number of degrees of freedom* $\mathcal{N}_{\Omega,\mathcal{Y}}$ *of the form*

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{q,\boldsymbol{r}}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{q,\boldsymbol{r}})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim \exp(-b'\mathcal{N}^{1/4}_{\Omega,\mathcal{Y}}) \tag{7.8}$$

for some $b' > 0$ *independent of* $\mathcal{N}_{\Omega,\mathcal{Y}}$*.*

Proof. Let $\mathscr{U}_M$ solve (6.1). We proceed in two steps.

Bounds on the semidiscretization error $\mathscr{U} - \mathscr{U}_M$*:* By the assumption of analyticity of f, there exist constants C_f, K_f such that (7.4) holds. We thus have that $f \in \mathbb{H}^{1/2-\delta}(\Omega)$ for any $\delta > 0$. Consequently, an application of Lemma 6.2. reveals that for a sufficiently large slope $\mathfrak{s}$ of the linear degree vector $\boldsymbol{r}$ (depending on the constants K_f in the analytic regularity bound (7.4) of the data f) there exists $b > 0$ such that

$$\|\nabla(\mathscr{U} - \mathscr{U}_M)\|_{L^2(y^\alpha,\mathcal{C})} \lesssim e^{-bM}.$$

Bounds on the errors $\|U_i - \Pi_i U_i\|_{\mu_i,\Omega}$*:* We first notice that Lemma 6.3 immediately yields $\mathfrak{s}^{-2}M^{-1}\sigma^M \lesssim \mu_i$. This, in view of the assumption (7.6), implies that $\sigma^{2L} \lesssim \mu_i$. Consequently, given that f is analytic on $\overline{\Omega}$, we apply Proposition 7.1 (more precisely, the refinement (7.5) to obtain that

$$\|U_i - \Pi_i U_i\|_{\mu_i,\Omega} \lesssim \mathcal{Y} e^{-bq} \lesssim M e^{-bq}, \tag{7.9}$$

Fig. 7.1. *Anisotropic geometric mesh (see Definition 7.6). Left: geometric refinement of the reference patch. Right: Example of mesh with $N = 5$ and $n = 4$. Solid lines indicate patches, dashed lines represent mesh lines introduced by refinement of reference patches.*

where we have also used that $\mu_i \lesssim M^2 \lesssim \mathscr{Y}^2$, which follows, again, from Lemma 6.3 and the condition $c_1 M \leq \mathscr{Y} \leq c_2 M$. We recall that $\|\cdot\|_{\mu_i,\Omega}$ is defined as in (6.5). Finally, combining (7.9) with (6.8) and recalling that $\mathcal{M} \lesssim M^2$ give

$$\|\mathscr{U}_M - \mathscr{U}_{h,M}\|^2_{L^2(y^\alpha,\mathcal{C})} \lesssim \mathcal{M} M^2 e^{-2bq} \lesssim M^4 e^{-bq}.$$

This concludes the proof. ☐

Remark 7.4 (other operators). *Theorem 7.3 also holds for $0 < c \in \mathbb{R}$ by arguing as in the proof of Theorem 7.7 ahead.* ■

Remark 7.5 (mesh gradings Ω). *The condition* (7.6) *is a sufficient condition ensuring that the smallest boundary layer length scale (characterized by $\min_i \mu_i$) that arises from the diagonalization is resolved by the mesh $\mathcal{T}^{1D,L}_{geo,\sigma}$. More generally, if the geometric mesh of* (7.3) *were based on the mesh grading factor $\sigma_{x'} \in (0,1)$ (distinct from the factor σ in the mesh in the extended variable y), then condition* (7.6) *could be replaced with $\sigma^{2L}_{x'} \lesssim \mathscr{Y}(\mathfrak{s}M)^{-2}\sigma^M$ for some constant independent of $L, M, \mathscr{Y}$.* ■

7.1.2. Exponential convergence of *hp*-FEM in two dimensions. Let us now discuss the extension of the ideas of Section 7.1.1 to the two dimensional case. As it is structurally similar to the univariate case, we proceed briefly. For domains $\Omega \subset \mathbb{R}^d$, $d > 1$, with *smooth* boundary, the boundary layers presented in the solutions U_i of the singularly perturbed problems (6.4) can be resolved by meshes that are *anisotropically* refined towards the boundary $\partial\Omega$. A two dimensional analogue of the meshes $\mathcal{T}^{1D}_{geo,L}$ of Proposition 7.1 is presented in [36, Section 3.4.3] and illustrated in Figure 7.1 (right). These *anisotropic geometric meshes* $\mathcal{T}^{2D,L}_{geo,\sigma}$ are created as push-forwards of anistropically refined geometric meshes on references patches as detailed in the following definition, where we follow the notation employed in [36, Section 3.4.3].

Definition 7.6 (anisotropic geometric meshes $\mathcal{T}^{2D,L}_{geo,\sigma}$). *Denote by $S = [0,1]^2$ the reference element. Let Ω_i, $i = 1, \ldots, N$, be a fixed mesh on $\Omega \subset \mathbb{R}^2$ consisting of curvilinear quadrilaterals with bijective element maps $M_i : S \to \Omega_i$ satisfying the "usual" conditions for H^1-conforming triangulations (see [36, (M1)–(M3) in Section 3.1] for the precise definition). The elements Ω_i are called* patches *and the associated maps M_i* patch maps*. Let Ω_i, $i = 1, \ldots, n \leq N$, be such that the left edge $e := \{0\} \times (0,1)$ of S is mapped to $\partial\Omega$, i.e., $M_i(e_1) \subset \partial\Omega$, and that $M_i(\partial S \setminus e) \cap \partial\Omega = \emptyset$. Assume that the remaining elements Ω_i, $i = n+1, \ldots, N$ satisfy $\overline{\Omega}_i \cap \partial\Omega = \emptyset$.*

Subdivide the reference element S into $L+1$ rectangles S^ℓ, $\ell = 0, \ldots, L$, as follows *for chosen grading factor* $\sigma \in (0,1)$*:*

$$S^0 = (0,\sigma^L)\times(0,1), \qquad S^\ell = (\sigma^{L+1-\ell},\sigma^{L-\ell})\times(0,1), \qquad \ell = 1,\ldots,L. \tag{7.10}$$

Define elements Ω_i^ℓ, $i=1,\ldots,n$, $\ell = 0,\ldots,L$, *and the corresponding element maps* $M_i^\ell : S \to \Omega_i^\ell$ *by*

$$\begin{aligned}
\Omega_i^0 &:= M_i(S^0), \qquad & M_i^0(\xi,\eta) &:= M_i(\xi\sigma^L,\eta),\\
\Omega_i^\ell &:= M_i(S^\ell), \qquad & M_i^\ell(\xi,\eta) &:= M_i(\sigma^{L+1-\ell}+\xi\sigma^{L-\ell},\eta), \qquad \ell = 1,\ldots,L.
\end{aligned}$$

The mesh $\mathcal{T}^{2D,L}_{geo,\sigma}$ *given by the elements* $\{\Omega_i^\ell : i=1,\ldots,n, \ell=0,\ldots,L\}\cup\{\Omega_j : j = n+1,\ldots,N\}$ *with corresponding element maps introduced above is a triangulation of* Ω *that satisfies the "usual" conditions of* H^1*-conforming triangulations, i.e., conditions [36, (M1)–(M3) in Section 3.1]. For* $\mathcal{T}^{2D,L}_{geo,\sigma}$ *the FE-space is given by the standard* $H^1_0(\Omega)$*-conforming space of mapped polynomials of degree* q*:*

$$S_0^q(\mathcal{T}^{2D,L}_{geo,\sigma}) := \{u \in H^1_0(\Omega) : u|_K \circ F_K \in \mathbb{Q}_q(S) \quad \forall K \in \mathcal{T}^{2D,L}_{geo,\sigma}\}, \tag{7.11}$$

where $F_K : S \to K$ *is the element map of* $K \in \mathcal{T}^{2D,L}_{geo,\sigma}$ *and* $\mathbb{Q}_q(S)$ *is the space of polynomials of degree* q *in each variable on* S*.*

For such anisotropically refined meshes, we have the following exponential convergence result.

Theorem 7.7 (exponential convergence). *Let* $u \in \mathbb{H}^s(\Omega)$ *and* $\mathscr{U} \in \mathring{H}^1(y^\alpha,\mathcal{C})$ *solve* (1.1) *and* (1.2)*, respectively, with* $\Omega \subset \mathbb{R}^2$ *having an analytic boundary,* $A = I$*,* $0 \le c \in \mathbb{R}$*, and* f *satisfying the regularity requirement* (7.1) (7.1)*. Given fixed constants* c_1*,* $c_2 > 0$*, let* $\mathcal{G}^M_{geo,\sigma}$ *be a geometric mesh on* $[0,\mathcal{Y}]$ *with grading factor* $\sigma \in (0,1)$ *and such that* $c_1 M \le \mathcal{Y} \le c_2 M$*. Let* $\boldsymbol{r}$*, on* $\mathcal{G}^M_{geo,\sigma}$*, be the linear degree vector with slope* $\mathfrak{s}$*. Assume that* L *is chosen such that* (7.6) *holds. Let* $\mathcal{T}^{2D,L}_{geo,\sigma}$ *be an anisotropic geometric mesh with* L *layers as described in Definition 7.6 where, additionally, the patch maps* M_i*,* $i=1,\ldots,N$ *are assumed to be analytic. Then, there are constants* C*,* b*,* $\mathfrak{s}_{min} > 0$ *independent of* M *and* $\mathcal{Y}$ *such that for* $\mathfrak{s} \ge \mathfrak{s}_{min}$ *the Galerkin approximation* $\mathscr{U}_{q,\boldsymbol{r}} \in S^q_0(\Omega,\mathcal{T}^{2D,L}_{geo,\sigma})\otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M_{geo,\sigma})$ *to* $\mathscr{U}$ *satisfies*

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{q,\boldsymbol{r}}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U}-\mathscr{U}_{q,\boldsymbol{r}})\|_{L^2(y^\alpha,\mathcal{C})} \le C\left(M^2 e^{-bq} + e^{-bM}\right). \tag{7.12}$$

Furthermore, as $M \to \infty$*, with* L *related to* M *by* (7.6)*, we have that, uniformly in* $q \in \mathbb{N}$*, the total number of degrees of freedom behaves like*

$$\mathcal{N}_{\Omega,\mathcal{Y}} := \dim S^q_0(\Omega,\mathcal{T}^{2D,L}_{geo,\sigma})\otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M_{\sigma}) = \mathcal{O}(q^2M^3).$$

Choosing, in particular, $q \sim M$ *yields a convergence rate bound in terms of the total number of degrees of freedom* $\mathcal{N}_{\Omega,\mathcal{Y}}$ *of the form*

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{q,\boldsymbol{r}}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U}-\mathscr{U}_{q,\boldsymbol{r}})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim \exp(-b'\mathcal{N}^{1/5}_{\Omega,\mathcal{Y}}) \tag{7.13}$$

for some $b' > 0$ *independent of* $\mathcal{N}_{\Omega,\mathcal{Y}}$*.*

Proof. The proof parallels that of Theorem 7.3. We start with the case $c = 0$. By the arguments in [36, Section 3.4.3] the meshes $\mathcal{T}^{2D,L}_{geo,\sigma}$ allow for estimates of the form

$$\inf_{v \in S^q_0(\Omega,\mathcal{T}^{2D,L}_{geo,\sigma})} \|U_i - v\|_{\mu_i,\Omega} \lesssim e^{-bq} \tag{7.14}$$

for the solutions U_i of (6.4), provided L and μ_i satisfy $\sigma^{2L} \lesssim \mu_i$, which is ensured by assumption (7.6). Here, the implied constant and $b > 0$ depend on f, $\partial\Omega$, and the analyticity of the patch maps M_i, $i = 1, \ldots, N$. The estimates (7.14) then allow us to conclude the proof for $c = 0$ as in Theorem 7.3.

For $c \neq 0$, we observe that the singularly perturbed problems (6.4) in Ω take the form

$$-\mu_i \Delta U_i + (1 + c\mu_i)U_i = f \quad \text{on } \Omega, \qquad U_i|_{\partial\Omega} = 0.$$

This can be transformed to the case $c = 0$ by rewriting it in terms of $\widetilde{\mu}_i := \mu_i/(1+c\mu_i)$ as

$$-\widetilde{\mu}_i \Delta U_i + U_i = \widetilde{f} := \frac{1}{1 + c\mu_i} f \quad \text{on } \Omega, \qquad U_i|_{\partial\Omega} = 0.$$

The approximation result (7.14) holds again (with μ_i replaced with $\widetilde{\mu}_i$ there). □

Remark 7.8 (limitations and extensions). *The result of Theorem 7.7 warrants the following remarks:*

(i) Theorem 7.7 is restricted to $A = I$ and to the coefficient c being constant, as it relies on [36], which in turn builds on the regularity theory developed in [37]. The results of [36] can be generalized to A and c that satisfy (7.1) *using the results from [35]. In turn, Theorem 7.7 could be generalized to this setting as well.*

(ii) Theorem 7.7 can be expected to generalize to $\Omega \subset \mathbb{R}^d$ with $d > 2$ if $\partial\Omega$ is analytic. The underlying reason for this is that the boundary layers are structurally a one dimensional phenomenon, which can be resolved with anisotropic refinement towards $\partial\Omega$. The approximation result (7.12) *can therefore be expected to hold, however, the complexity is then $\mathcal{N}_{\Omega,\mathcal{Y}} = \mathcal{O}(qM^{d+2})$, resulting in an exponential convergence bound of* $\exp(-b'\mathcal{N}_{\Omega,\mathcal{Y}}^{1/(d+3)})$.

(iii) Theorem 7.7 does generalize to so-called "bounded, curvilinear polygonal domains" $\Omega \subset \mathbb{R}^2$. The analogue of Proposition 7.1, i.e., a rigorous convergence analysis of hp-FEM in Ω for the single-scale reaction diffusion problem with the appropriate mesh refinement towards the corners of Ω is available in [35]. ■

8. Numerical experiments. We consider $A = I$ and $c = 0$, i.e., $\mathcal{L}^s = (-\Delta)^s$. Most of the numerical experiments will be performed on the so-called L-shaped polygonal domain $\Omega \subset \mathbb{R}^2$ determined by the vertices

$$\boldsymbol{c} \in \{(0,0), (1,0), (1,1), (-1,1), (-1,-1), (0,-1)\}.$$

For validation purposes again, we consider the following smooth exact solution with the corresponding right-hand side (recall $x' = (x_1, x_2) \in \Omega$)

$$u(x_1, x_2) = \sin \pi x_1 \sin \pi x_2, \quad f(x_1, x_2) = (2\pi^2)^s \sin \pi x_1 \sin \pi x_2. \tag{8.1}$$

To investigate the effect of mesh refinement in Ω, we also consider

$$f(x_1, x_2) \equiv 1\ . \tag{8.2}$$

Notice that, in this case, $f \in \mathcal{A}(\overline{\Omega}, \mathbb{R})$, but $f \in \mathbb{H}^{1-s}(\Omega)$ only for $s > 1/2$ due to boundary incompatibility. The exact solution is not known, so that the error will be estimated numerically, with reference to an accurate numerical solution. The error measure will always be the energy norm

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{h,M}\|^2_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{h,M})\|^2_{L^2(y^\alpha,\mathcal{C})} = d_s \int_\Omega f(u - \operatorname{tr}_\Omega \mathscr{U}_{h,M}),$$

where $\mathscr{U}_{h,M}$ denotes the discrete solution in $\mathcal{C}_\mathcal{Y}$.

Finally, a one-dimensional example $\Omega = (0,1)$ will be described to illustrate *hp*-FEM in $\Omega \times (0, \mathcal{Y})$.

Remark 8.1 (implementation). *Let us provide some algorithmic details of the methods used in practical computations. For the chosen discrete spaces the mass and stiffness matrices in* Ω *and* $(0,\mathcal{Y})$ *are computed. We then numerically solve the generalized eigenvalue problem* (6.3)*, thereby arriving at* $\mathcal{M}$ *decoupled linear systems: Find* $U_i \in S^q_0(\Omega, \mathcal{T})$ *such that*

$$a_{\mu_i,\Omega}(U_i, V) = d_s v^i(0) \int_\Omega f V \,\mathrm{d}x' \qquad \forall V \in S^q_0(\Omega, \mathcal{T}), \tag{8.3}$$

where $a_{\mu_i,\Omega}$ *is defined in* (6.4)*. Following* (6.7)*, the solution is then obtained by*

$$\mathscr{U}_{h,M}(x', y) = \sum_{i=1}^{\mathcal{M}} v^i(y) U_i(x').$$

The implementation was done in Matlab R2017a, with the generalized eigenvalue problem solved with `eig` *and the decoupled linear systems by a direct solver, i.e., Matlab's "backslash" operator.* ∎

8.1. P_1**-FEM in** Ω **with radical meshes in** $(0,\mathcal{Y})$**.** In the following examples we make use of the family of graded meshes $\mathcal{G}^k_{gr,\eta}$ as described in Section 5.4.2 with particular choices $\eta = 2/s$, $k = h/2$, and $\mathcal{Y} = |\log h|$, where h denotes the mesh width of the mesh in Ω to be described next.

8.1.1. Smooth solution. For the first experiment we investigate the smooth solution (8.1). We use the P_1-FEM in Ω on a hierarchy of uniformly refined meshes $\mathcal{T}^\ell$. The results are displayed in Figure 8.1. As the theory predicts we see linear convergence in the energy norm with respect to the meshwidth h.

8.1.2. Mesh refinement at $(0,0)$**.** In the next experiment we consider the case $f \equiv 1 \in \mathbb{H}^{1-s}(\Omega)$ for $s \in (1/2, 1)$. As above we use the graded mesh $\mathcal{G}^k_{gr,\eta}$ in $(0,\mathcal{Y})$, whereas we now use a hierarchy $\{\mathcal{T}^\ell_\beta\}_{\ell\geq 0}$ of bisection–tree meshes in Ω that are refined towards the re-entrant corner at $(0,0)$ as constructed in [26]. In Figure 8.2 we see linear convergence with respect to the mesh width as predicted by Theorem 5.9 and in contrast to the results obtained with uniformly refined meshes.

To the best of the authors' knowledge, the nature of the geometric singularity of the solution at the re-entrant corner of the L-shaped domain for general $0 < s < 1$ is not known.

8.1.3. Sparse grid P_1**-FEM with mesh refinement at** $(0,0)$**.** With the above described discrete spaces we are able to obtain optimal order convergence with respect to the number of degrees of freedom $\mathcal{N}_\Omega$. Nevertheless, the number of degrees of freedom in the extended problem is of size $\mathcal{O}(\mathcal{N}_\Omega^{1+1/2} \log\log \mathcal{N}_\Omega)$, i.e., it grows superlinearly with respect to $\mathcal{N}_\Omega$. To reduce the complexity to nearly linear, we use sparse grids as explained in Section 5.4.4; see in particular the combination formula

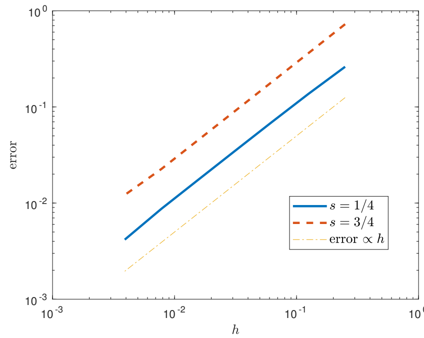

FIG. 8.1. *Convergence of the error in the energy norm versus the meshwidth in* Ω *with the (smooth) exact solution given by* (8.1). *A* P_1*-FEM on uniformly refined meshes in* Ω *and* P_1*-FEM on radical meshes in* $(0, \mathcal{Y})$ *is used.*

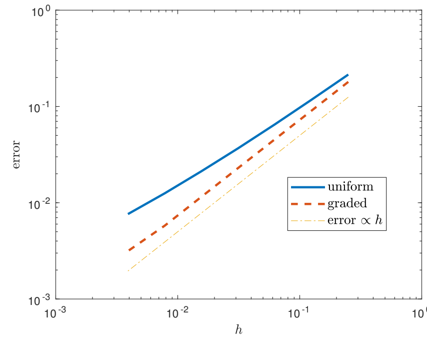

FIG. 8.2. *Convergence of the error in the energy norm versus meshwidth in* Ω *with the right-hand side* $f \equiv 1$ *and* $s = 3/4$*, leading to a solution with singular behavior near the re-entrant corner* $(0, 0)$*. Error graphs are shown for a* P_1*-FEM on uniformly refined meshes in* Ω *and on meshes refined towards the corner.*

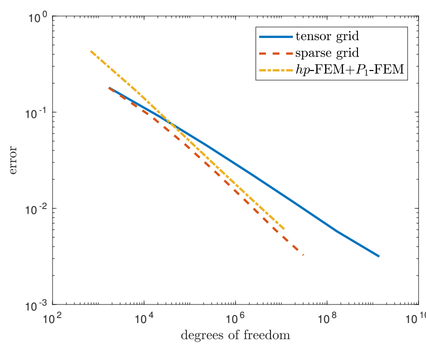

FIG. 8.3. *Convergence of the error in the energy norm versus the number of degrees of freedom of the extended problem with the right-hand side* $f \equiv 1$ *and* $s = 3/4$*.* P_1*-FEM on corner-refined, regular simplicial meshes is used in* Ω*. We compare hp-FEM in* $(0, \mathcal{Y})$ *with tensor grid and sparse grids, the latter two employing radical meshes in* $(0, \mathcal{Y})$*.*

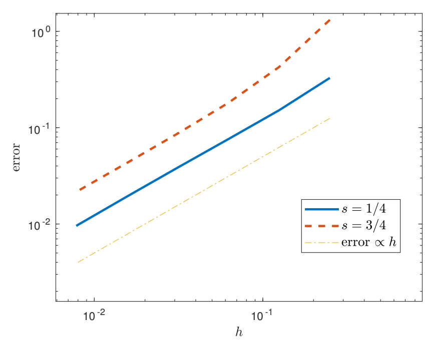

FIG. 8.4. *Convergence of the error in the energy norm versus the meshwidth in* Ω *with the (smooth) exact solution given by* (8.1) *for two different values of* s*. A* P_1*-FEM on uniformly refined meshes in* Ω *and hp-FEM in* $(0, \mathcal{Y})$ *is used.*

described in Remark 5.11. The results are shown in Figure 8.3. These show that the use of sparse grids dramatically reduces the number of degrees of freedom and is comparable to hp-FEM, which is described next.

8.2. P_1**-FEM in** Ω **with** hp**-FEM in** $(0, \mathcal{Y})$**.** We again start with the smooth solution (8.1). P_1-FEM on uniformly refined meshes is used in Ω, whereas in the extended direction y we use hp-discretization on the geometric meshes $\mathcal{G}^M_{geo,\sigma}$ on $[0, \mathcal{Y}]$. We use $\mathcal{Y} = \frac{1}{3}|\log_2 h|$, $M = |\log_2(h/2)|$, $\sigma = 0.05$, and linear degree vector $\boldsymbol{r}$ with slope $\mathfrak{s} = 2$. Linear convergence, as predicted by theory, can be seen in Figure 8.4. We also consider the right-hand side $f \equiv 1$ for $s = 0.75$. This time we show convergence versus the number of degrees of freedom $\mathcal{N}_{\Omega,\mathcal{Y}}$ in the extended problem and compare

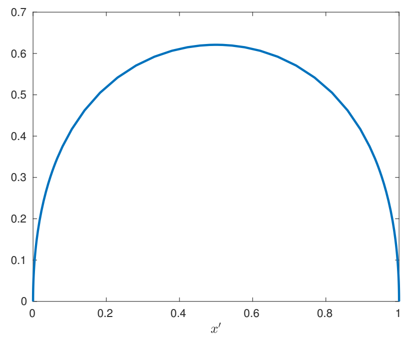

FIG. 8.5. *Solution on $\Omega = (0,1)$ with algebraic boundary singularity for $s = 0.25$ and $f \equiv 1$.*

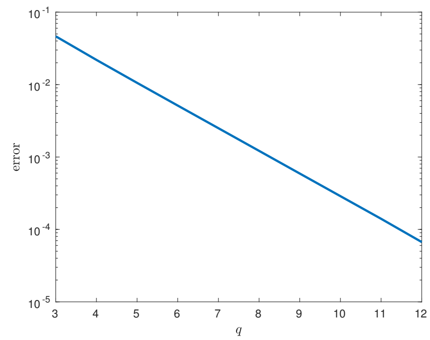

FIG. 8.6. *Convergence of error in energy norm of the hp-FEM on $\Omega \times (0, \mathcal{Y})$ against polynomial order q for $s = 0.25$ and $f \equiv 1$.*

with P_1-FEM in Ω on so-called *radical meshes*. We obtain nearly optimal complexity as predicted by theory, but interestingly in this example slightly worse behavior compared with sparse grids. This is reported in Figure 8.3.

8.3. ***hp*-FEM in $(0,1) \times (0, \mathcal{Y})$.** We consider an example in one space dimension where $\Omega = (0,1)$, with smooth, but incompatible right-hand side $f \equiv 1$. We comment that, according to the regularity results presented in [16], the solution behaves like

$$u(x') \sim \begin{cases} \mathrm{dist}(x', \partial\Omega) + v(x') & \text{for} \quad s > 1/2\,, \\ \mathrm{dist}(x', \partial\Omega)^{2s} + v(x') & \text{for} \quad 0 < s < 1/2\,, \end{cases} \tag{8.4}$$

with v denoting a smoother remainder. Here, the singular support of u is $\partial\Omega$, i.e. u exhibits an *algebraic boundary singularity* (distinct from the smooth exponential boundary layers arising in linear, elliptic-elliptic singular perturbations) near the boundary of Ω; see Figure 8.5.

Again, as the exact solution is not known, we compare the numerical solution with an accurate solution obtained on a finer grid.

In $(0, \mathcal{Y})$, we use the same geometric *hp*-FEM space $\mathcal{G}^M_{geo,\sigma}$ as in the previous section. The *hp*-FEM space $S^q_0(\Omega, \mathcal{T}_L)$ is as described in Section 7.1, where $q = M$ and $L = M$. Exponential convergence with respect to the polynomial degree q as predicted by the theory is shown in Figure 8.6.

In Figure 8.7 we illustrate the behavior of the solution given by (8.4). We also investigate numerically the borderline case $s = 1/2$ in Figure 8.8. Even if the domain Ω is smooth, u exhibits in general a *boundary singularity* with singular support $\partial\Omega$. For $s = 1/2$ and polygonal Ω, this boundary singularity is the trace, at $y = 0$, of an *edge singularity* of the solution $\mathscr{U}$ of the extended problem (1.2) in $\mathcal{C}$ whose structure is known; see, for instance, [20] and the references therein. Here, *hp*-FE approximations with geometric boundary layer meshes in Ω naturally appear as $y = 0$ slices of $d+1$-dimensional geometric meshes in $\mathcal{C}_{\mathcal{Y}}$ as developed in [50].

9. Conclusions and generalizations. In the course of this work, we introduced and analyzed four different types of *local* FEM discretizations for the numerical approximation of the spectral fractional diffusion problem (1.1) in a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with straight sides (or a bounded interval $\Omega \subset \mathbb{R}$), subject to homogeneous Dirichlet boundary conditions. Our local FEM schemes are based on

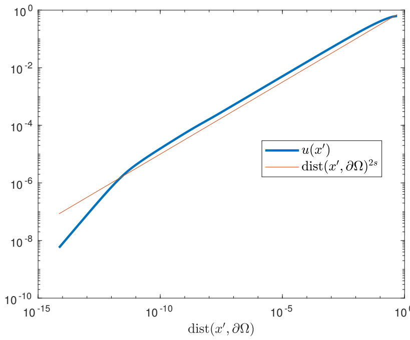

Fig. 8.7. *Numerical verification of the algebraic boundary singularity* (8.4) *for* $x' \in (0, 1/2)$ *and* $s = 1/4$. *Note that the change in the slope (from* $1/2$ *to* 1*) near the boundary is a numerical artifact – as the approximation is improved, the kink moves to the left.*

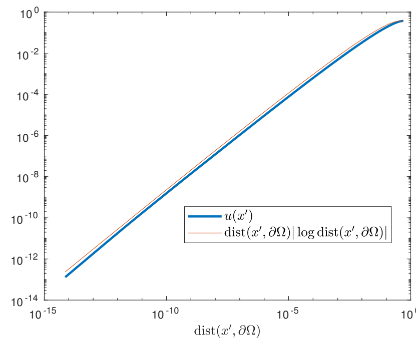

Fig. 8.8. *Boundary behavior for* $s = 1/2$. *Here the numerical solution is compared with* $\mathrm{dist}(x', \partial\Omega)|\log \mathrm{dist}(x', \partial\Omega)|$.

the Caffarelli-Silvestre extension of (1.1) from Ω to $\mathcal{C}$. Our main contributions are the following.

- **General operators and nonconvex domains.** We proposed a tensor product argument for continuous, piecewise linear FEM in both $(0, \infty)$, and in Ω with proper mesh refinement towards $y = 0$ and the corners $\boldsymbol{c}$ of Ω. Assuming that A and c are as in Proposition 5.3, we showed that the approximate solution to problem (1.1) exhibits a near optimal asymptotic convergence rate $\mathcal{O}(h_\Omega |\log h_\Omega|)$ subject to the optimal regularity $f \in \mathbb{H}^{1-s}(\Omega)$. However, if $\mathcal{N}_\Omega$ denotes the number of degrees of freedom in the discretization in Ω, then the total number of degrees of freedom grows asymptotically as $\mathcal{O}(\mathcal{N}_\Omega^{3/2})$ (ignoring logarithmic factors).
  This result is analogous to the bounds obtained in [41] for convex domains Ω, thus generalizing these results to nonconvex, polygonal domains $\Omega \subset \mathbb{R}^2$. The error analysis proceeded by a suitable form of quasi-optimality in Lemma 5.1 and the construction of a tensor product FEM interpolant in the truncated cylinder $\mathcal{C}_{\mathcal{Y}}$. This interpolant was constructed from a nodal, continuous and piecewise linear interpolant $\pi_\eta^{1,\ell}$ with respect to the extended variable $y \in (0, \mathcal{Y})$ on a radical-geometric mesh, and from an $L^2(\Omega)$ projection Π_β^ℓ in Ω onto the space of continuous, piecewise linears on a suitable sequence $\{\mathcal{T}_\beta^\ell\}_{\ell \geq 0}$ of regular nested, bisection-tree, simplicial meshes with refinement towards the corners $\boldsymbol{c}$ of Ω. A novel result from [25] implies that Π_β^ℓ is also uniformly $H^1(\Omega)$-stable with respect to the refinement level ℓ. The present construction would likewise work with any other concurrently $L^2(\Omega)$ and $H^1(\Omega)$ stable family of quasi-interpolation operators, e.g. those of [54].
- **Sparse tensor grids.** While the regularity requirement $f \in \mathbb{H}^{1-s}(\Omega)$ is, essentially, minimal for first order convergence in Ω, the complexity $\mathcal{O}(\mathcal{N}_\Omega^{3/2})$ due to the extra degrees of freedom in the extended variable results in superlinear work with respect to $\mathcal{N}_\Omega$. We therefore proposed in Section 5.4.4 a *novel, sparse tensor product FE discretization* of the truncated, extended problem. Using novel regularity results for the extended solution in $\mathcal{C}$ in weighted spaces and sparse tensor product constructions of the interpolation operators $\pi_\eta^{1,\ell}$ and Π_β^ℓ in Ω, we proved that this approach still delivers FEM solutions of (1.1) with essentially first order

convergence rates (i.e., up to logarithmic factors), under the slightly more stringent regularity $f \in \mathbb{H}^{1-s+\nu}(\Omega)$, $\nu > 0$, while requiring essentially only $\mathcal{O}(\mathcal{N}_\Omega)$ many degrees of freedom.

- ***hp*-FE approximation in the extended variable.** The solution of the extended problem being analytic with respect to the extended variable $y > 0$ allows for designing hp-FE approximations with respect to the variable y on geometric meshes and proving *exponential convergence rates* even under finite regularity of A, c and f as specified in Proposition 5.3. The proof is based on a novel framework of countably normed, weighted Bochner spaces in $(0, \infty)$ to quantify the analytic regularity with respect to y. We also developed a corresponding family of hp-interpolation operators *that affords exponential convergence rates* in the extended variable.
  Upon tensorization with the projectors Π^ℓ_β onto spaces of continuous, piecewise linear finite elements on simplicial, bisection-tree meshes with corner refinement in Ω, we obtained a class of FE schemes that afford essentially optimal, linear convergence rate in Ω under the regularity $f \in \mathbb{H}^{1-s}(\Omega)$, also for nonconstant coefficients and nonconvex polygonal domains Ω, thereby generalizing [33]. We remark that the convergence rate bounds essentially equal the results of so-called wavelet Galerkin discretizations for the integral fractional Laplacian (see [48, 47] and the references therein). Wavelet Galerkin methods are based on direct, "nonlocal" Galerkin discretization of integro-differential operators, which entail numerical evaluation of singular integrals and dense stiffness matrices, neither of which occurs in the present local FE approach. However, these methods can also cope with variable exponent $s(x')$, which seems to be beyond reach with the present approach; see [49, 19] and the references therein. We also point out that the *boundary compatibility of f*, which is implicit in the assumption $f \in \mathbb{H}^{1-s}(\Omega)$, is essential in the arguments in Section 5 as well as in the results of [41, 33, 11].
- **Diagonalization.** We developed a novel diagonalization approach which allows us to decouple the second order elliptic system in $\mathcal{C}_\mathcal{Y}$, resulting from *any* Galerkin semidiscretization in the extended variable y (either of h-FEM or of hp-FEM type) of the truncated problem, into a finite number of decoupled, singularly perturbed, second order elliptic problems in Ω. This approach is instrumental for both the design of hp-FEMs in Ω in Section 7 as well as the implementation of parallel and inexact solvers in Section 8.
- ***hp*-FEMs.** Exploiting results on *robust exponential convergence of hp-FEMs* for second order, singularly perturbed problems [37, 36, 34, 35], and tensorization with the exponentially convergent hp-FEM in $(0, \mathcal{Y})$ resulted in *exponential convergence for analytic input data A, c, f, and Ω* for *incompatible* forcing f (i.e. $f \in H^{1-s}(\Omega)$ but $f \notin \mathbb{H}^{1-s}(\Omega)$). The boundary incompatibility of f leads to the formation of a strong boundary singularity for $0 < s \leq 1/2$ and a weaker one for $s > 1/2$ with $\partial\Omega$ analytic, which is a genuine fractional diffusion effect. Our analysis in Section 7.1.2 revealed that for incompatible data f in space dimension $d > 1$, *anisotropic, geometric meshes in* Ω capable of resolving boundary layers over a wide range of length scales, are generally indispensable, even if $\partial\Omega$ is smooth. Section 8 displays an example.

The following generalizations of the results of the present work suggest themselves.

- **Boundary conditions.** The present analysis was limited to polygonal domains in two space dimensions and to homogeneous Dirichlet boundary conditions. The extension (1.2) is also available for homogeneous Neumann boundary conditions in [16, Section 7] and for combinations of Dirichlet and Neumann boundary conditions on parts of $\partial\Omega$. Solutions $\mathscr{U}$ of these extensions also admit the representation (4.1), so that the analytic regularity results in Section 4 extend almost verbatim. Likewise, all regularity results in Section 5, being based on [8], extend verbatim to homogeneous Neumann and Dirichlet-Neumann boundary conditions on polygonal domains.

- **Higher dimensions and elements of degree $q \geq 2$ in Ω.** Analogous results as in Section 5 hold for polyhedral domains $\Omega \subset \mathbb{R}^3$ with plane faces, using corresponding regularity results for the Dirichlet Laplacian in weighted spaces in the polyhedron Ω, combined with corresponding FE projections on anisotropically refined FE meshes (with corner and edge-refinements in Ω), as described in [4].
  Returning to polygons, if we consider piecewise polynomials of degree $q \geq 2$ on families of simplicial meshes which are sufficiently refined towards the vertices $\boldsymbol{c}$ of Ω, we expect algebraic convergence rates higher than for linear elements *provided* the forcing $f \in \mathbb{H}^{q-s}(\Omega)$. This implies, in particular, that f should satisfy besides $f \in H^{q-s}_{loc}(\Omega)$ also certain higher-order boundary compatibility on $\partial\Omega$, a consequence of the eigenfunction expansions used in our regularity analysis.

Appendix A. Proof of Lemma 5.13. We will only show (5.45), (5.46) as the estimates (5.47), (5.48) are proved using similar arguments; see, for instance, the proof of [5, Theorem 8]. We distinguish between the first element I_1, the terminal element I_M, and the remaining ones. We write $h_i = |I_i|$. We simplify the exposition by assuming $X = \mathbb{R}$. It is convenient to define, for each interval I_i, $i = 2, \dots, M$, the quantity C_i by

$$C_i^2 := \sum_{\ell=1}^{\infty} (2K_u)^{-\ell} \frac{1}{\ell!^2} \|u^{(\ell)}\|^2_{L^2(\omega_{\alpha+2\ell-2\beta,\gamma}, I_i)}. \tag{A.1}$$

We observe that, since $u \in \mathcal{B}^1_{\beta,\gamma}(C_u, K_u)$,

$$\sum_{i=2}^{M} C_i^2 \leq 2C_u^2, \tag{A.2}$$

where, we recall that the space $\mathcal{B}^1_{\beta,\gamma}(C_u, K_u)$ corresponds to a class of analytic functions and is defined as in (5.43). We begin the proof with an auxiliary result about linear interpolation on the reference element.

Lemma A.1 (linear interpolant). *Let X be a Hilbert space, $\widehat{K} = (0,1)$, and let $\widetilde{\pi}_1$ be the linear interpolant in the points $1/2$, 1. Let $\alpha > -1$ and $\delta \leq 1$. Then, for $u \in C((0,1];X)$ and provided the terms on the right-hand side are finite, we have*

$$\int_{\widehat{K}} y^{\alpha} \|u - \widetilde{\pi}_1 u\|_X^2 \, \mathrm{d}y \lesssim \int_{\widehat{K}} y^{\alpha+2\delta} \|u'\|_X^2 \, \mathrm{d}y, \tag{A.3}$$

$$\int_{\widehat{K}} y^{\alpha} \|(u - \widetilde{\pi}_1 u)'\|_X^2 \, \mathrm{d}y \lesssim \int_{\widehat{K}} y^{\alpha+2\delta} \|u''\|_X^2 \, \mathrm{d}y, \tag{A.4}$$

where the hidden constant is independent of u.

Proof. For notational simplicity, we will prove the lemma only for the case $X = \mathbb{R}$.

We begin with the proof of (A.3). Since $(u - \widetilde{\pi}_1 u)(1) = 0$ we have, for $y \in \widehat{K}$,

$$(u - \widetilde{\pi}_1 u)(y) = \int_1^y (u - \widetilde{\pi}_1 u)'(t) \, \mathrm{d}t,$$

so that

$$\int_0^1 y^\alpha |u-\widetilde{\pi}_1 u|^2 \,\mathrm{d}y \le 2\int_0^1 y^\alpha \left|\int_y^1 |u'(t)|\,\mathrm{d}t\right|^2 \,\mathrm{d}y + 2\int_0^1 y^\alpha \left|\int_y^1 |(\widetilde{\pi}_1 u)'(t)|\,\mathrm{d}t\right|^2 \,\mathrm{d}y.$$

From Hardy's inequality (e.g., [22, Chapter 2, Theorem 3.1]) we infer

$$\int_0^1 y^\alpha \left|\int_y^1 |u'(t)|\,\mathrm{d}t\right|^2 \,\mathrm{d}y \le (\alpha+1)^{-2}\int_0^1 y^{\alpha+2}|u'(y)|^2\,\mathrm{d}y.$$

From $(\widetilde{\pi}_1 u)' = 2\int_{1/2}^1 u'(t)\,\mathrm{d}t$ we obtain $|(\widetilde{\pi}_1 u)'|^2 \le C\int_{1/2}^1 t^{\alpha+2\delta}|u'(t)|^2\,\mathrm{d}t$ and therefore, in view of $\alpha > -1$, the estimate

$$\int_0^1 y^\alpha |(\widetilde{\pi}_1 u)'|^2\,\mathrm{d}y \lesssim \int_0^1 y^{\alpha+2}|u'(y)|^2\,\mathrm{d}y.$$

This concludes the proof of (A.3) for the case $\delta = 1$. Since the integration range is $y \in \widehat{K} = (0,1)$, we may replace $y^{\alpha+2}$ by $y^{\alpha+2\delta}$.

Let us now prove (A.4). Again, it suffices to consider the limiting case $\delta = 1$ and use Hardy's inequality. We write

$$\begin{aligned}(u-\widetilde{\pi}_1 u)'(y) &= u'(y) - 2\int_{1/2}^1 u'(t)\,\mathrm{d}t = 2\int_{1/2}^1 \left(u'(y)-u'(t)\right)\,\mathrm{d}t \\ &= 2\int_{1/2}^1\int_t^y u''(\tau)\,\mathrm{d}\tau\,\mathrm{d}t.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 y^\alpha |(u-\widetilde{\pi}_1 u)'(y)|^2\,\mathrm{d}y &= 4\int_0^1 y^\alpha \left|\int_{1/2}^1\int_t^y u''(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\right|^2\,\mathrm{d}y \\ &\le 2\int_{1/2}^1\int_0^1 y^\alpha \left|\int_t^y |u''(\tau)|^2\,\mathrm{d}\tau\right|^2\,\mathrm{d}y\,\mathrm{d}t \\ &\lesssim \int_{1/2}^1\int_0^1 y^\alpha \left[\left|\int_y^1 |u''(\tau)|^2\,\mathrm{d}\tau\right|^2 + \left|\int_t^1 |u''(\tau)|^2\,\mathrm{d}\tau\right|^2\right]\mathrm{d}y\,\mathrm{d}t \\ &\lesssim \int_0^1 y^{\alpha+2}|u''(y)|^2\,\mathrm{d}y + \int_{1/2}^1 y^{\alpha+2}|u''(y)|^2\,\mathrm{d}y,\end{aligned}$$

where, in the last step we applied Hardy's inequality.

The Lemma is thus proved. □

With this auxiliary result at hand we can estimate I_1 as follows: scaling the estimate (A.3) gives

$$\|u-\pi_y^{\boldsymbol{r}} u\|_{L^2(\omega_{\alpha,0},I_1)} \le C h_1^\beta \|u'\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)}. \tag{A.5}$$

The assumption $|I_1| = \sigma^M \mathcal{Y} \le 1$ implies that we may insert the weight $e^{\gamma y}$ on both sides of (A.5).

We now proceed the estimation over the elements away from the origin, i.e., on I_i, $i = 2, \ldots, M$. These elements satisfy $h_i\sigma/(1-\sigma) = \operatorname{dist}(I_i, 0)$. For $I_i = (y_{i-1}, y_i)$ the pull-back $\widehat{u}_i := u|_{I_i} \circ F_{I_i}$ satisfies

$$
\begin{aligned}
&\|\widehat{u}_i^{(\ell+1)}\|^2_{L^2(-1,1)} = (h_i/2)^{-1+2(\ell+1)}\|u^{(\ell+1)}\|^2_{L^2(I_i)} \\
&\leq (h_i/2)^{-1+2(\ell+1)} e^{-\gamma y_{i-1}} \max_{y\in I_i} y^{-\alpha-2(\ell+1)+2\beta}\|u^{(\ell+1)}\|^2_{L^2(\omega_{\alpha+2(\ell+1)-2\beta,\gamma},I_i)} \\
&\lesssim e^{-\gamma y_{i-1}} h_i^{-1+2(\ell+1)} h_i^{-\alpha-2(\ell+1)+2\beta}(2(1-\sigma))^{-2(\ell+1)} C_i^2 (2K_u)^{2(\ell+1)}(\ell+1)!^2,
\end{aligned}
$$

where in the last step we have used (A.1). The assumption on the operator $\widehat{\Pi}_r$, defined on the reference element, then yields the existence of a $b > 0$ that depends solely on K_u and σ, for which

$$
\|\widehat{u} - \widehat{\Pi}_{\boldsymbol{r}_i}\widehat{u}\|_{L^2(-1,1)} \lesssim C_i e^{-\gamma y_{i-1}} e^{-b\boldsymbol{r}_i} h_i^{-(1+\alpha)/2+\beta} .
$$

Scaling back to I_i and using again $h_i \sim \operatorname{dist}(I_i, 0)$ yields

$$
\|u - \pi_y^{\boldsymbol{r}} u\|^2_{L^2(\omega_{\alpha,\gamma},I_i)} \leq C h_i^{2\beta} C_i^2 e^{-2b\boldsymbol{r}_i}.
$$

Summation over i and taking the slope of the linear degree vector sufficiently large (see, for instance, the proof of [5, Theorem 8] for details) gives

$$
\sum_{i=2}^{M} \|u - \pi_y^{\boldsymbol{r}} u\|^2_{L^2(\omega_{\alpha,\gamma},I_i)} \lesssim \mathcal{Y}^{2\beta} e^{-2b'M}
$$

for suitable $b' > 0$. Combining this with (A.5) gives the desired (5.45).

It remains to prove (5.46). We begin with a preparatory result.

Lemma A.2 (exponential decay). *Let X be a Hilbert space and let $\delta \in \mathbb{R}$, $\gamma > 0$, $\mathcal{Y}_0 > 0$. Then the following holds for $u \in C^1((\mathcal{Y}_0,\infty); X)$ in items (i), (ii) and for $u \in C^2((\mathcal{Y}_0,\infty); X)$ in items (iii), (iv) with implied constants depending solely on δ, γ, and $\mathcal{Y}_0$:*

(i) If $\lim_{y\to\infty} u(y) = 0$ and $\|u'\|_{L^2(\omega_{\delta,\gamma},(\mathcal{Y}_0,\infty);X)} < \infty$, then

$$
\|u(\mathcal{Y})\|_X \lesssim \mathcal{Y}^{-\delta/2}\exp(-\mathcal{Y}\gamma/2)\|u'\|_{L^2(\omega_{\delta,\gamma},(\mathcal{Y},\infty);X)} \qquad \forall \mathcal{Y} \geq \mathcal{Y}_0. \tag{A.6}
$$

(ii) If $\sum_{j=0}^{1}\|u^{(j)}\|_{L^2(\omega_{\delta,\gamma},(\mathcal{Y}_0,\infty);X)} < \infty$, then $\lim_{y\to\infty} u(y) = 0$.

(iii) If $\lim_{y\to\infty} u^{(j)}(y) = 0$ for $j = 0, 1$ and $\|u''\|_{L^2(\omega_{\delta,\gamma},(\mathcal{Y}_0,\infty);X)} < \infty$, then

$$
\|u(\mathcal{Y})\|_X \lesssim \mathcal{Y}^{-\delta/2}\exp(-\mathcal{Y}\gamma/2)\|u''\|_{L^2(\omega_{\delta,\gamma},(\mathcal{Y},\infty);X)} \qquad \forall \mathcal{Y} \geq \mathcal{Y}_0. \tag{A.7}
$$

(iv) If $\sum_{j=0}^{2}\|u^{(j)}\|_{L^2(\omega_{\delta,\gamma},(\mathcal{Y}_0,\infty);X)} < \infty$, then $\lim_{y\to\infty} u(y) = \lim_{y\to\infty} u'(y) = 0$.

Proof. We will only prove items (i) and (ii) as the remaining two are proved by similar arguments.

We begin the proof with the following observation: There is a constant that depends only on δ, $\mathcal{Y}_0$, and γ such that

$$
\int_{\mathcal{Y}}^{\infty} y^{-\delta}\exp(-\gamma y)\,\mathrm{d}y \lesssim \mathcal{Y}^{-\delta}\exp(-\gamma\mathcal{Y}). \tag{A.8}
$$

For $\delta \geq 0$, this is immediate. For $\delta < 0$, one integrates by parts once to discover that the leading order asymptotics (as $\mathcal{Y} \to \infty$) of the integral is $\gamma^{-1}\exp(-\gamma\mathcal{Y})\mathcal{Y}^{-\delta}$.

We now proceed with the proof of (A.6): Since $\gamma > 0$, we can write

$$\|-u(\mathcal{Y})\|_X = \left\|\int_{\mathcal{Y}}^{\infty} u'(y)\,\mathrm{d}y\right\|_X \le \sqrt{\int_{\mathcal{Y}}^{\infty} y^{-\delta}\exp(-\gamma y)\,\mathrm{d}y}\|u'\|_{L^2(\omega_{\delta,\gamma},(\mathcal{Y},\infty))},$$

and (A.6) follows from (A.8). The assertion of item (ii) follows by a similar argument, starting from $u(y) = u(\eta)+\int_\eta^y u'(t)\,\mathrm{d}t$, squaring, multiplying by $\exp(\gamma'\eta)$ for arbitrary $0 < \gamma' < \gamma$, and integrating in η. □

To prove (5.46) we have to estimate $u(\mathcal{Y})$. Lemma A.2 shows

$$\|u(\mathcal{Y})\|_X \lesssim \mathcal{Y}^{-\alpha/2-(1-\beta)}\exp(-\mathcal{Y}\gamma/2)C_u.$$

With this estimate in hand, we can show (5.46), recalling that $|I_M| \sim \mathcal{Y}$.

Appendix B. Analysis of the decoupling eigenvalue problem.

Lemma B.1 (weighted Poincaré). *Let $\mathcal{Y} > 0$ and $\alpha \in (-1,1)$. Then, for $v \in C^1((0,\mathcal{Y}])$ with $v(\mathcal{Y}) = 0$ there holds*

$$\|v\|_{L^\infty(0,\mathcal{Y})} \le \mathcal{Y}^{(1-\alpha)/2}(1-\alpha)^{-1/2}\|v'\|_{L^2(y^\alpha,(0,\mathcal{Y}))}. \tag{B.1}$$

Proof. From $v(\mathcal{Y}) = 0$ we get $v(y) = -\int_y^{\mathcal{Y}} v'(t)\,\mathrm{d}t$. Hence, for $y \in (0,\mathcal{Y})$,

$$\begin{aligned}|v(y)| &= \left|\int_y^{\mathcal{Y}} v'(t)\,\mathrm{d}t\right| = \left|\int_y^{\mathcal{Y}} t^{-\alpha/2}t^{\alpha/2}v'(t)\,\mathrm{d}t\right| \le \left(\int_y^{\mathcal{Y}} t^{-\alpha}\,\mathrm{d}t\right)^{1/2}\|v'\|_{L^2(y^\alpha,(0,\mathcal{Y}))}\\ &\le \mathcal{Y}^{(1-\alpha)/2}(1-\alpha)^{-1/2}\|v'\|_{L^2(y^\alpha,(0,\mathcal{Y}))},\end{aligned}$$

which finishes the proof. □

Lemma B.2 (eigenvalue upper bound). *Let $\mathcal{Y} > 0$ and $\alpha \in (-1,1)$. Assume that (v,μ) satisfy*

$$\|v'\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} = 1, \qquad \|v\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} = \mu, \qquad v(\mathcal{Y}) = 0. \tag{B.2}$$

Then, $0 < \mu \le \mathcal{Y}^2(1-\alpha^2)^{-1}$.

Proof. We compute, using Lemma B.1

$$\begin{aligned}\mu = \|v\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} &= \int_0^{\mathcal{Y}} t^\alpha|v(t)|^2\,\mathrm{d}t \le \|v\|^2_{L^\infty(0,\mathcal{Y})}\mathcal{Y}^{1+\alpha}(1+\alpha)^{-1}\\ &\le \mathcal{Y}^{1+\alpha}\mathcal{Y}^{1-\alpha}(1+\alpha)^{-1}(1-\alpha)^{-1}\|v'\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} = \mathcal{Y}^2(1-\alpha^2)^{-1},\end{aligned}$$

which finishes the proof. □

We also need lower bounds for eigenvalues.

Lemma B.3 (eigenvalue lower bound). *Let $\alpha > -1$. Let $\mathcal{G}^M$ be an arbitrary mesh on $(0,\mathcal{Y})$ with the property that for all elements I_i, $i = 2,\dots,M$, not abutting $y = 0$ there holds $|I_i| \le C_{geo}\operatorname{dist}(I_i,0)$. Let $V_h \subset H^1(y^\alpha,(0,\mathcal{Y}))$ be a subspace of the space of piecewise polynomials of degree q on $\mathcal{G}^M$. Then, with h_{min} denoting the smallest element size,*

$$\|v'\|_{L^2(y^\alpha,(0,\mathcal{Y}))} \lesssim h_{min}^{-1}q^2\|v\|_{L^2(y^\alpha,(0,\mathcal{Y}))}, \qquad \forall v \in V_h, \tag{B.3}$$

where the hidden constant depends solely on C_{geo} and α.

Proof. We emphasize that the condition $h_i \le C_{geo}\,\mathrm{dist}(I_i, 0)$ is satisfied for all meshes where neighboring elements have comparable size. We also remark that (slightly) sharper estimates (in the dependence on the polynomial degree q) are possible on geometric meshes with linear degree vector. We write $h_i = |I_i|$. We note the polynomial inverse estimate

$$\int_{-1}^{1} (1+y)^\alpha w'(y)^2 \,\mathrm{d}x \lesssim q^4 \int_{-1}^{1} (1+y)^\alpha w^2(y)\,\mathrm{d}y \quad \forall w \in \mathbb{P}_q(\widehat{K}). \tag{B.4}$$

For the first element $I_1 = (0, y_1)$ we calculate for $v \in V_h$ and its pull-back $\widehat{v} := v|_{I_1} \circ F_{I_1}$

$$\begin{aligned} \|v'\|^2_{L^2(y^\alpha,\widehat{K})} &= (h_1/2)^{\alpha+1-2} \int_{-1}^{1} (1+y)^\alpha |\widehat{v}'(y)|^2 \,\mathrm{d}y \\ &\lesssim h_1^{\alpha+1-2} q^4 \int_{-1}^{1} (1+y)^\alpha |\widehat{v}(y)|^2 \,\mathrm{d}y \sim h_1^{-2} q^4 \|v\|^2_{L^2(y^\alpha, I_1)}, \end{aligned} \tag{B.5}$$

where, in the last step, we used the inverse estimate (B.4). For the remaining elements I_i, we exploit that the assumption $h_i \ge C_{geo}\,\mathrm{dist}(I_i, 0)$ to obtain that the weight is slowly varying over them, i.e.,

$$\max_{y\in I_i} y^\alpha \le (1 + C_{geo})^{|\alpha|} \min_{y\in I_i} y^\alpha, \qquad i = 2, \ldots, M.$$

Hence, the polynomial inverse estimate (B.4) (with $\alpha = 0$ there) yields by scaling arguments

$$\|v'\|_{L^2(y^\alpha, I_i)} \le C h_i^{-1} q^2 \|v\|_{L^2(y^\alpha, I_i)}. \tag{B.6}$$

Combining (B.5), (B.6) yields the result. ∎

REFERENCES

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] G. Acosta and J.P. Borthagaray. A fractional Laplace equation: regularity of solutions and finite element approximations. *SIAM J. Numer. Anal.*, 55(2):472–495, 2017.

[3] L.V. Ahlfors. *Complex analysis*. McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[4] T. Apel. Interpolation of non-smooth functions on anisotropic finite element meshes. *M2AN Math. Model. Numer. Anal.*, 33(6):1149–1185, 1999.

[5] T. Apel and J.M. Melenk. interpolation and quasi-interpolation in h and hp-finite element spaces. Technical report. to appear in: encyclopedia of computational mechanics, extended version available at: `http://www.asc.tuwien.ac.at/preprint/2015/asc39x2015.pdf`.

[6] M. Aurada, M. Feischl, T. Führer, M. Karkulik, and D. Praetorius. Energy norm based error estimators for adaptive BEM for hypersingular integral equations. *Appl. Numer. Math.*, 95:15–35, 2015.

[7] I. Babuška and B.Q. Guo. The *h-p* version of the finite element method for domains with curved boundaries. *SIAM J. Numer. Anal.*, 25(4):837–861, 1988.

[8] C. Bacuta, H. Li, and V. Nistor. Differential operators on domains with conical points: precise uniform regularity estimates. HAL Preprint, to appear in Revue Romaine de Mathématiques Pures et Appliqués, 2016.

[9] M.Š. Birman and M.Z. Solomjak. *Spektralnaya teoriya samosopryazhennykh operatorov v gilbertovom prostranstve*. Leningrad. Univ., Leningrad, 1980.

[10] A. Bonito, J.P. Borthagaray, R.H. Nochetto, E. Otárola, and A.J. Salgado. Numerical methods for fractional diffusion. Technical report, 2017. arXiv:1707.01566.
[11] A. Bonito and J.E. Pasciak. Numerical approximation of fractional powers of elliptic operators. *Math. Comp.*, 84(295):2083–2110, 2015.
[12] C. Brändle, E. Colorado, A. de Pablo, and U. Sánchez. A concave-convex elliptic problem involving the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A*, 143(1):39–71, 2013.
[13] X. Cabré and Y. Sire. Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions. *Trans. Amer. Math. Soc.*, 367(2):911–941, 2015.
[14] X. Cabré and J. Tan. Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math.*, 224(5):2052–2093, 2010.
[15] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Part. Diff. Eqs.*, 32(7-9):1245–1260, 2007.
[16] L.A. Caffarelli and P.R. Stinga. Fractional elliptic equations, Caccioppoli estimates and regularity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(3):767–807, 2016.
[17] A. Capella, J. Dávila, L. Dupaigne, and Y. Sire. Regularity of radial extremal solutions for some non-local semilinear equations. *Comm. Partial Differential Equations*, 36(8):1353–1384, 2011.
[18] H. Cartan. *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*. Avec le concours de Reiji Takahashi. Enseignement des Sciences. Hermann, Paris, 1961.
[19] X. Chen, F. Zeng, and G.E. Karniadakis. A tunable finite difference method for fractional differential equations with non-smooth solutions. *Comput. Methods Appl. Mech. Engrg.*, 318:193–214, 2017.
[20] M. Costabel, M. Dauge, and S. Nicaise. Analytic regularity for linear elliptic systems in polygons and polyhedra. *Math. Meths. Appl. Sci.*, 22(8), 2012.
[21] M. D'Elia and M. Gunzburger. The fractional Laplacian operator on bounded domains as a special case of the nonlocal diffusion operator. *Comput. Math. Appl.*, 66(7):1245 – 1260, 2013.
[22] R.A. DeVore and G.G. Lorentz. *Constructive Approximation*. Springer Verlag, 1993.
[23] J. Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.
[24] E.B. Fabes, C.E. Kenig, and R.P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Part. Diff. Eqs.*, 7(1):77–116, 1982.
[25] F.D. Gaspoz, C.-J. Heine, and K.G. Siebert. Optimal grading of the newest vertex bisection and H^1-stability of the L_2-projection. *IMA J. Numer. Anal.*, 36(3):1217–1241, 2016.
[26] F.D. Gaspoz and P. Morin. Convergence rates for adaptive finite elements. *IMA J. Numer. Anal.*, 29(4):917–936, 2009.
[27] V. Gol′dshtein and A. Ukhlov. Weighted Sobolev spaces and embedding theorems. *Trans. Amer. Math. Soc.*, 361(7):3829–3850, 2009.
[28] W. Gui and I. Babuška. The h, p and h-p versions of the finite element method in 1 dimension. II. The error analysis of the h- and h-p versions. *Numer. Math.*, 49(6):613–657, 1986.
[29] H. Harbrecht, M. Peters, and M. Siebenmorgen. Combination technique based k-th moment analysis of elliptic problems with random diffusion. *J. Comput. Phys.*, 252:128–141, 2013.
[30] B.N. Khoromskij and J.M. Melenk. Boundary concentrated finite element methods. *SIAM J. Numer. Anal.*, 41(1):1–36, 2003.
[31] A. Kufner and B. Opic. How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carolin.*, 25(3):537–554, 1984.
[32] N.S. Landkof. *Foundations of modern potential theory*. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.
[33] D. Meidner, J. Pfefferer, K. Schürholz, and B. Vexler. *hp*-finite elements for fractional diffusion. Technical report, 2017. arxiv:1706.04066v1.
[34] J.M. Melenk. On the robust exponential convergence of *hp* finite element method for problems with boundary layers. *IMA J. Numer. Anal.*, 17(4):577–601, 1997.
[35] J.M. Melenk. *hp-finite element methods for singular perturbations*, volume 1796 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.
[36] J.M. Melenk and C. Schwab. HP FEM for reaction-diffusion equations. I. Robust exponential convergence. *SIAM J. Numer. Anal.*, 35(4):1520–1557, 1998.
[37] J.M. Melenk and C. Schwab. Analytic regularity for a singularly perturbed problem. *SIAM J. Math. Anal.*, 30(2):379–400, 1999.
[38] K.S. Miller and S.G. Samko. Completely monotonic functions. *Integral Transform. Spec. Funct.*,

12(4):389–402, 2001.

[39] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[40] F. Müller, D. Schötzau, and C. Schwab. Symmetric interior penalty discontinuous galerkin methods for elliptic problems in polygons. Technical Report 2017-15, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2017.

[41] R.H. Nochetto, E. Otárola, and A.J. Salgado. A PDE approach to fractional diffusion in general domains: a priori error analysis. *Found. Comput. Math.*, 15(3):733–791, 2015.

[42] R.H. Nochetto, E. Otárola, and A.J. Salgado. Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces and applications. *Numer. Math.*, 132(1):85–130, 2016.

[43] R.H. Nochetto and A. Veeser. Primer of adaptive finite element methods. In *Multiscale and adaptivity: modeling, numerics and applications*, volume 2040 of *Lecture Notes in Math.*, pages 125–225. Springer, Heidelberg, 2012.

[44] M.A. Olshanskii and A. Reusken. On the convergence of a multigrid method for linear reaction-diffusion problems. *Computing*, 65(3):193–202, 2000.

[45] E. Otárola. *A PDE approach to numerical fractional diffusion*. PhD thesis, University of Maryland, College Park, 2014.

[46] H.-G. Roos, M. Stynes, and L. Tobiska. *Numerical methods for singularly perturbed differential equations*, volume 24 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1996. Convection-diffusion and flow problems.

[47] S.A. Sauter and C. Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2011. Translated and expanded from the 2004 German original.

[48] R. Schneider. *Multiskalen- und Wavelet-Matrixkompression*. Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 1998. Analysisbasierte Methoden zur effizienten Lösung großer vollbesetzter Gleichungssysteme. [Analysis-based methods for the efficient solution of large nonsparse systems of equations].

[49] R. Schneider, O. Reichmann, and C. Schwab. Wavelet solution of variable order pseudodifferential equations. *Calcolo*, 47(2):65–101, 2010.

[50] D. Schötzau and C. Schwab. Exponential convergence for *hp*-version and spectral finite element methods for elliptic problems in polyhedra. *M3AS*, 25(9):1617–1661, 2015.

[51] C. Schwab. *p- and hp-finite element methods*. Numerical Mathematics and Scientific Computation. The Clarendon Press, Oxford University Press, New York, 1998. Theory and applications in solid and fluid mechanics.

[52] C. Schwab and M. Suri. The *p* and *hp* versions of the finite element method for problems with boundary layers. *Math. Comp.*, 65(216):1403–1429, 1996.

[53] C. Schwab, M. Suri, and C. Xenophontos. The *hp* finite element method for problems in mechanics with boundary layers. *Comput. Methods Appl. Mech. Engrg.*, 157(3-4):311–333, 1998. Seventh Conference on Numerical Methods and Computational Mechanics in Science and Engineering (NMCM 96) (Miskolc).

[54] L.R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[55] P.R. Stinga and J.L. Torrea. Extension problem and Harnack's inequality for some fractional operators. *Comm. Partial Differential Equations*, 35(11):2092–2122, 2010.

[56] B.O. Turesson. *Nonlinear potential theory and weighted Sobolev spaces*, volume 1736 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.