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J.M. Melenk, S.A. Sauter

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

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J.M. Melenk* S.A. Sauter†

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Abstract

The time-harmonic Maxwell equations at high wavenumber k are discretized by edge elements of degree p on a mesh of width h. For the case of a ball and exact, transparent boundary conditions, we show quasi-optimality of the Galerkin method under the k-explicit scale resolution condition that a) kh/p is sufficient small and b) $p/\log k$ is sufficiently large.

Contents

*(melenk@tuwien.ac.at), Institut für Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstrasse 8-10, A-1040 Wien, Austria
†(stas@math.uzh.ch), Institut für Mathematik, Universität Zürich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland

Glossary and Notation

general

$k \geq 1 > 0$	wavenumber
i	imaginary unit $\sqrt{-1}$
$A \lesssim B$	$A \leq CB$ for some C independent of k, h, p, and functions that appear in A and B; see Rem. 1.2

geometry

$B_1(0)$	unit ball in $\mathbb{R}^3$
B_r^+	half-balls in $\mathbb{R}^3$
Ω	domain in $\mathbb{R}^3$ or unit ball $B_1(0)$ in $\mathbb{R}^3$
Ω^+	$\mathbb{R}^3 \setminus \overline{\Omega}$
$\Gamma = \partial\Omega$	boundary of Ω
$\mathbf{n}$	unit normal vector on Γ pointing into Ω^+
$\mathbf{n}^*$	constant extension of $\mathbf{n}$ into tubular neighborhood of Γ

spaces

$\mathbf{X} := \mathbf{H}(\Omega, \mathrm{curl})$	(2.8)
$\mathbf{X}_0 := \mathbf{H}_0(\Omega, \mathrm{curl})$	(4.7)
$\mathbf{H}(\Omega, \mathrm{curl})$, $\mathbf{H}(\Omega, \mathrm{div})$	(2.8), (2.11)
$\mathbf{L}^2(\Omega)$	space of vector-valued L^2-functions
$H^s(\Omega)$, $H^s(\Gamma)$	scalar-valued Sobolev spaces on Ω, Γ, Sec. 2.3.1, (2.18)
$\mathbf{H}^s(\Omega)$	vector-valued Sobolev spaces on Ω
$\mathbf{L}^2_T(\Gamma)$, $\mathbf{H}^s_T(\Gamma)$	Sobolev space of tangential fields on Γ, (2.12), (2.20)
$\mathbf{H}^{-1/2}_{\mathrm{div}}(\Gamma)$	(2.24)
$\mathbf{H}^{-1/2}_{\mathrm{curl}}(\Gamma)$	(2.24)
$\mathbf{V}_0$, $\mathbf{V}_0^*$	spaces of divergence-free functions, see (4.21), (4.22)
$\mathcal{A}(Ck^\alpha, \gamma, D)$, $\mathcal{A}^\infty(Ck^\alpha, \gamma, D)$, $\mathcal{A}(Ck^\alpha, \gamma, \Gamma)$	classes of analytic fcts., Def. 2.5; C, γ, α are independent of k

functions

$\mathbf{E}$, $\mathbf{H}$, $\mathbf{E}^+$, $\mathbf{H}^+$	electric and magnetic fields in Ω and in Ω^+
Y_ℓ^m, λ_ℓ	eigenfunctions of Laplace-Beltrami, (2.17);
$\tilde{Y}_\ell^m$	analytic extension of Y_ℓ^m into tubular neighborhood $\mathcal{U}$ of Γ, (5.11) for $\Gamma = \partial B_1(0)$, the spherical harmonics
$\mathbf{T}_\ell^m := \overrightarrow{\mathrm{curl}_\Gamma} Y_\ell^m$	$\mathbf{T}_\ell^m$, $\nabla_\Gamma Y_{\ell'}^{m'}$ are $L^2_T(\Gamma)$-orthgonal basis, cf. [47, Thm. 2.4.8]
ι_ℓ	index set of indices for eigenvalue λ_ℓ, (2.17); for the unit sphere, $\iota_\ell = \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$
g_k	Helmholtz fundamental solution, (7.9)
$\mathbf{G}_k$	Maxwell fundamental solution, (7.9)

sesquilinear forms, norms

$(\cdot,\cdot)$, $(\cdot,\cdot)_\Gamma$	$L^2(\Omega)$-inner prod. and $L^2(\Gamma)$-inner prod. (or duality pairing)
a_k, A_k, b_k	sesquilinear forms associated with Maxwell's equations, (2.28), (4.3)
b_k^{low}, b_k^{high},	low- and high-frequency parts of bilinear form b_k, (4.11)
$((\cdot,\cdot))$	$((\cdot,\cdot)) = k^2(\cdot,\cdot)_{L^2(\Omega)} + \mathrm{i}\,k b_k((\cdot)^\nabla, (\cdot)^\nabla)$ $= k^2(\cdot,\cdot)_{L^2(\Omega)} + \mathrm{i}\,k(T_k(\cdot)_T, (\cdot)_T)_\Gamma$; see (4.1)
$(\cdot,\cdot)_{\mathrm{curl},\Omega,k}$	$(\mathrm{curl}\,\cdot, \mathrm{curl}\,\cdot) + k^2(\cdot,\cdot)$; see (2.9)
$\|\cdot\|_{-1/2,\mathrm{curl}_\Gamma}$, $\|\cdot\|_{-1/2,\mathrm{div}_\Gamma}$	norms on $\mathbf{H}^{-1/2}_{\mathrm{curl}}(\Gamma)$, on $\mathbf{H}^{-1/2}_{\mathrm{div}}(\Gamma)$, (2.24)

$\|\cdot\|_{\mathrm{curl},\Omega,k,\lambda}$	see (5.52)
$\|\cdot\|_{\mathcal{H},\omega}$	$\|\cdot\|^2_{\mathcal{H},\omega} = \|\nabla\cdot\|^2_{L^2(\omega)} + k^2\|\cdot\|^2_{L^2(\omega)}$
$\langle\cdot,\cdot\rangle$,	Euclidean scalar prod. with complex conjugation in second argument (cf. (2.10))
$N'_{R,p}$, $N'_{R,p,q}$,	
$M'_{R,p}$, $M'_{R,p,q}$, $H_{R,p}$	seminorms to control high order derivatives, (D.21), (D.2.1), (D.38)

discrete spaces, meshes

$\widehat{K}$	reference tetrahedron
$\mathcal{T}_h$, F_K, F_K, A_K	triangulation, element maps, Sec. 3.2, Ass. 3.1
S_h	(discrete) subspace of $H^1(\Omega)$; we require $\nabla S_h \subset \mathbf{X}_h$ and exact seq. property (1.8), (3.2)
$\mathbf{X}_h$	(discrete) subspace of $\mathbf{H}(\Omega, \mathrm{curl})$
h, h_K, p	global and local meshwidth (Thm. 4.17, (3.3)), polyn. deg. p
$\mathcal{P}_p$, $\boldsymbol{\mathcal{P}}_p$	space of $\mathbb{R}$-valued and $\mathbb{R}^3$-valued polynomials of degree p, (3.4)
$\boldsymbol{\mathcal{N}}^I_p(\widehat{K})$	Nédélec type I space on reference tetrahedron $\widehat{K}$, (3.5)
$\mathbf{RT}_p(\widehat{K})$	Raviart-Thomas elements on reference tetrahedron $\widehat{K}$, (3.6)
$S_{p+1}(\mathcal{T}_h)$, $\boldsymbol{\mathcal{N}}^I_p(\mathcal{T}_h)$,	
$\mathbf{RT}_p(\mathcal{T}_h)$, $Z_p(\mathcal{T}_h)$	polyn. spaces on $\mathcal{T}_h$: $H^1(\Omega)$-, $\mathbf{H}(\mathrm{curl},\Omega)$-, $\mathbf{H}(\mathrm{div},\Omega)$-, and $L^2(\Omega)$-conforming

operators

curl, div	3D curl and divergence operators
curl_Γ, div_Γ	2D scalar curl and divergence operators on the surface Γ, (2.14)
$\overrightarrow{\mathrm{curl}}_\Gamma$, ∇_Γ,	2D vectorial curl and surface gradient operators on Γ, (2.13)
Δ_Γ	surface Laplace-Beltrami operator, (2.15)
T_k	(Maxwell) capacity operator
T_k^{low}, T_k^{high}	low- and high-frequency part of capacity operator, (4.11)
$\mathcal{E}_{\mathrm{curl}}$, $\mathcal{E}_{\mathrm{div}}$,	lifting operators (see Thm. 2.4)
Π_T, Π_T^+, γ_T, γ_T^+	trace operators for Ω and Ω^+, (2.3), Thm. 2.4
$(\cdot)_T$	subscript T indicates tangential trace: $\mathbf{u}_T = \Pi_\tau \mathbf{u}$
$(\cdot)^{\mathrm{high}}$, $(\cdot)^{\mathrm{low}}$	$\mathbf{v}^{\mathrm{high}} = H_\Omega \mathbf{v}$, $\mathbf{v}^{\mathrm{low}} = L_\Omega \mathbf{v}$
$(\cdot)^\nabla$	gradient part of functions on Γ, (2.21)
$(\cdot)^{\mathrm{curl}}$	curl part of functions on Γ, (2.21)
$[\cdot]_{0,\Gamma}$, $[\cdot]_{1,\Gamma}$	jump operators across Γ, (2.4)
L_Ω, $H_\Omega = \mathrm{I} - L_\Omega$,	
L_Γ, $H_\Gamma = \mathrm{I} - L_\Gamma$	high and low frequency operators with cut-off parameter $\lambda > 1$, Def. 4.2, (4.9) for the case $\Omega = B_1(0)$, one has $\|L_\Omega\|_{\mathrm{curl},\Omega,k} \le 1$, and $\|H_\Omega\|_{\mathrm{curl},\Omega,k} \le 2$, (5.27)
$\mathcal{S}^{\mathrm{Hh}}_{-k}$	Helmholtz single layer operator, (7.11)
$\mathcal{N}^{\mathrm{Hh}}_{-k}$	Helmholtz Newton potential, (7.12)
$\mathcal{S}^{\mathrm{Mw}}_{-k}$	Maxwell single layer operator, (7.14)
T_Δ	Laplace Dirichlet-to-Neumann operator for $B_1(0)$; Sec. 7.1.2
Π_h^E	abstract form of $\Pi_p^{curl,c}$
Π_h^F	abstract form of $\Pi_p^{div,c}$
$\Pi_p^{curl,c}$	$\mathbf{H}(\mathrm{curl})$-conf. commuting diagram projector (matches $\Pi_{p+1}^{grad,c}$)
$\widehat{\Pi}_p^{curl,c}$	the operator $\Pi_p^{curl,c}$ on the reference tetrahedron
$\Pi_p^{curl,s}$	$\mathbf{H}(\mathrm{curl})$-conforming approx. operator, optimal p-rates *simultaneously* in L^2 and $\mathbf{H}(\mathrm{curl})$
$\widehat{\Pi}_p^{curl,s}$	the operator $\Pi_p^{curl,s}$ on the reference tetrahedron
$\Pi_{p+1}^{grad,c}$	H^1-conf. commuting diagram projector (matches $\Pi_p^{curl,c}$)
Π^∇, $\Pi^{\nabla,*}$, Π_V^∇, $\Pi_V^{\nabla,*}$	projection onto $\nabla H^1(\Omega)$ or V w.r.t. $((\cdot,\cdot))$ (Lemma 4.7)
Π_h^∇, $\Pi_h^{\nabla,*}$	projection onto ∇S_h w.r.t. $((\cdot,\cdot))$ (Lemma 4.7)
Π^{curl}, $\Pi^{\mathrm{curl},*}$	$\mathrm{I} - \Pi^\nabla$ and $\mathrm{I} - \Pi^{\nabla,*}$, see Def. 4.9
Π_h^{curl}, $\Pi_h^{\mathrm{curl},*}$	$\mathrm{I} - \Pi_h^\nabla$ and $\mathrm{I} - \Pi_h^{\nabla,*}$, see Def. 4.9

$\Pi^{\mathrm{comp},*} := L_\Omega + \Pi^{\mathrm{curl},*} H_\Omega$	see Def. 4.9
$\Pi_h^{\mathrm{comp},*} := L_\Omega + \Pi_h^{\mathrm{curl},*} H_\Omega$	see Def. 4.9

constants

C_{affine}, C_{metric}	constants measuring the quality of the mesh (Assumption 3.1)
C_Γ	continuity of tangential trace operator (2.26) (bounded uniformly in k)
$C_k^{L,\Omega}$, $C_k^{H,\Omega}$	$\|L_\Omega\|_{\mathrm{curl},\Omega,k}$, $\|H_\Omega\|_{\mathrm{curl},\Omega,k}$, see (4.6); for general domains, $C_k^{L,\Omega}$, $C_k^{H,\Omega} = O(k)$; for $\Omega = B_1$, $C_k^{L,\Omega}$,$C_k^{H,\Omega} = O(1)$ (cf. Cor. 5.13)
$C_{b,k}^{\nabla,high}$	continuity of constant of b_k^{high}, see (4.12a); for $\Omega = B_1$: $C_{b,k}^{\nabla,high} = O(1)$ by Cor. 5.13
$C_{b,k}^{\mathrm{curl},high}$	continuity const. of $b_k^{\mathrm{curl,high}}$, see (4.12a); for $\Omega = B_1$: $C_{b,k}^{\mathrm{curl},high} = O(1)$ by Cor. 5.13
$C_{\mathrm{DtN},k}$	norm of capacity operator T_k, (4.13); for $\Omega = B_1$: $C_{\mathrm{DtN},k} = O(k^2)$ by Cor. 5.13
$C_{\mathrm{cont},k}$	cont. const. of A_k and of $((\cdot,\cdot))$, see (4.6);
$C_{\mathrm{cont},k}^{\mathrm{high}}$	cont. const. of $A_k(H_\Omega\cdot,\cdot)$, see (4.16); for $\Omega = B_1$: $C_{\mathrm{cont},k} = O(k^3)$ by Cor. 5.13
$C_{b,k}^{\mathrm{high}}$	cont. const. of $((\cdot, H_\Omega\cdot))$ and $((H_\Omega\cdot,\cdot))$, see (4.15); for $\Omega = B_1$: $C_{b,k}^{\mathrm{high}} = O(1)$ by Cor. 5.13
$C_{\Omega,k}$	the embedding constant $\mathbf{V}_0 \subset \mathbf{H}^1(\Omega)$, see (4.32); for $\Omega = B_1(0)$, $C_{\Omega,k} = 1$ by Lemma B.1
C_k^I	constant in fundamental approximation result, (4.43), (4.44)
$C_{r,k}$	see (6.2),
$C_{\#,k}$	see (6.2),
$C_{\#\#,k}$	see (6.15),
α_j, $C_{\mathcal{A},j}$, $\gamma_{\mathcal{A},j}$,	constants characterizing k-dependence in analyticity classes, (4.61)
C_b, C_b',	$O(1)$ constants related to the bilinear form b_k; see Props. 5.7, 5.8
$\widetilde{C}_b$	$O(1)$ continuity constant of $(\!(\cdot,\cdot)\!)$ for $B_1(0)$, if one argument is a high frequency function of form $H_\Omega\mathbf{v}$, cf. Prop. 5.12
C_{rough}	an $O(1)$ constant associated with adjoint solution operator $\mathcal{N}_2$, Prop. 7.2

dual problems and approximation properties

$\widehat{\mathcal{N}}$	(4.39), (7.15)
$\mathcal{N}_1^{\mathcal{A}}$	solution of an adjoint problem with analytic data, (4.40)
$\tilde{\eta}_1^{\mathrm{exp}}$	approximation property related to $\mathcal{N}_1^{\mathcal{A}}$, (4.41)
$\mathcal{N}_2$	adjoint sol. operator, right-hand sides finite regularity (4.50)
$\mathcal{N}_3^{\mathcal{A}}$, $\mathcal{N}_4^{\mathcal{A}}$,	adjoint sol. operator, analytic data, (4.51), (4.52)
$\tilde{\eta}_i^{\mathrm{alg}}$, $\tilde{\eta}_i^{\mathrm{exp}}$, η_i^{alg}, η_i^{exp}	see (4.53)—(4.58), (4.65)—(4.70), a tilde indicates that an adjoint sol. operator $\mathcal{N}$ is involved; η without a tilde indicates a pure approximation property, superscript "exp" indicates that exponential convergence of hp-FEM is expected; superscript "alg" indicates that algebraic convergence of hp-FEM is expected

1 Introduction

High-frequency electromagnetic scattering problems are often modelled by the time-harmonic Maxwell equations (2.1), and the high-frequency case is characterized by a large wavenumber $k > 0$. The solution is then highly oscillatory, and its numerical resolution requires fine meshes. Besides this natural condition on the discretization, a second, more subtle issue arises in the high-frequency regime, namely, the difficulty of Galerkin discretizations to control dispersion errors. That is, in fixed order methods the discrepancy between the best approximation from the discrete space and the Galerkin error widens as the wavenumber k increases. It is the purpose of the present paper to show for a model problem that high order methods are able to control these dispersion errors and can lead to quasi-optimality for a fixed (but sufficiently large) number of degrees of freedom per wavelength.
For the related, simpler case of high-frequency acoustic scattering, which is modelled by the Helmholtz equation, substantial progress in the understanding of the dispersive properties of low order and high order methods has been made in the last decades. We mention the dispersion analyses on regular grids for fixed order Galerkin methods [7, 27–29], the works [2, 5, 6], for high order methods and [4] for a non-conforming discretization and refer the reader to [21, 42] for a more detailed discussion. These analyses on regular grids give strong arguments for the numerical observation that high order discretizations are much better suited to control dispersion errors than low-order methods. For general meshes, a rigorous argument in favor of high order (conforming and non-conforming) methods is put forward in the works [21, 34, 38, 41, 42], where stability and convergence analyses that are explicit in the mesh size h, the approximation order p, and the wavenumber k are provided for several classes of Helmholtz problems. The underlying principles in these works are not restricted to FEM discretizations; indeed, [31] applies these techniques in a Helmholtz BEM context.
The numerical analysis focussing on the dispersive properties of high order methods for the time-harmonic Maxwell equations is to date significantly less developed. An analysis on regular grids that is explicit in the polynomial degree p is available in [3]. A convergence analysis for a Maxwell problem on general grids that is explicit in the mesh size h, the polynomial degree p, and the wavenumber k is the purpose of the present work. To fix ideas we consider as a model problem the time-harmonic Maxwell equations (2.1) in full space $\mathbb{R}^3$. Since a (high order) finite element method (FEM) is our goal, we consider the equivalent reformulation of the full space problem as a problem in the unit ball $\Omega = B_1(0)$ complemented with transparent boundary conditions on $\Gamma = \partial\Omega$ (cf. (2.7)). As we study conforming Galerkin discretizations, the starting point for the discretization is the variational formulation (2.28). For this model problem, our main result is Theorem 4.17, which establishes quasi-optimality of the Galerkin method based on Nédélec type I elements of degree p under the scale resolution conditions

$$kh/p \leq c \qquad \text{and} \qquad p \geq C \log k \tag{1.1}$$

for some constants c, $C > 0$ independent of h, k, and p.
We focus here on a conforming Galerkin discretization, which will require the scale resolution condition (1.1) to ensure existence of the discrete solution. It is worth pointing out that alternatives to conforming Galerkin methods have been proposed in the literature. Without attempting completeness and restricting ourselves to approaches based on higher order polynomials, we mention stabilized methods for Helmholtz [22, 23, 25, 53] and Maxwell [24, 32] problems; hybridizable methods [14]; least-squares type methods [15] and Discontinous Petrov Galerkin methods, [20, 49]. In convex domains, H^1-conforming discretizations for Maxwell problems can be employed instead of $\mathbf{H}(\mathrm{curl})$-conforming ones; a k-explicit regularity theory in convex polyhedra with subsequent fixed-order H^1-conforming convergence analysis is given in [48].
We close this introduction by emphasizing that, as in the case of the Helmholtz equation, the techniques employed in the present work are not restricted to the model problem under consideration here; in the forthcoming [40], we apply the techniques developed here to Maxwell's equations equipped with impedance boundary conditions. Finally, a general note on notation is warranted: as we aim at a k-explicit theory, we indicate constants that (possibly) depend on the wavenumber k by a subscript k.

1.1 Road Map: Setting

Our k-explicit convergence analysis of high order FEM for Maxwell's equations requires a variety of tools including compactness arguments, k-explicit regularity based on decomposing the solution into parts with finite regularity and analytic parts as developed for the Helmholtz equation, and commuting diagram operators that are explicit in the polynomial degree p. It may therefore be useful to provide here an outline of the key steps.

The reformulation of the original full space problem (2.1) as the problem (2.7) in a bounded domain $\Omega \subset \mathbb{R}^3$ uses transparent boundary conditions, which are expressed in terms of the capacity operator T_k (see Section 2.2 and (5.7) for its explicit series representation in the case of the unit ball $\Omega = B_1(0)$). The pertinent sesquilinear form that we consider in this work is then

$$A_k(\mathbf{u}, \mathbf{v}) = (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) - k^2(\mathbf{u}, \mathbf{v}) - \mathrm{i}\, k(T_k \mathbf{u}_T, \mathbf{v}_T)_\Gamma.$$

Here, $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product and $(\cdot, \cdot)_\Gamma$ the $L^2(\Gamma)$ inner product with $\Gamma = \partial\Omega$. The subscript T indicates that the tangential component of the trace is considered. For $\Omega = B_1(0)$, our analysis will be explicit in the wavenumber k and we therefore focus on this case in this introduction.

1.2 Road Map: the Maxwell Aspect

Let us first discuss the key issues that are specific to discretizations of Maxwell's equations; in the following Section 1.3, we will focus on the additional difficulties arising from making the error analysis explicit in k. The arguments that we highlight in the current Section 1.2 are essentially those of [8, 12, 26, 44] and [43, Sec. 7.2]. To understand the Galerkin error for Maxwell's equations, it is imperative to decompose the various fields in gradient fields and solenoidal fields, both in Ω and on the surface Γ. The tangential field $\mathbf{u}_T$ is decomposed as a gradient part $\mathbf{u}^\nabla$ and a (surface) divergence-free part $\mathbf{u}^{\text{curl}}$. The decomposition $\mathbf{u}_T = \mathbf{u}^\nabla + \mathbf{u}^{\text{curl}}$ leads to the decomposition of the sesquilinear form A_k as (cf. (4.3))

$$A_k(\mathbf{u}, \mathbf{v}) = (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) - \mathrm{i}\, k \left(T_k \mathbf{u}^{\text{curl}}, \mathbf{v}^{\text{curl}}\right)_\Gamma - \underbrace{\left(k^2 (\mathbf{u}, \mathbf{v}) + \mathrm{i}\, k \left(T_k \mathbf{u}^\nabla, \mathbf{v}^\nabla\right)_\Gamma\right)}_{=:((\mathbf{u},\mathbf{v}))}$$

By [47, Thm. 5.3.6], we have for $\Omega = B_1(0)$ sign properties of the expressions $\mathrm{i}\, k(T_k \mathbf{u}^{\text{curl}}, \mathbf{u}^{\text{curl}})_\Gamma$ and $((\mathbf{u}, \mathbf{u}))$. Furthermore, the curl-part $\mathbf{u}^{\text{curl}}$ of the tangential trace $\mathbf{u}_T$ vanishes for gradient fields $\mathbf{u} = \nabla\varphi$, $\varphi \in H^1(\Omega)$. Collecting these observations, we have:

(I) $\operatorname{Re}\left((\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{u}) - \mathrm{i}\, k(T_k \mathbf{u}^{\text{curl}}, \mathbf{u}^{\text{curl}})_\Gamma\right) \geq \|\operatorname{curl} \mathbf{u}\|^2 \quad \forall \mathbf{u} \in \mathbf{X} := \mathbf{H}(\Omega, \operatorname{curl})$, (cf. [47, Thm. 5.3.6], Lemma 5.2);

(II) $\operatorname{Re}((\nabla\varphi, \nabla\varphi)) \geq (k\|\nabla\varphi\|)^2 \quad \forall \varphi \in H^1(\Omega)$, (cf. (4.20));

(III) $A_k(\mathbf{u}, \nabla\varphi) = -((\mathbf{u}, \nabla\varphi)) \quad \forall \varphi \in H^1(\Omega),\ \mathbf{u} \in \mathbf{H}(\Omega, \operatorname{curl})$, (cf. (4.3) in conjunction with Rem. 2.3).

Let $\mathbf{u} \in \mathbf{X} = \mathbf{H}(\Omega, \operatorname{curl})$ and $\mathbf{u}_h \in \mathbf{X}_h \subset \mathbf{X}$ be its Galerkin approximation. Then, for arbitrary $\mathbf{w}_h \in \mathbf{X}_h$ we get for the Galerkin error $\mathbf{e}_h := \mathbf{u} - \mathbf{u}_h$

$$\begin{aligned} \|\mathbf{e}_h\|^2_{\text{curl},\Omega,k} &:= \|\operatorname{curl} \mathbf{e}_h\|^2 + k^2\|\mathbf{e}_h\|^2 \leq \operatorname{Re} A_k(\mathbf{e}_h, \mathbf{e}_h) + 2\operatorname{Re}((\mathbf{e}_h, \mathbf{e}_h)) \qquad (1.2)\\ &= \operatorname{Re} A_k(\mathbf{e}_h, \mathbf{u} - \mathbf{w}_h) + 2\operatorname{Re}((\mathbf{e}_h, \mathbf{u} - \mathbf{w}_h)) + 2\operatorname{Re}((\mathbf{e}_h, \mathbf{w}_h - \mathbf{u}_h)) \\ &\leq \underbrace{\operatorname{Re}\left(A_k(\mathbf{e}_h, \mathbf{u} - \mathbf{w}_h) + 2((\mathbf{e}_h, \mathbf{u} - \mathbf{w}_h))\right)}_{=:T_1} + 2 \sup_{\mathbf{v}_h \in \mathbf{X}_h \setminus \{0\}} \frac{\operatorname{Re}((\mathbf{e}_h, \mathbf{v}_h))}{\|\mathbf{v}_h\|_{\text{curl},\Omega,k}} \underbrace{\|\mathbf{u}_h - \mathbf{w}_h\|_{\text{curl},\Omega,k}}_{\leq \|\mathbf{e}_h\|_{\text{curl},\Omega,k} + \|\mathbf{u}-\mathbf{w}_h\|_{\text{curl},\Omega,k}}. \qquad (1.3) \end{aligned}$$

Assuming continuity of A_k and $((\cdot, \cdot))$ with respect to the norm $\|\cdot\|_{\text{curl},\Omega,k}$ (defined in (1.2)) this analysis shows that quasi-optimality of the Galerkin method can be achieved *provided* one can ensure

$$2 \sup_{\mathbf{v}_h \in \mathbf{X}_h \setminus \{0\}} \frac{\operatorname{Re}((\mathbf{e}_h, \mathbf{v}_h))}{\|\mathbf{v}_h\|_{\text{curl},\Omega,k} \|\mathbf{e}_h\|_{\text{curl},\Omega,k}} < 1. \qquad (1.4)$$

It is tempting to treat this term by a duality argument. However, the duality argument cannot be applied directly since the map $\mathbf{X} \ni \mathbf{v} \mapsto ((\cdot, \mathbf{v})) \in \mathbf{X}'$ is not necessarily compact. In the numerical analysis of Maxwell's equations, this lack of compactness is addressed by suitable "continuous" and "discrete" Helmholtz decompositions, thereby exploiting that $\mathbf{v}_h$ is from the discrete space $\mathbf{X}_h$. Specifically, we decompose $\mathbf{v}_h \in \mathbf{X}_h$ in two ways ("continuous Helmholtz decomposition" and "discrete Helmholtz decomposition") into a divergence-free part and a gradient part:

$$\mathbf{v}_h = \Pi^{\text{curl},*}\mathbf{v}_h + \Pi^{\nabla,*}\mathbf{v}_h \qquad \text{(with "continuous" } \Pi^{\text{curl},*}\mathbf{v}_h \in \mathbf{X},\ \Pi^{\nabla,*}\mathbf{v}_h \in \mathbf{X} \cap \nabla H^1(\Omega)\text{; see (IV))}, \qquad (1.5)$$

$$\mathbf{v}_h = \Pi_h^{\mathrm{curl},*}\mathbf{v}_h + \Pi_h^{\nabla,*}\mathbf{v}_h \qquad \text{(with "discrete" } \Pi_h^{\mathrm{curl},*}\mathbf{v}_h \in \mathbf{X}_h,\ \Pi_h^{\nabla,*}\mathbf{v}_h \in \mathbf{X}_h \cap \nabla H^1(\Omega)\text{; see (V)).} \tag{1.6}$$

Since, by construction, $\Pi_h^{\nabla,*}\mathbf{v}_h \in \mathbf{X}_h$ is a gradient, the Galerkin orthogonality and the observation (III) imply $\left(\left(\mathbf{e}_h, \Pi_h^{\nabla,*}\mathbf{v}_h\right)\right) = 0$. Hence, we can write using both decompositions (1.5), (1.6)

$$((\mathbf{e}_h, \mathbf{v}_h)) = \left(\left(\mathbf{e}_h, \Pi^{\mathrm{curl},*}\mathbf{v}_h\right)\right) + \left(\left(\mathbf{e}_h, \Pi_h^{\mathrm{curl},*}\mathbf{v}_h - \Pi^{\mathrm{curl},*}\mathbf{v}_h\right)\right) =: T_2 + T_3. \tag{1.7}$$

The convergence analysis based on this decomposition then relies on a) the fact that the term $T_2 = \left(\left(\mathbf{e}_h, \Pi^{\mathrm{curl},*}\mathbf{v}_h\right)\right)$ can be estimated with a duality argument and b) that $\widehat{\Pi}^{\mathrm{curl},*}\mathbf{v}_h - \Pi_h^{\mathrm{curl},*}\mathbf{v}_h$ is shown to be small.
The continuous and continuous and discrete Helmholtz decompositions (1.5), (1.6) are defined as follows:

(IV) (decomposition in gradient part and divergence-free part) The gradient part $\Pi^{\nabla,*}\mathbf{v} \in \nabla H^1(\Omega)$ is defined by the "orthogonality" condition

$$\left(\left(\nabla\psi, \Pi^{\nabla,*}\mathbf{v}\right)\right) = ((\nabla\psi, \mathbf{v})) \qquad \forall\psi \in H^1(\Omega),$$

which is well posed by (II). We set $\Pi^{\mathrm{curl},*} := I - \Pi^{\nabla,*}$ and denote its range by $\mathbf{V}_0^*$. We note that the operators $\Pi^{\nabla,*}$ and Π^{curl} effect a stable decomposition of the direct sum $\mathbf{X} = \mathbf{V}_0^* \oplus \nabla H^1(\Omega)$. The above mentioned duality argument for T_2 relies on the compactness of $\mathbf{X} \ni \mathbf{v} \mapsto \left(\left(\cdot, \Pi^{\mathrm{curl},*}\mathbf{v}\right)\right) \in \mathbf{X}'$, which is shown in Lemma 4.12 and ultimately relies on the embedding $\mathbf{V}_0^* \subset \mathbf{H}^1(\Omega)$.

(V) (decomposition of discrete functions in gradient part and discrete divergence-free part) Let $S_h \subset H^1(\Omega)$ be defined by the requirement that the following (discrete) *exact sequence* property holds:

$$S_h \xrightarrow{\nabla} \mathbf{X}_h \xrightarrow{\mathrm{curl}} \mathrm{curl}\,\mathbf{X}_h \tag{1.8}$$

(cf. (3.8) for the specific example of *hp*-FEM). We define the discrete version $\Pi_h^{\nabla,*} : \mathbf{X} \to \nabla S_h$ of $\Pi^{\nabla,*}$ by the "orthogonality" condition

$$\left(\left(\nabla\psi, \Pi_h^{\nabla,*}\mathbf{v}\right)\right) = ((\nabla\psi, \mathbf{v})) \qquad \forall\psi \in S_h$$

and set $\Pi_h^{\mathrm{curl},*} := I - \Pi_h^{\nabla,*}$.

While the term T_2 in (1.7) is treated by a duality argument, control of the term T_3 in (1.7) relies on the existence of an interpolating projector Π_h^E (and a companion operator Π_h^F) with a commuting diagram property:

(VI) (commuting diagram projector) Define $\mathbf{V}_{0,h}^* := \{\mathbf{v} \in \mathbf{V}_0^* \mid \mathrm{curl}\,\mathbf{v} \in \mathrm{curl}\,\mathbf{X}_h\}$. We require the existence of an operator $\Pi_h^E : \mathbf{V}_{0,h}^* + \mathbf{X}_h \to \mathbf{X}_h$ with the following properties:

  (a) Π_h^E is a projector.
  (b) There is a companion operator Π_h^F defined on $\mathrm{curl}\,\mathbf{X}_h$ with the commuting diagram property $\mathrm{curl}\,\Pi_h^E = \Pi_h^F\,\mathrm{curl}$.
  (c) Π_h^E has some approximation properties in $L^2(\Omega)$:

$$k\|\mathbf{v} - \Pi_h^E\mathbf{v}\| \leq \eta_6^{\mathrm{alg}}\|\mathbf{v}\|_{\mathrm{curl},\Omega,k} \qquad \forall\mathbf{v} \in \mathbf{V}_{0,h}^*, \tag{1.9}$$

  where the parameter η_6^{alg} quantifies certain the approximation properties of $\mathbf{X}_h$ (e.g., in terms of the mesh size h and polynomial degree p).

Remark 1.1 *In the case of hp-FEM, the operators Π_h^E and Π_h^F will be constructed in an element-by-element fashion (cf. Def. 8.1) from the operators $\widehat{\Pi}_p^{\mathrm{curl},c}$ and $\widehat{\Pi}_p^{\mathrm{div},c}$ (cf. Theorem 8.3) that are defined on the reference tetrahedron $\widehat{K}$. In the hp-FEM setting, the quantity η_6^{alg} in (1.9) is estimated via Lemma 8.6, (iii) by[1] $\eta_6^{\mathrm{alg}} \lesssim kh/p$; see (4.77).* ∎

[1] $A \lesssim B$ is shorthand for $A \leq CB$ for some $C > 0$ that is independent of the wavenumber k, the mesh size h, the polynomial degree p, as well as functions appearing in A and B.

Remark 1.2 *Various approximation properties η_ℓ will appear in our analysis, which depend on the subspace $\mathbf{X}_h$. In the context of hp-finite elements, these quantities η_ℓ will depend on the mesh width h, the polynomial order p of approximation, and the regularity of the functions involved. Given that we focus on high order FEM with the potential of exponential convergence, we employ the following notational convention: If some η_ℓ is (generically) algebraically small in p, we employ the superscript "alg" while we use the superscript "exp" if the quantity is exponentially small.* ∎

The use of the properties of Π_h^E required in (VI) become apparent if we observe the following arguments for estimating T_3:

(i) The definition of $\Pi^{\mathrm{curl},*}$ and $\Pi_h^{\mathrm{curl},*}$ implies the "orthogonality"

$$\left(\left(\nabla\widetilde{\psi}_h, \left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right)\right) = 0 \qquad \forall \widetilde{\psi}_h \in S_h. \tag{1.10}$$

(ii) From $\operatorname{curl}\Pi^{\mathrm{curl},*} = \operatorname{curl}\Pi_h^{\mathrm{curl},*} = \operatorname{curl}$ on $\mathbf{X}_h$ by (1.5), (1.6) we get for any $\mathbf{v}_h \in \mathbf{X}_h$

$$\begin{aligned}\operatorname{curl}\left(\Pi_h^{\mathrm{curl},*}\mathbf{v}_h - \Pi_h^E\Pi^{\mathrm{curl},*}\mathbf{v}_h\right) &\overset{\text{(VIb)}}{=} \operatorname{curl}\left(\Pi_h^{\mathrm{curl},*}\mathbf{v}_h\right) - \Pi_h^F \operatorname{curl}\left(\Pi^{\mathrm{curl},*}\mathbf{v}_h\right) = \operatorname{curl}\mathbf{v}_h - \Pi_h^F\operatorname{curl}\mathbf{v}_h \\ &\overset{\text{(VIb)}}{=} \operatorname{curl}\mathbf{v}_h - \operatorname{curl}\Pi_h^E\mathbf{v}_h \overset{\text{(VIa)}}{=} \operatorname{curl}\left(\mathbf{v}_h - \mathbf{v}_h\right) = 0. \end{aligned}\tag{1.11}$$

(iii) By the exact sequence property, the observation (1.11) implies that $\Pi_h^{\mathrm{curl},*}\mathbf{v}_h - \Pi_h^E\Pi^{\mathrm{curl},*}\mathbf{v}_h$ is the gradient of an element of S_h, i.e., $\Pi_h^{\mathrm{curl},*}\mathbf{v}_h - \Pi_h^E\Pi^{\mathrm{curl},*}\mathbf{v}_h = \nabla\psi_h$ for some $\psi_h \in S_h$.

(iv) Combining (II), (iii), (1.10) yields

$$\begin{aligned} k^2\left\|\left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right\|^2 &\overset{\text{(II)}}{\leq} \operatorname{Re}\left(\left(\left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h, \left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right)\right) \\ &\overset{(1.10),\,\text{(iii)}}{=} \operatorname{Re}\left(\left(\left(I - \Pi_h^E\right)\Pi^{\mathrm{curl},*}\mathbf{v}_h, \left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right)\right). \end{aligned}\tag{1.12}$$

(v) The continuity of $((\cdot,\cdot))$ (cf. (4.14), Prop. 5.12) and using $\operatorname{curl}\left(\left(I - \Pi_h^E\right)\Pi^{\mathrm{curl},*}\mathbf{v}_h\right) = 0 = \operatorname{curl}\left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h$ (as a consequence of the above calculation), gives $\left\|\left(I - \Pi_h^E\right)\Pi^{\mathrm{curl},*}\mathbf{v}_h\right\|_{\mathrm{curl},\Omega,k} = k\left\|\left(I - \Pi_h^E\right)\Pi^{\mathrm{curl},*}\mathbf{v}_h\right\|$ so that we may continue the estimate (1.12):

$$\begin{aligned} k^2\left\|\left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right\|^2 &\leq C_{\mathrm{cont},k}\left\|\left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right\|_{\mathrm{curl},\Omega,k}\left\|\left(I - \Pi_h^E\right)\Pi^{\mathrm{curl},*}\mathbf{v}_h\right\|_{\mathrm{curl},\Omega,k} \\ &= C_{\mathrm{cont},k}\left(k\left\|\left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right\|\right)\left(k\left\|\left(I - \Pi_h^E\right)\Pi^{\mathrm{curl},*}\mathbf{v}_h\right\|\right). \end{aligned}$$

Here, the constant $C_{\mathrm{cont},k}$ could depend on k.

(vi) The final step in treating T_3 uses the continuity of $((\cdot,\cdot))$, the above steps, and the stability of the map $\mathbf{v}_h \mapsto \Pi^{\mathrm{curl},*}\mathbf{v}_h$:

$$\begin{aligned} |T_3| &= \left|\left(\left(\mathbf{e}_h, \left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right)\right)\right| \leq C_{\mathrm{cont},k}\|\mathbf{e}_h\|_{\mathrm{curl},\Omega,k}\left\|\left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*}\right)\mathbf{v}_h\right\|_{\mathrm{curl},\Omega,k} \\ &\leq C_k\|\mathbf{e}_h\|_{\mathrm{curl},\Omega,k}\left(k\left\|\left(I - \Pi_h^E\right)\Pi^{\mathrm{curl},*}\mathbf{v}_h\right\|\right) \leq C_k\eta_6^{\mathrm{alg}}\|\mathbf{e}_h\|_{\mathrm{curl},\Omega,k}\left\|\Pi^{\mathrm{curl},*}\mathbf{v}_h\right\|_{\mathrm{curl},\Omega,k} \\ &\leq C_k\eta_6^{\mathrm{alg}}\|\mathbf{e}_h\|_{\mathrm{curl},\Omega,k}\|\mathbf{v}_h\|_{\mathrm{curl},\Omega,k}. \end{aligned}$$

Here, the constant C_k may depend on k (and is, of course, different in each occurrence). Recalling our starting point (1.4), we discover that the approximation space $\mathbf{X}_h$ and the operator Π_h^E should be such that η_6^{alg} can be made sufficiently small (see (4.77)).

A few more comments concerning the above procedure are in order:

Remark 1.3 *(a) The basic estimate (1.3) is formulated in such a way that one is led to study* $((\mathbf{e}_h, \mathbf{v}_h))$ *with* $\mathbf{v}_h \in \mathbf{X}_h$ *in the* discrete *space* $\mathbf{X}_h$*. This seemingly innocuous choice has far reaching ramifications. First, one has* $\operatorname{curl} \Pi^{\mathrm{curl},*} \mathbf{v}_h = \operatorname{curl} \mathbf{v}_h = \operatorname{curl} \Pi_h^{\mathrm{curl},*} \mathbf{v}_h$*, which allows one to replace the stronger* $\|\cdot\|_{\mathrm{curl},\Omega,k}$ *norm by the weaker* L^2*-norm in the estimates of Step (v):* $\left\| \left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*} \right) \mathbf{v}_h \right\|_{\mathrm{curl},\Omega,k} = k \left\| \left(\Pi^{\mathrm{curl},*} - \Pi_h^{\mathrm{curl},*} \right) \mathbf{v}_h \right\|$ *and* $\left\| \left(I - \Pi_h^E \right) \Pi^{\mathrm{curl},*} \mathbf{v}_h \right\|_{\mathrm{curl},\Omega,k} = k \left\| \left(I - \Pi_h^E \right) \Pi^{\mathrm{curl},*} \mathbf{v}_h \right\|$*. Second, the commuting diagram property of* Π_h^E *and the (discrete) exact sequence property (1.8) are responsible for the "orthogonality" (1.10) (cf. Steps (i)—(iii)).*

(b) The L^2*-approximation properties of* Π_h^E *stipulated in (VIc) can be met because of the special structure of the space* $\mathbf{V}_{0,h}^*$*: first, as we discovered in (IV), functions from* $\mathbf{V}_0^*$ *are in fact in* $\mathbf{H}^1(\Omega)$*. Second, for functions* $\mathbf{v} \in \mathbf{V}_{0,h}^*$ *one has that* $\operatorname{curl} \mathbf{v} \in \operatorname{curl} \mathbf{X}_h$ *is a discrete object. For the specific case of Nédélec Type I elements of degree p, an operator* Π_h^E *is constructed on the reference tetrahedron in Theorem 8.3 (called* $\widehat{\Pi}_p^{\mathrm{curl},c}$ *there) that exploits these properties and leads to the quantitative estimate* $\eta_6^{\mathrm{alg}} = O(hk/p)$*. We flag at this point that, while the space* $\mathbf{V}_0^*$ *is a space of divergence-free functions, the operator* $\widehat{\Pi}_p^{\mathrm{curl},c}$ *is additionally defined for (elementwise) smooth (actually, elementwise* $\mathbf{H}^1(\mathrm{curl})$*) functions. This property will be needed in Section 1.3 below to argue the benefits of high order methods.* ■

1.3 Road Map: k-explicit Estimates

The argument outlined above does not take into account how the wavenumber k enters the estimates, which occurs in various places, for example, in the continuity of A_k and $((\cdot,\cdot))$, the stability of the map $\Pi^{\mathrm{curl},*}$, and the regularity properties of the solution $\mathbf{z}$ of the dual problem $A_k(\cdot, \mathbf{z}) = \left((\cdot, \Pi^{\mathrm{curl},*} \mathbf{v}_h)\right)$. Indeed, care is required as we only have the k-dependent continuity bounds (cf. Cor. 5.13)

$$|((\mathbf{v}, \mathbf{w}))| + |A_k(\mathbf{v}, \mathbf{w})| \le C k^3 \|\mathbf{v}\|_{\mathrm{curl},\Omega,k} \|\mathbf{w}\|_{\mathrm{curl},\Omega,k}. \tag{1.13}$$

1.3.1 Continuity of A_k, $((\cdot,\cdot))$ and Treatment of T_1

The fundamental ingredient for k-explicit bounds that are useful for the analysis of high-order FEM is the ability to decompose functions $\mathbf{u} \in \mathbf{X}$ into "high-frequency" parts $H_\Omega \mathbf{u}$ and "low-frequency" parts $L_\Omega \mathbf{u}$. An overaching theme of the present work is that the high-frequency component $H_\Omega \mathbf{u}$ leads to estimates *uniform* in k in the expected Sobolev norms; the low-frequency component $L_\Omega \mathbf{u}$ involves k-dependencies, but is smooth (even analytic), which can be exploited by high order approximation spaces. We note that such decompositions $\mathbf{u} = H_\Omega \mathbf{u} + L_\Omega \mathbf{u}$ of functions entail corresponding decompositions of sesquilinear forms such as A_k and $((\cdot,\cdot))$. The frequency splitting operators L_Ω and H_Ω are motivated by an analysis of the k-dependence of the continuity constants of A_k and $((\cdot,\cdot))$, e.g., in the bound $|A_k(\mathbf{u}, \mathbf{v})| \le C_{\mathrm{cont},k} \|\mathbf{u}\|_{\mathrm{curl},\Omega,k} \|\mathbf{v}\|_{\mathrm{curl},\Omega,k}$. One discovers that it is the capacity operator T_k that introduces a k-dependence in $C_{\mathrm{cont},k}$. Inspection of the series expansion of T_k in (5.7) (see in particular Lemma 5.3, which gives sharp bounds for the symbol of the operator T_k) shows that the k-dependence is due to the low-frequency parts of $\mathbf{u}_T$. Having identified these components as the culprits for unfavorable k-dependencies, we introduce in Definition 4.2 the low-frequency operator $L_\Omega : \mathbf{X} \to \mathbf{X}$ and the high-frequency operator $H_\Omega = I - L_\Omega$ that have, for the case $\Omega = B_1(0)$ considered here, the following properties:

(VII) (stability) $\|L_\Omega \mathbf{u}\|_{\mathrm{curl},\Omega,k} \le \|\mathbf{u}\|_{\mathrm{curl},\Omega,k}$ and $\|H_\Omega \mathbf{u}\|_{\mathrm{curl},\Omega,k} \le 2 \|\mathbf{u}\|_{\mathrm{curl},\Omega,k}$ (cf. (5.27))

(VIII) (smoothness) $L_\Omega \mathbf{u}$ is analytic. Specifically, there are C, α, $\gamma > 0$ independent of k and $\mathbf{u}$ such that $L_\Omega \mathbf{u} \in \mathcal{A}(C k^\alpha \|\mathbf{u}\|_{\mathrm{curl},\Omega,k}, \gamma, \Omega)$ with the analyticity class $\mathcal{A}$ given by Def. 2.5 (cf. Theorem 5.9).

(IX) (k-uniform continuity at the expense of a compact perturbation) For some $C > 0$ independent of k (cf. Prop. 5.12 and Lemma 4.6 in conjunction with Cor. 5.13):

$$|((H_\Omega \mathbf{u}, \mathbf{v}))| + |((\mathbf{v}, H_\Omega \mathbf{u}))| \le C \|\mathbf{u}\|_{\mathrm{curl},\Omega,k} \|\mathbf{v}\|_{\mathrm{curl},\Omega,k}, \tag{1.14}$$

$$|A_k(H_\Omega \mathbf{u}, \mathbf{v})| + |A_k(\mathbf{v}, H_\Omega \mathbf{u})| \le C \|\mathbf{u}\|_{\mathrm{curl},\Omega,k} \|\mathbf{v}\|_{\mathrm{curl},\Omega,k}. \tag{1.15}$$

The refined continuity properties of A_k and $((\cdot,\cdot))$ given in (IX) allow us to estimate the terms T_1 in the basic error estimate (1.3) explicitly in k. Abbreviating $\mathbf{v} := \mathbf{u} - \mathbf{w}_h$ and decomposing $\mathbf{v}^{\mathrm{low}} := L_\Omega \mathbf{v}$ and $\mathbf{v}^{\mathrm{high}} := H_\Omega \mathbf{v}$

we write

$$T_1 = \operatorname{Re}\left(A_k(\mathbf{e}_h,\mathbf{v}) + 2\left((\mathbf{e}_h,\mathbf{v})\right)\right) = \operatorname{Re}\left(A_k(\mathbf{e}_h,H_\Omega\mathbf{v}) + 2\left((\mathbf{e}_h,H_\Omega\mathbf{v})\right)\right) + \operatorname{Re}\left(A_k(\mathbf{e}_h,L_\Omega\mathbf{v}) + 2\left((\mathbf{e}_h,L_\Omega\mathbf{v})\right)\right)$$
$$= \underbrace{\operatorname{Re}\left(A_k(\mathbf{e}_h,H_\Omega\mathbf{v}) + 2\left((\mathbf{e}_h,H_\Omega\mathbf{v})\right)\right)}_{=:T_{1,1}} + \underbrace{\operatorname{Re}(\operatorname{curl}\mathbf{e}_h,\operatorname{curl}L_\Omega\mathbf{v})}_{=:T_{1,2}} + \operatorname{Re}\Big(\underbrace{-\,\mathrm{i}\,k\left(T_k\mathbf{e}_h^{\operatorname{curl}},(L_\Omega\mathbf{v})^{\operatorname{curl}}\right)_\Gamma + ((\mathbf{e}_h,L_\Omega\mathbf{v}))}_{=:T_{1,3}(\mathbf{e}_h,L_\Omega\mathbf{v})}\Big).$$

The sesquilinear forms in $T_{1,1}$ and $T_{1,2}$ have good continuity properties (cf. (IX) and (VII) respectively) and can be estimated with k-independent constants. The term $T_{1,3}$ is amenable to a treatment by a duality argument: Let $\boldsymbol{\psi}\in\mathbf{X}$ solve $A_k(\cdot,\boldsymbol{\psi}) = T_{1,3}(\cdot,L_\Omega\mathbf{v})$. By Galerkin orthogonality satisfied by $\mathbf{e}_h$ and the stability estimate (1.13), one arrives at

$$|T_{1,3}\left(L_\Omega\left(\mathbf{e}_h\right),\mathbf{u}-\mathbf{w}_h\right)| = |A_k(\mathbf{e}_h,\boldsymbol{\psi})| \le Ck^3\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k}\inf_{\boldsymbol{\psi}_h\in\mathbf{X}_h}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{\operatorname{curl},\Omega,k}. \tag{1.16}$$

Since $L_\Omega(\mathbf{u}-\mathbf{w}_h)$ is an analytic function by (VIII) and the geometry $\Gamma=\partial B_1(0)$ is analytic so is the dual solution $\boldsymbol{\psi}$. As discussed in Proposition 7.5, one has the following analytic regularity assertion:

(X) Given $\mathbf{r}\in\mathbf{X}$, the solutions $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2\in\mathbf{X}$ of $A_k(\cdot,\boldsymbol{\psi}_1) = T_{1,3}(\cdot,L_\Omega\mathbf{r})$ and $A_k(\cdot,\boldsymbol{\psi}_2) = ((\cdot,L_\Omega\mathbf{r}))$ are analytic in $\overline{\Omega}$ and satisfy $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2\in\mathcal{A}(Ck^\alpha\|\mathbf{r}\|_{\operatorname{curl},\Omega,k},\gamma)$ for some C, γ, $\alpha\ge 0$ independent of k and $\mathbf{r}$. The analyticity classes $\mathcal{A}$ are introduced in Def. 2.5.

Since, by (X), the solution $\boldsymbol{\psi}$ in (1.16) is analytic, exponential approximation properties of hp-FEM spaces will be able to offset the algebraic factor k^3 in (1.16). Indeed, we will show in Lemma 8.5, (ii) for Nédélec elements of degree p that the infimum in (1.16) decays exponentially in p (provided that kh/p is sufficient small).

1.3.2 Treatment of T_2: the k-explicit Duality Argument for $\Pi^{\operatorname{curl},*}\mathbf{v}_h$

The analysis of the terms $T_2 = \left((\mathbf{e}_h,\Pi^{\operatorname{curl},*}\mathbf{v}_h)\right)$ and $T_3 = \left(\left(\mathbf{e}_h,\Pi^{\operatorname{curl},*}\mathbf{v}_h - \Pi_h^{\operatorname{curl},*}\mathbf{v}_h\right)\right)$ and arising in (1.7) requires us to make the decompositions $\mathbf{v}_h = \Pi^{\operatorname{curl},*}\mathbf{v}_h + \Pi^{\nabla,*}\mathbf{v}_h = \Pi_h^{\operatorname{curl},*}\mathbf{v}_h + \Pi_h^{\nabla,*}\mathbf{v}_h$ in a more careful, k-dependent way. The stability property (IX) implies $\|\Pi^{\nabla,*}H_\Omega\mathbf{v}\|_{\operatorname{curl},\Omega,k}\le C\|H_\Omega\mathbf{v}\|_{\operatorname{curl},\Omega,k}\le C\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}$ with $C>0$ independent of k so that

$$\|\Pi^{\operatorname{curl},*}H_\Omega\mathbf{v}\|_{\operatorname{curl},\Omega,k}\le C\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}, \tag{1.17}$$

again with $C>0$ independent of k (cf. also (4.24b)). These favorable estimates for $H_\Omega\mathbf{v}$ instead of $\mathbf{v}$ directly suggest that we should study, for $\mathbf{v}_h\in\mathbf{X}_h$, the following decompositions instead of (1.5) (1.6):

$$\mathbf{v}_h = \Pi^{\operatorname{comp},*}\mathbf{v}_h + \Pi^{\nabla,*}H_\Omega\mathbf{v}_h \quad\text{with}\quad \Pi^{\operatorname{comp},*} := L_\Omega + \Pi^{\operatorname{curl},*}H_\Omega, \tag{1.18}$$
$$\mathbf{v}_h = \Pi_h^{\operatorname{comp},*}\mathbf{v}_h + \Pi_h^{\nabla,*}H_\Omega\mathbf{v}_h \quad\text{with}\quad \Pi_h^{\operatorname{comp},*} := L_\Omega + \Pi_h^{\operatorname{curl},*}H_\Omega. \tag{1.19}$$

The duality argument for $T_2 = ((\mathbf{e}_h,\Pi^{\operatorname{comp},*}\mathbf{v}_h)) = ((\mathbf{e}_h,L_\Omega\mathbf{v}_h)) + \left((\mathbf{e}_h,\Pi^{\operatorname{curl},*}H_\Omega\mathbf{v}_h)\right)$ is split into two duality arguments. For the first term, one observes again that $L_\Omega\mathbf{v}_h$ is analytic and so will be the appropriate dual solution by (X), which in turn means that exponential approximability of hp-FEM space can be brought to bear. For the second term, the duality argument requires much more care since $\Pi^{\operatorname{curl},*}H_\Omega\mathbf{v}_h$ has only finite regularity. We have (cf. Prop. 7.2):

(XI) The solution $\boldsymbol{\psi}$ of $A_k(\cdot,\boldsymbol{\psi}) = \left((\cdot,\Pi^{\operatorname{curl},*}H_\Omega\mathbf{v}_h)\right)$ can be decomposed as $\boldsymbol{\psi} = \boldsymbol{\psi}_{H^2} + \boldsymbol{\psi}_{\mathcal{A}}$ with $k^2\|\boldsymbol{\psi}_{H^2}\| + \|\boldsymbol{\psi}_{H^2}\|_{\mathbf{H}^2(\Omega)}\le C\|H_\Omega\mathbf{v}_h\|_{\operatorname{curl},\Omega,k}\le C\|\mathbf{v}_h\|_{\operatorname{curl},\Omega,k}$ and $\boldsymbol{\psi}_{\mathcal{A}}\in\mathcal{A}(Ck^\alpha\|H_\Omega\mathbf{v}_h\|_{\operatorname{curl},\Omega,k},\gamma,\Omega)$ for some C, γ, $\alpha\ge 0$ independent of k (cf. Def. 2.5 for the definition of the analyticity class $\mathcal{A}$).

The decomposition of (XI) into a part $\boldsymbol{\psi}_{H^2}$ with finite regularity in conjunction with k-uniform control of the second derivatives and an analytic part $\boldsymbol{\psi}_{\mathcal{A}}$ is shown in Section 7.2; it relies on a solution formula based on Green's function for the Helmholtz equation and the decomposition is then inferred from the one developed in [41].

1.3.3 Treatment of T_3: Estimating $\left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h$

For the final term, $T_3 = \left(\left(\mathbf{e}_h, \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right)$, a new type of duality argument appears. We start by writing

$$T_3 = \left(\left(\mathbf{e}_h, \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right) = \left(\left(\mathbf{e}_h, L_\Omega \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right) + \left(\left(\mathbf{e}_h, H_\Omega \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right)$$
$$=: T_{3,1} + T_{3,2}.$$

Exploiting the analyticity of $L_\Omega \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h$, the first term, $T_{3,1}$ can be treated by a duality argument as in Section 1.3.1. For the second term, $T_{3,2}$, we use (IX) to estimate

$$\begin{aligned}|T_{3,2}| &= \left|\left(\left(\mathbf{e}_h, H_\Omega \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right)\right| \overset{(1.14)}{\leq} C\|\mathbf{e}_h\|_{\text{curl},\Omega,k}\| \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\|_{\text{curl},\Omega,k} \\ &= C\|\mathbf{e}_h\|_{\text{curl},\Omega,k} \left(k \left\|\left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right\|\right),\end{aligned}$$

where, in the last step, we used $\operatorname{curl}\left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h = 0$. The term $k \left\|\left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right\|$ is estimated by

$$\begin{aligned}\left(k \left\|\left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right\|\right)^2 &\overset{(\text{II})}{\leq} \operatorname{Re}\left(\left(\left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h, \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right) \\ &\overset{(1.10)}{=} \operatorname{Re}\left(\left(\left(I - \Pi_h^E\right) \Pi^{\text{comp},*}\mathbf{v}_h, \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right) \\ &= \operatorname{Re}\left(\left(H_\Omega \left(I - \Pi_h^E\right) \Pi^{\text{comp},*}\mathbf{v}_h, \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right) \\ &\quad + \operatorname{Re}\left(\left(L_\Omega \left(I - \Pi_h^E\right) \Pi^{\text{comp},*}\mathbf{v}_h, \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\right)\right) =: T_{4,1} + T_{4,2}.\end{aligned}$$

From (1.14) in (IX), we get $|T_{4,1}| \leq C \left(k\| \left(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h\|\right) \left(k\| \left(I - \Pi_h^E\right) \Pi^{\text{comp},*}\mathbf{v}_h\|\right)$. We remark that the above argument glossed over a minor point: In view of the modified definition of the decomposition (1.18), (1.19), we have to require that the operator Π_h^E is additionally defined on the space of smooth functions (in (VI), the operator Π_h^E is only defined on $\mathbf{V}_{0,h}^* + \mathbf{X}_h$) and satisfy some appropriate stability properties. The term $\left\|\left(I - \Pi_h^E\right) \Pi^{\text{comp},*}\mathbf{v}_h\right\|$ can be estimated as follows in view of the definition (1.18):

$$\left\|\left(I - \Pi_h^E\right) \Pi^{\text{comp},*}\mathbf{v}_h\right\| \leq \left\|\left(I - \Pi_h^E\right) L_\Omega\mathbf{v}_h\right\| + \left\|\left(I - \Pi_h^E\right) \Pi^{\text{curl},*} H_\Omega\mathbf{v}_h\right\| =: T_{5,1} + T_{5,2}.$$

For Nédélec elements of degree p, Theorem 8.3 provides an operator Π_h^E (its restriction to the reference element $\widehat{K}$ is denoted there $\widehat{\Pi}_p^{\text{curl},c}$) that is also defined on (elementwise) smooth functions and has good polynomial approximation properties. In particular, by the analyticity of $L_\Omega\mathbf{v}_h$, the term $T_{5,1}$ is exponentially small in the polynomial degree p for Nédélec elements. The term $T_{5,2}$ can be controlled by the assumption (VIc) and the stability bound (1.17) as

$$T_{5,2} \overset{(VIc)}{\leq} \eta_6^{\text{alg}}\|\Pi^{\text{curl},*} H_\Omega\mathbf{v}_h\|_{\text{curl},\Omega,k} \overset{(1.17)}{\leq} \eta_6^{\text{alg}} C\|\mathbf{v}_h\|_{\text{curl},\Omega,k}.$$

The term $T_{4,2}$ requires a duality argument that exploits the orthogonality property (1.10). Specifically, the dual problem is to find $\psi \in H^1(\Omega)$ such that

$$\left(\left(\nabla\psi, \nabla\widetilde{\psi}\right)\right) = \left(\left(L_\Omega \left(I - \Pi_h^E\right) \Pi^{\text{comp},*}\mathbf{v}_h, \nabla\widetilde{\psi}\right)\right) \qquad \forall\widetilde{\psi} \in H^1(\Omega). \tag{1.20}$$

Solvability is ensured by (II). The analyticity of $L_\Omega \left(I - \Pi_h^E\right) \Pi^{\text{comp},*}\mathbf{v}_h$ and $\partial\Omega$ give that ψ is analytic; we have by Proposition 7.4 (problem (1.20) is of Type 2 discussed in Sec. 7.1):

(XII) The solution ψ of the problem (1.20) belongs to an analyticity class $\psi \in \mathcal{A}(Ck^\alpha\|\mathbf{v}_h\|_{\text{curl},\Omega,k}, \gamma, \Omega)$ for some C, α, $\gamma \geq 0$ independent of k

We obtain, noting that $(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*})\mathbf{v}_h$ satisfies the same orthogonality condition (1.10) as the difference $(\Pi^{\text{curl},*} - \Pi_h^{\text{curl},*})\mathbf{v}_h$,

$$\begin{aligned}T_{4,2} &= \left(\left(\nabla\psi, (\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*})\mathbf{v}_h\right)\right) \overset{(1.10)}{=} \inf_{\psi_h\in S_h} \left(\left(\nabla(\psi - \psi_h), (\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*})\mathbf{v}_h\right)\right) \\ &\overset{(1.13)}{\leq} Ck^3 \inf_{\psi_h\in S_h} \|\nabla(\psi - \psi_h)\|_{\text{curl},\Omega,k}\|(\Pi^{\text{comp},*} - \Pi_h^{\text{comp},*})\mathbf{v}_h\|_{\text{curl},\Omega,k}\end{aligned}$$

$$= Ck^3 \inf_{\psi_h \in S_h} (k\|\nabla(\psi - \psi_h)\|)(k\|(\Pi^{\mathrm{comp},*} - \Pi_h^{\mathrm{comp},*})\mathbf{v}_h\|);$$

a more detailed argument can be found in the proof of Prop. 6.1.

The main result of the present work is quasi-optimality of the $\mathbf{H}(\Omega, \mathrm{curl})$-conforming discretization: In Theorem 4.15, we present a fairly abstract convergence result (which is not fully explicit in k). In Theorem 4.17 we consider high order Nédélec elements and the specific situation of the unit ball $B_1(0)$ and show quasi-optimality of the Galerkin discretization under the scale resolution condition (1.1).

2 Maxwell's Equations

In Sections 2.1 and 2.2 we introduce the strong form of the Maxwell problem first in the full domain and then in an equivalent way on a bounded domain. At this stage we are vague concerning the precise function spaces and mapping properties of trace operators. The variational formulation of the problem in a bounded domain is given in Section 2.4, where also the appropriate function spaces are introduced.

2.1 Maxwell's Equations in the Full Space $\mathbb{R}^3$

We consider the solution of the Maxwell equations in the full space $\mathbb{R}^3$ with Silver-Müller radiation conditions at infinity. The angular frequency is denoted by ω, the electric permittivity by ε, and the magnetic permeability by μ. We formulate the problem in terms of the wavenumber $k = \omega\sqrt{\varepsilon\mu}$, the scaled magnetic field $\tilde{\mathbf{H}} = \sqrt{\frac{\mu}{\varepsilon}}\mathbf{H}$, and the scaled electric charge density $\tilde{\mathbf{j}} = \sqrt{\mu/\varepsilon}\mathbf{j}$: Find the electric field $\mathbf{E}$ and the scaled magnetic field $\tilde{\mathbf{H}}$ such that

$$\begin{aligned} &\operatorname{curl}\mathbf{E} - \mathrm{i}\,k\tilde{\mathbf{H}} = \mathbf{0} \quad \text{and} \quad \operatorname{curl}\tilde{\mathbf{H}} + \mathrm{i}\,k\mathbf{E} = \tilde{\mathbf{j}} \quad \text{in } \mathbb{R}^3, \\ &\left|\mathbf{E} - \tilde{\mathbf{H}} \times \frac{\mathbf{x}}{r}\right| \leq \frac{c}{r^2} \quad \text{and} \quad \left|\mathbf{E} \times \frac{\mathbf{x}}{r} + \tilde{\mathbf{H}}\right| \leq \frac{c}{r^2} \quad \text{for } r = \|\mathbf{x}\| \to \infty \end{aligned} \tag{2.1}$$

is satisfied in a weak sense. Throughout the paper we assume that the data $\tilde{\mathbf{j}}$ satisfies the following Assumption 2.1a) which is sufficient to prove quasi-optimality of the Galerkin discretization (cf. Theorems 4.15, 4.17) while further assumptions on $\tilde{\mathbf{j}}$ are needed to prove convergence *rates* (cf. Corollary 4.18).

Assumption 2.1 *a) The scaled electric charge density* $\tilde{\mathbf{j}}$ *is a compactly supported distribution (functional on the space* $\mathbf{H}_{\mathrm{loc}}\left(\mathrm{curl}, \mathbb{R}^3\right)$ *defined in Section 2.3) in the sense that there exists a bounded, smooth Lipschitz domain* $\Omega \subset \mathbb{R}^3$ *with simply connected boundary* $\Gamma := \partial\Omega$ *that satisfies* $\operatorname{supp}\tilde{\mathbf{j}} \subset \Omega$. *We denote by* $\mathbf{n}$ *the unit normal vector on the boundary* Γ *oriented such that it points into the unbounded exterior* $\Omega^+ := \mathbb{R}^3\backslash\overline{\Omega}$.
b) The wavenumber k *is considered as a real parameter in the range*[2]

$$k \geq 1. \tag{2.2}$$

2.2 Reformulation on a Bounded Domain

Assumption 2.1 allows us to formulate problem (2.1) in an equivalent way as a transmission problem. For this we have to introduce in (2.3) the trace operators Π_T and γ_T, which map sufficiently smooth functions $\mathbf{u}$ in $\overline{\Omega}$ to tangential fields on the surface Γ while the trace operators Π_T^+ and γ_T^+ denote the corresponding traces for function $\mathbf{u}^+$ in the exterior domain $\overline{\Omega^+}$:

$$\begin{array}{ll} \Pi_T : \mathbf{u} \mapsto \mathbf{n} \times (\mathbf{u}|_\Gamma \times \mathbf{n}), & \gamma_T : \mathbf{u} \mapsto \mathbf{u}|_\Gamma \times \mathbf{n}, \\ \Pi_T^+ : \mathbf{u}^+ \mapsto \mathbf{n} \times (\mathbf{u}^+|_\Gamma \times \mathbf{n}), & \gamma_T^+ : \mathbf{u}^+ \mapsto \mathbf{u}^+|_\Gamma \times \mathbf{n}. \end{array} \tag{2.3}$$

This allows us to define the jumps for a sufficiently smooth functions $\mathbf{w}$ in the interior and $\mathbf{w}^+$ in the exterior domain:

$$\left[\left(\mathbf{w}, \mathbf{w}^+\right)\right]_{0,\Gamma} := \gamma_T\mathbf{w} - \gamma_T^+\mathbf{w}^+, \quad \left[\left(\mathbf{w}, \mathbf{w}^+\right)\right]_{1,\Gamma} := \gamma_T \operatorname{curl}\mathbf{w} - \gamma_T^+ \operatorname{curl}\mathbf{w}^+. \tag{2.4}$$

[2]The condition $k \geq 1$ can be replaced by $k \geq k_0 > 0$. Our estimates remain valid for all choices of $k_0 > 0$. The constants in the estimates are uniform for all $k \geq k_0$ while they depend continuously on k_0 and, possibly, become large as $k_0 \to 0$. We use (2.2) simply to reduce technicalities.

With this notation, the problem (2.1) takes the form: Find $\mathbf{E}$, $\mathbf{E}^+$, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{H}}^+$ such that

$$\begin{aligned}
\operatorname{curl}\mathbf{E} - \mathrm{i}\,k\tilde{\mathbf{H}} &= \mathbf{0}, & \operatorname{curl}\tilde{\mathbf{H}} + \mathrm{i}\,k\mathbf{E} &= \tilde{\mathbf{j}} \quad \text{in } \Omega, && \text{(2.5a)}\\
\operatorname{curl}\mathbf{E}^+ - \mathrm{i}\,k\tilde{\mathbf{H}}^+ &= \mathbf{0}, & \operatorname{curl}\tilde{\mathbf{H}}^+ + \mathrm{i}\,k\mathbf{E}^+ &= \mathbf{0} \quad \text{in } \Omega^+, && \text{(2.5b)}\\
\left[\left(\mathbf{E},\mathbf{E}^+\right)\right]_{0,\Gamma} &= 0, & \left[\left(\mathbf{E},\mathbf{E}^+\right)\right]_{1,\Gamma} &= \mathbf{0}, && \text{(2.5c)}\\
\left|\mathbf{E}^+ - \tilde{\mathbf{H}}^+ \times \frac{\mathbf{x}}{r}\right| &\le \frac{c}{r^2}, & \left|\mathbf{E}^+ \times \frac{\mathbf{x}}{r} + \tilde{\mathbf{H}}^+\right| &\le \frac{c}{r^2} \quad \text{for } r = \|\mathbf{x}\| \to \infty. && \text{(2.5d)}
\end{aligned}$$

The key role for formulating this problem as an equation on the bounded domain Ω is played by the capacity operator T_k. This operator associates to $\mathbf{g}_T \in \mathbf{H}_{\text{curl}}^{-1/2}(\Gamma)$ the value of $\gamma_T^+\tilde{\mathbf{H}}^+$ on Γ where $\left(\mathbf{E}^+, \tilde{\mathbf{H}}^+\right)$ solves the homogeneous Maxwell problem in the exterior domain Ω^+ with Silver-Müller radiation conditions at ∞ (i.e., (2.5b), (2.5d)) together with Dirichlet boundary conditions $\gamma_T\mathbf{E}^+ = \mathbf{g}_T \times \mathbf{n}$. That is, $T_k\mathbf{g}_T := \gamma_T^+\tilde{\mathbf{H}}^+$.

Remark 2.2 *From [47, Lemma 5.4.3, Thm. 5.4.6][3] we conclude that the exterior homogeneous Maxwell equations with given Dirichlet data* $\mathbf{g} \in \mathbf{H}_{\text{div}}^{-1/2}(\Gamma)$, *i.e.,* $\gamma_T^+\mathbf{E}^+ = \mathbf{g}$ *on* Γ, *for the electric field has a weak solution* $\mathbf{E}^+ \in \mathbf{H}_{\text{loc}}(\text{curl}, \Omega^+)$, *which is unique and satisfies*

$$\left\|\mathbf{E}^+\right\|_{\text{curl},B_R(0)\cap\Omega^+,1} \le C_{R,\Omega}\,\|\mathbf{g}\|_{\mathbf{H}_{\text{div}}^{-1/2}(\Gamma)},$$

where $B_R(0)$ *is a ball with radius* R *centered at* 0 *such that* $\overline{\Omega} \subset B_R(0)$ *and* $C_{R,\Omega}$ *is a constant which only depends on* Ω *and* R.
This implies that the capacity operator $T_k : \mathbf{H}_{\text{curl}}^{-1/2}(\Gamma) \to \mathbf{H}_{\text{div}}^{-1/2}(\Gamma)$ *is continuous.* ■

The Maxwell equations on the bounded domain are given by

$$\begin{aligned}
&\operatorname{curl}\mathbf{E} - \mathrm{i}\,k\tilde{\mathbf{H}} = \mathbf{0}, \qquad \operatorname{curl}\tilde{\mathbf{H}} + \mathrm{i}\,k\mathbf{E} = \tilde{\mathbf{j}} \quad \text{in } \Omega,\\
&\gamma_T \operatorname{curl}\mathbf{E} - \mathrm{i}\,kT_k\Pi_T\mathbf{E} = \mathbf{0} \qquad\qquad\qquad \text{on } \Gamma.
\end{aligned}$$

Eliminating $\tilde{\mathbf{H}}$ from these equations we arrive at the Maxwell equations for the electric field on a bounded domain Ω

$$\begin{aligned}
\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} &= \mathrm{i}\,k\tilde{\mathbf{j}} && \text{in } \Omega,\\
\gamma_T \operatorname{curl}\mathbf{E} - \mathrm{i}\,kT_k\Pi_T\mathbf{E} &= \mathbf{0} && \text{on } \Gamma.
\end{aligned} \tag{2.7}$$

2.3 Sobolev Spaces in Ω and on Γ

We introduce the pertinent function spaces.

2.3.1 Sobolev Spaces in Ω

By $H^s(\Omega)$ we denote the usual Sobolev spaces of index $s \ge 0$ with norm $\|\cdot\|_{H^s(\Omega)}$. The closure of $C_0^\infty(\Omega)$ functions with respect to $\|\cdot\|_{H^s(\Omega)}$ is denoted by $H_0^s(\Omega)$. For $s \ge 0$, the dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. If the functions are vector-valued we indicate this by writing $\mathbf{H}^s(\Omega)$, $\mathbf{H}_0^s(\Omega)$. For details we refer to [1].
The *energy space* for the electric field is given by

$$\mathbf{X} := \mathbf{H}(\Omega, \text{curl}) := \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{curl}\mathbf{u} \in \mathbf{L}^2(\Omega)\right\} \tag{2.8}$$

equipped with the indexed scalar product and norm

$$(\mathbf{f},\mathbf{g})_{\text{curl},\Omega,k} := (\operatorname{curl}\mathbf{f}, \operatorname{curl}\mathbf{g}) + k^2(\mathbf{f},\mathbf{g}) \quad \text{and} \quad \|\mathbf{f}\|_{\text{curl},\Omega,k} := (\mathbf{f},\mathbf{f})_{\text{curl},\Omega,k}^{1/2}, \tag{2.9}$$

where $(\cdot,\cdot)$ denotes the $\mathbf{L}^2(\Omega)$-scalar product

$$(\mathbf{f},\mathbf{g}) := \int_\Omega \langle\mathbf{f},\mathbf{g}\rangle. \tag{2.10}$$

Here, $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{C}^3$ (with complex conjugation in the second argument). We also introduce the space

$$\mathbf{H}(\Omega, \text{div}) := \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{u} \in L^2(\Omega)\right\}. \tag{2.11}$$

For *unbounded* domains $D \subset \mathbb{R}^3$ we denote $\mathbf{H}_{\text{loc}}(D, \text{curl})$ the space of all distributions $\mathbf{f}$ with the property that $\varphi\mathbf{f} \in \mathbf{H}(D, \text{curl})$ for all smooth, compactly supported functions $\varphi \in C_0^\infty(\mathbb{R}^3)$.

[3] The function spaces appearing in these statements will be introduced in Section 2.3.

2.3.2 Sobolev Spaces on Γ

The Sobolev spaces on the boundary Γ are denoted by $H^s(\Gamma)$ for scalar-valued functions and by $\mathbf{H}^s(\Gamma)$ for vector-valued functions. A formal definition may be found in [33]; however, below and throughout this work, we will use the characterization in terms of expansions via eigenfunctions of the Laplace–Beltrami operator. We will need the space $\mathbf{L}^2_T(\Gamma)$ of tangential vector fields given by

$$\mathbf{L}^2_T(\Gamma) := \{\mathbf{v} \in \mathbf{L}^2(\Gamma) \mid \langle \mathbf{n}, \mathbf{v}\rangle = 0 \text{ on } \Gamma\}. \tag{2.12}$$

For a sufficiently smooth scalar-valued function u and vector-valued function $\mathbf{v}$ on Γ, the constant (along the normal direction) extensions into a sufficiently small three-dimensional neighborhood $\mathcal{U}$ of Γ is denoted by u^* and $\mathbf{v}^*$. The *surface gradient* ∇_Γ, the *tangential curl* $\overrightarrow{\operatorname{curl}}_\Gamma$, and the *surface divergence* $\operatorname{div}_\Gamma$ are defined by (cf., e.g., [47], [9])

$$\nabla_\Gamma u := (\nabla u^\star)|_\Gamma, \quad \overrightarrow{\operatorname{curl}}_\Gamma u := \nabla_\Gamma u \times \mathbf{n}, \quad \text{and} \quad \operatorname{div}_\Gamma \mathbf{v} = (\operatorname{div} \mathbf{v}^*)|_\Gamma \qquad \text{on } \Gamma. \tag{2.13}$$

The scalar counterpart of the tangential curl, $\overrightarrow{\operatorname{curl}}_\Gamma$, is the *surface curl*

$$\operatorname{curl}_\Gamma \mathbf{v} := \langle (\operatorname{curl} \mathbf{v}^*)|_\Gamma, \mathbf{n}\rangle \qquad \text{on } \Gamma. \tag{2.14}$$

The composition of the surface divergence and surface gradient leads to the *scalar Laplace-Beltrami operator*

$$\Delta_\Gamma u = \operatorname{div}_\Gamma \nabla_\Gamma u. \tag{2.15}$$

From [47, (2.5.197)] we have the relation

$$\operatorname{div}_\Gamma (\mathbf{v} \times \mathbf{n}) = \operatorname{curl}_\Gamma \mathbf{v}. \tag{2.16}$$

The operator Δ_Γ is self-adjoint with respect to the $L^2(\Gamma)$ scalar product $(\cdot,\cdot)_\Gamma$ and positive semidefinite. It admits a countable sequence of eigenfunctions in $L^2(\Gamma)$ denoted by Y_ℓ^m such that

$$-\Delta_\Gamma Y_\ell^m = \lambda_\ell Y_\ell^m \qquad \text{for } \ell = 0, 1, \ldots \text{and } m \in \iota_\ell. \tag{2.17}$$

Here, ι_ℓ is a finite index set whose cardinality equals the multiplicity of the eigenvalue λ_ℓ, and we always assume that the eigenvalues λ_ℓ are distinct and ordered increasingly. We have $\lambda_0 = 0$ and for $\ell \geq 1$, they are real and positive and accumulate at infinity. By Assumption 2.1 the surface Γ is simply connected so that $\lambda_0 = 0$ is a simple eigenvalue. From [47, Sec. 5.4] we know that any distribution w, defined on the surface Γ, can formally be expanded with respect to the basis Y_ℓ^m as

$$w = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} w_\ell^m Y_\ell^m.$$

The space $H^s(\Gamma)$ can be characterized by

$$H^s(\Gamma) = \left\{ w \in (C^\infty(\Gamma))' \mid \|w\|^2_{H^s(\Gamma)} := \sum_{\ell=0}^{\infty} (\delta_{\ell,0} + \lambda_\ell)^s \sum_{m \in \iota_\ell} |w_\ell^m|^2 < \infty \right\} \tag{2.18}$$

with Kronecker's $\delta_{m,\ell}$. A norm on $H^s(\Gamma)$ is given by $\|\cdot\|_{H^s(\Gamma)}$.

Next, we define spaces of vector-valued functions. By [47, Sec. 5.4.1], every function $\mathbf{v}_T \in \mathbf{L}^2_T(\Gamma)$ can be written in the form

$$\mathbf{v}_T = \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} \left(v_\ell^m \overrightarrow{\operatorname{curl}}_\Gamma Y_\ell^m + V_\ell^m \nabla_\Gamma Y_\ell^m \right), \tag{2.19}$$

where the coefficients satisfy $\sum_{\ell=1}^{\infty} \lambda_\ell \sum_{m \in \iota_\ell} \left(|v_\ell^m|^2 + |V_\ell^m|^2 \right) < \infty$. We set

$$\|\mathbf{v}_T\|^2_{\mathbf{H}^s_T(\Gamma)} := \sum_{\ell=1}^{\infty} \lambda_\ell^{s+1} \sum_{m \in \iota_\ell} \left(|v_\ell^m|^2 + |V_\ell^m|^2 \right). \tag{2.20}$$

A tangential vector field $\mathbf{v}_T$ can be decomposed into a surface gradient and a surface curl part:

$$\mathbf{v}^\nabla := \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} V_\ell^m \nabla_\Gamma Y_\ell^m \quad \text{and} \quad \mathbf{v}^{\operatorname{curl}} := \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} v_\ell^m \overrightarrow{\operatorname{curl}}_\Gamma Y_\ell^m$$
$$\text{so that} \quad \mathbf{v}_T = \mathbf{v}^{\operatorname{curl}} + \mathbf{v}^\nabla. \tag{2.21}$$

Remark 2.3 *For gradient fields* $\nabla\varphi$ *we have* $(\nabla\varphi)^{\operatorname{curl}} = 0$ *and* $(\nabla\varphi)^{\nabla} = \nabla_\Gamma\varphi$. ■

The decomposition (2.21) allows us to express the operators $\operatorname{curl}_\Gamma$ and $\operatorname{div}_\Gamma$ and the corresponding norms in terms of the Fourier coefficients: for a tangential field $\mathbf{v}_T$ of the form (2.19), the surface divergence and surface gradient are defined (formally) as in [47, (5.4.18)-(5.4.21)]

$$\operatorname{div}_\Gamma \mathbf{v}_T = \sum_{\ell=1}^{\infty} \lambda_\ell \sum_{m\in\iota_\ell} V_\ell^m Y_\ell^m \quad \text{and} \quad \operatorname{curl}_\Gamma \mathbf{v}_T = \sum_{\ell=1}^{\infty} \lambda_\ell \sum_{m\in\iota_\ell} v_\ell^m Y_\ell^m. \tag{2.22}$$

The $H^s(\Gamma)$ norm (cf. (2.18)) of $\operatorname{curl}_\Gamma(\cdot)$ and $\operatorname{div}_\Gamma(\cdot)$ can accordingly be expressed in terms of the Fourier expansions:

$$\|\operatorname{curl}_\Gamma \mathbf{v}_T\|^2_{H^s(\Gamma)} = \sum_{\ell=1}^{\infty} \lambda_\ell^{s+2} \sum_{m\in\iota_\ell} |v_\ell^m|^2 \quad \text{and} \quad \|\operatorname{div}_\Gamma \mathbf{v}_T\|^2_{H^s(\Gamma)} = \sum_{\ell=1}^{\infty} \lambda_\ell^{s+2} \sum_{m\in\iota_\ell} |V_\ell^m|^2. \tag{2.23}$$

We define

$$\|\mathbf{v}_T\|^2_{-1/2,\operatorname{curl}_\Gamma} = \sum_{\ell=1}^{\infty} \lambda_\ell^{1/2} \sum_{m\in\iota_\ell} \left((1+\lambda_\ell)|v_\ell^m|^2 + |V_\ell^m|^2\right), \tag{2.24a}$$

$$\|\mathbf{v}_T\|^2_{-1/2,\operatorname{div}_\Gamma} = \sum_{\ell=1}^{\infty} \lambda_\ell^{1/2} \sum_{m\in\iota_\ell} \left(|v_\ell^m|^2 + (1+\lambda_\ell)|V_\ell^m|^2\right). \tag{2.24b}$$

The spaces $\mathbf{H}^{-1/2}_{\operatorname{curl}}(\Gamma)$ and $\mathbf{H}^{-1/2}_{\operatorname{div}}(\Gamma)$ allow for orthogonal decompositions on the surface Γ. From [47, (5.4.20), (5.4.21)] we conclude that

$$\begin{aligned} \mathbf{v}_T \in \mathbf{H}^{-1/2}_{\operatorname{div}}(\Gamma) &\iff \mathbf{v}_T \text{ is of the form (2.19) and } \|\mathbf{v}_T\|_{-1/2,\operatorname{div}_\Gamma} < \infty, \\ \mathbf{v}_T \in \mathbf{H}^{-1/2}_{\operatorname{curl}}(\Gamma) &\iff \mathbf{v}_T \text{ is of the form (2.19) and } \|\mathbf{v}_T\|_{-1/2,\operatorname{curl}_\Gamma} < \infty \end{aligned}$$

holds. The system $\left\{\nabla_\Gamma Y_\ell^m, \overrightarrow{\operatorname{curl}_\Gamma} Y_\ell^m\right\}$ forms an orthogonal basis in $\mathbf{L}^2_T(\Gamma)$ (cf. [47, § after (5.4.12)]) so that

$$\left(\mathbf{v}^\nabla, \mathbf{v}^{\operatorname{curl}}\right)_{\mathbf{L}^2_T(\Gamma)} = 0 \qquad \forall \mathbf{v} \in \mathbf{L}^2_T(\Gamma). \tag{2.25}$$

The following theorem shows that $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and $\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$ are the correct spaces to define continuous trace operators.

Theorem 2.4 *The trace mappings*

$$\Pi_T : \mathbf{X} \to \mathbf{H}^{-1/2}_{\operatorname{curl}}(\Gamma), \gamma_T : \mathbf{X} \to \mathbf{H}^{-1/2}_{\operatorname{div}}(\Gamma)$$

are continuous and surjective. Moreover, there exist continuous liftings $\mathcal{E}_{\operatorname{curl},}$: $\mathbf{H}^{-1/2}_{\operatorname{curl}}(\Gamma) \to \mathbf{X}$ *and* $\mathcal{E}_{\operatorname{div},}$: $\mathbf{H}^{-1/2}_{\operatorname{div}}(\Gamma) \to \mathbf{X}$ *for these trace spaces which are divergence-free.*

For a proof we refer to [13], [47, Thm. 5.4.2]. For a vector field $\mathbf{u} \in \mathbf{X}$, we will employ frequently the notation $\mathbf{u}_T := \Pi_T\mathbf{u}$. The continuity constant of Π_T is

$$C_\Gamma := \sup_{\mathbf{v}\in\mathbf{X}\setminus\{0\}} \frac{\|\Pi_T\mathbf{v}\|_{-1/2,\operatorname{curl}_\Gamma}}{\|\mathbf{v}\|_{\operatorname{curl},\Omega,1}}. \tag{2.26}$$

2.3.3 The Analyticity Classes $\mathcal{A}$

We introduce classes of analytic functions whose growth of the derivatives (as the order of differentiation grows) is controlled explicitly in terms of the wavenumber k. For smooth tensor-valued functions $\mathbf{u} = (\mathbf{u}_\mathbf{i})_{\mathbf{i}\in\mathbf{I}}$ on a subset $\omega \subset \mathbb{R}^d$, where $\mathbf{I}$ is a suitable finite index set and using the usual multi-index conventions $\boldsymbol{\alpha} = (\alpha_s)_{s=1}^d$, we set $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_d$, and abbreviate

$$|\nabla^n \mathbf{u}(x)|^2 = \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^d \\ |\boldsymbol{\alpha}|=n}} \sum_{\mathbf{i}\in\mathbf{I}} \binom{n}{\boldsymbol{\alpha}} \left|D^{\boldsymbol{\alpha}\boldsymbol{\alpha}} \mathbf{u}_\mathbf{i}(x)\right|^2, \qquad \binom{n}{\boldsymbol{\alpha}} = \frac{n!}{\alpha_1!\cdots\alpha_d!}, \quad D^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_d^{\alpha_d}. \tag{2.27}$$

We then define:

Definition 2.5 *For an open set $\omega \subset \mathbb{R}^d$ and constants C_1, $\gamma_1 > 0$, and wavenumber $k \geq 1$ (cf. (2.2)), we set*

$$\mathcal{A}(C_1, \gamma_1, \omega) := \left\{ \mathbf{u} \in L^2(\omega) \mid \|\nabla^n \mathbf{u}\|_{L^2(\omega)} \leq C_1 \gamma_1^n \max\{n+1, k\}^n \ \forall n \in \mathbb{N}_0 \right\},$$

$$\mathcal{A}^\infty(C_1, \gamma_1, \omega) := \left\{ \mathbf{u} \in L^\infty(\omega) \mid \|\nabla^n \mathbf{u}\|_{L^\infty(\omega)} \leq C_1 \gamma_1^n n! \quad \forall n \in \mathbb{N}_0 \right\}.$$

For the unit sphere Γ in $\mathbb{R}^3$ and constants C_1, γ_1, and the wavenumber $k \geq 1$, we set

$$\mathcal{A}(C_1, \gamma_1, \Gamma) := \left\{ \mathbf{f} \in \mathbf{L}^2_T(\Gamma) \mid \|\nabla^n_\Gamma \mathbf{f}\|_{L^2(\Gamma)} \leq C_1 \gamma_1^n \max\{n+1, k\}^n \ \forall n \in \mathbb{N}_0 \right\},$$

where ∇_Γ denotes the surface gradient as in (2.13) and the application of ∇^n_Γ to $\mathbf{f}$ is defined componentwise.

Membership in the analyticity class $\mathcal{A}$ is invariant under analytic changes of variables and multiplication by analytic functions:

Lemma 2.6 *Let $d \in \mathbb{N}$ and ω_1, $\omega_2 \subset \mathbb{R}^d$ be bounded, open sets. Let $g : \omega_1 \to \omega_2$ be a bijection and analytic on the closure $\overline{\omega_1}$: there are constants C_g, $C_{g,\mathrm{inv}}$, γ_g such that*

$$g \in \mathcal{A}^\infty(C_g, \gamma_g, \omega_1) \quad \text{and} \quad \|(g')^{-1}\|_{L^\infty(\omega_1)} \leq C_{g,\mathrm{inv}}.$$

Let f be analytic on the closure $\overline{\omega_2}$, i.e., $f \in \mathcal{A}^\infty(C_f, \gamma_f, \omega_2)$ for some C_f, γ_f. Let $\mathbf{u} \in \mathcal{A}(C_\mathbf{u}, \gamma_\mathbf{u}, \omega_2)$ for some $C_\mathbf{u}$, $\gamma_\mathbf{u}$. Then there are constants C', $\gamma' > 0$ depending solely on C_g, γ_g, $C_{g,\mathrm{inv}}$, $\gamma_\mathbf{u}$, γ_f, and d, such that $\widetilde{\mathbf{u}} := f \cdot (\mathbf{u} \circ g)$ satisfies $\widetilde{\mathbf{u}} \in \mathcal{A}(C' C_f C_\mathbf{u}, \gamma', \omega_1)$.

Proof. The case $d = 2$ is proved in [35, Lemma 4.3.1]. Inspection of the proof shows, as was already observed in [41, Lemma C.1], that it generalizes to arbitrary $d \in \mathbb{N}$. ∎

2.4 Variational Formulation of the Electric Maxwell Equations

We formulate (2.7) as a variational problem. We introduce the sesquilinear forms a_k, b_k, $A_k : \mathbf{X} \times \mathbf{X} \to \mathbb{C}$ by

$$\begin{aligned} &a_k(\mathbf{u}, \mathbf{v}) := (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) - k^2 (\mathbf{u}, \mathbf{v}), \quad b_k(\mathbf{u}_T, \mathbf{v}_T) := (T_k \mathbf{u}_T, \mathbf{v}_T)_\Gamma, \\ &A_k(\cdot, \cdot) := a_k(\cdot, \cdot) - \mathrm{i}\, k b_k(\Pi_T \cdot, \Pi_T \cdot). \end{aligned} \tag{2.28a}$$

Then, the weak form of the electric Maxwell equations on a bounded domain $\Omega \subset \mathbb{R}^3$ with transparent boundary conditions reads:

$$\text{given } \mathbf{F} \in \mathbf{X}' \quad \text{find } \mathbf{E} \in \mathbf{X} \quad \text{such that} \quad A_k(\mathbf{E}, \mathbf{v}) = F(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{X}. \tag{2.28b}$$

Note that the strong formulation (2.7) corresponds to the choice $F(\mathbf{v}) = \left(\mathrm{i}\, k \tilde{\mathbf{j}}, \mathbf{v}\right)$ in (2.28b).

Theorem 2.7 *Let Assumption 2.1 be satisfied. Let A_k have the form (2.28a). Then, for every $F \in \mathbf{X}'$, problem (2.28b) has a unique solution.*

Proof. Let $B_R(0)$ denote a ball centered at the origin with sufficiently large radius R such that $\overline{\Omega} \subset B_R(0)$. We consider the electric Maxwell equation in $B_R(0)$ of the form: Find $\mathbf{E}_R \in \mathbf{H}(B_R(0), \operatorname{curl})$ such that for all $\mathbf{v} \in \mathbf{H}(B_R(0), \operatorname{curl})$

$$(\operatorname{curl} \mathbf{E}_R, \operatorname{curl} \mathbf{v})_{L^2(B_R(0))} - k^2 (\mathbf{E}_R, \mathbf{v})_{L^2(B_R(0))} - \mathrm{i}\, k (T_{k,R} \mathbf{E}_{R,T}, \mathbf{v}_T)_{L^2(\partial B_R(0))} = F_R(\mathbf{v}), \tag{2.29}$$

where $T_{k,R}$ is the capacity operator for the exterior domain $\mathbb{R}^3 \backslash \overline{B_R(0)}$ and F_R is the extension of F by zero, i.e., $F_R(\mathbf{v}) := F(\mathbf{v}|_\Omega)$. In [47, Lem. 5.4.4 (with $\Gamma = \emptyset$ therein)] an ansatz $\mathbf{E}_R = \mathbf{u}_R + \nabla p_R$ is employed, where $\mathbf{u}_R$ and p_R are the solutions of a variational saddle point problem. In [47, Thm. 5.4.6], an inf-sup condition is proved for this saddle point problem which implies the well-posedness of (2.29). The construction implies that $\mathbf{E} := \mathbf{E}_R|_\Omega$ then satisfies (2.28b). On the other hand, every solution $\mathbf{E}$ of (2.28b) can be extended to a solution of (2.29) by employing the well-posedness of the exterior Dirichlet problem, [47, Thm. 5.4.6]. Since (2.29) has a unique solution also the solution of (2.28b) is unique. ∎

3 Discretization

3.1 Abstract Galerkin Discretization

Let $\mathbf{X}_h \subset \mathbf{X}$ denote a finite dimensional subspace. The Galerkin discretization of (2.28) reads: Find $\mathbf{E}_h \in \mathbf{X}_h$ such that

$$A_k(\mathbf{E}_h, \mathbf{v}_h) = F(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{X}_h. \tag{3.1}$$

For the error analysis we will impose an assumption (Assumption 4.14) on the space $\mathbf{X}_h$ by requiring the existence of a suitable projection onto the space $\mathbf{X}_h$. Also for the error analysis we will make use of a space S_h such that the following *exact sequence property* holds:

$$S_h \xrightarrow{\nabla} \mathbf{X}_h \xrightarrow{\operatorname{curl}} \operatorname{curl} \mathbf{X}_h \tag{3.2}$$

In the next section we will introduce the Nédélec space $\boldsymbol{\mathcal{N}}_p^I(\mathcal{T}_h)$; for the choice $\mathbf{X}_h = \boldsymbol{\mathcal{N}}_p^I(\mathcal{T}_h)$, we will perform the error analysis explicitly in the wavenumber k, the mesh width h and the polynomial degree p.

3.2 Curl-Conforming *hp*-Finite Element Spaces

The classical example of curl-conforming FE spaces are the Nédélec elements, [46]. We restrict our attention here to so-called "type I" elements (sometimes also referred to as the Nédélec-Raviart-Thomas element) on tetrahedra. These spaces are based on a regular, shape-regular triangulation $\mathcal{T}_h$ of $\Omega \subset \mathbb{R}^3$. That is, $\mathcal{T}_h$ satisfies:

(i) The (open) elements $K \in \mathcal{T}_h$ cover Ω, i.e., $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} \overline{K}$.

(ii) Associated with each element K is the *element map*, a C^1-diffeomorphism $F_K : \overline{\widehat{K}} \to \overline{K}$. The set $\widehat{K}$ is the *reference tetrahedron*.

(iii) Denoting $h_K = \operatorname{diam} K$, there holds, with some *shape-regularity constant* γ,

$$h_K^{-1} \|F_K'\|_{L^\infty(\widehat{K})} + h_K \|(F_K')^{-1}\|_{L^\infty(\widehat{K})} \leq \gamma. \tag{3.3}$$

(iv) The intersection of two elements is only empty, a vertex, an edge, a face, or they coincide (here, vertices, edges, and faces are the images of the corresponding entities on the reference tetrahedron $\widehat{K}$). The parametrization of common edges or faces are compatible. That is, if two elements K, K' share an edge (i.e., $F_K(e) = F_K(e')$ for edges e, e' of $\widehat{K}$) or a face (i.e., $F_K(f) = F_K(f')$ for faces f, f' of $\widehat{K}$), then $F_K^{-1} \circ F_{K'} : f' \to f$ is an affine isomorphism.

The following assumption assumes that the element map F_K can be decomposed as a composition of an affine scaling with an h-independent mapping. We adopt the setting of [41, Sec. 5] and assume that the element maps F_K of the regular, γ-shape regular triangulation $\mathcal{T}_h$ satisfy the following additional requirements:

Assumption 3.1 (normalizable regular triangulation) *Each element map F_K can be written as $F_K = R_K \circ A_K$, where A_K is an* affine *map and the maps R_K and A_K satisfy for constants C_{affine}, C_{metric}, $\gamma > 0$ independent of K:*

$$\begin{aligned} &\|A_K'\|_{L^\infty(\widehat{K})} \leq C_{\text{affine}} h_K, && \|(A_K')^{-1}\|_{L^\infty(\widehat{K})} \leq C_{\text{affine}} h_K^{-1} \\ &\|(R_K')^{-1}\|_{L^\infty(\widetilde{K})} \leq C_{\text{metric}}, && \|\nabla^n R_K\|_{L^\infty(\widetilde{K})} \leq C_{\text{metric}} \gamma^n n! \qquad \forall n \in \mathbb{N}_0. \end{aligned}$$

Here, $\widetilde{K} = A_K(\widehat{K})$ and $h_K > 0$ is the element diameter.

Remark 3.2 *A prime example of meshes that satisfy Assumption 3.1 are those patchwise structured meshes as described, for example, in [41, Ex. 5.1] or [35, Sec. 3.3.2]. These meshes are obtained by first fixing a macrotriangulation of Ω; the actual triangulation is then obtained as images of affine triangulations of the reference element.* ∎

On the reference tetrahedron $\widehat{K}$ we introduce the classical Nédélec type I and Raviart-Thomas elements of degree $p \geq 0$ (see, e.g., [43]):

$$\mathcal{P}_p(\widehat{K}) := \operatorname{span}\{x^\alpha \,|\, |\alpha| \leq p\}, \tag{3.4}$$

$$\boldsymbol{\mathcal{N}}_p^{\mathrm{I}}(\widehat{K}) := \{\mathbf{p}(x) + x \times \mathbf{q}(x) \,|\, \mathbf{p}, \mathbf{q} \in (\mathcal{P}_p(\widehat{K}))^3\}, \tag{3.5}$$

$$\mathbf{RT}_p(\widehat{K}) := \{\mathbf{p}(x) + xq(x) \,|\, \mathbf{p} \in (\mathcal{P}_p(\widehat{K}))^3, q \in \mathcal{P}_p(\widehat{K})\}. \tag{3.6}$$

The spaces $S_{p+1}(\mathcal{T}_h)$, $\boldsymbol{\mathcal{N}}_p^{\mathrm{I}}(\mathcal{T}_h)$, $\mathbf{RT}_p(\mathcal{T}_h)$, and $Z_p(\mathcal{T}_h)$ are then defined as in [43, (3.76), (3.77)] by transforming covariantly $\boldsymbol{\mathcal{N}}_p^{\mathrm{I}}(\widehat{K})$ and $\mathbf{RT}_p(\widehat{K})$ with the aid of the Piola transform:

$$S_{p+1}(\mathcal{T}_h) := \{u \in H^1(\Omega) \,|\, u|_K \circ F_K \in \mathcal{P}_{p+1}(\widehat{K})\}, \tag{3.7a}$$

$$\boldsymbol{\mathcal{N}}_p^{\mathrm{I}}(\mathcal{T}_h) := \{\mathbf{u} \in \mathbf{H}(\Omega, \operatorname{curl}) \,|\, (F_K')^T \mathbf{u}|_K \circ F_K \in \boldsymbol{\mathcal{N}}_p^{\mathrm{I}}(\widehat{K})\}, \tag{3.7b}$$

$$\mathbf{RT}_p(\mathcal{T}_h) := \{\mathbf{u} \in \mathbf{H}(\Omega, \operatorname{div}) \,|\, (\det F_K')(F_K')^{-1} \mathbf{u}|_K \circ F_K \in \mathbf{RT}_p(\widehat{K})\}, \tag{3.7c}$$

$$Z_p(\mathcal{T}_h) := \{u \in L^2(\Omega) \,|\, u|_K \circ F_K \in \mathcal{P}_p(\widehat{K})\}. \tag{3.7d}$$

A key property of these spaces is that we have the following exact sequence:

$$\mathbb{R} \longrightarrow S_{p+1}(\mathcal{T}_h) \xrightarrow{\nabla} \boldsymbol{\mathcal{N}}_p^{I}(\mathcal{T}_h) \xrightarrow{\operatorname{curl}} \mathbf{RT}_p(\mathcal{T}_p) \xrightarrow{\operatorname{div}} Z_p(\mathcal{T}_h) \tag{3.8}$$

4 Stability and Error Analysis

4.1 The Basic Error Estimate

4.1.1 Preliminaries

The basic error estimates for curl-conforming Galerkin discretization involve some k-dependent sesquilinear forms and corresponding k-dependent norms which, in turn, are based on Helmholtz decompositions on the surface Γ. With start this section with these preliminaries. For the proof of the basic error estimate (Thm. 4.13), we introduce the sesquilinear form $((\cdot,\cdot)) : \mathbf{X} \times \mathbf{X} \to \mathbb{C}$ by

$$((\mathbf{u}, \mathbf{v})) := k^2 (\mathbf{u}, \mathbf{v}) + \mathrm{i}\, k b_k \left(\mathbf{u}^\nabla, \mathbf{v}^\nabla\right). \tag{4.1}$$

We need some definiteness assumptions for the sesquilinear form $b_k(\cdot,\cdot)$. Throughout the paper, we will assume:

Assumption 4.1 *The sesquilinear form $b_k : \mathbf{X} \times \mathbf{X} \to \mathbb{C}$ of (2.28a) satisfies*

$$\begin{array}{lr} \operatorname{Im} b_k\left(\mathbf{u}^\nabla, \mathbf{u}^\nabla\right) \leq 0 \quad \text{and} \quad \operatorname{Im} b_k\left(\mathbf{u}^{\operatorname{curl}}, \mathbf{u}^{\operatorname{curl}}\right) \geq 0 & \forall \mathbf{u} \in \mathbf{X}, \\ \operatorname{Re} b_k(\mathbf{v}_T, \mathbf{v}_T) > 0 & \forall \mathbf{v} \in \mathbf{X} \backslash \{\mathbf{0}\} \end{array} \tag{4.2a}$$

and

$$b_k\left(\mathbf{u}^\nabla, \mathbf{v}^{\operatorname{curl}}\right) = b_k\left(\mathbf{u}^{\operatorname{curl}}, \mathbf{v}^\nabla\right) = 0 \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}. \tag{4.2b}$$

For Ω being the unit ball, the statements in Assumption 4.1 are proved in [47, Sec. 5.3.2]. Assumption (4.2) implies in particular:

$$A_k(\mathbf{u}, \mathbf{v}) = (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) - \mathrm{i}\, k b_k\left(\mathbf{u}^{\operatorname{curl}}, \mathbf{v}^{\operatorname{curl}}\right) - ((\mathbf{u}, \mathbf{v})), \tag{4.3}$$

$$A_k(\mathbf{u}, \nabla\varphi) = -((\mathbf{u}, \nabla\varphi)) \qquad \forall \mathbf{u} \in \mathbf{X}, \quad \varphi \in H^1(\Omega). \tag{4.4}$$

The stability and convergence analysis of the Galerkin discretization (3.1) involve a) some frequency splittings on the surface Γ and in the domain Ω as well as b), some Helmholtz decomposition for the space $\mathbf{X}$. These splittings will be defined next while their analysis (for the case of the unit ball) is postponed to Section 5.

Definition 4.2 (frequency splittings) *Let $\lambda > 1$ be a parameter. For a tangential field with an expansion of the form (2.19), the low-frequency operator L_Γ and high-frequency operator H_Γ are given by*

$$L_\Gamma \mathbf{v}_T := \sum_{1 \leq \ell \leq \lambda k} \sum_{m \in \iota_\ell} \left(v_\ell^m \overrightarrow{\operatorname{curl}}_\Gamma Y_\ell^m + V_\ell^m \nabla_\Gamma Y_\ell^m\right) \quad \text{and} \quad H_\Gamma := I - L_\Gamma.$$

The mapping $L_\Omega : \mathbf{X} \to \mathbf{X}$ is the solution operator of the minimization problem:

$$\|L_\Omega \mathbf{u}\|_{\operatorname{curl},\Omega,k} = \min_{\substack{\mathbf{v}\in\mathbf{X} \\ \Pi_T\mathbf{v}=L_\Gamma\mathbf{u}_T}} \|\mathbf{v}\|_{\operatorname{curl},\Omega,k}\,. \tag{4.5}$$

Set $H_\Omega := I - L_\Omega$. We introduce the notation

$$C_k^{L,\Omega} := \sup_{\mathbf{u}\in\mathbf{X}\backslash\{0\}} \frac{\|L_\Omega\mathbf{u}\|_{\operatorname{curl},\Omega,k}}{\|\mathbf{u}\|_{\operatorname{curl},\Omega,k}} \quad \text{and} \quad C_k^{H,\Omega} := \sup_{\mathbf{u}\in\mathbf{X}\backslash\{0\}} \frac{\|H_\Omega\mathbf{u}\|_{\operatorname{curl},\Omega,k}}{\|\mathbf{u}\|_{\operatorname{curl},\Omega,k}} \tag{4.6}$$

Remark 4.3 *Since*

$$\mathbf{X}_0 := \{\mathbf{w}\in\mathbf{X} \mid \Pi_T\mathbf{w} = 0\}\,. \tag{4.7}$$

is a Hilbert space with respect to $\|\cdot\|_{\operatorname{curl},\Omega,k}$, the operator $L_\Omega : \mathbf{X}\to\mathbf{X}$ is well-defined and bounded and linear (see also [51] and [52, Lemma 3.3]). The function $L_\Omega\mathbf{u}$ can be characterized equivalently to (4.5) as the solution of the following variational problem: Find $L_\Omega\mathbf{u}\in\mathbf{X}$ with $\Pi_T L_\Omega\mathbf{u} = L_\Gamma\mathbf{u}_T$ such that

$$(\operatorname{curl} L_\Omega\mathbf{u}, \operatorname{curl}\mathbf{w}) + k^2(L_\Omega\mathbf{u},\mathbf{w}) = 0 \qquad \forall\mathbf{w}\in\mathbf{X}_0. \tag{4.8}$$

Selecting gradients as test functions, i.e., $\mathbf{w} = \nabla\varphi$ for $\varphi\in H_0^1(\Omega)$ yields the equation $\operatorname{div} L_\Omega\mathbf{u} = 0$. The strong formulation of (4.8) is thus

$$\operatorname{curl}\operatorname{curl} L_\Omega\mathbf{u} + k^2 L_\Omega\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{4.9a}$$
$$\operatorname{div} L_\Omega\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{4.9b}$$
$$\Pi_T L_\Omega\mathbf{u} = L_\Gamma\mathbf{u}_T \qquad \text{on } \partial\Omega. \tag{4.9c}$$

■

Clearly the following commuting properties are valid

$$\Pi_T L_\Omega = L_\Gamma\Pi_T \quad \text{and} \quad \Pi_T H_\Omega = \Pi_T - L_\Gamma\Pi_T = (I - L_\Gamma)\,\Pi_T = H_\Gamma\Pi_T. \tag{4.10}$$

Remark 4.4 *For the special case of a ball $\Omega = B_1(0)$, we will derive in Section 5.3 k-independent estimates for the continuity constants $C_k^{L,\Omega}$ and $C_k^{H,\Omega}$. In the general case, one can show estimates of the form $C_k^{L,\Omega} \leq \tilde{C}k$ and $C_k^{L,\Omega} \leq 1+\tilde{C}k$ for some $\tilde{C} > 0$ independent of k by the following argument based on the (k-independent) lifting operator $\mathcal{E}_{\operatorname{curl}} : \mathbf{H}_{\operatorname{curl}}^{-1/2}(\Gamma) \to \mathbf{H}(\Omega,\operatorname{curl})$ provided by Theorem 2.4: The ansatz $L_\Omega\mathbf{u} = \mathbf{U} - \mathbf{U}_0$ with $\mathbf{U} = \mathcal{E}_{\operatorname{curl}} L_\Gamma\mathbf{u}_T$ leads to the equation*

$$\operatorname{curl}\operatorname{curl}\mathbf{U}_0 + k^2\mathbf{U}_0 = \operatorname{curl}\operatorname{curl}\mathbf{U} + k^2\mathbf{U} \quad \text{in } \Omega, \qquad\qquad \Pi_T\mathbf{U}_0 = 0 \quad \text{on } \partial\Omega.$$

Hence,

$$\|\mathbf{U}_0\|_{\operatorname{curl},\Omega,k} \leq \|\mathbf{U}\|_{\operatorname{curl},\Omega,k} \leq Ck\,\|L_\Gamma\mathbf{u}_T\|_{-1/2,\operatorname{curl}_\Gamma} \leq Ck\,\|\mathbf{u}_T\|_{-1/2,\operatorname{curl}_\Gamma} \leq CC_\Gamma k\,\|\mathbf{u}\|_{\operatorname{curl},\Omega,1} \leq CC_\Gamma k\,\|\mathbf{u}\|_{\operatorname{curl},\Omega,k}$$

from which we get $\|L_\Omega\mathbf{u}\|_{\operatorname{curl},\Omega,k} \leq \tilde{C}k\,\|\mathbf{u}\|_{\operatorname{curl},\Omega,k}$, i.e., $C_k^{L,\Omega} \leq \tilde{C}k$. The triangle inequality gives $C_k^{H,\Omega} \leq 1 + C_k^{H,\Omega}$. ■

The operators L_Γ and L_Ω map into low frequency modes which correspond to smooth functions and, hence, can be approximated well by *hp* finite elements. We also use the operators L_Γ and H_Γ to define the high- and low frequency parts of the sesquilinear form b_k.

Definition 4.5 *The low- and high-frequency parts of the capacity operator and the sesquilinear form b_k are given by*

$$\begin{array}{ll} T_k^{\text{low}} := T_k L_\Gamma, & T_k^{\text{high}} := T_k H_\Gamma, \\ b_k^{\text{low}}(\cdot,\cdot) := b_k(\cdot, L_\Gamma\cdot)\,, & b_k^{\text{high}}(\cdot,\cdot) := b_k(\cdot, H_\Gamma\cdot)\,. \end{array} \tag{4.11}$$

The continuity constants of the high-frequency parts of b_k are given by

$$C_{b,k}^{\nabla,\text{high}} := k \sup_{\mathbf{u},\mathbf{v}\in\mathbf{X}\backslash\{0\}} \frac{\max\left\{\left|b_k\left(\mathbf{u}^\nabla, (H_\Omega\mathbf{v})^\nabla\right)\right|, \left|b_k\left((H_\Omega\mathbf{u})^\nabla, \mathbf{v}^\nabla\right)\right|\right\}}{\|\mathbf{u}\|_{\operatorname{curl},\Omega,k}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}}, \tag{4.12a}$$

$$C_{b,k}^{\operatorname{curl},\text{high}} := \sup_{\mathbf{u},\mathbf{v}\in\mathbf{X}\backslash\{0\}} \frac{k\left|b_k\left(\mathbf{u}^{\operatorname{curl}}, (H_\Omega\mathbf{v})^{\operatorname{curl}}\right)\right|}{\|\mathbf{u}\|_{\operatorname{curl},\Omega,k}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}}. \tag{4.12b}$$

Lemma 4.6 *The capacity operator* $T_k : \mathbf{H}_{\text{curl}}^{-1/2}(\Gamma) \to \mathbf{H}_{\text{div}}^{-1/2}(\Gamma)$ *is continuous with continuity constant*

$$C_{\text{DtN},k} := \|T_k\|_{\mathbf{H}_{\text{div}}^{-1/2}(\Gamma)\leftarrow\mathbf{H}_{\text{curl}}^{-1/2}(\Gamma)} < \infty. \tag{4.13}$$

The sesquilinear form $A_k : \mathbf{X} \times \mathbf{X} \to \mathbb{C}$ *is continuous. For all* $\mathbf{u}, \mathbf{v} \in \mathbf{X}$ *it holds*

$$\max\{|A_k(\mathbf{u},\mathbf{v})|\ ,\ |((\mathbf{u},\mathbf{v}))|\} \leq C_{\text{cont},k} \|\mathbf{u}\|_{\text{curl},\Omega,1} \|\mathbf{v}\|_{\text{curl},\Omega,1} \quad \text{with } C_{\text{cont},k} := k^2 + C_\Gamma^2 C_{\text{DtN},k} k, \tag{4.14}$$

$$\max\{|((\mathbf{u},H_\Omega\mathbf{v}))|\ ,\ |((H_\Omega\mathbf{u},\mathbf{v}))|\} \leq C_{b,k}^{\text{high}} \|\mathbf{u}\|_{\text{curl},\Omega,k} \|\mathbf{v}\|_{\text{curl},\Omega,k} \quad \text{with } C_{b,k}^{\text{high}} := C_k^{H,\Omega} + C_{b,k}^{\nabla,\text{high}}, \tag{4.15}$$

$$\max\{|A_k(H_\Omega\mathbf{u},\mathbf{v})|, |A_k(\mathbf{u},H_\Omega\mathbf{v})|\} \leq C_{\text{cont},k}^{\text{high}} \|\mathbf{u}\|_{\text{curl},\Omega,k} \|\mathbf{v}\|_{\text{curl},\Omega,k} \tag{4.16}$$

with $C_{\text{cont},k}^{\text{high}} := C_k^{H,\Omega} + C_{b,k}^{\text{curl},\text{high}} + C_{b,k}^{\nabla,\text{high}}$.

Proof. The continuity of $T_k : \mathbf{H}_{\text{curl}}^{-1/2}(\Gamma) \to \mathbf{H}_{\text{div}}^{-1/2}(\Gamma)$ is asserted in Remark 2.2. For the sesquilinear form A_k we employ

$$|A_k(\mathbf{u},\mathbf{v})| \leq \|\mathbf{u}\|_{\text{curl},\Omega,k} \|\mathbf{v}\|_{\text{curl},\Omega,k} + k|b_k(\mathbf{u}_T,\mathbf{v}_T)|.$$

For the last term, we obtain

$$\begin{aligned} k|b_k(\mathbf{u}_T,\mathbf{v}_T)| &= k|(T_k\mathbf{u}_T,\mathbf{v}_T)_\Gamma| \leq k\|T_k\mathbf{u}_T\|_{-1/2,\text{div}_\Gamma} \|\mathbf{v}_T\|_{-1/2,\text{curl}_\Gamma} \\ &\leq C_{\text{DtN},k} k \|\mathbf{u}_T\|_{-1/2,\text{curl}_\Gamma} \|\mathbf{v}_T\|_{-1/2,\text{curl}_\Gamma} \overset{(2.26)}{\leq} C_\Gamma^2 C_{\text{DtN},k} k \|\mathbf{u}\|_{\text{curl},\Omega,1} \|\mathbf{v}\|_{\text{curl},\Omega,1}. \end{aligned} \tag{4.17}$$

For the continuity bound of the sesquilinear form $((\cdot,\cdot))$ we obtain similarly as before

$$\begin{aligned} |((\mathbf{u},\mathbf{v}))| &\leq k^2\|\mathbf{u}\|^2\|\mathbf{v}\|^2 + C_{\text{DtN},k} k \left\|\mathbf{u}^\nabla\right\|_{-1/2,\text{curl}_\Gamma} \left\|\mathbf{v}^\nabla\right\|_{-1/2,\text{curl}_\Gamma} \\ &\overset{(2.25)}{\leq} k^2\|\mathbf{u}\|^2\|\mathbf{v}\|^2 + C_{\text{DtN},k} k \|\mathbf{u}_T\|_{-1/2,\text{curl}_\Gamma} \|\mathbf{v}_T\|_{-1/2,\text{curl}_\Gamma} \\ &\leq C_{\text{cont},k} \|\mathbf{u}\|_{\text{curl},\Omega,1} \|\mathbf{v}\|_{\text{curl},\Omega,1}. \end{aligned}$$

For the high frequency estimate of $((\cdot,\cdot))$ we employ

$$\begin{aligned} |((\mathbf{u},H_\Omega\mathbf{v}))| &\leq (k\|\mathbf{u}\|)\left((k\|H_\Omega\mathbf{v}\|)\right) + k\left|b_k\left(\mathbf{u}^\nabla,(H_\Omega\mathbf{v})^\nabla\right)\right| \\ &\leq (k\|\mathbf{u}\|)\, C_k^{H,\Omega}\|\mathbf{v}\|_{\text{curl},\Omega,k} + C_{b,k}^{\nabla,\text{high}} \|\mathbf{u}\|_{\text{curl},\Omega,k} \|\mathbf{v}\|_{\text{curl},\Omega,k} \\ &\leq \left(C_k^{H,\Omega} + C_{b,k}^{\nabla,\text{high}}\right) \|\mathbf{u}\|_{\text{curl},\Omega,k} \|\mathbf{v}\|_{\text{curl},\Omega,k}. \end{aligned}$$

The estimates with interchanged arguments follow along the same lines. The bound (4.16) follows similarly. ■

Next, we introduce frequency-dependent Helmholtz decompositions for the space $\mathbf{X}$. Let $V \subset H^1(\Omega)$ be a closed subspace (the choice $V = H^1(\Omega)$ is allowed). Note that this implies $\nabla V \subset \mathbf{X}$. Consider the problems

$$\text{Given } \mathbf{w} \in \mathbf{X}, \text{ find } \Pi_V^\nabla\mathbf{w} \in \nabla V \quad \text{s.t.} \quad ((\Pi_V^\nabla\mathbf{w},\boldsymbol{\xi})) = ((\mathbf{w},\boldsymbol{\xi})) \qquad \forall\boldsymbol{\xi} \in \nabla V. \tag{4.18}$$

$$\text{Given } \mathbf{w} \in \mathbf{X}, \text{ find } \Pi_V^{\nabla,*}\mathbf{w} \in \nabla V \quad \text{s.t.} \quad \left(\left(\boldsymbol{\xi},\Pi_V^{\nabla,*}\mathbf{w}\right)\right) = ((\boldsymbol{\xi},\mathbf{w})) \qquad \forall\boldsymbol{\xi} \in \nabla V. \tag{4.19}$$

Lemma 4.7 *Let assumption (4.2a) be satisfied. Let* $V \subset H^1(\Omega)$ *be a closed subspace. Then, problems (4.18) and (4.19) are both uniquely solvable. Thus, the operators* Π_V^∇ *and* $\Pi_V^{\nabla,*}$ *are well defined.*

Proof. The definiteness of $\operatorname{Im} b_k\left((\cdot)^\nabla,(\cdot)^\nabla\right)$ (cf. (4.2a)) leads to

$$\operatorname{Re}((\nabla\xi,\nabla\xi)) = (k\|\nabla\xi\|)^2 - k\operatorname{Im} b_k\left((\nabla\xi)^\nabla,(\nabla\xi)^\nabla\right) \geq (k\|\nabla\xi\|)^2 \quad \forall\xi \in H^1(\Omega). \tag{4.20}$$

From (4.14) we furthermore get $|((\mathbf{w},\nabla\xi))| \leq C_{\text{cont},k}\|\mathbf{w}\|_{\text{curl},\Omega,1}\|\nabla\xi\|_{\text{curl},\Omega,1} = C_{\text{cont},k}\|\mathbf{w}\|_{\text{curl},\Omega,1}(k\|\nabla\xi\|)$, which shows the well-posedness of Π_V^∇. The well-posedness of $\Pi_V^{\nabla,*}$ is shown analogously. ■

For $V = S_h$, we write short Π_h^∇ for $\Pi_{S_h}^\nabla$ and $\Pi_h^{\nabla,*}$ for $\Pi_{S_h}^{\nabla,*}$ while, for $V = H^1(\Omega)$ we use the shorthands Π^∇ for $\Pi_{H^1(\Omega)}^\nabla$ and $\Pi^{\nabla,*}$ for $\Pi_{H^1(\Omega)}^{\nabla,*}$.

A central role of the analysis is played by the spaces (cf. [47, p.220])

$$\mathbf{V}_0 := \left\{\mathbf{u} \in \mathbf{X} \mid ((\mathbf{u}, \nabla\xi)) = 0 \quad \forall\xi \in H^1(\Omega)\right\} \overset{(4.4)}{=} \left\{\mathbf{u} \in \mathbf{X} \mid A_k(\mathbf{u}, \nabla\xi) = 0 \quad \forall\xi \in H^1(\Omega)\right\}, \tag{4.21}$$

$$\mathbf{V}_0^* := \left\{\mathbf{u} \in \mathbf{X} \mid ((\nabla\xi, \mathbf{u})) = 0 \quad \forall\xi \in H^1(\Omega)\right\} \overset{(4.4)}{=} \left\{\mathbf{u} \in \mathbf{X} \mid A_k(\nabla\xi, \mathbf{u}) = 0 \quad \forall\xi \in H^1(\Omega)\right\}. \tag{4.22}$$

These spaces of divergence-free functions are the ranges of the operators Π^{curl} and $\Pi^{\mathrm{curl},*}$ given by

$$\Pi^{\mathrm{curl}} := I - \Pi^{\nabla}, \quad \Pi^{\mathrm{curl},*} := I - \Pi^{\nabla,*}. \tag{4.23}$$

Lemma 4.8 *Let Assumption 4.1 be satisfied. For any* $\mathbf{v} \in \mathbf{X}$

$$\left\|\Pi^{\nabla,*} H_\Omega \mathbf{v}\right\|_{\mathrm{curl},\Omega,k} \le C_{b,k}^{\mathrm{high}} \left\|\mathbf{v}\right\|_{\mathrm{curl},\Omega,k}, \tag{4.24a}$$

$$\left\|\Pi^{\mathrm{curl},*} H_\Omega \mathbf{v}\right\|_{\mathrm{curl},\Omega,k} \le \left(C_k^{H,\Omega} + C_{b,k}^{\nabla,\mathrm{high}}\right) \left\|\mathbf{v}\right\|_{\mathrm{curl},\Omega,k}. \tag{4.24b}$$

Proof. We employ (4.20) to obtain

$$\left(k\left\|\Pi^{\nabla,*} H_\Omega \mathbf{v}\right\|\right)^2 \le \operatorname{Re}\left((\Pi^{\nabla,*} H_\Omega \mathbf{v}, \Pi^{\nabla,*} H_\Omega \mathbf{v})\right) = \operatorname{Re}\left((\Pi^{\nabla,*} H_\Omega \mathbf{v}, H_\Omega \mathbf{v})\right) \overset{(4.15)}{\le} C_{b,k}^{\mathrm{high}} k \left\|\Pi^{\nabla,*} H_\Omega \mathbf{v}\right\| \left\|\mathbf{v}\right\|_{\mathrm{curl},\Omega,k}.$$

Since $\operatorname{curl} \Pi^{\nabla,*} H_\Omega \mathbf{v}^{\mathrm{high}} = 0$, the estimate (4.24a) follows. Estimate (4.24b) is obtained from (4.24a) and the triangle inequality using $\Pi^{\mathrm{curl},*} = I - \Pi^{\nabla,*}$:

$$\left\|\Pi^{\mathrm{curl},*} H_\Omega \mathbf{v}\right\|_{\mathrm{curl},\Omega,k} \le \left\|H_\Omega \mathbf{v}\right\|_{\mathrm{curl},\Omega,k} + \left\|\Pi^{\nabla,*} H_\Omega \mathbf{v}\right\|_{\mathrm{curl},\Omega,k} \overset{(4.15)}{=} \left(C_k^{H,\Omega} + C_{b,k}^{\nabla,\mathrm{high}}\right) \left\|\mathbf{v}\right\|_{\mathrm{curl},\Omega,k}$$

■

It is finally convenient to introduce the discrete counterparts of these operators:

$$\Pi_h^{\mathrm{curl}} := I - \Pi_h^{\nabla}, \quad \Pi_h^{\mathrm{curl},*} := I - \Pi_h^{\nabla,*}. \tag{4.25}$$

The operators Π^{∇} and Π^{curl} (analogously $\Pi^{\nabla,*}$ and $\Pi^{\mathrm{curl},*}$) can be used to define a Helmholtz decomposition of $\mathbf{u} \in \mathbf{X}$ into a gradient part and a divergence-free part. Since favorable stability properties of Π^{∇} (and thus also of Π^{curl}) will only be available for high-frequency functions, the decomposition (4.26) below involves additionally the frequency-splitting operators H_Ω and L_Ω.

Definition 4.9 (Helmholtz decompositions) *For* $\mathbf{u}$, $\mathbf{v} \in \mathbf{X}$ *we set*

$$\mathbf{u} = \Pi^{\mathrm{comp}} \mathbf{u} + \Pi^{\nabla} H_\Omega \mathbf{u} \quad \text{with } \Pi^{\mathrm{comp}} := L_\Omega + \Pi^{\mathrm{curl}} H_\Omega, \tag{4.26a}$$

The adjoint splitting is based on the operator $\Pi^{\nabla,*}$ *and given by*

$$\mathbf{v} = \Pi^{\mathrm{comp},*} \mathbf{v} + \Pi^{\nabla,*} H_\Omega \mathbf{v} \quad \text{with } \Pi^{\mathrm{comp},*} := L_\Omega + \Pi^{\mathrm{curl},*} H_\Omega, \tag{4.26b}$$

The discrete counterparts of these splittings are

$$\begin{aligned} \mathbf{u} &= \Pi_h^{\mathrm{comp}} \mathbf{u} + \Pi_h^{\nabla} H_\Omega \mathbf{u} && \text{with } \Pi_h^{\mathrm{comp}} := L_\Omega + \Pi_h^{\mathrm{curl}} H_\Omega, \\ \mathbf{v} &= \Pi_h^{\mathrm{comp},*} \mathbf{v} + \Pi_h^{\nabla,*} H_\Omega \mathbf{v} && \text{with } \Pi_h^{\mathrm{comp},*} := L_\Omega + \Pi_h^{\mathrm{curl},*} H_\Omega. \end{aligned} \tag{4.27}$$

The next lemma characterizes the spaces $\mathbf{V}_0$ and $\mathbf{V}_0^\star$ in terms of the capacity operator T_k:

Lemma 4.10 *Let Assumption 4.1 be satisfied. Then:* $\mathbf{u} \in \mathbf{V}_0$ *if and only if*

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } L^2(\Omega) \wedge \mathrm{i}\, k \langle \mathbf{u}, \mathbf{n} \rangle + \operatorname{div}_\Gamma T_k \Pi_T \mathbf{u} = 0 \quad \text{in } H^{-1/2}(\Gamma). \tag{4.28}$$

Furthermore, $\mathbf{v} \in \mathbf{V}_0^*$ *if and only if* [4]

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } L^2(\Omega) \wedge \mathrm{i}\, k \langle \mathbf{v}, \mathbf{n} \rangle + \operatorname{div}_\Gamma T_{-k} \Pi_T \mathbf{v} = 0 \quad \text{in } H^{-1/2}(\Gamma), \tag{4.29}$$

where $T_k^* : \mathbf{H}_{\mathrm{curl}}^{-1/2}(\Gamma) \to \mathbf{H}_{\mathrm{div}}^{-1/2}(\Gamma)$ *is defined by*

$$(T_k \boldsymbol{\phi}, \boldsymbol{\psi})_\Gamma = (\boldsymbol{\phi}, T_k^* \boldsymbol{\psi})_\Gamma \qquad \forall \boldsymbol{\phi}, \boldsymbol{\psi} \in \mathbf{H}_{\mathrm{curl}}^{-1/2}(\Gamma). \tag{4.30}$$

[4] It holds $(\mathrm{i}\, k T_k)^* = -\mathrm{i}\, k T_k^* = -\mathrm{i}\, k T_{-k}$. This follows by representing T_k by trace operators and boundary/volume potentials for the electric Maxwell equation as, e.g., explained in [10], and by applying the rules for computing the adjoint of composite operators.

Proof. We only show the equivalence (4.29), since (4.28) follows by the same reasoning. Integration by parts applied to the condition $((\nabla\xi, \mathbf{v})) = 0$ yields, for all $\xi \in H^1(\Omega)$,

$$\begin{aligned}0 = ((\nabla\xi, \mathbf{v})) &= k^2 (\nabla\xi, \mathbf{v}) + \mathrm{i}\, k b_k \left((\nabla\xi)^{\nabla}, \mathbf{v}^{\nabla}\right) = -k^2 (\xi, \operatorname{div} \mathbf{v}) + k^2 (\xi, \langle \mathbf{v}, \mathbf{n}\rangle)_\Gamma + \mathrm{i}\, k \left(T_k (\nabla\xi)_T, \mathbf{v}_T\right)_\Gamma \\ &= -k^2 (\xi, \operatorname{div} \mathbf{v}) + k^2 (\xi, \langle \mathbf{v}, \mathbf{n}\rangle)_\Gamma + \mathrm{i}\, k \left((\nabla\xi)_T, T_k^* \mathbf{v}_T\right)_\Gamma \\ &= -k^2 (\xi, \operatorname{div} \mathbf{v}) + \mathrm{i}\, k \left(\xi, \mathrm{i}\, k \langle \mathbf{v}, \mathbf{n}\rangle + \operatorname{div}_\Gamma T_k^* \Pi_T \mathbf{v}\right)_\Gamma .\end{aligned}$$

This is equivalent to (4.29). ■

Corollary 4.11 *Let the right-hand side in (2.28b) be defined by* $F(\mathbf{v}) = \left(\mathrm{i}\, k\tilde{\mathbf{j}}, \mathbf{v}\right)$ *for some* $\tilde{\mathbf{j}} \in \mathbf{H}(\Omega, \operatorname{div})$ *with* $\operatorname{div} \tilde{\mathbf{j}} = 0$ *and* $\tilde{\mathbf{f}} \cdot \mathbf{n} = 0$ *on* Γ. *Then the solution* $\mathbf{E}$ *satisfies* $\mathbf{E} \in \mathbf{V}_0$.

Proof.
Next, we will prove that the spaces $\mathbf{V}_0$ and $\mathbf{V}_0^*$ are subspaces of $\mathbf{H}^1(\Omega)$. For the special case of Γ being the unit sphere, the constants in the norm equivalences can be determined explicitly – these details can be found in Lemma B.1.

Lemma 4.12 *Let* $\mathbf{V}_0$, $\mathbf{V}_0^*$ *be defined as in (4.1.1). Then,*

$$\mathbf{V}_0 \cup \mathbf{V}_0^* \subset \mathbf{H}^1(\Omega). \tag{4.31}$$

There exists a constant $C_{\Omega,k} > 0$ *such that*

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C_{\Omega,k} \|\mathbf{u}\|_{\operatorname{curl},\Omega,1} \qquad \forall \mathbf{u} \in \mathbf{V}_0 \cup \mathbf{V}_0^*. \tag{4.32}$$

Moreover, for any $\mathbf{v}_0^* \in \mathbf{V}_0^*$ *and* $\mathbf{u}_0 \in \mathbf{V}_0$, *the mappings* $\mathbf{X} \ni \mathbf{u} \mapsto \left((\Pi^{\operatorname{curl}} \mathbf{u}, \cdot)\right) \in \mathbf{X}'$, *and* $\mathbf{X} \ni \mathbf{v} \mapsto \left((\cdot, \Pi^{\operatorname{curl},*} \mathbf{v})\right)$ *are compact.*

Proof. Let $\mathbf{u} \in \mathbf{V}_0$. The function $T_k \Pi_T \mathbf{u}$ is computed by first solving the exterior problem (cf. Remark 2.2)

$$\begin{array}{ll} \operatorname{curl}\operatorname{curl} \mathbf{u}^+ - k^2 \mathbf{u}^+ = 0 & \text{in } \Omega^+, \\ \left[(\mathbf{u}, \mathbf{u}^+)\right]_{0,\Gamma} = 0 & \text{on } \Gamma \end{array} \tag{4.33}$$

with Silver-Müller radiation conditions and then setting $T_k \Pi_T \mathbf{u} = \frac{1}{\mathrm{i}\, k} \gamma_T^+ \operatorname{curl} \mathbf{u}^+$. Since the tangential components of $\mathbf{u}$ and $\mathbf{u}^+$ coincide on Γ, the function $\mathbf{U} : \mathbb{R}^3 \to \mathbb{C}$ defined by $\mathbf{U}|_\Omega = \mathbf{u}$ and $\mathbf{U}|_{\Omega^+} = \mathbf{u}^+$ (and Γ considered as a set of measure zero) is in $\mathbf{H}_{\mathrm{loc}}\left(\operatorname{curl}, \mathbb{R}^3\right)$. Then, for all $\mathbf{v} \in \mathbf{C}_0^\infty\left(\mathbb{R}^3\right)$ it holds

$$\begin{aligned}(\operatorname{curl} \mathbf{U}, \operatorname{curl} \mathbf{v})_{\mathbf{L}^2(\mathbb{R}^3\setminus\Gamma)} - k^2 (\mathbf{U}, \mathbf{v})_{\mathbf{L}^2(\mathbb{R}^3)} &= a_k(\mathbf{u}, \mathbf{v}) + \left(\operatorname{curl} \mathbf{u}^+, \operatorname{curl} \mathbf{v}\right)_{\mathbf{L}^2(\Omega^+)} - k^2 \left(\mathbf{u}^+, \mathbf{v}\right)_{\mathbf{L}^2(\Omega^+)} \\ &= a_k(\mathbf{u}, \mathbf{v}) + \left(\operatorname{curl}\operatorname{curl} \mathbf{u}^+ - k^2 \mathbf{u}^+, \mathbf{v}\right)_{\mathbf{L}^2(\Omega^+)} - \left(\gamma_T \operatorname{curl} \mathbf{u}^+, \mathbf{v}\right)_\Gamma \\ &= a_k(\mathbf{u}, \mathbf{v}) - \left(\gamma_T \operatorname{curl} \mathbf{u}^+, \mathbf{v}\right)_\Gamma \\ &= a_k(\mathbf{u}, \mathbf{v}) - \mathrm{i}\, k \left(T_k \mathbf{u}_T, \mathbf{v}_T\right)_\Gamma = a_k(\mathbf{u}, \mathbf{v}) - \mathrm{i}\, k b_k (\mathbf{u}_T, \mathbf{v}_T).\end{aligned} \tag{4.34}$$

If we test with gradients $\mathbf{v} = \nabla\varphi$ for $\varphi \in C_0^\infty\left(\mathbb{R}^3\right)$ we obtain

$$\begin{aligned}(\operatorname{curl} \mathbf{U}, \operatorname{curl} \nabla\varphi)_{\mathbf{L}^2(\mathbb{R}^3\setminus\Gamma)} - k^2 (\mathbf{U}, \nabla\varphi)_{\mathbf{L}^2(\mathbb{R}^3)} &= -k^2 (\mathbf{U}, \nabla\varphi)_{\mathbf{L}^2(\mathbb{R}^3)} = k^2 (\operatorname{div} \mathbf{U}, \varphi)_{L^2(\mathbb{R}^3)}, \\ (\operatorname{curl} \mathbf{U}, \operatorname{curl} \nabla\varphi)_{\mathbf{L}^2(\mathbb{R}^3\setminus\Gamma)} - k^2 (\mathbf{U}, \nabla\varphi)_{\mathbf{L}^2(\mathbb{R}^3)} &\overset{(4.34)}{=} a_k(\mathbf{u}, \nabla\varphi) - \mathrm{i}\, k b_k \left(\mathbf{u}_T, (\nabla\varphi)_T\right) \overset{\text{Rem. 2.3}}{=} -((\mathbf{u}, \nabla\varphi)).\end{aligned}$$

Since $\mathbf{u} \in \mathbf{V}_0$ implies $((\mathbf{u}, \nabla\varphi)) = 0$, the combination of the previous two equations leads to $\operatorname{div} \mathbf{U} = 0$ in $\mathbb{R}^3$. Hence

$$\mathbf{U} \in \mathbf{H}_{\mathrm{loc}}\left(\mathbb{R}^3, \operatorname{div}\right) \cap \mathbf{H}_{\mathrm{loc}}\left(\mathbb{R}^3, \operatorname{curl}\right). \tag{4.35}$$

Let $B_R(0)$ denote the ball with radius $0 < R < \infty$ and centered at 0 such that $\overline{\Omega} \subset B_R(0)$. We use (4.35) to conclude from [16, p. 157] that $\mathbf{U} \in \mathbf{H}^1(B_R(0))$ and that there exists a constant $C_R > 0$ such that

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq \|\mathbf{U}\|_{\mathbf{H}^1(B_R(0))} \leq C_R \left(\|\operatorname{curl} \mathbf{U}\|_{\operatorname{curl}, B_R(0), 1} + \|\operatorname{div} \mathbf{U}\|_{\mathbf{L}^2(B_R(0))}\right).$$

We already know that $\operatorname{div}\mathbf{U}=0$ in $\mathbb{R}^3$ so that

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C_R \left(\|\mathbf{u}\|_{\mathrm{curl},\Omega,1} + \left\|\mathbf{u}^+\right\|_{\mathrm{curl},\Omega^+\cap B_R(0),1} \right). \tag{4.36}$$

An inspection of the proof of [47, Thm. 5.4.6] implies that

$$\|\mathbf{u}_+\|_{\mathrm{curl},\Omega^+\cap B_R(0),1} \leq C_k \left\|\gamma_T^+ \mathbf{u}_+\right\|_{H_{\mathrm{div}}^{-1/2}(\Gamma)} \overset{(4.33)}{=} C_k \left\|\gamma_T \mathbf{u}\right\|_{H_{\mathrm{div}}^{-1/2}(\Gamma)} \overset{\text{Thm. 2.4}}{\leq} C_k' \|\mathbf{u}\|_{\mathrm{curl},\Omega,1}.$$

The combination with (4.36) leads to (4.32) for $\mathbf{u}\in\mathbf{V}_0$ with a constant $C_{\Omega,k}$, possibly depending on Ω and k. The inclusion $\mathbf{V}_0^* \subset \mathbf{H}^1(\Omega)$ in (4.31) and (4.32) for $\mathbf{u}\in\mathbf{V}_0^*$ follow by the same reasoning.
Next we prove that the mapping $\mathbf{X}\ni\mathbf{u}\mapsto\left((\Pi^{\mathrm{curl}}\mathbf{u},\cdot)\right)\in\mathbf{X}'$ is compact. The $\mathbf{L}^2(\Omega)$ part of this mapping is compact since $\Pi^{\mathrm{curl}}\mathbf{u}\in\mathbf{V}_0\subset\mathbf{H}^1(\Omega)\overset{\mathrm{comp}}{\hookrightarrow}\mathbf{L}^2(\Omega)$. Hence, it remains to prove the compactness of

$$\mathbf{X}\ni\mathbf{u}\mapsto\left(T_k\left(\Pi^{\mathrm{curl}}\mathbf{u}\right)^{\nabla},(\cdot)^{\nabla}\right)_{\Gamma}\in\mathbf{X}'. \tag{4.37}$$

We set $\mathbf{u}_0:=\Pi^{\mathrm{curl}}\mathbf{u}$ and write $\Pi_T\mathbf{u}_0=:\mathbf{u}_0^{\mathrm{curl}}+\mathbf{u}_0^{\nabla}$ according to (2.21). We observe

$$\left(T_k\Pi_T\mathbf{u}_0,\mathbf{v}^{\nabla}\right)_{\Gamma}=\left(T_k\mathbf{u}_0^{\nabla},\mathbf{v}^{\nabla}\right)_{\Gamma}.$$

From

$$\operatorname{div}_{\Gamma}T_k\mathbf{u}_0^{\nabla}=\operatorname{div}_{\Gamma}T_k\Pi_T\mathbf{u}_0\overset{(4.28)}{=}-\mathrm{i}\,k\left\langle\mathbf{u}_0,\mathbf{n}\right\rangle\in H^{1/2}(\Gamma).$$

we conclude that $T_k\mathbf{u}_0^{\nabla}\in H_{\mathrm{div}}^{1/2}(\Gamma)$. For any $\mathbf{v}\in\mathbf{X}$ we write $\mathbf{v}_T=\mathbf{v}^{\mathrm{curl}}+\nabla_{\Gamma}\varphi$ for some $\varphi\in H^{1/2}(\Gamma)$. Then

$$\left(T_k\mathbf{u}_0^{\nabla},\mathbf{v}^{\nabla}\right)=\left(T_k\mathbf{u}_0^{\nabla},\nabla_{\Gamma}\varphi\right)_{\Gamma}=-\left(\operatorname{div}_{\Gamma}T_k\mathbf{u}_0^{\nabla},\varphi\right)_{\Gamma}=\left(\mathrm{i}\,k\left\langle\mathbf{u}_0,\mathbf{n}\right\rangle,\varphi\right)_{\Gamma}\leq k\left\|\left\langle\mathbf{u}_0,\mathbf{n}\right\rangle\right\|_{H^{1/2}(\Gamma)}\|\varphi\|_{H^{-1/2}(\Gamma)}.$$

Since $\varphi\in H^{1/2}(\Gamma)\overset{\mathrm{comp}}{\hookrightarrow}H^{-1/2}(\Gamma)$ the compactness of the mapping (4.37) follows.
The compactness of the mapping $\mathbf{X}\ni\mathbf{v}\mapsto\left((\cdot)^{\nabla},T_k^*\left(\Pi^{\mathrm{curl},*}\mathbf{v}\right)^{\nabla}\right)_{\Gamma}\in\mathbf{X}'$ follows analogously. ■

4.1.2 Abstract Error Estimate

We have collected all ingredients to prove the quasi-optimal error estimate for the Galerkin solution in the following Theorem 4.13. It is the "Maxwell generalization" of the Galerkin convergence theory for sesquilinear forms satisfying a Gårding inequality, going back to [44]; various generalizations of this technique can be found in [8,26]. We follow [43, Sec. 7.2]. For $\mathbf{w}_h\in\mathbf{X}_h\backslash\{0\}$ we introduce the quantity

$$\delta_k(\mathbf{w}_h):=\sup_{\mathbf{v}\in\mathbf{X}_h\backslash\{0\}}\left(2\frac{\operatorname{Re}\left((\mathbf{w}_h,\mathbf{v}_h)\right)}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega,k}\|\mathbf{v}_h\|_{\mathrm{curl},\Omega,k}}\right). \tag{4.38}$$

We need an adjoint approximation property $\tilde{\eta}_1^{\exp}$ defined via the following dual problem: For given $\mathbf{w},\mathbf{h}\in\mathbf{X}$, find $\widehat{\mathcal{N}}(\mathbf{w},\mathbf{h})\in\mathbf{X}$ such that

$$A_k\left(\mathbf{v},\widehat{\mathcal{N}}(\mathbf{w},\mathbf{h})\right)=((\mathbf{v},\mathbf{w}))-\mathrm{i}\,kb_k\left(\mathbf{v}^{\mathrm{curl}},\mathbf{h}^{\mathrm{curl}}\right)\qquad\forall\mathbf{v}\in\mathbf{X}. \tag{4.39}$$

In (7.15) we will present an explicit solution formula for this problem which directly implies existence of a solution. The operator $\mathcal{N}_1^{\mathcal{A}}:\mathbf{X}\to\mathbf{X}$ then is given by $\mathcal{N}_1^{\mathcal{A}}(\mathbf{w}):=\widehat{\mathcal{N}}(L_\Omega\mathbf{w},L_\Omega\mathbf{w})$, i.e.,

$$A_k(\mathbf{v},\mathcal{N}_1\mathbf{w})=((\mathbf{v},L_\Omega\mathbf{w}))-\mathrm{i}\,kb_k\left(\mathbf{v}^{\mathrm{curl}},(L_\Omega\mathbf{w})^{\mathrm{curl}}\right)\qquad\forall\mathbf{v}\in\mathbf{X}. \tag{4.40}$$

The adjoint approximation property $\tilde{\eta}_1^{\exp}$ is defined by

$$\tilde{\eta}_1^{\exp}:=\tilde{\eta}_1^{\exp}(\mathbf{X}_h):=\sup_{\mathbf{w}\in\mathbf{X}\backslash\{0\}}\inf_{\mathbf{z}_h\in\mathbf{X}_h}\frac{\left\|\mathcal{N}_1^{\mathcal{A}}\mathbf{w}-\mathbf{z}_h\right\|_{\mathrm{curl},\Omega,k}}{\|\mathbf{w}\|_{\mathrm{curl},\Omega,k}}. \tag{4.41}$$

Theorem 4.13 *Let (4.2) be satisfied. Let* $\mathbf{E} \in \mathbf{X}$ *and* $\mathbf{E}_h \in \mathbf{X}_h$ *satisfy*

$$A_k(\mathbf{E} - \mathbf{E}_h, \mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{X}_h. \tag{4.42}$$

Assume that $\delta_k(\mathbf{e}_h) < 1$ *for* $\mathbf{e}_h := \mathbf{E} - \mathbf{E}_h$*. Then,* $\mathbf{e}_h$ *satisfies, for all* $\mathbf{w}_h \in \mathbf{X}_h$*, the quasi-optimal error estimate*

$$\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k} \leq \frac{C_k^{\mathrm{I}} + \delta_k(\mathbf{e}_h)}{1 - \delta_k(\mathbf{e}_h)} \|\mathbf{E} - \mathbf{w}_h\|_{\operatorname{curl},\Omega,k} \tag{4.43}$$

with

$$C_k^{\mathrm{I}} := 1 + C_{b,k}^{\mathrm{high}} + C_{b,k}^{\mathrm{curl,high}} + C_{\mathrm{cont},k}\tilde{\eta}_1^{\mathrm{exp}} \tag{4.44}$$

Proof. The assumed sign conditions of T_k (cf. (4.2a)) imply

$$\begin{aligned}\|\mathbf{e}_h\|^2_{\operatorname{curl},\Omega,k} &\leq (\operatorname{curl}\mathbf{e}_h, \operatorname{curl}\mathbf{e}_h) + k^2(\mathbf{e}_h, \mathbf{e}_h) - k \operatorname{Im} b_k\left(\mathbf{e}_h^{\nabla}, \mathbf{e}_h^{\nabla}\right) + k \operatorname{Im} b_k\left(\mathbf{e}_h^{\operatorname{curl}}, \mathbf{e}_h^{\operatorname{curl}}\right) \\ &= \operatorname{Re} A_k(\mathbf{e}_h, \mathbf{e}_h) + 2\operatorname{Re}((\mathbf{e}_h, \mathbf{e}_h)).\end{aligned}$$

We employ Galerkin orthogonality for the first term to obtain for any $\mathbf{w}_h \in \mathbf{X}_h$

$$\|\mathbf{e}_h\|^2_{\operatorname{curl},\Omega,k} \leq \operatorname{Re} A_k(\mathbf{e}_h, \mathbf{E} - \mathbf{w}_h) + 2\operatorname{Re}((\mathbf{e}_h, \mathbf{E} - \mathbf{w}_h)) + \delta_k(\mathbf{e}_h)\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k} \underbrace{\|\mathbf{E}_h - \mathbf{w}_h\|_{\operatorname{curl},\Omega,k}}_{\leq \|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k} + \|\mathbf{E} - \mathbf{w}_h\|_{\operatorname{curl},\Omega,k}}.$$

We write A_k in the form (4.3) so that

$$\begin{aligned}(1 - \delta_k(\mathbf{e}_h))\|\mathbf{e}_h\|^2_{\operatorname{curl},\Omega,k} &\leq |(\operatorname{curl}\mathbf{e}_h, \operatorname{curl}(\mathbf{E} - \mathbf{w}_h))| + \operatorname{Re}\left\{((\mathbf{e}_h, \mathbf{E} - \mathbf{w}_h)) - \mathrm{i}\, k b_k\left(\mathbf{e}_h^{\operatorname{curl}}, (\mathbf{E} - \mathbf{w}_h)^{\operatorname{curl}}\right)\right\} \\ &\quad + \delta_k(\mathbf{e}_h)\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k}\|\mathbf{E} - \mathbf{w}_h\|_{\operatorname{curl},\Omega,k}.\end{aligned} \tag{4.45}$$

It turns out that the sesquilinear forms in $\{\ldots\}$ allow for good continuity constants when applied to high frequency functions while these constants grow with k when being applied to low frequency functions. For a function $\mathbf{v} \in \mathbf{X}$ we introduce the splitting into a high-frequency and low-frequency part $\mathbf{v} = \mathbf{v}^{\mathrm{high}} + \mathbf{v}^{\mathrm{low}} := H_\Omega \mathbf{v} + L_\Omega \mathbf{v}$ and get by using (4.40)

$$\begin{aligned}((\mathbf{e}_h, \mathbf{v})) - \mathrm{i}\, k b_k\left(\mathbf{e}_h^{\operatorname{curl}}, \mathbf{v}^{\operatorname{curl}}\right) &= \left((\mathbf{e}_h, \mathbf{v}^{\mathrm{high}})\right) - \mathrm{i}\, k b_k\left(\mathbf{e}_h^{\operatorname{curl}}, \left(\mathbf{v}^{\mathrm{high}}\right)^{\operatorname{curl}}\right) + \left((\mathbf{e}_h, \mathbf{v}^{\mathrm{low}})\right) - \mathrm{i}\, k b_k\left(\mathbf{e}_h^{\operatorname{curl}}, \left(\mathbf{v}^{\mathrm{low}}\right)^{\operatorname{curl}}\right) \\ &= \left((\mathbf{e}_h, \mathbf{v}^{\mathrm{high}})\right) - \mathrm{i}\, k b_k\left(\mathbf{e}_h^{\operatorname{curl}}, \left(\mathbf{v}^{\mathrm{high}}\right)^{\operatorname{curl}}\right) + A_k\left(\mathbf{e}_h, \mathcal{N}_1^{\mathcal{A}}\mathbf{v}\right).\end{aligned} \tag{4.46}$$

We employ the continuity estimate of (4.12a) to get

$$\begin{aligned}\left|\left((\mathbf{e}_h, \mathbf{v}^{\mathrm{high}})\right)\right| &\leq (k\|\mathbf{e}_h\|)\left(k\left\|\mathbf{v}^{\mathrm{high}}\right\|\right) + k\left|b_k\left(\mathbf{e}_h^{\nabla}, \left(\mathbf{v}^{\mathrm{high}}\right)^{\nabla}\right)\right| \\ &\leq \|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k}\|H_\Omega \mathbf{v}\|_{\operatorname{curl},\Omega,k} + C_{b,k}^{\nabla,\mathrm{high}}\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k} \leq C_{b,k}^{\mathrm{high}}\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}.\end{aligned}$$

For the second term in (4.46) we use (4.12b) and obtain in a similar fashion

$$\left|k b_k\left(\mathbf{e}_h^{\operatorname{curl}}, \left(\mathbf{v}^{\mathrm{high}}\right)^{\operatorname{curl}}\right)\right| \leq C_{b,k}^{\mathrm{curl,high}}\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}.$$

For the last term in (4.46), the combination of Galerkin orthogonality, the continuity estimate (4.14) and the definition of $\tilde{\eta}_1^{\mathrm{exp}}$ in (4.41) gives

$$\left|A_k\left(\mathbf{e}_h, \mathcal{N}_1^{\mathcal{A}}\mathbf{v}\right)\right| = \inf_{\mathbf{w}_h \in \mathbf{X}_h}\left|A_k\left(\mathbf{e}_h, \mathcal{N}_1^{\mathcal{A}}\mathbf{v} - \mathbf{w}_h\right)\right| \leq \tilde{\eta}_1^{\mathrm{exp}} C_{\mathrm{cont},k}\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}.$$

Thus

$$\left|((\mathbf{e}_h, \mathbf{v})) - \mathrm{i}\, k b_k\left(\mathbf{e}_h^{\operatorname{curl}}, \mathbf{v}^{\operatorname{curl}}\right)\right| \leq \left(C_{b,k}^{\mathrm{high}} + C_{b,k}^{\mathrm{curl,high}} + \tilde{\eta}_1^{\mathrm{exp}} C_{\mathrm{cont},k}\right)\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}.$$

This allows us to continue the error estimation in (4.45) resulting in

$$(1 - \delta_k(\mathbf{e}_h))\|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k} \leq \left(1 + C_{b,k}^{\mathrm{high}} + C_{b,k}^{\mathrm{curl,high}} + \delta_k(\mathbf{e}_h) + \tilde{\eta}_1^{\mathrm{exp}} C_{\mathrm{cont},k}\right)\|\mathbf{E} - \mathbf{w}_h\|_{\operatorname{curl},\Omega,k}.$$

■
This theorem implies that quasi-optimality of the Galerkin method is ensured if $\delta(\mathbf{e}_h) < 1$. As will be shown in Theorem 4.15 below, this condition also implies existence and uniqueness of the Galerkin approximation $\mathbf{E}_h$. In the following, we will focus on estimating $\delta(\mathbf{e}_h)$, heavily exploiting the Galerkin orthogonality (4.42). For the case $\Omega = B_1(0)$ we will derive k-explicit estimates for the constants in (4.43) in Corollary 5.13. In this case, the constants $C_{b,k}^{\text{high}}$, $C_{b,k}^{\text{curl,high}}$ are independent of k; $C_{\text{cont},k} = O(k^3)$ grows algebraically in k, which can be offset by controlling $\tilde{\eta}_1^{\text{exp}}$.

4.2 Splittings of $\mathbf{v}_h$ and a basic estimate for $\delta(\mathbf{e}_h)$

It remains to estimate $\delta(\mathbf{e}_h)$ in (4.38). In this section, we will introduce some frequency-dependent Helmholtz decompositions for a splitting of the term $((\mathbf{e}_h, \mathbf{v}_h))$.
For $\mathbf{v} \in \mathbf{X}$ we introduce two decompositions according to Definition 4.9. Let $\mathbf{v}^{\text{low}} := L_\Omega \mathbf{v}$ and $\mathbf{v}^{\text{high}} := H_\Omega \mathbf{v}$. Then,

$$\begin{aligned} \mathbf{v} &= \Pi_h^{\text{comp},*}\mathbf{v} + \Pi_h^{\nabla,\star}\mathbf{v}^{\text{high}} \quad \text{with } \Pi_h^{\text{comp},*} \text{ as in (4.27)},\\ \mathbf{v} &= \Pi^{\text{comp},*}\mathbf{v} + \Pi^{\nabla,\star}\mathbf{v}^{\text{high}} \quad \text{with } \Pi^{\text{comp},*} \text{ as in (4.26b)}. \end{aligned} \tag{4.47}$$

An important point to note is that for $\mathbf{v}_h \in \mathbf{X}_h$ we have $\Pi_h^{\text{comp},*}\mathbf{v}_h \in \mathbf{X}_h$ and, for any $\mathbf{v} \in \mathbf{X}$, we have $\Pi_h^{\nabla,\star}\mathbf{v}^{\text{high}} \in \nabla S_h \subset \mathbf{X}_h$. However, $\Pi^{\text{comp}}\mathbf{v}_h$ and $\Pi^{\nabla}\mathbf{v}_h^{\text{high}}$ are only in $\mathbf{X}$ and $\nabla H^1(\Omega)$. From $\operatorname{curl}\left(\Pi_h^{\nabla,\star}\mathbf{v}_h^{\text{high}}\right) = 0$ and Galerkin orthogonality we conclude that

$$\left(\left(\mathbf{e}_h, \Pi_h^{\nabla,\star}\mathbf{v}_h^{\text{high}}\right)\right) \overset{(4.3),\ \text{Rem. } 2.3}{=} -A_k\left(\mathbf{e}_h, \Pi_h^{\nabla,\star}\mathbf{v}_h^{\text{high}}\right) = 0 \tag{4.48}$$

since $\Pi_h^{\nabla,\star}\mathbf{v}_h^{\text{high}} \in \nabla S_h \subset \mathbf{X}_h$. We employ the splitting

$$\mathbf{v}_h = \Pi^{\text{comp},*}\mathbf{v}_h + \left(\Pi_h^{\text{comp},*} - \Pi^{\text{comp},*}\right)\mathbf{v}_h + \Pi_h^{\nabla,\star}\mathbf{v}_h^{\text{high}}$$

and arrive via (4.48) at our main splitting

$$((\mathbf{e}_h, \mathbf{v}_h)) = \left(\left(\mathbf{e}_h, \left(\Pi_h^{\text{comp},*} - \Pi^{\text{comp},*}\right)\mathbf{v}_h\right)\right) + ((\mathbf{e}_h, \Pi^{\text{comp},*}\mathbf{v}_h)) \tag{4.49a}$$

$$= \left(\left(\mathbf{e}_h, \left(\Pi_h^{\text{comp},*}\mathbf{v}_h - \Pi^{\text{comp},*}\mathbf{v}_h\right)^{\text{high}}\right)\right) + \left(\left(\mathbf{e}_h, \left(\Pi_h^{\text{comp},*}\mathbf{v}_h - \Pi^{\text{comp},*}\mathbf{v}_h\right)^{\text{low}}\right)\right) \tag{4.49b}$$

$$+ ((\mathbf{e}_h, \mathbf{v}_h^{\text{low}})) + \left(\left(\mathbf{e}_h, \Pi^{\text{curl},*}\mathbf{v}_h^{\text{high}}\right)\right). \tag{4.49c}$$

4.3 Adjoint Approximation Properties

The error analysis involve solution operators for some adjoint problems and we introduce here corresponding approximation properties which measure how well these adjoint solutions can be approximated by functions in the Galerkin space $\mathbf{X}_h$ and its companion space S_h. One of these approximation properties involve the existence of an interpolating projector which will also be introduced in this section.
Recall the definition of $\mathbf{V}_0^*$ of (4.22). We set

$$\mathbf{V}_{0,h}^* := \{\mathbf{v} \in \mathbf{V}_0^* \mid \operatorname{curl}\mathbf{v} \in \operatorname{curl}\mathbf{X}_h\}.$$

The following assumption stipulates the existence of a projector $\Pi_h^E : \mathbf{V}_{0,h}^* + \operatorname{Range}(L_\Omega) + \mathbf{X}_h \to \mathbf{X}_h$.

Assumption 4.14 *There exists a linear operator* $\Pi_h^E : \mathbf{V}_{0,h}^* + \operatorname{Range}(L_\Omega) + \mathbf{X}_h \to \mathbf{X}_h$ *with the following properties:*

a. Π_h^E *is a projection, i.e., the restriction* $\Pi_h^E\big|_{\mathbf{X}_h}$ *is the identity on* $\mathbf{X}_h$.

b. There exists a companion operator $\Pi_h^F : \operatorname{curl}\mathbf{X}_h \to \operatorname{curl}\mathbf{X}_h$ *with the property* $\operatorname{curl}\Pi_h^E = \Pi_h^F \operatorname{curl}$.

Now we formulate the arising adjoint problems along their solution operators: We introduce the solution operators $\mathcal{N}_2$, $\mathcal{N}_3^{\mathcal{A}}$ for the following adjoint problems

$$A_k(\mathbf{w}, \mathcal{N}_2\mathbf{r}) = ((\mathbf{w}, \mathbf{r})) \qquad \forall \mathbf{w} \in \mathbf{X}, \quad \forall \mathbf{r} \in \mathbf{V}_0^*, \tag{4.50}$$

$$A_k\left(\mathbf{w},\mathcal{N}_3^{\mathcal{A}}\mathbf{r}\right)=((\mathbf{w},L_\Omega\mathbf{r}))\qquad\forall\mathbf{w}\in\mathbf{X},\quad\forall\mathbf{r}\in\mathbf{X},\tag{4.51}$$

i.e.,

$$\mathcal{N}_2\mathbf{r}:=\widehat{\mathcal{N}}\left(\mathbf{r},\mathbf{0}\right)\quad\text{and}\quad\mathcal{N}_3^{\mathcal{A}}\mathbf{r}:=\mathcal{N}_2\left(L_\Omega\mathbf{r}\right)=\widehat{\mathcal{N}}\left(L_\Omega\mathbf{r},\mathbf{0}\right).$$

The solution operator $\mathcal{N}_4^{\mathcal{A}}:\mathbf{X}\to H^1\left(\Omega\right)/\mathbb{R}$ is related to some Poisson problem and explicitly given by

$$-A_k(\mathcal{N}_4^{\mathcal{A}}\mathbf{r},\nabla\xi)\overset{(4.4)}{=}\left(\left(\nabla\mathcal{N}_4^{\mathcal{A}}\mathbf{r},\nabla\xi\right)\right)=((L_\Omega\mathbf{r},\nabla\xi))\qquad\forall\xi\in H^1\left(\Omega\right).\tag{4.52}$$

We introduce the adjoint approximation properties[5]

$$\tilde{\eta}_2^{\text{alg}}:=\tilde{\eta}_2^{\text{alg}}\left(\mathbf{X}_h\right):=\sup_{\mathbf{v}_0\in\mathbf{V}_0^*\backslash\{0\}}\inf_{\mathbf{w}_h\in\mathbf{X}_h}\frac{\left\|\mathcal{N}_2\mathbf{v}_0-\mathbf{w}_h\right\|_{\operatorname{curl},\Omega,k}}{\left\|\mathbf{v}_0\right\|_{\operatorname{curl},\Omega,k}},\tag{4.53}$$

$$\tilde{\eta}_3^{\exp}:=\tilde{\eta}_3^{\exp}\left(\mathbf{X}_h\right):=\sup_{\mathbf{r}\in\mathbf{X}\backslash\{0\}}\inf_{\mathbf{w}_h\in\mathbf{X}_h}\frac{\left\|\mathcal{N}_3^{\mathcal{A}}\mathbf{r}-\mathbf{w}_h\right\|_{\operatorname{curl},\Omega,k}}{\left\|\mathbf{r}\right\|_{\operatorname{curl},\Omega,k}},\tag{4.54}$$

$$\tilde{\eta}_4^{\exp}:=\tilde{\eta}_4^{\exp}\left(S_h\right):=\sup_{\mathbf{r}\in\mathbf{X}\backslash\{0\}}\inf_{v_h\in S_h}\frac{\left\|\nabla\left(\mathcal{N}_4^{\mathcal{A}}\mathbf{r}-v_h\right)\right\|}{\left\|\mathbf{r}\right\|_{\operatorname{curl},\Omega,1}},\tag{4.55}$$

$$\tilde{\eta}_5^{\exp}:=\tilde{\eta}_5^{\exp}\left(\mathbf{X}_h\right):=\sup_{\mathbf{r}\in\mathbf{X}\backslash\{0\}}\inf_{\mathbf{w}_h\in\mathbf{X}_h}\frac{\left\|L_\Omega\mathbf{r}-\mathbf{w}_h\right\|_{\operatorname{curl},\Omega,k}}{\left\|\mathbf{r}\right\|_{\operatorname{curl},\Omega,k}},\tag{4.56}$$

$$\eta_6^{\text{alg}}:=\eta_6^{\text{alg}}\left(\mathbf{X}_h,\Pi_h^E\right):=\sup_{\substack{\mathbf{w}\in\mathbf{V}_0^*\backslash\{0\}:\\ \operatorname{curl}\mathbf{w}\in\operatorname{curl}\mathbf{X}_h}}\frac{k\left\|\mathbf{w}-\Pi_h^E\mathbf{w}\right\|}{\left\|\mathbf{w}\right\|_{\mathbf{H}^1(\Omega)}},\tag{4.57}$$

$$\tilde{\eta}_7^{\exp}:=\tilde{\eta}_7^{\exp}\left(\mathbf{X}_h,\Pi_h^E\right):=\sup_{\mathbf{r}\in\mathbf{X}\backslash\{0\}}\frac{k\left\|L_\Omega\mathbf{r}-\Pi_h^E L_\Omega\mathbf{r}\right\|}{\left\|\mathbf{r}\right\|_{\operatorname{curl},\Omega,k}}.\tag{4.58}$$

In Section 6 we will derive the following estimates for the terms in (4.49c). Let $\mathbf{r}:=\Pi_h^{\text{comp},*}\mathbf{v}_h-\Pi^{\text{comp},*}\mathbf{v}_h$. Then

$$\begin{aligned}\left|\left(\left(\mathbf{e}_h,\mathbf{r}^{\text{high}}\right)\right)\right|&\overset{\text{Prop. 6.1}}{\leq}C_{b,k}^{\text{high}}C_{r,k}\left\|\mathbf{e}_h\right\|_{\operatorname{curl},\Omega,k}\left\|\mathbf{v}_h\right\|_{\operatorname{curl},\Omega,k},\\ \left|\left(\left(\mathbf{e}_h,\Pi^{\operatorname{curl},*}\mathbf{v}_h^{\text{high}}\right)\right)\right|&\overset{\text{Prop. 6.2}}{\leq}C_{\#\#,k}\left(C_{\#\#,k}+C_{b,k}^{\operatorname{curl},\text{high}}+C_{\text{cont},k}\tilde{\eta}_5^{\exp}\right)\tilde{\eta}_2^{\text{alg}}\left\|\mathbf{e}_h\right\|_{\operatorname{curl},\Omega,k}\left\|\mathbf{v}_h\right\|_{\operatorname{curl},\Omega,k},\\ \left|\left(\left(\mathbf{e}_h,L_\Omega\mathbf{r}\right)\right)\right|+\left|\left(\left(\mathbf{e}_h,L_\Omega\mathbf{v}_h\right)\right)\right|&\overset{\text{Prop. 6.3}}{\leq}C_{\text{cont},k}\tilde{\eta}_3^{\exp}\left(1+C_{r,k}\right)\left\|\mathbf{e}_h\right\|_{\operatorname{curl},\Omega,k}\left\|\mathbf{v}_h\right\|_{\operatorname{curl},\Omega,k}.\end{aligned}$$

We combine this together with (4.49) and (4.38) to obtain

$$\delta_k\left(\mathbf{e}_h\right)\leq\delta_k^{\text{I}}:=2\left(C_{b,k}^{\text{high}}C_{r,k}+C_{\#\#,k}\left(C_{\#\#,k}+C_{b,k}^{\operatorname{curl},\text{high}}+C_{\text{cont},k}\tilde{\eta}_5^{\exp}\right)\tilde{\eta}_2^{\text{alg}}+C_{\text{cont},k}\tilde{\eta}_3^{\exp}\left(1+C_{r,k}\right)\right).\tag{4.59}$$

Theorem 4.15 *Let Assumption 2.1 be satisfied and let* $\mathbf{E}$ *be the solution of Maxwell's equations (2.28b). Assume that* δ_k^{I} *in (4.59) is smaller than* 1*. Then the discrete problem (3.1) has a unique solution* $\mathbf{E}_h$*, which satisfies the quasi-optimal error estimate*

$$\left\|\mathbf{e}_h\right\|_{\operatorname{curl},\Omega,k}\leq\frac{C_k^{\text{I}}+\delta_k^{\text{I}}}{1-\delta_k^{\text{I}}}\inf_{\mathbf{w}_h\in\mathbf{X}_h}\left\|\mathbf{E}-\mathbf{w}_h\right\|_{\operatorname{curl},\Omega,k}.\tag{4.60}$$

Proof. The proof uses the same arguments as the proof of [31, Thm. 3.9]. Under the assumption that a solution exists, the quasi-optimal error estimate (4.60) follows from (4.43) and the assumption $\delta_k^{\text{I}}<1$. Next, we will prove uniqueness of problem (3.1). We show that if $\mathbf{E}_h$ solves

$$A_k\left(\mathbf{E}_h,\mathbf{v}_h\right)=0\quad\forall\mathbf{v}_h\in\mathbf{X}_h,$$

[5] We write $\tilde{\eta}_\ell$ for an approximation property which involves a *solution operator and* η_ℓ for a "pure" approximation property for a given space/set of functions.

then $\mathbf{E}_h = 0$. This is the Galerkin discretization of the continuous problem: Find $\mathbf{E} \in \mathbf{X}$ such that $A_k(\mathbf{E}, \mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{X}$. Theorem 2.7 implies that $\mathbf{E} = \mathbf{0}$ is the unique solution. Hence $\mathbf{e}_h = \mathbf{E} - \mathbf{E}_h = -\mathbf{E}_h$ satisfies the error estimate

$$\|\mathbf{E}_h\|_{\mathrm{curl},\Omega,k} = \|\mathbf{e}_h\|_{\mathrm{curl},\Omega,k} \leq \frac{C_k^{\mathrm{I}} + \delta_k^{\mathrm{I}}}{1 - \delta_k^{\mathrm{I}}} \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\mathbf{E} - \mathbf{w}_h\|_{\mathrm{curl},\Omega,k} = 0$$

since $\mathbf{E} = \mathbf{0}$. Hence $\mathbf{E}_h = 0$. Since (3.1) is finite dimensional, uniqueness implies existence. ■

4.4 k-explicit hp-FEM

In this section, we show that the choice $(\mathbf{X}_h, S_h) := \left(\boldsymbol{\mathcal{N}}_p^{\mathrm{I}}(\mathcal{T}_h), S_{p+1}(\mathcal{T})\right)$ for properly chosen mesh size h and k-dependent polynomial degree $p \geq 1$ leads to a *k-independent* quasi-optimality constant in (4.43). We adopt the setting described in Section 3.2. That is, we let $\mathcal{T}_h$ be a mesh satisfying the assumptions of Section 3.2 and Assumption 3.1. We have postponed the definition of the operators Π_h^E and Π_h^F whose existence is required in Section 3.1 to Sections 8–8.3 (and chosen to be $\Pi_p^{\mathrm{curl},c}$ and $\Pi_p^{\mathrm{div},c}$; cf. Theorem 8.3).

4.4.1 Applications to the Case $\Omega = B_1(0)$

The adjoint approximation properties $\tilde{\eta}_\ell^{\mathrm{alg}}$, $\tilde{\eta}_\ell^{\mathrm{exp}}$ involve solution operators whose regularity are investigated in Sections 5.3 and 7 for the unit ball $\Omega = B_1(0)$. In particular, we show in Proposition 7.2 that the solution operator $\mathcal{N}_2$ allows for a stable additive splitting $\mathcal{N}_2 = \mathcal{N}_2^{\mathrm{rough}} + \mathcal{N}_2^{\mathcal{A}}$, where $\mathcal{N}_2^{\mathcal{A}}$ maps into some analyticity class and $\mathcal{N}_2^{\mathrm{rough}} : \mathbf{V}_0^* \to \mathbf{H}^2(\Omega)$ satisfies the estimate $\left\|\mathcal{N}_2^{\mathrm{rough}}\mathbf{v}_0\right\|_{\mathbf{H}^2(\Omega)} \leq C_{\mathrm{rough}} k \|\mathbf{v}_0\|_{\mathrm{curl},\Omega,k}$. In Theorem 5.9 and Propositions 7.2, 7.3, 7.4, 7.5 we show that all other solution operators map into some analyticity class, more precisely, for all $\mathbf{r} \in \mathbf{X}$ and $\mathbf{v}_0 \in \mathbf{V}_0^*$, it holds[6] with $\alpha_1 = 3$, $\alpha_2 = 3$, $\alpha_3 = 3$, $\alpha_4 = 5/2$, $\alpha_5 = 3/2$

$$\mathcal{N}_j^{\mathcal{A}}\mathbf{r} \in \mathcal{A}\left(C_{\mathcal{A},j} k^{\alpha_j} \|\mathbf{r}\|_{\mathrm{curl},\Omega,k}, \gamma_{\mathcal{A},j}, \Omega\right), \quad j = 1, 3, \tag{4.61}$$

$$\mathcal{N}_2^{\mathcal{A}}\mathbf{v}_0 \in \mathcal{A}\left(C_{\mathcal{A},2} k^{\alpha_2} \|\mathbf{v}_0\|_{\mathrm{curl},\Omega,k}, \gamma_{\mathcal{A},2}, \Omega\right), \tag{4.62}$$

$$\nabla\mathcal{N}_4^{\mathcal{A}}\mathbf{r} \in \mathcal{A}\left(C_{\mathcal{A},4} k^{\alpha_4} \|\mathbf{r}\|_{\mathrm{curl},\Omega,1}, \gamma_{\mathcal{A},4}, \Omega\right), \tag{4.63}$$

$$L_\Omega\mathbf{r} \in \mathcal{A}\left(C_{\mathcal{A},5} k^{\alpha_5} \|\mathbf{r}\|_{\mathrm{curl},\Omega,1}, \gamma_{\mathcal{A},5}, \Omega\right), \tag{4.64}$$

This allows us to estimate those adjoint approximations that involve solution operators by simpler approximation properties, which we will introduce next. We set

$$\eta_1^{\mathrm{exp}}(\gamma, \mathbf{X}_h) := \sup_{\mathbf{z} \in \mathcal{A}(1,\gamma,\Omega)} \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\mathbf{z} - \mathbf{w}_h\|_{\mathrm{curl},\Omega,k}, \tag{4.65}$$

$$\eta_2^{\mathrm{alg}}(\mathbf{X}_h) := \sup_{\substack{\mathbf{z} \in \mathbf{H}^2(\Omega) \\ \|\mathbf{z}\|_{\mathbf{H}^2(\Omega)} \leq k}} \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\mathbf{z} - \mathbf{w}_h\|_{\mathrm{curl},\Omega,k}, \tag{4.66}$$

$$\eta_4^{\mathrm{exp}}(\gamma, S_h) := \sup_{\nabla z \in \mathcal{A}(1,\gamma,\Omega)} \inf_{v_h \in S_h} \|\nabla(z - v_h)\|, \tag{4.67}$$

$$\eta_7^{\mathrm{exp}}(\gamma, \mathbf{X}_h) := k \sup_{\mathbf{z} \in \mathcal{A}(1,\gamma,\Omega)} \left\|\mathbf{z} - \Pi_h^E\mathbf{z}\right\|, \tag{4.68}$$

and obtain

$$\begin{array}{ll} \tilde{\eta}_1^{\mathrm{exp}} \leq C_{\mathcal{A},1} k^{\alpha_1} \eta_1^{\mathrm{exp}}(\gamma_{\mathcal{A},1}, \mathbf{X}_h), & \tilde{\eta}_2^{\mathrm{alg}} \leq C_{\mathrm{rough}} \eta_2^{\mathrm{alg}}(\mathbf{X}_h) + C_{\mathcal{A},2} k^{\alpha_2} \eta_1^{\mathrm{exp}}(\gamma_{\mathcal{A},2}, \mathbf{X}_h), \\ \tilde{\eta}_3^{\mathrm{exp}} \leq C_{\mathcal{A},3} k^{\alpha_3} \eta_1^{\mathrm{exp}}(\gamma_{\mathcal{A},3}, \mathbf{X}_h), & \tilde{\eta}_4^{\mathrm{exp}} \leq C_{\mathcal{A},4} k^{\alpha_4} \eta_4^{\mathrm{exp}}(\gamma_{\mathcal{A},4}, S_h), \\ \tilde{\eta}_5^{\mathrm{exp}} \leq C_{\mathcal{A},5} k^{\alpha_5} \eta_1^{\mathrm{exp}}(\gamma_{\mathcal{A},5}, \mathbf{X}_h), & \tilde{\eta}_7^{\mathrm{exp}} \leq C_{\mathcal{A},5} k^{\alpha_5} \eta_7^{\mathrm{exp}}(\gamma_{\mathcal{A},5}, \mathbf{X}_h). \end{array} \tag{4.69}$$

Corollary 4.16 *Let $\Omega = B_1(0)$ and recall the definition of α_ℓ before (4.4.1). Define*

$$\begin{array}{ll} \eta_1^{\mathrm{exp}} := \max_{j \in \{1,2,3,5\}} C_{\mathcal{A},j} \eta_1^{\mathrm{exp}}(\gamma_{\mathcal{A},j}, \mathbf{X}_h), & \eta_2^{\mathrm{alg}} := C_{\mathrm{rough}} \eta_2^{\mathrm{alg}}(\mathbf{X}_h), \\ \eta_4^{\mathrm{exp}} := C_{\mathcal{A},4} \eta_4^{\mathrm{exp}}(\gamma_{\mathcal{A},4}, S_h), & \eta_7^{\mathrm{exp}} := C_{\mathcal{A},5} \eta_7^{\mathrm{exp}}(\gamma_{\mathcal{A},5}, \mathbf{X}_h). \end{array} \tag{4.70}$$

[6]For the last relation, we have estimated $\|\cdot\|_{\mathrm{curl},\Omega,1} \leq \|\cdot\|_{\mathrm{curl},\Omega,k}$ in (5.29) (using (2.2)) to simplify technicalities.

For $0<\tau\leq 1$ sufficiently small but independent of k, and any $0<\varepsilon_\ell\leq\tau$, $\ell\in\{1,2,4,6,7\}$, select the mesh size h and the polynomial degree p for the hp-finite element space $\mathbf{X}_h$ such that $\mathbf{X}_h$ and its companion space S_h (cf. (3.2)) satisfy Assumption 4.14 and the approximation properties:

$$k^{\alpha_3+3}\eta_1^{\exp}\leq\varepsilon_1,\quad \eta_2^{\mathrm{alg}}\leq\varepsilon_2,\quad k^{\alpha_4+1}\eta_4^{\exp}\leq\varepsilon_4,\quad \eta_6^{\mathrm{alg}}\leq\varepsilon_6,\quad k^{\alpha_5}\eta_7^{\exp}\leq\varepsilon_7. \tag{4.71}$$

Then, the quantity δ_k^{I} in (4.59), (4.60) can be estimated by $\delta_k^{\mathrm{I}}<1/2$, and the discrete problem (3.1) has a unique solution $\mathbf{E}_h$, which satisfies the quasi-optimal error estimate

$$\|\mathbf{e}_h\|_{\mathrm{curl},\Omega,k}\leq C\inf_{\mathbf{w}_h\in\mathbf{X}_h}\|\mathbf{E}-\mathbf{w}_h\|_{\mathrm{curl},\Omega,k} \tag{4.72}$$

for a constant C independent of k.

Proof. We estimate δ_k^{I} of (4.59) termwise by using (4.69), (4.70), and the values of α_j. From Corollary 5.13, we deduce that the constants $C_{\#,k}$, $C_{\#\#,k}$ in (6.2) and (6.15) are in fact bounded uniformly in k. Hence

$$C_{r,k}\leq C\left(\varepsilon_6+\varepsilon_7\right)$$

for a constant C independent of k. Again from Corollary 5.13 and (4.69), it follows that

$$\delta_k^{\mathrm{I}}\leq C\left(\varepsilon_6+\varepsilon_7+\left(1+k^{\alpha_5+3}\eta_1^{\exp}\right)\left(\eta_2^{\mathrm{alg}}+k^{\alpha_2}\eta_1^{\exp}\right)+k^{\alpha_3+3}\eta_1^{\exp}\left(1+\varepsilon_6+\varepsilon_7\right)\right)$$

for a constant C independent of k. We use $\alpha_3+3\geq\max\{\alpha_1+3,\alpha_5+3,\alpha_2\}$ and the conditions in (4.71) along with $\varepsilon_\ell\leq\tau\leq 1$ to obtain

$$\delta_k^{\mathrm{I}}\leq C\left(\varepsilon_1+\varepsilon_2+\varepsilon_6+\varepsilon_7\right)\leq\tilde{C}\tau$$

for a constant $\tilde{C}$ independent of k. Hence, the condition $0<\tau<\left(2\tilde{C}\right)^{-1}$ implies $\delta_k^{\mathrm{I}}<1/2$ and existence and uniqueness of the discrete solution follow from Theorem 4.15.
To prove that the quasi-optimality constant C in (4.72) is independent of k we use (4.60) so that it remains to prove that C_k^{I} in (4.60) (cf. (4.44)) is bounded independently of k. This, in turn, is a direct consequence of Corollary 5.13 and

$$C_{\mathrm{cont},k}\tilde{\eta}_1^{\exp}\overset{\text{Cor. 5.13}}{\leq}Ck^3\tilde{\eta}_1^{\exp}\overset{(4.69)}{\leq}Ck^{\alpha_1+3}\eta_1^{\exp}\leq Ck^{\alpha_3+3}\eta_1^{\exp}\overset{(4.71)}{\leq}C\varepsilon_1\leq C\tau\leq C$$

independent of k. ■

4.4.2 *hp*-FEM: Results

Theorem 4.17 *Let $\Omega=B_1(0)$ be the unit ball and let $\mathbf{E}$ denote the exact solution of (2.28b). Let the mesh $\mathcal{T}_h$ satisfy Assumption 3.1 and set $h:=\max_{K\in\mathcal{T}}h_K$. Let $S_h=S_{p+1}(\mathcal{T}_h)$ and $\mathbf{X}_h=\boldsymbol{\mathcal{N}}_p^{\mathrm{I}}(\mathcal{T}_h)$. Then there exist constants c_1, c_2, $C>0$ depending solely on the constants C_{affine}, C_{metric}, γ of Assumption 3.1 such the following holds: If $k\geq 1$, $p\geq 1$, $h>0$ satisfy*

$$\frac{kh}{p}\leq c_1\quad\text{and}\quad p\geq c_2\log k, \tag{4.73}$$

then the Galerkin approximation $\mathbf{E}_h\in\mathbf{X}_h$ (cf. (3.1)) exists and satisfies

$$\|\mathbf{E}-\mathbf{E}_h\|_{\Omega,\mathrm{curl},k}\leq C\inf_{\mathbf{w}_h\in\mathbf{X}_h}\|\mathbf{E}-\mathbf{w}_h\|_{\Omega,\mathrm{curl},k}. \tag{4.74}$$

Proof. The proof consists in checking the conditions of Corollary 4.16.The infima in η_2^{alg}, $\eta_j^{\exp}$, $j\in\{1,4\}$ are estimated with the aid the specific approximation operator $\Pi^{\mathrm{curl},s}$ analyzed in Lemma 8.5:

$$\eta_2^{\mathrm{alg}}\leq\sup_{\substack{\mathbf{z}\in\mathbf{H}^2(\Omega)\\ \|\mathbf{z}\|_{\mathbf{H}^2(\Omega)}\leq k}}\left\|\mathbf{z}-\Pi_p^{\mathrm{curl},s}\mathbf{z}\right\|_{\mathrm{curl},\Omega,k}\overset{\text{Lemma 8.5, (i)}}{\lesssim}\left(\frac{h}{p}+\frac{h^2}{p^2}k\right)k=\frac{kh}{p}+\left(\frac{kh}{p}\right)^2. \tag{4.75}$$

The terms $\eta_j^{\exp}$, $j \in \{1,4\}$ involve the approximation of analytic functions: The term $\eta_1^{\exp}$ involves the approximation from $\mathbf{X}_h = \boldsymbol{\mathcal{N}}_p^I(\mathcal{T}_h)$ and is estimated with Lemma 8.5, (ii); the term $\eta_4^{\exp}$ involves the approximation from $S_h = S_{p+1}(\mathcal{T}_h)$ and is taken from the proof of [41, Thm. 5.5]:

$$\sum_{j\in\{1,4\}} \eta_j^{\exp} \lesssim \left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p + k\left\{\left(\frac{h}{h+\sigma}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1}\right\}. \tag{4.76}$$

The terms η_6^{alg}, $\eta_7^{\exp}$ involve the operator $\Pi_p^{\text{curl},c}$. These are estimated in Lemma 8.6. Specifically, η_6^{alg} is controlled with Lemma 8.6, (iii) and $\eta_7^{\exp}$ is controlled with Lemma 8.6, (ii) after the observation (4.61) that $L_\Omega \mathbf{v}$ is in an analyticity class:

$$\eta_6^{\text{alg}} \lesssim \frac{hk}{p}, \tag{4.77}$$

$$\eta_7^{\exp} \lesssim k\left(\left(\frac{h_K}{h_K+\sigma}\right)^{p+1} + \left(\frac{kh_K}{\sigma p}\right)^{p+1}\right). \tag{4.78}$$

In total, we observe that an exponential convergence in p can absorb algebraic growth in k, which concludes the proof. ■

Corollary 4.18 *Let Assumption 2.1 be satisfied, and let the right-hand side in (2.28b) be defined by* $F(\mathbf{v}) = \left(\mathrm{i}\, k\tilde{\mathbf{j}}, \mathbf{v}\right)$ *for some* $\tilde{\mathbf{j}} \in \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{u} = 0 \wedge \langle \mathbf{u}|_\Gamma, \mathbf{n}\rangle = 0\right\}$. *Let the assumptions of Theorem 4.17 be satisfied. Then under the scale resolution condition (4.73), the Galerkin approximation* $\mathbf{E}_h \in \mathbf{X}_h$ *(cf. (3.1)) exists and satisfies*

$$\|\mathbf{E} - \mathbf{E}_h\|_{\Omega,\text{curl},k} \le C\frac{kh}{p}\left\|\tilde{\mathbf{j}}\right\|_{L^2(\Omega)}. \tag{4.79}$$

Proof. Under the assumption of this corollary the solution $\mathbf{E}$ is the restriction of the electric field of the full space problem (2.1) (with right-hand side defined as the extension of $\tilde{\mathbf{j}}$ to $\mathbb{R}^3$ by zero). In Section 7.1, we will derive a solution formula (7.15) for an adjoint Maxwell problem which can be easily adapted to the original Maxwell problem and to our assumption on the data $\tilde{\mathbf{j}}$. We obtain

$$\mathbf{E}(\mathbf{x}) = \mathrm{i}\, k\int_\Omega g_k(\|\mathbf{x}-\mathbf{y}\|)\,\tilde{\mathbf{j}}(\mathbf{y})\, d\mathbf{y} \quad \forall \mathbf{x} \in \Omega,$$

where g_k is the fundamental solution of the Helmholtz equation (7.9). From [41, Lemma 3.5], we know that there exist constants c, $C > 0$ independent of k and $\tilde{\mathbf{j}}$ such that, for every $\mu > 1$, there exists a μ- and k-dependent splitting $\mathbf{E} = \mathbf{E}_{H^2} + \mathbf{E}_{\mathcal{A}}$ with

$$\|\nabla^p \mathbf{E}_{H^2}\|_{L^2(\Omega)} \le C\left(1 + \frac{1}{\mu^2-1}\right)(\mu k)^{m-1}\left\|\tilde{\mathbf{j}}\right\|_{L^2(\Omega)} \qquad \forall m \in \{0,1,2\}, \tag{4.80a}$$

$$\|\nabla^p \mathbf{E}_{\mathcal{A}}\|_{L^2(\Omega)} \le C\mu\,(\gamma\mu k)^p\,\|\tilde{\mathbf{j}}\|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0. \tag{4.80b}$$

As in (4.75) and (4.76) we obtain constants C, $\sigma > 0$ independent of k, h, p, and $\tilde{\mathbf{j}}$:

$$\left\|\mathbf{E}_{H^2} - \Pi_p^{\text{curl},s}\mathbf{E}_{H^2}\right\|_{\text{curl},\Omega,k} \le C\frac{kh}{p}\left\|\tilde{\mathbf{j}}\right\|_{L^2(\Omega)},$$

$$\left\|\mathbf{E}_{\mathcal{A}} - \Pi_p^{\text{curl},s}\mathbf{E}_{\mathcal{A}}\right\|_{\text{curl},\Omega,k} \le C\left(\left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p + k\left(\frac{h}{h+\sigma}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1}\right)\left\|\tilde{\mathbf{j}}\right\|_{L^2(\Omega)}.$$

Suitably choosing c_1, c_2 in condition (4.73) implies the result. ■

5 k-explicit Analysis of Operators for $\Omega = B_1(0)$

A key ingredient of wavenumber-explicit estimates for the terms in the splitting (4.49b,c) of $((\mathbf{e}_h, \mathbf{v}_h))$ are k-explicit estimates of the capacity operator T_k for the low- and high-frequency parts of the arguments as these, in turn, allow for a k-explicit analysis of the continuity properties of the sesquilinear form $((\cdot,\cdot))$, the operators $\Pi^{\nabla,*}$, $\Pi^{\text{comp},*}$, and A_k. Our analysis of the operator T_k is based on the explicit knowledge of the Fourier coefficients and hence we restrict in this section to the case that $\Omega = B_1(0)$ is the ball with radius 1 centered at the origin. These estimates will be derived in Section 6 and applied to the different terms of the splitting of $((\mathbf{e}_h, \mathbf{v}_h))$ in Sections 6.1–6.3.

We also analyze in the present section the operator L_Ω and show that it maps into an analyticity class. The fact that we consider $\Omega = B_1(0)$ here implies the *a priori* bound $\|L_\Omega \mathbf{v}\|_{\text{curl},\Omega,k} \leq \|\mathbf{v}\|_{\text{curl},\Omega,k}$ which, in turn, leads to the quantitative assertion $L_\Omega \mathbf{v} \in \mathcal{A}(Ck^{3/2}\|\mathbf{v}\|_{\text{curl},\Omega,1}, \gamma, \Omega)$ in Theorem 5.9.

5.1 The Capacity Operator T_k on the Sphere

We restrict to the case that $\Omega^+ := \mathbb{R}^3 \backslash \overline{\Omega}$, where $\Omega = B_1(0)$ is the open unit ball with boundary Γ. Let $T_k : H_{\text{curl}}^{-1/2}(\Gamma) \to H_{\text{div}}^{-1/2}(\Gamma)$ be the capacity operator which was introduced in the paragraph before Remark 2.2. In the case of the sphere the eigenfunctions of the negative Laplace-Beltrami operator "$-\Delta_\Gamma$" are given by the spherical harmonics Y_ℓ^m (cf. [47, Sec. 2.4.1]) with eigenvalues $\lambda_\ell = \ell(\ell+1)$. In this case, the index set ι_ℓ in (2.17) is given by

$$\iota_\ell = \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}. \tag{5.1}$$

We introduce the decomposition of $\mathbf{E}_T$ according to (2.21) (cf. [47, (5.3.87)])

$$\mathbf{E}_T = \mathbf{E}^{\text{curl}} + \mathbf{E}^{\nabla}, \tag{5.2a}$$

where

$$\mathbf{E}^{\text{curl}} := \sum_{\ell=1}^{\infty} \sum_{m\in\iota_\ell} u_\ell^m \mathbf{T}_\ell^m \quad \text{and} \quad \mathbf{E}^{\nabla} = \nabla_\Gamma p \quad \text{with} \quad p := \sum_{\ell=1}^{\infty} \sum_{m\in\iota_\ell} U_\ell^m Y_\ell^m \tag{5.2b}$$

with the vectorial spherical harmonics $\mathbf{T}_\ell^m := \overrightarrow{\text{curl}_\Gamma} Y_\ell^m$ (cf. [47, (2.4.152), (2.4.173)]). This implies $\operatorname{div}_\Gamma \mathbf{E}_T^{\text{curl}} = 0$.

Remark 5.1 *For the expansion of a tangential field, e.g.,* $\mathbf{E}_T$ *the summation starts with* $\ell = 1$ *since* $\mathbf{T}_0 = \nabla_\Gamma Y_0^0 = \mathbf{0}$, *i.e.,*

$$\mathbf{E}_T = \sum_{\ell=1}^{\infty} \sum_{m\in\iota_\ell} \left(u_\ell^m \mathbf{T}_\ell^m + U_\ell^m \nabla Y_\ell^m\right). \tag{5.3}$$

Lemma 5.2 *Let* $\mathbf{E}_T \in H_{\text{curl}}^{-1/2}(\Gamma)$ *be decomposed as in (5.2). Then*

$$\operatorname{div}_\Gamma \mathbf{E}^{\text{curl}} = 0 \quad \textit{and} \quad \left(\mathbf{E}^{\text{curl}}, \mathbf{E}^{\nabla}\right)_\Gamma = b_k\left(\mathbf{E}^{\text{curl}}, \mathbf{E}^{\nabla}\right) = b_k\left(\mathbf{E}^{\nabla}, \mathbf{E}^{\text{curl}}\right) = 0. \tag{5.4}$$

Furthermore, we have the definiteness relations: For all $\mathbf{E} \in \mathbf{X}$ *it holds*

$$\operatorname{Im} b_k\left(\mathbf{E}^{\text{curl}}, \mathbf{E}^{\text{curl}}\right) \geq 0 \quad \textit{and} \quad \operatorname{Im} b_k\left(\mathbf{E}^{\nabla}, \mathbf{E}^{\nabla}\right) \leq 0. \tag{5.5}$$

Proof. It follows from [47, (5.3.87) and (5.3.91)] that the first term in (5.4) is zero. Integration by parts to the second term in (5.4) and using $\operatorname{div}_\Gamma \mathbf{E}^{\text{curl}} = 0$ shows that the second term vanishes. The third term in (5.4) vanishes as a consequence of $\operatorname{div}_\Gamma \mathbf{E}^{\text{curl}} = 0$ and [47, (5.3.109)]. The last term is zero since $T_k \mathbf{E}^{\nabla}$ is a linear combination of $\nabla_\Gamma Y_\ell^m$ (cf. [47, (5.3.87) and (5.3.88)]) and $\left(\nabla_\Gamma Y_\ell^m, \mathbf{E}^{\text{curl}}\right)_\Gamma = \left(Y_\ell^m, \operatorname{div}_\Gamma \mathbf{E}^{\text{curl}}\right)_\Gamma = 0$. The first inequality in (5.5) follows from [47, (5.3.107)] and the second one is a consequence of [47, (5.3.106)]. ■

Any tangential vector field $\mathbf{u}_T \in \mathbf{H}_{\text{curl}}^{-1/2}(\Gamma)$ can be expanded in terms of surface gradients of spherical harmonics Y_ℓ^m and vectorial spherical harmonics $\mathbf{T}_\ell^m$ via

$$\mathbf{u}_T = \mathbf{u}^{\text{curl}} + \mathbf{u}^{\nabla} \tag{5.6}$$

with

$$\mathbf{u}^{\text{curl}} := \sum_{\ell=1}^{\infty} \sum_{m\in\iota_\ell} u_\ell^m \mathbf{T}_\ell^m \quad \text{and} \quad \mathbf{u}^{\nabla} := \nabla_\Gamma p \quad \text{with} \quad p := \sum_{\ell=1}^{\infty} \sum_{m\in\iota_\ell} U_\ell^m Y_\ell^m.$$

The application of the capacity operator T_k to $\mathbf{u}_T$ has the explicit form (cf. [47, (5.3.88)])

$$T_k \mathbf{u}_T = \sum_{\ell=1}^{\infty} \frac{z_\ell(k)+1}{\mathrm{i}\,k} \sum_{m\in\iota_\ell} u_\ell^m \mathbf{T}_\ell^m + \sum_{\ell=1}^{\infty} \frac{\mathrm{i}\,k}{z_\ell(k)+1} \sum_{m\in\iota_\ell} U_\ell^m \nabla_\Gamma Y_\ell^m, \tag{5.7}$$

where

$$z_\ell(r) := r\frac{\left(h_\ell^{(1)}\right)'(r)}{h_\ell^{(1)}(r)} = -\frac{p_\ell\left(r^{-2}\right)}{q_\ell\left(r^{-2}\right)} + \mathrm{i}\frac{r}{q_\ell\left(r^{-2}\right)}, \tag{5.8}$$

with the spherical Hankel functions $h_\ell^{(1)}$, and p_ℓ, q_ℓ are polynomials of degree ℓ with real coefficients (cf. [47, (2.6.19)-(2.6.22)]).

Lemma 5.3 *Let $\lambda_0 > 1$ arbitrary but fixed. Then there exists C_0 depending only on λ_0 such that for any $\lambda \geq \lambda_0$:*

$$\frac{k}{|z_n(k)+1|} \leq \begin{cases} 2\sqrt{2}k & \forall n \in \mathbb{N}_0, \\ 2\sqrt{2}\left(\frac{2}{\lambda_0}+1\right)\frac{k}{(n+1)} & n > \lambda k^2, \\ C_0\frac{k}{n+1} & n \geq \lambda k. \end{cases} \tag{5.9}$$

It holds

$$\frac{|z_n(k)+1|}{k} \leq 1 + \frac{n}{k}. \tag{5.10}$$

Estimate (5.10) follows from [47, (5.3.95)]. The proof of (5.9) is rather technical and postponed to the Appendix A.

Remark 5.4 *From (5.5), (5.7), and (5.8) we conclude that Assumption 4.1 is satisfied for the sphere.* ∎

5.2 Analysis of Frequency Splittings L_Γ, H_Γ on the Surface of the Sphere

5.2.1 Analyticity of L_Γ

Lemma 5.5 *Let $\Omega = B_1(0)$.*

(i) There exists a fixed tubular neighborhood $\mathcal{U}_\Gamma$ of Γ and constants C_2, γ_2 independent of k (but dependent on Γ, λ) such that for each $\mathbf{u} \in \mathbf{H}(\mathrm{curl}, \Omega)$ there is an extension $\mathbf{U} \in \mathcal{A}(C_2 k^{3/2}\|\mathbf{u}\|_{\mathrm{curl},\Omega,1}, \gamma_2, \mathcal{U}_\Gamma)$ of $L_\Gamma \mathbf{u}_T$ to $\mathcal{U}_\Gamma$.

(ii) The function $L_\Gamma \Pi_T \mathbf{u}$ belongs to the class $\mathcal{A}\left(C_1 k^{3/2}\|\mathbf{u}\|_{\mathrm{curl},\Omega,1}, \gamma_1, \Gamma\right)$, where C_1, γ_1 are constants which are independent of k and $\mathbf{u}$. In particular, $\|L_\Gamma \Pi_T \mathbf{u}\|_{H^{1/2}(\Gamma)} \leq C_1' k^2 \|\mathbf{u}\|_{\mathrm{curl},\Omega,1}$.

Proof. Before proving Lemma 5.5, we mention that the algebraic growth rates with respect to k are likely suboptimal. However, sharper estimates would require more technicalities. We start by noting that the analyticity of Γ provides that the eigenfunctions Y_ℓ^m of the Laplace-Beltrami operator have analytic extensions $\tilde{Y}_\ell^m$ to a tubular neighborhood $\mathcal{U}_\Gamma$ of Γ. A quantitative bound in terms of the eigenvalue λ_ℓ is given in [37, Lemma C.1]

$$\left\|\nabla^n \tilde{Y}_\ell^m\right\|_{L^2(\mathcal{U}_\Gamma)} \leq C_S \max\left\{\sqrt{\lambda_\ell}, n\right\}^n \gamma_S^n \qquad \forall n \in \mathbb{N}_0 \tag{5.11}$$

for some C_S, γ_S depending solely on Γ. We recall specifically that the eigenvalues λ_ℓ of the Laplace-Beltrami operator on the sphere are $\lambda_\ell = \ell(\ell+1)$.

Proof of (i): Let $\mathbf{u}_T$ denote a tangential field on the sphere with the representation (cf. (5.3))

$$\mathbf{u}_T = \sum_{\ell=1}^{\infty}\sum_{m\in\iota_\ell}\left(u_\ell^m \mathbf{T}_\ell^m + U_\ell^m \nabla_\Gamma Y_\ell^m\right), \qquad \mathbf{T}_\ell^m = \overrightarrow{\mathrm{curl}_\Gamma} Y_\ell^m = \nabla_\Gamma Y_\ell^m \times \mathbf{n}.$$

With the extension $\mathbf{n}^*$ of the normal vector $\mathbf{n}$ that is constant in normal direction, we may define the extension $\mathbf{U}$ of $L_\Gamma \mathbf{u}_T$ as

$$\mathbf{U} = \sum_{1\leq\ell\leq\lambda k}\sum_{m\in\iota_\ell} u_\ell^m \nabla\tilde{Y}_\ell^m \times \mathbf{n}^* + U_\ell^m \nabla\tilde{Y}_\ell^m.$$

By the analyticity of $\mathbf{n}^*$ we get from Lemma 2.6 and (5.11) that, for some C', $\tilde{\gamma} > 0$ depending solely on Γ,

$$\|\nabla^n(\nabla\tilde{Y}_\ell^m \times \mathbf{n}^*)\|_{L^2(\mathcal{U}_\Gamma)} + \|\nabla^n(\nabla\tilde{Y}_\ell^m)\|_{L^2(\mathcal{U}_\Gamma)} \leq C'\sqrt{\lambda_\ell}\,\tilde{\gamma}^n \max\{\sqrt{\lambda_\ell}, n\}^n \qquad \forall n \in \mathbb{N}_0. \tag{5.12}$$

We take $\ell \leq \lambda k$ into account (and writing $\lambda_\ell = \ell(\ell+1)$) which allows us to estimate $\mathbf{U}$ by

$$
\begin{aligned}
\|\nabla^n \mathbf{U}\|_{L^2(\mathcal{U}_\Gamma)} &\leq \sum_{1\leq \ell \leq \lambda k} \sum_{m \in \iota_\ell} \left(|u_\ell^m| \left\| \nabla^n (\nabla \tilde{Y}_\ell^m \times \mathbf{n}^*) \right\|_{L^2(\mathcal{U}_\Gamma)} + |U_\ell^m| \left\| \nabla^{n+1} \tilde{Y}_\ell^m \right\|_{L^2(\mathcal{U}_\Gamma)} \right) \\
&\lesssim \tilde{\gamma}^n \sum_{1\leq \ell \leq \lambda k} \max\left\{ \sqrt{\lambda_\ell}, n \right\}^n \lambda_\ell^{1/4} \lambda_\ell^{1/4} \sum_{m=-\ell}^{\ell} \left(|u_\ell^m| + |U_\ell^m| \right) \\
&\lesssim \tilde{\gamma}^n \left(\sum_{1\leq \ell \leq \lambda k} \sum_{m=-\ell}^{\ell} \max\left\{ \sqrt{\lambda_\ell}, n \right\}^{2n} \lambda_\ell^{1/2} \right)^{1/2} \left(\sum_{1\leq \ell \leq \lambda k} \sum_{m=-\ell}^{\ell} \lambda_\ell^{1/2} \left(|u_\ell^m| + |U_\ell^m| \right)^2 \right)^{1/2} \\
&\overset{(2.24a)}{\lesssim} \tilde{\gamma}^n (\lambda k + 1)^{3/2} \max\{\lambda k + 1, n\}^n \|L_\Gamma \mathbf{u}_T\|_{-1/2, \operatorname{curl}_\Gamma}.
\end{aligned} \tag{5.13}
$$

Since $\|L_\Gamma \mathbf{u}_T\|_{-1/2,\operatorname{curl},\Gamma} \leq \|\mathbf{u}_T\|_{-1/2,\operatorname{curl},\Gamma} \lesssim \|\mathbf{u}\|_{\operatorname{curl},\Omega,1}$, the proof of (i) is complete.
Proof of (ii): An application of the multiplicative trace inequality would allow us to infer from (i) the assertion $L_\Gamma \Pi_T \mathbf{u} \in \mathcal{A}(C_1 k^2 \|\mathbf{u}\|_{\operatorname{curl},\Omega,1}, \gamma_1, \Gamma)$ for suitable C_1, γ_1. The sharper statement follows by repeating the arguments of (i) starting with the assertion of [37, Lemma C.1] that

$$
\|\nabla_\Gamma^n Y_\ell^m\|_{L^2(\Gamma)} \leq C_S \gamma_S^n \max\{\sqrt{\lambda_\ell}, n\}^n \qquad \forall n \in \mathbb{N}_0. \tag{5.14}
$$

■

5.2.2 Estimates for High and Low Frequency Parts of the Capacity Operator

In this section we derive continuity estimates for the sesquilinear form b_k. The k-dependence is different for the low- and high-frequency parts of the tangential fields and for the summands in the splitting $\mathbf{u}_T = \mathbf{u}^{\operatorname{curl}} + \mathbf{u}^\nabla$. In Proposition 5.7 we derive such estimates for the tangential fields while these estimates are lifted to the space $\mathbf{X}$ and some subspaces thereof in Proposition 5.8.

Remark 5.6 *If Γ is the surface of the unit ball there holds for all $\mathbf{u}_T, \mathbf{v}_T \in \mathbf{H}^{-1/2}_{\operatorname{curl}}(\Gamma)$*

$$
\begin{aligned}
b_k^{\text{high}}(\mathbf{u}_T, \mathbf{v}_T) &\overset{Def.\ 4.5}{=} (T_k \mathbf{u}_T, H_\Gamma \mathbf{v}_T)_\Gamma \overset{Def.\ 4.2}{=} (H_\Gamma T_k \mathbf{u}_T, \mathbf{v}_T)_\Gamma \\
&\overset{(5.7)}{=} (T_k H_\Gamma \mathbf{u}_T, \mathbf{v}_T)_\Gamma \overset{Def.\ 4.5}{=} \left(T_k^{\text{high}} \mathbf{u}_T, \mathbf{v}_T \right)_\Gamma.
\end{aligned}
$$

Analogous relations hold for the low frequency part b_k^{low}. ■

Proposition 5.7 *With the frequency filters L_Γ, H_Γ of Definition 4.2 given by a cut-off parameter $\lambda \geq \lambda_0 > 1$ the sesquilinear form b_k can be written as*

$$
b_k(\mathbf{u}_T, \mathbf{v}_T) = b_k\left(\mathbf{u}^{\operatorname{curl}}, \mathbf{v}^{\operatorname{curl}}\right) + b_k^{\text{high}}\left(\mathbf{u}^\nabla, \mathbf{v}^\nabla\right) + b_k^{\text{low}}\left(\mathbf{u}^\nabla, \mathbf{v}^\nabla\right) \quad \forall \mathbf{u}_T, \mathbf{v}_T \in \mathbf{H}^{-1/2}_{\operatorname{curl}}(\Gamma), \tag{5.15}
$$

and there is $C_b > 0$ depending solely on λ_0 such that the following holds:

$$
\begin{aligned}
\left| b_k\left(\mathbf{u}^{\operatorname{curl}}, \mathbf{v}^{\operatorname{curl}}\right) \right| &\leq C_b \left(\frac{1}{k} \|\operatorname{curl}_\Gamma \mathbf{u}_T\|_{H^{-1/2}(\Gamma)} \|\operatorname{curl}_\Gamma \mathbf{v}_T\|_{H^{-1/2}(\Gamma)} \right. \\
&\qquad \left. + (1+\lambda) \|\operatorname{curl}_\Gamma L_\Gamma \mathbf{u}_T\|_{H^{-1}(\Gamma)} \|\operatorname{curl}_\Gamma L_\Gamma \mathbf{v}_T\|_{H^{-1}(\Gamma)} \right), \\
\left| b_k^{\text{high}}\left(\mathbf{u}^{\operatorname{curl}}, \mathbf{v}^{\operatorname{curl}}\right) \right| &\leq C_b \frac{1}{k} \|\operatorname{curl}_\Gamma \mathbf{u}_T\|_{H^{-1/2}(\Gamma)} \|\operatorname{curl}_\Gamma \mathbf{v}_T\|_{H^{-1/2}(\Gamma)}, \\
\left| b_k^{\text{low}}\left(\mathbf{u}^\nabla, \mathbf{v}^\nabla\right) \right| &\leq C_b \lambda^\rho k^{\rho+1} \|\operatorname{div}_\Gamma L_\Gamma \mathbf{u}_T\|_{H^{-1-\rho/2}(\Gamma)} \|\operatorname{div}_\Gamma L_\Gamma \mathbf{v}_T\|_{H^{-1-\rho/2}(\Gamma)}
\end{aligned} \tag{5.16}
$$

for $0 \leq \rho \leq 2$. If $\operatorname{div}_\Gamma \mathbf{u}_T \in H^{\rho_1}(\Gamma)$ and $\operatorname{div} \mathbf{v}_T \in H^{\rho_2}(\Gamma)$ for some $\rho_1 + \rho_2 + 3 \geq 0$ we have

$$
\left| b_k^{\text{high}}\left(\mathbf{u}^\nabla, \mathbf{v}^\nabla\right) \right| \leq C_b \frac{k}{(\lambda k)^{\rho_1+\rho_2+3}} \|\operatorname{div}_\Gamma \mathbf{u}_T\|_{H^{\rho_1}(\Gamma)} \|\operatorname{div}_\Gamma \mathbf{v}_T\|_{H^{\rho_2}(\Gamma)}. \tag{5.17}
$$

Proof. The equality (5.15) follows from Lemma 5.2.
By using the orthogonality relations of $\mathbf{T}_\ell^m$ and $\nabla_\Gamma Y_\ell^m$, the representations in [47, Sec. 5.3.2] give us

$$
\begin{aligned}
\left|b_k\left(\mathbf{u}^{\mathrm{curl}},\mathbf{v}^{\mathrm{curl}}\right)\right| &= \left|\mathrm{i}\sum_{\ell=1}^{\infty}\left(-\frac{z_\ell(k)+1}{k}\right)\sum_{m=-\ell}^{\ell}u_\ell^m\overline{v_\ell^m}\left(\mathbf{T}_\ell^m,\mathbf{T}_\ell^m\right)_\Gamma\right| \\
&\overset{[47,\ (2.4.155)]}{\leq} \left|\mathrm{i}\sum_{\ell=1}^{\infty}\ell(\ell+1)\left(-\frac{z_\ell(k)+1}{k}\right)\sum_{m=-\ell}^{\ell}u_\ell^m\overline{v_\ell^m}\right| \\
&\overset{(5.10)}{\leq} \left|\sum_{\ell=1}^{\infty}\ell(\ell+1)\left(1+\frac{\ell}{k}\right)\sum_{m\in\iota_\ell}u_\ell^m\overline{v_\ell^m}\right| \\
&\overset{(2.23)}{\leq} 2\left(\frac{1}{k}\left\|\mathrm{curl}_\Gamma\,\mathbf{u}_T\right\|_{H^{-1/2}(\Gamma)}\left\|\mathrm{curl}_\Gamma\,\mathbf{v}_T\right\|_{H^{-1/2}(\Gamma)}\right. \\
&\qquad\left.+(1+\lambda)\left\|\mathrm{curl}_\Gamma\,L_\Gamma\mathbf{u}_T\right\|_{H^{-1}(\Gamma)}\left\|\mathrm{curl}_\Gamma\,L_\Gamma\mathbf{v}_T\right\|_{H^{-1}(\Gamma)}\right).
\end{aligned}
$$

This leads to the first estimate in (5.16). In a similar way we obtain for the high-frequency part

$$
\begin{aligned}
\left|b_k^{\mathrm{high}}\left(\mathbf{u}^{\mathrm{curl}},\mathbf{v}^{\mathrm{curl}}\right)\right| &\leq \left|\sum_{\ell\geq\lambda k}^{\infty}\ell(\ell+1)\left(1+\frac{\ell}{k}\right)\sum_{m\in\iota_\ell}u_\ell^m\overline{v_\ell^m}\right| \\
&\leq \frac{2}{k}\sum_{\ell\geq\lambda k}^{\infty}\ell^2(\ell+1)\sum_{m\in\iota_\ell}\left|u_\ell^m\right|\left|v_\ell^m\right| \\
&\overset{(2.23)}{\leq} \frac{2}{k}\left\|\mathrm{curl}_\Gamma\,H_\Gamma\mathbf{u}_T\right\|_{H^{-1/2}(\Gamma)}\left\|\mathrm{curl}_\Gamma\,H_\Gamma\mathbf{v}_T\right\|_{H^{-1/2}(\Gamma)}
\end{aligned}
$$

so that the second estimate in (5.16) follows. For the third one and (5.17) we obtain

$$
\begin{aligned}
b_k^{\mathrm{low}}\left(\mathbf{u}^\nabla,\mathbf{v}^\nabla\right) &= \mathrm{i}\sum_{1\leq\ell\leq\lambda k}\ell(\ell+1)\sum_{m\in\iota_\ell}\left(\frac{k}{z_\ell(k)+1}U_\ell^m\overline{V_\ell^m}\right), \\
b_k^{\mathrm{high}}\left(\mathbf{u}^\nabla,\mathbf{v}^\nabla\right) &= \mathrm{i}\sum_{\ell>\lambda k}\ell(\ell+1)\sum_{m\in\iota_\ell}\left(\frac{k}{z_\ell(k)+1}U_\ell^m\overline{V_\ell^m}\right).
\end{aligned}
$$

By using (5.3) and (5.9) we get for any $0\leq\rho\leq 2$

$$
\begin{aligned}
\left|b_k^{\mathrm{low}}\left(\mathbf{u}^\nabla,\mathbf{v}^\nabla\right)\right| &\leq 2\sqrt{2}k\sum_{1\leq\ell\leq\lambda k}\ell(\ell+1)\sum_{m\in\iota_\ell}\left|U_\ell^m\right|\left|V_\ell^m\right| \\
&\leq 4\sqrt{2}k(\lambda k)^\rho\sum_{1\leq\ell\leq\lambda k}\ell^{2-\rho}\sum_{m\in\iota_\ell}\left|U_\ell^m\right|\left|V_\ell^m\right| \\
&\leq 16\sqrt{2}\lambda^\rho k^{\rho+1}\sum_{1\leq\ell\leq\lambda k}\ell^2(\ell+1)^{-\rho}\sum_{m\in\iota_\ell}\left|U_\ell^m\right|\left|V_\ell^m\right| \\
&\overset{(2.23)}{\leq} 16\sqrt{2}\lambda^\rho k^{\rho+1}\left\|\mathrm{div}_\Gamma\,L_\Gamma\mathbf{u}_T\right\|_{H^{-1-\rho/2}}\left\|\mathrm{div}_\Gamma\,L_\Gamma\mathbf{u}_T\right\|_{H^{-1-\rho/2}}.
\end{aligned}
$$

For $\rho_1+\rho_2+3\geq 0$ we get from (5.9)

$$
\begin{aligned}
\left|b_k^{\mathrm{high}}\left(\mathbf{u}^\nabla,\mathbf{v}^\nabla\right)\right| &\leq C_0 k\sum_{\ell>\lambda k}\ell\sum_{m\in\iota_\ell}\left|U_\ell^m\right|\left|V_\ell^m\right| \\
&\leq \frac{C_0 k}{(\lambda k)^{\rho_1+\rho_2+3}}\sum_{\ell>\lambda k}\ell^{4+\rho_1+\rho_2}\sum_{m\in\iota_\ell}\left|U_\ell^m\right|\left|V_\ell^m\right| \\
&\overset{(2.23)}{\leq} C\frac{k}{(\lambda k)^{\rho_1+\rho_2+3}}\left\|\mathrm{div}_\Gamma\,\mathbf{u}_T\right\|_{H^{\rho_1}(\Gamma)}\left\|\mathrm{div}_\Gamma\,\mathbf{v}_T\right\|_{H^{\rho_2}(\Gamma)}.
\end{aligned}
$$

■

Proposition 5.8 *There is a constant $C_b' > 0$ depending solely on λ_0 such that the following holds: Let $\mathbf{u}, \mathbf{v} \in \mathbf{X}$. Then:*

$$\left|b_k\left(\mathbf{u}^{\nabla}, \mathbf{v}^{\nabla}\right)\right| \leq C_b' k^2 \|\mathbf{u}\|_{\text{curl},\Omega,1} \|\mathbf{v}\|_{\text{curl},\Omega,1}. \tag{5.18}$$

Let $\mathbf{u}_0 \in \mathbf{V}_0$ and $\mathbf{v} \in \mathbf{X}$. Then:

$$\left|k b_k^{\text{high}}\left(\mathbf{u}_0^{\nabla}, \mathbf{v}^{\nabla}\right)\right| \leq C_b' \frac{k}{\lambda} \|\mathbf{u}_0\|_{\text{curl},\Omega,1} \|\mathbf{v}\|_{\text{curl},\Omega,1}, \tag{5.19a}$$

$$\left|k b_k^{\text{low}}\left(\mathbf{u}_0^{\nabla}, \mathbf{v}^{\nabla}\right)\right| \leq C_b' \lambda k^3 \|\operatorname{div}_\Gamma L_\Gamma \mathbf{u}_{0,T}\|_{H^{-3/2}(\Gamma)} \|\operatorname{div}_\Gamma L_\Gamma \mathbf{v}_T\|_{H^{-3/2}(\Gamma)}. \tag{5.19b}$$

For $\mathbf{u} \in \mathbf{X}$ and $\mathbf{v}_0 \in \mathbf{V}_0^*$ it holds

$$\left|k b_k^{\text{high}}\left(\mathbf{u}^{\nabla}, \mathbf{v}_0^{\nabla}\right)\right| \leq C_b' \frac{k}{\lambda} \|\mathbf{u}\|_{\text{curl},\Omega,1} \|\mathbf{v}_0\|_{\text{curl},\Omega,1}, \tag{5.20a}$$

$$\left|k b_k^{\text{low}}\left(\mathbf{u}^{\nabla}, \mathbf{v}_0^{\nabla}\right)\right| \leq C_b' \lambda k^3 \|\operatorname{div}_\Gamma L_\Gamma \mathbf{u}_T\|_{H^{-3/2}(\Gamma)} \|\operatorname{div}_\Gamma L_\Gamma \mathbf{v}_{0,T}\|_{H^{-3/2}(\Gamma)}. \tag{5.20b}$$

For $\mathbf{u}_0 \in \mathbf{V}_0$, $\mathbf{v}_0 \in \mathbf{V}_0^*$ and $p, q \in H^1(\Omega)$ it holds

$$\left|k b_k^{\text{high}}\left(\mathbf{u}_0^{\nabla}, (\nabla p)^{\nabla}\right)\right| \leq \frac{C_b'}{\lambda} \|\mathbf{u}_0\|_{\text{curl},\Omega,1} \|\nabla p\|_{\text{curl},\Omega,k}, \tag{5.21a}$$

$$\left|k b_k^{\text{high}}\left((\nabla p)^{\nabla}, \mathbf{v}_0^{\nabla}\right)\right| \leq \frac{C_b'}{\lambda} \|\nabla p\|_{\text{curl},\Omega,k} \|\mathbf{v}_0\|_{\text{curl},\Omega,1}, \tag{5.21b}$$

$$\left|k b_k^{\text{high}}\left((\nabla p)^{\nabla}, (\nabla q)^{\nabla}\right)\right| \leq C_b' \|\nabla p\|_{\text{curl},\Omega,k} \|\nabla q\|_{\text{curl},\Omega,k}. \tag{5.21c}$$

For $\mathbf{u}_0 \in \mathbf{V}_0$ and $\mathbf{v}_0 \in \mathbf{V}_0^*$ it holds

$$\left|k b_k^{\text{high}}\left(\mathbf{u}_0^{\nabla}, \mathbf{v}_0^{\nabla}\right)\right| \leq \frac{C_b'}{\lambda^2} \|\mathbf{u}_0\|_{\text{curl},\Omega,1} \|\mathbf{v}_0\|_{\text{curl},\Omega,1}, \tag{5.22a}$$

$$\left|k b_k^{\text{low}}\left(\mathbf{u}_0^{\nabla}, \mathbf{v}_0^{\nabla}\right)\right| \leq C_b' \lambda k^3 \|\mathbf{u}_0\|_{\text{curl},\Omega,1} \|\mathbf{v}_0\|_{\text{curl},\Omega,1}. \tag{5.22b}$$

For $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\mathbf{w} \in \mathbf{X}$, it holds

$$\left|k b_k^{\text{high}}\left(\mathbf{u}^{\nabla}, \mathbf{v}^{\nabla}\right)\right| \leq \frac{C_b'}{\lambda^2} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}, \tag{5.23a}$$

$$\left|k b_k^{\text{high}}\left(\mathbf{w}^{\nabla}, \mathbf{v}^{\nabla}\right)\right| \leq C_b' \frac{k}{\lambda} \|\mathbf{w}\|_{\text{curl},\Omega,1} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}. \tag{5.23b}$$

Proof. *Proof of (5.18):* We combine the last estimate in (5.16) (for $\rho = 1$ and $\lambda = \lambda_0$) with (5.17) (for $\rho_1 = \rho_2 = -3/2$ and $\lambda = \lambda_0$) and obtain

$$\begin{aligned}
\left|b_k\left(\mathbf{u}^{\nabla}, \mathbf{v}^{\nabla}\right)\right| &\leq \left|b_k^{\text{low}}\left(\mathbf{u}^{\nabla}, \mathbf{v}^{\nabla}\right)\right| + \left|b_k^{\text{high}}\left(\mathbf{u}^{\nabla}, \mathbf{v}^{\nabla}\right)\right| \\
&\leq C_b \Big(\lambda_0 k^2 \|\operatorname{div}_\Gamma L_\Gamma \mathbf{u}_T\|_{H^{-3/2}(\Gamma)} \|\operatorname{div}_\Gamma L_\Gamma \mathbf{v}_T\|_{H^{-3/2}(\Gamma)} \\
&\quad + k \|\operatorname{div}_\Gamma \mathbf{u}_T\|_{H^{-3/2}(\Gamma)} \|\operatorname{div}_\Gamma \mathbf{v}_T\|_{H^{-3/2}(\Gamma)}\Big) \\
&\leq C(1 + \lambda_0 k) k \|\operatorname{div}_\Gamma \mathbf{u}_T\|_{H^{-3/2}(\Gamma)} \|\operatorname{div}_\Gamma \mathbf{v}_T\|_{H^{-3/2}(\Gamma)} \\
&\leq C(1 + \lambda_0 k) k \|\mathbf{u}_T\|_{H^{-1/2}(\Gamma)} \|\mathbf{v}_T\|_{H^{-1/2}(\Gamma)} \\
&\leq C(1 + \lambda_0 k) k \|\mathbf{u}\|_{\text{curl},\Omega,1} \|\mathbf{v}\|_{\text{curl},\Omega,1}.
\end{aligned} \tag{5.24}$$

Proof of (5.19), (5.20), (5.23b): Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $\mathbf{v} \in \mathbf{X}$. Choose $\rho_1 = -1/2$ and $\rho_2 = -3/2$ in (5.17) to obtain

$$\begin{aligned}
\left|k b_k^{\text{high}}\left(\mathbf{u}^{\nabla}, \mathbf{v}^{\nabla}\right)\right| &\leq C_b \frac{k}{\lambda} \|\operatorname{div}_\Gamma \mathbf{u}_T\|_{H^{-1/2}(\Gamma)} \|\operatorname{div}_\Gamma \mathbf{v}_T\|_{H^{-3/2}(\Gamma)} \\
&\leq C \frac{k}{\lambda} \|\mathbf{u}_T\|_{\mathbf{H}^{1/2}(\Gamma)} \|\mathbf{v}_T\|_{\mathbf{H}^{-1/2}(\Gamma)} \\
&\leq C \frac{k}{\lambda} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\text{curl},\Omega,1}.
\end{aligned} \tag{5.25}$$

This shows (up to interchanging the roles of $\mathbf{u}$ and $\mathbf{v}$) the estimate (5.23b). Since $\mathbf{V}_0 \subset \mathbf{H}^1(\Omega)$, we may apply estimate (5.25) to $\mathbf{u} \in \mathbf{V}_0$ and $\mathbf{v} \in \mathbf{X}$. Lemma B.1 implies the estimate $\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq \|\mathbf{u}\|_{\mathrm{curl},\Omega,1}$ so that (5.19a) follows. For the low frequency part we get from (5.16) for $\rho = 1$ the estimate

$$\left|kb_k^{\mathrm{low}}\left(\mathbf{u}_0^{\nabla}, \mathbf{v}^{\nabla}\right)\right| \leq C_b \lambda k^3 \left\|\operatorname{div}_\Gamma L_\Gamma \mathbf{u}_{0,T}\right\|_{H^{-3/2}(\Gamma)} \left\|\operatorname{div}_\Gamma L_\Gamma \mathbf{v}_T\right\|_{H^{-3/2}(\Gamma)},$$

which is (5.19b). For $\mathbf{u} \in \mathbf{X}$ and $\mathbf{v}_0 \in \mathbf{V}_0^*$, estimates (5.20) follow by the same arguments and interchanging the roles of $\mathbf{u}$ and $\mathbf{v}$.
Proof of (5.21): For $\mathbf{u}_0 \in \mathbf{V}_0$ and $p \in H^1(\Omega)$ we employ (5.19a) with $\mathbf{v} = \nabla p$ and $\operatorname{curl} \nabla p = 0$ so that

$$\begin{aligned}\left|kb_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, (\nabla p)^{\nabla}\right)\right| &\leq C\frac{k}{\lambda} \|\mathbf{u}_0\|_{\mathrm{curl},\Omega,1} \|\nabla p\|_{\mathrm{curl},\Omega,1} = \frac{C}{\lambda} \|\mathbf{u}_0\|_{\mathrm{curl},\Omega,1} (k \|\nabla p\|) \\ &\leq \frac{C}{\lambda} \|\mathbf{u}_0\|_{\mathrm{curl},\Omega,1} \|\nabla p\|_{\mathrm{curl}\,\Omega,k},\end{aligned}$$

which shows (5.21a). The proof of (5.21b) is just a repetition of the previous arguments while the proof of (5.21c) uses (5.17) for $\rho_1 = \rho_2 = -3/2$:

$$\begin{aligned}\left|kb_k^{\mathrm{high}}\left((\nabla p)^{\nabla}, (\nabla q)^{\nabla}\right)\right| &\leq C_b k^2 \left\|\operatorname{div}_\Gamma (\nabla p)_T\right\|_{H^{-3/2}(\Gamma)} \left\|\operatorname{div}_\Gamma (\nabla q)_T\right\|_{H^{-3/2}(\Gamma)} \\ &\leq Ck^2 \|\nabla p\|_{\mathrm{curl},\Omega,1} \|\nabla q\|_{\mathrm{curl},\Omega,1} \\ &= Ck^2 \|\nabla p\| \|\nabla q\| = C \|\nabla p\|_{\mathrm{curl},\Omega,k} \|\nabla q\|_{\mathrm{curl},\Omega,k},\end{aligned}$$

where the second step uses the same arguments as in (5.24).
Proof of (5.22), (5.23a): For any $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$ we may choose $\rho_1 = \rho_2 = -1/2$ in (5.17) to obtain

$$\begin{aligned}\left|kb_k^{\mathrm{high}}\left(\mathbf{u}^{\nabla}, \mathbf{v}^{\nabla}\right)\right| &\leq \frac{C_b}{\lambda^2} \left\|\operatorname{div}_\Gamma \mathbf{u}_T\right\|_{H^{-1/2}(\Gamma)} \left\|\operatorname{div}_\Gamma \mathbf{v}_T\right\|_{H^{-1/2}(\Gamma)} \\ &\leq \frac{C}{\lambda^2} \|\mathbf{u}_T\|_{\mathbf{H}^{1/2}(\Gamma)} \|\mathbf{v}_T\|_{\mathbf{H}^{1/2}(\Gamma)} \leq \frac{C}{\lambda^2} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}.\end{aligned} \tag{5.26}$$

This proves (5.23a). If we assume in addition $\mathbf{u}_0 \in \mathbf{V}_0$ and $\mathbf{v}_0 \in \mathbf{V}_0^*$ we can appeal to Lemma B.1 to get

$$\left|kb_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \mathbf{v}_0^{\nabla}\right)\right| \leq \frac{C}{\lambda^2} \|\mathbf{u}_0\|_{\mathrm{curl},\Omega,1} \|\mathbf{v}_0\|_{\mathrm{curl},\Omega,1},$$

i.e., (5.22a). For (5.22b) we employ the last equation in (5.16) for $\rho = 1$ and proceed as for (5.22a). ■

5.3 Analysis of Frequency Splittings L_Ω, H_Ω for the Case $\Omega = B_1(0)$

The operator L_Ω is defined in Definition 4.2 as the minimum norm extension from the boundary with respect to the norm $\|\cdot\|_{\mathrm{curl},\Omega,k}$. From Lemma C.1 we have the following stability estimate for the case $\Omega = B_1(0)$:

$$\|L_\Omega \mathbf{u}\|_{\mathrm{curl},\Omega,k} \leq \|\mathbf{u}\|_{\mathrm{curl},\Omega,k}. \tag{5.27a}$$

By the triangle inequality we infer that also H_Ω is stable

$$\|H_\Omega \mathbf{u}\|_{\mathrm{curl},\Omega,k} \leq 2 \|\mathbf{u}\|_{\mathrm{curl},\Omega,k}. \tag{5.27b}$$

For the present case of analytic Γ we have the following result.

Theorem 5.9 *Let $\Omega = B_1(0)$. Then the low-frequency part $L_\Omega \mathbf{u}$ satisfies*

$$\|L_\Omega \mathbf{u}\|_{\mathrm{curl},\Omega,k} \leq \|\mathbf{u}\|_{\mathrm{curl},\Omega,k} \quad \text{and} \quad \operatorname{div} L_\Omega \mathbf{u} = 0. \tag{5.28}$$

Furthermore, $L_\Omega \mathbf{u} \in \mathcal{A}(C_{\mathcal{A},5} C_u'', \gamma_{\mathcal{A},5}, \Omega)$ with

$$C_u'' = k^{3/2} \|\mathbf{u}\|_{\mathrm{curl},\Omega,1}. \tag{5.29}$$

The constants $C_{\mathcal{A},5}$, $\gamma_{\mathcal{A},5}$ are independent of k and $\mathbf{u}$ but depend on Γ and the choice of cut-off parameter λ. Furthermore, there exists a tubular neighborhood $\mathcal{U}_\Gamma$ of Γ such that $L_\Omega \mathbf{u}$ is analytic on $\Omega \cup \mathcal{U}_\Gamma$ with $L_\Omega \mathbf{u} \in \mathcal{A}(C'_{\mathcal{A},5} C_u'', \gamma'_{\mathcal{A},5}, \Omega \cup \mathcal{U}_\Gamma)$.

Proof. *1. step (interior regularity):* Using the vector identity

$$\operatorname{curl}\operatorname{curl} = -\Delta + \nabla \operatorname{div} \tag{5.30}$$

we infer from (4.9) that $-\Delta L_\Omega \mathbf{u} + k^2 L_\Omega \mathbf{u} = 0$ in Ω. Interior regularity in the form [35, Prop. 5.5.1] then gives $L_\Omega \mathbf{u} \in \mathcal{A}(C_R k^{-1} \|L_\Omega \mathbf{u}\|_{\operatorname{curl},\Omega,k}, \gamma_R, B_R)$ for any ball $B_R \subset \Omega$, where the constants C_R, γ_R are independent of k and $\mathbf{u}$ (but depend on R). Noting (5.27) shows the desired analyticity assertion for the interior of Ω.
2. step (smoothness up to the boundary and $\mathbf{H}^1$-estimates): Let the tubular neighborhood $\mathcal{U}_\Gamma$ of Γ and the extension $\mathbf{U} \in \mathcal{A}(Ck^{3/2}\|\mathbf{u}\|_{\operatorname{curl},\Omega,1}, \gamma_2, \mathcal{U}_\Gamma)$ of $L_\Gamma \mathbf{u}_T$ be given by Lemma 5.5 and write $L_\Omega \mathbf{u} = \mathbf{U} + \widetilde{\mathbf{u}}$. By the triangle inequality we have

$$k\|\widetilde{\mathbf{u}}\|_{\mathbf{L}^2(\mathcal{U}_\Gamma)} \leq \|L_\Omega \mathbf{u}\|_{\operatorname{curl},\Omega,k} + \|\mathbf{U}\|_{\operatorname{curl},\mathcal{U}_\Gamma,k} \leq Ck^{5/2}\|\mathbf{u}\|_{\operatorname{curl},\Omega,1}, \tag{5.3a}$$

$$\|\operatorname{curl}\widetilde{\mathbf{u}}\|_{\mathbf{L}^2(\mathcal{U}_\Gamma)} \leq \|L_\Omega \mathbf{u}\|_{\operatorname{curl},\Omega,k} + \|\mathbf{U}\|_{\operatorname{curl},\mathcal{U}_\Gamma,k} \leq Ck^{5/2}\|\mathbf{u}\|_{\operatorname{curl},\Omega,1}. \tag{5.3b}$$

In view of (4.9) $\widetilde{\mathbf{u}}$ satisfies

$$\operatorname{curl}\operatorname{curl}\widetilde{\mathbf{u}} + k^2\widetilde{\mathbf{u}} = \mathbf{f} := -\operatorname{curl}\operatorname{curl}\mathbf{U} - k^2\mathbf{U} \quad \text{in } \mathcal{U}_\Gamma \cap \Omega, \tag{5.3a}$$

$$\operatorname{div}\widetilde{\mathbf{u}} = G := -\operatorname{div}\mathbf{U} \quad \text{in } \mathcal{U}_\Gamma \cap \Omega, \tag{5.3b}$$

$$\Pi_T \widetilde{\mathbf{u}} = 0 \quad \text{on } \Gamma. \tag{5.3c}$$

We have (suitably adjusting the constants γ_2)

$$\mathbf{f} \in \mathcal{A}(Ck^{7/2}\|\mathbf{u}\|_{\operatorname{curl},\Omega,1}, \gamma_2, \mathcal{U}_\Gamma \cap \Omega), \qquad G \in \mathcal{A}(Ck^{5/2}\|\mathbf{u}\|_{\operatorname{curl},\Omega,1}, \gamma_2, \mathcal{U}_\Gamma \cap \Omega). \tag{5.31}$$

The analyticity of $\mathbf{U}$, Lemma D.1, and a simple induction argument (to deal with the presence of the lower order term $k^2\widetilde{\mathbf{u}}$) shows that $\widetilde{\mathbf{u}}$ is in $C^\infty(\mathcal{U}_\Gamma \cap \Omega)$. Additionally, by suitably localizing, Lemma D.1, (i) gives for a suitable subset $\mathcal{U}' \subset \mathcal{U}_\Gamma$

$$\|\widetilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{U}'_\Gamma)} \leq C\left[\|\widetilde{\mathbf{u}}\|_{\mathbf{L}^2(\mathcal{U}_\Gamma)} + \|\operatorname{curl}\widetilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega\cap\mathcal{U}_\Gamma)} + \|\operatorname{div}\widetilde{\mathbf{u}}\|_{L^2(\Omega\cap\mathcal{U}_\Gamma)}\right] \overset{(5.3),(5.3b)}{\leq} Ck^{5/2}\|\mathbf{u}\|_{\operatorname{curl},\Omega,1}. \tag{5.32}$$

For notational convenience, we will henceforth denote $\mathcal{U}'_\Gamma$ again by $\mathcal{U}_\Gamma$.
3. step (analytic regularity of $\widetilde{\mathbf{u}}$): Quantitative bounds for higher derivatives of $\widetilde{\mathbf{u}}$ are obtained by locally flattening the boundary. By the analyticity of Γ and the compactness of Γ there are R_0, C_χ, $\gamma_\chi > 0$ such that for each $\mathbf{x}_0 \in \Gamma$ we can find a parametrization $\chi_{\mathbf{x}_0} \in \mathcal{A}^\infty(C_\chi, \gamma_\chi, B_{R_0}(0))$ with the following properties[7]:

1. $\chi_{\mathbf{x}_0}(0) = \mathbf{x}_0$ and, for $B^+_{R_0} := \{\widehat{\mathbf{x}} \in B_{R_0}(0) \,|\, \widehat{\mathbf{x}}_3 > 0\}$ and $\widehat{\Gamma}_{R_0} := \{\widehat{\mathbf{x}} \in B_{R_0}(0) \,|\, \mathbf{x}_3 = 0\}$, we have $V_{\mathbf{x}_0} := \chi_{\mathbf{x}_0}(B^+_{R_0}) \subset \Omega$ as well as $\chi_{\mathbf{x}_0}(\widehat{\Gamma}_{R_0}) \subset \Gamma$.
2. For $\widehat{\mathbf{x}} \in \widehat{\Gamma}_{R_0}$ the vectors $\mathbf{t}^i_{\mathbf{x}_0} := \partial_i \chi_{\mathbf{x}_0}(\widehat{\mathbf{x}})$, $i \in \{1, 2\}$, span the tangent plane of Γ at $\chi_{\mathbf{x}_0}(\widehat{\mathbf{x}})$ and $\mathbf{n}(\mathbf{x}) := -\partial_3 \chi_{\mathbf{x}_0}(\widehat{\mathbf{x}})$ is the outward normal vector.
3. The Jacobian $D\chi_{\mathbf{x}_0}(0) \in \mathbb{R}^{3\times 3}$ is orthogonal, i.e., $(D\chi_{\mathbf{x}_0}(0))^T(D\chi_{\mathbf{x}_0}(0)) = \mathrm{I}$.

The transformation of the system (5.3) on $V_{\mathbf{x}_0}$ to the half-ball $B^+_{R_0}$ is effected with a covariant transformation of the dependent variable $\widetilde{\mathbf{u}}$ by setting $\widetilde{\mathbf{u}}^{\text{cov}} := (D\chi_{\mathbf{x}_0})^\intercal \widetilde{\mathbf{u}} \circ \chi_{\mathbf{x}_0}$. We recall the formula (see, e.g., [43, Cor. 3.58])

$$\frac{1}{\det(D\chi_{\mathbf{x}_0})}(D\chi_{\mathbf{x}_0})\operatorname{curl}\mathbf{w}^{\text{cov}} = (\operatorname{curl}\mathbf{w}) \circ \chi_{\mathbf{x}_0}$$

and introduce the two pointwise symmetric positive definite matrices

$$\mathbf{A} := \frac{1}{\det(D\chi_{\mathbf{x}_0})}(D\chi_{\mathbf{x}_0})^\intercal(D\chi_{\mathbf{x}_0}), \qquad \mathbf{B} := (\det(D\chi_{\mathbf{x}_0}))\,(D\chi_{\mathbf{x}_0})^{-1}(D\chi_{\mathbf{x}_0})^{-T}; \tag{5.33}$$

note that $\mathbf{A}, \mathbf{B} \in \mathcal{A}^\infty(C', \gamma', B^+_{R_0})$ for some constants C', γ' that depend solely on Γ. We also note that, since $D\chi_{\mathbf{x}_0}(0)$ is assumed to be orthogonal, we have

$$\mathbf{A}(0) = \mathbf{B}(0) = \mathrm{I} \in \mathbb{R}^{3\times 3}. \tag{5.34}$$

[7] The third condition is not essential but leads to a significant simplification as the ensuing (5.34) effects a decoupling of the elliptic system (5.3) into three scalar problems at 0.

From (5.3a) we obtain for all $\mathbf{V} \in C_0^\infty(B_{R_0}^+)$

$$\int_{B_{R_0}^+} \left(\frac{1}{\det D\chi_{\mathbf{x}_0}} \langle (D\chi_{\mathbf{x}_0}) \operatorname{curl} \widetilde{\mathbf{u}}^{\mathrm{cov}}, (D\chi_{\mathbf{x}_0}) \operatorname{curl} \mathbf{V} \rangle + (\det D\chi_{\mathbf{x}_0})\, k^2 \left\langle \widetilde{\mathbf{u}}^{\mathrm{cov}}, (D\chi_{\mathbf{x}_0})^{-1} (D\chi_{\mathbf{x}_0})^{-\intercal} \mathbf{V} \right\rangle \right) = \int_{B_{R_0}^+} \left\langle \widehat{\mathbf{f}}, \mathbf{V} \right\rangle$$

with $\widehat{\mathbf{f}} := \det(D\chi_{\mathbf{x}_0})(D\chi_{\mathbf{x}_0})^{-1} \mathbf{f} \circ \chi_{\mathbf{x}_0}$. The strong form of this equation is

$$\operatorname{curl} (\mathbf{A} \operatorname{curl} \widetilde{\mathbf{u}}^{\mathrm{conv}}) + k^2 \mathbf{B} \widetilde{\mathbf{u}}^{\mathrm{conv}} = \widehat{\mathbf{f}} \quad \text{in } B_{R_0}^+. \tag{5.3a}$$

The transformation of the divergence condition (5.3b) to $B_{R_0}^+$ is:

$$\operatorname{div} (\mathbf{B} \widetilde{\mathbf{u}}^{\mathrm{cov}}) = \widehat{G} := \det(D\chi_{\mathbf{x}_0}) G \circ \chi_{\mathbf{x}_0} \quad \text{in } B_{R_0}^+. \tag{5.3b}$$

The covariant transformation leaves the homogeneous tangential trace (5.3c) invariant:

$$\Pi_T \widetilde{\mathbf{u}}^{\mathrm{cov}} = 0 \quad \text{on } \widehat{\Gamma}_{R_0}. \tag{5.3c}$$

We rewrite the equations (5.3) in the form (D.2). To that end, we note that the solution $\widetilde{\mathbf{u}}^{\mathrm{cov}}$ is smooth (up to the boundary $\widehat{\Gamma}_{R_0}$) by Step 2 so that the manipulations are admissible; we also note $\mathbf{A}(0) = \mathbf{B}(0) = \mathrm{I}$ by (5.34). Adding the gradient of equation (5.3b) to equation (5.3a) and taking the trace of (5.3b) on $\widehat{\Gamma}_{R_0}$ as well as taking note of (5.3c) gives a system of the following form:

$$-\sum_{\alpha,\beta,j=1}^{3} \partial_\alpha \left(A_{\alpha\beta}^{ij} \partial_\beta \widetilde{\mathbf{u}}_j^{\mathrm{cov}} \right) + \sum_{j,\beta=1}^{3} B_\beta^{ij} \partial_\beta \widetilde{\mathbf{u}}_j^{cov} + \sum_{j=1}^{3} \left(C^{ij} + k^2 \mathbf{B}_{ij} \right) \widetilde{\mathbf{u}}_j^{\mathrm{cov}} = \widehat{\mathbf{f}}_i + \partial_i \widehat{G}, \quad \text{on } B_{R_0}^+, \quad i = 1,2,3, \tag{5.35}$$

$$\widetilde{\mathbf{u}}_i^{\mathrm{cov}} = 0 \quad \text{on } \Gamma_{R_0}, \quad i = 1,2, \tag{5.36}$$

$$\partial_3 \widetilde{\mathbf{u}}_3^{\mathrm{cov}} = \widehat{G} - (\sum_{i=1}^{3} \partial_i \mathbf{B}_{i3}) \widetilde{\mathbf{u}}_3^{\mathrm{cov}} - \sum_{i=1}^{2} \mathbf{B}_{i3} \partial_i \widetilde{\mathbf{u}}_3^{\mathrm{cov}} - \sum_{j=1}^{2} \mathbf{B}_{3j} \partial_3 \widetilde{\mathbf{u}}_j^{\mathrm{cov}} - (\mathbf{B}_{33} - 1) \widetilde{\mathbf{u}}_3^{\mathrm{cov}}. \quad \text{on } \widehat{\Gamma}_{R_0}. \tag{5.37}$$

The tensors $(A_{\alpha\beta}^{ij})_{i,j,\alpha,\beta}$, $(B_\beta^{ij})_{i,j,\beta}$, and $(C^{ij})_{i,j}$ are analytic on $B_{R_0}^+$ and, with constants C'', γ'', depending solely on Γ, we have $(A_{\alpha\beta}^{ij})_{i,j,\alpha,\beta}$, $(B_\beta^{ij})_{i,j,\beta}$, $(C^{ij})_{i,j} \in \mathcal{A}^\infty(C'', \gamma'', B_{R_0}^+)$. Additionally, we have the structural property (cf. (5.30) and (5.34))

$$A_{\alpha\beta}^{ij}(0) = \delta_{ij} \delta_{\alpha\beta}, \qquad \mathbf{B}_{j3}(0) = \mathbf{B}_{3j}(0) = 0 \quad \text{for } j \in \{1,2,3\}. \tag{5.38}$$

Lemma 2.6 and (5.31) imply, for suitable constants C, γ_3,

$$\widehat{f} \in \mathcal{A}(Ck^{7/2} \|\mathbf{u}\|_{\mathrm{curl},\Omega,1}, \gamma_3, B_{R_0}^+), \qquad \widehat{G} \in \mathcal{A}(Ck^{5/2} \|\mathbf{u}\|_{\mathrm{curl},\Omega,1}, \gamma_3, B_{R_0}^+), \tag{5.39}$$

$$k \|\widetilde{\mathbf{u}}^{\mathrm{cov}}\|_{\mathbf{L}^2(B_{R_0}^+)} + \|\widetilde{\mathbf{u}}^{\mathrm{cov}}\|_{\mathbf{H}^1(B_{R_0}^+)} \overset{(5.3a),(5.32)}{\leq} Ck^{5/2} \|\mathbf{u}\|_{\mathrm{curl},\Omega,1}. \tag{5.40}$$

Dividing (5.3) by k^2 makes Theorem D.5 applicable with $\varepsilon = k^{-1}$ and the constants C_f, C_G there of the form $C_f = O(k^{3/2} \|\mathbf{u}\|_{\mathrm{curl},\Omega,1})$ and $C_G = O(k^{3/2} \|\mathbf{u}\|_{\mathrm{curl},\Omega,1})$. Theorem D.5 yields a $R > 0$ (depending only on Γ) and constants C, γ such that for $B_R^+ := \{\widehat{\mathbf{x}} \in B_R(0) \,|\, \widehat{\mathbf{x}}_3 > 0\}$ we have $\widetilde{\mathbf{u}}^{\mathrm{cov}} \in \mathcal{A}(C_u, \gamma, B_R^+)$, where

$$C_u = k^{3/2} \|\mathbf{u}\|_{\mathrm{curl},\Omega,1}.$$

Transforming back using again Lemma 2.6 gives for $V_R := \chi_{\mathbf{x}_0}(B_R^+)$ the analytic regularity assertion $\widetilde{\mathbf{u}} \in \mathcal{A}(CC_u, \gamma, V_R)$ for suitable constants C, γ. A covering argument completes the estimate of $\widetilde{\mathbf{u}}$ on $\mathcal{U}_\Gamma$. ■
The normal trace $\langle L_\Omega \mathbf{v}, \mathbf{n} \rangle$ is also analytic. We have:

Lemma 5.10 *Let $\Omega = B_1(0)$. There is a tubular neighborhood $\mathcal{U}_\Gamma$ of Γ and there are constants $C_{\mathcal{A},\Gamma}$, $\gamma_{\mathcal{A},\Gamma}$, $C'_{\mathcal{A},\Gamma}$, $\gamma'_{\mathcal{A},\Gamma}$, $C''_{\mathcal{A},\Gamma}$, $\gamma''_{\mathcal{A},\Gamma}$, $b > 0$ depending only on Γ and choice of cut-off parameter λ such that for any $\mathbf{v} \in \mathbf{H}(\mathrm{curl}, \Omega)$ the normal trace $\langle L_\Omega \mathbf{v}, \mathbf{n} \rangle$ on Γ satisfies the following:*

(i) $g_1 := \langle L_\Omega \mathbf{v}, \mathbf{n} \rangle$ has an analytic extension g_1^* to $\mathcal{U}_\Gamma$ with $g_1^* \in \mathcal{A}(C_{\mathcal{A},\Gamma} k^{3/2} \|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \gamma_{\mathcal{A},\Gamma}, \mathcal{U}_\Gamma)$.

(ii) $\langle L_\Omega \mathbf{v}, \mathbf{n}\rangle \in \mathcal{A}(C_{\mathcal{A},\Gamma} k^2 \|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \gamma_{\mathcal{A},\Gamma}, \Gamma)$.

(iii) The expansion coefficients κ_ℓ^m *of*

$$\langle L_\Omega \mathbf{v}, \mathbf{n}\rangle = \sum_{\ell=1}^{\infty} \sum_{m\in\iota_\ell} \kappa_\ell^m Y_\ell^m \tag{5.41}$$

satisfy

$$|\kappa_\ell^m| \le C'_{\mathcal{A},\Gamma} k^2 \|\mathbf{v}\|_{\mathrm{curl},\Omega,1} \begin{cases} 1 & \text{if } \ell \le \gamma'_{\mathcal{A},\Gamma} k \\ e^{-b\ell} & \text{if } \ell > \gamma'_{\mathcal{A},\Gamma} k. \end{cases} \tag{5.42}$$

$$\sum_{\ell \le k\gamma'_{\mathcal{A},\Gamma}} \sum_{m\in\iota_\ell} |\kappa_\ell^m| \le C''_{\mathcal{A},\Gamma} k^{-1/2} \|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \tag{5.43}$$

$$\sum_{\ell > k\gamma'_{\mathcal{A},\Gamma}} \sum_{m\in\iota_\ell} |\kappa_\ell^m| (\ell+1)^\alpha \le C''_{\mathcal{A},\Gamma} k^2 (\gamma''_{\mathcal{A},\Gamma})^{\alpha+1} (\alpha+1)^{\alpha+1} \|\mathbf{v}\|_{\mathrm{curl},\Omega,1} \qquad \forall \alpha \ge 0. \tag{5.44}$$

Proof. *Proof of (i):* From Theorem 5.9 we infer for suitable C, γ that $L_\Omega \mathbf{v}$ is in fact analytic on $\Omega \cup \mathcal{U}_\Gamma$ and satisfies there

$$L_\Omega \mathbf{v} \in \mathcal{A}(C k^{3/2} \|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \gamma, \Omega \cup \mathcal{U}_\Gamma) \tag{5.45}$$

The extension g_1^* of $g_1 = \langle L_\Omega \mathbf{v}, \mathbf{n}\rangle$ into $\mathcal{U}_\Gamma$ is taken as $g_1^* := \langle L_\Omega \mathbf{v}, \mathbf{n}^*\rangle$ where $\mathbf{n}^*(\mathbf{x}) := \mathbf{x}/\|\mathbf{x}\|$ is the extension of the normal vector $\mathbf{n}$ to $\mathcal{U}_\Gamma$. By the analyticity of $\mathbf{n}^*$ and (5.45) we may apply Lemma 2.6 to get with suitable constants $\tilde{C}$, $\tilde{\gamma}$ independent of k and $\mathbf{v}$,

$$g_1^* \in \mathcal{A}\left(\tilde{C} k^{3/2} \|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \tilde{\gamma}, \mathcal{U}_\Gamma\right). \tag{5.46}$$

Proof of (ii): Since for smooth w we have the pointwise bound $|\nabla_\Gamma w| \le |(\nabla w)|_\Gamma|$, we get from a multiplicative trace inequality (see, e.g., [36, Thm. A.2])

$$\|\nabla_\Gamma^n g_1\|_\Gamma \le C \left(\|\nabla^n g_1^*\|_{L^2(\mathcal{U}_\Gamma)} \|\nabla^n g_1^*\|_{H^1(\mathcal{U}_\Gamma)} \right)^{1/2} \qquad \forall n \in \mathbb{N}_0$$

so that $g_1 \in \mathcal{A}\left(C_1 k^2 \|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \gamma_1, \Gamma\right)$ for suitable C_1, γ_1; this is the second statement.
Proof of (iii): 1. step: By [47, (2.5.212)], the Laplace-Beltrami operator can be expressed in terms of differential operators in ambient space: $\Delta u = \Delta_\Gamma u + 2H\partial_n u + \partial_n^2 u$, where $H \equiv 1$ is the mean curvature of the unit sphere. Applying this to $u = g_1^*$ implies for some C, $\gamma_2 > 0$ independent of k and j again with the trace inequality

$$\|\Delta_\Gamma^j g_1\|_{L^2(\Gamma)} \le C k^2 \|\mathbf{v}\|_{\mathrm{curl},\Omega,1} \gamma_2^{2j} \max\{k, 2j\}^{2j}, \tag{5.47}$$

2. step: By orthonormality of the Y_ℓ^m, the expansion coefficients κ_ℓ^m are given by $\kappa_\ell^m = (g_1, Y_\ell^m)_\Gamma$. By orthonormality of Y_ℓ^m we have

$$\left(\sum_{\ell=1}^{\infty} \sum_{m\in\iota_\ell} |\kappa_\ell^m|^2 \right)^{1/2} = \|g_1\|_{L^2(\Gamma)} \le C k^2 \|\mathbf{v}\|_{\mathrm{curl},\Omega,1} \tag{5.48}$$

3. step: The minimum of $x \mapsto x^x$ is attained at $1/e$ with value $\mathrm{e}^{-1/\mathrm{e}} < 1$. Hence, there are $q \in (0,1)$ and $\delta > 0$ such that

$$x^x \le q < 1 \qquad \forall x \in [1/\mathrm{e} - \delta, 1/\mathrm{e} + \delta]. \tag{5.49}$$

4. step: There is $\gamma_3 > 0$ independent of k such that the following implication holds:

$$\ell \ge \gamma_3 k \quad \Longrightarrow \quad j := \left\lfloor \frac{\ell}{2\gamma_2 \mathrm{e}} \right\rfloor \quad \text{satisfies } j \ge k \quad \text{and} \quad \frac{2j\gamma_2}{\ell} \in [1/\mathrm{e} - \delta, 1/\mathrm{e} + \delta]. \tag{5.50}$$

5. step: Given $\ell \ge \gamma_3 k$ we select j as in (5.50). Using the orthonormality of the Y_ℓ^m with the eigenvalues $\lambda_\ell = \ell(\ell+1)$ of $-\Delta_\Gamma$, we compute

$$|\kappa_\ell^m| = \lambda_\ell^{-j} \left| \left(g_1, (-\Delta_\Gamma)^j Y_\ell^m\right)_\Gamma \right| = \lambda_\ell^{-j} \left| \left((-\Delta_\Gamma)^j g_1, Y_\ell^m\right)_\Gamma \right| \overset{(5.47)}{\le} C k^2 \|\mathbf{v}\|_{\mathrm{curl},\Omega,1} \gamma_2^{2j} (\ell(\ell+1))^{-j} \max\{k, 2j\}^{2j}$$

$$\leq Ck^2\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}\gamma_2^{2j}(\ell(\ell+1))^{-j}(2j)^{2j} \leq Ck^2\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}\left((2j\gamma_2/\ell)^{2j\gamma_2/\ell}\right)^{\ell/\gamma_2} \stackrel{(5.50),(5.49)}{\leq} Ck^2\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}q^{\ell/\gamma_2}.$$

This shows the bound (5.43) for κ_ℓ^m.

6. step: We show (5.44). Recall that by (5.1) we have $\operatorname{card}\iota_\ell = 2\ell+1$ and that, by (5.28), $\|L_\Omega\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\Omega)} = \|L_\Omega\mathbf{v}\|_{L^2(\Omega)} \leq k^{-2}\|L_\Omega\mathbf{v}\|_{\mathrm{curl},\Omega,k} \leq k^{-2}\|\mathbf{v}\|_{\mathrm{curl},\Omega,k}$. We estimate

$$\sum_{\ell\leq k\gamma'_{\mathcal{A},\Gamma}}\sum_{m\in\iota_\ell}|\kappa_\ell^m| \leq \left(\sum_{\ell\leq k\gamma'_{\mathcal{A},\Gamma}}\sum_{m\in\iota_\ell}|\kappa_\ell^m|^2\lambda_\ell^{-1/2}\right)^{1/2}\left(\sum_{\ell\leq k\gamma'_{\mathcal{A},\Gamma}}\sum_{m\in\iota_\ell}\lambda_\ell^{1/2}\right)^{1/2} \lesssim \|\langle L_\Omega\mathbf{v},\mathbf{n}\rangle\|_{H^{-1/2}(\Gamma)}k^{3/2}$$
$$\lesssim k^{3/2}\|L_\Omega\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\Omega)} \lesssim k^{-1/2}\|\mathbf{v}\|_{\mathrm{curl},\Omega,k}.$$

7. step: We show (5.42). We start with the observation

$$\sup_{x>0} x^\alpha \,\mathrm{e}^{-x} \leq \alpha^\alpha\,\mathrm{e}^{-\alpha} \qquad \forall\alpha\geq 0. \tag{5.51}$$

Then,

$$\sum_{\ell\geq k\gamma'_{\mathcal{A},\Gamma}}\sum_{m\in\iota_\ell}|\kappa_\ell^m|(\ell+1)^\alpha \stackrel{(5.43)}{\leq} C'_{\mathcal{A},\Gamma}k^2\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}\sum_{\ell>k\gamma'_{\mathcal{A},\Gamma}}(\ell+1)^\alpha(2\ell+1)e^{-b\ell}$$
$$\lesssim C'_{\mathcal{A},\Gamma}k^2\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}\sum_{\ell=1}^\infty(\ell+1)^{\alpha+1}\,\mathrm{e}^{-b(\ell+1)}.$$

Upon writing

$$(\ell+1)^{\alpha+1}\,\mathrm{e}^{-b(\ell+1)} = \left(\frac{(\ell+1)b}{2}\right)^{\alpha+1}\mathrm{e}^{-b(\ell+1)/2}\left(\frac{2}{b}\right)^{\alpha+1}\mathrm{e}^{-b(\ell+1)/2} \stackrel{(5.51)}{\leq} (\alpha+1)^{\alpha+1}\left(\frac{2}{b}\right)^{\alpha+1}\mathrm{e}^{-b(\ell+1)/2}$$

we see that the infinite sum can be controlled in the desired fashion. ■

5.4 Helmholtz Decomposition

The stability properties of the operators Π^{comp}, Π^∇, Π_h^{comp}, Π_h^∇ and the splittings induced by them in Definition 4.9 are characterized in Lemma 4.8 in terms of the constants $C_{b,k}^{\mathrm{high}}$, $C_k^{H,\Omega}$, $C_{b,k}^{\nabla,\mathrm{high}}$. For the case of the unit ball $B_1(0)$ we show in Lemma 5.11 that these constants can be bounded uniformly in k. We furthermore track the dependence of these constants on the cut-off parameter $\lambda>1$ that enters the definition of L_Ω and H_Ω (cf. Definition 4.2). We track the λ-dependence with the aid of the norm

$$\|\mathbf{u}\|_{\mathrm{curl},\Omega,k,\lambda} := \left(k^2\|\mathbf{u}\|^2 + \frac{1}{\lambda^2}\|\operatorname{curl}\mathbf{u}\|^2\right)^{1/2}. \tag{5.52}$$

Lemma 5.11 (Stability of the splitting) *Let $\Omega = B_1(0)$ and $\lambda\geq\lambda_0>1$. Then there exists $C>0$ depending solely on λ_0 such that the following holds: The decomposition of $\mathbf{u}$, $\mathbf{v}\in\mathbf{X}$ as*

$$\mathbf{u} = \Pi^{\mathrm{comp}}\mathbf{u} + \Pi^\nabla H_\Omega\mathbf{u} = (\mathbf{u}^{\mathrm{low}} + \Pi^{\mathrm{curl}}\mathbf{u}^{\mathrm{high}}) + \Pi^\nabla\mathbf{u}^{\mathrm{high}},$$
$$\mathbf{v} = \Pi^{\mathrm{comp},*}\mathbf{v} + \Pi^{\nabla,*}H_\Omega\mathbf{v} = (\mathbf{v}^{\mathrm{low}} + \Pi^{\mathrm{curl},*}\mathbf{v}^{\mathrm{high}}) + \Pi^{\nabla,*}\mathbf{v}^{\mathrm{high}},$$

where $\mathbf{u}^{\mathrm{low}} := L_\Omega\mathbf{u}$, $\mathbf{u}^{\mathrm{high}} := H_\Omega\mathbf{u}$, $\mathbf{v}^{\mathrm{low}} = L_\Omega\mathbf{v}$, and $\mathbf{v}^{\mathrm{high}} := H_\Omega\mathbf{v}$ satisfies:

$$k\left\|\Pi^{\mathrm{curl}}\mathbf{u}^{\mathrm{high}}\right\| + k\left\|\Pi^\nabla\mathbf{u}^{\mathrm{high}}\right\| \leq C\left\|\mathbf{u}^{\mathrm{high}}\right\|_{\mathrm{curl},\Omega,k,\lambda} \leq C\|\mathbf{u}\|_{\mathrm{curl},\Omega,k}, \tag{5.11a}$$
$$\left\|\operatorname{curl}\left(\Pi^{\mathrm{curl}}\mathbf{u}^{\mathrm{high}}\right)\right\| = \left\|\operatorname{curl}\mathbf{u}^{\mathrm{high}}\right\| \leq 2\|\mathbf{u}\|_{\mathrm{curl},\Omega,k}, \qquad \left\|\operatorname{curl}\left(\Pi^\nabla\mathbf{u}^{\mathrm{high}}\right)\right\| = 0, \tag{5.11b}$$
$$\|\Pi^{\mathrm{curl}}\mathbf{u}^{\mathrm{high}}\|_{\mathbf{H}^1(\Omega)} \leq C\|\mathbf{u}\|_{\mathrm{curl},\Omega,k}. \tag{5.11c}$$

Analogous estimates hold for $\Pi^{\mathrm{curl},*}\mathbf{v}^{\mathrm{high}}$ and $\Pi^{\nabla,*}\mathbf{v}^{\mathrm{high}}$.

Proof. For $\mathbf{u} \in \mathbf{X}$, choose $p \in H^1(\Omega)/\mathbb{R}$ such that $\Pi^{\nabla}\mathbf{u}^{\mathrm{high}} = \nabla p$, and $\mathbf{u}_0 := \Pi^{\mathrm{curl}}\mathbf{u}^{\mathrm{high}} \in \mathbf{V}_0$. We first collect some simple facts about this splitting.
1) The definition of the space $\mathbf{V}_0^*$ implies $0 = ((\nabla p, \mathbf{v}_0)) = \left(\left(\mathbf{u}^{\mathrm{high}} - \mathbf{u}_0, \mathbf{v}_0\right)\right)$ for all $\mathbf{v}_0 \in \mathbf{V}_0^*$.
2) In the Appendix, Lemma B.1, we will prove for the unit ball

$$\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \leq \|\mathbf{u}_0\|_{\mathrm{curl},\Omega,1}. \tag{5.53}$$

3) $\operatorname{curl}\nabla p = 0$ implies

$$\operatorname{curl}\mathbf{u}^{\mathrm{high}} = \operatorname{curl}\mathbf{u}_0. \tag{5.54}$$

The combination with (5.27) leads to the first relation in (5.11b).
4) From Lemma 5.2) we have

$$\operatorname{Re}((\mathbf{u}_0, \mathbf{u}_0)) = \operatorname{Re}\left(\left(\mathbf{u}_0, \mathbf{u}^{\mathrm{high}}\right)\right) - \operatorname{Re}((\mathbf{u}_0, \nabla p)) = \operatorname{Re}\left(\left(\mathbf{u}_0, \mathbf{u}^{\mathrm{high}}\right)\right). \tag{5.55}$$

5) Finally, we estimate the weighted $\mathbf{L}^2(\Omega)$-norm of $\mathbf{u}_0$ via

$$k^2\|\mathbf{u}_0\|^2 \overset{(5.5)}{\leq} k^2\|\mathbf{u}_0\|^2 - \operatorname{Im} k b_k\left(\mathbf{u}_0^{\nabla}, \mathbf{u}_0^{\nabla}\right) = \operatorname{Re}((\mathbf{u}_0, \mathbf{u}_0)) \overset{(5.55)}{=} \operatorname{Re}\left(\left(\mathbf{u}_0, \mathbf{u}^{\mathrm{high}}\right)\right) \tag{5.56}$$

$$\begin{aligned} &= \operatorname{Re}\left(k^2\left(\mathbf{u}_0, \mathbf{u}^{\mathrm{high}}\right) + \mathrm{i}\,k\left(b_k^{\mathrm{low}}\left(\mathbf{u}_0^{\nabla}, \left(\mathbf{u}^{\mathrm{high}}\right)^{\nabla}\right) + b_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \left(\mathbf{u}^{\mathrm{high}}\right)^{\nabla}\right)\right)\right) \\ &\leq \frac{1}{2}(k\|\mathbf{u}_0\|)^2 + \frac{1}{2}\left(k\left\|\mathbf{u}^{\mathrm{high}}\right\|\right)^2 + k\left|b_k^{\mathrm{low}}\left(\mathbf{u}_0^{\nabla}, \left(\mathbf{u}^{\mathrm{high}}\right)^{\nabla}\right)\right| + k\left|b_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \left(\mathbf{u}^{\mathrm{high}}\right)^{\nabla}\right)\right|. \end{aligned} \tag{5.57}$$

From (4.10), we conclude that $\left(\mathbf{u}^{\mathrm{high}}\right)^{\nabla} = \sum_{\ell > \lambda k}\sum_{m \in \iota_\ell} U_\ell^m \nabla_\Gamma Y_\ell^m$ and it follows from the definition of b_k^{low} (4.11) that $b_k^{\mathrm{low}}\left(\mathbf{u}_0^{\nabla}, \left(\mathbf{u}^{\mathrm{high}}\right)^{\nabla}\right) = 0$.
Next, we estimate the last term in (5.57). Our decomposition $\mathbf{u}^{\mathrm{high}} = \mathbf{u}_0 + \nabla p$ leads to

$$k\left|b_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \left(\mathbf{u}^{\mathrm{high}}\right)^{\nabla}\right)\right| \leq k\left|b_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \mathbf{u}_0^{\nabla}\right)\right| + k\left|b_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \nabla_\Gamma p\right)\right|. \tag{5.58}$$

The first term can be estimated by using (5.23a):

$$k\left|b_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \mathbf{u}_0^{\nabla}\right)\right| \leq \frac{C_b'}{\lambda^2}\|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} \overset{(5.53)}{\leq} \frac{C_b'}{\lambda^2}\|\mathbf{u}_0\|^2_{\mathrm{curl},\Omega,1} = \frac{C_b'}{\lambda^2}\left(\|\mathbf{u}_0\|^2 + \left\|\operatorname{curl}\mathbf{u}^{\mathrm{high}}\right\|^2\right). \tag{5.59}$$

For the second term in the right-hand side of (5.58) we assume first that $p \in C^\infty\left(\overline{\Omega}\right)$ while the result for general $p \in H^1(\Omega)$ follows by a density argument. We obtain

$$\begin{aligned} k\left|b_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \nabla_\Gamma p\right)\right| &\overset{(5.23b)}{\leq} \frac{C_b'}{\lambda}k\|\nabla p\|_{\mathrm{curl},\Omega,1}\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \overset{\left(\substack{(5.53)\\ \operatorname{curl}\nabla p = 0}\right)}{\leq} \frac{C_b'}{\lambda}k\|\nabla p\|\left(\|\mathbf{u}_0\|^2 + \left\|\operatorname{curl}\mathbf{u}^{\mathrm{high}}\right\|^2\right)^{1/2} \\ &\leq \frac{C_b'}{\lambda}\left(k\left\|\mathbf{u}^{\mathrm{high}}\right\| + k\|\mathbf{u}_0\|\right)\left(\|\mathbf{u}_0\| + \left\|\operatorname{curl}\mathbf{u}^{\mathrm{high}}\right\|\right). \end{aligned} \tag{5.60}$$

Inserting (5.59), (5.60) into (5.58) and employing Cauchy-Schwarz inequalities with $\eta > 0$ leads to

$$k\left|b_k^{\mathrm{high}}\left(\mathbf{u}_0^{\nabla}, \left(\mathbf{u}^{\mathrm{high}}\right)^{\nabla}\right)\right| \leq C_b'\left(\left(\frac{3}{2\lambda^2k^2} + \frac{1}{\lambda k} + \frac{\eta}{2}\right)(k\|\mathbf{u}_0\|)^2 + \left(k\left\|\mathbf{u}^{\mathrm{high}}\right\|\right)^2 + \left(\frac{3+\eta^{-1}}{2}\right)\left(\frac{\left\|\operatorname{curl}\mathbf{u}^{\mathrm{high}}\right\|}{\lambda}\right)^2\right).$$

We combine this estimate with (5.56) and absorb the first term on the right-hand side of (5.57) into the left-hand side of (5.56) to obtain

$$\frac{k^2}{2}\|\mathbf{u}_0\|^2 \leq C_b'\left(\frac{5}{2\lambda k} + \frac{\eta}{2}\right)k^2\|\mathbf{u}_0\|^2 + \left(\frac{1}{2} + C_b'\right)\left(k\left\|\mathbf{u}^{\mathrm{high}}\right\|\right)^2 + C_b'\left(\frac{3+\eta^{-1}}{2}\right)\left(\frac{\left\|\operatorname{curl}\mathbf{u}^{\mathrm{high}}\right\|}{\lambda}\right)^2.$$

We first consider the case $k \geq \max\left\{1, \frac{20C_b'}{\lambda}\right\}$ and choose $\eta = \frac{1}{4C_b'}$. This leads to

$$\frac{k^2}{4}\|\mathbf{u}_0\|^2 \leq C_1k^2\left\|\mathbf{u}^{\mathrm{high}}\right\|^2 + \frac{C_2}{\lambda^2}\left\|\operatorname{curl}\mathbf{u}^{\mathrm{high}}\right\|^2 \tag{5.61}$$

$$\text{with} \quad C_1 := \left(\frac{1}{2} + C_b'\right), \quad C_2 := C_b' \left(\frac{3+4C_b'}{2}\right).$$

This and the stability of the frequency splitting (5.27) yields the first estimate in (5.11).
For $1 \leq k \leq \max\left\{1, \frac{20C_b'}{\lambda}\right\}$, we estimate the term $k\left|b_k^{\text{high}}\left(\mathbf{u}_0^\nabla, \left(\mathbf{u}^{\text{high}}\right)^\nabla\right)\right|$ in (5.57) by using (5.19a) and $\eta > 0$

$$\begin{aligned}
k\left|b_k^{\text{high}}\left(\mathbf{u}_0^\nabla, \left(\mathbf{u}^{\text{high}}\right)^\nabla\right)\right| &\leq C_b' \frac{k}{\lambda} \|\mathbf{u}_0\|_{\text{curl},\Omega,1} \left\|\mathbf{u}^{\text{high}}\right\|_{\text{curl},\Omega,1} \\
&\leq C_b' \frac{k}{2\lambda}\left(\eta \|\mathbf{u}_0\|_{\text{curl},\Omega,1}^2 + \frac{1}{\eta}\left\|\mathbf{u}^{\text{high}}\right\|_{\text{curl},\Omega,1}^2\right) \\
&\overset{(5.54)}{\leq} C_b' \frac{k}{2\lambda}\left(\eta \|\mathbf{u}_0\|^2 + \left(\eta + \frac{1}{\eta}\right)\left\|\mathbf{u}^{\text{high}}\right\|_{\text{curl},\Omega,1}^2\right).
\end{aligned}$$

This allows us to estimate the right-hand side in (5.56) from above. Recall $\lambda k > 1$. The choice $\eta = \frac{1}{2C_b'}$ leads to

$$k^2 \|\mathbf{u}_0\|^2 \leq 2\left(k\left\|\mathbf{u}^{\text{high}}\right\|\right)^2 + \frac{k}{\lambda}\left(1 + 4(C_b')^2\right)\left\|\mathbf{u}^{\text{high}}\right\|_{\text{curl},\Omega,1}^2. \tag{5.62}$$

Since $k \leq \frac{20C_b'}{\lambda}$ we get

$$k^2 \|\mathbf{u}_0\|^2 \leq C\left\|\mathbf{u}^{\text{high}}\right\|_{\text{curl},\Omega,k,\lambda}^2 \leq C\|\mathbf{u}\|_{\text{curl},\Omega,k}^2.$$

The L^2 estimate for ∇p follows by a triangle inequality:

$$k\|\nabla p\| \leq k\left(\left\|\mathbf{u}^{\text{high}}\right\| + \|\mathbf{u}_0\|\right) \overset{(5.27),\ (5.61)}{\leq} C_3\left(k\left\|\mathbf{u}^{\text{high}}\right\| + \lambda^{-1}\left\|\operatorname{curl}\mathbf{u}^{\text{high}}\right\|\right) \leq C_3'\left\|\mathbf{u}^{\text{high}}\right\|_{\text{curl},\Omega,k,\lambda} \leq C_3''\|\mathbf{u}\|_{\text{curl},\Omega,k}.$$

The estimates for $\mathbf{v}_0$ and ∇q are derived by repeating the arguments above. ■
By similar techniques we will prove next that if one argument in $((\cdot,\cdot))$ has only high-frequency components then we get k-independent continuity estimates (cf. also (4.15) for the general case):

Proposition 5.12 *Let $\Omega = B_1(0)$ and $\lambda \geq \lambda_0 > 1$. Then there exists $\widetilde{C}_b > 0$ depending solely on λ_0 such that for all $\mathbf{u}$, $\mathbf{v} \in \mathbf{X}$*

$$|((H_\Omega \mathbf{u}, \mathbf{v}))| + |((\mathbf{u}, H_\Omega \mathbf{v}))| \leq \widetilde{C}_b \|\mathbf{u}\|_{\text{curl},\Omega,k,\lambda}\|\mathbf{v}\|_{\text{curl},\Omega,k,\lambda}, \tag{5.63}$$

$$|((\mathbf{u}, \mathbf{v}))| \leq C_{\text{cont},k}\|\mathbf{u}\|_{\text{curl},\Omega,1}\|\mathbf{v}\|_{\text{curl},\Omega,1}, \tag{5.64}$$

where $C_{\text{cont},k} \leq \widetilde{C}_b k^3$.

Proof. For $\mathbf{u}$, $\mathbf{v} \in \mathbf{X}$, write $\mathbf{u}^{\text{high}} := H_\Omega \mathbf{u}$, $\mathbf{v}^{\text{high}} := H_\Omega \mathbf{v}$. Choose p, $q \in H^1(\Omega)$ such that $\Pi^\nabla \mathbf{u}^{\text{high}} = \nabla p$, $\Pi^{\nabla,*}\mathbf{v}^{\text{high}} = \nabla q$ and set $\mathbf{u}_0 = \mathbf{u}^{\text{high}} - \nabla p$, $\mathbf{v}_0 = \mathbf{v}^{\text{high}} - \nabla q$. Since $\Pi_T H_\Omega = H_\Gamma \Pi_T$ (cf. (4.10)) we have

$$\left|\left(\left(\mathbf{u}^{\text{high}}, \mathbf{v}\right)\right)\right| \leq \left(k\left\|\mathbf{u}^{\text{high}}\right\|\right)(k\|\mathbf{v}\|) + \left|kb_k^{\text{high}}\left(\left(\mathbf{u}^{\text{high}}\right)^\nabla, \mathbf{v}^\nabla\right)\right|.$$

For the boundary term, we get

$$\begin{aligned}
\left|kb_k^{\text{high}}\left(\left(\mathbf{u}^{\text{high}}\right)^\nabla, \mathbf{v}^\nabla\right)\right| &\leq \left|kb_k^{\text{high}}\left(\mathbf{u}_0^\nabla, \mathbf{v}_0^\nabla\right)\right| + \left|kb_k^{\text{high}}\left((\nabla p)^\nabla, \mathbf{v}_0^\nabla\right)\right| \\
&\quad + \left|kb_k^{\text{high}}\left(\mathbf{u}_0^\nabla, (\nabla q)^\nabla\right)\right| + \left|kb_k^{\text{high}}\left((\nabla p)^\nabla, (\nabla q)^\nabla\right)\right| \\
&\overset{(5.21),\ (5.22a)}{\leq} \frac{C_b'}{\lambda^2}\|\mathbf{u}_0\|_{\text{curl},\Omega,1}\|\mathbf{v}_0\|_{\text{curl},\Omega,1} + \frac{C_b'}{\lambda}(k\|\nabla p\|)\|\mathbf{v}_0\|_{\text{curl},\Omega,1} \\
&\quad + \frac{C_b'}{\lambda}\|\mathbf{u}_0\|_{\text{curl},\Omega,1} k\|\nabla q\| + C_b'(k\|\nabla p\|)(k\|\nabla q\|) \\
&\overset{(5.11)}{\leq} \widetilde{C}_b\|\mathbf{u}\|_{\text{curl},\Omega,k,\lambda}\|\mathbf{v}\|_{\text{curl},\Omega,k,\lambda}.
\end{aligned} \tag{5.65}$$

The estimate for $((\mathbf{u}, \mathbf{v}^{\text{high}}))$ follows from the same arguments.
It remains to prove estimate (5.64). We choose $\lambda = \lambda_0 = O(1)$ in all splittings and estimates and start with

$$|((L_\Omega \mathbf{u}, \mathbf{v}))| \leq k^2|(L_\Omega \mathbf{u}, \mathbf{v})| + \left|kb_k\left((L_\Omega \mathbf{u})^\nabla, \mathbf{v}^\nabla\right)\right| \leq (k\|L_\Omega \mathbf{u}\|)(k\|\mathbf{v}\|) + \left|kb_k^{\text{low}}\left(\mathbf{u}^\nabla, \mathbf{v}^\nabla\right)\right|.$$

We employ (5.16) with $\rho = +1$ to obtain with Lemma 5.5, (ii)

$$\begin{aligned}\left|b_k^{\text{low}}\left(\mathbf{u}^{\nabla},\mathbf{v}^{\nabla}\right)\right| &\leq C_b k^2 \left\|\operatorname{div}_\Gamma L_\Gamma \mathbf{u}_T\right\|_{H^{-3/2}(\Gamma)} \left\|\operatorname{div}_\Gamma L_\Gamma \mathbf{v}_T\right\|_{H^{-3/2}(\Gamma)} \\ &\leq C k^2 \left\|L_\Gamma \mathbf{u}_T\right\|_{\mathbf{H}^{-1/2}(\Gamma)} \left\|L_\Gamma \mathbf{v}_T\right\|_{\mathbf{H}^{-1/2}(\Gamma)} \overset{\text{Lem. 5.5, (ii)}}{\leq} C k^2 \|\mathbf{u}\|_{\operatorname{curl},\Omega,1} \|\mathbf{v}\|_{\operatorname{curl},\Omega,1}. \end{aligned} \tag{5.66}$$

Combining (5.65) and (5.66) leads to

$$\begin{aligned}\left|k b_k\left(\mathbf{u}^{\nabla},\mathbf{v}^{\nabla}\right)\right| &\leq C \|\mathbf{u}\|_{\operatorname{curl},\Omega,k,\lambda} \|\mathbf{v}\|_{\operatorname{curl},\Omega,k,\lambda} + C k^3 \|\mathbf{u}\|_{\operatorname{curl},\Omega,1} \|\mathbf{v}\|_{\operatorname{curl},\Omega,1} \\ &\leq C k^3 \|\mathbf{u}\|_{\operatorname{curl},\Omega,1} \|\mathbf{v}\|_{\operatorname{curl},\Omega,1}.\end{aligned}$$

Taking into account the $L^2(\Omega)$ part in $((\cdot,\cdot))$ results in the estimate (5.64). ■

Corollary 5.13 *For $\Omega = B_1(0)$, the constants in (4.13), (4.14), (4.12), (4.6), and (4.15) can be estimated by*

$$C_{\mathrm{DtN},k} \leq C k^2, \quad C_{\mathrm{cont},k} \leq \widetilde{C}_b k^3, \quad C_{b,k}^{\nabla,\mathrm{high}} \leq \widetilde{C}_b, \quad C_{b,k}^{\mathrm{curl,high}} \leq C_b C_\Gamma^2, \quad C_k^{H,\Omega} \leq 2, \quad C_{b,k}^{\mathrm{high}} \leq 2 + \widetilde{C}_b \tag{5.67}$$

with k-independent constants C, C_b (cf. Prop. 5.7), $\widetilde{C}_b$ (cf. Prop. 5.12), and C_Γ.

Proof. The estimate of $C_{\mathrm{DtN},k}$ follows by combining (5.16) and (5.24). Proposition 5.12 implies the bound for $C_{\mathrm{cont},k}$. Estimate (5.65) implies the estimate of $C_{b,k}^{\nabla,\mathrm{high}}$ as in (4.12a). For $C_{b,k}^{\mathrm{curl,high}}$ we use (5.16) to obtain

$$\begin{aligned} k\left|b_k\left(\mathbf{u}^{\operatorname{curl}},\left(\mathbf{v}^{\mathrm{high}}\right)^{\operatorname{curl}}\right)\right| = k\left|b_k^{\mathrm{high}}\left(\mathbf{u}^{\operatorname{curl}},\mathbf{v}^{\operatorname{curl}}\right)\right| &\leq C_b \left\|\operatorname{curl}_\Gamma \mathbf{u}_T\right\|_{H^{-1/2}(\Gamma)} \left\|\operatorname{curl}_\Gamma \mathbf{v}_T\right\|_{H^{-1/2}(\Gamma)} \\ &\leq C_b C_\Gamma^2 \|\mathbf{u}\|_{\operatorname{curl},\Omega,1} \|\mathbf{v}\|_{\operatorname{curl},\Omega,1}\end{aligned}$$

so that the estimate for $C_{b,k}^{\mathrm{curl,high}}$ is shown. Finally, $C_k^{H,\Omega} \leq 2$ is proved in (5.27b) and the estimate of $C_b^{\mathrm{high}} = C_k^{H,\Omega} + C_{b,k}^{\nabla,\mathrm{high}}$ follows by combining the previous estimates. ■

6 Estimating the Terms in the Splitting (4.49b,c) of $((\mathbf{e}_h, \mathbf{v}_h))$

6.1 Estimate of $\left(\left(\mathbf{e}_h, \left(\left(\Pi_h^{\mathrm{comp},*} - \Pi^{\mathrm{comp},*}\right)\mathbf{v}_h\right)^{\mathrm{high}}\right)\right)$ in (4.49b,c)

In this section, we will prove the following Proposition 6.1. Recall the definition of $\tilde{\eta}_4^{\exp}$, η_6^{alg}, $\tilde{\eta}_7^{\exp}$ in (4.55), (4.57), (4.58), which involve the operator Π_h^E as in Assumption 4.14.

Proposition 6.1 *Let $\mathbf{e}_h = \mathbf{E} - \mathbf{E}_h$ denote the Galerkin error and for $\mathbf{v}_h \in \mathbf{X}_h$ let $\Pi_h^{\mathrm{comp},*}$, $\Pi^{\mathrm{comp},*}$ be defined as in Definition 4.9. Let Assumption 4.14 be satisfied. Then*

$$\left|\left(\left(\mathbf{e}_h, \left(\left(\Pi^{\mathrm{comp},*} - \Pi_h^{\mathrm{comp},*}\right)\mathbf{v}_h\right)^{\mathrm{high}}\right)\right)\right| \leq C_{b,k}^{\mathrm{high}} C_{r,k} \|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k} \|\mathbf{v}_h\|_{\operatorname{curl},\Omega,k}. \tag{6.1}$$

with

$$C_{r,k} := \left(C_{b,k}^{\mathrm{high}} + \frac{C_{\mathrm{cont},k}}{k^2}\tilde{\eta}_4^{\exp}\right)\left(\tilde{\eta}_7^{\exp} + C_{\#,k}\eta_6^{\mathrm{alg}}\right) \quad \text{and} \quad C_{\#,k} := \left(C_k^{H,\Omega} + C_{b,k}^{\nabla,\mathrm{high}}\right) C_{\Omega,k}. \tag{6.2}$$

The constant $C_{b,k}^{\mathrm{high}}$ is as in (4.15), $C_{\mathrm{cont},k}$ as in (4.14), and $C_{\Omega,k}$ as in (4.32).
For the case $\Omega = B_1(0)$ we have $C_{\mathrm{cont},k} \leq Ck^3$ while $C_{b,k}^{\mathrm{high}}$, $C_{r,k}$, and $C_{\#,k}$ are bounded independently of k.

Proof. From (4.47) we conclude that

$$\left.\begin{array}{rl} & \operatorname{curl}\Pi_h^{\mathrm{comp},*}\mathbf{v} = \operatorname{curl}\Pi^{\mathrm{comp},*}\mathbf{v} = \operatorname{curl}\mathbf{v} \\ \text{and, in turn,} & \operatorname{curl}\Pi_h^{\mathrm{curl},*}H_\Omega\mathbf{v} = \operatorname{curl}\Pi^{\mathrm{curl},*}H_\Omega\mathbf{v} = \operatorname{curl}H_\Omega\mathbf{v}\end{array}\right\} \quad \forall \mathbf{v} \in \mathbf{X}. \tag{6.3}$$

Let $\mathbf{r} := \left(\Pi^{\mathrm{comp},*} - \Pi_h^{\mathrm{comp},*}\right)\mathbf{v}_h$ and let $\mathbf{q} := \left(I - \Pi_h^E\right)\Pi^{\mathrm{comp},*}\mathbf{v}_h$. First we prove some curl-free properties. It holds

$$\operatorname{curl}\left(\Pi_h^E \Pi^{\mathrm{comp},*} - \Pi_h^{\mathrm{comp},*}\right)\mathbf{v}_h \overset{(4.47),\text{ A. }4.14}{=} \operatorname{curl}\left(\Pi_h^E - I\right) L_\Omega \mathbf{v}_h + \left(\Pi_h^F \operatorname{curl}\Pi^{\mathrm{curl},*} - \operatorname{curl}\Pi_h^{\mathrm{curl},*}\right) H_\Omega \mathbf{v}_h \tag{6.4}$$

$$= \operatorname{curl}\left(\Pi_h^E - I\right) L_\Omega \mathbf{v}_h + \Pi_h^F \operatorname{curl} H_\Omega \mathbf{v}_h - \operatorname{curl} H_\Omega \mathbf{v}_h$$
$$= \operatorname{curl}\left(\Pi_h^E - I\right) L_\Omega \mathbf{v}_h + \operatorname{curl}\left(\Pi_h^E - I\right) H_\Omega \mathbf{v}_h = \operatorname{curl}\left(\Pi_h^E - I\right) \mathbf{v}_h$$
$$\overset{\text{A. 4.14(a)}}{=} \operatorname{curl}\left(\mathbf{v}_h - \mathbf{v}_h\right) = 0,$$

and also

$$\operatorname{curl} \mathbf{r} \overset{(6.3)}{=} 0, \tag{6.5}$$
$$\operatorname{curl} \mathbf{q} = \operatorname{curl}\left(\Pi^{\text{comp},*} - \Pi_h^E \Pi^{\text{comp},*}\right) \mathbf{v}_h \overset{(6.3)}{=} \operatorname{curl}\left(\Pi_h^{\text{comp},*} - \Pi_h^E \Pi^{\text{comp},*}\right) \mathbf{v}_h \overset{(6.4)}{=} 0. \tag{6.6}$$

We start our estimate with a continuity bound for the sesquilinear form $((\cdot, H_\Omega \cdot))$ and employ (4.15) to get

$$\left|\left(\left(\mathbf{e}_h, \mathbf{r}^{\text{high}}\right)\right)\right| \leq C_{b,k}^{\text{high}} \|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k} \|\mathbf{r}\|_{\operatorname{curl},\Omega,k} \overset{(6.3)}{=} C_{b,k}^{\text{high}} \|\mathbf{e}_h\|_{\operatorname{curl},\Omega,k} \left(k \|\mathbf{r}\|\right). \tag{6.7}$$

The coercivity of $((\cdot, \cdot))$ in the form (4.20) leads to

$$\left(k \|\mathbf{r}\|\right)^2 \leq \operatorname{Re}\left((\mathbf{r}, \mathbf{r})\right) = \operatorname{Re}\left((\mathbf{q}, \mathbf{r})\right) + \operatorname{Re}\left(\left(\left(\Pi_h^E \Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h, \mathbf{r}\right)\right). \tag{6.8}$$

We use the definition of $\Pi^{\nabla,*}$, $\Pi^{\text{curl},*}$, $\Pi^{\text{comp},*}$ and its discrete versions as in (4.19) and Definition 4.9 to get

$$((\mathbf{w}_h, \mathbf{r})) = \left(\left(\mathbf{w}_h, \left(\Pi^{\text{curl},*} - \Pi_h^{\text{curl},*}\right) H_\Omega \mathbf{v}_h\right)\right) = 0 \quad \forall \mathbf{w}_h \in \nabla S_h. \tag{6.9}$$

From (6.4) and the exact sequence property (3.2) we conclude that $\left(\Pi_h^E \Pi^{\text{comp},*} - \Pi_h^{\text{comp},*}\right) \mathbf{v}_h = \nabla \psi_h$ for some $\psi_h \in S_h$. The combination of this with (6.9) for $\mathbf{w}_h = \nabla \psi_h$ implies that the last term in (6.8) vanishes. Hence,

$$\left(k \|\mathbf{r}\|\right)^2 \leq \operatorname{Re}\left((\mathbf{q}, \mathbf{r})\right) = \operatorname{Re}\left((H_\Omega \mathbf{q}, \mathbf{r})\right) + \operatorname{Re}\left((L_\Omega \mathbf{q}, \mathbf{r})\right). \tag{6.10}$$

For the high-frequency part on the right-hand side we employ again (4.15) and obtain

$$\operatorname{Re}\left((H_\Omega \mathbf{q}, \mathbf{r})\right) \leq C_{b,k}^{\text{high}} \|\mathbf{q}\|_{\operatorname{curl},\Omega,k} \|\mathbf{r}\|_{\operatorname{curl},\Omega,k} \overset{(6.1)}{=} C_{b,k}^{\text{high}} \left(k \|\mathbf{q}\|\right) \left(k \|\mathbf{r}\|\right). \tag{6.11}$$

The term $\|\mathbf{q}\|$ can be estimated by using the definition of $\Pi^{\text{comp},*}$ as in Definition 4.9

$$\begin{aligned} k \|\mathbf{q}\| &\leq k \left\|\left(I - \Pi_h^E\right) L_\Omega \mathbf{v}_h\right\| + k \left\|\left(I - \Pi_h^E\right) \Pi^{\text{curl},*} H_\Omega \mathbf{v}_h\right\| \\ &\leq \tilde{\eta}_7^{\exp} \|\mathbf{v}_h\|_{\operatorname{curl},\Omega,k} + \eta_6^{\text{alg}} \left\|\Pi^{\text{curl},*} H_\Omega \mathbf{v}_h\right\|_{\mathbf{H}^1(\Omega)} \\ &\overset{\text{Lem. 4.12}}{\leq} \tilde{\eta}_7^{\exp} \|\mathbf{v}_h\|_{\operatorname{curl},\Omega,k} + C_{\Omega,k} \eta_6^{\text{alg}} \left\|\Pi^{\text{curl},*} H_\Omega \mathbf{v}_h\right\|_{\operatorname{curl},\Omega,1} \\ &\overset{\text{Lem. 4.8}}{\leq} \left(\tilde{\eta}_7^{\exp} + C_{\#,k} \eta_6^{\text{alg}}\right) \|\mathbf{v}_h\|_{\operatorname{curl},\Omega,k}. \end{aligned} \tag{6.12}$$

To estimate the low frequency part in (6.10) we observe that $\boldsymbol{\zeta} := \Pi^\nabla L_\Omega \mathbf{q} = \nabla \mathcal{N}_4^{\mathcal{A}} \mathbf{q}$ (cf. (4.52)) satisfies

$$((\boldsymbol{\zeta}, \boldsymbol{\xi})) = ((L_\Omega \mathbf{q}, \boldsymbol{\xi})) \quad \forall \boldsymbol{\xi} \in \nabla H^1(\Omega).$$

By choosing $\boldsymbol{\xi} = \mathbf{r}$ we can use a Galerkin orthogonality in the form (6.9) to obtain for any $\mathbf{w}_h \in \mathbf{X}_h$

$$\operatorname{Re}\left((L_\Omega \mathbf{q}, \mathbf{r})\right) = \operatorname{Re}\left((\boldsymbol{\zeta}, \mathbf{r})\right) = \operatorname{Re}\left((\boldsymbol{\zeta} - \mathbf{w}_h, \mathbf{r})\right) \leq C_{\text{cont},k} \|\mathbf{r}\|_{\operatorname{curl},\Omega,1} \|\boldsymbol{\zeta} - \mathbf{w}_h\|_{\operatorname{curl},\Omega,1}.$$

The last factor can be estimated by using (4.55), (6.12), and the definition of $\boldsymbol{\zeta}$:

$$\begin{aligned} \inf_{v_h \in S_h} \left\|\nabla\left(\mathcal{N}_4^{\mathcal{A}} \mathbf{q} - v_h\right)\right\|_{\operatorname{curl},\Omega,1} &= \inf_{v_h \in S_h} \left\|\nabla\left(\mathcal{N}_4^{\mathcal{A}} \mathbf{q} - v_h\right)\right\| \leq \tilde{\eta}_4^{\exp} \|\mathbf{q}\|_{\operatorname{curl},\Omega,1} \overset{(6.1)}{=} \tilde{\eta}_4^{\exp} \|\mathbf{q}\| \\ &\overset{(6.12)}{\leq} \frac{\tilde{\eta}_4^{\exp}}{k} \left(\tilde{\eta}_7^{\exp} + C_{\#,k} \eta_6^{\text{alg}}\right) \|\mathbf{v}_h\|_{\operatorname{curl},\Omega,k}. \end{aligned} \tag{6.13}$$

Finally, we combine this estimate with (6.10), (6.11), (6.12) to bound the last factor in (6.7)

$$k \|\mathbf{r}\| \leq C_{r,k} \|\mathbf{v}_h\|_{\operatorname{curl},\Omega,k}. \tag{6.14}$$

We insert (6.14) into (6.7) and arrive at the assertion.
The bounds for the constants are stated in Corollary 5.13. ■

6.2 Estimate of $\left(\left(\mathbf{e}_h, \Pi^{\text{curl},*}\mathbf{v}_h^{\text{high}}\right)\right)$ in (4.49b,c)

In this section, we investigate the second term in the right-hand side in (4.49c).

Proposition 6.2 *Let $\mathbf{e}_h = \mathbf{E} - \mathbf{E}_h$ denote the Galerkin error with splitting of $\mathbf{v}_h \in \mathbf{X}_h$ as in (4.47). Recall the definition of the adjoint solution operators (cf. (4.3)). Let Assumption 4.14 be satisfied. Then*

$$\left|\left(\left(\mathbf{e}_h, \Pi^{\text{curl},*}\mathbf{v}_h^{\text{high}}\right)\right)\right| \leq C_{\#\#,k}\left(C_{\#\#,k} + C_{b,k}^{\text{curl,high}} + C_{\text{cont},k}\tilde{\eta}_5^{\text{exp}}\right)\tilde{\eta}_2^{\text{alg}}\|\mathbf{e}_h\|_{\text{curl},\Omega,k}\|\mathbf{v}_h\|_{\text{curl},\Omega,k} \tag{6.15}$$

with $C_{\#\#,k} := C_k^{H,\Omega} + C_{b,k}^{\text{high}}$. For $\Omega = B_1(0)$, it holds $C_{\text{cont},k} \leq Ck^3$ while all other constants are bounded independently of k.

Proof. Note that $\mathbf{s} := \Pi^{\text{curl},*}\mathbf{v}_h^{\text{high}} \in \mathbf{V}_0^*$. We consider the adjoint problem (cf. (4.50)) with solution operator $\mathcal{N}_2$ and set $\mathbf{z} := \mathcal{N}_2\mathbf{s}$. Galerkin orthogonality with arbitrary $\mathbf{z}_h \in \mathbf{X}_h$gives

$$((\mathbf{e}_h, \mathbf{s})) = A_k(\mathbf{e}_h, \mathbf{z}) = A_k(\mathbf{e}_h, \mathbf{z} - \mathbf{z}_h) = A_k(\mathbf{e}_h, H_\Omega(\mathbf{z} - \mathbf{z}_h)) + A_k(\mathbf{e}_h, L_\Omega(\mathbf{z} - \mathbf{z}_h)). \tag{6.16}$$

For the first term we obtain

$$|A_k(\mathbf{e}_h, H_\Omega(\mathbf{z} - \mathbf{z}_h))| \leq \|\operatorname{curl}\mathbf{e}_h\|\,\|\operatorname{curl}(H_\Omega(\mathbf{z} - \mathbf{z}_h))\| + |((\mathbf{e}_h, H_\Omega(\mathbf{z} - \mathbf{z}_h)))| + \left|kb_k\left(\mathbf{e}_h^{\text{curl}}, (H_\Omega(\mathbf{z} - \mathbf{z}_h))^{\text{curl}}\right)\right|.$$

The three terms on the right-hand side can be estimated by using the constants in (4.12), (4.6), (4.15):

$$\begin{aligned}
\|\operatorname{curl}(H_\Omega(\mathbf{z} - \mathbf{z}_h))\| &\leq \|H_\Omega(\mathbf{z} - \mathbf{z}_h)\|_{\text{curl},\Omega,k} \leq C_k^{H,\Omega}\|\mathbf{z} - \mathbf{z}_h\|_{\text{curl},\Omega,k}, \\
\left|kb_k\left(\mathbf{e}_h^{\text{curl}}, (H_\Omega(\mathbf{z} - \mathbf{z}_h))^{\text{curl}}\right)\right| &\leq C_{b,k}^{\text{curl,high}}\|\mathbf{e}_h\|_{\text{curl},\Omega,k}\|\mathbf{z} - \mathbf{z}_h\|_{\text{curl},\Omega,k}, \\
|((\mathbf{e}_h, H_\Omega(\mathbf{z} - \mathbf{z}_h)))| &\leq C_{b,k}^{\text{high}}\|\mathbf{e}_h\|_{\text{curl},\Omega,k}\|\mathbf{z} - \mathbf{z}_h\|_{\text{curl},\Omega,k}.
\end{aligned}$$

This leads to

$$|A_k(\mathbf{e}_h, H_\Omega(\mathbf{z} - \mathbf{z}_h))| \leq \left(C_{\#\#,k} + C_{b,k}^{\text{curl,high}}\right)\|\mathbf{e}_h\|_{\text{curl},\Omega,k}\|\mathbf{z} - \mathbf{z}_h\|_{\text{curl},\Omega,k}.$$

For the second term in (6.16) we obtain for arbitrary $\tilde{\mathbf{z}}_h \in \mathbf{X}_h$

$$\begin{aligned}
|A_k(\mathbf{e}_h, L_\Omega(\mathbf{z} - \mathbf{z}_h))| &\leq |A_k(\mathbf{e}_h, L_\Omega(\mathbf{z} - \mathbf{z}_h) - \tilde{\mathbf{z}}_h)| \\
&\overset{(4.14)}{\leq} C_{\text{cont},k}\|\mathbf{e}_h\|_{\text{curl},\Omega,1}\|L_\Omega(\mathbf{z} - \mathbf{z}_h) - \tilde{\mathbf{z}}_h\|_{\text{curl},\Omega,1}.
\end{aligned} \tag{6.17}$$

This leads to the estimate

$$|((\mathbf{e}_h, \mathbf{s}))| \leq \left(C_{\#\#,k} + C_{b,k}^{\text{curl,high}}\right)\|\mathbf{e}_h\|_{\text{curl},\Omega,k}\|\mathbf{z} - \mathbf{z}_h\|_{\text{curl},\Omega,k} + C_{\text{cont},k}\|\mathbf{e}_h\|_{\text{curl},\Omega,1}\|L_\Omega(\mathbf{z} - \mathbf{z}_h) - \tilde{\mathbf{z}}_h\|_{\text{curl},\Omega,1}. \tag{6.18}$$

With the definition of the adjoint approximation properties (cf. Sec. 4.3) we arrive at

$$\inf_{\mathbf{z}_h \in \mathbf{X}_h}\|\mathbf{z} - \mathbf{z}_h\|_{\text{curl},\Omega,k} \overset{(4.53)}{\leq} \tilde{\eta}_2^{\text{alg}}\left\|\Pi^{\text{curl},*}\mathbf{v}_h^{\text{high}}\right\|_{\text{curl},\Omega,k} \tag{6.19}$$

$$\inf_{\mathbf{z}_h}\inf_{\tilde{\mathbf{z}}_h}\|L_\Omega(\mathbf{z} - \mathbf{z}_h) - \tilde{\mathbf{z}}_h\|_{\text{curl},\Omega,k} \overset{(4.56)}{\leq} \tilde{\eta}_5^{\text{exp}}\inf_{\mathbf{z}_h}\|\mathbf{z} - \mathbf{z}_h\|_{\text{curl},\Omega,k} \overset{(6.19)}{\leq} \tilde{\eta}_2^{\text{alg}}\tilde{\eta}_5^{\text{exp}}\left\|\Pi^{\text{curl},*}\mathbf{v}_h^{\text{high}}\right\|_{\text{curl},\Omega,k}. \tag{6.20}$$

The combination of these estimates with Lemma 4.8 leads to (6.15).
The estimates of the constants for the case $\Omega = B_1(0)$ are stated in Corollary 5.13. ■

6.3 Estimate of $((\mathbf{e}_h, L_\Omega(\Pi_h^{\text{comp},*}\mathbf{v}_h - \Pi^{\text{comp},*}\mathbf{v}_h)))$ and $((\mathbf{e}_h, L_\Omega\mathbf{v}_h))$ in (4.49b,c)

Next, we investigate the first and last term of the right-hand side in (4.49c).

Proposition 6.3 *Let $\mathbf{e}_h = \mathbf{E} - \mathbf{E}_h$ denote the Galerkin error with splitting of $\mathbf{v}_h \in \mathbf{X}_h$ as in (4.47) and let Assumption 4.14 be satisfied. Then:*

$$|((\mathbf{e}_h, L_\Omega\mathbf{r}))| + |((\mathbf{e}_h, L_\Omega\mathbf{v}_h))| \leq C_{\text{cont},k}\tilde{\eta}_3^{\text{exp}}(1 + C_{r,k})\|\mathbf{e}_h\|_{\text{curl},\Omega,k}\|\mathbf{v}_h\|_{\text{curl},\Omega,k} \tag{6.21}$$

with $\mathbf{r} := \Pi_h^{\text{comp},*}\mathbf{v}_h - \Pi^{\text{comp},*}\mathbf{v}_h$ and $C_{r,k}$ as in (6.2).

Proof. Recall the definition of the solution operator $\mathcal{N}_3^{\mathcal{A}}$ from (4.51) satisfying for given $\mathbf{s} \in \mathbf{X}$

$$A_k\left(\mathbf{w}, \mathcal{N}_3^{\mathcal{A}}\mathbf{s}\right) = ((\mathbf{w}, L_\Omega \mathbf{s})) \quad \forall \mathbf{w} \in \mathbf{X}.$$

For the first term in (6.21) we get in a similar fashion as in (6.17)

$$\begin{aligned}|((\mathbf{e}_h, L_\Omega \mathbf{s}))| = \inf_{\mathbf{z}_h \in \mathbf{X}_h} \left|A_k\left(\mathbf{e}_h, \mathcal{N}_3^{\mathcal{A}}\mathbf{s} - \mathbf{z}_h\right)\right| &\overset{(4.14)}{\leq} C_{\mathrm{cont},k} \|\mathbf{e}_h\|_{\mathrm{curl},\Omega,1} \inf_{\mathbf{z}_h \in \mathbf{X}_h} \left\|\mathcal{N}_3^{\mathcal{A}}\mathbf{s} - \mathbf{z}_h\right\|_{\mathrm{curl},\Omega,1} \\ &\overset{(4.54)}{\leq} C_{\mathrm{cont},k}\tilde{\eta}_3^{\exp} \|\mathbf{e}_h\|_{\mathrm{curl},\Omega,k} \|\mathbf{s}\|_{\mathrm{curl},\Omega,k}.\end{aligned}$$

This leads directly to the estimate of the second term in (6.21) by choosing $\mathbf{s} = \mathbf{v}_h$. For the choice $\mathbf{s} = \mathbf{r}$, we combine (6.5) with (6.14) to get $\|\mathbf{r}\|_{\mathrm{curl},\Omega,k} = k\|\mathbf{r}\| \leq C_{r,k}\|\mathbf{v}_h\|_{\mathrm{curl},\Omega,k}$. ■

7 Analysis of the Dual Problems

For the stability and convergence analysis, we have introduced various adjoint approximation properties in Sec. 4.3. In this section, we analyze the regularity of the adjoint solutions in Sec. 7.2 based on a solution formula which we will derive in Sec. 7.1. The quantitative convergence rates require interpolation operators for hp finite element spaces that will be presented in Sections 8.3.

7.1 Solution Formulae

In this section, we will develop a regularity theory to estimate the solutions of the dual problems which have been introduced in Section 4.3. They belong to one of the following two types.
Type 1:

$$\begin{aligned}&\text{Given } \mathbf{v} \in \mathbf{H}(\Omega, \mathrm{div}), \quad \mathbf{g}, \mathbf{h} \in \mathbf{X} \quad \text{find } \mathbf{z} \in \mathbf{X} \text{ s.t.} \\ &A_k(\mathbf{w}, \mathbf{z}) = k^2(\mathbf{w}, \mathbf{v}) + \mathrm{i}\,k\left(b_k\left(\mathbf{w}^\nabla, \mathbf{g}^\nabla\right) - \mathrm{i}\,k b_k\left(\mathbf{w}^{\mathrm{curl}}, \mathbf{h}^{\mathrm{curl}}\right)\right) \quad \forall \mathbf{w} \in \mathbf{X}.\end{aligned} \tag{7.1}$$

This is problem (4.50) with $\mathbf{v} := \mathbf{g} := \mathbf{r}$ and $\mathbf{h} := \mathbf{0}$, problem (4.51) with $\mathbf{v} := \mathbf{g} := L_\Omega \mathbf{r}$ and $\mathbf{h} := \mathbf{0}$, and problem (4.40) with $\mathbf{v} = \mathbf{h} = \mathbf{g} := L_\Omega \mathbf{w}$.
Type 2:

$$\text{Given } \mathbf{r} \in \mathbf{X} \quad \text{find } Z \in H^1(\Omega)/\mathbb{R} \text{ s.t.} \quad ((\nabla Z, \nabla \xi)) = ((L_\Omega \mathbf{r}, \nabla \xi)) \qquad \forall \xi \in H^1(\Omega). \tag{7.2}$$

This is problem (4.52).

7.1.1 Solution Formula for Problems of Type 1

Integration by parts in the sesquilinear form $A_k(\cdot,\cdot)$ gives

$$\begin{aligned}A_k(\mathbf{w}, \mathbf{z}) &= (\mathrm{curl}\,\mathbf{w}, \mathrm{curl}\,\mathbf{z}) - k^2(\mathbf{w}, \mathbf{z}) - \mathrm{i}\,k\,(T_k \mathbf{w}_T, \mathbf{z}_T)_\Gamma \\ &= \left(\mathbf{w}, \mathrm{curl}\,\mathrm{curl}\,\mathbf{z} - k^2\mathbf{z}\right) - (\gamma_T \mathbf{w}, \Pi_T\,\mathrm{curl}\,\mathbf{z})_\Gamma + (\mathbf{w}_T, \mathrm{i}\,k T_{-k}\mathbf{z}_T)_\Gamma \\ &= \left(\mathbf{w}, \mathrm{curl}\,\mathrm{curl}\,\mathbf{z} - k^2\mathbf{z}\right) + (\mathbf{w}_T, \gamma_T\,\mathrm{curl}\,\mathbf{z})_\Gamma + (\mathbf{w}_T, \mathrm{i}\,k T_{-k}\mathbf{z}_T)_\Gamma.\end{aligned} \tag{7.3}$$

In a similar way, we can express the right-hand side in (7.1) by

$$\begin{aligned}\text{r.h.s.} &= k^2(\mathbf{w}, \mathbf{v}) + \mathrm{i}\,k\left(\left(T_k \mathbf{w}^\nabla, \mathbf{g}^\nabla\right)_\Gamma - \left(T_k \mathbf{w}^{\mathrm{curl}}, \mathbf{h}^{\mathrm{curl}}\right)_\Gamma\right) \\ &= k^2(\mathbf{w}, \mathbf{v}) + \left(\mathbf{w}^\nabla, (\mathrm{i}\,k T_k)^* \mathbf{g}^\nabla\right)_\Gamma - \left(\mathbf{w}^{\mathrm{curl}}, (\mathrm{i}\,k T_k)^* \mathbf{h}^{\mathrm{curl}}\right)_\Gamma \\ &= k^2(\mathbf{w}, \mathbf{v}) + \left(\mathbf{w}_T, -\mathrm{i}\,k T_{-k}\left(\mathbf{g}^\nabla - \mathbf{h}^{\mathrm{curl}}\right)\right)_\Gamma.\end{aligned} \tag{7.4}$$

The right-hand sides in (7.3) and (7.4) must be equal which leads to

$$\begin{aligned}\mathrm{curl}\,\mathrm{curl}\,\mathbf{z} - k^2\mathbf{z} &= k^2\mathbf{v} && \text{in } \Omega, \\ \gamma_T\,\mathrm{curl}\,\mathbf{z} + \mathrm{i}\,k T_{-k}\mathbf{z}_T &= -\mathrm{i}\,k T_{-k}\left(\mathbf{g}^\nabla - \mathbf{h}^{\mathrm{curl}}\right) && \text{on } \Gamma.\end{aligned} \tag{7.5}$$

In the next step, we eliminate the capacity operator T_{-k} by considering a full space problem with transmission condition. Note that for any given $\mathbf{q}_T \in \mathbf{H}_{\mathrm{curl}}^{-1/2}(\Gamma)$ the adjoint capacity operator $T_{-k}\mathbf{q}_T$ is computed by first solving the exterior problem

$$\left.\begin{array}{rl} -\mathrm{i}\,k\mathbf{z}^+ + \operatorname{curl}\tilde{\mathbf{H}} = 0 & \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \\ \mathrm{i}\,k\tilde{\mathbf{H}} + \operatorname{curl}\mathbf{z}^+ = 0 & \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \\ \gamma_T^+\mathbf{z}^+ = \mathbf{q}_T \times \mathbf{n} & \text{on } \Gamma, \\ \left.\begin{array}{r} \left|\mathbf{z}^+(\mathbf{x})\right| \le c/r \\ \left|\tilde{\mathbf{H}}(\mathbf{x})\right| \le c/r \\ \left|\mathbf{z}^+ - \tilde{\mathbf{H}} \times \frac{\mathbf{x}}{r}\right| \le c/r^2 \end{array}\right\} & \text{as } r = \|\mathbf{x}\| \to \infty \end{array}\right. \tag{7.6}$$

so that $\gamma_T^+\tilde{\mathbf{H}} = T_{-k}\mathbf{q}_T$. In the following we always choose $\mathbf{q}_T = \Pi_T\mathbf{z}$ in (7.6) with $\mathbf{z}$ being the solution of (7.1).
From the third equation in (7.6) we obtain $[(\mathbf{z},\mathbf{z}^+)]_{0,\Gamma} = 0$ and from the second equation in (7.6)

$$\gamma_T^+ \operatorname{curl}\mathbf{z}^+ = -\mathrm{i}\,k\gamma_T^+\tilde{\mathbf{H}} = -\mathrm{i}\,kT_{-k}\mathbf{z}_T. \tag{7.7}$$

Hence,

$$\left[\left(\mathbf{z},\mathbf{z}^+\right)\right]_{1,\Gamma} \overset{(2.4)}{=} \gamma_T \operatorname{curl}\mathbf{z} - \gamma_T^+ \operatorname{curl}\mathbf{z}^+ \overset{(7.5),\,(7.7)}{=} -\mathrm{i}\,kT_{-k}\left(\mathbf{g}^\nabla - \mathbf{h}^{\mathrm{curl}}\right).$$

Let $\mathbf{v}_{\mathrm{zero}}$ denote the extension of $\mathbf{v}$ to the full space by 0 and define $\mathbf{Z} \in \mathbf{H}_{\mathrm{loc}}\left(\mathbb{R}^3,\operatorname{curl}\right)$ by $\mathbf{Z}|_\Omega = \mathbf{z}$ and $\mathbf{Z}|_{\Omega^+} = \mathbf{z}^+$. The combination with the second equation in (7.5) leads to (see [47, (5.2.22)] for the radiation condition)

$$\begin{array}{rll} \operatorname{curl}\operatorname{curl}\mathbf{Z} - k^2\mathbf{Z} & = k^2\mathbf{v}_{\mathrm{zero}} & \text{in } \mathbb{R}^3\backslash\Gamma, \\ \left[(\mathbf{z},\mathbf{z}^+)\right]_{0,\Gamma} & = 0, & \\ \left[(\mathbf{z},\mathbf{z}^+)\right]_{1,\Gamma} & = -\mathrm{i}\,kT_{-k}\left(\mathbf{g}^\nabla - \mathbf{h}^{\mathrm{curl}}\right), & \\ \left|\partial_r\mathbf{z}^+(\mathbf{x}) + \mathrm{i}\,k\mathbf{z}^+(\mathbf{x})\right| & \le c/r^2, & \text{as } r = \|\mathbf{x}\| \to \infty. \end{array} \tag{7.8}$$

We first construct a particular solution for the corresponding full space problem by ignoring the transmission conditions. Then we adjust this solution to satisfy the transmission condition.
For this purpose we need the fundamental solution for the electric part of the Maxwell problem in the full space:

$$\begin{array}{rll} \operatorname{curl}\operatorname{curl}\mathbf{G}_k - k^2\mathbf{G}_k & = \delta\mathbf{I} & \text{in } \mathbb{R}^3, \\ \left|\partial_r\mathbf{G}_k(\mathbf{x}) - \mathrm{i}\,k\mathbf{G}_k(\mathbf{x})\right| & \le c/r^2 & \text{as } r = \|\mathbf{x}\| \to \infty. \end{array}$$

We eliminate in [47, (5.2.1)] the magnetic field to get the equations

$$\begin{array}{rll} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} & = \delta\mathbf{I} & \text{in } \mathbb{R}^3, \\ \left|\partial_r\mathbf{E}(\mathbf{x}) - \mathrm{i}\,k\mathbf{E}(\mathbf{x})\right| & \le c/r^2 & \text{as } r = \|\mathbf{x}\| \to \infty. \end{array}$$

Hence, the fundamental solution is obtained by dividing the one in [47, (5.2.8)] by $(\mathrm{i}\,\omega\mu)$ to obtain

$$\mathbf{G}_k(\mathbf{x}) = g_k(\|\mathbf{x}\|)\,\mathbf{I} + \frac{1}{k^2}\nabla\nabla^\intercal g_k(\|\mathbf{x}\|) \quad \text{with} \quad g_k(r) := \frac{\mathrm{e}^{\mathrm{i}\,kr}}{4\pi r}. \tag{7.9}$$

The second term in the sum is understood as a distribution, i.e., the convolution with a function $\mathbf{f} \in C_{\mathrm{comp}}^\infty\left(\mathbb{R}^3,\mathbb{C}^3\right)$ is defined by

$$(\mathbf{G}_k \star \mathbf{f})(x) = \int_{\mathbb{R}^3} g_k(\|x-y\|)\,\mathbf{f}(y)\,dy + \frac{1}{k^2}\nabla\int_{\mathbb{R}^3} g_k(\|\mathbf{x}-\mathbf{y}\|)\operatorname{div}\mathbf{f}(\mathbf{y})\,d\mathbf{y}. \tag{7.10}$$

From (7.10) we conclude that

$$\mathbf{z}_1 = k^2\int_\Omega g_{-k}(\|\cdot-\mathbf{y}\|)\,\mathbf{v}(\mathbf{y})\,d\mathbf{y} + \nabla\int_{\mathbb{R}^3} g_{-k}(\|\mathbf{x}-\mathbf{y}\|)\,(\operatorname{div}\mathbf{v}_{\mathrm{zero}})(\mathbf{y})\,d\mathbf{y} \quad \text{in } \mathbb{R}^3$$

solves the differential equation (first line in (7.8)) in $\mathbb{R}^3\backslash\Gamma$ and the radiation condition. The function $\mathbf{v}_{\mathrm{zero}}$ has a jump across Γ and it is easy to verify that the distributional divergence is given by

$$(\operatorname{div}_{\mathbb{R}^3}\mathbf{v}_{\mathrm{zero}})(\psi) = \int_\Omega (\operatorname{div}\mathbf{v})\,\psi - \int_\Gamma \langle\mathbf{v},\mathbf{n}\rangle\,\psi \qquad \forall\psi \in C_{\mathrm{comp}}^\infty\left(\mathbb{R}^3\right).$$

Hence,

$$\mathbf{z}_1 = k^2\mathcal{N}_{-k}^{\mathrm{Hh}}(\mathbf{v}) + \nabla\mathcal{N}_{-k}^{\mathrm{Hh}}(\operatorname{div}\mathbf{v}) - \nabla\mathcal{S}_{-k}^{\mathrm{Hh}}(\langle\mathbf{v},\mathbf{n}\rangle) =: \mathbf{z}_{1,1} + \mathbf{z}_{1,2} + \mathbf{z}_{1,3}$$

with acoustic single layer potential

$$\mathcal{S}_k^{\mathrm{Hh}}\phi := \int_\Gamma g_k(\|\cdot - \mathbf{y}\|)\,\phi(\mathbf{y})\,d\Gamma_{\mathbf{y}} \tag{7.11}$$

and the acoustic Newton potential

$$\mathcal{N}_k^{\mathrm{Hh}}w := \int_\Omega g_k(\|\cdot - \mathbf{y}\|)\,w(\mathbf{y})\,d\Gamma_{\mathbf{y}}. \tag{7.12}$$

We assumed $\mathbf{v} \in \mathbf{H}(\Omega, \operatorname{div})$. Well-known mapping properties of $\mathcal{S}_k^{\mathrm{Hh}}$ and $\mathcal{N}_k^{\mathrm{Hh}}$ (cf. [50]) imply that

$$\mathbf{z}_{1,1} \in \mathbf{H}_{\mathrm{loc}}^2(\mathbb{R}^3) \text{ so that } [\mathbf{z}_{1,1}]_{0,\Gamma} = 0 \quad \text{and} \quad [\mathbf{z}_{1,1}]_{1,\Gamma} = 0.$$

By the same reasoning we know that $\mathbf{z}_{1,2} \in \mathbf{H}_{\mathrm{loc}}^1(\mathbb{R}^3)$ and also $\operatorname{curl}\mathbf{z}_{1,2} = 0$. Hence $[\mathbf{z}_{1,2}]_{0,\Gamma} = [\mathbf{z}_{1,2}]_{1,\Gamma} = 0$. Since $\langle\mathbf{v},\mathbf{n}\rangle \in H^{-1/2}(\Gamma)$ we know that $\mathcal{S}_{-k}^{\mathrm{Hh}}(\langle\mathbf{v},\mathbf{n}\rangle) \in \mathbf{H}_{\mathrm{loc}}^1(\mathbb{R}^3)$ and $\operatorname{curl}\nabla\mathcal{S}_{-k}^{\mathrm{Hh}}(\langle\mathbf{v},\mathbf{n}\rangle) = 0$ so that $[\mathbf{z}_{1,3}]_{1,\Gamma} = 0$. Since $\gamma_\tau\nabla$ is a tangential differential operator its jump vanishes on functions in $\mathbf{H}_{\mathrm{loc}}^1(\mathbb{R}^3)$. This implies that

$$[\mathbf{z}_1]_{0,\Gamma} = 0 \quad \text{and} \quad [\mathbf{z}_1]_{1,\Gamma} = 0. \tag{7.13}$$

To obtain the full solution we introduce the single layer operator for the Maxwell problem (cf. [11, (3.11)]) by

$$\mathcal{S}_k^{\mathrm{Mw}}(\boldsymbol{\phi}) = \mathcal{S}_k^{\mathrm{Hh}}(\boldsymbol{\phi}) + \frac{1}{k^2}\nabla\mathcal{S}_k^{\mathrm{Hh}}(\operatorname{div}_\Gamma\boldsymbol{\phi}). \tag{7.14}$$

From [47, (5.5.29)] we get that

$$\left[\mathcal{S}_k^{\mathrm{Mw}}\boldsymbol{\phi}\right]_{1,\Gamma} = -\boldsymbol{\phi}.$$

The combination of this, the third equation in (7.8), and (7.13) show that

$$\mathbf{z}_2 := \mathcal{S}_{-k}^{\mathrm{Mw}}\left(\mathrm{i}\,kT_{-k}\left(\mathbf{g}^\nabla - \mathbf{h}^{\mathrm{curl}}\right)\right)$$

satisfies $\operatorname{curl}\operatorname{curl}\mathbf{z}_2 - k^2\mathbf{z}_2 = 0$ in $\mathbb{R}^3\backslash\Gamma$, the transmission condition (2nd and 3rd equation in (7.8)), and the Silver-Müller radiation conditions for the dual problem. Next we give a formula for the full solution of (7.8)

$$\begin{aligned}
\mathbf{Z} = {} & k^2\int_\Omega g_{-k}(\|\cdot - \mathbf{y}\|)\,\mathbf{v}(\mathbf{y})\,d\mathbf{y} + \nabla\int_\Omega g_{-k}(\|\cdot - \mathbf{y}\|)\,(\operatorname{div}\mathbf{v})(\mathbf{y})\,d\mathbf{y} \\
& - \nabla\int_\Gamma g_{-k}(\|\cdot - \mathbf{y}\|)\,\langle\mathbf{v},\mathbf{n}\rangle(\mathbf{y})\,d\mathbf{y} \\
& + (\mathrm{i}\,k)\int_\Gamma g_{-k}(\|\cdot - \mathbf{y}\|)\,T_{-k}\left(\mathbf{g}^\nabla - \mathbf{h}^{\mathrm{curl}}\right)(\mathbf{y})\,d\Gamma_{\mathbf{y}} - \frac{1}{\mathrm{i}\,k}\nabla\int_\Gamma g_{-k}(\|\cdot - \mathbf{y}\|)\operatorname{div}_\Gamma T_{-k}\mathbf{g}^\nabla(\mathbf{y})\,d\Gamma_{\mathbf{y}},
\end{aligned} \tag{7.15}$$

where we used $\operatorname{div}_\Gamma T_{-k}\mathbf{h}^{\mathrm{curl}} = 0$ (cf. (2.22)).

Theorem 7.1

1. *For $\mathbf{v} \in \mathbf{V}_0^*$, $\mathbf{g} = \mathbf{v}$, and $\mathbf{h} = \mathbf{0}$, the solution of (7.1) is given by*

$$\mathbf{z} = k^2\int_\Omega g_{-k}(\|\cdot - \mathbf{y}\|)\,\mathbf{v}(\mathbf{y})\,d\mathbf{y} + \mathrm{i}\,k\int_\Gamma g_{-k}(\|\cdot - \mathbf{y}\|)\,T_{-k}\mathbf{v}^\nabla(\mathbf{y})\,d\Gamma_{\mathbf{y}}. \tag{7.16}$$

2. *For $\mathbf{v} = \mathbf{0}$, formula (7.15) simplifies to a combined layer potential*

$$\mathbf{z} = \mathrm{i}\,k\int_\Gamma g_{-k}(\|\cdot - \mathbf{y}\|)\,T_{-k}\left(\mathbf{g}^\nabla - \mathbf{h}^{\mathrm{curl}}\right)(\mathbf{y})\,d\Gamma_{\mathbf{y}} - \frac{1}{\mathrm{i}\,k}\nabla\int_\Gamma g_{-k}(\|\cdot - \mathbf{y}\|)\operatorname{div}_\Gamma T_{-k}\mathbf{g}^\nabla(\mathbf{y})\,d\Gamma_{\mathbf{y}}. \tag{7.17}$$

Proof. For the choices as in (7.16), the properties (4.29) allow us to simplify (7.15) and to obtain (7.16). Formula (7.17) follows simply by setting $\mathbf{v} = \mathbf{0}$ in (7.15). ■

7.1.2 Solution Formula for Type 2 Problems in the Unit Ball $B_1(0)$

The problem of Type 2 (cf. (7.2)) is a Poisson-type problem. Integration by parts leads to its strong formulation. We recall $\operatorname{div} L_\Omega \mathbf{r} = 0$ by (4.9b) so that

$$\begin{aligned} -\Delta Z &= 0 && \text{in } \Omega, \\ \frac{\partial Z}{\partial \mathbf{n}} - \frac{\mathrm{i}}{k} \operatorname{div}_\Gamma T_k \nabla_\Gamma Z &= \langle L_\Omega \mathbf{r}, \mathbf{n} \rangle - \frac{\mathrm{i}}{k} \operatorname{div}_\Gamma T_k^{\mathrm{low}} \mathbf{r}_T && \text{on } \Gamma. \end{aligned} \tag{7.18}$$

To analyze problem, we introduce the Dirichlet-to-Neumann operator $T_\Delta : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ that maps $g \in H^{1/2}(\Gamma)$ to $\partial_n u$, where u is the (weak) solution of

$$\Delta u = 0, \qquad u = g \quad \text{on } \Gamma.$$

This allows us to formulate (7.18) as follows (with L_Γ as in Def. 4.2)

$$\begin{aligned} -\Delta Z &= 0 && \text{in } \Omega, \\ T_\Delta Z - \frac{\mathrm{i}}{k} \operatorname{div}_\Gamma T_k \nabla_\Gamma Z &= \langle L_\Omega \mathbf{r}, \mathbf{n} \rangle - \frac{\mathrm{i}}{k} \operatorname{div}_\Gamma T_k^{\mathrm{low}} \mathbf{r}_T && \text{on } \Gamma. \end{aligned} \tag{7.19}$$

We employ expansions of $\langle L_\Omega \mathbf{r}, \mathbf{n} \rangle$ and $\mathbf{r}_T$ in the forms

$$\langle L_\Omega \mathbf{r}, \mathbf{n} \rangle = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} \kappa_\ell^m Y_\ell^m \quad \text{and} \quad \mathbf{r}_T = \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} \left(r_\ell^m \mathbf{T}_\ell^m + R_\ell^m \nabla_\Gamma Y_\ell^m \right) \tag{7.20}$$

so that the right-hand side in the second equation of (7.19) is

$$\begin{aligned} \langle L_\Omega \mathbf{r}, \mathbf{n} \rangle - \frac{\mathrm{i}}{k} \operatorname{div}_\Gamma T_k^{\mathrm{low}} \mathbf{r}_T &= \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} \left(\kappa_\ell^m Y_\ell^m - \frac{\mathrm{i}}{k} \operatorname{div}_\Gamma T_k^{\mathrm{low}} \left(r_\ell^m \mathbf{T}_\ell^m + R_\ell^m \nabla_\Gamma Y_\ell^m \right) \right) \\ &\overset{[47,\ (2.4.173),\ (5.3.93)]}{=} \sum_{\ell > \lambda k}^{\infty} \sum_{m \in \iota_\ell} \kappa_\ell^m Y_\ell^m + \sum_{\ell \leq \lambda k} \sum_{m \in \iota_\ell} \left(\kappa_\ell^m - \frac{\ell (\ell+1)}{z_\ell(k)+1} R_\ell^m \right) Y_\ell^m. \end{aligned} \tag{7.21}$$

Note that $Y_0^0 = \frac{1}{\sqrt{4\pi}}$ is constant and hence $\kappa_0^0 = \left(\langle L_\Omega \mathbf{r}, \mathbf{n} \rangle, Y_0^0 \right)_\Gamma = \left(\operatorname{div} L_\Omega \mathbf{r}, \frac{1}{\sqrt{4\pi}} \right)_\Omega = 0$. Hence the summation index for the second sum in (7.21) can be restricted to $1 \leq \ell \leq \lambda k$. The representation (7.21) motivates the ansatz for the trace of Z

$$Z|_\Gamma = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} Z_\ell^m Y_\ell^m.$$

The left-hand side in the second equation of (7.19) becomes

$$\begin{aligned} T_\Delta Z - \frac{\mathrm{i}}{k} \operatorname{div}_\Gamma T_k \nabla_\Gamma Z &= \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} Z_\ell^m \left(T_\Delta Y_\ell^m - \frac{\mathrm{i}}{k} \operatorname{div}_\Gamma T_k \nabla_\Gamma Y_\ell^m \right) \\ &\overset{[47,\ (2.5.9),\ (5.3.93)]}{=} \sum_{\ell=1}^{\infty} \ell \left(1 - \frac{\ell+1}{z_\ell(k)+1} \right) \sum_{m \in \iota_\ell} Z_\ell^m Y_\ell^m. \end{aligned} \tag{7.22}$$

The right-hand sides in (7.21) and (7.22) must be equal. Thus

$$Z_\ell^m = \begin{cases} \varphi_\ell^m := \dfrac{1}{\ell} \left(\dfrac{z_\ell(k)+1}{z_\ell(k) - \ell} \right) \kappa_\ell^m - \dfrac{\ell+1}{z_\ell(k) - \ell} R_\ell^m & \ell \leq \lambda k, \\ \Phi_\ell^m := \dfrac{1}{\ell} \left(\dfrac{z_\ell(k)+1}{z_\ell(k) - \ell} \right) \kappa_\ell^m & \ell > \lambda k. \end{cases} \tag{7.23}$$

Hence, the solution Z of (7.19) is the solution of the following Laplace equation with non-homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\Delta Z &= 0 && \text{in } \Omega, \\ Z &= g_D && \text{on } \Gamma, \\ \text{with} \quad g_D &:= \sum_{\ell \leq \lambda k} \sum_{m \in \iota_\ell} \varphi_\ell^m Y_\ell^m + \sum_{\ell > \lambda k}^{\infty} \sum_{m \in \iota_\ell} \Phi_\ell^m Y_\ell^m. \end{aligned} \tag{7.24}$$

7.2 Regularity of the Dual Problems

7.2.1 The High-Frequency Case

We consider the regularity of the solution in (4.50) for a right-hand side $\mathbf{r} \leftarrow \mathbf{v}_0^h \in \mathbf{V}_0^*$. Recall the definition of ∇^p in (2.27).

Proposition 7.2 *Let $\mathbf{v}_0 \in \mathbf{V}_0^*$ and $\mathbf{z} = \mathcal{N}_2 \mathbf{v}_0$ with $\mathcal{N}_2$ given by (4.50). There exists a k-dependent splitting $\mathcal{N}_2 = \mathcal{N}_2^{\text{rough}} + \mathcal{N}_2^{\mathcal{A}}$ such that*

$$\begin{aligned} \left\| \mathcal{N}_2^{\text{rough}} \mathbf{v}_0 \right\|_{\mathbf{H}^2(\Omega)} &\leq C_{\text{rough}} k \left\| \mathbf{v}_0 \right\|_{\text{curl},\Omega,k}, \\ \left\| \nabla^p \mathcal{N}_2^{\mathcal{A}} \mathbf{v}_0 \right\| &\leq C_{\mathcal{A},2} k^3 \gamma_{\mathcal{A},2} \left(\max\left\{ p+1, k \right\} \right)^p \left\| \mathbf{v}_0 \right\|_{\text{curl},\Omega,k} \quad \forall p \in \mathbb{N}_0, \end{aligned} \tag{7.25}$$

where C_{rough}, $C_{\mathcal{A},2}$, $\gamma_{\mathcal{A},2} > 0$ are constants independent of k and $\mathbf{v}$.

Proof. The solution of the dual problem (4.50) is given (cf. (7.16), (4.2b)) by

$$\begin{aligned} &\mathbf{z} = \left(-\mathrm{i}\, k \mathbf{z}_1 + \mathbf{z}_2 \right) \mathrm{i}\, k \\ \text{with} \quad &\mathbf{z}_1 := \int_\Omega g_{-k}\left(\|\cdot - \mathbf{y}\| \right) \mathbf{v}_0\left(\mathbf{y} \right) d\mathbf{y} \quad \text{and} \quad \mathbf{z}_2 := \int_\Gamma g_{-k}\left(\|\cdot - \mathbf{y}\| \right) T_{-k} \mathbf{v}_0^{\nabla}\left(\mathbf{y} \right) d\Gamma_{\mathbf{y}}. \end{aligned}$$

From the decomposition lemma in [41, Lemma 3.5] we get a k-dependent additive splitting $\mathbf{z}_1 := \mathbf{z}_1^{\text{rough}} + \mathbf{z}_1^{\mathcal{A}}$ such that

$$\begin{aligned} \left\| \nabla^m \mathbf{z}_1^{\text{rough}} \right\| &\leq C k^{m-2} \left\| \mathbf{v}_0 \right\| \quad \forall m \in \{0,1,2\}, \\ \left\| \nabla^p \mathbf{z}_1^{\mathcal{A}} \right\| &\leq C k^{p-1} \left\| \mathbf{v}_0 \right\| \quad \forall p \in \mathbb{N}_0 \end{aligned} \tag{7.26}$$

for a constant C independent of k and $\mathbf{v}_0$. For the function $\mathbf{z}_2$ we employ the splitting

$$\mathbf{v}_0^{\nabla,\text{low}} := L_\Gamma\left(\mathbf{v}_0^{\nabla} \right) \quad \text{and} \quad \mathbf{v}_0^{\nabla,\text{high}} := H_\Gamma\left(\mathbf{v}_0^{\nabla} \right)$$

and define $\mathbf{z}_2^{\text{low}} := \mathcal{S}_{-k}^{\text{Hh}}\left(T_{-k} \mathbf{v}_0^{\nabla,\text{low}} \right)$ and $\mathbf{z}_2^{\text{high}} := \mathbf{z}_2 - \mathbf{z}_2^{\text{low}}$. From [37, Lem. 3.4, Thm. 5.3] we conclude that there exists a splitting $\mathbf{z}_2^{\text{high}} = \mathbf{z}_2^{\text{rough}} + \mathbf{z}_2^{\mathcal{A}}$ such that, for $\mathbf{w} := T_{-k} \mathbf{v}_0^{\nabla,\text{high}}$,

$$\begin{aligned} \left\| \nabla^m \mathbf{z}_2^{\text{rough}} \right\| &\leq C \left\| \mathbf{w} \right\|_{\mathbf{H}_T^{1/2}(\Gamma)} && \forall m = 0,1,2, \\ \left\| \nabla^p \mathbf{z}_2^{\mathcal{A}} \right\| &\leq \tilde{C} \tilde{\gamma}^p \max\left\{ p+1, k \right\}^{p+1} \left\| \mathbf{w} \right\|_{\mathbf{H}_T^{-3/2}(\Gamma)} && \forall p \in \mathbb{N}_0. \end{aligned} \tag{7.27}$$

Here the constants C, $\tilde{C}$, $\tilde{\gamma}$ are independent of k and $\mathbf{w}$. This motivates the definition of the operator $\mathcal{N}_2^{\text{rough}} : \mathbf{V}_0^* \to \mathbf{H}^2(\Omega)$ by

$$\mathcal{N}_2^{\text{rough}} \mathbf{v}_0 := \mathbf{z}_1^{\text{rough}} + \mathbf{z}_2^{\text{rough}}. \tag{7.28}$$

To estimate the norms of $\mathbf{w}$ in (7.27) we employ the third estimate in Lemma 5.3 for $s \leq 3/2$ (we also use that (5.8) gives $z_\ell(-k) = \overline{z_\ell(k)}$ and $\left| \frac{-k}{z_\ell(-k)+1} \right| = \left| \frac{k}{z_\ell(k)+1} \right|$):

$$\begin{aligned} \left\| \mathbf{w} \right\|_{\mathbf{H}_T^s(\Gamma)}^2 &\leq \sum_{\ell > \lambda k} \sum_{m \in \iota_\ell} \left(\ell(\ell+1) \right)^{s+1} \left| \frac{k}{z_\ell(k)+1} \right|^2 \left| V_\ell^m \right|^2 \leq C k^2 \sum_{\ell > \lambda k} \sum_{m \in \iota_\ell} (\ell+1)^{2s-3} \left(\ell(\ell+1) \right)^{3/2} \left| V_\ell^m \right|^2 \\ &\leq C k^{2s-1} \sum_{\ell > \lambda k} \sum_{m \in \iota_\ell} \left(\ell(\ell+1) \right)^{3/2} \left| V_\ell^m \right|^2 \overset{(2.23)}{\leq} C k^{2s-1} \left\| \operatorname{div}_\Gamma \mathbf{v}_0^{\nabla} \right\|_{H^{-1/2}(\Gamma)}^2 \leq C k^{2s-1} \left\| \mathbf{v}_0^{\nabla} \right\|_{\mathbf{H}_T^{1/2}(\Gamma)}^2 \\ &\leq C k^{2s-1} \left\| \mathbf{v}_0 \right\|_{\mathbf{H}_T^1(\Omega)}^2 \overset{\operatorname{div} \mathbf{v}_0 = 0}{\leq} C k^{2s-1} \left\| \mathbf{v}_0 \right\|_{\text{curl},\Omega,1}^2. \end{aligned} \tag{7.29}$$

We set $s = 1/2$ in (7.29) to derive

$$\left\| \mathbf{w} \right\|_{\mathbf{H}_T^{1/2}(\Gamma)} \leq C \left\| \mathbf{v}_0 \right\|_{\text{curl},\Omega,k}. \tag{7.30}$$

The combination of the first lines in (7.26) and (7.27) with (7.30) leads to the first estimate in (7.25). To estimate $\left\| \mathbf{w} \right\|_{\mathbf{H}_T^{-3/2}(\Gamma)}$ we employ (7.29) for $s = -3/2$ and obtain

$$\left\| \mathbf{w} \right\|_{\mathbf{H}_T^{-3/2}(\Gamma)} \leq C k^{-2} \left\| \mathbf{v}_0 \right\|_{\text{curl},\Omega,1}. \tag{7.31}$$

Taking into account the second estimate in (7.27) results in

$$\left\|\nabla^p \mathbf{z}_2^{\mathcal{A}}\right\| \leq C\gamma^p \max\{p+1,k\}^{p-1} \|\mathbf{v}_0\|_{\operatorname{curl},\Omega,1}.$$

The term $\mathbf{z}_2^{\text{low}}$ is defined as the acoustic single layer potential applied to the function $T_{-k}\mathbf{v}_0^{\nabla,\text{low}}$. The analysis of such a term will be carried out in Section 7.2.2 and it follows from (7.40) (where the function c corresponds to $\mathbf{z}_2^{\text{low}}$) that

$$\mathbf{z}_2^{\text{low}} \in \mathcal{A}\left(Ck^2 \|\mathbf{v}_0\|_{\operatorname{curl},\Omega,1}, \gamma, \Omega\right), \tag{7.32}$$

where C and γ are positive constants independent of k and $\mathbf{v}_0$. The combination of the second estimates in (7.26), (7.27) with (7.31) and (7.32) leads to the second estimate in (7.25). ■

7.2.2 The Low-Frequency Cases

First, we study the regularity of the solution operator $\mathcal{N}_3^{\mathcal{A}}$ as in (4.51) which is of Type 1 with $\mathbf{r} = \mathbf{g} = L_\Omega \mathbf{v}$ and $\mathbf{h} = 0$. Since $L_\Omega \mathbf{v}$ is, in general, not in $\mathbf{V}_0^*$ we have to employ the solution formula (7.15), where the second summand can be dropped due to $\operatorname{div} L_\Omega \mathbf{v} = 0$ (cf. (4.9b)). We set

$$\begin{array}{ll} a := \mathcal{N}_{-k}^{\text{Hh}}\left(L_\Omega \mathbf{v}\right), & b := \mathcal{S}_{-k}^{\text{Hh}}\left(\langle L_\Omega \mathbf{v}, \mathbf{n}\rangle\right), \\ c := \mathcal{S}_{-k}^{\text{Hh}}\left(T_{-k}\left(L_\Omega \mathbf{v}\right)^{\nabla}\right), & d := \mathcal{S}_{-k}^{\text{Hh}}\left(\operatorname{div}_\Gamma T_{-k}\left(L_\Omega \mathbf{v}\right)^{\nabla}\right). \end{array} \tag{7.33}$$

so that

$$\mathbf{z} := \mathcal{N}_3^{\mathcal{A}} \mathbf{v}_h = k^2 a - \nabla b + \mathrm{i}\, kc + \frac{\mathrm{i}}{k}\nabla d. \tag{7.34}$$

Proposition 7.3 *For any $\mathbf{v} \in \mathbf{X}$ There exist positive constants $C_{\mathcal{A},3}$ and $\gamma_{\mathcal{A},3}$ independent of k such that for any $\mathbf{v} \in \mathbf{X}$*

$$\mathcal{N}_3^{\mathcal{A}} \mathbf{v} \in \mathcal{A}\left(C_{\mathcal{A},3} k^3 \|\mathbf{v}\|_{\operatorname{curl},\Omega,k}, \gamma_{\mathcal{A},3}, \Omega\right).$$

Proof. We determine the analyticity classes for the functions in the splitting (7.34), distinguishing between the terms related to the acoustic Newton potential $\mathcal{N}_{-k}^{\text{Hh}}$ and the acoustic single layer operator.
@Newton potential. We start by writing a function $q = \mathcal{N}_{-k}^{\text{Hh}}(g)$ as a solution of a transmission problem: Let

$$\begin{aligned} -\Delta q - k^2 q = g_{\text{zero}} \text{ in } \mathbb{R}^3 \backslash \Gamma \quad &\text{with} \quad g_{\text{zero}} := \begin{cases} g & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^3 \backslash \overline{\Omega}. \end{cases} \\ [q]_\Gamma = \left[\frac{\partial q}{\partial \mathbf{n}}\right]_\Gamma = 0, & \\ \left|\frac{\partial q}{\partial r} + \mathrm{i}\, kq\right| = o\left(\|\mathbf{x}\|^{-1}\right) \quad &\text{as } \|\mathbf{x}\| \to \infty. \end{aligned}$$

Next, we will determine the class of analyticity for the function q by using the results in [37]. For this, we have to investigate the analyticity class of $g = L_\Omega \mathbf{v}$. From Theorem 5.9 we conclude with C_1, γ_1 independent of k and $\mathbf{v}$

$$L_\Omega \mathbf{v} \in \mathcal{A}\left(C_{\mathbf{v},1}, \gamma_1, \Omega\right) \quad \text{with} \quad C_{\mathbf{v},1} := C_1 k^{3/2} \|\mathbf{v}\|_{\operatorname{curl},\Omega,k}. \tag{7.35}$$

This allows us to use [37, Thm. B.4] to deduce

$$\left.\mathcal{N}_{-k}^{\text{Hh}}\left(L_\Omega \mathbf{v}\right)\right|_\Omega \in \mathcal{A}\left(C_{\mathbf{v},2}, \gamma_3, \Omega\right),$$

with

$$C_{\mathbf{v},2} := C_3\left(k^{-2} C_{\mathbf{v},1} + k^{-1}\left\|\mathcal{N}_{-k}^{\text{Hh}}\left(L_\Omega \mathbf{v}\right)\right\|_{\mathcal{H},B_R(0)}\right);$$

here, $B_R(0)$ is an (arbitrarily chosen) ball containing $\overline{\Omega}$. From [41, Lemma 3.5], we get $\left\|\mathcal{N}_{-k}^{\text{Hh}}\left(L_\Omega \mathbf{v}\right)\right\|_{\mathcal{H},B_R(0)} \leq C\left\|L_\Omega \mathbf{v}\right\|$ so that

$$C_{\mathbf{v},2} \leq CC_3\left(k^{-2} C_{\mathbf{v},1} + k^{-1}\left\|L_\Omega \mathbf{v}\right\|\right) \leq CC_3\left(C_1 k^{-1/2}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k} + k^{-2}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}\right) \leq C_4 k^{-1/2}\|\mathbf{v}\|_{\operatorname{curl},\Omega,k}.$$

Hence

$$k^2 a \in \mathcal{A}\left(C_4 k^{3/2} \|\mathbf{v}\|_{\operatorname{curl},\Omega,k}, \gamma_4, \Omega\right). \tag{7.36}$$

@Single layer potential: We write a function $q = \mathcal{S}_{-k}^{\mathrm{Hh}}(g)$ as a solution of a transmission problem: Let γ_0 denote the standard, one-sided trace operator for Γ from the interior and γ_0^+ the one from the exterior. The one-sided normal trace (from the interior) is denoted by $\gamma_1 := \partial/\partial\mathbf{n}$ and by γ_1^+ from the exterior. The respective jumps are $[u]_\Gamma = \gamma_0^+ u - \gamma_0 u$ and $[u]_{\mathbf{n},\Gamma} = \gamma_1^+ u - \gamma_1 u$. The well-known jump relations for the single layer potential yield for the potential $q = \mathcal{S}_{-k}^{\mathrm{Hh}}(g)$

$$\begin{aligned}
-\Delta q - k^2 q = 0 \text{ in } \mathbb{R}^3\backslash\Gamma,\\
[q]_\Gamma = 0 \quad \text{and} \quad [q]_{\mathbf{n},\Gamma} = -g,\\
\left|\frac{\partial q}{\partial r} + \mathrm{i}\, kq\right| = o\left(\|\mathbf{x}\|^{-1}\right) \quad \text{as } \|\mathbf{x}\| \to \infty.
\end{aligned}$$

The essential part of the regularity estimates are those near the boundary/interface Γ, where the analyticity of the jump g and the geometry Γ come into play. We follow the standard procedure of locally flattening Γ so that [35, Thm. 5.5.4] becomes applicable. In view of (7.33) we have to analyze the transmission problem for 3 different choices of g:

$$g \in \{g_1, g_2, \mathbf{g}_3\} \quad \text{with} \quad g_1 := \langle L_\Omega \mathbf{v}, \mathbf{n}\rangle, \quad g_2 := \operatorname{div}_\Gamma T_{-k}(L_\Omega \mathbf{v})^\nabla, \quad \mathbf{g}_3 := T_{-k}(L_\Omega \mathbf{v})^\nabla. \tag{7.37}$$

1. step (analyticity classes of g): In the following $\mathcal{U}_\Gamma$ is a sufficiently small neighborhood of Γ whose size depends solely on Γ. Lemma 5.10 directly implies the existence of an extension g_1^* of g_1 into $\mathcal{U}_\Gamma$

$$g_1^* \in \mathcal{A}\left(\tilde{C} k^{3/2} \|\mathbf{v}\|_{\operatorname{curl},\Omega,k}, \tilde{\gamma}, \mathcal{U}_\Gamma\right). \tag{7.38}$$

To define extensions of g_2, $\mathbf{g}_3$ we repeat the arguments of Lemma 5.5. From the expansion

$$\Pi_T \mathbf{v} := \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} \left(v_\ell^m \overrightarrow{\operatorname{curl}_\Gamma} Y_\ell^m + V_\ell^m \nabla_\Gamma Y_\ell^m\right)$$

we get

$$\begin{aligned}
\mathbf{g}_3 = T_{-k}(L_\Omega \mathbf{v})^\nabla &\overset{(5.7)}{=} \sum_{1 \le \ell \le \lambda k} \overline{\left(\frac{\mathrm{i}\, k}{z_\ell(k)+1}\right)} \sum_{m \in \iota_\ell} V_\ell^m \nabla_\Gamma Y_\ell^m,\\
g_2 = \operatorname{div}_\Gamma T_{-k}(L_\Omega \mathbf{v})^\nabla &\overset{(2.22)}{=} \sum_{1 \le \ell \le \lambda k} \overline{\left(\frac{\mathrm{i}\, k\ell(\ell+1)}{z_\ell(k)+1}\right)} \sum_{m \in \iota_\ell} V_\ell^m Y_\ell^m.
\end{aligned}$$

Recall the analytic extension $\tilde{Y}_\ell^m$ of Y_ℓ^m with the property (5.11) from the proof of Lemma 5.5. We define the analytic extensions of g_2, $\mathbf{g}_3$ by

$$g_2^* := \sum_{1 \le \ell \le \lambda k} \overline{\left(\frac{\mathrm{i}\, k\ell(\ell+1)}{z_\ell(k)+1}\right)} \sum_{m \in \iota_\ell} V_\ell^m \tilde{Y}_\ell^m \quad \text{and} \quad \mathbf{g}_3^* := \sum_{1 \le \ell \le \lambda k} \overline{\left(\frac{\mathrm{i}\, k}{z_\ell(k)+1}\right)} \sum_{m \in \iota_\ell} V_\ell^m \nabla \tilde{Y}_\ell^m.$$

We obtain by using Cauchy-Schwarz inequalities

$$\begin{aligned}
\|\nabla^n g_2^*\|_{L^2(\mathcal{U}_\Gamma)} &\le \sum_{1 \le \ell \le \lambda k} \left|\frac{\mathrm{i}\, k\ell(\ell+1)}{z_\ell(k)+1}\right| \sum_{m \in \iota_\ell} |V_\ell^m| \left\|\nabla^n \tilde{Y}_\ell^m\right\|_{L^2(\mathcal{U}_\Gamma)} \overset{(5.9),\,(2.24a)}{\le} C k^{7/2} \gamma^n \max\{k,n\}^n \|\mathbf{v}_T\|_{-1/2,\operatorname{curl}_\Gamma},\\
\|\nabla^n \mathbf{g}_3^*\|_{L^2(\mathcal{U}_\Gamma)} &\le \sum_{1 \le \ell \le \lambda k} \left|\frac{\mathrm{i}\, k}{z_\ell(k)+1}\right| \sum_{m \in \iota_\ell} |V_\ell^m| \left\|\nabla^{n+1} \tilde{Y}_\ell^m\right\|_{L^2(\mathcal{U}_\Gamma)}\\
&\le \tilde{C} \tilde{\gamma}^{n+1} k \max\{k, n+1\}^n \sum_{1 \le \ell \le \lambda k} \sqrt{\ell(\ell+1)} \sum_{m \in \iota_\ell} |V_\ell^m|\\
&\le \hat{C} \hat{\gamma}^n k^{5/2} \max\{k, n+1\}^n \|\mathbf{v}_T\|_{-1/2,\operatorname{curl}_\Gamma}.
\end{aligned}$$

We combine this with Theorem 2.4 and have proved that the extensions g_1^*, g_2^*, $\mathbf{g}_3^*$ belong to the analyticity classes

$$g_1^* \in \mathcal{A}\left(C_1 k^{3/2} \|\mathbf{v}\|_{\mathrm{curl},\Omega,k}, \gamma_1, \mathcal{U}_\Gamma\right), \quad g_2^* \in \mathcal{A}\left(C_2 k^{7/2} \|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \gamma_2, \mathcal{U}_\Gamma\right), \quad \mathbf{g}_3^* \in \mathcal{A}\left(C_3 k^{5/2} \|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \gamma_3, \mathcal{U}_\Gamma\right),$$

where C_j, γ_j are independent of k and $\mathbf{v}$.
2. step (a priori bounds for potential q): Note that, for an (arbitrary) fixed ball $B_R(0)$ with $\overline{\Omega} \subset B_R(0)$ [37, Lemma 3.3, Thm. 5.3] imply

$$\left\|\mathcal{S}_{-k}^{\mathrm{Hh}}(\mathbf{g}_3)\right\|_{\mathcal{H},B_R(0)} \leq Ck^2 \|\mathbf{g}_3\|_{\mathbf{H}_T^{-3/2}(\Gamma)} \quad \text{and} \quad \left\|\mathcal{S}_{-k}^{\mathrm{Hh}}(g_i)\right\|_{\mathcal{H},B_R(0)} \leq Ck^2 \|g_i\|_{H^{-3/2}(\Gamma)}, \quad i = 1,2. \tag{7.39}$$

The $\|\cdot\|_{H^{-3/2}(\Gamma)}$ norm of g_i can be estimated as follows:
@g_1 :

$$\begin{aligned}\|\langle L_\Omega \mathbf{v}, \mathbf{n}\rangle\|_{H^{-3/2}(\Gamma)} &\leq C \|\langle L_\Omega \mathbf{v}, \mathbf{n}\rangle\|_{H^{-1/2}(\Gamma)} \leq C \|L_\Omega \mathbf{v}\|_{\mathbf{H}(\Omega,\mathrm{div})} \\ &\overset{(4.9b)}{=} C\|L_\Omega \mathbf{v}\| \leq \frac{C}{k} \|L_\Omega \mathbf{v}\|_{\mathrm{curl},\Omega,k} \leq Ck^{-1} \|\mathbf{v}\|_{\mathrm{curl},\Omega,k}.\end{aligned}$$

@g_2 :

$$\begin{aligned}\|g_2\|^2_{H^{-3/2}(\Gamma)} &\overset{(2.18)}{\leq} \sum_{1\leq\ell\leq\lambda k} (\ell(\ell+1))^{-3/2} \left|\frac{\mathrm{i}\,k\ell(\ell+1)}{z_\ell(k)+1}\right|^2 \sum_{m\in\iota_\ell} |V_\ell^m|^2 \\ &\overset{(5.9)}{\leq} Ck^2 \sum_{1\leq\ell\leq\lambda k} (\ell(\ell+1))^{1/2} \sum_{m\in\iota_\ell} |V_\ell^m|^2 \leq Ck^2 \|\mathbf{v}\|^2_{\mathrm{curl},\Omega,1}.\end{aligned}$$

@g_3 :

$$\begin{aligned}\|\mathbf{g}_3\|^2_{\mathbf{H}_T^{-3/2}(\Gamma)} &\overset{(2.20)}{\leq} \sum_{1\leq\ell\leq\lambda k} (\ell(\ell+1))^{-1/2} \left|\frac{\mathrm{i}\,k}{z_\ell(k)+1}\right|^2 \sum_{m\in\iota_\ell} |V_\ell^m|^2 \\ &\leq Ck^2 \sum_{1\leq\ell\leq\lambda k} (\ell(\ell+1))^{-1/2} \sum_{m\in\iota_\ell} |V_\ell^m|^2 \leq Ck^2 \|\mathbf{v}\|^2_{\mathrm{curl},\Omega,1}.\end{aligned}$$

The combination with (7.39) leads to

$$\left\|\mathcal{S}_{-k}^{\mathrm{Hh}}(g_1)\right\|_{\mathcal{H},B_R(0)} \leq Ck\|\mathbf{v}\|_{\mathrm{curl},\Omega,k}, \quad \left\|\mathcal{S}_{-k}^{\mathrm{Hh}}(g_2)\right\|_{\mathcal{H},B_R(0)} \leq Ck^3\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \quad \left\|\mathcal{S}_{-k}^{\mathrm{Hh}}(\mathbf{g}_3)\right\|_{\mathcal{H},B_R(0)} \leq Ck^3\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}.$$

3. step (analyticity of potential q): The above steps and [35, Thm. 5.5.4] give

$$\nabla b \in \mathcal{A}\left(Ck^{3/2}\|\mathbf{v}\|_{\mathrm{curl},\Omega,k}, \gamma, \Omega\right), \quad kc \in \mathcal{A}\left(Ck^3\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \gamma, \Omega\right), \quad \frac{1}{k}\nabla d \in \mathcal{A}\left(Ck^{5/2}\|\mathbf{v}\|_{\mathrm{curl},\Omega,1}, \gamma, \Omega\right). \tag{7.40}$$

From the decomposition (7.34) and (7.36), (7.40) we conclude that

$$\mathcal{N}_3^{\mathcal{A}}\mathbf{v} \in \mathcal{A}\left(C_{\mathcal{A},3}k^3\|\mathbf{v}\|_{\mathrm{curl},\Omega,k}, \gamma_{\mathcal{A},3}, \Omega\right)$$

for constants $C_{\mathcal{A},3}$, $\gamma_{\mathcal{A},3}$ independent of k and $\mathbf{v}$. ■
Next, we analyze the regularity of the solution operator $\mathcal{N}_4^{\mathcal{A}}$ of (4.52).

Proposition 7.4 *Let $\Omega = B_1(0)$. There exist positive constants $C_{\mathcal{A},4}$ and $\gamma_{\mathcal{A},4}$ depending only on Γ and the cut-off parameter λ such that for any $\mathbf{r} \in \mathbf{X}$*

$$\nabla\mathcal{N}_4^{\mathcal{A}}\mathbf{r} \in \mathcal{A}\left(C_{\mathcal{A},4}k^{5/2}\|\mathbf{r}\|_{\mathrm{curl},\Omega,1}, \gamma_{\mathcal{A},4}, \Omega\right).$$

Proof. We first analyze g_D of (7.24) (in Steps 1–3) and subsequently the solution Z of (7.24) in Step 4. As in the proof of Proposition 7.3, we let $\mathcal{U}_\Gamma$ be a sufficiently small neighborhood of Γ.

1. step (analyticity class of g_D^*): With the analytic extensions $\tilde{Y}_\ell^m$ of the eigenfunctions Y_ℓ^m (cf. (5.11)) we extend g_D to $\mathcal{U}_\Gamma$ by

$$g_D^* := \sum_{\ell=1}^{\infty}\sum_{m\in\iota_\ell} Z_\ell^m \tilde{Y}_\ell^m \overset{(7.23)}{=} \sum_{\ell=1}^{\infty}\sum_{m\in\iota_\ell} \frac{1}{\ell}\left(\frac{z_\ell(k)+1}{z_\ell(k)-\ell}\right)\kappa_\ell^m \tilde{Y}_\ell^m - \sum_{\ell\le\lambda k}\sum_{m\in\iota_\ell}\frac{\ell+1}{z_\ell(k)-\ell}R_\ell^m\tilde{Y}_\ell^m,$$

where κ_ℓ^m and R_ℓ^m are given by (7.20). We note that the coefficients κ_ℓ^m are controlled by Lemma 5.10. For the coefficients R_ℓ^m we estimate

$$\sum_{\ell\le\lambda k}\sum_{m\in\iota_\ell}\ell|R_\ell^m| \lesssim k^{3/2}\left(\sum_{\ell\le\lambda k}\sum_{m\in\iota_\ell}\ell|R_\ell^m|^2\right)^{1/2} \overset{(2.24a)}{\lesssim} k^{3/2}\|\mathbf{r}_T\|_{-1/2,\mathrm{curl}_\Gamma} \lesssim k^{3/2}\|\mathbf{r}\|_{\mathrm{curl},\Omega,1}. \tag{7.41}$$

2. step (symbol estimates): We have

$$\left|\frac{1}{\ell}\left(\frac{z_\ell(k)+1}{z_\ell(k)-\ell}\right)\right| \le 2\left(\frac{1}{\ell+1}+\frac{1}{\sqrt{|\mathrm{Re}\,(z_\ell(k))-\ell|^2+|\mathrm{Im}\,z_\ell(k)|^2}}\right)$$

$$\overset{[47,\ (2.6.23)]}{\le} 2\left(\frac{1}{\ell+1}+\frac{1}{|z_\ell(k)+1|}\right) \overset{\text{Lem. } 5.3}{\le} C\begin{cases}1 & \ell\le\lambda k,\\ \frac{1}{\ell+1} & \ell\ge\lambda k\end{cases}$$

and

$$\left|\frac{\ell}{z_\ell(k)-\ell}\right| \le \frac{\ell}{|z_\ell(k)+1|} \le C\begin{cases}\ell & \ell\le\lambda k,\\ 1 & \ell\ge\lambda k.\end{cases}$$

3. step (analyticity classes of g_D^*): We claim: there are C, $\gamma'>0$ independent of k and $\mathbf{r}$ such that

$$g_D^* \in \mathcal{A}(Ck^{3/2}\|\mathbf{r}\|_{\mathrm{curl},\Omega,1},\gamma',\mathcal{U}_\Gamma) \tag{7.42}$$

Using (5.11), the symbol estimates of Step 2 we estimate with the abbreviation $\lambda_\ell = \ell(\ell+1)$

$$\|\nabla^n g_D^*\|_{L^2(\mathcal{U}_\Gamma)} \lesssim \gamma^n\Big\{\sum_{\ell\le k\gamma'_{\mathcal{A},\Gamma}}\sum_{m\in\iota_\ell}|\kappa_\ell^m|\max\left\{\sqrt{\lambda_\ell},n\right\}^n + \sum_{\ell>k\gamma'_{\mathcal{A},\Gamma}}\sum_{m\in\iota_\ell}\frac{1}{\ell+1}|\kappa_\ell^m|\max\left\{\sqrt{\lambda_\ell},n\right\}^n \tag{7.43}$$

$$+\sum_{\ell\le\lambda k}\ell|R_\ell^m|\max\left\{\sqrt{\lambda_\ell},n\right\}^n\Big\} =: \gamma^n\Big\{\cdots\Big\}.$$

We estimate the expression $\{\cdot\}$ in curly braces further with Lemma 5.10 :

$$\Big\{\cdots\Big\} \lesssim k^{3/2}\|\mathbf{r}\|_{\mathrm{curl},\Omega,1} + \sum_{\ell>k\gamma'_{\mathcal{A},\Gamma}}\sum_{m\in\iota_\ell}\frac{1}{\ell+1}|\kappa_\ell^m|\left[\lambda_\ell^{n/2}+n^n\right] \tag{7.44}$$

$$\lesssim k^{3/2}\|\mathbf{r}\|_{\mathrm{curl},\Omega,1} + k^{-1}\sum_{\ell>k\gamma'_{\mathcal{A},\Gamma}}\sum_{m\in\iota_\ell}|\kappa_\ell^m|\left[\lambda_\ell^{n/2}+n^n\right] \tag{7.45}$$

$$\lesssim k^{3/2}\|\mathbf{r}\|_{\mathrm{curl},\Omega,1} + k\tilde{\gamma}^n n^n\|\mathbf{r}\|_{\mathrm{curl},\Omega,1}, \tag{7.46}$$

for suitable $\tilde{\gamma}>0$; in the last step, we employed (5.42) once with $\alpha=n$ and once with $\alpha=0$. We also note

$$\|g_D\|_{H^{1/2}(\Gamma)} \lesssim \|g_D^*\|_{H^1(\mathcal{U})} \overset{(7.42)}{\lesssim} k^{5/2}\|\mathbf{r}\|_{\mathrm{curl},\Omega,1}. \tag{7.47}$$

3. step (interior regularity): Given $\mathbf{r}\in\mathbf{X}$, the function $\mathbf{z}=\mathcal{N}_4^{\mathcal{A}}\mathbf{r}$ solves (7.24). First, interior regularity as derived in [35, Prop. 5.5.1] gives

$$\|\nabla^n Z\|_{L^2(\Omega\setminus\mathcal{U}_\Gamma)} \le C\gamma^n(n+1)^n\|\nabla Z\| \le C\gamma^n(n+1)^n\|g_D\|_{H^{1/2}(\Gamma)} \qquad \forall n\in\mathbb{N}_0, \tag{7.48}$$

This is the desired bound away from Γ in view of (7.47).
4. step: For the behavior of Z near Γ, we write $Z=Z_0-g_D^*$. Near Γ, the function Z_0 satisfies

$$-\Delta Z_0 = -\Delta g_D^* \text{ in } \mathcal{U}_\Gamma \quad \text{and } Z_0|_\Gamma = 0. \tag{7.49}$$

From (7.42) we get $\Delta g_1^* \in \mathcal{A}(Ck^{7/2}, \gamma, \mathcal{U}_\Gamma)$ for suitably adjusted constants C, $\gamma > 0$. Also we have

$$\|\nabla Z_0\|_{L^2(\mathcal{U}_\Gamma)} \leq \|\nabla Z\| + \|\nabla g_D^*\|_{L^2(\mathcal{U}_\Gamma)} \lesssim \|g_D\|_{H^{1/2}(\Gamma)} + Ck^{5/2}\|\mathbf{r}\|_{\text{curl},\Omega,1} \overset{(7.47)}{\lesssim} k^{5/2}\|\mathbf{r}\|_{\text{curl},\Omega,1}. \tag{7.50}$$

One concludes with the aid of Theorem E.2 (and suitable localization as well as flattening of the boundary) that Z_0 in (7.49) satisfies

$$\nabla Z_0 \in \mathcal{A}\left(Ck^{5/2}\|\mathbf{r}\|_{\text{curl},\Omega,1}, \gamma, \mathcal{U}_\Gamma\right),$$

again with adjusted constants C, γ. This in turn implies $\nabla Z|_{\mathcal{U}_\Gamma} \in \mathcal{A}\left(Ck^{5/2}\|\mathbf{r}\|_{\text{curl},\Omega,1}, \gamma, \mathcal{U}_\Gamma\right)$. ■

Proposition 7.5 *Let $\Omega = B_1(0)$. There exist positive constants $C_{\mathcal{A},1}$ and $\gamma_{\mathcal{A},1}$ depending only on Γ and and the cut-off parameter λ such that for any $\mathbf{v} \in \mathbf{X}$*

$$\mathcal{N}_1^{\mathcal{A}}\mathbf{v} \in \mathcal{A}\left(C_{\mathcal{A},1}k^3\|\mathbf{v}\|_{\text{curl},\Omega,k}, \gamma_{\mathcal{A},1}, \Omega\right).$$

Proof. For given $\mathbf{v} \in \mathbf{X}$, the solution $\mathbf{z} := \mathcal{N}_1^{\mathcal{A}}\mathbf{v}$ can be split into

$$\mathbf{z} = \mathcal{N}_3^{\mathcal{A}}\mathbf{v} + \tilde{\mathbf{z}}$$

with the solution $\tilde{\mathbf{z}} \in \mathbf{X}$ of

$$A_k(\mathbf{w}, \tilde{\mathbf{z}}) = -\mathrm{i}\, k b_k\left(\mathbf{w}^{\text{curl}}, (L_\Omega\mathbf{v})^{\text{curl}}\right) \qquad \forall \mathbf{w} \in \mathbf{X}.$$

From (7.15) we get the following representation of the solution

$$\tilde{\mathbf{z}} = -\mathrm{i}\, k\int_\Gamma g_{-k}(\|\cdot - \mathbf{y}\|)\, T_{-k}\left((L_\Omega\mathbf{v})^{\text{curl}}\right)(\mathbf{y})\, d\Gamma_{\mathbf{y}}.$$

Fourier expansion of $(L_\Gamma\mathbf{v}_T)^{\text{curl}}$ leads to (cf. (5.7))

$$\boldsymbol{\mu} := T_{-k}(L_\Omega\mathbf{v})^{\text{curl}} = \sum_{1\leq\ell\leq\lambda k}\overline{\left(\frac{z_\ell(k)+1}{\mathrm{i}\, k}\right)}\sum_{m\in\iota_\ell} v_\ell^m \mathbf{T}_\ell^m.$$

An extension of $\boldsymbol{\mu}^*$ is given by

$$\boldsymbol{\mu}^* := \sum_{1\leq\ell\leq\lambda k}\overline{\left(\frac{z_\ell(k)+1}{\mathrm{i}\, k}\right)}\sum_{m\in\iota_\ell} v_\ell^m \tilde{\mathbf{T}}_\ell^m,$$

where $\tilde{\mathbf{T}}_\ell^m := \nabla\tilde{Y}_\ell^m \times \mathbf{n}^*$ with $\mathbf{n}^*(\mathbf{x}) := \mathbf{x}/\|\mathbf{x}\|$ and $\tilde{Y}_\ell^m$ as in (5.11). Now we proceed as in the proof of Proposition 7.3. First, we derive the estimates

$$\begin{aligned}\|\boldsymbol{\mu}\|^2_{\mathbf{H}_T^{-3/2}(\Gamma)} &= \sum_{1\leq\ell\leq\lambda k}(\ell(\ell+1))^{-1/2}\left|\frac{z_\ell(k)+1}{\mathrm{i}\, k}\right|^2\sum_{m\in\iota_\ell}|v_\ell^m|^2 \\ &\overset{\text{Lem. 5.3}}{\leq} \sum_{1\leq\ell\leq\lambda k}(\ell(\ell+1))^{-1/2}\left(1+\frac{\ell}{k}\right)^2\sum_{m\in\iota_\ell}|v_\ell^m|^2 \leq C\|\mathbf{v}_T\|^2_{\mathbf{H}_{\text{curl}}^{-1/2}(\Gamma)} \leq C\|\mathbf{v}\|^2_{\text{curl},\Omega,1}\end{aligned}$$

and

$$\begin{aligned}\|\nabla^n\boldsymbol{\mu}^*\|_{L^2(\mathcal{U}_\Gamma)} &\leq \sum_{1\leq\ell\leq\lambda k}\left|\left(\frac{z_\ell(k)+1}{\mathrm{i}\, k}\right)\right|\sum_{m\in\iota_\ell}|v_\ell^m|\left\|\nabla^n\tilde{\mathbf{T}}_\ell^m\right\|_{L^2(\mathcal{U}_\Gamma)} \\ &\overset{(5.12)}{\leq} C\max\{k, n+1\}^{n+1}\gamma^n\sum_{1\leq\ell\leq\lambda k}\left(1+\frac{\ell}{k}\right)\sum_{m\in\iota_\ell}|v_\ell^m| \leq \tilde{C}k^2\max\{k, n+1\}^n\tilde{\gamma}^n\|\mathbf{v}\|_{\text{curl},\Omega,1}.\end{aligned}$$

The application of $\mathcal{S}_{-k}^{\text{Hh}}$ to $\boldsymbol{\mu}$ can then be estimated by

$$\left\|k\mathcal{S}_{-k}^{\text{Hh}}\left(T_{-k}(L_\Omega\mathbf{v})^{\text{curl}}\right)\right\|_{\mathcal{H}} \overset{(7.39)}{\leq} Ck^3\|\boldsymbol{\mu}\|_{\mathbf{H}_T^{-3/2}(\Gamma)} \leq Ck^3\|\mathbf{v}\|_{\text{curl},\Omega,1}$$

and $k\mathcal{S}_{-k}^{\text{Hh}}\left(T_{-k}(L_\Omega\mathbf{v})^{\text{curl}}\right) \in \mathcal{A}\left(Ck^2\|\mathbf{v}\|_{\text{curl},\Omega,k}, \gamma, \Omega\right)$. The combination with Proposition 7.3 leads to the assertion. ■

$$
\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\nabla} & H(\Omega,\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\Omega,\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\uparrow \iota & & \uparrow \iota & & \uparrow \iota & & \uparrow \iota & & \uparrow \iota \\
\mathbb{R} & \longrightarrow & S_{p+1}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}^I_p(\mathcal{T}_h) & \xrightarrow{\mathrm{curl}} & \mathbf{RT}_p(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & Z_p(\mathcal{T}_h)
\end{array}
$$

Figure 1: Continuous and discrete exact sequences.

8 Approximation Operators for $S_{p+1}(\mathcal{T}_h)$ and $\boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\mathcal{T}_h)$

The relevant *hp* finite element spaces have been introduced in Section 3.2. A key property of these spaces is that both lines in the diagram in Fig. 1 are exact sequences, [26,43,46]. In particular, therefore, (3.2) is satisfied for the pair $(S_h, \mathbf{X}_h) = (S_{p+1}(\mathcal{T}_h), \boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\mathcal{T}_h))$. The operators Π^E_h and Π^F_h of Assumption 4.14 are constructed to satisfy the stronger "commuting diagram property" that make the diagram in Fig. 2 commute. In that case, the operator Π^E_h is defined on the space $\prod_{K\in\mathcal{T}_h} \mathbf{H}^1(K,\mathrm{curl}) \cap \mathbf{X} \supset \{\mathbf{u} \in \mathbf{H}^1(\Omega) \mid \mathrm{curl}\,\mathbf{u} \in \mathrm{curl}\,\mathbf{X}_h\}$.

8.1 Optimal Simultaneous *hp*-Approximation in L^2 and $\mathbf{H}(\mathrm{curl})$

We restrict our attention to approximation operators that are constructed element-by-element.

Definition 8.1 (element-by-element construction) *An operator* $\widehat{\Pi}^{\mathrm{grad}} : H^2(\widehat{K}) \to \mathcal{P}_{p+1}$ *is said to admit* element-by-element construction *if the operator* $\Pi^{\mathrm{grad}} : H^1(\Omega) \cap \prod_{K\in\mathcal{T}_h} H^2(K)$ *defined elementwise by* $(\Pi^{\mathrm{grad}} u)|_K := (\widehat{\Pi}^{\mathrm{grad}}(u \circ F_K)) \circ F_K^{-1}$ *maps into the conforming subspace* $S^{p+1}(\mathcal{T}_h) \subset H^1(\Omega)$.
An operator $\widehat{\Pi}^{\mathrm{curl}} : \mathbf{H}^1(\widehat{K},\mathrm{curl}) \to \boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\widehat{K})$ *is said to admit* element-by-element construction *if the operator* $\Pi^{\mathrm{curl}} : \mathbf{H}(\Omega,\mathrm{curl}) \cap \prod_{K\in\mathcal{T}_h} \mathbf{H}^1(K,\mathrm{curl})$ *defined elementwise by* $(\Pi^{\mathrm{curl}} u)|_K := (F_K')^{-T}(\widehat{\Pi}^{\mathrm{curl}}((F_K')^T \mathbf{u} \circ F_K)) \circ F_K^{-1}$ *maps into the conforming subspace* $\boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\mathcal{T}_h) \subset \mathbf{H}(\Omega,\mathrm{curl})$.
An operator $\widehat{\Pi}^{\mathrm{div}} : \mathbf{H}^1(\widehat{K},\mathrm{div}) \to \mathbf{RT}_p(\widehat{K})$ *is said to admit* element-by-element construction *if the operator* $\Pi^{\mathrm{div}} : \mathbf{H}(\Omega,\mathrm{div}) \cap \prod_{K\in\mathcal{T}_h} \mathbf{H}^1(K,\mathrm{div})$ *defined elementwise by*

$$(\Pi^{\mathrm{div}} u)|_K := (\det(F_K'))^{-1} F_K' (\widehat{\Pi}^{\mathrm{div}}(\det F_K')(F_K')^{-1} \mathbf{u} \circ F_K)) \circ F_K^{-1}$$

maps into the conforming subspace $\mathbf{RT}_p(\mathcal{T}_h) \subset \mathbf{H}(\Omega,\mathrm{div})$. *Finally, any operator* $\widehat{\Pi}^{L^2} : L^2(\widehat{K}) \to \mathcal{P}_p(\widehat{K})$ *leads to a globally defined* $L^2(\Omega)$*-conforming operator by the following* element-by-element *construction:* $(\Pi^{L^2} u)|_K := (\widehat{\Pi}^{L^2}(\mathbf{u} \circ F_K)) \circ F_K^{-1}$

As it is typical, we will construct such operators on the reference tetrahedron $\widehat{K}$ in such a way that the value of the operator restricted to a lower-dimensional entity (i.e., a vertex, an edge, or a face) is completely determined by the value of the function on that entity. For scalar functions the operator Π_p of [41, Def. 5.3, Thm. B.4] is an example that we will build on; it can be viewed as a variant of the projection-based interpolation technique of [18] that also underlies the construction of the operator Π^E_h. Important features of the construction of Π_p are: $(\Pi_p u)(V) = u(V)$ for all vertices V; it has the property that $(\Pi_p u)|_e$ is the projection of $u|_e$ onto a space of polynomials of degree p on each edge e under the constraint that $\Pi_p u$ has already been fixed in the vertices; it has the property that $(\Pi_p u)|_f$ is the (constrained) projection of $u|_f$ onto a space of polynomials of degree p on each face f under the constraint that $\Pi_p u$ has already been fixed on edges. We note that the fact that Π_p is a (constrained) projection on polynomial spaces for the edges and faces makes the definition independent of the parametrization of the edges and faces of the reference tetrahedron.
We need approximation operators suitable for the approximation in the norm $\|\cdot\|_{\mathrm{curl},\Omega,k}$. Such an operator can be defined in an element-by-element fashion on the reference tetrahedron:

Lemma 8.2 *Let* $s > 3/2$. *There exist operators* $\widehat{\Pi}^{\mathrm{curl},s}_p : \mathbf{H}^s(\widehat{K}) \to \boldsymbol{\mathcal{N}}^{\mathrm{I}}_p(\widehat{K})$ *with the following properties:*

(i) $\widehat{\Pi}^{\mathrm{curl},s}_p$ *admits an element-by-element construction as in Definition 8.1.*

(ii) For $p \geq s-1$ *we have*

$$(p+1)\|\mathbf{u} - \widehat{\Pi}^{\mathrm{curl},s}_p \mathbf{u}\|_{\mathbf{L}^2(\widehat{K})} + \|\mathbf{u} - \widehat{\Pi}^{\mathrm{curl},s}_p \mathbf{u}\|_{\mathbf{H}^1(\widehat{K})} \leq C p^{-(s-1)} |\mathbf{u}|_{\mathbf{H}^s(\widehat{K})}. \tag{8.1}$$

(iii) Let $\mathbf{u}$ *satisfy, for some* $C_{\mathbf{u}}$, $\overline{\gamma}$, $h > 0$, *and* $\kappa \geq 1$

$$\|\nabla^n \mathbf{u}\|_{\mathbf{L}^2(\widehat{K})} \leq C_{\mathbf{u}} (\overline{\gamma} h)^n \max\{n, \kappa\}^n \qquad \forall n \in \mathbb{N}, \quad n \geq 2. \tag{8.2}$$

Assume furthermore

$$h + \kappa h/p \leq \widetilde{C}. \tag{8.3}$$

Then there exist constants C, $\sigma > 0$ *depending solely on* $\widetilde{C}$ *and* $\overline{\gamma}$ *such that*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},s} \mathbf{u}\|_{W^{2,\infty}(\widehat{K})} \leq C C_{\mathbf{u}} \left[\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{\kappa h}{\sigma p} \right)^{p+1} \right]. \tag{8.4}$$

Proof. Let $\Pi_p : H^s(\widehat{K}) \to \mathcal{P}_p$, $s > 3/2$, be the scalar polynomial approximation operator[8] of [41, Def. 5.3, Thm. B.4]. A key property of Π_p is that, as described above, one has that the restriction of $\Pi_p u$ to a vertex, edge, or face is completely determined by u restricted to that entity. We write, e.g., for a face f: $\Pi_p \left(u|_f \right) := (\Pi_p u)|_f$. We define the operator $\widehat{\Pi}_p^{\mathrm{curl},s} : \mathbf{H}^s(\widehat{K}) \to (\mathcal{P}_p)^3 \subset \boldsymbol{\mathcal{N}}_p^{\mathrm{I}}(\widehat{K})$ by componentwise application to $\mathbf{u} = (u_i)_{i=1}^3$, i.e.,

$$\Pi^{\mathrm{curl},s} \mathbf{u} := (\Pi_p u_i)_{i=1}^3 .$$

1. step: We show that $\widehat{\Pi}_p^{\mathrm{curl},s}$ admits an element-by-element construction. We show this by asserting that the tangential component $\Pi_T \widehat{\Pi}_p^{\mathrm{curl},s} \mathbf{u}$ depends solely on the tangential component $\Pi_T \mathbf{u}$. Fix a face f of $\widehat{K}$ with normal $\mathbf{n}_f$. Note that $\mathbf{n}_f$ is constant on f. The tangential component of $\Pi^{\mathrm{curl},s}\mathbf{u}$ on f is

$$\left(\Pi_T \left(\Pi^{\mathrm{curl},s} \mathbf{u}\right)\right)\big|_f = \left((\Pi_p u_i)_{i=1}^3 \right)\Big|_f - \left(\sum_{j=1}^3 n_j \Pi_p u_j \right) \mathbf{n} \Bigg|_f .$$

Using that $(\Pi_p \mathbf{u})|_f$ is completely determined by the values of $\mathbf{u}$ on f and using that the normal vector $\mathbf{n}$ is constant on f, we infer with the understanding that Π_p acts componentwise on a vector-valued object

$$\left(\Pi_T \left(\Pi^{\mathrm{curl},s} \mathbf{u}\right)\right)\big|_f = \Pi_p \left(\mathbf{u}|_f\right) - \Pi_p \left((\mathbf{n} \cdot \mathbf{u}|_f) \, \mathbf{n} \right) = \Pi_p \left(\mathbf{u}|_f - ((\mathbf{n} \cdot \mathbf{u}|_f) \, \mathbf{n}) \right) = \Pi_p (\Pi_T \mathbf{u})|_f ,$$

which is the desired claim.
2. step: Estimate (8.1) then follows from [41, Thm. B.4].
3. step: From [41, Lemma C.2], we conclude that (8.4) holds. ■

8.2 Projection Operators with Commuting Diagram Property

The operator $\Pi_p^{\mathrm{curl},s}$, which is obtained by an elementwise use of $\widehat{\Pi}_p^{\mathrm{curl},s}$ of Lemma 8.2 (cf. Definition 8.1 for the transformation rule) has (p-optimal) approximation properties in $\|\cdot\|_{\mathrm{curl},\Omega,k}$ as it has simultaneously p-optimal approximation properties in L^2 and H^1. However, it is not a projection and does not have the commuting diagram property. We therefore present a second operator, $\widehat{\Pi}^{\mathrm{curl},c}$, in Theorem 8.3 with this property. The construction is given in [39] and similar to that in [18, 19]. We point out that the difference between Theorem 8.3 from [39] and the works [18,19] is that, by assuming $H^2(\widehat{K})$- and $\mathbf{H}^1(\widehat{K}, \mathrm{curl})$-regularity, Theorem 8.3 features the optimal p-dependence, thus avoiding the factors of $\log p$ present in [18, 19].

Theorem 8.3 ([39]) *There are linear* projection *operators* $\widehat{\Pi}_{p+1}^{\mathrm{grad},c}$, $\widehat{\Pi}_p^{\mathrm{curl},c}$, $\widehat{\Pi}_p^{\mathrm{div},c}$, $\widehat{\Pi}_p^{L^2}$ *such that the following holds:*

(i) The diagram in Fig. 2 commutes.

(ii) The operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},c}$, $\widehat{\Pi}_p^{\mathrm{curl},c}$, $\widehat{\Pi}_p^{\mathrm{div},c}$, $\widehat{\Pi}_p^{L^2}$ *admit element-by-element constructions as in Definition 8.1. The global operators* $\Pi_{p+1}^{\mathrm{grad},c}$, $\Pi_p^{\mathrm{curl},c}$, $\Pi_p^{\mathrm{div},c}$, $\Pi_p^{L^2}$ *obtained from the operators* $\widehat{\Pi}_{p+1}^{\mathrm{grad},c}$, $\widehat{\Pi}_p^{\mathrm{curl},c}$, $\widehat{\Pi}_p^{\mathrm{div},c}$, $\widehat{\Pi}_p^{L^2}$ *by an element-by-element construction are also linear projection operators and the diagram in Fig. 3 commutes.*

[8] In [41, Def. 5.3, Thm. B.4] the element-by-element construction of the polynomial approximation on the reference element only fixes Π_p on $\partial\widehat{K}$. The operator Π_p is fully determined by adding a final minimization step to fix the interior degrees of freedom on the reference element.

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^2(\widehat{K}) & \xrightarrow{\nabla} & \mathbf{H}^1(\widehat{K},\operatorname{curl}) & \xrightarrow{\operatorname{curl}} & \mathbf{H}^1(\widehat{K},\operatorname{div}) & \xrightarrow{\operatorname{div}} & H^1(\widehat{K}) \\
\big\downarrow & & \big\downarrow \widehat{\Pi}^{\mathrm{grad},c}_{p+1} & & \big\downarrow \widehat{\Pi}^{\mathrm{curl},c}_{p} & & \big\downarrow \widehat{\Pi}^{\mathrm{div},c}_{p} & & \big\downarrow \widehat{\Pi}^{L^2}_{p} \\
\mathbb{R} & \longrightarrow & \mathcal{P}_{p+1}(\widehat{K}) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}^I_p(\widehat{K}) & \xrightarrow{\operatorname{curl}} & \mathbf{RT}_p(\widehat{K}) & \xrightarrow{\operatorname{div}} & \mathcal{P}_p(\widehat{K})
\end{array}$$

Figure 2: Commuting diagram on reference element $\widehat{K}$.

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^2(\Omega) & \xrightarrow{\nabla} & \mathbf{H}^1(\Omega,\operatorname{curl}) & \xrightarrow{\operatorname{curl}} & \mathbf{H}^1(\Omega,\operatorname{div}) & \xrightarrow{\operatorname{div}} & H^1(\Omega) \\
\big\downarrow & & \big\downarrow \Pi^{\mathrm{grad},c}_{p+1} & & \big\downarrow \Pi^{\mathrm{curl},c}_{p} & & \big\downarrow \Pi^{\mathrm{div},c}_{p} & & \big\downarrow \Pi^{L^2}_{p} \\
\mathbb{R} & \longrightarrow & S_{p+1}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}^I_p(\mathcal{T}_h) & \xrightarrow{\operatorname{curl}} & \mathbf{RT}_p(\mathcal{T}_h) & \xrightarrow{\operatorname{div}} & Z_p(\mathcal{T}_h)
\end{array}$$

Figure 3: Commuting diagram on mesh $\mathcal{T}_h$

(iii) For all $\varphi \in H^2(\widehat{K})$ *there holds*

$$\|\varphi - \widehat{\Pi}^{\mathrm{grad},c}_{p+1}\varphi\|_{H^s(\widehat{K})} \le C_s p^{-1-(1-s)} \inf_{v \in \mathcal{P}_{p+1}(\widehat{K})} \|\varphi - v\|_{H^2(\widehat{K})}, \qquad s \in [0,1].$$

(iv) For all $\mathbf{u} \in \mathbf{H}^1(\widehat{K},\operatorname{curl})$ *there holds*

$$\|\mathbf{u} - \widehat{\Pi}^{\mathrm{curl},c}_{p}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{curl})} \le C p^{-1} \inf_{\mathbf{v} \in \boldsymbol{\mathcal{N}}^I_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathrm{curl})}.$$

(v) For all $k \ge 1$ *and all* $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ *with* $\operatorname{curl}\mathbf{u} \in \boldsymbol{\mathcal{P}}_p$ *there holds*

$$\|\mathbf{u} - \widehat{\Pi}^{\mathrm{curl},c}_{p}\mathbf{u}\|_{\mathbf{L}^2(\widehat{K})} \le C_k p^{-k} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}. \tag{8.5}$$

If $p \ge k-1$*, then the full norm* $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ *can be replaced with the seminorm* $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

8.3 *hp*-FEM Approximation

Our *hp*-FEM convergence result will be formulated for the specific class of meshes which have been introduced in Section 3.2. For such meshes, we can formulate approximation results for both, the operators $\Pi^{\mathrm{curl},s}_p$ and $\Pi^{\mathrm{curl},c}_p$. In both cases, we will need to relate functions defined on K to their pull-back to the reference tetrahedron $\widehat{K}$. The appropriate transformations are described in Definition 3.1: For scalar functions φ defined on K and vector-valued functions $\mathbf{u}$ defined on K, we let

$$\widehat{\varphi} = \varphi \circ F_K, \qquad \widehat{\mathbf{u}} = (F_K')^\top (\mathbf{u} \circ F_K). \tag{8.6}$$

Lemma 8.4 *Let the regular mesh* $\mathcal{T}_h$ *satisfy Assumption 3.1.*

(i) With implied constants depending only on C_{affine}*,* C_{metric}*,* γ *there holds for all* $K \in \mathcal{T}_h$

$$|\widehat{\varphi}|_{H^j(\widehat{K})} \sim h_K^{j-3/2}|\varphi|_{H^j(K)}, \quad j \in \{0,1\}, \qquad |\widehat{\varphi}|_{H^2(\widehat{K})} \lesssim h_K^{2-3/2}\|\varphi\|_{H^2(K)}, \tag{8.7}$$

$$\|\widehat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{K})} \sim h_K^{1-3/2}\|\mathbf{u}\|_{\mathbf{L}^2(K)}, \quad \|\operatorname{curl}\widehat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{K})} \sim h_K^{2-3/2}\|\operatorname{curl}\mathbf{u}\|_{\mathbf{L}^2(K)}, \quad |\widehat{\mathbf{u}}|_{\mathbf{H}^2(\widehat{K})} \lesssim h_K^{3-3/2}\|\mathbf{u}\|_{\mathbf{H}^2(K)}. \tag{8.8}$$

(ii) Let $\overline{\gamma} > 0$*. Then there exist* γ'*,* $C > 0$ *depending only on* $\overline{\gamma}$ *and the constants of Assumption 3.1 such that*

$$\|\nabla^n\varphi\|_{\mathbf{L}^2(K)} \le C_\varphi \overline{\gamma}^n \max\{n,k\}^n \quad \forall n \in \mathbb{N}_0 \quad \Longrightarrow \quad \|\nabla^n\widehat{\varphi}\|_{\mathbf{L}^2(\widehat{K})} \le C C_\varphi h_K^{-3/2}(h_k\gamma')^n \max\{n,k\}^n \quad \forall n \in \mathbb{N}_0, \tag{8.9}$$

$$\|\nabla^n\mathbf{u}\|_{\mathbf{L}^2(K)} \le C_{\mathbf{u}} \overline{\gamma}^n \max\{n,k\}^n \quad \forall n \in \mathbb{N}_0 \quad \Longrightarrow \quad \|\nabla^n\widehat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{K})} \le C h_K^{1-3/2} C_{\mathbf{u}} (h_K\gamma')^n \max\{n,k\}^n \quad \forall n \in \mathbb{N}_0. \tag{8.10}$$

Proof. We will not show (8.7). For (8.8), the first and third estimate in (8.8) follow by inspection, the second equivalence follows from (cf., e.g., [43, Cor. 3.58])

$$F_K' \operatorname{curl} \widehat{\mathbf{u}} = (\det F_K')(\operatorname{curl} \mathbf{u}) \circ F_K.$$

The implications (8.9), (8.10) are obtained by similar arguments. We will therefore focus on (8.10). Recalling that the element map F_K has the form $F_K = R_K \circ A_K$, we introduce the function $\widetilde{\mathbf{u}} := (R_K')^\top \mathbf{u} \circ R_K$, which is defined on $\widetilde{K} := A_K(\widehat{K})$. Using [35, Lemma 4.3.1] (and noting as in the proof [41, Lemma C.1] that the original 2d arguments extends to 3d), we get the existence of C, $\widetilde{\gamma}$, which depends solely on the constants of Assumption 3.1 and on $\overline{\gamma}$, such that

$$\|\nabla^n \widetilde{\mathbf{u}}\|_{\mathbf{L}^2(\widetilde{K})} \leq C C_{\mathbf{u}} (\widetilde{\gamma})^n \max\{n,k\}^n \quad \forall n \in \mathbb{N}_0.$$

Next, we observe $\widehat{\mathbf{u}} = (A_K')^\top \widetilde{\mathbf{u}} \circ A_K$. Using that A_K is affine, it is easy to deduce

$$\|\nabla^n \widehat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{K})} \leq C C_{\mathbf{u}} \max\{n,k\}^n \left(h_K \gamma'\right)^n \quad \forall n \in \mathbb{N}_0,$$

which is the desired estimate. ■

Lemma 8.5 *Let $\mathcal{T}_h$ be a regular mesh satisfying Assumption 3.1 and assume $p \geq 1$.*

(i)

$$\|\mathbf{u} - \Pi_p^{\mathrm{curl},s} \mathbf{u}\|_{\mathbf{L}^2(K)} + h_K p^{-1} \|\operatorname{curl}(\mathbf{u} - \Pi_p^{\mathrm{curl},s} \mathbf{u})\|_{\mathbf{L}^2(K)} \lesssim h_K^2 p^{-2} \|\mathbf{u}\|_{\mathbf{H}^2(K)}.$$

(ii) Let $\widetilde{C} > 0$ be given. If $\mathbf{u}$ satisfies (8.10), then there exist C, $\sigma > 0$ depending only on $\widetilde{C}$ and $\overline{\gamma}$ and the constants of Assumption 3.1 such that under the side constraint

$$h_K + \frac{k h_K}{p} \leq \widetilde{C} \tag{8.11}$$

the following approximation result holds:

$$\|\mathbf{u} - \Pi_p^{\mathrm{curl},s} \mathbf{u}\|_{\mathbf{L}^2(K)} + h_K p^{-1} \|\operatorname{curl}(\mathbf{u} - \Pi_p^{\mathrm{curl},s} \mathbf{u})\|_{\mathbf{L}^2(K)} \lesssim \left(\left(\frac{h_K}{h_K + \sigma} \right)^{p+1} + \left(\frac{k h_K}{\sigma p} \right)^{p+1} \right). \tag{8.12}$$

Proof. *Proof of (i):* From Lemma 8.2 with $s = 2$ we have on the reference tetrahedron

$$p \|\widehat{\mathbf{u}} - \widehat{\Pi}_p^{\mathrm{curl},s} \widehat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{K})} + \|\widehat{\mathbf{u}} - \widehat{\Pi}_p^{\mathrm{curl},s} \widehat{\mathbf{u}}\|_{\mathbf{H}^1(\widehat{K})} \lesssim p^{-1} |\widehat{\mathbf{u}}|_{\mathbf{H}^2(\widehat{K})}.$$

Hence, using (8.8) we infer

$$p h_K^{1-3/2} \|\mathbf{u} - \Pi_p^{\mathrm{curl},s} \mathbf{u}\|_{\mathbf{L}^2(K)} + h_K^{2-3/2} \|\operatorname{curl}(\mathbf{u} - \Pi_p^{\mathrm{curl},s} \mathbf{u})\|_{\mathbf{L}^2(K)} \lesssim h_K^{3-3/2} p^{-1} \|\mathbf{u}\|_{\mathbf{H}^2(K)}.$$

Proof of (ii): We proceed as above. The transformation rules of Lemma 8.4 and Lemma 8.2 give

$$\left\| \widehat{\mathbf{u}} - \widehat{\Pi}_p^{\mathrm{curl},s} \widehat{\mathbf{u}} \right\|_{W^{2,\infty}(\widehat{K})} \leq C C_{\mathbf{u}} h_K^{1-3/2} \left(\left(\frac{h_K}{h_K + \sigma} \right)^{p+1} + \left(\frac{k h_K}{\sigma p} \right)^{p+1} \right). \tag{8.13}$$

Since the norm $\|\cdot\|_{W^{2,\infty}(\widehat{K})}$ is stronger than $\|\cdot\|_{\mathbf{L}^2(\widehat{K})}$ and $\|\operatorname{curl} \cdot\|_{\mathbf{H}^1(\widehat{K})}$ the result follows by transforming back to K using Lemma 8.4. ■

For the operator $\Pi_p^{\mathrm{curl},c}$ we have the following approximation results:

Lemma 8.6 *Let $\mathcal{T}_h$ be a regular mesh satisfying Assumption 3.1. Then for $p \geq 1$:*

(i)

$$h_K^{-1} \|\mathbf{u} - \Pi_p^{\mathrm{curl},c} \mathbf{u}\|_{\mathbf{L}^2(K)} + \|\operatorname{curl}(\mathbf{u} - \Pi_p^{\mathrm{curl},c} \mathbf{u})\|_{\mathbf{L}^2(K)} \leq C h_K (p+1)^{-1} \|\mathbf{u}\|_{\mathbf{H}^2(K)}. \tag{8.14}$$

(ii) Assume the hypotheses of Lemma 8.5, (ii). Then

$$h_K^{-1} \|\mathbf{u} - \Pi_p^{\mathrm{curl},c} \mathbf{u}\|_{\mathbf{L}^2(K)} + \|\operatorname{curl}(\mathbf{u} - \Pi_p^{\mathrm{curl},c} \mathbf{u})\|_{\mathbf{L}^2(K)} \lesssim \left(\left(\frac{h_K}{h_K + \sigma} \right)^p + \frac{k}{p} \left(\frac{k h_K}{\sigma p} \right)^p \right).$$

(iii) For $\mathbf{u} \in \mathbf{H}^1(K, \operatorname{curl})$ *with* $\operatorname{curl} \widehat{\mathbf{u}} \in (\mathcal{P}_p(\widehat{K}))^3$ *there holds*

$$\|\mathbf{u} - \Pi_p^{\operatorname{curl},c}\mathbf{u}\|_{\mathbf{L}^2(K)} \leq C h_K p^{-1} \|\mathbf{u}\|_{\mathbf{H}^1(K)}.$$

Proof. *Proof of (i):* Using Lemma 8.4, we get from Theorem 8.3 and the assumption $p \geq 1$

$$\begin{aligned} h_K^{-1}\|\mathbf{u} - \Pi_p^{\operatorname{curl},c}\mathbf{u}\|_{\mathbf{L}^2(K)} + \|\operatorname{curl}(\mathbf{u} - \Pi_p^{\operatorname{curl},c}\mathbf{u})\|_{\mathbf{L}^2(K)} &\sim h_K^{-2+3/2}\|\widehat{\mathbf{u}} - \widehat{\Pi}_p^{\operatorname{curl},c}\widehat{\mathbf{u}}\|_{\mathbf{H}(\widehat{K},\operatorname{curl})} \\ \lesssim p^{-1}h_K^{-2+3/2} \inf_{\mathbf{v}\in\mathcal{P}_1^3} \|\widehat{\mathbf{u}} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\operatorname{curl})} &\lesssim p^{-1}h_K^{-2+3/2}|\widehat{\mathbf{u}}|_{\mathbf{H}^2(\widehat{K})} \lesssim p^{-1}h_K\|\mathbf{u}\|_{\mathbf{H}^2(K)} \end{aligned}$$

Proof of (ii): We start as above. The novel aspect is that $\inf_{\mathbf{v}\in\boldsymbol{\mathcal{N}}_p^{\text{I}}(\widehat{K})} \|\widehat{\mathbf{u}} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\operatorname{curl})}$ can be estimated as in the proof of Lemma 8.5

$$\begin{aligned} h_K^{-1}\left\|\mathbf{u} - \Pi_p^{\operatorname{curl},c}\mathbf{u}\right\|_{\mathbf{L}^2(K)} + \left\|\operatorname{curl}\left(\mathbf{u} - \Pi_p^{\operatorname{curl},c}\mathbf{u}\right)\right\|_{\mathbf{L}^2(K)} &\lesssim p^{-1}h_K^{-2+3/2} \inf_{\mathbf{v}\in\mathcal{P}_1^3} \|\widehat{\mathbf{u}} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\operatorname{curl})} \\ &\overset{(8.13)}{\lesssim} CC_{\mathbf{u}}\left(\left(\frac{h_K}{h_K+\sigma}\right)^p + \frac{k}{p}\left(\frac{kh_K}{\sigma p}\right)^p\right) \end{aligned}$$

Proof of (iii): With Lemma 8.4 and Theorem 8.3, (v) we estimate

$$\|\mathbf{u} - \Pi_p^{\operatorname{curl},c}\mathbf{u}\|_{\mathbf{L}^2(K)} \sim h^{-1+3/2}\|\widehat{\mathbf{u}} - \widehat{\Pi}_p^{\operatorname{curl},c}\widehat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{K})} \lesssim h^{-1+3/2}p^{-1}|\widehat{\mathbf{u}}|_{\mathbf{H}^1(\widehat{K})} \lesssim h^{-1+3/2}p^{-1}h_K^{2-3/2}\|\mathbf{u}\|_{\mathbf{H}^1(K)},$$

which completes the proof. ■

A Proof of Lemma 5.3

In this appendix we prove Lemma 5.3. The first two estimates in (5.9) are proved in the following lemma.

Lemma A.1 *For any* $\lambda > 1$ *there holds*

$$\frac{k}{|z_n(k) + 1|} \leq \begin{cases} 2\sqrt{2}k & n \in \mathbb{N}_0, \\ 2\sqrt{2}\left(\frac{2}{\lambda} + 1\right)\dfrac{k}{(n+1)} & n > \lambda k^2. \end{cases}$$

Proof. We follow the reasoning in [47, Thm. 2.6.1]. The coefficient $z_n(k)$ can be expressed by

$$z_n(k) = -\frac{\left(m_n^2\right)'}{m_n^2} + k\frac{\mathrm{i}}{m_n^2},$$

where

$$\mu = (2n+1)^2 \quad \text{and} \quad \begin{cases} m_n^2 = \displaystyle\sum_{m=0}^n \frac{\delta_m(\mu)}{k^{2m}}, & \left(m_n^2\right)' = \displaystyle\sum_{m=0}^n (m+1)\frac{\delta_m(\mu)}{k^{2m}}, \\ \delta_m(\mu) = \dfrac{(2m)!}{(m!)^2\,16^m}\gamma_m\left((2n+1)^2\right), & \gamma_m(\mu) := \displaystyle\prod_{s=1}^m \left(\mu - (2s-1)^2\right). \end{cases}$$

Define

$$a_{m,n} := \delta_m\left((2n+1)^2\right) = \frac{(2m)!\,(n+m)!}{(m!)^2\,4^m\,(n-m)!}.$$

With the function

$$\rho_n(k) := \frac{\displaystyle\sum_{m=0}^n \frac{a_{m,n}}{k^{2m}}}{\displaystyle\sum_{m=0}^n (m+1)\frac{a_{m,n}}{k^{2m}}} \tag{A.1}$$

we estimate

$$\frac{k}{|z_n(k)+1|} = \frac{k m_n^2}{\left|m_n^2 - (m_n^2)' + k\,\mathrm{i}\right|} \overset{\mu=(2n+1)^2}{\leq} \sqrt{2}k \frac{\sum_{m=0}^{n} \frac{a_{m,n}}{k^{2m}}}{k + \sum_{m=1}^{n} m \frac{a_{m,n}}{k^{2m}}} \tag{A.2}$$

$$\leq 2\sqrt{2}k\rho_n(k) \overset{\text{ansatz}}{\leq} 2\sqrt{2}k\left(\frac{k^2+\beta}{k^2+C_n\beta}\right). \tag{A.3}$$

The ansatz (A.3) is equivalent to

$$\left(k^2 + C_n\beta\right)\sum_{m=0}^{n}\frac{a_{m,n}}{k^{2m}} \leq \left(k^2+\beta\right)\sum_{m=0}^{n}(m+1)\frac{a_{m,n}}{k^{2m}},$$

which, by multiplying out and rearranging terms, is equivalent to

$$k^2 a_{0,n} + \sum_{m=0}^{n-1}\left(a_{m+1,n} + C_n\beta a_{m,n}\right)\frac{1}{k^{2m}} + C_n\beta\frac{a_{n,n}}{k^{2n}}$$
$$\leq k^2 a_{0,n} + \sum_{m=0}^{n-1}\left((m+2)\,a_{m+1,n} + \beta(m+1)\,a_{m,n}\right)\frac{1}{k^{2m}} + \beta(n+1)\frac{a_{n,n}}{k^{2n}}.$$

Hence, we have to stipulate

$$\left(a_{m+1,n} + C_n\beta a_{m,n}\right) \leq \left((m+2)\,a_{m+1,n} + \beta(m+1)\,a_{m,n}\right), \qquad m = 0,\ldots,n,$$
$$C_n \leq n+1.$$

We select $C_n := (n+1)$ and insert this in the left-hand side of the first condition to obtain

$$0 \leq (m+1)\,a_{m+1,n} + \beta\left(m+1-(n+1)\right)a_{m,n}.$$

Inserting the definitions of $a_{m,n}$ leads to

$$0 \leq (m+1)\frac{(2m+2)!\,(n+m+1)!}{((m+1)!)^2\,4^{m+1}\,(n-(m+1))!} + \beta\left(m+1-(n+1)\right)\frac{(2m)!\,(n+m)!}{(m!)^2\,4^m\,(n-m)!}.$$

This implies

$$\beta(n-m)\frac{(2m)!\,(n+m)!}{(m!)^2\,4^m\,(n-m)!} \leq (m+1)\frac{(2m+2)!\,(n+m+1)!}{((m+1)!)^2\,4^{m+1}\,(n-(m+1))!}$$

and in turn

$$\beta \leq (m+1)\frac{(2m+1)(2m+2)(n+m+1)}{(m+1)^2\,4} \leq \left(m+\frac{1}{2}\right)(n+m+1), \qquad m = 0,\ldots,m.$$

We select $\beta = \frac{n+1}{2}$, which finally leads to

$$\frac{k}{|z_n(k)+1|} \leq 2\sqrt{2}k\left(\frac{2k^2+n+1}{2k^2+(n+1)^2}\right) \leq \begin{cases} 2\sqrt{2}k & \forall n \in \mathbb{N}_0 \\ 2\sqrt{2}\left(\frac{2}{\lambda}+1\right)\frac{k}{(n+1)}, & n+1 > \lambda k^2. \end{cases}$$

■

The proof of the third estimate in (5.9) is more technical and the assertion of the next lemma.

Lemma A.2 *For every $\lambda_0 > 1$ there is $C_0 > 0$ depending only on λ_0 such that*

$$\frac{n+1}{|z_n(k)+1|} \leq C_0 \qquad \forall n \geq \lambda_0 k.$$

Proof. Recall the definition of the function ρ_n as in (A.1). We will prove

$$(n+1)\,\rho_n(k) \leq \tilde{C}_0 \qquad \forall n \geq \lambda_0 k$$

from which the statement (cf. (A.2)) follows.
Step 1: We claim that ρ_n is monotone increasing with respect to k. To see this, we compute

$$\rho_n'(k) = \frac{-\left(\sum_{m=0}^{n}(m+1)\frac{a_{m,n}}{k^{2m}}\right)\left(\sum_{m=0}^{n}2m\frac{a_{m,n}}{k^{2m+1}}\right)+\sum_{m=0}^{n}\left(\frac{a_{m,n}}{k^{2m}}\right)\sum_{m=0}^{n}(2m)(m+1)\frac{a_{m,n}}{k^{2m+1}}}{\left(\sum_{m=0}^{n}(m+1)\frac{a_{m,n}}{k^{2m}}\right)^2}.$$

Thus, it is sufficient to prove that the numerator (denoted by $d_n(k)$) is positive. We write

$$d_n(k) = 2\sum_{m=0}^{n}\sum_{\ell=0}^{n}\ell(\ell-m)\frac{a_{\ell,n}a_{m,n}}{k^{2m+2\ell+1}}.$$

We now exploit the fact that the coefficients $a_{\ell,n}$ are non-negative. The double sum on the right-hand side can be interpreted as a quadratic form. Note that we have, for vectors $\mathbf{x}$ and matrices $\mathbf{B}$,

$$2\mathbf{x}^\intercal\mathbf{B}\mathbf{x} = \mathbf{x}^\intercal(\mathbf{B}^\intercal+\mathbf{B})\mathbf{x} \geq 0$$

if the vector $\mathbf{x}$ has non-negative entries and the symmetric part $1/2(\mathbf{B}^\intercal+\mathbf{B})$ of the matrix $\mathbf{B}$ has non-negative entries. For $\mathbf{B}_{\ell,m} := \ell(\ell-m)$ we compute

$$\mathbf{B}_{\ell,m}+\mathbf{B}_{m,\ell} = (\ell-m)^2 \geq 0.$$

Step 2: The monotonicity of ρ_n shown in Step 1 implies for $n \geq \lambda_0 k$

$$\rho_n(k) \leq \rho_n(n/\lambda_0) = \frac{\sum_{m=0}^{n}a_{m,n}\left(\frac{\lambda_0}{n}\right)^{2m}}{\sum_{m=0}^{n}(m+1)\,a_{m,n}\left(\frac{\lambda_0}{n}\right)^{2m}} =: \rho_n^{\mathrm{I}}. \tag{A.4}$$

We next show that the dominant contribution to the sums in (A.4) arises from few coefficients with index m close to $n\sqrt{1-\lambda_0^{-2}}$. To that end, we analyze the coefficients $a_{m,n}$ with Stirling's formula in the form

$$\sqrt{2\pi}\exp\left(\frac{1}{12}\right)\sqrt{n+1}\left(\frac{n}{\mathrm{e}}\right)^n \geq n! \geq \sqrt{n+1}\left(\frac{n}{\mathrm{e}}\right)^n.$$

Upon setting $C_1 := 2\pi\exp(1/6)$ and $C_2 := (2\pi)^{-3/2}\exp(-1/4)$, we get

$$a_{m,n} = \frac{(2m)!\,(n+m)!}{(m!)^2\,4^m\,(n-m)!} \leq C_1\frac{\sqrt{n+m+1}}{\sqrt{n-m+1}}\frac{\sqrt{2m+1}}{m+1}\frac{(n+m)^{n+m}}{(n-m)^{n-m}\,\mathrm{e}^{2m}} \tag{A.5}$$

$$\leq 2C_1\frac{(n+m)^{n+m}}{(n-m)^{n-m}\,\mathrm{e}^{2m}}, \tag{A.6}$$

$$a_{m,n} \geq C_2\frac{\sqrt{2m+1}\sqrt{n+m+1}}{(m+1)\sqrt{n-m+1}}\frac{(n+m)^{n+m}}{(n-m)^{n-m}\,\mathrm{e}^{2m}}. \tag{A.7}$$

The dominant contribution of $a_{m,n}(\lambda_0/n)^{2m}$ is

$$b_{m,n} := \frac{(n+m)^{n+m}}{(n-m)^{n-m}\,\mathrm{e}^{2m}}\left(\frac{\lambda_0}{n}\right)^{2m}.$$

The maximum of $m \mapsto b_{m,n}$ in the interval $[0,n] \subset \mathbb{R}$ is attained at $\tilde{m} = n\mu_0$ with $\mu_0 = \sqrt{1-\lambda_0^{-2}}$ and value

$$\tilde{b}_n = c_{\mu_0}^n \quad \text{with } c_{\mu_0} = \frac{(1+\mu_0)^{1+\mu_0}}{(1-\mu_0)^{1-\mu_0}} \left(\frac{\lambda_0}{\mathrm{e}}\right)^{2\mu_0}.$$

We also introduce the factor

$$f_{m,n} := \frac{\sqrt{n+m+1}}{\sqrt{n-m+1}} \frac{\sqrt{2m+1}}{m+1},$$

so as to be able to describe $a_{m,n}\left(\frac{\lambda_0}{n}\right)^{2m} \sim f_{m,n} b_{m,n}$ uniformly in m, n.
Case 1: We consider the range

$$0 \le n \le \max\{\frac{2}{1-\mu_0}, \frac{2}{\mu_0}, \frac{\lambda_0^2}{\mu_0 c_0}, c_5\},$$

where the parameter c_0 is given by Lemma A.3 (with $\lambda = \lambda_0$ there) and c_5 is defined in (A.14); both constants depend solely on λ_0. This is a finite set so

$$\sup_{0 \le n \le \max\{\frac{2}{1-\mu_0}, \frac{2}{\mu_0}, \frac{\lambda_0^2}{\mu_0 c_0}, c_5\}} (n+1)\rho_n^{\mathrm{I}} =: \tilde{C}_1 < \infty$$

depends solely on λ_0.
Case 2: We assume

$$n \ge \max\{\frac{2}{1-\mu_0}, \frac{2}{\mu_0}, \frac{\lambda_0^2}{\mu_0 c_0}, c_5\}. \tag{A.8}$$

We split the summations $\sum_{m=0}^n$ in the representation of ρ_n^{I} (cf. (A.4)) as $S_n^{\mathrm{I}} + S_n^{\mathrm{II}}$ with

$$S_n^{\mathrm{I}} := \sum_{n\tilde{\delta}_0 \le m \le n} \frac{a_{m,n}}{\tilde{b}_n}\left(\frac{\lambda_0}{n}\right)^{2m}, \qquad S_n^{\mathrm{II}} := \sum_{0 \le m < n\tilde{\delta}_0} \frac{a_{m,n}}{\tilde{b}_n}\left(\frac{\lambda_0}{n}\right)^{2m},$$

where

$$\tilde{\delta}_0 := \mu_0^3. \tag{A.9}$$

In view of

$$\min\left\{m+1 : m \ge n\tilde{\delta}_0\right\} \ge 1 + n\tilde{\delta}_0$$

we have

$$\rho_n^{\mathrm{I}} \le \frac{S_n^{\mathrm{I}} + S_n^{\mathrm{II}}}{\left(1+n\tilde{\delta}_0\right) S_n^{\mathrm{I}}} =: \rho_n^{\mathrm{II}}.$$

In order to estimate the terms S_n^{I}, S_n^{II}, we have to investigate the behavior of $a_{m,n}\left(\frac{\lambda_0}{n}\right)^{2m}/\tilde{b}_n$ depending on the distance of m to $\tilde{m}$. We write $m = n\mu_0(1+\varepsilon)$ for some $\varepsilon \in \mathbb{R}$ that satisfies $0 < \mu_0(1+\varepsilon) < 1$. This gives

$$C_2 f_{m,n} \left(\gamma_{\lambda_0}(\varepsilon)\right)^n \le \frac{a_{m,n}\left(\frac{\lambda_0}{n}\right)^{2m}}{\tilde{b}_n} \le 2C_1 \left(\gamma_{\lambda_0}(\varepsilon)\right)^n \tag{A.10}$$

$$\text{with} \quad \gamma_{\lambda_0}(\varepsilon) := \frac{(1+\mu_0(1+\varepsilon))^{1+\mu_0(1+\varepsilon)}}{(1-\mu_0(1+\varepsilon))^{1-\mu_0(1+\varepsilon)}} \frac{(1-\mu_0)^{1-\mu_0}}{(1+\mu_0)^{1+\mu_0}} \left(\frac{\lambda_0}{\mathrm{e}}\right)^{2\mu_0\varepsilon}.$$

Estimate of S_n^{I}: The dominant contribution in ρ_n^{II} is S_n^{I}, for which we therefore need a lower bound. Our strategy is to estimate this sum by a single summand, namely, the summand corresponding to an integer m close to $\tilde{m} = n\mu_0$. For $m \in \{\lfloor n\mu_0 \rfloor, \lceil n\mu_0 \rceil\}$ we have

$$m - n\mu_0 = n\mu_0\varepsilon_m \text{ with } \varepsilon_m \in \left\{-\frac{n\mu_0 - \lfloor n\mu_0 \rfloor}{n\mu_0}, \frac{\lceil n\mu_0 \rceil - n\mu_0}{n\mu_0}\right\}.$$

For these two values of m (in fact, we will only need the one with $m \le \mu_0 n$), we have $m = n\mu_0(1+\varepsilon_m)$ with $|\varepsilon_m| \le (n\mu_0)^{-1}$ and (cf. (A.8))

$$\frac{\mu_0}{2} n \le n\mu_0 - 1 \le m \le n\mu_0 + 1 \le \frac{1+\mu_0}{2} n, \tag{A.11}$$

$$\lambda_0^2 |\varepsilon_m| \le \frac{\lambda_0^2}{n\mu_0} \le c_0. \tag{A.12}$$

The estimate (A.11) makes Lemma A.3 applicable, which gives

$$1 \ge \gamma_{\lambda_0}(\varepsilon_m) \ge 1 - c_2\lambda_0^2\varepsilon_m^2 \ge 1 - c_2 c_0|\varepsilon_m| \ge 1 - \frac{c_6}{n} \quad \text{with } c_6 = \frac{c_2 c_0}{\mu_0}. \tag{A.13}$$

The estimate (A.11) leads to two-sided bounds for $f_{m,n}$:

$$f_{m,n} \le \frac{2n+1}{n\mu_0/2\sqrt{n(1-\mu_0)/2}} \overset{n\ge 1}{\le} \frac{6\sqrt{2}}{\mu_0\sqrt{1-\mu_0}} n^{-1/2} =: c_7 n^{-1/2},$$

$$f_{m,n} \ge \frac{\sqrt{n+\mu_0 n}\sqrt{\mu_0 n}}{\sqrt{n-\mu_0 n}(1+\mu_0)n} =: c_8 n^{-1/2}.$$

Define $c_5 > 0$ such that

$$n \ge c_5 \quad \Longrightarrow (1 - c_6/n)^n \ge \frac{1}{2}\,\mathrm{e}^{-c_6}. \tag{A.14}$$

This leads to

$$S_n^{\mathrm{I}} \ge C_2 f_{m,n}(\gamma_{\lambda_0}(\varepsilon_m))^n \ge C_2 c_8 n^{-1/2}\left(1 - \frac{c_6}{n}\right)^n \ge \frac{1}{2} C_2 c_8\,\mathrm{e}^{-c_6}\, n^{-1/2}. \tag{A.15}$$

Estimate of S_n^{II}: Let $c_0 \in (0,1)$ be the constant in Lemma A.3 (note that we may assume, without loss of generally, $c_0 < 1$). Upon writing $m \in \{0, \ldots, \lfloor n\tilde{\delta}_0 \rfloor\}$ in the form $m = \mu_0 n(1+\varepsilon_m)$, we find in view of $\tilde{\delta}_0 = \mu_0^3$ that $|\varepsilon_m| \ge \lambda_0^{-2}$. Hence, the monotonicity properties of the function γ_{λ_0} of Lemma A.3 imply

$$\gamma_{\lambda_0}(\varepsilon_m) \le 1 - \frac{c_2}{c_0}\lambda_0^{-2}. \tag{A.16}$$

We therefore get

$$S_n^{\mathrm{II}} = \sum_{0\le m\le \lfloor \tilde{\delta}_0 n \rfloor} \frac{a_{m,n}}{\tilde{b}_n}\left(\frac{\lambda_0}{n}\right)^{2m} \le 2C_1 \sum_{0\le m\le \lfloor \tilde{\delta}_0 n \rfloor}\left(1 - \frac{c_2}{c_0}\lambda_0^{-2}\right)^n \le 2C_1(n+1)\left(1 - \frac{c_2}{c_0}\lambda_0^{-2}\right)^n \tag{A.17}$$

The combination of (A.17) and (A.15) shows $S_n^{\mathrm{I}} + S_n^{\mathrm{II}} \le C S_n^{\mathrm{I}}$ for some constant $C > 0$ that depends solely on λ_0. This concludes the proof. ■

Lemma A.3 *For $\lambda > 1$ and $\mu := \sqrt{1-\lambda^{-2}}$ introduce the function*

$$(-1, \mu^{-1} - 1) \ni \varepsilon \mapsto \gamma_\lambda(\varepsilon) := \frac{(1+\mu(1+\varepsilon))^{1+\mu(1+\varepsilon)}}{(1-\mu(1+\varepsilon))^{1-\mu(1+\varepsilon)}} \frac{(1-\mu)^{1-\mu}}{(1+\mu)^{1+\mu}}\left(\frac{\lambda}{\mathrm{e}}\right)^{2\mu\varepsilon}.$$

Let $\lambda_0 > 1$. Then there are constants c_0, c_1, $c_2 > 0$ depending solely on λ_0 such that the following holds for every $\lambda \ge \lambda_0$: For every ε satisfying

$$|\varepsilon|\lambda^2 \le c_0 \tag{A.18}$$

the function γ_λ satisfies

$$1 - c_1\lambda^2\varepsilon^2 \le \gamma_\lambda(\varepsilon) \le 1 - c_2\lambda^2\varepsilon^2. \tag{A.19}$$

Furthermore, the function γ_λ is monotone increasing on $(-1,0)$ and monotone decreasing on $(0, \mu^{-1}-1)$. In particular, therefore,

$$0 < \gamma_\lambda(\varepsilon) \le 1 - \frac{c_2}{c_0}\lambda^{-2} \qquad \forall \varepsilon \in (-1, \mu^{-1}-1) \setminus (-c_0\lambda^{-2}, c_0\lambda^{-2}). \tag{A.20}$$

Proof. Define the function

$$g_\lambda(\varepsilon) := \ln\left(\left(1 - \mu^2(1+\varepsilon)^2\right)\lambda^2\right) \tag{A.21}$$

and observe

$$g_\lambda'(\varepsilon) = -2\mu^2 \frac{1+\varepsilon}{1-\mu^2(1+\varepsilon)^2}, \qquad g_\lambda''(\varepsilon) = -2\mu^2\frac{1+\mu^2(\varepsilon+1)^2}{\left(1-\mu^2(1+\varepsilon)^2\right)^2}, \tag{A.22}$$

$$\gamma_\lambda' = \mu\gamma_\lambda g_\lambda, \qquad \gamma_\lambda'' = \mu\gamma_\lambda\left(\mu g_\lambda^2 + g_\lambda'\right), \qquad \gamma_\lambda''' = \mu\gamma_\lambda\left(\mu^2 g_\lambda^3 + 3\mu g_\lambda g_\lambda' + g_\lambda''\right). \tag{A.23}$$

Step 1: (monotonicity properties of γ_λ) The function γ_λ is defined in the interval $(-1, \mu^{-1} - 1)$.
Claim: γ_λ is strictly increasing on $(-1, 0)$, strictly decreasing on $(0, \mu^{-1} - 1)$, and thus has a proper maximum at $\varepsilon = 0$. To see the monotonicity properties, we note that $\gamma_\lambda \geq 0$ and that $g_\lambda(\varepsilon) < 0$ for $\varepsilon < 0$ and $g_\lambda(\varepsilon) > 0$ for $\varepsilon > 0$. We calculate

$$\gamma_\lambda(0) = 1, \qquad \gamma_\lambda'(0) = 0, \qquad \gamma_\lambda''(0) = -2(\lambda^2 - 1)^{3/2}\lambda^{-1}. \tag{A.24}$$

Step 2: Use $\mu = \sqrt{1 - \lambda^{-2}}$ to write

$$\left(1 - \mu^2(1+\varepsilon)^2\right)\lambda^2 = 1 - (\lambda^2 - 1)(2\varepsilon + \varepsilon^2). \tag{A.25}$$

Fix $q \in (0, 1)$ and consider ε satisfying

$$0 < \mu(1+\varepsilon) < 1 \quad \text{and} \quad (\lambda^2 - 1)|2\varepsilon + \varepsilon^2| \leq q < 1. \tag{A.26}$$

From (A.25) and (A.26) we infer

$$(1-q)\lambda^{-2} \leq 1 - \mu^2(1+\varepsilon)^2 \leq (1+q)\lambda^{-2}.$$

This, together with $0 < \mu(1+\varepsilon) < 1$ and $\mu \in (0, 1)$ implies

$$|g_\lambda(\varepsilon)| \leq \max\{|\ln(1-q)|, \ln(1+q)\}, \qquad |g_\lambda'(\varepsilon)| \leq \frac{2\lambda^2}{1-q}, \qquad |g_\lambda''(\varepsilon)| \leq \frac{4\lambda^4}{(1-q)^2}. \tag{A.27}$$

Taylor's theorem now implies for every ε satisfying (A.26) the existence of an ε' in the interval $\langle 0, \varepsilon\rangle$ with endpoints 0 and ε such that

$$\gamma_\lambda(\varepsilon) = \gamma_\lambda(0) + \gamma_\lambda'(0)\varepsilon + \frac{1}{2}\gamma_\lambda''(0)\varepsilon^2 + \frac{1}{3!}\gamma_\lambda'''(\varepsilon')\varepsilon^3 = 1 - 2(\lambda^2-1)^{3/2}\lambda^{-1}\varepsilon^2 + \frac{1}{3!}\gamma_\lambda'''(\varepsilon')\varepsilon^3. \tag{A.28}$$

The remainder term $\gamma_\lambda'''(\varepsilon')$ is estimated using (A.27) as follows (note that $\gamma_\lambda \geq 0$ and has maximum 1) as

$$|\gamma_\lambda'''(\varepsilon')| \leq \max\{\ln(1+q), |\ln(1-q)|\}^3 + 6\lambda^2 \max\{\ln(1+q), |\ln(1-q)|\}(1-q)^{-1} + 4\lambda^4(1-q)^{-2} \leq C_1\lambda^4$$

for a constant C_1 that depends solely on $\lambda_0 > 1$ and the chosen q. Finally, there are constants C_2, $C_3 > 0$ depending solely on $\lambda_0 > 1$ such that

$$C_2\lambda^2 \leq 2(\lambda^2 - 1)^{3/2}\lambda^{-1} \leq C_3\lambda^2. \tag{A.29}$$

We conclude for ε satisfying (A.26)

$$1 - C_2\lambda^2\varepsilon^2 - \frac{C_1}{3!}\lambda^4\varepsilon^3 \leq \gamma_\lambda(\varepsilon) \leq 1 - C_3\lambda^2\varepsilon^2 + \frac{C_1}{3!}\lambda^4\varepsilon^3.$$

The two-sided bound (A.19) now follows if we assume (A.18) for c_0 sufficiently small so that the terms $\lambda^4\varepsilon^3$ are small compared to the terms involving $\lambda^2\varepsilon^2$. We note that the condition (A.18) for sufficiently small c_0 also implies (A.26). Finally, the estimate (A.20) is a consequence of (A.19) and the monotonicity properties of γ_λ. ∎

B Equivalence of $\|\cdot\|_{\mathbf{H}^1(\Omega)}$ and $\|\cdot\|_{\operatorname{curl},\Omega,1}$ in $\mathbf{V}_0$ and $\mathbf{V}_0^*$

The spaces $\mathbf{V}_0$ and $\mathbf{V}_0^*$ as in (4.1.1) involve the capacity operator (cf. Lemma 4.10). For the case that Γ is the surface of the ball, they are subspaces of $\mathbf{H}^1(\Omega)$ as shown in the following lemma. In contrast to Lemma 4.12 we obtain k-independent explicit bounds for the norm estimates.

Lemma B.1 *Let $\Omega = B_1(0)$ and let $\mathbf{V}_0$, $\mathbf{V}_0^*$ be defined as in (4.1.1). Then, $\mathbf{V}_0 \cup \mathbf{V}_0^* \subset \mathbf{H}^1(\Omega)$ and*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq \|\mathbf{u}\|_{\operatorname{curl},\Omega,1} \qquad \forall \mathbf{u} \in \mathbf{V}_0 \cup \mathbf{V}_0^*, \tag{B.1}$$

i.e., the constant $C_{\Omega,k}$ in Lemma 4.12 equals 1 for $\Omega = B_1(0)$.

Proof. The inclusion $\mathbf{V}_0 \cup \mathbf{V}_0^* \subset \mathbf{H}^1(\Omega)$ follows from Lemma 4.12 and it remains to prove the norm estimates. Let $\mathbf{u} \in \mathbf{V}_0$. Then, from [47, (2.5.151), (2.5.152), Lemma 5.4.2] we have

$$\begin{aligned}&(\nabla \mathbf{u}, \nabla \mathbf{v}) - (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) - (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) \\ &\quad = -\left(\operatorname{div}_\Gamma \mathbf{u}_T, \langle \mathbf{v}, \mathbf{n}\rangle\right)_\Gamma - \left(\langle \mathbf{u}, \mathbf{n}\rangle, \operatorname{div}_\Gamma \mathbf{v}_T\right)_\Gamma - 2\left(\langle \mathbf{u}, \mathbf{n}\rangle, \langle \mathbf{v}, \mathbf{n}\rangle\right)_\Gamma - \left(\mathbf{u}_T, \mathbf{v}_T\right)_\Gamma .\end{aligned}$$

We choose $\mathbf{v} = \mathbf{u}$ and employ (4.28) to obtain after rearranging terms

$$\begin{aligned}\|\nabla \mathbf{u}\|^2 &= \|\operatorname{curl}\mathbf{u}\|^2 - 2\operatorname{Re}\left(\operatorname{div}_\Gamma \mathbf{u}_T, \langle \mathbf{u}, \mathbf{n}\rangle\right)_\Gamma - 2\|\langle \mathbf{u}, \mathbf{n}\rangle\|_\Gamma^2 - \|\mathbf{u}_T\|_\Gamma^2 \\ &\overset{(4.28)}{=} \|\operatorname{curl}\mathbf{u}\|^2 + \frac{2}{k}\operatorname{Im}\left(\operatorname{div}_\Gamma T_k \mathbf{u}_T, \operatorname{div}_\Gamma \mathbf{u}_T\right)_\Gamma - 2\|\langle \mathbf{u}, \mathbf{n}\rangle\|_\Gamma^2 - \|\mathbf{u}_T\|_\Gamma^2 \\ &\leq \|\operatorname{curl}\mathbf{u}\|^2 + \frac{2}{k}\operatorname{Im}\left(\operatorname{div}_\Gamma T_k \mathbf{u}_T, \operatorname{div}_\Gamma \mathbf{u}_T\right)_\Gamma .\end{aligned} \tag{B.2}$$

From [47, (5.3.91), (5.3.93)] we conclude that

$$\left(\left(\operatorname{div}_\Gamma T_k \mathbf{u}_T\right), \operatorname{div}_\Gamma \overline{\mathbf{u}}_T)_{\Gamma)} = \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} \mathrm{i}\, \ell^2 (\ell+1)^2 \frac{k}{z_\ell(k)+1} \left|U_\ell^m\right|^2 .$$

Since

$$\operatorname{Im}\left(\frac{\mathrm{i}}{z_\ell(k)+1}\right) = \frac{\operatorname{Im}\left(\mathrm{i}\left(\overline{z_\ell(k)}+1\right)\right)}{\left|z_\ell(k)+1\right|^2} = \frac{\operatorname{Re}\left(z_\ell(k)+1\right)}{\left|z_\ell(k)+1\right|^2} \overset{[47,\ (2.6.23)]}{\leq} 0,$$

the second summand in (B.2) is non-positive so that$\|\nabla \mathbf{u}\| \leq \|\operatorname{curl}\mathbf{u}\|$. This implies the first estimate in (B.1) while the statement about $\mathbf{u} \in \mathbf{V}_0^*$ is simply a repetition of these arguments. ■

C Vector Spherical Harmonics

For $\mathbf{x} \in \mathbb{R}^3$, $r = \|\mathbf{x}\|$, and $\hat{\mathbf{x}} := \mathbf{x}/r$ we introduce the vectorial spherical harmonics (VSH) as in [30, Thm. 2.46] (with a different scaling)

$$\mathbf{Y}_\ell^m(\hat{\mathbf{x}}) := \hat{\mathbf{x}} Y_\ell^m(\hat{\mathbf{x}}), \quad \mathbf{U}_\ell^m(\hat{\mathbf{x}}) := \nabla_\Gamma Y_\ell^m(\hat{\mathbf{x}}), \quad \mathbf{V}_\ell^m(\hat{\mathbf{x}}) := \nabla_\Gamma Y_\ell^m(\hat{\mathbf{x}}) \times \hat{\mathbf{x}}.$$

From [30, Thm. 5.36] we conclude that any $\mathbf{u} \in \mathbf{X}$ has an expansion of the form

$$\mathbf{u}(r\hat{\mathbf{x}}) = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} \left(u_\ell^m(r)\, \mathbf{Y}_\ell^m(\hat{\mathbf{x}}) + v_\ell^m(r)\, \mathbf{U}_\ell^m(\hat{\mathbf{x}}) + w_\ell^m(r)\, \mathbf{V}_\ell^m(\hat{\mathbf{x}})\right). \tag{C.1}$$

We use the relations (cf. [30, p.271])[9]

$$\begin{aligned}&\operatorname{curl}\left(u_\ell^m(r)\, \mathbf{Y}_\ell^m(\hat{\mathbf{x}})\right) = \frac{u_\ell^m(r)}{r} \mathbf{V}_\ell^m(\hat{\mathbf{x}}), \quad \operatorname{curl}\left(v_\ell^m(r)\, \mathbf{U}_\ell^m(\hat{\mathbf{x}})\right) = -\frac{1}{r}\left(r v_\ell^m(r)\right)' \mathbf{V}_\ell^m(\hat{\mathbf{x}}), \\ &\operatorname{curl}\left(w_\ell^m(r)\, \mathbf{V}_\ell^m(\hat{\mathbf{x}})\right) = \frac{1}{r}\left(r w_\ell^m(r)\right)' \mathbf{U}_\ell^m(\hat{\mathbf{x}}) + w_\ell^m(r) \frac{\ell(\ell+1)}{r} \mathbf{Y}_\ell^m(\hat{\mathbf{x}}),\end{aligned}$$

so that $\operatorname{curl}\mathbf{u}$ is given by

$$\operatorname{curl}\mathbf{u}(r\hat{\mathbf{x}}) = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} \frac{1}{r}\left(\left(u_\ell^m(r) - \left(r v_\ell^m(r)\right)'\right) \mathbf{V}_\ell^m(\hat{\mathbf{x}}) + \left(r w_\ell^m(r)\right)' \mathbf{U}_\ell^m(\hat{\mathbf{x}}) + w_\ell^m(r)\, \ell(\ell+1)\, \mathbf{Y}_\ell^m(\hat{\mathbf{x}})\right).$$

Using the orthogonality relations of the vectorial spherical harmonics we get

$$\|\mathbf{u}\|^2 = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} \int_{\mathbb{R}} r^2 \left(\left|u_\ell^m(r)\right|^2 + \ell(\ell+1)\left(\left|v_\ell^m(r)\right|^2 + \left|w_\ell^m(r)\right|^2\right)\right) dr, \tag{C.2}$$

$$\|\operatorname{curl}\mathbf{u}\|^2 = \sum_{m \in \iota_\ell} \int_{\mathbb{R}} \ell(\ell+1) \left(\sum_{\ell=0}^{\infty} \left|u_\ell^m(r) - \left(r v_\ell^m(r)\right)'\right|^2 + \left|\left(r w_\ell^m(r)\right)'\right|^2 + \ell(\ell+1)\left|w_\ell^m(r)\right|^2\right) dr. \tag{C.3}$$

[9]There is a sign error in the second last relation on [30, p.271].

For $a > 0$, we introduce an operator $L_a^{\mathrm{VSH}} : \mathbf{X} \to \mathbf{X}$ and $H_a^{\mathrm{VSH}} : \mathbf{X} \to \mathbf{X}$ for a function $\mathbf{u}$ as in (C.1) by

$$L_a^{\mathrm{VSH}}\mathbf{u} = \sum_{\ell:0\leq\ell\leq a}\sum_{m\in\iota_\ell}\left(u_\ell^m(r)\,\mathbf{Y}_\ell^m(\hat{\mathbf{x}}) + v_\ell^m(r)\,\mathbf{U}_\ell^m(\hat{\mathbf{x}}) + w_\ell^m(r)\,\mathbf{V}_\ell^m(\hat{\mathbf{x}})\right), \quad H_a^{\mathrm{VSH}}\mathbf{u} = \mathbf{u} - L_a^{\mathrm{VSH}}\mathbf{u}. \tag{C.4}$$

From (C.2), (C.3) we conclude the stability of the splitting

$$\begin{array}{ll}\left\|L_a^{\mathrm{VSH}}\mathbf{u}\right\| \leq \|\mathbf{u}\|, & \left\|\operatorname{curl} L_a^{\mathrm{VSH}}\mathbf{u}\right\| \leq \|\operatorname{curl}\mathbf{u}\|, \\ \left\|H_a^{\mathrm{VSH}}\mathbf{u}\right\| \leq \|\mathbf{u}\|, & \left\|\operatorname{curl} H_a^{\mathrm{VSH}}\mathbf{u}\right\| \leq \|\operatorname{curl}\mathbf{u}\|.\end{array} \tag{C.5}$$

In addition the splitting is orthogonal

$$\left(L_a^{\mathrm{VSH}}\mathbf{u}, H_a^{\mathrm{VSH}}\mathbf{u}\right) = \left(\operatorname{curl} L_a^{\mathrm{VSH}}\mathbf{u}, \operatorname{curl} H_a^{\mathrm{VSH}}\mathbf{u}\right) = 0.$$

Note that on the unit sphere, it holds

$$\begin{aligned}
&\Pi_T\mathbf{Y}_\ell^m(\hat{\mathbf{x}}) = \hat{\mathbf{x}} \times (Y_{\ell,m}(\hat{\mathbf{x}})\,\hat{\mathbf{x}} \times \hat{\mathbf{x}}) = 0, \\
&\Pi_T\mathbf{U}_\ell^m(\hat{\mathbf{x}}) = \hat{\mathbf{x}} \times (\nabla_\Gamma Y_\ell^m(\hat{\mathbf{x}}) \times \hat{\mathbf{x}}) = \nabla_\Gamma Y_\ell^m(\hat{\mathbf{x}}), \\
&\Pi_T\mathbf{V}_\ell^m(\hat{\mathbf{x}}) = \hat{\mathbf{x}} \times ((\nabla_\Gamma Y_\ell^m(\hat{\mathbf{x}}) \times \hat{\mathbf{x}}) \times \hat{\mathbf{x}}) = -\hat{\mathbf{x}} \times \nabla_\Gamma Y_\ell^m(\hat{\mathbf{x}}) = \mathbf{T}_\ell^m,
\end{aligned}$$

where $\mathbf{T}_\ell^m$ is as in [47, (2.4.173)]. Hence, the application of the trace map Π_T yields

$$\mathbf{u}_T = \Pi_T\mathbf{u} = \sum_{\ell=0}^{\infty}\sum_{m\in\iota_\ell}\left(v_\ell^m\nabla_\Gamma Y_\ell^m + w_\ell^m\mathbf{T}_\ell^m\right), \tag{C.6}$$

where $v_\ell^m = v_\ell^m(1)$, $w_\ell^m := w_\ell^m(1)$. A key observation for the case of the unit sphere is that for any $\mathbf{u} \in \mathbf{X}$, the function $L_{\lambda k}^{\mathrm{VSH}}\mathbf{u}$ satisfies $\Pi_T L_{\lambda k}^{\mathrm{VSH}}\mathbf{u} = L_\Gamma\Pi_T\mathbf{u}$, where L_Γ was introduced in Definition 4.2.

Lemma C.1 *Let $\Omega = B_1(0)$ and L_Ω be as in Definition 4.2. Then: $\Pi_T L_{\lambda k}^{\mathrm{VSH}} = L_\Gamma\Pi_T$ and*

$$\|L_\Omega\mathbf{u}\|_{\operatorname{curl},\Omega,k} \leq \|\mathbf{u}\|_{\operatorname{curl},\Omega,k} \qquad \forall\mathbf{u} \in \mathbf{X}.$$

Furthermore, the stability constants in (4.6) satisfy $C_k^{L,\Omega} \leq 1$ and $C_k^{H,\Omega} \leq 2$.

Proof. Since L_Ω is the minimum norm extension (cf. Definition 4.2) the bound (C.5) lead to

$$\|L_\Omega\mathbf{u}\|_{\operatorname{curl},\Omega,k}^2 \leq \|L_{\lambda k}^{\mathrm{VSH}}\mathbf{u}\|_{\operatorname{curl},\Omega,k}^2 = k^2\|L_{\lambda k}^{\mathrm{VSH}}\mathbf{u}\|^2 + \|\operatorname{curl} L_{\lambda k}^{\mathrm{VSH}}\mathbf{u}\|^2 \leq k^2\|\mathbf{u}\|^2 + \|\operatorname{curl}\mathbf{u}\|^2 \leq \|\mathbf{u}\|_{\operatorname{curl},\Omega,k}^2.$$

■

D Analytic regularity of Maxwell and Maxwell-like Problems

D.1 Local Smoothness

Consider for a bounded Lipschitz domain $\omega \subset \mathbb{R}^3$

$$\operatorname{curl}(\mathbf{A}(x)\operatorname{curl}\mathbf{u}) = \mathbf{f} \quad \text{in } \omega, \tag{D.1a}$$

$$\operatorname{div}(\mathbf{B}(x)\mathbf{u}) = g \quad \text{in } \omega, \tag{D.1b}$$

$$\Pi_T\mathbf{u} = 0 \quad \text{on } \partial\omega. \tag{D.1c}$$

We have smoothness of $\mathbf{u}$ under regularity assumptions on the right-hand sides:

Lemma D.1 *Let $\partial\omega$ be a smooth bounded Lipschitz domain that is star-shaped with respect to a ball. Let $\mathbf{A}$, $\mathbf{B} \in C^\infty(\overline{\omega})$ be pointwise symmetric positive definite. Then:*

(i) If $\mathbf{u} \in \mathbf{H}_0(\omega, \operatorname{curl})$ and $\operatorname{div}(\mathbf{B}\mathbf{u}) \in L^2(\omega)$, then $\mathbf{u} \in \mathbf{H}^1(\omega)$ with

$$\|\mathbf{u}\|_{\mathbf{H}^1(\omega)} \leq C\left[\|\operatorname{div}(\mathbf{B}\mathbf{u})\|_{L^2(\omega)} + \|\operatorname{curl}\mathbf{u}\|_{\mathbf{L}^2(\omega)}\right].$$

(ii) If $\mathbf{u} \in \mathbf{H}_0(\omega, \mathrm{curl})$ *satisfies (D.1) for some* $\mathbf{f} \in \mathbf{H}^s(\omega)$, $g \in H^{s+1}(\omega)$, $s \in \mathbb{N}_0$, *then* $\mathbf{u} \in \mathbf{H}^{s+2}(\omega)$ *and*

$$\|\mathbf{u}\|_{\mathbf{H}^{s+2}(\omega)} \le C_s \left[\|\mathbf{f}\|_{\mathbf{H}^s(\omega)} + \|g\|_{H^{s+1}(\omega)}\right].$$

Proof. We use the right inverse R^{curl} of the curl-operator and use its mapping properties due to [17] as formulated in [39, Lemma 5.4]; specifically, we employ $R^{\mathrm{curl}} : \mathbf{H}^s(\omega) \to \mathbf{H}^{s+1}(\omega)$ for any $s \in \mathbb{N}_0$. We will also repeatedly use decompositions formulated in [39, Lemma 5.5], i.e., for $s \in \mathbb{N}_0$ and $\mathbf{v} \in \mathbf{H}^s(\omega, \mathrm{curl})$ there is $\varphi \in H^{s+1}(\omega)$ such that

$$\mathbf{v} = \nabla\varphi + R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{v}). \tag{D.1}$$

Proof of (i): Using (D.1), we write

$$\mathbf{u} = \nabla\varphi + R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{u}). \tag{D.2}$$

The mapping property $R^{\mathrm{curl}} : \mathbf{L}^2(\omega) \to \mathbf{H}^1(\omega)$ implies $R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{u}) \in \mathbf{H}^1(\omega)$. Using $\Pi_T\mathbf{u} = 0$, we infer $\nabla_{\partial\omega}\varphi = -\Pi_T R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{u}) \in \mathbf{H}^{1/2}_T(\partial\omega)$ so that, by the smoothness of $\partial\omega$, we have $g_D := \varphi|_{\partial\omega} \in H^{3/2}(\partial\omega)$. Multiplying (D.2) by $\mathbf{B}$ and applying the divergence reveals that φ solves

$$g = \mathrm{div}\,(\mathbf{B}\mathbf{u}) = \mathrm{div}\,(\mathbf{B}\nabla\varphi) + \mathrm{div}\,\big(\mathbf{B}R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{u})\big) \quad \text{in } \omega, \qquad \varphi = g_D \quad \text{on } \partial\omega. \tag{D.3}$$

This is a standard Poisson problem for φ, and the smoothness of $\partial\omega$ and $\mathbf{B}$ then imply $\varphi \in H^2(\omega)$ with

$$\|\varphi\|_{H^2(\omega)} \lesssim \|g - \mathrm{div}\,\big(\mathbf{B}R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{u})\big)\|_{L^2(\omega)} + \|g_D\|_{H^{3/2}(\partial\omega)} \lesssim \|g\|_{L^2(\omega)} + \|\mathrm{curl}\,\mathbf{u}\|_{L^2(\omega)}. \tag{D.4}$$

Proof of (ii): We set $\mathbf{w} := \mathrm{curl}\,\mathbf{u}$ and note

$$\mathrm{div}\,\mathbf{w} = 0, \qquad \mathbf{n}\cdot\mathbf{w} = \mathbf{n}\cdot\mathrm{curl}\,\mathbf{u} = \mathrm{curl}_{\partial\omega}\,\Pi_T\mathbf{u} = 0. \tag{D.5}$$

1. step: From (D.1) we see that we can we write, for some $\varphi \in H^1(\omega)$,

$$\mathbf{A}\mathbf{w} = \nabla\varphi + R^{\mathrm{curl}}(\mathrm{curl}(\mathbf{A}\mathbf{w})) \overset{\text{(D.1a)}}{=} \nabla\varphi + R^{\mathrm{curl}}(\mathbf{f}). \tag{D.6}$$

Hence, $\mathbf{w} = \mathbf{A}^{-1}\left(R^{\mathrm{curl}}(\mathbf{f}) + \nabla\varphi\right)$ and we get from (D.5) that φ satisfies

$$-\mathrm{div}\,\big(\mathbf{A}^{-1}\nabla\varphi\big) = \mathrm{div}\,\big(\mathbf{A}^{-1}R^{\mathrm{curl}}(\mathbf{f})\big) \quad \text{in } \omega, \qquad \mathbf{n}\cdot\mathbf{A}^{-1}\nabla\varphi = -\mathbf{n}\cdot\mathbf{A}^{-1}R^{\mathrm{curl}}(\mathbf{f}) \quad \text{on } \partial\omega. \tag{D.7}$$

The mapping properties of $R^{\mathrm{curl}} : \mathbf{H}^s(\omega) \to \mathbf{H}^{s+1}(\omega)$ give $R^{\mathrm{curl}}(\mathbf{f}) \in \mathbf{H}^{s+1}(\omega)$ so that the scalar shift theorem for Poisson problems gives in fact $\varphi \in H^{s+2}(\omega)$ with $\|\varphi\|_{H^{s+2}(\omega)} \le C\|\mathbf{f}\|_{\mathbf{H}^s(\omega)}$. Inserting this regularity information in (D.6) provides $\mathbf{w} \in \mathbf{H}^{s+1}(\omega)$ with

$$\|\mathbf{w}\|_{\mathbf{H}^{s+1}(\omega)} \le C\|\mathbf{f}\|_{\mathbf{H}^s(\omega)}. \tag{D.8}$$

2. step: From (i) we have $\mathbf{u} \in \mathbf{H}^1(\omega)$ and from the first step we get $\mathrm{curl}\,\mathbf{u} \in \mathbf{H}^{s+1}(\omega)$. In particular, $\mathbf{u} \in \mathbf{H}^1(\omega, \mathrm{curl})$. Hence, (D.1) allows us to write, for some $\varphi \in H^2(\omega)$

$$\mathbf{u} = \nabla\varphi + R^{\mathrm{curl}}(\underbrace{\mathrm{curl}\,\mathbf{u}}_{\in\mathbf{H}^{s+1}(\omega)}). \tag{D.9}$$

3. step: An equation for φ is obtained in two steps: using $\Pi_T\mathbf{u} = 0$, we see again that

$$\nabla_{\partial\omega}\varphi = -\Pi_T R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{u}) \in \mathbf{H}^{s+3/2}(\partial\omega),$$

where we used the trace estimate and the mapping properties of R^{curl}. We conclude $g_D := \varphi|_{\partial\omega} \in H^{s+5/2}(\partial\omega)$. Multiplying (D.9) with $\mathbf{B}$ and applying the divergence operator reveals a Poisson problem for φ:

$$g = \mathrm{div}\,\mathbf{B}\mathbf{u} = \mathrm{div}\,(\mathbf{B}\nabla\varphi) + \mathrm{div}\,\big(\mathbf{B}R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{u})\big) \quad \text{in } \omega, \qquad \varphi = g_D \quad \text{on } \partial\omega. \tag{D.10}$$

By standard elliptic regularity in view of the smoothness of $\partial\omega$ and $\mathbf{B}$, we get $\varphi \in H^{s+3}(\omega)$ with

$$\begin{aligned}\|\varphi\|_{H^{s+3}(\omega)} &\lesssim \|g - \mathrm{div}\,\big(\mathbf{B}R^{\mathrm{curl}}(\mathrm{curl}\,\mathbf{u})\big)\|_{H^{s+1}(\omega)} + \|g_D\|_{H^{s+5/2}(\partial\omega)} \lesssim \|g\|_{H^{s+1}(\omega)} + \|\mathrm{curl}\,\mathbf{u}\|_{\mathbf{H}^{s+1}(\omega)} \\ &\lesssim \|g\|_{H^{s+1}(\omega)} + \|\mathbf{f}\|_{\mathbf{H}^s(\omega)}. \end{aligned} \tag{D.11}$$

4. step: Inserting the information (D.11) in (D.9) implies $\mathbf{u} \in \mathbf{H}^{s+2}(\omega)$ together with $\|\mathbf{u}\|_{\mathbf{H}^{s+2}(\omega)} \lesssim \|g\|_{H^{s+1}(\omega)} + \|\mathbf{f}\|_{\mathbf{H}^s(\omega)}$.
■

D.2 Local Analytic Regularity

We show analytic regularity of solutions of elliptic systems of the form (D.2) on half-balls $B_r^+ := \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, |\mathbf{x}| < r, x_3 > 0\}$. We denote $\Gamma_R := \{\mathbf{x} \in B_R(0) \,|\, \mathbf{x}_3 = 0\}$.
On B_R^+ with $R \leq 1$ we consider smooth functions $\mathbf{u}$ that satisfy the following equations for some $\varepsilon > 0$:

$$-\varepsilon^2 \sum_{\alpha,\beta,i=1}^{3} \partial_\alpha \left(A^{ij}_{\alpha\beta} \partial_\beta \mathbf{u}_j \right) + \varepsilon \sum_{\beta,j=1}^{3} B^{ij}_\beta \partial_\beta \mathbf{u}_j + \sum_{j=1}^{3} C^{ij} \mathbf{u}_j = \mathbf{f}_i, \qquad i = 1, 2, 3, \tag{D.12}$$

$$\mathbf{u}_1 = \mathbf{u}_2 = 0 \quad \text{on } \Gamma_R, \tag{D.13}$$

$$\partial_3 \mathbf{u}_3 = \varepsilon^{-1}(G + b\mathbf{u}_3) + \sum_{j=1}^{2} d_j \partial_j \mathbf{u}_3 + \sum_{j=1}^{2} e_j \partial_3 \mathbf{u}_j \quad \text{on } \Gamma_R. \tag{D.14}$$

We assume that the coefficients are analytic, i.e., (cf. Def. 2.5)

$$(A^{ij}_{\alpha\beta})_{i,j,\alpha,\beta} \in \mathcal{A}^\infty(C_A, \gamma_A, B_R^+), \quad (B^{ij}_\beta)_{i,j,\beta} \in \mathcal{A}^\infty(C_B, \gamma_B, B_R^+), \quad (C^{ij})_{i,j} \in \mathcal{A}^\infty(C_C, \gamma_C, B_R^+), \tag{D.15}$$

$$b \in \mathcal{A}^\infty(C_b, \gamma_b, B_R^+), \qquad (d_j)_j \in \mathcal{A}^\infty(C_d, \gamma_d, B_R^+), \qquad (e_j)_j \in \mathcal{A}^\infty(C_e, \gamma_e, B_R^+) \tag{D.16}$$

here, we have written, e.g., $(d_j)_j$ to emphasize that the objects are tensor-valued and the multiindex notation is understood as in (2.27). Concerning the tensor $A^{ij}_{\alpha\beta}$ and the coefficients d_j, e_j we will furthermore make the following structural assumption:

$$A^{ij}_{\alpha\beta}(0) = \delta_{ij}\delta_{\alpha\beta}, \qquad d_j(0) = 0, \qquad e_j(0) = 0.$$

This structural assumption implies that the leading order differential operator in (D.2) reduces to a block Laplace operator at the origin and that the boundary conditions for the third component $\mathbf{u}_3$ reduce to Neumann boundary conditions. In other words: the system decouples at the origin.

Remark D.2 *The structural assumption on $A^{ij}_{\alpha\beta}$ implies the "very strong ellipticity"/Legendre condition for the leading order differential operator (near the origin). No sign conditions are imposed on the coefficients B^{ij}_α, C^{ij}, b_j, d_j, which could even by complex. The condition $\varepsilon > 0$ can always be enforced by a scaling so that mutatis mutandis the ensuing theory is also valid for complex ε.* ▪

It is convenient to introduce $\mathcal{E} \in (0, 1]$ by

$$\mathcal{E}^{-1} := \frac{C_B}{\varepsilon} + \frac{\sqrt{C_C}}{\varepsilon} + \frac{C_b}{\varepsilon} + 1, \tag{D.17}$$

which implies the estimates

$$\frac{C_C}{\varepsilon^2} \leq \mathcal{E}^{-1}, \qquad \frac{C_B}{\varepsilon} \leq \mathcal{E}^{-1}, \qquad \frac{C_b}{\varepsilon} \leq \mathcal{E}^{-1}. \tag{D.18}$$

We will make the following assumptions on the right-hand sides

$$\|\nabla^p \mathbf{f}\|_{L^2(B_R)} \leq C_f \gamma_f^p \max\{p/R, \mathcal{E}^{-1}\}^p \qquad \forall p \in \mathbb{N}_0, \tag{D.19}$$

$$\|\nabla^p G\|_{L^2(B_R)} \leq C_G \gamma_G^p \max\{p/R, \mathcal{E}^{-1}\}^p \qquad \forall p \in \mathbb{N}_0. \tag{D.20}$$

Given the special role of the variable x_3, we will interchangeably use the notation $\mathbf{x} = (x, y)$ with $x = (\mathbf{x}_1, \mathbf{x}_2)$ and $y = \mathbf{x}_3$. Analytic regularity of the solution of (D.2) will be characterized in Theorem D.5 by the following seminorms:

$$N'_{R,p,q}(v) = \frac{1}{[p+q]!} \sup_{R/2 \leq r < R} (R - r)^{p+q+2} \|\partial_y^{q+2} \nabla_x^p v\|_{L^2(B_r^+)}, \quad p \geq 0, q \geq -2. \tag{D.21}$$

Our procedure to control $N'_{R,p,q}(\mathbf{u})$ is the standard one by first controlling tangential derivatives and then using the differential equation to control normal derivatives. We follow [35, Sec. 5.5]. In the proofs, we implicitly assume that the solution $\mathbf{u} \in C^\infty(B_R^+)$. This could be proved by carefully arguing with the difference quotient method or, alternatively, by asserting the smoothness of the solution by a separate argument (this is how we proceed in the present application of Theorem D.5).

D.2.1 Control of Tangential Derivatives

We introduce the following auxiliary notation suitable for controlling tangential derivatives (cf. [35, Sec. 5.5])

$$[p] := \max\{1, p\}, \tag{D.22}$$

$$M'_{R,p}(v) = \frac{1}{p!} \sup_{R/2 \le r < R} (R-r)^{p+2} \|\nabla_x^p v\|_{L^2(B_r^+)}, \tag{D.23}$$

$$N'_{R,p}(v) = \begin{cases} \frac{1}{p!} \sup\limits_{R/2 \le r < R} (R-r)^{p+2} \|\nabla^2 \nabla_x^p v\|_{L^2(B_r^+)} & \text{if } p \ge 0 \\ \sup\limits_{R/2 \le r < R} (R-r)^{p+2} \|\nabla^{2+p} v\|_{L^2(B_r^+)} & \text{if } p = -2, -1, \end{cases} \tag{D.24}$$

$$H_{R,p}(v) := \frac{1}{[p-1]!} \sup_{R/2 \le r < R} (R-r)^{p+1} \left[\|\nabla_x^p v\|_{L^2(B_r^+)} + \frac{R-r}{[p]} \|\nabla_x^p \nabla v\|_{L^2(B_r^+)} \right]. \tag{D.25}$$

Lemma D.3 *There exists a universal constant $C_I > 0$ such that for f, G sufficiently smooth, there holds:*

(i) Let u solve $-\Delta u = f$ on B_R^+ and $u|_{\Gamma_R} = 0$.

$$N'_{R,p}(u) \le C_I \left[M'_{R,p}(f) + N'_{R,p-1}(u) + N'_{R,p-2}(u) \right] \qquad \forall p \ge 0. \tag{D.26}$$

For $p = 0$, we have the sharper estimate $N'_{R,0}(u) \le C_I \left[M'_{R,0}(f) + N'_{R,-1}(u) \right]$.

(ii) Let u solve $-\Delta u = f$ on B_R^+ and $\partial_y u|_{\Gamma_R} = G$. Then

$$N'_{R,p}(u) \le C_I \left[M'_{R,p}(f) + H_{R,p}(G) + N'_{R,p-1}(u) + N'_{R,p-2}(u) \right] \qquad \forall p \ge 0. \tag{D.27}$$

For $p = 0$, we have the sharper estimate $N'_{R,0}(u) \le C_I \left[M'_{R,0}(f) + H_{R,0}(G) + N'_{R,-1}(u) \right]$.

Proof. For the proof of (i), see [35, Lemma 5.5.15] or [45, Lemma 5.7.3']. Statement (ii) is essentially taken from [35, Lemma 5.5.23]. The special cases $p = 0$ follow from the general case and the first Poincaré inequality in the case (i) and the second Poincaré inequality in the case (ii). ∎

Lemma D.4 *Let $\mathbf{u}$ satisfy (D.2) with coefficients and data satisfying (D.2) and (D.2). Let C_I be given by Lemma D.3. Let $R \le 1$ be such that*

$$3C_I \left(C_A \gamma_A + C_d \gamma_d + C_e \gamma_e \right) R \le \frac{1}{2}. \tag{D.28}$$

Then there is $K > 1$ depending only on the constants appearing in (D.2) and on γ_f, γ_G such that

$$N'_{R,p}(\mathbf{u}) \le C_{\mathbf{u}} K^{p+2} \frac{\max\{R/\mathcal{E}, p+3\}^{p+2}}{p!}, \qquad p \ge -1, \tag{D.29}$$

$$\begin{aligned} C_{\mathbf{u}} &= \min\{1, R/\mathcal{E}\}(1 + \mathcal{E} C_A \gamma_A) \mathcal{E} \|\nabla \mathbf{u}\|_{L^2(B_R^+)} + \min\{1, R/\mathcal{E}\}^2 (\mathcal{E}/\varepsilon)^2 \left[C_f + C_C \|\mathbf{u}\|_{L^2(B_R^+)} \right] \\ &\quad + C_G(1 + \gamma_G) \min\{1, R/\mathcal{E}\}(\mathcal{E}/\varepsilon) \\ &\quad + C_b(1 + \gamma_b R) \min\{1, R/\mathcal{E}\}(\mathcal{E}/\varepsilon) \|\mathbf{u}\|_{L^2(B_R^+)} + C_b \min\{1, R/\mathcal{E}\}^2 (\mathcal{E}/\varepsilon) \mathcal{E} \|\nabla \mathbf{u}\|_{L^2(B_R^+)} \\ &\quad + (C_d \gamma_d R + C_e \gamma_e R) \min\{1, R/\mathcal{E}\} \mathcal{E} \|\nabla \mathbf{u}\|_{L^2(B_R^+)} \end{aligned}$$

Proof. We start with the observation

$$\min\{1, R/\mathcal{E}\} \max\{1, R/\mathcal{E}\} = R/\mathcal{E}. \tag{D.30}$$

The proof will be by induction on p and we will employ Lemma D.3. To that end, recall $A_{\alpha\beta}^{ij}(0) = \delta_{\alpha\beta}\delta_{ij}$ from (5.38) we write (D.2) as

$$-\Delta \mathbf{u}_i = \varepsilon^{-2} \mathbf{f}_i - \varepsilon^{-2} \sum_{j=1}^{3} C^{ij} \mathbf{u}_j - \varepsilon^{-1} \sum_{\beta,j=1}^{3} \widetilde{B}_\beta^{ij} \partial_\beta \mathbf{u}_j + \sum_{\alpha,\beta,j=1}^{3} \underbrace{\left(A_{\alpha\beta}^{ij} - A_{\alpha\beta}^{ij}(0) \right)}_{=: \widetilde{A}_{\alpha\beta}^{ij}} \partial_\alpha \partial_\beta \mathbf{u}_j, \tag{D.2.1a}$$

$$\mathbf{u}_1 = \mathbf{u}_2 = 0 \quad \text{on } \Gamma, \tag{D.2.1b}$$

$$\partial_3 \mathbf{u}_3 = \varepsilon^{-1}(G + b\mathbf{u}_3) + \sum_{j=1}^{2} d_j \partial_j \mathbf{u}_3 + \sum_{j=1}^{2} e_j \partial_3 \mathbf{u}_j \quad \text{on } \Gamma, \tag{D.2.1c}$$

where the coefficient

$$\widetilde{B}^{ij}_\beta := B^{ij}_\beta + \varepsilon \sum_{\alpha=1}^{3} \partial_\alpha \left(A^{ij}_{\alpha\beta} - A^{ij}_{\alpha\beta}(0)\right) = B^{ij}_\beta + \varepsilon \sum_{\alpha=1}^{3} \partial_\alpha A^{ij}_{\alpha\beta}$$

is again an analytic function with $(\widetilde{B}^{ij}_\beta)_{i,j,\beta} \in \mathcal{A}^\infty(C_{\widetilde{B}}, \gamma_{\widetilde{B}}, B^+_R)$ with $C_{\widetilde{B}} := C_B + C_A\gamma_A\varepsilon$ and $\gamma_{\widetilde{B}} := \gamma_B + 2\gamma_A$ (note: $C_B\gamma_B^p + \varepsilon C_A \gamma_A^{p+1}(p+1) \le C_B\gamma_B^p + \varepsilon C_A\gamma_A(2\gamma_A)^p$). The system (D.2.1) is of the form analyzed in Lemma D.3. We therefore get

$$N'_{R,p}(\mathbf{u}) \le C_I \Big[\varepsilon^{-2} M'_{R,p}(\mathbf{f}) + \sum_{i=1}^{3} M'_{R,p}\Big(\varepsilon^{-2}\sum_j C^{ij}\mathbf{u}_j + \varepsilon^{-1}\sum_{\beta,j}\widetilde{B}^{ij}_\beta \partial_\beta \mathbf{u}_j + \sum_{\alpha,\beta,j}\widetilde{A}^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta \mathbf{u}_j\Big) \tag{D.31}$$
$$+ \varepsilon^{-1}H_{R,p}(G) + \varepsilon^{-1}H_{R,p}\big(b\mathbf{u}_3\big) + H_{R,p}\big(\sum_{j=1}^{2} d_j\partial_j\mathbf{u}_3\big) + H_{R,p}\big(\sum_{j=1}^{2} e_j\partial_3\mathbf{u}_j\big) + N'_{R,p-1}(\mathbf{u}) + N'_{R,p-2}(\mathbf{u})\Big].$$

1. step: For $p = -1$, the assertion (D.29) follows directly from $K \ge 1$, the definition of $C_{\mathbf{u}}$, and (D.30).
2. step: For $p = 0$, we employ the sharpened versions of Lemma D.3 which leads to (D.31) for $p = 0$ where the last term, $N_{p-2}(\mathbf{u})$, is dropped:

$$\begin{aligned}
N'_{R,0}(\mathbf{u}) \le{}& C_I\Big[\varepsilon^{-2}M'_{R,0}(\mathbf{f}) + \sum_{i=1}^{3} M'_{R,0}\Big(\varepsilon^{-2}\sum_j C^{ij}\mathbf{u}_j + \varepsilon^{-1}\sum_{\beta,j}\widetilde{B}^{ij}_\beta\partial_\beta\mathbf{u}_j + \sum_{\alpha,\beta,j}\widetilde{A}^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\mathbf{u}_j\Big)\\
&+ \varepsilon^{-1}H_{R,0}(G) + \varepsilon^{-1}H_{R,0}\big(b\mathbf{u}_3\big) + H_{R,0}\big(\sum_{j=1}^{2} d_j\partial_j\mathbf{u}_3\big) + H_{R,0}\big(\sum_{j=1}^{2} e_j\partial_3\mathbf{u}_j\big) + N'_{R,-1}(\mathbf{u})\Big]\\
\le{}& 3C_I\Big[(R/2)^2\varepsilon^{-2}C_f + (R/2)^2\varepsilon^{-2}C_C\|\mathbf{u}\|_{L^2(B^+_R)} + C_{\widetilde{B}}(R/2)^2\varepsilon^{-1}\|\nabla\mathbf{u}\|_{L^2(B^+_R)} + C_A\gamma_A R N'_{R,0}(\mathbf{u})\\
&+ C_G/2R\varepsilon^{-1} + C_G\gamma_G(R/2)^2\varepsilon^{-1}\max\{1/R,\mathcal{E}^{-1}\}\\
&+ \frac{C_b}{2}(1+\gamma_b R)R\varepsilon^{-1}\|\mathbf{u}\|_{L^2(B^+_R)} + \frac{C_b}{2}R^2\varepsilon^{-1}\|\nabla\mathbf{u}\|_{L^2(B^+_R)}\\
&+ 2C_d\gamma_d R^2\|\nabla\mathbf{u}\|_{L^2(B^+_R)} + C_d\gamma_d R N'_{R,0}(\mathbf{u}) + C_e\gamma_e R^2\|\nabla\mathbf{u}\|_{L^2(B^+_R)} + 2C_e\gamma_e R N'_{R,0}(\mathbf{u}) + N'_{R,-1}(\mathbf{u})\Big]\\
\le{}& 3C_I\Big[\frac{1}{4}(R/\mathcal{E})^2(\mathcal{E}/\varepsilon)^2\Big\{C_f + C_C\|\mathbf{u}\|_{L^2(B^+_R)}\Big\} + \frac{1}{4}(R/\mathcal{E})^2 C_{\widetilde{B}}(\mathcal{E}/\varepsilon)\mathcal{E}\|\nabla\mathbf{u}\|_{L^2(B^+_R)} + C_A\gamma_A R N'_{R,0}(\mathbf{u})\\
&+ \frac{C_G}{2}R/\mathcal{E}(\mathcal{E}/\varepsilon) + \frac{C_G}{4}\gamma_G R/\mathcal{E}(\mathcal{E}/\varepsilon)\max\{1,R/\mathcal{E}\}\\
&+ \frac{C_b}{2}(1+\gamma_b R)R/\mathcal{E}(\mathcal{E}/\varepsilon)\|\mathbf{u}\|_{L^2(B^+_R)} + \frac{C_b}{2}(R/\mathcal{E})^2(\mathcal{E}/\varepsilon)\mathcal{E}\|\nabla\mathbf{u}\|_{L^2(B^+_R)}\\
&+ 2C_d\gamma_d R(R/\mathcal{E})\mathcal{E}\|\nabla\mathbf{u}\|_{L^2(B^+_R)} + C_d\gamma_d R N'_{R,0}(\mathbf{u}) + 2C_e\gamma_e R(R/\mathcal{E})\mathcal{E}\|\nabla\mathbf{u}\|_{L^2(B^+_R)}\\
&+ C_e\gamma_e R N'_{R,0}(\mathbf{u}) + N'_{R,-1}(\mathbf{u})\Big].
\end{aligned}$$

Using (D.30) for $R/\mathcal{E} = \min\{1,R/\mathcal{E}\}\max\{1,R/\mathcal{E}\}$, inserting the definition of $C_{\mathbf{u}}$, using the condition (D.28), and assuming that K is sufficiently large shows the case $p = 0$.
3. step: For $p \ge 1$, we proceed by induction, assuming that (D.29) is valid up to $p-1$, i.e.,

$$N'_{R,p-q}(\mathbf{u}) \le C_{\mathbf{u}} K^{p+2-q}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}, \qquad q = 1,\dots,p+1. \tag{D.32}$$

We need to estimate the terms in (D.31). To bound the terms $M'_{R,p}(\sum_{j,\beta}\widetilde{B}^{ij}_\beta\partial_\beta\mathbf{u})$ in terms of $N'_{R,p-q-1}(\mathbf{u})$, it is useful to note the simple facts (cf. also [45, (5.7.19)])

$$|\nabla\nabla^p_x\mathbf{u}|^2 \le |\nabla^2\nabla^{p-1}_x\mathbf{u}|^2, \quad p \ge 1, \qquad |\nabla\nabla^p_x\mathbf{u}|^2 = |\nabla\mathbf{u}|^2, \quad p = 0. \tag{D.33}$$

To estimate these terms, we compute (cf. [35, Lemma 5.5.13] for similar calculations)

$$\varepsilon^{-2}M'_{R,p}\Big(\sum_j C^{ij}\mathbf{u}_j\Big) \le \frac{C_c}{4}\sum_{q=0}^{p}\left(\gamma_c\frac{R}{2}\right)^q\left(\frac{R}{\varepsilon}\right)^2\frac{[p-q-2]!}{(p-q)!}N'_{R,p-q-2}(\mathbf{u}),$$

$$\varepsilon^{-1}M'_{R,p}\Big(\sum_{j,\beta}\widetilde{B}^{ij}_{\beta}\partial_\beta\mathbf{u}_j\Big) \le \frac{C_{\widetilde{B}}}{2}\sum_{q=0}^{p}\left(\gamma_{\widetilde{B}}\frac{R}{2}\right)^q\frac{R}{\varepsilon}\frac{[p-q-1]!}{(p-q)!}N'_{R,p-q-1}(\mathbf{u}),$$

$$M'_{R,p}\Big(\sum_{j,\alpha,\beta}\widetilde{A}^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\mathbf{u}_j\Big) \le C_A\gamma_A R N'_{R,p}(\mathbf{u}) + C_A\sum_{q=1}^{p}\left(\gamma_A\frac{R}{2}\right)^q N'_{R,p-q}(\mathbf{u}),$$

$$\varepsilon^{-1}\frac{1}{[p-1]!}\sup_{R/2\le r<R}(R-r)^{p+1}\|\nabla_x^p(b\mathbf{u}_3)\|_{L^2(B_r^+)} \le \frac{C_b}{2}\frac{R[p]}{\varepsilon}\sum_{q=0}^{p}\left(\frac{\gamma_b R}{2}\right)^q\frac{[p-q-2]!}{(p-q)!}N'_{R,p-q-2}(\mathbf{u}_3),$$

$$\varepsilon^{-1}\frac{1}{[p]!}\sup_{R/2\le r<R}(R-r)^{p+2}\|\nabla_x^p\nabla(b\mathbf{u}_3)\|_{L^2(B_r^+)} \le \frac{C_b}{2}\frac{R(p+1)}{\varepsilon}\sum_{q=0}^{p+1}\left(\frac{\gamma_b R}{2}\right)^q\frac{[p-q-1]!}{(p-q+1)!}N'_{R,p-q-1}(\mathbf{u}_3),$$

$$\frac{1}{[p-1]!}\sup_{R/2\le r<R}(R-r)^{p+1}\|\nabla_x^p(\sum_{j=1}^{2}d_j\partial_j\mathbf{u}_3)\|_{L^2(B_r^+)} \le C_d\gamma_d R N'_{R,p-1}(\mathbf{u}_3) + C_d\sum_{q=1}^{p}\left(\frac{\gamma_d R}{2}\right)^q\frac{[p]}{[p-q]}N'_{R,p-q-1}(\mathbf{u}_3),$$

$$\frac{1}{[p]!}\sup_{R/2\le r<R}(R-r)^{p+2}\|\nabla_x^p\nabla(\sum_{j=1}^{2}d_j\partial_j\mathbf{u}_3)\|_{L^2(B_r^+)} \le C_d R\gamma_d N'_{R,p}(\mathbf{u}_3) + C_d\sum_{q=1}^{p+1}\left(\frac{\gamma_d R}{2}\right)^q\frac{p+1}{[p-q+1]}N'_{R,p-q}(\mathbf{u}_3),$$

$$\frac{1}{[p-1]!}\sup_{R/2\le r<R}(R-r)^{p+1}\|\nabla_x^p(\sum_{j=1}^{2}e_j\partial_3\mathbf{u}_j)\|_{L^2(B_r^+)} \le C_e\gamma_e R N'_{R,p-1}(\mathbf{u}) + C_e\sum_{q=1}^{p}\left(\frac{\gamma_e R}{2}\right)^q\frac{[p]}{[p-q]}N'_{R,p-q-1}(\mathbf{u}),$$

$$\frac{1}{[p]!}\sup_{R/2\le r<R}(R-r)^{p+2}\|\nabla_x^p\nabla(\sum_{j=1}^{2}e_j\partial_3\mathbf{u}_j)\|_{L^2(B_r^+)} \le C_e R\gamma_e N'_{R,p}(\mathbf{u}) + C_e\sum_{q=1}^{p+1}\left(\frac{\gamma_e R}{2}\right)^q\frac{p+1}{[p-q+1]}N'_{R,p-q}(\mathbf{u}).$$

We choose

$$K > \max\{1, \gamma_f/2, \gamma_G/2, \gamma_A/2, \gamma_{\widetilde{B}}/2, \gamma_C/2, \gamma_b/2, \gamma_d/2, \gamma_e/2\}$$

such that the expression in brackets $[\cdots]$ in (D.37) is smaller than $1/(6C_I)$ (and, of course, such that the case $p=0$ is proved). The calculation in [35, p. 206, bottom] for $M'_{R,p}(\mathbf{f})$ and similar calculations for $H_{R,p}(G)$ give

$$\varepsilon^{-2}M'_{R,p}(\mathbf{f}) \le C_f\min\{1, R/\mathcal{E}\}^2(\mathcal{E}/\varepsilon)^2K^{p+2}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}\frac{1}{4}K^{-2}\left(\frac{\gamma_f}{2K}\right)^p, \tag{D.34}$$

$$\varepsilon^{-1}H_{R,p}(G) \le C_G\min\{1, R/\varepsilon\}(\mathcal{E}/\varepsilon)K^{p+2}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}\left[K^{-2}\left(\frac{\gamma_G}{2K}\right)^p + \frac{1}{2K}\left(\frac{\gamma_G}{2K}\right)^{p+1}\right]. \tag{D.35}$$

We use the induction assumption (D.32), recall (D.18) as well as (D.30) (to deal with the cases where $N'_{R,-2}(\mathbf{u})$ is involved) to estimate

$$C_C R^2\varepsilon^{-2}\frac{[p-q-2]!}{(p-q)!}N'_{R,p-q-2}(\mathbf{u}) \le C_{\mathbf{u}}K^{p+2-q-2}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}, \qquad q = 0,\dots,p,$$

$$C_{\widetilde{B}}R\varepsilon^{-1}\frac{[p-q-1]!}{(p-q)!}N'_{R,p-q-1}(\mathbf{u}) \le C_{\mathbf{u}}(1+C_A\gamma_A\varepsilon)K^{p+2-q-1}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}, \qquad q = 0,\dots,p,$$

$$C_b[p]R/\varepsilon\frac{[p-q-2]!}{(p-q)!}N'_{R,p-q-2}(\mathbf{u}) \le C_{\mathbf{u}}K^{p+2-q-2}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}, \qquad q = 0,\dots,p,$$

$$C_b(p+1)R/\varepsilon\frac{[p-q-1]!}{(p-q+1)!}N'_{R,p-q-1}(\mathbf{u}) \le C_{\mathbf{u}}K^{p+2-q-1}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}, \qquad q = 0,\dots,p+1,$$

$$\frac{[p]}{[p-q]}N'_{R,p-q-1}(\mathbf{u}) \le C_{\mathbf{u}}K^{p+2-q-1}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}, \qquad q = 1,\dots,p,$$

$$\frac{[p+1]}{[p-q+1]}N'_{R,p-q}(\mathbf{u}) \le C_{\mathbf{u}}K^{p+2-q}\frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}, \qquad q = 1,\dots,p+1,$$

Inserting all of the above in (D.31) yields[10] together with the geometric series

$$N'_{R,p}(\mathbf{u}) \le 3C_I\,(C_A\gamma_A R + C_d\gamma_d R + C_e\gamma_e R)\,N'_{R,p}(\mathbf{u}) + C_{\mathbf{u}}K^{p+2}\frac{\max\{p+3,R/\varepsilon\}^{p+2}}{p!}$$
$$\times 3C_I\Big[\frac{1}{4K}\left(\frac{\gamma_f}{2K}\right)^p + 2K^{-2}\left(\frac{\gamma_G}{2K}\right)^p + \frac{1}{4K^2}\frac{1}{1-(\gamma_c R/(2K))} + K^{-1}\frac{1+C_A\gamma_A\varepsilon}{1-R\gamma_{\widetilde{B}}/(2K)} \qquad \text{(D.36)}$$
$$+\frac{C_A\gamma_A R}{2K}\frac{1}{1-\gamma_A R/(2K)} + \frac{1}{2K^2}\frac{1}{1-\gamma_b R/(2K)} + \frac{1}{2K}\frac{1}{1-\gamma_b R/(2K)}$$
$$+\frac{2C_d\gamma_d R}{K}\frac{1}{1-\gamma_d R/(2K)} + \frac{2C_e\gamma_e R}{K}\frac{1}{1-\gamma_d R/(2K)} + K^{-1} + K^{-2}\Big] \qquad \text{(D.37)}$$

By the choice of K, the expression in brackets, $[\cdots]$, is smaller than $1/(6C_I)$ and by (D.28) the expression $3C_I(C_A\gamma_A R + C_d\gamma_d R) \le 1/2$. Hence, the induction step is completed. ■

D.2.2 Control of Normal Derivatives

Define

$$M'_{R,p,q}(\mathbf{v}) := \frac{1}{[p+q]!}\sup_{R/2\le r<R}(R-r)^{p+q+2}\|\partial_y^q\nabla_x^p\mathbf{v}\|_{L^2(B_r^+)}. \qquad \text{(D.38)}$$

Theorem D.5 *Let* $\mathbf{u}$ *satisfy (D.2) with coefficients and data satisfying (D.2) and (D.2). Let* $R\le 1$ *be such that, with the universal constant* C_I *given by Lemma D.3,*

$$(3C_I+6)\,(C_A\gamma_A + C_d\gamma_d + C_e\gamma_e)\,R \le \frac{1}{2}.$$

Then there are K_1, $K_2>1$ *depending only on the constants appearing in (D.2) and on* γ_f, γ_G *such that with* $C_{\mathbf{u}}$ *given by (D.29)*

$$N'_{R,p,q}(\mathbf{u}) \le C_{\mathbf{u}}K_1^{p+2}K_2^{q+2}\max\{R/\varepsilon,p+q+3\}^{p+q+2} \qquad \forall p\ge 0, q\ge -2 \text{ with } (p,q)\ne(0,-2). \qquad \text{(D.39)}$$

Proof. With K given by Lemma D.4, we select $K_1 = K$. We select

$$K_2 > \max\{1,\gamma_f/2,\gamma_A/2,\gamma_{\widetilde{B}}/2,\gamma_C/2\}$$

such that the expression in brackets, $[\cdots]$, in (D.45) is smaller than $1/12$. The proof is by induction on q. For $q\in\{-2,-1,0\}$ and all $p\in\mathbb{N}_0$ (with the exception of the excluded case $(q,p)=(-2,0)$) the result follows directly from Lemma D.4. Let us assume that (D.39) holds (for all $p\in\mathbb{N}_0$) up to $q-1$ for some $q\ge 1$. Starting point is the observation that for a smooth solution $\widetilde{\mathbf{u}}$ of

$$-\partial_y^2\widetilde{\mathbf{u}} = \Delta_x\widetilde{\mathbf{u}} + \widetilde{\mathbf{f}} \quad \text{on } B_R^+ \qquad \text{(D.40)}$$

we have by the definition of the seminorms $N'_{R,r,s}$, $M'_{R,r,s}$ the estimate

$$N'_{R,p,q}(\widetilde{\mathbf{u}}) \le 2\left[N'_{R,p+2,q-2}(\widetilde{\mathbf{u}}) + M'_{p,q}(\widetilde{\mathbf{f}})\right], \qquad p\ge 0, q\ge 0. \qquad \text{(D.41)}$$

The system (D.2.1) is of the form (D.40) with

$$\widetilde{\mathbf{f}}_i = \sum_{j=1}^{3}\widetilde{A}_{33}^{ij}\partial_y^2\mathbf{u}_j + \varepsilon^{-2}\mathbf{f}_i - \varepsilon^{-2}\sum_j C^{ij}\mathbf{u}_j - \varepsilon^{-1}\sum_{j,\beta}\widetilde{B}_\beta^{ij}\partial_\beta\mathbf{u}_j + \sum_{\substack{j,\alpha,\beta\\(\alpha,\beta)\ne(3,3)}}\widetilde{A}_{\alpha\beta}^{ij}\partial_\alpha\partial_\beta\mathbf{u}_j. \qquad \text{(D.42)}$$

We estimate

$$\varepsilon^{-2}M'_{R,p,q}(\mathbf{f}) \le \frac{C_f}{4}(\mathcal{E}/\varepsilon)^2\left(\frac{\gamma_f}{2}\right)^{p+q}\frac{\max\{p+q+3,R/\mathcal{E}\}^{p+q+2}}{[p+q]!},$$

[10] the factor 3 in $3C_I$ is due to the summation over i and likely suboptimal

$$\varepsilon^{-2}M'_{R,p,q}(\sum_j C^{ij}\mathbf{u}_j) \le C_C\varepsilon^{-2}\sum_{r=0}^{p}\sum_{s=0}^{q}\binom{p}{r}\binom{q}{s}\gamma_C^{r+s}\frac{s!r!}{(p+q)!}\sup_{R/2\le r<R}(R-r)^{p+q+2}\|\partial_y^{q-s}\nabla_x^{p-r}\mathbf{u}\|_{L^2(B_r^+)}$$

$$\le \frac{C_C}{4}(R/\varepsilon)^2\sum_{r=0}^{p}\sum_{s=0}^{q}\binom{p}{r}\binom{q}{s}\left(\frac{\gamma_C R}{2}\right)^{r+s}\frac{s!r![p+q-r-s-2]!}{(p+q)!}N'_{R,p-r,q-s-2}(\mathbf{u}),$$

$$\varepsilon^{-1}M'_{R,p,q}(\sum_{j,\beta}\widetilde{B}^{ij}_\beta\partial_\beta\mathbf{u}_j) \le$$

$$\frac{C_{\widetilde{B}}}{2}(R/\varepsilon)\sum_{r=0}^{p}\sum_{s=0}^{q}\binom{p}{r}\binom{q}{s}\left(\frac{\gamma_{\widetilde{B}}R}{2}\right)^{r+s}\frac{s!r![p+q-r-s-1]!}{(p+q)!}\left[N'_{R,p-r,q-s-1}(\mathbf{u})+N'_{R,p+1-r,q-s-2}(\mathbf{u})\right],$$

$$M'_{R,p,q}(\sum_{\substack{j,\alpha,\beta\\(\alpha,\beta)\ne(3,3)}}\widetilde{A}^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\mathbf{u}_j) \le$$

$$C_A\sum_{r=0}^{p}\sum_{s=0}^{q}\binom{p}{r}\binom{q}{s}\left(\frac{\gamma_A R}{2}\right)^{r+s}\frac{s!r![p+q-r-s]!}{(p+q)!}\left[N'_{R,p+2-r,q-s-2}(\mathbf{u})+N'_{R,p+1-r,q-s-1}(\mathbf{u})\right],$$

$$M'_{R,p,q}((\sum_j\widetilde{A}^{ij}_{33}\partial_3^2\mathbf{u}_j)_i) \le C_A\gamma_A R N'_{R,p,q}(\mathbf{u})+$$

$$C_A\sum_{(r,s)\ne(0,0)}\binom{p}{r}\binom{q}{s}\left(\frac{\gamma_A R}{2}\right)^{r+s}\frac{s!r![p+q-r-s]!}{(p+q)!}\left[N'_{R,p+2-r,q-s-2}(\mathbf{u})+N'_{R,p+1-r,q-s-1}(\mathbf{u})\right].$$

We introduce the abbreviation

$$m_{p,q} := \max\{p+q+3, R/\mathcal{E}\}^{p+q+2}. \tag{D.43}$$

so that, for $q' \le q-1$ the induction hypothesis reads $[p'+q''_{R,p',q'}(\mathbf{u}) \le C_\mathbf{u}K_1^{p'+2}K_2^{q'+2}m_{p',q'}$. We have

$$\varepsilon^{-2}M'_{R,p,q}(\mathbf{f}) \le \left[\frac{C_f(\mathcal{E}/\varepsilon)^2}{4K_1^2K_2^2}\left(\frac{\gamma_f}{2K_1}\right)^p\left(\frac{\gamma_f}{K_2}\right)^q\right]K_1^{p+2}K_2^{q+2}\frac{m_{p,q}}{(p+q)!}.$$

We recall the elementary estimates

$$\binom{p}{r}r! \le p^r, \qquad 0\le r\le p. \tag{D.44}$$

Recalling (D.18) we get from the induction hypothesis

$$\varepsilon^{-2}M'_{R,p,q}\Big(\sum_j C^{ij}\mathbf{u}_j\Big) \le C_\mathbf{u}\frac{C_C}{4}\sum_{r,s}\binom{p}{r}\binom{q}{s}\left(\frac{\gamma_C R}{2}\right)^{r+s}\frac{r!s!(R/\varepsilon)^2}{[p+q]!}K_1^{p-r+2}K_2^{q-s}m_{p-r,q-s-2}$$

$$\le C_\mathbf{u}\frac{1}{4}\sum_{r,s}\left(\frac{\gamma_C R}{2}\right)^{r+s}K_1^{p-r+2}K_2^{q-s}\frac{m_{p,q}}{[p+q]!},$$

$$\varepsilon^{-1}M'_{R,p,q}\Big(\sum_{j,\beta}\partial_\beta\widetilde{B}^{ij}_\beta\mathbf{u}_j\Big)$$

$$\le C_\mathbf{u}C_{\widetilde{B}}\sum_{r,s}\binom{p}{r}\binom{q}{s}\left(\frac{\gamma_{\widetilde{B}}R}{2}\right)^{r+s}\frac{r!s!R/\varepsilon}{[p+q]!}\left[K_1^{p-r+2}K_2^{q-s+1}m_{p-r,q-s-1}+K_1^{p-r+3}K_2^{q-s}m_{p+1-r,q-s-2}\right]$$

$$\le C_\mathbf{u}(1+C_A\gamma_A\varepsilon)\sum_{r,s}\left(\frac{\gamma_{\widetilde{B}}R}{2}\right)^{r+s}K_1^{p-r+2}K_2^{q+2-s}\frac{m_{p,q}}{[p+q]!}\left[K_2^{-1}+K_1K_2^{-2}\right],$$

$$M'_{R,p,q}\Big(\sum_{\substack{j,\alpha,\beta\\(\alpha,\beta)\ne(3,3)}}\widetilde{A}^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\mathbf{u}_j\Big)$$

$$\le C_\mathbf{u}C_A\sum_{r,s}\binom{p}{r}\binom{q}{s}\left(\frac{\gamma_A R}{2}\right)^{r+s}\frac{r!s!}{[p+q]!}\left[K_1^{p-r+4}K_2^{q-s}m_{p+2-r,q-s-2}+K_1^{p-r+3}K_2^{q-s+1}m_{p+1-r,q-s-1}\right]$$

$$\le C_\mathbf{u}C_A\sum_{r,s}\left(\frac{\gamma_A R}{2}\right)^{r+s}K_1^{p-r+2}K_2^{q+2-s}\frac{m_{p,q}}{[p+q]!}\left[K_1^2K_2^{-2}+K_1K_2^{-1}\right],$$

$$M'_{R,p,q}\Big(\sum_j \widetilde{A}^{ij}_{33}\partial_3^2 \mathbf{u}_j\Big) \leq C_A\gamma_A R N'_{R,p,q}(\mathbf{u}) + C_{\mathbf{u}} C_A \sum_{r,s} \left(\frac{\gamma_A R}{2}\right)^{r+s} K_1^{p-r+2} K_2^{q+2-s} \frac{m_{p,q}}{[p+q]!} \left[K_1^2 K_2^{-2} + K_1 K_2^{-1}\right].$$

Inserting these estimates in (D.41) gives

$$N'_{R,p,q}(\mathbf{u}) \leq 6C_A\gamma_A R N'_{R,p,q}(\mathbf{u}) + C_{\mathbf{u}} K_1^{p+2} K_2^{q+2} \frac{m_{p,q}}{(p+q)!} 6\Big[\tag{D.45}$$
$$\frac{1}{4K_1^2K_2^2}\left(\frac{\gamma_f}{2K_1}\right)^p \left(\frac{\gamma_f}{K_2}\right)^q + \frac{K_1^2}{4K_2^2} \frac{1}{1-\gamma_C R/(2K_1)} \frac{1}{1-\gamma_C R/(2K_2)}$$
$$+ (1 + C_A\gamma_A\varepsilon)(K_2^{-1} + K_1^2K_2^{-2}) \frac{1}{1-\gamma_{\widetilde{B}} R/(2K_1)} \frac{1}{1-\gamma_{\widetilde{B}} R/(2K_2)}$$
$$+ 2C_A(K_1^2K_2^{-2} + K_1K_2^{-1}) \frac{1}{1-\gamma_A R/(2K_1)} \frac{1}{1-\gamma_A R/(2K_2)} + K_1^2K_2^{-2}\Big]$$

By the choice of K_2, the expression in brackets, $[\cdots]$, is smaller than $1/12$ and by assumption on R, the term $C_A\gamma_A R \leq 1/2$. Hence, the induction step is completed. ■

E Analytic regularity for Poisson Problems

We consider, on the half-ball B_R^+, solutions u of

$$-\operatorname{div}(A(x)\nabla u) = f \quad \text{in } B_R^+, \qquad u|_{\Gamma_R} = 0. \tag{E.1}$$

Here, the matrix A is pointwise symmetric positive definite and satisfies

$$A \in \mathcal{A}^\infty(C_A, \gamma_A, B_R^+), \qquad A \geq \lambda_{min} > 0. \tag{E.2}$$

The data f is assumed to satisfy, for some $\varepsilon \in (0,1]$

$$\|\nabla^p f\|_{L^2(B_R^+)} \leq C_f \gamma_f^p \max\{p/R, \varepsilon^{-1}\}^p \qquad \forall p \in \mathbb{N}_0. \tag{E.3}$$

Note that this problem has been considered in [35, Lemma 5.5.15] where a recursion for the tangential derivatives, i.e., for the seminorm $N'_{R,p}(u)$, is derived. We use this result here to derive the following estimate.

Lemma E.1 *Assume (E.2) and (E.3) and $R \leq 1$. Then there exists $K > 0$ depending solely on λ_{min}, C_A, γ_A, γ_f such that a solution u of (E.1) satisfies*

$$N'_{R,p}(u) \leq K^{p+2}\left[C_f R^2 \frac{\max\{p+1, R/\varepsilon\}^p}{p!} + (p+1)R\|\nabla u\|_{L^2(B_R^+)}\right] \qquad \forall p \geq 0. \tag{E.4}$$

Additionally, $N'_{R,-1}(u) \leq R/2\|\nabla u\|_{L^2(B_R^+)}$.

Proof. The estimate $N'_{R,-1}(u) \leq R/2\|\nabla u\|_{L^2(B_R^+)}$ is a direct consequence of the definition. The case $p = 0$ follows directly from [35, Lemma 5.5.15]. The case $p = 1$ follows from an inspection of the arguments below. For $p \geq 2$, the proof is by induction on p, assuming that (E.4) holds for all $p' \leq p-1$ for some $p \geq 2$. From [35, Lemma 5.5.15] we get

$$N'_{R,p}(u) \leq C'_B\Big[C_f\left(\frac{R}{2}\right)^2\left(\frac{\gamma_f}{2}\right)^p \frac{\max\{p, R/\varepsilon\}^p}{p!} + C_A(p+1)\left(\frac{\gamma_A R}{2}\right)^{p+1} N'_{R,-1}(u) \tag{E.5}$$
$$+ C_A\sum_{q=1}^p \frac{(p+1)!}{(p+1-q)!}\left(\frac{\gamma_A R}{2}\right)^q \frac{(p-q)!}{p!} N'_{R,p-q}(u) + N'_{R,p-1}(u) + N'_{R,p-2}(u)\Big].$$

The induction hypothesis gives for $q = 1, \ldots, p$

$$\frac{(p+1)!}{(p+1-q)!}\frac{(p-q)!}{p!} N'_{R,p-q}(u) \leq K^{p+2-q}\left[C_f R^2 \frac{\max\{p+1, R/\varepsilon\}^p}{p!} + (p+1)R\|\nabla u\|_{L^2(B_R^+)}\right] =: K^{p+2-q} B_p,$$

where B_p abbreviates the expression in brackets, $\big[\cdots\big]$. Inserting the above and the induction hypothesis in (E.5) gives, assuming $\gamma_A R/(2K) < 1$,

$$N'_{R,p}(u) \le C'_B B_p \Big[\frac{1}{4K^2}\left(\frac{\gamma_f}{2K}\right)^p + C_A K^{-2}\left(\frac{\gamma_A R}{2K}\right)^p + C_A \frac{\gamma_A R}{2}\frac{1}{1-\gamma_A R/2} + K^{-1} + K^{-2}\Big]. \tag{E.6}$$

Selecting K sufficiently large shows that the factor $C'_B[\cdots]$ can be made smaller than 1, which concludes the induction argument. ■

Theorem E.2 *Assume (E.2) and (E.3) and $R \le 1$. Then there exist K_1, $K_2 \ge 1$ depending solely on λ_{min}, C_A, γ_A, γ_f such that a solution u of (E.1) satisfies, for all $p \ge 0$, $q \ge -2$ with $(p,q) \ne (0,-2)$*

$$N'_{R,p,q}(u) \le K_1^{p+2} K_2^{q+2}\left[C_f R^2 \frac{\max\{p+q+3, R/\varepsilon\}^{p+q}}{(p+q)!} + (p+q+3)R\|\nabla u\|_{L^2(B_R^+)}\right]. \tag{E.7}$$

Proof. We control the normal derivatives as in the proof of Theorem D.5. Inspection of the arguments leading to [35, (5.5.30)] shows that we have

$$-\partial_y^2 u = \widetilde{f} + \widetilde{A}\nabla u + B : \nabla^2 u, \tag{E.8}$$

where, for C', $\gamma > 0$ depending solely on λ_{min}, C_A, γ_A, γ_f

$$\|\nabla^p \widetilde{f}\|_{L^2(B_R^+)} \le C' C_f \gamma^p \max\{p/R, \varepsilon^{-1}\}^p \qquad \forall p \in \mathbb{N}_0, \tag{E.9}$$

$$\widetilde{A}, B \in \mathcal{A}^\infty(C', \gamma, B_R^+), \quad B_{33} \equiv 0. \tag{E.10}$$

We abbreviate

$$M_{p,q} := \left[C_f R^2 \frac{\max\{p+q+3, R/\varepsilon\}^{p+q}}{(p+q)!} + (p+q+3)R\|\nabla u\|_{L^2(B_R^+)}\right].$$

The proof is by induction on q, the cases $q \in \{-2,-1,0\}$ being shown in Lemma E.1 if we select $K_1 = K$ with K given by Lemma E.1. Assume that (E.7) holds for all $q' \le q-1$ for some $q \ge 0$ and all p. From (E.8) we get

$$N'_{R,p,q}(u) \le M'_{R,p,q}(\widetilde{f}) + M'_{R,p,q}(\widetilde{A}\nabla u) + M'_{R,p,q}(B\nabla^2 u), \tag{E.11}$$

$$M'_{R,p,q}(\widetilde{f}) \le \frac{C'}{4}\left(\frac{\gamma}{2}\right)^{p+q} M_{p,q}, \tag{E.12}$$

where the estimate for $M'_{R,p,q}(\widetilde{f})$ follows from a direct calculation. The terms $M'_{R,p,q}(\widetilde{A}\nabla u)$ and $M'_{R,p,q}(B\nabla^2 u)$ are treated as in the proof of Theorem D.5. First, we note that

$$r \mapsto \frac{[p-r-\alpha]!}{(p-r)!} \qquad \text{is monotone decreasing for } r \in \{0,\ldots,p\} \text{ and } \alpha \ge 0. \tag{E.13}$$

The induction hypothesis and (E.13) imply

$$\binom{p}{r}\binom{q}{s}\frac{s!r![p+q-r-s-1]!}{(p+q)!}\left[N'_{R,p-r,q-s-1}(u) + N'_{R,p+1-r,q-s-2}(u)\right] \le K_1^{p-r+2}K_2^{q-s+2}\frac{M_{p,q}}{p+q}\left[K_2^{-2} + K_1 K_2^{-1}\right]$$

so that we get, as in the treatment of the terms $M'_{R,p,q}(\widetilde{B}^{ij}_\beta \partial_\beta \mathbf{u}_j)$ in the proof of Theorem D.5,

$$M'_{R,p,q}(\widetilde{A}\nabla u) \le K_1^{p+2}K_2^{q+2} M_{p,q}\left[K_2^{-1} + K_1 K_2^{-2}\right]\frac{C'R}{p+q}\sum_{r,s}\left(\frac{\gamma R}{2K_1}\right)^r\left(\frac{\gamma R}{2K_2}\right)^s. \tag{E.14}$$

Analogously, as in the treating of the term $M'_{R,p,q}(\sum_{j,\alpha,\beta:(\alpha,\beta)\ne(3,3)} \widetilde{A}^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta \mathbf{u}_j)$ in the proof of Theorem D.5 we get for $M'_{R,p,q}(B : \nabla^2 u)$ by observing that the assumption $B_{33} \equiv 0$ allows us to invoke the induction hypothesis for all terms

$$M'_{R,p,q}(B : \nabla^2 u) \le K_1^{p+2}K_2^{q+2} M_{p,q}\left[K_1^2 K_2^{-2} + K_1 K_2^{-1}\right] C' \sum_{r,s}\left(\frac{\gamma R}{2K_1}\right)^r\left(\frac{\gamma R}{2K_2}\right)^s. \tag{E.15}$$

Hence, by selecting K_2 sufficiently large depending solely on C' and γ, the induction step is completed. ■

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