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J.M. Melenk, A. Rieder

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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

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hp-FEM for the fractional heat equation

Jens Markus Melenk* Alexander Rieder†

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We consider a time dependent problem generated by a nonlocal operator in space. Applying a discretization scheme based on hp-Finite Elements and a Caffarelli-Silvestre extension we obtain a semidiscrete semigroup. The discretization in time is carried out by using hp-Discontinuous Galerkin based timestepping. We prove exponential convergence for such a method in an abstract framework for the discretization in the original domain Ω.

1. Introduction

For stationary fractional diffusion, numerical techniques have recently been proposed that provide exponential convergence of the error with respect to the computational effort, [BMN+18, BMS19]. The construction is based on hp-Finite Elements on appropriate geometric meshes. The purpose of the present article is to generalize these techniques to the time dependent setting. We consider the discretization of the time dependent problem (2.1), generated by a fractional power of an elliptic operator. The spatial discretization of the nonlocal operator is based on a reformulation using the Caffarelli-Silvestre extension, for which an hp-finite element discretization (FEM) is employed. The discretization in time is then carried out by a Discontinuous Galerkin method in the spirit of [SS00] of either fixed order or in its hp version. Our analysis hinges on two conditions, one related to stable liftings of the initial condition and the second one related to the ability to approximate solutions of singularly perturbed problems.

After establishing an abstract framework, we work out the case of hp-FEM in one spatial dimension and for a special case of constant coefficients in full detail. The reduction of scope to 1D mainly is done to keep the presentation to a reasonable length; we expect that it is possible to establish the assumptions of the abstract framework also for the case $\Omega \subset \mathbb{R}^d$, $d > 1$. We point out how and where our proofs would need modifications. Discretization schemes for the same model problem have already appeared in the literature. In [BLP17], the approximation is done by applying numerical quadrature to the Dunford-Taylor representation of the solution and using a low-order finite element method in space. The idea of treating the extension problem via finite elements is already well established for the case of elliptic problems, e.g. [NOS15] for the low-order FEM or [MPSV18] as well as [BMN+18] for using hp-based discretizations. The use of an extension problem in order to discretize a time-dependent problem was used in [NOS16], focusing on low order finite elements and time-stepping schemes, but allowing also for fractional

*Institute for Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria. melenk@tuwien.ac.at

†Institute for Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria. alexander.rieder@tuwien.ac.at

time derivatives. In the context of wave equations, such a discretization was recently analyzed in [BO18].

When dealing with parabolic problems, it is well-known that, if the initial condition does not satisfy certain compatibility conditions, so called startup singularities form. They need to be accounted for in the numerical method. We rigorously prove that, as long as the meshes are designed in a proper way, our discretization scheme delivers exponential convergence rate for the spatial discretization and optimal convergence rate in time, i.e., optimal order for fixed order timestepping like implicit Euler and exponential convergence for the *hp*-DG based method.

The paper is structured as follows: Section 2 presents the model problem and the functional analytic setting. In Section 3, we then perform a first discretization step with respect to the spatial variables. This yields a continuous in time/discrete in space approximation. In order to prove exponential convergence for this discretization, we take a small detour in Section 3.1 to analyze an auxiliary elliptic problem. This problem will allow us to lift a representation formula from the domain Ω to the extended cylinder $\Omega \times \mathbb{R}_+$ while allowing to reuse the techniques developed in [BMN$^+$18]. These preparations then allow us to prove exponential convergence for their space discretization in Section 3.2. The discretization in time is then carried out in Section 4 yielding a fully discrete scheme. This scheme was implemented and Section 5 confirms the exponential convergence. The appendices provide results that could not readily be cited from the literature: Appendix B generalizes results on *hp*-FEM for singularly perturbed problems to the case of complex perturbation parameters. Appendix C is concerned with the lifting of polynomials in Ω to piecewise polynomials on the cylinder $\Omega \times \mathbb{R}_+$ in a stable way.

We also would like to point out that using the Caffarelli-Silvestre extension is not the only approach to discretize the nonlocal operator which is able to yield an exponentially convergent scheme. We mention schemes based on sinc-quadrature and the Balakrishnan or Riesz-Taylor formulations of the fractional Laplacian (see [BLP17]). We expect that it is possible to combine such a scheme with *hp*-FEM in the space discretization and by combining [BLP17] with the techniques laid out in this paper it should be possible to show exponential convergence.

We close with a remark on notation. We write $A \lesssim B$ to mean there exists a constant $C > 0$, which is independent of the main quantities of interest, i.e., mesh size or polynomial degree used, etc., such that $A \leq CB$. We write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. The exact dependencies of the implied constant is specified in the context.

2. Model problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We consider the following model problem for $s \in (0,1)$:

$$\dot{u}(t) + \mathcal{L}^s u(t) = f(t) \qquad \text{in } \Omega, \ \forall t > 0 \tag{2.1a}$$

$$u(\cdot, t) = 0 \qquad \text{on } \Gamma, \ \forall t > 0 \tag{2.1b}$$

with initial condition $u(0) = u_0$ and right-hand side $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}$. We assume that the initial condition and right-hand side are analytic but do not require any compatibility or boundary conditions.

The operator $\mathcal{L}u := -\operatorname{div}(A\nabla u) + cu$ is a linear, elliptic and self-adjoint differential operator, where we assume that $A \in L^\infty\big(\Omega, \mathbb{R}^{d\times d}\big)$ is uniformly SPD in Ω and $c \in L^\infty(\Omega)$ satisfies $c \geq 0$.

The fractional power $\mathcal{L}^s$ is defined using the spectral decomposition

$$\mathcal{L}^s u := \sum_{j=0}^{\infty} \mu_j^s (u, \varphi_j)_{L^2(\Omega)} \varphi_j, \tag{2.2}$$

where $(\mu_j, \varphi_j)_{j\in\mathbb{N}_0}$ are eigenvalues and eigenfunctions of the operator $\mathcal{L}$ with homogeneous Dirichlet boundary conditions.

Using the Caffarelli-Silvestre extension one can localize the nonlocal operator $\mathcal{L}^s$ and rewrite (2.1) in the following form with $\alpha := 1 - 2s$:

$$-\operatorname{div}(y^\alpha A \nabla \mathscr{U}) + y^\alpha c \mathscr{U} = 0 \qquad \text{on } \mathcal{C} \times (0,T), \tag{2.3a}$$

$$d_s \operatorname{tr} \dot{\mathscr{U}} + \partial_\nu^\alpha \mathscr{U} = d_s f \qquad \text{on } \Omega \times \{0\} \times (0,T), \tag{2.3b}$$

$$\mathscr{U} = 0 \qquad \text{on } \partial_L \mathcal{C} \times (0,T). \tag{2.3c}$$

Here $\mathcal{C}$ denotes the cylinder $\Omega \times \mathbb{R}_+$, $d_s := 2^\alpha \Gamma(1-s)/\Gamma(s)$. The lateral boundary is defined as $\partial_L \mathcal{C} := \partial\Omega \times \mathbb{R}_+$ and

$$\partial_\nu^\alpha \mathscr{U} := -\lim_{y\to 0^+} y^\alpha \partial_y \mathscr{U}(\cdot, y), \qquad \text{and} \qquad \operatorname{tr} \mathscr{U} := \mathscr{U}(\cdot, 0)$$

is the conormal derivative and boundary trace at $y = 0$ respectively. The connection to u is then given by $\operatorname{tr} \mathscr{U}(t) = u(t)$.

In order to treat this extended problem, we introduce the following weighted Sobolev spaces:

$$H^1(y^\alpha, D) := \left\{ w \in L^2(y^\alpha, D) : |\nabla w| \in L^2(y^\alpha, D) \right\}, \tag{2.4}$$

$$\mathring{H}^1(y^\alpha, D) := \left\{ w \in H^1(y^\alpha, D) : u = 0 \text{ on } \partial_L \mathcal{C} \right\}. \tag{2.5}$$

The space $\mathring{H}^1(y^\alpha, \mathcal{C})$ is equipped with the norm $\|\mathscr{U}\|^2_{\mathring{H}^1(y^\alpha,\mathcal{C})} := \int_{\mathcal{C}} y^\alpha |\nabla \mathscr{U}|^2$.

We also define the bilinear form corresponding to the weak form of (2.3a) as:

$$\mathcal{A}(\mathscr{U}, \mathscr{V}) := \int_{\mathcal{C}} y^\alpha \big(A \nabla \mathscr{U}\big) \cdot \nabla \mathscr{V} + y^\alpha c\, \mathscr{U} \mathscr{V}.$$

Throughout this paper, we will make use of fractional Sobolev and interpolation spaces. We define for two Banach spaces $X_1 \subseteq X_0$ and $\theta \in (0,1)$:

$$\|u\|^2_{[X_0,X_1]_{\theta,2}} := \int_{t=0}^{\infty} t^{-2\theta} \left(\inf_{v\in X_1} \|u - v\|_0 + t\|v\|_1 \right)^2 \frac{dt}{t},$$

$$[X_0, X_1]_{\theta,2} := \Big\{ u \in X_0 : \|u\|_{[X_0,X_1]_{\theta,2}} < \infty \Big\}.$$

For the endpoints we set $[X_0, X_1]_{0,2} := X_0$ and $[X_0, X_1]_{1,2} := X_1$. Fractional Sobolev spaces are defined as

$$H^s(\Omega) := \big[L^2(\Omega), H^1(\Omega) \big]_{s,2},$$

and the spaces with zero boundary conditions are defined as

$$\widetilde{H}^s(\Omega) := \big[L^2(\Omega), H^1_0(\Omega) \big]_{s,2}.$$

The boundary condition in (2.1) is understood in the sense of $u(t) \in \widetilde{H}^s(\Omega)$ for all $t > 0$. That is, for $s < 1/2$ no boundary condition is imposed, while for $s > 1/2$ it is imposed in the sense of traces. For $s = 1/2$ the boundary condition is imposed as membership in the Lions-Magenes space, often also denoted $H^{1/2}_{00}(\Omega)$.

Sometimes it is useful to work with a different scale of spaces, characterized using the eigendecomposition of $\mathcal{L}$, as

$$\mathbb{H}^s(\Omega) := \left\{ u \in L^2(\Omega) : \sum_{j=0}^{\infty} \mu_j^s |(u, \varphi_j)_{L^2(\Omega)}|^2 < \infty \right\}.$$

For $s \in [0, 1]$, the spaces coincide, i.e., $\widetilde{H}^s(\Omega) = \mathbb{H}^s(\Omega)$ with equivalent norms.

We consider the discretization in two separate steps. We semidiscretize in space and subsequently discretize in time, i.e.,

1. discretize in space using tensor product hp-FEM in Ω and the artificial variable y,

2. discretize in time by a discontinuous Galerkin method.

3. Discretization in space – the semidiscrete scheme

In this section we investigate the convergence of a semidiscrete semigroup to the solution of (2.1). We consider finite dimensional subspaces $\mathbb{V}_h^{\mathcal{X}} \subseteq H_0^1(\Omega)$ and $\{0\} \neq \mathbb{V}_h^{\mathcal{Y}} \subseteq H^1(y^\alpha, \mathbb{R}_+)$, and set $\mathbb{V}_h^{\mathcal{X},\mathcal{Y}} := \mathbb{V}_h^{\mathcal{X}} \otimes \mathbb{V}_h^{\mathcal{Y}} \subseteq \mathring{H}^1(y^\alpha, \mathcal{C})$ as our approximation space. We will keep most of our analysis as general as possible, but will provide concrete examples on how to implement these spaces in Sections 3.1.1 and 3.1.2. Throughout the paper, we will write

$$\mathcal{N}_\Omega := \dim(\mathbb{V}_h^{\mathcal{X}}) \qquad \text{and} \qquad \mathcal{N}_\Omega := \dim(\mathbb{V}_h^{\mathcal{Y}}).$$

While we will give a detailed construction of $\mathbb{V}_h^{\mathcal{Y}}$ later on, for now we just assume that there exists $v \in \mathbb{V}_h^{\mathcal{Y}}$ with $v(0) = 1$ in order to be able to solve Dirichlet problems.

We define the Galerkin approximation $\mathcal{L}_h^s : \mathbb{V}_h^{\mathcal{X}} \to \mathbb{V}_h^{\mathcal{X}}$ to the operator $\mathcal{L}^s$ via the relation:

$$(\mathcal{L}_h^s u, v)_{L^2(\Omega)} := \frac{1}{d_s} \mathcal{A}(\mathscr{L}_h u, \mathscr{L}_h v), \tag{3.1}$$

where $\mathscr{L}_h : \mathbb{V}_h^{\mathcal{X}} \to \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ denotes the solution to the following "lifting problem":

$$\mathcal{A}(\mathscr{L}_h u, \mathscr{V}_h) = 0 \qquad \forall \mathscr{V}_h \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}} \text{ s.t. } \operatorname{tr} \mathscr{V}_h = 0, \tag{3.2a}$$
$$\operatorname{tr} \mathscr{L}_h u = u. \tag{3.2b}$$

We also introduce the notation $\mathscr{L}$ for the solution to

$$\mathcal{A}(\mathscr{L} u, \mathscr{V}) = 0 \qquad \forall \mathscr{V} \in \mathring{H}^1(y^\alpha, \mathcal{C}) \text{ s.t. } \operatorname{tr} \mathscr{V} = 0, \tag{3.3a}$$
$$\operatorname{tr} \mathscr{L} u = u. \tag{3.3b}$$

Remark 3.1. *We note that by [NOS15, Proposition 2.5] and the ellipticity of $\mathcal{A}$, the operator $\mathscr{L}$ is bounded with respect to the $\widetilde{H}^s(\Omega) \to \mathring{H}^1(y^\alpha, \mathcal{C})$-norm. It is non-trivial to show that $\mathscr{L}_h$ is is bounded, especially for anisotropic meshes. See Appendix C for this result in a simplified setting. See also [MKR18] for a related problem.*

Theorem 3.2. *The operator $-\mathcal{L}_h^s$ is the generator of an analytic semigroup on $\left(\mathbb{V}_h^{\mathcal{X}}, \|\cdot\|_{L^2(\Omega)}\right)$.*

Proof. The operator $\mathcal{L}_h^s$ is symmetric due to the symmetry of $\mathcal{A}$. By [Paz83, Section 2.5, Theorem 5.2], it remains to show the estimate

$$\left\|(\lambda \mathrm{I} + \mathcal{L}_h^s)^{-1} f\right\|_{L^2(\Omega)} \leq \frac{M}{1+|\lambda|} \|f\|_{L^2(\Omega)}$$

for $\mathrm{Re}(\lambda) \geq 0$ and a constant M, independent of u and λ. It is easy to see that $(\lambda \mathrm{I} + \mathcal{L}_h^s)^{-1} f = \operatorname{tr} \mathscr{U}_\lambda$ where $\mathscr{U}_\lambda \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ solves

$$(\lambda d_s \operatorname{tr} \mathscr{U}_\lambda, \operatorname{tr} \mathscr{V}_h)_{L^2(\Omega)} + \mathcal{A}(\mathscr{U}_\lambda, \mathscr{V}_h) = (d_s f, \operatorname{tr} \mathscr{V}_h)_{L^2(\Omega)} \qquad \forall \mathscr{V}_h \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}.$$

Existence of the inverse follows from the coercivity of the bilinear form on the left-hand side. The *a priori* estimate follows by testing with $\mathscr{V}_h := \mathscr{U}_\lambda$ to get:

$$\lambda d_s \|\operatorname{tr} \mathscr{U}_\lambda\|^2 + \mathcal{A}(\mathscr{U}_\lambda, \mathscr{U}_\lambda) \leq d_s \|f\|_{L^2(\Omega)} \|\operatorname{tr} \mathscr{U}_\lambda\|_{L^2(\Omega)}.$$

Since $\mathcal{A}(\mathscr{U}_\lambda, \mathscr{U}_\lambda)$ is non-negative this concludes the proof for $\lambda \neq 0$. For $\lambda = 0$ we use the continuity of the trace operator $\|u\|_{L^2(\Omega)} \lesssim \|\mathscr{U}_\lambda\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim \|f\|_{L^2(\Omega)}$. □

Lemma 3.3. *If we equip the space $\mathbb{V}_h^{\mathcal{X}}$ with the norm*

$$\|u\|_{\mathbb{V}_h^{\mathcal{X}}} := \|\mathscr{L}_h u\|_{\mathring{H}^1(y^\alpha,\mathcal{C})}, \tag{3.4}$$

the operator $\mathcal{L}_h^s$ is elliptic, i.e.,

$$c_1 \|u\|_{\mathbb{V}_h^{\mathcal{X}}}^2 \leq (\mathcal{L}_h^s u, u)_{L^2(\Omega)} \leq c_2 \|u\|_{\mathbb{V}_h^{\mathcal{X}}}^2.$$

We also have the following estimate of the $\widetilde{H}^s(\Omega)$-norm:

$$c_3 \|u\|_{\widetilde{H}^s(\Omega)}^2 \leq (\mathcal{L}_h^s u, u)_{L^2(\Omega)}.$$

The constants c_i are independent of the spaces $\mathbb{V}_h^{\mathcal{Y}}$ and $\mathbb{V}_h^{\mathcal{Y}}$ and depend only on Ω, α, and $\mathcal{L}$.

Proof. By the trace estimate [NOS15, Proposition 2.5], we get

$$\|u\|_{\widetilde{H}^s(\Omega)}^2 \lesssim \|\mathscr{L}_h u\|_{\mathring{H}^1(y^\alpha,\mathcal{C})}^2 \lesssim \mathcal{A}(\mathscr{L}_h u, \mathscr{L}_h u) = d_s (\mathcal{L}_h^s u, u)_{L^2(\Omega)}.$$

On the other hand we get:

$$(\mathcal{L}_h^s u, u)_{L^2(\Omega)} = d_s^{-1} \mathcal{A}(\mathscr{L}_h u, \mathscr{L}_h u) \lesssim \|\mathscr{L}_h u\|_{\mathring{H}^1(y^\alpha,\mathcal{C})}^2 = \|u\|_{\mathbb{V}_h^{\mathcal{X}}}^2.$$ □

The operator $\mathcal{L}_h^s$ gives rise to the semidiscrete problem posed in $\mathbb{V}_h^{\mathcal{X}}$:

$$\dot{u}_h + \mathcal{L}_h^s u_h = \Pi_{L^2} f, \tag{3.5a}$$
$$u_h(0) = u_{h,0}, \tag{3.5b}$$

where $\Pi_{L^2} : L^2(\Omega) \to \mathbb{V}_h^{\mathcal{X}}$ denotes the L^2-orthogonal projection and $u_{h,0} \in \mathbb{V}_h^{\mathcal{X}}$ denotes some approximation to the initial condition.

By Duhamel's principle, u and u_h can be written as

$$u(t) = \mathcal{E}(t)u_0 + \int_0^t \mathcal{E}(\tau)f(t-\tau)\,d\tau \quad \text{and} \quad u_h(t) = \mathcal{E}_h(t)u_0 + \int_0^t \mathcal{E}_h(\tau)f(t-\tau)\,d\tau,$$

where $\mathcal{E} : \mathbb{R}_+ \to \mathscr{B}(L^2(\Omega), L^2(\Omega))$ and $\mathcal{E}_h : \mathbb{R}_+ \to \mathscr{B}(\mathbb{V}_h^{\mathcal{X}}, \mathbb{V}_h^{\mathcal{X}})$ are the semigroups generated by $-\mathcal{L}^s$ and $-\mathcal{L}_h^s$ respectively.

When considering the discrete flow for initial conditions without compatibility conditions, the right spaces will be the following:

Definition 3.4. *Let $\beta \in (0,1)$. Recall that the space $\mathbb{V}_h^{\mathcal{X}}$ is equipped with the norm $\|u\|_{\mathbb{V}_h^{\mathcal{X}}} := \|\mathscr{L}_h u\|_{\mathring{H}^1(y^\alpha,\mathcal{C})}$. We define the interpolation spaces*

$$\mathbb{V}_{h,\beta}^{\mathcal{X}} := \left[\left(\mathbb{V}_h^{\mathcal{X}}, \|\cdot\|_{L^2(\Omega)}\right), \left(\mathbb{V}_h^{\mathcal{X}}, \|\cdot\|_{\mathbb{V}_h^{\mathcal{X}}}\right)\right]_{\beta,2}.$$

We employ the convention $\|\cdot\|_{\mathbb{V}_{h,0}^{\mathcal{X}}} = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{\mathbb{V}_{h,1}^{\mathcal{X}}} = \|\cdot\|_{\mathbb{V}_h^{\mathcal{X}}}$ for the endpoints.

Throughout this paper, we will work with abstract spaces $\mathbb{V}_h^{\mathcal{X}}$. Exponential convergence of the numerical method relies on the following Assumptions 3.5, 3.9:

Assumption 3.5. *There exist constants β, b, $\mu > 0$, such that for all u_0 that are analytic on a fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, there exists a function $u_{h,0} \in \mathbb{V}_h^{\mathcal{X}}$ and a constant $C > 0$ such that*

$$\|u_{h,0}\|_{\mathbb{V}_{h,\beta}^{\mathcal{X}}} \le C\,\|u_0\|_{H^{s\beta}(\Omega)} \qquad \text{and} \qquad \|u_0 - u_{h,0}\|_{L^2(\Omega)} \le Ce^{-b\mathcal{N}_\Omega^\mu},$$

where $\mathcal{N}_\Omega := \dim\left(\mathbb{V}_h^{\mathcal{X}}\right)$.

When considering the Riesz-Dunford representation of u, the contour lies in the set of values for which $\mathcal{L} - z$ is elliptic. Therefore we consider the set of complex numbers up to a cone containing the part of the positive real axis for which $\mathcal{L} - z$ is no longer elliptic.

Definition 3.6. *With the Poincaré constant C_P of Ω and fixed $0 < \varepsilon_0 < z_0 \le \min\left(\frac{1}{2C_P}, 1\right)^2$, we define*

$$\mathscr{S} := \mathbb{C} \setminus \left[\left\{z_0 + z : |\mathrm{Arg}(z)| \le \frac{\pi}{8}, \mathrm{Re}(z) \ge 0\right\} \cup B_{\varepsilon_0}(0)\right].$$

Remark 3.7. *The set $\mathscr{S}$ is chosen in such a way that it contains the contour $\mathcal{C}$ used in the proof of Theorem 3.23. Namely, it contains the rays $re^{i\pi/4}$, $re^{-i\pi/4}$ as well as the circular arc $z_0e^{i\theta}$ connecting the two rays. The ball $B_\varepsilon(0)$ is removed in order to avoid problems at 0 when dividing by z. See Figure 3.1.*

Definition 3.8. *A function $f : [0,T] \to L^\infty(\Omega)$ is said to be* uniformly analytic *if:*

(i) For all $t \in [0,T]$, $f(t)$ is analytic in a fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$,

(ii) there exist constants $C_f, \gamma_f > 0$, the analyticity constants of f, *such that for all $t \in [0,T]$ and $p \in \mathbb{N}_0$,*

$$\|\nabla^p f(t)\|_{L^\infty(\widetilde{\Omega})} \le C_f \gamma^p p!\,.$$

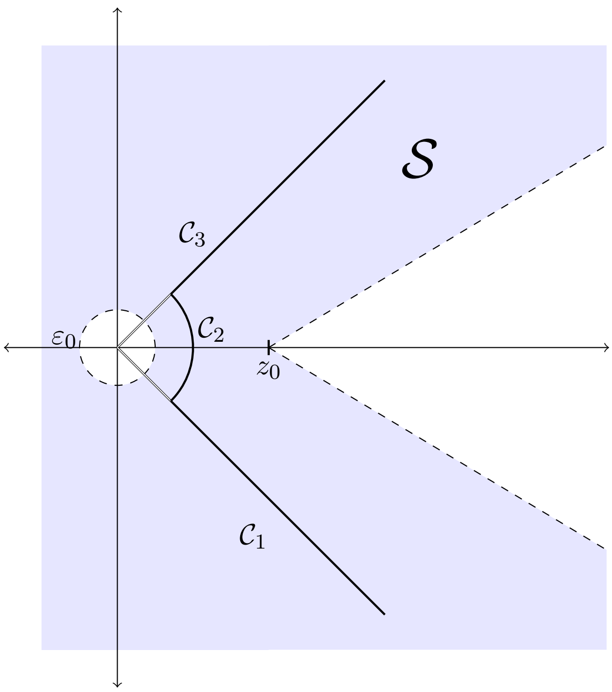


Figure 3.1: Geometric configuration of Definition 3.6

The second assumption we have to make is that for a certain class of singularly perturbed elliptic problems, the solution can be approximated exponentially well. We formalize this as follows.

Assumption 3.9. *A function space $\mathbb{V}_h^{\mathcal{X}}$ is said to resolve the scale $\varepsilon > 0$ if for all $z \in \mathscr{S}$ with $|z|^{-1/2} \geq \varepsilon$ and for all $f \in L^2(\Omega)$ that are analytic on a fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, the solutions to the elliptic problem*

$$-z^{-1}\mathcal{L}u + u = f$$

can be approximated exponentially well from it. That is, there exist constants $C(f)$, b and $\mu > 0$ such that

$$\inf_{v_h \in \mathbb{V}_h^{\mathcal{X}}} \left[|z|^{-1} \|\nabla u - \nabla v\|_{L^2(\Omega)}^2 + \|u - v\|_{L^2(\Omega)}^2 \right] \lesssim C(f) e^{-b\mathcal{N}_\Omega^\mu},$$

where $\mathcal{N}_\Omega := \dim(\mathbb{V}_h^{\mathcal{X}})$. The constant $C(f)$ may depend only on $\widetilde{\Omega}$, the analyticity constants of f, on A, c, Ω, z_0 and ε_0, while the constants b and μ depend only on A, c, $\widetilde{\Omega}$, Ω, z_0 and ε_0 Most notably the constants are independent of z, ε and $\mathcal{N}_\Omega$.

For simplicity of notation, we assume that the constants b and μ in Assumption 3.5 and 3.9 coincide. All our results will hold for general spaces $\mathbb{V}_h^{\mathcal{X}}$, as long as they resolve specific scales. We will later provide a concrete example of constructing such spaces in $1D$, see also [BMN$^+$18] and [BMS19].

The next lemma collects some facts about the time evolution. These results are well known for the case of the heat equation, and their proof easily carries over to our setting.

Lemma 3.10. *The following statements hold for the continuous and the semidiscrete problems:*

(i) The maps $t \mapsto u(t)$ and $t \mapsto u_h(t)$ are in $C\big([0,\infty), L^2(\Omega)\big)$.

(ii) For all $t > 0$ and $\ell \in \mathbb{N}_0$, $\beta \in (0,1)$ and $\gamma \in [0,1]$ such that $2\ell + \gamma - \beta \geq 0$,

$$\|\mathcal{E}(t)u_0\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} \qquad \text{and} \qquad \left\|[\mathcal{E}(\cdot)u_0]^{(\ell)}(t)\right\|_{\widetilde{H}^{s\gamma}(\Omega)} \lesssim t^{-\ell+\frac{\beta-\gamma}{2}} \|u_0\|_{\widetilde{H}^{s\beta}(\Omega)},$$

provided that the right-hand side is finite.

(iii) In the discrete setting, these estimates read as

$$\|\mathcal{E}_h(t)u_0\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} \qquad \text{and} \qquad \left\|[\mathcal{E}_h(\cdot)u_0]^{(\ell)}(t)\right\|_{\widetilde{H}^{s\gamma}(\Omega)} \lesssim t^{-\ell+\frac{\beta-\gamma}{2}} \|u_0\|_{\mathbb{V}^{\mathcal{X}}_{h,\beta}},$$

provided that the right-hand side is finite.

(iv) Set $w_h := \int_0^t \mathcal{E}_h(\tau)\Pi_{L^2} f(t-\tau)\, d\tau$. Then the following estimates hold:

$$\|w_h(t)\|^2_{L^2(\Omega)} \lesssim t \int_0^t \|\Pi_{L^2} f(\tau)\|^2_{L^2(\Omega)}\, d\tau \ \ \text{and} \ \int_0^t \|\dot{w}_h(\tau)\|^2_{L^2(\Omega)}\, d\tau \lesssim \int_0^t \|\Pi_{L^2} f(\tau)\|^2_{L^2(\Omega)}\, d\tau.$$

Proof. Statement (i) is one of the defining properties of a C_0-semigroup. Thus it follows from Theorem 3.2.

Proof of (ii): Using the representation (2.2), we write $u(t) = \sum_{j=0}^{\infty} e^{-\mu_j^s t} (u_0, \varphi)_{L^2(\Omega)} \varphi_j$. This allows us to estimate:

$$\begin{aligned}\left\|u^{(\ell)}(t)\right\|^2_{\widetilde{H}^{s\gamma}(\Omega)} &= \sum_{j=0}^{\infty} \mu_j^{s\gamma} \mu_j^{2s\ell} e^{-2\mu_j^s t} \left|(u_0, \varphi_j)_{L^2(\Omega)}\right|^2 \\ &\lesssim t^{-2\ell+\beta-\gamma} \sum_{j=0}^{\infty} (t\mu_j^s)^{2\ell-\beta+\gamma} e^{-2\mu_j^s t} \mu_j^{s\beta} \left|(u_0, \varphi_j)_{L^2(\Omega)}\right|^2 \leq t^{-2\ell+\beta-\gamma} \|u_0\|^2_{\widetilde{H}^{s\beta}(\Omega)},\end{aligned}$$

where, in the last step, we used $\sup_{x>0} x^{2\ell-\beta+\gamma} e^{-2x} < \infty$ as long as $2\ell - \beta + \gamma \geq 0$. The L^2-estimate of (ii) is just a special case with $\ell = \beta = \gamma = 0$.

For the semidiscrete semigroup T_h in (iii), the same calculation can be done. We use a basis of eigenvectors of the operator $\mathcal{L}_h^s$, denoted by $(\widetilde{\varphi}_j)_{j\in\mathbb{N}_0}$, instead of $(\varphi_j)_{j\in\mathbb{N}_0}$, and replace μ_j^s with the eigenvalue $\widetilde{\mu}_j$. What needs to be shown is the final norm equivalence

$$\sum_{j=0}^{\infty} \widetilde{\mu}_j^{\beta} \left|(u_0, \widetilde{\varphi}_j)_{L^2(\Omega)}\right|^2 \lesssim \|u_0\|^2_{\mathbb{V}^{\mathcal{X}}_{h,\beta}}.$$

The case $\beta = 0$ is clear. In the case $\beta = 1$, we get

$$\sum_{j=0}^{\infty} \widetilde{\mu}_j \left|(u_0, \widetilde{\varphi}_j)_{L^2(\Omega)}\right|^2 = (\mathcal{L}_h^s u_0, u_0)_{L^2(\Omega)} \leq \|u_0\|^2_{\mathbb{V}^{\mathcal{X}}_h},$$

where in the last step we used Lemma 3.3. The general case then follows by interpolation.

Proof of (iv): We use an energy argument. The function w_h solves $\dot{w}_h + \mathcal{L}_h^s w_h = f$ with $w_h(0) = 0$. Testing this equation with w_h gives:

$$\frac{1}{2}\frac{d}{dt} \|w_h(t)\|^2_{L^2(\Omega)} + (\mathcal{L}_h^s w_h(t), w_h(t))_{L^2(\Omega)} = (\Pi_{L^2} f(t), w_h(t))_{L^2(\Omega)}.$$

Integrating and using the fact that $\mathcal{L}_h^s$ is nonnegative then gives the first estimate after applying Gronwall's Lemma. For the second estimate, we test with $\dot{w}_h$ and get due to the symmetry of $\mathcal{L}_h^s$:

$$\|\dot{w}_h\|_{L^2(\Omega)}^2 + \frac{1}{2}\frac{d}{dt}\left(\mathcal{L}_h^s w_h, w_h\right)_{L^2(\Omega)} = \left(\Pi f(t), \dot{w}_h\right)_{L^2(\Omega)},$$

or, after integrating and applying Cauchy-Schwarz:

$$\int_0^t \|\dot{w}_h(\tau)\|_{L^2(\Omega)}^2 \, d\tau + \frac{1}{2}\left(\mathcal{L}_h^s w_h, w_h\right)_{L^2(\Omega)} \leq \left(\int_0^t \|\Pi_{L^2} f(\tau)\|_{L^2(\Omega)}^2 \, d\tau\right)^{1/2} \left(\int_0^t \|\dot{w}_h(\tau)\|_{L^2(\Omega)}^2 \, d\tau\right)^{1/2}.$$

Again using the fact that $\mathcal{L}_h^s$ is non-negative concludes the proof. □

Corresponding to the operator $\mathcal{L}_h^s$, we define the Ritz approximation $\Pi_h : \operatorname{dom}(\mathcal{L}^s) \to \mathbb{V}_h^{\mathcal{X}}$ via

$$\left(\mathcal{L}_h^s \Pi_h u, v\right)_{L^2(\Omega)} = \left(\mathcal{L}^s u, v\right)_{L^2(\Omega)} \qquad \forall v \in \mathbb{V}_h^{\mathcal{X}}. \tag{3.6}$$

(Note: unlike in the heat equation case, the operator Π_h is not a projection). Since the bilinear form on the left-hand side is elliptic by Lemma 3.3 and $(\mathcal{L}^s u, v)_{L^2(\Omega)}$ is a linear functional in v, $\Pi_h u$ exists and is well defined. (Since $\mathbb{V}_h^{\mathcal{X}}$ is finite dimensional we do not have to worry about the norms involved.)

Lemma 3.11. *Let u solve* (2.1)*, and u_h solve* (3.5)*. Define $\rho := u - \Pi_h u$ and $\theta := \Pi_h u - u_h$. Then θ satisfies the following semidiscrete equation for all $t > 0$:*

$$\dot{\theta}(t) + \mathcal{L}_h^s \theta(t) = \dot{\rho}(t), \qquad \theta(0) = u0 - u_{h,0}. \tag{3.7}$$

Proof. We compute for $v_h \in \mathbb{V}_h^{\mathcal{X}}$, ignoring the dependence on t in the notation:

$$\begin{aligned}\left(\frac{d}{dt}\left(\Pi_h u\right), v_h\right)_{L^2(\Omega)} + \left(\mathcal{L}_h^s \Pi_h u, v_h\right)_{L^2(\Omega)} &= \left(\Pi_h \dot{u} - \dot{u}, v_h\right)_{L^2(\Omega)} + \left(\dot{u}, v_h\right)_{L^2(\Omega)} + \left(\mathcal{L}^s u, v_h\right)_{L^2(\Omega)} \\ &= -\left(\dot{\rho}, v_h\right)_{L^2(\Omega)} + \left(f, v_h\right)_{L^2(\Omega)}.\end{aligned}$$

Since $v_h \in \mathbb{V}_h^{\mathcal{X}}$, we can replace f with $\Pi_{L^2} f$. Subtracting this from (3.5) then gives (3.7). □

The following proposition holds:

Proposition 3.12. *Let u solve* (2.1)*, and u_h solve* (3.5)*. Define $\rho := u - \Pi_h u$ and $\theta := \Pi_h u - u_h$. Then the following estimates hold for all $t > 0$:*

$$\int_0^t \|\theta(\tau)\|_{L^2(\Omega)}^2 \, d\tau \lesssim t \left\|u_0 - u_{h,0}\right\|_{L^2(\Omega)}^2 + \int_0^t \|\rho(\tau)\|_{L^2(\tau)}^2 \, d\tau, \tag{3.8}$$

$$\begin{aligned} t \|\theta(t)\|_{L^2(\Omega)}^2 + \int_0^t \tau \|\theta(\tau)\|_{\widetilde{H}^s(\Omega)}^2 \, d\tau \lesssim t \left\|u_0 - u_{h,0}\right\|_{L^2(\Omega)}^2 + \int_0^t \tau^2 \|\dot{\rho}(\tau)\|_{L^2(\Omega)}^2 + \|\rho\|_{L^2(\Omega)}^2 \, d\tau \\ + \sup_{\tau \in (0,t)} \left(\tau \|\rho(\tau)\|_{L^2(\Omega)}^2\right), \end{aligned} \tag{3.9}$$

Proof. These estimates are well known for the case of the heat equation. Similar results and techniques can be found, for example, in [Tho06, Chapter 3]. The use of the backward parabolic problem goes back at least to [LR82]. For completeness, we provide a proof in Appendix A. □

The previous results mean that it is sufficient to analyze the behavior of the Ritz approximation when applied to u. We start this endeavor by showing that the Ritz approximation is quasi-optimal.

Lemma 3.13. *Let $u \in \operatorname{dom}(\mathcal{L}^s)$, and let $\mathscr{L}u$ denotes its lifting to $\mathring{H}^1(y^\alpha, \mathcal{C})$ defined in (3.3). Then the following estimate holds:*

$$\|u - \Pi_h u\|_{\widetilde{H}^s(\Omega)} \lesssim \inf_{\mathscr{V}_h \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}} \|\mathscr{L}u - \mathscr{V}_h\|_{\mathring{H}^1(y^\alpha,\mathcal{C})}.$$

Proof. We set $u_h := \Pi_h u$, and show Galerkin orthogonality $\mathcal{A}(\mathscr{L}u - \mathscr{L}_h u_h, \mathscr{V}_h) = 0$ for all $\mathscr{V}_h \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$. We first note that $\mathcal{A}(\mathscr{L}u, \mathscr{V}_h)$ and $\mathcal{A}(\mathscr{L}_h u_h, \mathscr{V}_h)$ depend only on the trace of $\mathscr{V}_h$. By the definition of the liftings (see (3.2a) and (3.3a) respectively), we have for $\mathscr{W}_h \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ with $\operatorname{tr} \mathscr{W}_h = \operatorname{tr} \mathscr{V}_h$:

$$\mathcal{A}(\mathscr{L}u, \mathscr{V}_h - \mathscr{W}_h) = 0 \qquad \text{and} \qquad \mathcal{A}(\mathscr{L}_h u_h, \mathscr{V}_h - \mathscr{W}_h) = 0.$$

Therefore, we get by inserting the definition of $u_h = \Pi_h u$ and (3.6):

$$\begin{aligned}\mathcal{A}(\mathscr{L}u - \mathscr{L}_h u_h, \mathscr{V}_h) &= \mathcal{A}(\mathscr{L}u - \mathscr{L}_h u_h, \mathscr{L}_h \operatorname{tr} \mathscr{V}_h) \\ &= \mathcal{A}(\mathscr{L}u, \mathscr{L}_h \operatorname{tr} \mathscr{V}_h) - \mathcal{A}(\mathscr{L}u, \mathscr{L} \operatorname{tr} \mathscr{V}_h) = 0.\end{aligned}$$

The approximation result then follows easily from the boundedness of the trace operator and the ellipticity of $\mathcal{A}$. □

The combination of Proposition 3.12 and Lemma 3.13 shows that we need to study the best approximation of $\mathscr{U}(t) = \mathscr{L}[u(t)]$ in the space $\mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$. This will be done in the next sections.

3.1. A related elliptic problem

In this section, we analyze a family of elliptic problems that will allow us to pass from the function $u \in \widetilde{H}^s(\Omega)$ to $\mathcal{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$.

Instead of using the more intuitive lifting $\mathscr{L}_h$, we use one in the form of a Neumann problem. This is done so as to be able to reuse the techniques developed in [BMN+18] instead of having to analyze a Dirichlet problem from scratch.

Definition 3.14. *Let $\lambda > 0$ be fixed. For $f \in L^2(\Omega)$, we define the solution operator $\mathcal{G}^\lambda f$ by:*

$$\begin{aligned} -\operatorname{div}(y^\alpha A \nabla \mathcal{G}^\lambda f) + y^\alpha c \mathcal{G}^\lambda f &= 0 && \textit{in } \mathcal{C}, \\ d_s \lambda \operatorname{tr} \mathcal{G}^\lambda f + \partial_\nu^\alpha \mathcal{G}^\lambda f &= d_s f && \textit{on } \Omega \times \{0\}, \\ \mathcal{G}^\lambda f &= 0 && \textit{on } \partial_L \mathcal{C}. \end{aligned}$$

Lemma 3.15. *The following stability estimate holds:*

$$\left\|\mathcal{G}^\lambda f\right\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim \lambda^{-1/2} \|f\|_{L^2(\Omega)}. \tag{3.10}$$

The implied constant depends only on c, A, and Ω but is independent of λ and f.

Proof. We note that

$$\left\|\mathcal{G}^\lambda f\right\|^2_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim \mathcal{A}(\mathcal{G}^\lambda f,\mathcal{G}^\lambda f) \lesssim \mathcal{A}(\mathcal{G}^\lambda f,\mathcal{G}^\lambda f) + d_s\lambda\left(\operatorname{tr}\mathcal{G}^\lambda f,\operatorname{tr}\mathcal{G}^\lambda f\right)_{L^2(\Omega)}.$$

Inserting the definition of $\mathcal{G}^\lambda$ gives:

$$\begin{aligned}\mathcal{A}(\mathcal{G}^\lambda f,\mathcal{G}^\lambda f) + d_s\lambda\left(\operatorname{tr}\mathcal{G}^\lambda f,\operatorname{tr}\mathcal{G}^\lambda f\right)_{L^2(\Omega)} &= d_s\left(f,\operatorname{tr}\mathcal{G}^\lambda f\right)_{L^2(\Omega)} \\ &\lesssim \lambda^{-1/2}\|f\|_{L^2(\Omega)}\left[\mathcal{A}(\mathcal{G}^\lambda f,\mathcal{G}^\lambda f) + \lambda d_s\left(\operatorname{tr}\mathcal{G}^\lambda f,\operatorname{tr}\mathcal{G}^\lambda f\right)_{L^2(\Omega)}\right]^{1/2}.\end{aligned}$$

□

Remark 3.16. *This "damping property" of the factor $\lambda^{-1/2}$ in* (3.10) *is the main motivation for considering such operators, compared to the more intuitive $\lambda = 0$ case, which is the operator analyzed in [BMN+18]. It will allow us to better control the behavior of $\mathscr{U}$ for small times t by choosing $\lambda \sim 1/t$, see Section 3.2. It is also the operator which needs to be inverted when discretizing using a implicit Euler timestepping scheme, where λ^{-1} is the timestep size, see Section 4. We also point out the strong relation of the operator $\mathcal{G}^\lambda$ to the resolvent $(\lambda+\mathcal{L})^{-1}$, see the proof of Theorem 3.2.*

3.1.1. Discretization of the extended variable y

***hp*-fem in 1d:** In this section, we introduce the basics of *hp*-Finite Elements in 1D. This will provide us with the discretization scheme for the extended variable y. Additionally, it will serve as a model construction for satisfying Assumptions 3.5 and 3.9.

We introduce the notion of a geometrically refined mesh. For a grading factor $0 < \sigma < 1$ and $L \in \mathbb{N}$ layers, the geometric mesh on the domain $(-1,1)$ refined towards -1, denoted by $\mathcal{T}^L_{(-\mathbf{1},1)} := (x_i)_{i=0}^{L+1}$ is given by

$$x_0 := -1,\ x_i := -1 + \sigma^{L-i+1},\ i = 1,\dots L,\ x_{L+1} := 1.$$

Analogously we define the geometric mesh refined towards 1 and denote it by $\mathcal{T}^L_{(-1,\mathbf{1})}$, and the mesh geometrically refined towards both endpoints $\mathcal{T}^L_{(-\mathbf{1},\mathbf{1})}$ with nodes at

$$\begin{aligned}&x_0 := -1,\ x_i := -1 + \sigma^{L-i+1},\ i = 1,\dots L,\\ &x_i := 1 - \sigma^{i-L},\ i = L+1,\dots 2L, \quad x_{2L+1} := 1.\end{aligned}$$

In general, triangulations on (a,b), for example denoted by $\mathcal{T}^L_{(\mathbf{a},b)}$ are obtained by an affine mapping of $\mathcal{T}^L_{(-\mathbf{1},1)}$ etc.

Let $\mathcal{T}$ be a triangulation of a domain Ω. For a polynomial degree distribution $\mathbf{r} \in \mathbb{N}_0^{|\mathcal{T}|}$, we define the space of piecewise polynomials

$$\mathcal{S}^{\mathbf{r},1}(\mathcal{T}) := \{u \in C(\Omega) : u|_{K_i} \text{ is a polynomial of degree } \mathbf{r}_i \ \ \forall K_i \in \mathcal{T}\}.$$

For the discontinuous case, we define:

$$\mathcal{S}^{\mathbf{r},0}(\mathcal{T}) := \{u : \Omega \to \mathbb{R},\ u|_{K_i} \text{ is a polynomial of degree } \mathbf{r}_i \ \ \forall K_i \in \mathcal{T}\}.$$

To simplify the notation, we write $\mathcal{S}^{p,1}(\mathcal{T}) := \mathcal{S}^{(p,\dots,p),1}(\mathcal{T})$ for the case of constant polynomial degree p, and analogously for $\mathcal{S}^{p,0}(\mathcal{T})$.

We also need to impose Dirichlet conditions on parts of the boundary. We write

$$\mathcal{S}^{\mathbf{r},1}_0(\mathcal{T}) := \{u \in \mathcal{S}^{\mathbf{r},1}(\mathcal{T}) : u|_{\partial\Omega} = 0\}.$$

The space $\mathbb{V}_h^{\mathcal{Y}}$: We now give the precise construction for the space $\mathbb{V}_h^{\mathcal{Y}}$. It is based on an *hp*-FEM on a graded mesh. The details are laid out in the next definition.

Definition 3.17. *Fix $\mathcal{Y} > 0$. Let $\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})}$ be a geometric mesh on $(0,\mathcal{Y})$, refined towards 0 with L levels and a grading factor $\sigma \in (0,1)$, i.e., given by the nodes $\{0, \mathcal{Y}\sigma^i \,|\, i = 0,\dots,L\}$. Assume that $\mathcal{Y} \sim L$. Let $\mathbb{V}_h^{\mathcal{Y}} := \mathcal{S}^{\mathbf{r},1}(\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})}) \cap \{u : u(\mathcal{Y}) = 0\}$ be the space of piecewise polynomials with degree distribution vector $\mathbf{r}$ which vanish at the endpoint $\mathcal{Y}$.*

Using the eigenpairs $(\varphi_j, \mu_j)_{j=0}^{\infty}$ from (2.2), we have the following representation of $\mathscr{U} := \mathcal{G}^\lambda f$:

$$\mathscr{U}(x,y) = \sum_{j=0}^{\infty} u_j \varphi_j(x)\psi_j(y) \qquad \text{with} \qquad u_j := \left(\lambda + \mu_j^s\right)^{-1} (f,\varphi_j)_{L^2(\Omega)}.$$

Here ψ_j are the functions from [BMN+18, Formula (4.2)]. They satisfy the differential equation:

$$\psi_j'' + \frac{\alpha}{y}\psi_j' - \mu_j \psi_j = 0 \qquad \text{in } (0,\infty),$$
$$\psi_j(0) = 1, \qquad \lim_{y\to\infty} \psi_j(y) = 0.$$

Lemma 3.18. *The coefficients u_j satisfy the follwing a priori estimate:*

$$|u_j|^2 \lesssim \lambda^{-2} \left|(f,\varphi_j)_{L^2(\Omega)}\right|^2 \quad \text{and} \quad \mu_j^s |u_j|^2 \lesssim \lambda^{-1} \left|(f,\varphi_j)_{L^2(\Omega)}\right|^2.$$

Proof. From the definition, we get by multiplying with u_j:

$$\lambda |u_j|^2 + \mu_i^s |u_j|^2 = (f,\varphi_j)_{L^2(\Omega)}\, u_j \le |(f,\varphi_j)_{L^2(\Omega)}|\, |u_j|,$$

which implies $\lambda |u_j| \le |(f,\varphi_j)_{L^2(\Omega)}|$. Inserting this knowledge gives:

$$\mu_i^s |u_j|^2 \le \lambda^{-1} |(f,\varphi_j)_{L^2(\Omega)}| \lambda |u_j| \le \lambda^{-1} |(f,\varphi_j)^2_{L^2(\Omega)}. \qquad \square$$

Lemma 3.19. *Let $\Pi_{\mathcal{Y}}$ denote the Galerkin projection onto the space $H_0^1(\Omega) \otimes \mathbb{V}_h^{\mathcal{Y}}$. Then the following estimate holds for all $f \in L^2(\Omega)$:*

$$\left\| \mathcal{G}^\lambda f - \Pi_{\mathcal{Y}} \mathcal{G}^\lambda f \right\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim \lambda^{-1/2} e^{-bp} \|f\|_{L^2(\Omega)}.$$

Proof. We follow the argument of [BMN+18]. By Galerkin orthogonality, we are only concerned with proving an estimate for the best approximation to $\mathcal{G}^\lambda f$. The functions ψ_j all decay exponentially for $y \to \infty$. We can bound

$$\left\| \mathcal{G}^\lambda f(y) - \mathscr{V}^{\text{cutoff}} \right\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim C e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \sqrt{\sum_{j=0} \mu_j^s |u_j|^2} \lesssim C e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \lambda^{-1/2} \|f\|_{L^2(\Omega)},$$

where $\lambda_1 > 0$ denotes the smallest eigenvalue of the operator $\mathcal{L}$ on Ω, see [NOS15, Lemma 3.3] for details, the proof can be taken verbatim, just replacing the definition of the coefficients u_j. It is thus sufficient to study the approximation on the finite cylinder $\Omega \times (0,\mathcal{Y})$.

We define the weights $\omega_{\beta,\gamma} := y^\beta e^{\gamma y}$, and the weighted L^2-norms

$$\|v\|^2_{L^2(\omega_{\beta,\gamma},\mathcal{C})} := \int_0^\infty \int_\Omega \omega_{\beta,\gamma}(y)\, |v(x,y)|^2 \, dx\, dy.$$

We note that the function $\mathcal{G}^\lambda u$ satisfies the following *a priori* estimates:

$$\left\|\partial_y^{\ell+1}\mathcal{G}^\lambda f\right\|_{L^2(\omega_{\alpha+2\ell},\mathcal{C})} \lesssim \lambda^{-1/2}\kappa^{\ell+1}(\ell+1)!\,\|f\|_{L^2(\Omega)} \quad \forall \ell\in\mathbb{N}_0,$$
$$\left\|\nabla_x\partial_y^{\ell+1}\mathcal{G}^\lambda f\right\|_{L^2(\omega_{\alpha+2(\ell+1)},\mathcal{C})} \lesssim \lambda^{-1/2}\kappa^{\ell+1}(\ell+1)!\,\|f\|_{L^2(\Omega)} \quad \forall \ell\in\mathbb{N}_0.$$

Again, this follows [BMN$^+$18, Theorem 4.7] verbatim, only plugging in the stronger estimate for the coefficients u_j to regain the factor $\lambda^{-1/2}$. This in turn implies that $\mathcal{G}^\lambda f$ is in some Banach-space valued countably normed spaces. Invoking the interpolation operator $\Pi^{\mathbf{r}}_{y,\{\mathcal{Y}\}}$ from [BMN$^+$18, Section 5.5.1] then shows the stated result. □

3.1.2. Discretization in x

In this section, we study the discretization error due to the choice of space $\mathbb{V}_h^{\mathcal{X}}$. We will show that the requirement that $\mathbb{V}_h^{\mathcal{X}}$ resolve appropriate scales (see Assumption 3.9) suffices to show exponential convergence.

Before we prove an approximation result for $\mathcal{G}^\lambda$, we need the following result on the solution of singularly perturbed problems, generalizing the theory developed in, e.g., [Mel97, Mel02] (for real singular perturbation parameters) to the case where the right hand side is itself the solution to a singularly perturbed problem:

Lemma 3.20. *Let $\varepsilon > 0$ and $z \in \mathscr{S}$ with $\operatorname{Re}(z) \geq 0$. Assume that the space $\mathbb{V}_h^{\mathcal{X}}$ resolves the scale ε and $|z|^{-1/2}$, as defined in Assumption 3.9.*

Let $u_z \in H_0^1(\Omega)$ denote the solution to $(\mathcal{L}-z)u_z = z\,f$, where $f \in L^2(\Omega)$ is analytic on $\overline{\Omega}$.

Let $u \in H_0^1(\Omega)$ solve

$$\varepsilon^2\mathcal{L}u + u = u_z. \tag{3.11}$$

Then the following best approximation result holds:

$$\inf_{v_h\in\mathbb{V}_h^{\mathcal{X}}}\left[\varepsilon^2\|\nabla u-\nabla v_h\|^2_{L^2(\Omega)} + \|u-v_h\|^2_{L^2(\Omega)}\right] \leq Ce^{-b\mathcal{N}_\Omega^\mu}.$$

The implied constant depends on $\mathscr{S}$, the constants of ellipticity of f, and the constants from Assumption 3.9 but not on ε or z.

Proof. We make the ansatz $u = \alpha u_z - w$, for $\alpha\in\mathbb{C}$ and some function $w\in H_0^1(\Omega)$. Plugging this decomposition into (3.11) and using the PDE for u_z, we get the conditions $\alpha = \frac{1}{1+\varepsilon^2 z}$ and w solves

$$\varepsilon^2\mathcal{L}w + w = \frac{\varepsilon^2 z}{1+\varepsilon^2 z}\,f.$$

Since we assumed $\operatorname{Re}(z)\geq 0$, the coefficient α is bounded independently of ε and z. We also compute

$$\left|1+\varepsilon^2 z\right|^2 = (1+\varepsilon^2\operatorname{Re}(z))^2 + \varepsilon^4\operatorname{Im}(z)^2 > \varepsilon^4|z|^2,$$

which shows that $\frac{\varepsilon^2 z}{1+\varepsilon^2 z}$ is also uniformly bounded.

Since we assumed that the mesh resolves the scale ε, we can apply Assumption 3.9 to w to get the estimate:

$$\inf_{v_h\in\mathbb{V}_h^{\mathcal{X}}}\left[\varepsilon^2\|\nabla w-\nabla v_h\|^2_{L^2(\Omega)} + \|w-v_h\|^2_{L^2(\Omega)}\right] \leq Ce^{-b\mathcal{N}_\Omega^\mu}.$$

We also assumed that the mesh resolves the scale $|z|^{-1/2}$. Thus we get an exponential approximation property for u_z in the $|z|^{-1/2}$ weighted norm. In order to get the estimate in the ε-weighted norm, we note that for $\varepsilon < |z|^{-1/2}$ we get the estimate trivially. For $\varepsilon > |z|^{-1/2}$ we note that

$$\begin{aligned}\varepsilon^2 \|\alpha \nabla u_z\|^2_{L^2(\Omega)} &\leq \left(\varepsilon\,|z|^{1/2}\,\alpha\right)^2 |z|^{-1} \|\nabla u_z\|^2_{L^2(\Omega)} \lesssim \left(\varepsilon^2\,|z|\,\alpha\right)^2 |z|^{-1} \|\nabla u_z\|^2_{L^2(\Omega)} \\ &\lesssim |z|^{-1} \|\nabla u_z\|^2_{L^2(\Omega)} .\end{aligned}$$

This means we can approximate $\alpha\, u_z$ in the ε-weighted norm at an exponential rate, which concludes the proof. □

We now repeat the construction in [BMN+18]. Let $(v_i)_{i=0}^{\mathcal{M}} \subseteq \mathbb{V}_h^{\mathcal{Y}}$ denote a basis with the following properties:

$$d_s \lambda v_i(0) v_j(0) + \int_0^{\mathcal{Y}} y^\alpha\, v_i' v_j' = \delta_{ij} \qquad \text{and} \qquad \int_0^{\mathcal{Y}} y^\alpha\, v_i v_j = \kappa_i \delta_{ij},$$

for coefficients $\kappa_i > 0$. Since the bilinear forms are SPD, such a basis exists. On Ω, we define the bilinear forms

$$a_{\kappa_i}(U,V) := \kappa_i \left[(\nabla U, \nabla V)_{L^2(\Omega)} + c\,(U,V)_{L^2(\Omega)} \right] + (U,V)_{L^2(\Omega)}\,, \tag{3.12}$$

and note that the following norm equivalence holds on $H_0^1(\Omega) \otimes \mathbb{V}_h^{\mathcal{Y}}$ for all $\mathscr{V} := \sum_{i=0}^{\mathcal{M}} \mathscr{V}_i v_i$:

$$\lambda \|\mathrm{tr}\, \mathscr{V}\|^2_{L^2(\Omega)} + \|\mathscr{V}\|^2_{\mathring{H}^1(y^\alpha, \mathcal{C})} \sim \sum_{j=0}^{\mathcal{M}} a_{\kappa_i}(\mathscr{V}_i, \mathscr{V}_i). \tag{3.13}$$

(3.13) shows that estimates in the $\mathring{H}^1(y^\alpha, \mathcal{C})$ norm can also be obtained from bounds on each component in the corresponding κ_i-weighted H^1-norm.

The bilinear forms a_{κ_i} correspond to singularly perturbed problems. We want to apply Assumption 3.9. For this we need bounds for κ_i as well as $v_i(0)$.

Lemma 3.21. *Let $h_{\min} > 0$ denote the smallest element size in $\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})}$ and p the maximal polynomial degree used for $\mathbb{V}_h^{\mathcal{Y}}$. Then following estimates hold for $j = 0, \ldots, \mathcal{M}$:*

$$\lambda^{-1} \frac{h_{\min}^2}{p^4} \leq \kappa_i \leq C \mathcal{Y}^2 (1-\alpha^2)^{-1}, \tag{3.14}$$

$$|v_i(0)| \leq \lambda^{-1/2}. \tag{3.15}$$

Proof. By definition we have $1 = d_s \lambda v_i(0)^2 + \int_0^{\mathcal{Y}} y^\alpha |v_i'|^2 = \kappa_i^{-1} \int_0^{\mathcal{Y}} y^\alpha |v_i|^2$, or $\kappa_i = \int_0^{\mathcal{Y}} y^\alpha |v_i|^2$. By [BMN+18, Lemma B.2] we can estimate

$$\|v_i\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} \lesssim \mathcal{Y}^2 (1-\alpha^2)^{-1} \left\|v_i'\right\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} \lesssim \mathcal{Y}^2 (1-\alpha^2)^{-1}.$$

On the other hand, the inverse estimate from [BMN+18, Lemma B.3], gives:

$$\begin{aligned}\lambda\,|v_i(0)|^2 + \left\|v_i'\right\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} &\lesssim (1+\lambda C) \left[\left\|v_i'\right\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} + \left\|v_i'\right\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} \right] \\ &\lesssim \frac{h_{\min}^{-2}}{p^4} \lambda \|v_i\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))}\,.\end{aligned}$$

To see (3.15), we calculate:

$$|v_i(0)|^2 \leq \lambda^{-1} d_s^{-1} \left[\lambda d_s\, |v_i(0)|^2 + \left\|v_i'\right\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} \right] = \lambda^{-1}.$$

□

Lemma 3.22. *Let $u \in L^2(\Omega)$ be either holomorphic in $\overline{\Omega}$ or solution to the singularly perturbed problem $-z^{-1}\mathcal{L}u + u = f$ for holomorphic $f \in L^2(\Omega)$ and $z \in \mathscr{S}$ with $\operatorname{Re}(z) \geq 0$. Assume that $\mathbb{V}_h^{\mathcal{X}}$ resolves the scales $|z|^{-1/2}$ and $\sqrt{\kappa_i}$ for all $i = 0, \dots, \mathcal{M}$.*

Then the following best approximation result holds:

$$\inf_{\mathscr{V}_h \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}} \left\| \mathcal{G}^\lambda u - \mathscr{V}_h \right\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim \lambda^{-1/2} \left(e^{-b\mathcal{N}_\Omega^\mu} + e^{-b\sqrt{\mathcal{N}_\mathcal{Y}}} \right).$$

where μ is the exponent for $\mathbb{V}_h^{\mathcal{X}}$ in Assumption 3.29.

Proof. By Lemma 3.19, it is sufficient to consider a semidiscrete functions $\mathscr{U}_y^h := \Pi_\mathcal{Y}\mathcal{G}^\lambda f \in H_0^1(\Omega) \otimes \mathbb{V}_h^{\mathcal{Y}}$ and their approximation in $\mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$. Using the basis $(v_j)_{j=0}^{\mathcal{M}}$, the function $\mathscr{U}_y^h =: \sum_{i=0}^{\mathcal{M}} U_i v_i$ from Lemma 3.19 solves:

$$a_{\kappa_i}(U_i, V) = d_s v_i(0)\,(u, V)_{L^2(\Omega)} \qquad \forall V \in H_0^1(\Omega).$$

This is just the weak formulation of the singularly perturbed problems from Assumption 3.9, with $\varepsilon = \sqrt{\kappa_i}$. Since we assumed that the scales are resolved, we can apply Lemma 3.20 to get the following estimate for the best approximations $\Pi U_i \in \mathbb{V}_h^{\mathcal{X}}$:

$$\kappa_i \left\| \nabla [U_i - \Pi U_i] \right\|_{L^2(\Omega)}^2 + \left\| U_i - \Pi U_i \right\|_{L^2(\Omega)}^2 \lesssim C(f)\lambda^{-1} e^{-b\mathcal{N}_\Omega^\mu},$$

the norm equivalence (3.13) then concludes the proof. $\square$

3.2. Returning to the semidiscretization

We are now in a position to show exponential convergence for the best approximation (and thus also the Ritz approximation) of the exact solution $\mathscr{U}$. We first consider positive times t bounded away from 0. In this regime, our finite element mesh is assumed to resolve the pertinent scales. The smaller times, for which the scales are not resolved, are treated separately later on.

Theorem 3.23. *Let $t \geq t_0 > 0$ be fixed. Let u_0 be analytic on a fixed neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$ (but we do not assume boundary conditions, i.e., $u_0 \notin \widetilde{H}^s(\Omega)$ is allowed), and assume homogeneous right-hand side, i.e., $f = 0$. Also assume that $\mathbb{V}_h^{\mathcal{X}}$ resolves the scales $z_{hf}^{-1/2}$ for a fixed "high frequency" cutoff $z_{hf} > z_0 > 0$. Then, for each $\ell \in \mathbb{N}_0$, there exists a function $\mathscr{V}_h(t) \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ such that the following estimate holds:*

$$\left\| \mathscr{U}^{(\ell)}(t) - \mathscr{V}_h(t) \right\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim t^{-1/2-\ell} \max\left(1, -\log(t)^{1-\min(\ell,1)}\right) \left(e^{-b_1\mathcal{N}_\Omega^\mu} + e^{-b_2\sqrt{\mathcal{N}_\mathcal{Y}}} + e^{-\frac{\gamma}{2} z_{hf}^s t_0} \right). \tag{3.16}$$

The implied constant depends on Ω, s, the constants of analyticity of u_0, z_0, and the constants from Assumption 3.9, but is independent of t and t_0. The rate b_2 also depends on the mesh grading for y. b_1 in addition depends on the constants from Assumption 3.9. γ can be chosen to depend on s only.

Proof. Since we assumed homogeneous right hand side, we only need to investigate $\mathscr{U} = \mathcal{E}(t)u_0$. We use the representation of $\mathcal{E}(t)u$ via the Riesz-Dunford calculus (following what is done in [BLP17, Section 2]), to write:

$$\mathcal{E}(t)u_0 = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{-tz^s}\,(z - \mathcal{L})^{-1} u_0\,dz, \qquad (\mathcal{E}(t)u_0)^{(\ell)} = \frac{(-1)^\ell}{2\pi i}\int_{\mathcal{C}} z^{\ell s} e^{-tz^s}\,(z - \mathcal{L})^{-1} u_0\,dz,$$

where $\mathcal{C}$ is the following contour consisting of three segments:

$$\begin{cases} \mathcal{C}_1 := \left\{z(r) = re^{-i\frac{\pi}{4}} \quad | \quad r \in (r_0, \infty)\right\} \\ \mathcal{C}_2 := \left\{z(\theta) := r_0 e^{i\theta} \quad | \quad \theta \in (-\pi/4, \pi/4)\right\} \\ \mathcal{C}_3 := \left\{z(r) := re^{i\frac{\pi}{4}} \quad | \quad r \in (r_0, \infty)\right\} \end{cases}$$

and $z^s := e^{s\log(z)}$ with the logarithm defined with the branch cut along the negative real axis. The parameter $r_0 \in (0, z_0)$ is fixed such that the whole path lies in the domain of ellipticity $\mathscr{S}$, as defined in Definition 3.6; see Figure 3.1.

By adding the term $\frac{1}{t} d_s \operatorname{tr} \mathscr{U}$ to both sides of (2.3b), we get that $\mathscr{U}$ solves

$$\begin{aligned} -\operatorname{div}(y^\alpha A \nabla \mathscr{U}) + y^\alpha c \mathscr{U} &= 0 && \text{on } \mathcal{C} \times \mathbb{R}_+, \\ \frac{d_s}{t} \operatorname{tr} \mathscr{U} + \partial_\nu^\alpha \mathscr{U} &= \frac{d_s}{t} \operatorname{tr} \mathscr{U} - d_s \operatorname{tr} \dot{\mathscr{U}} && \text{on } \omega \times \{0\} \times (0, T), \\ \mathscr{U} &= 0 && \text{on } \partial_L \mathcal{C}. \end{aligned}$$

Using the operator $\mathcal{G}^{1/t}$, we can therefore write the function $\mathscr{U}$ as

$$\mathscr{U} = -\mathcal{G}^{1/t} \operatorname{tr} \dot{\mathscr{U}} + \frac{1}{t} \mathcal{G}^{1/t} \operatorname{tr} \mathscr{U},$$

or using the Riesz-Dunford calculus:

$$\mathscr{U}(t) = -\frac{1}{2\pi i} \int_{\mathcal{C}} e^{-tz^s} z^s \mathcal{G}^{1/t} [z - \mathcal{L}]^{-1} u_0 \, dz + \frac{1}{t} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-tz^s} \mathcal{G}^{1/t} [z - \mathcal{L}]^{-1} u_0 \, dz.$$

For the derivatives, a similar formula holds:

$$\frac{d^\ell}{dt^\ell} \mathscr{U}(t) = \frac{(-1)^{\ell+1}}{2\pi i} \int_{\mathcal{C}} e^{-tz^s} z^{(\ell+1)s} \mathcal{G}^{1/t} [z - \mathcal{L}]^{-1} u_0 \, dz + \frac{1}{t} \frac{(-1)^\ell}{2\pi i} \int_{\mathcal{C}} e^{-tz^s} z^{\ell s} \mathcal{G}^{1/t} [z - \mathcal{L}]^{-1} u_0 \, dz.$$

Hence, we have to study integrals of the form

$$I_m := \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-tz^s} z^{m\, s} \mathcal{G}^{1/t} [z - \mathcal{L}]^{-1} u_0 \, dz, \qquad m \in \mathbb{N}_0, \tag{3.17}$$

and their best approximation, paying attention to the dependence on t.

If $|z| < z_{\mathrm{hf}}$, the function $\widehat{u}(z) := \left(\mathrm{I} - z^{-1}\mathcal{L}\right)^{-1} u_0$ can be approximated exponentially well by Assumption 3.9. By the results in Section 3.1 this implies for $|z| \in (\varepsilon_0, z_{\mathrm{hf}})$:

$$\left\| \mathcal{G}^{1/t} \left(\mathrm{I} - z^{-1}\mathcal{L}\right)^{-1} u_0 - \widehat{\mathscr{V}}_h(z) \right\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} \lesssim t^{1/2} \left(e^{-b_1 \mathcal{N}_\Omega^\mu} + e^{-b_2 \sqrt{\mathcal{N}_{\mathcal{Y}}}}\right) \tag{3.18}$$

for some function $\widehat{\mathscr{V}}_h(z) \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$. On $\mathcal{C}_2$, we can therefore estimate:

$$\left\| \int_{\mathcal{C}_2} e^{-tz^s} z^{m\, s-1} \mathcal{G}^{1/t} \left[\left(\mathrm{I} - z^{-1}\mathcal{L}\right)^{-1} u_0 - \widehat{\mathscr{V}}(z)\right] dz \right\| \leq C t^{1/2} \left(e^{-b_1 \mathcal{N}_\Omega^\mu} + e^{-b_2 \sqrt{\mathcal{N}_{\mathcal{Y}}}}\right).$$

The more interesting case are the paths $\mathcal{C}_1$ and $\mathcal{C}_2$. We focus on $\mathcal{C}_1$, and consider two cases, namely, $|z| \leq z_{\mathrm{hf}}$ and $|z| > z_{\mathrm{hf}}$. In the first case, the mesh resolves the underlying scales of $|z|^{-1/2}$ and we can apply Lemma 3.22. Setting $\gamma := \cos(\pi\, s/4)$ we estimate:

$$\begin{aligned} I_m^1 &:= \left\| \int_{\mathcal{C}_1 \cap |z| \leq z_{\mathrm{hf}}} e^{-tz^s} z^{m\, s-1} \left(\mathcal{G}^{1/t} (z - \mathcal{L})^{-1} (z u_0) - \widehat{\mathscr{V}}_h(z)\right) dz \right\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} \\ &\lesssim t^{1/2} \left(e^{-b_1 \mathcal{N}_\Omega^\mu} + e^{-b_2 \sqrt{\mathcal{N}_{\mathcal{Y}}}}\right) \int_{r_0}^{z_{hf}} e^{-\gamma t r^s} r^{m\, s-1} dr. \end{aligned}$$

Making the substitution $\gamma t\, r^s =: y$, we get:

$$\int_{r_0}^{z_{hf}} e^{-tr^s} r^{m\,s-1} dr = s^{-1} t^{-m} \gamma^{-m} \int_{\gamma t r_0^s}^{\gamma t z_{hf}^s} e^{-y} y^{m-1}\, dr.$$

We need to consider the case $m = 0$ separately, as the integrand then has a singularity at $r = 0$. Splitting the integration we get:

$$s^{-1} t^0 \int_{\gamma t r_0^s}^{\gamma t z_{hf}^s} e^{-y} y^{m-1}\, dr \lesssim \int_{\gamma t r_0^s}^{1} e^{-y} y^{-1}\, dr + \int_1^\infty e^{-y} y^{-1}\, dr \lesssim \int_{\gamma t r_0^s}^{1} y^{-1}\, dr + \int_1^\infty e^{-y}\, dr$$
$$\lesssim -\log(\gamma t r_0^s) + e^{-1} \sim 1 - \log(t r_0^s).$$

For $m > 0$, we do not get the logarithmic growth for small times, since:

$$\gamma^{-m} s^{-1} t^{-m} \int_{y_0}^{y_1} e^{-y} y^{m-1}\, dr \lesssim t^{-m} \int_0^\infty e^{-y} y^{m-1}\, dr = t^{-m} \Gamma(m).$$

Overall, this gives the estimate:

$$I_m^1 \lesssim t^{1/2-m} \max\left(1, -\log(t)^{1-\min(m,1)}\right)\left(e^{-b_1 \mathcal{N}_\Omega^\mu} + e^{-b_2\sqrt{\mathcal{N}_\mathcal{Y}}}\right).$$

In the case $r > z_{\mathrm{hf}}$, we set $\widehat{\mathscr{V}_h} := 0$ and use the stability estimate (3.10) and the uniform stability of the operator $(z - \mathcal{L})^{-1} z$ (see Lemma B.2). For $m > 0$, we estimate :

$$\left\| \int_{\mathcal{C}_1 \cap |z| > z_{\mathrm{hf}}} e^{-tz^s} z^{ms-1} \mathcal{G}^{1/t} \left[z - \mathcal{L}\right]^{-1} (z u_0)\, dz \right\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} \lesssim \|u_0\|_{L^2(\Omega)}\, t^{1/2} e^{-\frac{\gamma}{2} t\, z_{\mathrm{hf}}^s} \int_{r > z_{\mathrm{hf}}} e^{-\gamma t r^s/2} r^{m\,s-1}\, dr$$
$$\lesssim \|u_0\|_{L^2(\Omega)}\, t^{1/2} e^{-\frac{\gamma}{2} t\, z_{\mathrm{hf}}^s}\, t^{-m} \int_0^\infty e^{-y} y^{m-1}\, dr \;\lesssim\; \|u_0\|_{L^2(\Omega)}\, e^{-\frac{\gamma}{2} t\, z_{\mathrm{hf}}^s}\, t^{1/2-m}\, \Gamma(m).$$

For $m = 0$, the same calculation can be done, but picking up an extra logarithmic term from the integral where $y = z_{hf}^s t \lesssim 1$.

The same argument can be repeated for $\mathcal{C}_3$. The stated estimates then follow easily by setting $m = 0$ and $m = 1$ to estimate $\mathscr{U}$ (this term involves the logarithmic contributions) and $m = \ell$ and $m = \ell + 1$ to estimate higher derivatives. □

For small $t < t_0$, we cannot hope to retain exponential convergence, as it would require our mesh to resolve infinitely small scales. Instead, we rely on on our ability to control the behavior of the solution near $t = 0$ using some smoothness of u_0.

Lemma 3.24. *Let $u_0 \in H^\theta(\Omega)$ for $0 < \theta < 1/2$, and assume homogeneous right hand-side, i.e., $f = 0$. For all $\ell \in \mathbb{N}_0$, the following estimate holds for $t > 0$:*

$$\left\| \mathscr{U}^{(\ell)}(t) \right\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} \lesssim t^{-\ell - 1/2 + \min(\frac{\theta}{2s}, 1)} \|u_0\|_{H^\theta(\Omega)}\,. \tag{3.19}$$

The constant depends on Ω, θ, s and the coefficients A, c.

Proof. For simplicity we assume additionally $\theta \le 2s$. We note that for $\theta \in (0, 1/2)$, the spaces $\widetilde{H}^\theta(\Omega)$ and $H^\theta(\Omega)$ coincide with equivalent norms (see [Tri06, Section 1.11.6] or [McL00, Theorem 3.33, Theorem B.9, Theorem 3.40]).

Hence, we get $u_0 \in \widetilde{H}^\theta(\Omega)$. By Lemma 3.10, this implies for $\ell \in \mathbb{N}_0$:

$$\left\| u^{(\ell)} \right\|_{\widetilde{H}^s(\Omega)} \lesssim t^{-\ell + \frac{\theta}{2s} - 1/2} \left\| u_0 \right\|_{H^\theta(\Omega)}. \tag{3.20}$$

We write $\mathscr{U}(t) = \mathscr{L} u(t)$ using the lifting operator from (3.3). Since the lifting on the continuous level is bounded (see Remark 3.1), we can estimate:

$$\left\| \mathscr{U}^{(\ell)}(t) \right\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} = \left\| \mathscr{L} u^{(\ell)}(t) \right\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} \lesssim \left\| u^{(\ell)}(t) \right\|_{\widetilde{H}^s(\Omega)} \lesssim t^{-\ell + \frac{\theta}{2s} - 1/2} \left\| u_0 \right\|_{H^\theta(\Omega)}. \qquad \square$$

As a final step before showing convergence of the semidiscrete approximation, we remove the restriction to homogeneous right-hand sides f. This is a simple consequence of the previous results and Duhamel's principle.

Corollary 3.25. *Let $t_0 > 0$ and $\delta > 0$ be fixed. Let u_0 be analytic on $\overline{\Omega}$ and assume that f is ℓ times continuously differentiable with respect to t such that the functions $f^{(j)}$, $j = 0, \dots, \ell$ are uniformly analytic in the sense of Definition 3.8.*

Assume that $\mathbb{V}_h^{\mathcal{X}}$ resolves the scales $z_{hf}^{-1/2}$ for a fixed "high frequency" cutoff $z_{hf} > z_0 > 0$. Then, for each $\ell \in \mathbb{N}_0$, there exists a function $\mathscr{V}_h(t) \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ such that the following estimates holds for all $t \in (0, T)$:

$$\left\| \mathscr{U}^{(\ell)}(t) - \mathscr{V}_h(t) \right\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} \lesssim t^{-\ell - 1/2} \max\left(1, -\log(t)\right) \left(e^{-b_1 \mathcal{N}_\Omega^\mu} + e^{-b_2 \sqrt{\mathcal{N}_\mathcal{Y}}} + e^{-\frac{1}{2} z_{hf}^s t_0} \right) + t_0^{-\ell - 1/2 + \min(\frac{1}{4s} - \delta, 1)}. \tag{3.21}$$

The implied constant depends on the end time T, the data u_0, the constants of analyticity of $f^{(j)}$, δ, and the implied constants in Lemma 3.22, e.g., the mesh grading factor. It is independent of t, t_0, $\mathcal{N}_\Omega$ or $\mathcal{N}_\mathcal{Y}$. For $\ell = 0$ and $\ell = 1$ we can explicitly give $C(T) \lesssim \max(1, T)$.

Proof. For $f = 0$, this is just a collection of Lemma 3.23 and 3.24. For $f \neq 0$ we write

$$\mathscr{U}(t) = \mathscr{L}\left[\mathcal{E}(t) u_0 + \int_0^t \mathcal{E}(\tau) f(t - \tau)\, d\tau \right],$$
$$\dot{\mathscr{U}}(t) = \mathscr{L}\left[(\mathcal{E}(t) u_0)' + \mathcal{E}(t) f(0) + \int_0^t \mathcal{E}(t - \tau) \dot{f}(\tau)\, d\tau \right]$$

(see [Paz83, Section 4.2, Corollary 2.5] for the derivative of Duhamel's formula). The terms involving only $\mathcal{E}(t)$ are already covered by the results for the homogeneous problem. For fixed $\tau \in (0, t)$, the integrand in the last term corresponds to solving the homogeneous problem with initial condition $f(t - \tau)$ (or $\dot{f}(t - \tau)$ in the case of $\dot{\mathscr{U}}$). This means we can also apply Lemmas 3.23 and 3.24, only picking up an extra power of t due to the additional integration in τ. This gives the stated estimate for $\ell = 0$ and $\ell = 1$.

For higher derivatives, we proceed by induction and see that we can write $\mathscr{U}^{(\ell)}$ as

$$\mathscr{U}^{(\ell)}(t) = \mathscr{L}\left[(\mathcal{E}(t) u_0)^{(\ell)} + \sum_{j=0}^{\ell-1} \left(\frac{d}{dt} \right)^{\ell - j - 1} \left[\mathcal{E}(t) f^{(j)}(0) \right] + \int_0^t \mathcal{E}(t - \tau) f^{(\ell)}(\tau)\, d\tau \right].$$

All the terms can be estimated as before, where we estimate $t^{-j} \leq C(T) t^{-\ell}$ and only keep the dominant terms. $\square$

Theorem 3.26. *Assume that u_0 is analytic and f is uniformly analytic on a fixed neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$. Let $\mathbb{V}_h^{\mathcal{Y}}$ be given by Definition 3.17. Fix $t_0 > 0$, $\delta > 0$, and set $z_{hf} := t_0^{-1/s} L^{1/s}$, where L is the number of layers used for constructing the geometric mesh $\mathbb{V}_h^{\mathcal{Y}}$. Let the space $\mathbb{V}_h^{\mathcal{X}}$ resolve the scales up to*

$$\varepsilon_{\min} = \min\left(\sqrt{t_0}\frac{h_{\min}}{p^2}, |z_{hf}|^{-1/2}\right), \tag{3.22}$$

where $h_{\min}$ and p are the minimum element size and maximal polynomial degree of $\mathbb{V}_h^{\mathcal{Y}}$, and let Assumption 3.5 hold for the initial condition. Then the following estimate holds:

$$\int_0^t \|u(\tau) - u_h(\tau)\|^2_{L^2(\Omega)} \, d\tau \lesssim \max(1, t^2) \left(t_0^{\min(\frac{1}{2s}-\delta,1)} + |\log(t_0)|^2 \max(\log(t/t_0), 0) \left[e^{-b_1 \mathcal{N}_\Omega^\mu} + e^{-b_2\sqrt{\mathcal{N}_\Omega}}\right]\right).$$

Proof. We just collect all the previous results, most notably Proposition 3.12 and Corollary 3.25. Since we only need the best approximation estimate on $\mathscr{U}$ and $\dot{\mathscr{U}}$, we keep the dependence on the time t explicit. The error due to the different initial conditions is exponentially small by assumption. □

We can also obtain estimates in the energy norm or pointwise in time:

Theorem 3.27. *Assume that u_0 is analytic, f and $\dot{f}$ are uniformly analytic on a neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$, and that $u_{h,0} \in \mathbb{V}_{h,\beta}^{\mathcal{X}}$ is as in Assumption 3.5.*

Let L denote the number of layers used for $\mathbb{V}_h^{\mathcal{Y}}$, set $t_0 := e^{-L}$, and $z_{hf} := t_0^{-1/s} L^{1/s}$ and assume that the space $\mathbb{V}_h^{\mathcal{X}}$ resolves the scales up to (3.22).

Set $M := \min(L, \dim(\mathbb{V}_h^{\mathcal{X}})^\mu)$ with $\mu > 0$ from Assumptions 3.5 and 3.9.

Then there exists a constant b, independent of L, p and the specific choice of $\mathbb{V}_h^{\mathcal{X}}$, i.e. depending only on the constants from Assumptions 3.5 and 3.9 such that the following estimate holds:

$$\|u(t) - u_h(t)\|^2_{L^2(\Omega)} + \int_0^t \|u(\tau) - u_h(\tau)\|^2_{\widetilde{H}^s(\Omega)} \, d\tau \lesssim \max(1, t^2\log(t)) e^{-bM}.$$

Proof. Without loss of generality, we may assume $\beta s < 1/2$. Fix $t_1 > 0$ to be chosen later. We consider two regimes, $t \in (0, t_1)$ and $t \geq t_1$. For $t \leq t_1$, we use the stability estimates of Lemma 3.10 (ii) and (iii), together with the insight that $u_0 \in \widetilde{H}^{\beta s}(\Omega)$ for $\beta s < 1/2$ which was already used in Lemma 3.24.

We start with the energy norm estimate and use Lemma 3.10 to get:

$$\begin{aligned}\int_0^t \|u(\tau) - u_h(\tau)\|^2_{\widetilde{H}^s(\Omega)} d\tau &\lesssim \int_0^t \|u(\tau)\|^2_{\widetilde{H}^s(\Omega)} + \|u_h(\tau)\|^2_{\widetilde{H}^s(\Omega)} \, d\tau \\ &\lesssim \int_0^t \tau^{-1+\beta}\big(\|u_0\|^2_{\widetilde{H}^{\beta s}(\Omega)} + \|u_{h,0}\|^2_{\mathbb{V}_{h,\beta}^{\mathcal{X}}}\big) \, d\tau \lesssim t_1^\beta\big(\|u_0\|^2_{\widetilde{H}^{\beta s}(\Omega)} + \|u_{h,0}\|^2_{\mathbb{V}_{h,\beta}^{\mathcal{X}}}\big).\end{aligned}$$

For the pointwise estimate, we can write $u(t) = u_0 + \int_0^t \dot{u}(\tau) \, d\tau$ and $u_h(t) = u_{h,0} + \int_0^t \dot{u}(\tau) \, d\tau$ and obtain:

$$\begin{aligned}\|u(\tau) - u_h(\tau)\|_{L^2(\Omega)} &\lesssim \|u_0 - u_{h,0}\|_{L^2(\Omega)} + \int_0^t \|\dot{u}(\tau)\|_{L^2(\Omega)} + \|\dot{u}_h(\tau)\|_{L^2(\Omega)} \, d\tau \\ &\lesssim \|u_0 - u_{h,0}\|_{L^2(\Omega)} + t_1^{\beta/2}\big(\|u_0\|_{\widetilde{H}^{\beta s}(\Omega)} + \|u_{h,0}\|_{\mathbb{V}_{h,\beta}^{\mathcal{X}}}\big).\end{aligned}$$

For larger times $t > t_1$, we can establish the following bound by using (3.9) and plugging in the results on the best approximation from Corollary 3.25.

$$t \|u(t) - u_h(t)\|^2_{L^2(\Omega)} + \int_{t_1}^{t} \tau \|u(\tau) - u_h(\tau)\|^2_{\widetilde{H}^s(\Omega)} \, d\tau \lesssim \max(1, t^2 \log(t)) e^{-b'M} + t \|u_0 - u_{h,0}\|^2_{L^2(\Omega)} .$$

Or, since $\tau > t_1$:

$$\|u(t) - u_h(t)\|^2_{L^2(\Omega)} + \int_{t_1}^{t} \|u(\tau) - u_h(\tau)\|^2_{\widetilde{H}^s(\Omega)} \, d\tau \lesssim t_1^{-1} \max(1, t^2 \log(t)) \big(e^{-b'M} + \|u_0 - u_{h,0}\|^2_{L^2(\Omega)}\big).$$

Setting $t_1 \sim e^{-\frac{b'}{2}M}$ we get the stated exponential convergence with rate $b := -b'\beta/2$ after using Assumption 3.5 to estimate the error due to approximating the initial condition. □

3.3. A simpler model problem and example of a space $\mathbb{V}_h^{\mathcal{X}}$: hp-FEM in 1d

In this section, we verify that in the case of a simpler model problem in 1D, using an hp FEM for constructing $\mathbb{V}_h^{\mathcal{X}}$ meets our requirements. In other words, $\mathbb{V}_h^{\mathcal{X}}$ satisfies Assumptions 3.5 and 3.9.

Assumption 3.28. *$d = 1$, $\Omega := (-1, 1)$, $A := 1$, and $c \equiv const$.*

We start with the fact that we can resolve certain scales:

Theorem 3.29. *Let $\mathcal{T}^L_{(\mathbf{-1},\mathbf{1})}$ be a mesh on Ω that is geometrically refined towards both end points with grading factor $0 < \sigma < 1$ and L layers. Let $p \sim L$, and consider the space*

$$\mathbb{V}_h^{\mathcal{X}} := \mathcal{S}_0^{p,1}\big(\mathcal{T}^L_{(\mathbf{-1},\mathbf{1})}\big).$$

Then $\mathbb{V}_h^{\mathcal{X}}$ resolves the scales up to σ^L, i.e. there exist constants $C, b > 0$, such that for $z \in \mathscr{S}$ with $|z|^{-1/2} > \sigma^L$ and every f which is analytic on a neighborhood $\widetilde{\Omega}$ of Ω, the solution u_z to $(\mathcal{L} - z)u = zf$ can be approximated by $v_h \in \mathbb{V}_h^{\mathcal{X}}$ satisfying

$$|z|^{-1} \|\nabla u - \nabla v_h\|^2_{L^2(\Omega)} + \|u - v_h\|^2_{L^2(\Omega)} \leq Ce^{-bL} \sim e^{-b\sqrt{N_\Omega}}.$$

The constant b depends only on σ and Ω. The constant C also depends on the constants of analyticity of f.

Proof. See Appendix B. □

The hp-FEM spaces can also approximate the initial conditions at an exponential rate. But more importantly, they can do so in a way that is stable with respect to the non-standard $\mathbb{V}_{h,\beta}^{\mathcal{X}}$ norm. Since interpolation spaces between piecewise polynomials are non-trivial to handle, we use a set of easier subspaces.

Lemma 3.30. *Let $\mathcal{P}_0^p(\Omega) := \operatorname{span}\big\{x^i \; 0 \leq i \leq p\big\} \cap H_0^1(\Omega)$ denote the subspace of $\mathbb{V}_h^{\mathcal{X}}$ consisting of global polynomials. We equip the space $\mathcal{P}_0^p(\Omega)$ with the $\mathbb{V}_h^{\mathcal{X}}$ norm. Assume that the triangulation $\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})}$ used for the discretization in y satisfies $\sigma^L \lesssim p^{-2}$.*

Fix $\beta \in (0, 1)$. Then for all $u \in \mathcal{P}_0^p(\Omega)$ the following estimate holds:

$$\|u\|_{\mathbb{V}_{h,\beta}^{\mathcal{X}}} \lesssim \|u\|_{\widetilde{H}^{s\beta}(\Omega)} .$$

Proof. For $\beta = 0$ there is nothing to do. We consider the case $\beta = 1$. We first note that for any function $\mathscr{V} \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ with $\operatorname{tr}\mathscr{V} = u$, we can estimate $\|u\|_{\mathbb{V}_h^{\mathcal{X}}} \leq \|\mathscr{V}\|_{\mathring{H}^1(y^\alpha,\mathcal{C})}$. This follows from the fact that $\mathscr{L}_h u$ is the "minimum energy" lifting of u. We compute for $\mathscr{W} \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ with $\operatorname{tr}\mathscr{W} = 0$:

$$\begin{aligned}\mathcal{A}(\mathscr{L}_h u - \mathscr{W}, \mathscr{L}_h u - \mathscr{W}) &= \mathcal{A}(\mathscr{L}_h u, \mathscr{L}_h u) - 2\mathcal{A}(\mathscr{L}_h u, \mathscr{W}) + \mathcal{A}(\mathscr{W}, \mathscr{W}) \\ &= \mathcal{A}(\mathscr{L}_h u, \mathscr{L}_h u) + \mathcal{A}(\mathscr{W}, \mathscr{W}) \ \geq \ \mathcal{A}(\mathscr{L}_h u, \mathscr{L}_h u),\end{aligned}$$

where we used $\mathcal{A}(\mathscr{L}_h u, \mathscr{W}) = 0$ for $\mathscr{W} \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ with $\operatorname{tr}\mathscr{W} = 0$ by the definition of the lifting. Setting $\mathscr{W} := \mathscr{L}_h u - \mathscr{V}$ then shows the estimate $\|u\|_{\mathbb{V}_h^{\mathcal{X}}} \leq \|\mathscr{V}\|_{\mathring{H}^1(y^\alpha,\mathcal{C})}$.

Constructing a lifting $\mathscr{V}$ which is stable in the sense that $\|\mathscr{V}\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim \|u\|_{\widetilde{H}^s(\Omega)}$ is the content of Appendix C. The precise construction is carried out in Lemma C.3. This shows the case $\beta = 1$.

The general case follows by interpolation. We note the fact that by Proposition C.2 we can identify

$$\left[\left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{L^2(\Omega)}\right), \left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{H_0^1(\Omega)}\right)\right]_{\theta,2} = \left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{\widetilde{H}^\theta(\Omega)}\right).$$

(Note: in 2d for piecewise polynomials on shape regular meshes the analogous result is shown in [MKR18]).

Using the reiteration theorem [Tar07, Theorem 26.3], we further calculate

$$\begin{aligned}&\left[\left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{L^2(\Omega)}\right), \left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{\widetilde{H}^s(\Omega)}\right)\right]_{\beta,2} \\ &\qquad = \left[\left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{L^2(\Omega)}\right), \left[\left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{L^2(\Omega)}\right), \left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{H_0^1(\Omega)}\right)\right]_{s,2}\right]_{\beta,2} \\ &\qquad = \left[\left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{L^2(\Omega)}\right), \left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{H_0^1(\Omega)}\right)\right]_{s\beta,2} = \left(\mathcal{P}_0^p(\Omega), \|\cdot\|_{\widetilde{H}^{s\beta}(\Omega)}\right),\end{aligned}$$

equality to be understood in the sense of equivalent norms. This concludes the proof by interpolating the identity operator $\mathrm{I} : \left[\mathcal{P}_0^p(\Omega), \|\cdot\|_{\widetilde{H}^\mu(\Omega)}\right] \to \mathbb{V}_h^{\mathcal{X}}$ for $\mu := 0$ and $\mu := s$. ☐

Lemma 3.31. *Assume that the triangulation $\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})}$ used for the discretization in y satisfies $\sigma^L \lesssim p_x^{-2}$, where p_x denotes the (maximal) polynomial degree used for $\mathbb{V}_h^{\mathcal{X}}$.*

Let u_0 be analytic in a neighborhood $\widetilde{\Omega} \supset \overline{\Omega} = [-1, 1]$, and let $0 \leq \beta < 1$ such that $s\beta < 1/2$. Then there exists a function $u_{h,0} \in \mathbb{V}_h^{\mathcal{X}}$ such that

$$\|u_{h,0}\|_{\mathbb{V}_{h,\beta}^{\mathcal{X}}} \lesssim \|u_0\|_{H^{s\beta}(\Omega)} \qquad \text{and} \qquad \|u_{h,0} - u_0\|_{L^2(\Omega)} \lesssim e^{-b' p_x} \leq e^{-b\sqrt{\mathcal{N}_\Omega}}.$$

In other words $\mathbb{V}_h^{\mathcal{X}}$ satisfies Assumption 3.5 in this case.

Proof. Since u_0 is analytic, we do not need to approximate any boundary layers or singularities. We can therefore work with the space $\mathcal{P}_0^p(\Omega)$.

Let $\Pi_{s\beta} : \widetilde{H}^{s\beta}(\Omega) \to \mathcal{P}_0^p(\Omega)$ be the orthogonal projection in the $\widetilde{H}^{s\beta}(\Omega)$-inner product. Then we calculate using Lemma 3.30:

$$\|\Pi_{s\beta} u_0\|_{\mathbb{V}_{h,\beta}^{\mathcal{X}}} \leq \|\Pi_{s\beta} u_0\|_{\widetilde{H}^{s\beta}(\Omega)} \leq \|u_0\|_{\widetilde{H}^{s\beta}(\Omega)} \lesssim \|u_0\|_{H^{s\beta}(\Omega)},$$

where in the last step we used that $s\beta < 1/2$, and thus the $H^{s\beta}(\Omega)$ and $\widetilde{H}^{s\beta}(\Omega)$ spaces coincide with equivalent norms (see [Tri06, Section 1.11.6] or [McL00, Theorem 3.40]).

The approximation estimate then follows from the best approximation property of $\Pi_{s\beta}$ in $\widetilde{H}^{s\beta}$, and standard estimates for the approximation of analytic functions, e.g., [Sch98, Theorem 3.19], where we note that $\mathcal{N}_\Omega \sim p_x^2$. ☐

We can now give a more constructive characterization of how the triangulation of Ω must be chosen when working in 1D to get exponential convergence of the semidiscretization.

Corollary 3.32. *Let $\Omega = (-1,1)$, assume that u_0 is analytic and f is uniformly analytic in neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$. For $M \in \mathbb{N}$ and $\sigma \in (0,1)$, use a geometric mesh with M layers to discretize in x, i.e., $\mathbb{V}_h^{\mathcal{X}} := \mathcal{S}^{p,1}(\mathcal{T}^M_{(\mathbf{-1},\mathbf{1})})$. For discretizing in y, use L layers and a degree vector $\mathbf{r}$ with linear slope $\mathfrak{s}$, i.e., $\mathbb{V}_h^{\mathcal{Y}} := \mathcal{S}^{\mathbf{r},1}(\mathcal{T}^L_{(0,\mathcal{Y})})$. Assume that $\sigma^M \leq \mathcal{Y}(\mathfrak{s}L)^{-2}\sigma^{3L/2}$ and $u_{h,0}$ is as in Assumption 3.5.*

Then there exist constants $b_1, b_2 > 0$ independent of L, M, and p such that the following estimate holds:

$$\int_0^t \|u(\tau) - u_h(\tau)\|^2_{L^2(\Omega)}\, d\tau \lesssim \max(1, t^2\log(t)) \left(e^{-b_1 p} + e^{-b_2 L}\right).$$

Most notably for $M \sim \frac{3}{2}L$ and $p \sim L$, we get exponential convergence:

$$\int_0^t \|u(\tau) - u_h(\tau)\|^2_{L^2(\Omega)}\, d\tau \lesssim \max\left(1, t^2 \log(t)\right) e^{-b' \dim(\mathbb{V}_h^{\mathcal{X},\mathcal{Y}})^{1/4}}.$$

Proof. We choose $t_0 := \sigma^L$ and $z_{\mathrm{hf}} = \sigma^{L/s}L^{1/s}$ in Theorem 3.26. Assumption 3.5 is met via Lemma 3.31, since the condition $\sigma^L \leq p_x^{-2}$ is easily verified for such meshes. The assumptions on $\mathbb{V}_h^{\mathcal{X}}$ also imply that the necessary scales get resolved and we get:

$$\begin{aligned}\int_0^t &\|u(\tau) - u_h(\tau)\|^2_{L^2(\Omega)}\, d\tau \\ &\lesssim \max(1,t^2)\sigma^{\left(\frac{1}{2s}-\delta\right)M} + \max(1,t^2)\left|\log(t_0)\right|^2 \max\left(\log(t/t_0), 0\right)\left[e^{-b_1 p} + e^{-b_2 L} + e^{-L}\right].\end{aligned}$$

The explicit estimate then follows from the fact that $\dim(\mathbb{V}_h^{\mathcal{X}}) \sim \dim(\mathbb{V}_h^{\mathcal{Y}}) \sim L^2$ in this particular construction. We absorb the logarithmic terms $\log(\sigma^L) \sim L$ into the exponential by slightly reducing the rate b'. □

For the pointwise and energy errors, the corresponding concrete version reads:

Corollary 3.33. *Assume that u_0 is analytic and f, $\dot{f}$ are uniformly analytic in a neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$, and that the meshes and spaces are as in Corollary 3.32. Let $u_{h,0} \in \mathbb{V}^{\mathcal{X}}_{h,\beta}$ be as in Assumption 3.5 for $\beta > 0$.*

Then there exists a constant b, independent of L, M and p such that the following estimate holds:

$$\|u(t) - u_h(t)\|^2_{L^2(\Omega)} + \int_0^t \|u(\tau) - u_h(\tau)\|^2_{\widetilde{H}^s(\Omega)}\, d\tau \lesssim \max(1, t^2\log(t)) e^{-bL}.$$

Or in terms of degrees of freedom, we get

$$\|u(t) - u_h(t)\|^2_{L^2(\Omega)} + \int_0^t \|u(\tau) - u_h(\tau)\|^2_{\widetilde{H}^s(\Omega)}\, d\tau \lesssim \max\left(1, t^2\log(t)\right) e^{-b' \dim(\mathbb{V}_h^{\mathcal{X},\mathcal{Y}})^{1/4}}.$$

Proof. Follows from the fact that using the given parameters, the space $\mathbb{V}_h^{\mathcal{X}}$ satisfies the assumptions of Theorem 3.27. The estimate in terms of degrees of freedom follows easily. □

4. Discretization in t – the fully discrete scheme

In this section, we consider the discretization with respect to the time variable t. This can be done using mostly standard techniques. We focus on the case of using a discontinuous Galerkin type method. When applied in its hp-version, it will allow us to get an exponentially convergent fully discrete scheme, and thus it nicely complements our previous investigations. We follow the presentation in [SS00].

Let $\mathcal{T}_{(0,T)} := \{(t_{j-1}, t_j)\}_{j=1}^{M}$ be a partition of the time interval $[0,T]$ into subintervals with $0 \leq t_j < t_{j+1} \leq T$. We set $k_j := t_j - t_{j-1}$ and define the one-sided limits

$$u_j^+ := \lim_{h\to 0, h>0} u(t_j + h) \qquad \text{for } 0 \leq j \leq M-1,$$
$$u_j^- := \lim_{h\to 0, h>0} u(t_j - h) \qquad \text{for } 1 \leq j \leq M$$

as well as the jump $[u]_j := u_j^+ - u_j^-$. We define the DG-bilinear and linear forms:

$$B(\mathscr{U},\mathscr{V}) := \sum_{j=1}^{M}\int_{t_{j-1}}^{t_j} \Big(\operatorname{tr}\dot{\mathscr{U}}(t), \operatorname{tr}\mathscr{V}(t)\Big)_{L^2(\Omega)} + d_s^{-1}\mathcal{A}(\mathscr{U}(t),\mathscr{V}(t))\,dt$$
$$+ \sum_{j=2}^{M}\Big([\operatorname{tr}\mathscr{U}]_{j-1}, \operatorname{tr}\mathscr{V}_{j-1}^{+}\Big)_{L^2(\Omega)} + \big(\operatorname{tr}\mathscr{U}_0^{+}, \operatorname{tr}\mathscr{V}_0^{+}\big)_{L^2(\Omega)},$$
$$F(\mathscr{V}) := \sum_{j=1}^{M}\int_{t_{j-1}}^{t_j} (f(t), \operatorname{tr}\mathscr{V}(t))_{L^2(\Omega)}\;dt + \big(u_0, \operatorname{tr}\mathscr{V}_0^{+}\big)_{L^2(\Omega)}.$$

Then the DG-approximation is given as the solution to the following problem:

Problem 4.1. *Choose $\mathbf{r}_t \subseteq \mathbb{N}_0$ a polynomial degree distribution, and consider the space $\mathcal{S}^{\mathbf{r}_t,0}(\mathcal{T}_{(0,T)})$ of discontinuous piecewise polynomials. Set $\mathbb{V}_h^{\mathcal{X},\mathcal{Y},\mathcal{T}} := \mathcal{S}^{\mathbf{r}_t,0}(\mathcal{T}_{(0,T)}) \otimes \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$. Find $\mathscr{U}_{x,y,t}^{h} \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y},\mathcal{T}}$ such that*

$$B(\mathscr{U}_{x,y,t}^{h}, \mathscr{V}_h) = F(\mathscr{V}_h) \qquad \forall \mathscr{V}_h \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y},\mathcal{T}}. \tag{4.1}$$

Remark 4.2. *Note that we used the initial condition u_0 instead of the discrete initial condition $u_{h,0}$. This is due to the fact that we need assumptions on $u_{h,0}$ which make it non-computable in practice. When we talk about "equivalence to time discretization of the semidiscrete problem" we always mean "up to changing the initial condition", which incurs an additional (but easily treatable) error term.*

Lemma 4.3. *Problem 4.1 is equivalent to solving the "standard" DG-formulation for the semidiscrete semigroup (3.5), i.e. if we define*

$$\widetilde{B}(U,V) := \sum_{j=1}^{M}\int_{t_{j-1}}^{t_j} \Big(\dot{U}(t), V(t)\Big)_{L^2(\Omega)} + (\mathcal{L}_h^s U(t), V(t))_{L^2(\Omega)}\;dt$$
$$+ \sum_{j=2}^{M}\Big([U(t)]_{j-1}, V_{j-1}^{+}\Big)_{L^2(\Omega)} + \big(U_0^{+}, V_0^{+}\big)_{L^2(\Omega)},$$
$$\widetilde{F}(V) := \sum_{j=1}^{M}\int_{t_{j-1}}^{t_j} (f(t), V(t))_{L^2(\Omega)}\;dt + \big(u_0, V_0^{+}\big)_{L^2(\Omega)}.$$

Then $u_{h,k} := \operatorname{tr}(\mathscr{U}^h_{x,y,t}) \in \mathcal{S}^{\mathbf{r},0}(\mathcal{T}_{(0,T)}) \otimes \mathbb{V}^{\mathcal{X}}_h$ *solves*

$$\widetilde{B}(u_{h,k}, v_h) = \widetilde{F}(v_h) \qquad \forall v_h \in \mathcal{S}^{\mathbf{r},0}(\mathcal{T}_{(0,T)}) \otimes \mathbb{V}^{\mathcal{X}}_h. \tag{4.2}$$

On the other hand, we can recover the extended function by $\mathscr{U}^h_{x,y,t} := \mathscr{L}_h u_{h,k}$*.*

Proof. We first show that $\mathscr{L}_h u_{h,k}$ solves Problem 4.1.

Comparing the two formulations, the only interesting term is $\mathcal{A}(\mathscr{L}_h u_{h,k}, \mathscr{V})$. We note that we can write:

$$\begin{aligned}\mathcal{A}(\mathscr{L}_h u_{h,k}, \mathscr{V}_h) &= \mathcal{A}(\mathscr{L}_h u_{h,k}, \mathscr{V}_h - \mathscr{L}_h \operatorname{tr} \mathscr{V}_h) + \mathcal{A}(\mathscr{L}_h u_{h,k}, \mathscr{L}_h \operatorname{tr} \mathscr{V}_h) \\ &= \mathcal{A}(\mathscr{L}_h u_{h,k}, \mathscr{L}_h \operatorname{tr} \mathscr{V}_h) = (\mathcal{L}^s_h u_{h,k}, \operatorname{tr} \mathscr{V}_h)_{L^2(\Omega)},\end{aligned}$$

where we used that $\mathcal{A}(\mathscr{L}_h u_{h,k}, \mathscr{W}_h) = 0$ vanishes for functions with $\operatorname{tr} \mathscr{W}_h = 0$ by the definition of the lifting. Thus all the terms in the formulation directly correspond to each other.

We now show the other direction. Let $\mathscr{U}^h_{x,y,t}$ be a solution to Problem 4.1. We pick a function q, such that $q(t) = 0$ outside of a single interval (t_{j-1}, t_j) on which $q(t)$ is a polynomial. We then test (4.1) with functions of the form $\mathscr{V}_h(t) := q(t)\mathscr{V}_0$, where $\mathscr{V}_0 \in \mathbb{V}^{\mathcal{X},\mathcal{Y}}_h$ satisfies $\operatorname{tr} \mathscr{V}_0 = 0$. This means that $\mathscr{V}_h(t) \in \mathbb{V}^{\mathcal{X},\mathcal{Y},\mathcal{T}}_h$ and we get, since all the terms involving $\operatorname{tr} \mathscr{V}_h$ vanish:

$$\int_{t_{j-1}}^{t_j} \mathcal{A}(\mathscr{U}^h_{x,y,t}(t), \mathscr{V}_0) q(t)\, dt = 0.$$

Since $\mathscr{U}^h_{x,y,t}(t)$ is a polynomial of degree r_j in t, $\mathcal{A}(\mathscr{U}^h_{x,y,t}(t), \mathscr{V}_h)$ also is such a polynomial. Since the integral vanishes when tested with all similar polynomials, we get that $\mathcal{A}(\mathscr{U}(t), \mathscr{V}_h) = 0$ for all $t \in (t_{j-1}, t_j)$ and all admissible $\mathscr{V}_0$. This means we can write $\mathscr{U}^h_{x,y,t} = \mathscr{L}_h \operatorname{tr} \mathscr{U}^h_{x,y,t}$ and we can proceed as before to match all the terms in the formulation to their counterpart. $\square$

Theorem 4.4 (h-version)**.** *Let* u_h *denote the semidiscrete solution to* (3.5)*. Suppose that Assumption 3.5 is fulfilled with* $\beta > 0$*. Let* $\mathbf{r}_t = r \equiv$ *const be a fixed parameter. Choose* $\mathcal{T}_{(0,T)}$ *as a graded mesh with the grading function* $h(t) := t^{\beta(2r+3)}$*. Let* $N := \dim(\mathcal{S}^{\mathbf{r}_t,0}(\mathcal{T}_{(0,T)}))$*.*

Assume u_0 *is analytic in* $\overline{\Omega}$ *and that the right-hand side* f *satisfies*

$$\left\| f^{(\ell)}(t) \right\|_{L^2(\Omega)} \le C d^{\ell} \Gamma(\ell + 1) \qquad \forall t \in [0,T], \ell \in \mathbb{N}_0,$$

with constants C *and* d *independent of* ℓ *and* t*.*

Then the following error estimate holds:

$$\sqrt{\int_0^T \|u_h(t) - u_{h,k}(t)\|^2_{\widetilde{H}^s(\Omega)}} \lesssim N^{-(r+1)} + e^{-b\mathcal{N}^{\mu}_{\Omega}}.$$

The implied constant depends on u_0*,* f*,* r*, the terminal time* T*, and the constant from Assumption 3.5.*

Proof. We note that $u_{h,0} \in \mathcal{V}^{\mathcal{X}}_{h,\beta}$ by Assumption and also that the solution to DG-formulation depends continuously on the initial condition. This last statement can be easily seen from the coercivity of $\widetilde{B}$ as shown in [SS00, Lemma 2.7]. Thus, up to an additional error term $C(T)\,\|\Pi_{L^2} u_0 - u_{h,0}\|^2_{L^2(\Omega)}$ we may use $u_{h,0}$ as our initial condition. (This error term is exponentially small by Assumption 3.5).

We want to apply the results from [SS00] and translate our setting into their requirements. They require separable Hilbert spaces $X \subseteq H$ with continuous, dense and compact embedding and a bilinear form $a(\cdot,\cdot) : X \times X \to \mathbb{C}$, such that

$$|a(u,v)| \lesssim \|u\|_X \|v\|_X, \qquad \operatorname{Re}(a(u,u)) \geq c\|u\|_X^2, \qquad \text{and} \quad a(u,v) = \overline{a(v,u)}$$

for all $u, v \in X$. We set $H := \left(\mathbb{V}_h^{\mathcal{X}}, \|\cdot\|_{L^2(\Omega)}\right)$, $X := \left(\mathbb{V}_h^{\mathcal{X}}, \|\cdot\|_{\mathbb{V}_h^{\mathcal{X}}}\right)$ and $a(u,v) := (\mathcal{L}_h^s u, v)_{L^2(\Omega)}$ (extending the real valued bilinear form to a complex one in the canonical way). By Lemma 3.3 this bilinear form satisfies the boundedness and ellipticity conditions. The symmetry follows from the definition and the symmetry of $\mathcal{A}(\cdot,\cdot)$.

The stated result then is a consequence of [SS00, Theorem 5.10]. The main ingredient is the fact that the initial condition is in the interpolation space $\mathcal{V}_{h,\beta}^{\mathcal{X}}$ by Assumption 3.5. Note that [SS00, Theorem 5.10] gives an estimate in the $\mathbb{V}_h^{\mathcal{X}}$-norm. In order to get to the more natural $\widetilde{H}^s(\Omega)$-norm, we use Lemma 3.3. □

Remark 4.5. *For $r := 1$, the scheme in Theorem 4.4 is equivalent to the more common implicit Euler discretization, except that the right hand side is slightly modified. See [Tho06, Page 205] for details.*

Theorem 4.6 (hp-version)**.** *Let u_h denote the semidiscrete solution to* (3.5)*. Consider $\mathcal{T}_{(0,T)} := \mathcal{T}_{(\mathbf{0},t_1)}^M \cup \mathcal{T}_{(t_1,T)}$ to be a mesh on $(0,T)$ that is geometrically refined towards 0 and has constant size for larger times (t_1, T). We choose $\mathbf{r}_t$ such that it is linearly increasing on the geometrically refined part and constant afterwards. Let $N := \dim(\mathcal{S}^{\mathbf{r}_t,0}(\mathcal{T}_{(0,T)}))$.*

Assume that u_0 is analytic in $\overline{\Omega}$ and that the right-hand side f satisfies

$$\left\|f^{(\ell)}(t)\right\|_{L^2(\Omega)} \leq C d^\ell \Gamma(\ell+1) \qquad \forall t \in [0,T], \ell \in \mathbb{N}_0,$$

with constants C and d independent of ℓ and t. Suppose that Assumption 3.5 is satisfied.

Then the following error estimate holds:

$$\sqrt{\int_0^T \|u_h(t) - u_{h,k}(t)\|_{\widetilde{H}^s(\Omega)}^2} \lesssim e^{-bN^{1/2}} + e^{-b\mathcal{N}_\Omega^\mu}.$$

The implied constant depends on u_0, f, μ, the mesh grading and the terminal time T as well as the constants from Assumption 3.5.

Proof. The proof is completely analogous to Theorem 4.4, except we now invoke [SS00, Section 5.1.2]. □

For the simplified model problem, we can give explicit bounds for the full discretization.

Corollary 4.7. *Assume that we are in the simplified setting of Section 3.3 and let the spaces for $\mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ be designed as in Corollary 3.33. Denote the number of layers used in $\mathbb{V}_h^{\mathcal{Y}}$ as M. Assume that u_0 is analytic and f, $\dot{f}$ are uniformly analytic in a neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$.*

Let $\mathcal{T}_{(0,T)} := \mathcal{T}_{(\mathbf{0},t_1)}^M \cup \mathcal{T}_{(t_1,T)}$ be a mesh on $(0,T)$ which is geometrically refined towards 0 wit M layers and has constant size for larger times (t_1,T). We chose $\mathbf{r}_t$ such that it is linearly increasing on the geometrically refined part and constant afterwards. We take $M \sim L$, where L is the number of levels used for $\mathbb{V}_h^{\mathcal{Y}}$.

In addition, assume that the right-hand side f satisfies

$$\left\| f^{(\ell)}(t) \right\|_{L^2(\Omega)} \leq C d^{\ell} \Gamma(\ell+1) \qquad \forall t \in [0,T], \ell \in \mathbb{N}_0,$$

with constants C and d independent of ℓ and t.

Then there exist constants $C > 0$, $b > 0$ such that the following error estimate holds:

$$\sqrt{\int_0^T \|u(t) - u_{h,k}(t)\|^2_{\widetilde{H}^s(\Omega)}} \lesssim e^{-b\left[\dim(\mathbb{V}_h^{\mathcal{X},\mathcal{Y},\mathcal{T}})\right]^{1/6}}$$

The implied constant depends on u_0, f, end time T, the domain Ω, $\widetilde{\Omega}$, the mesh grading σ as well as on s.

Proof. Follows from Theorem 4.6, Theorem 3.32 and the fact that $\dim(\mathbb{V}_h^{\mathcal{X},\mathcal{Y},\mathcal{T}}) \sim \dim(\mathbb{V}_h^{\mathcal{X},\mathcal{Y}}) \cdot \dim\left(\mathcal{S}^{\mathbf{r}_t,0}(\mathcal{T}_{(0,T)})\right) \sim M^4 \cdot M^2$. □

4.1. Practical aspects

In order to efficiently implement the scheme presented, we combine the Schur-form based approach described in [SS00] with the ideas of [BMN+18] for dealing with the extended variable.

For each time-inteval, the Schur decomposition in time leads to a sequence of problems of the form

$$\sum_{j=0}^{r} T_{ij} w_j + \frac{k}{2} \mathcal{L}_h^s w_j = \text{r.h.s.}, \qquad i = 0, \ldots r$$

where $T \in \mathbb{C}^{r \times r}$ is an upper triangular matrix. These problems can be solved using a backward-substitution, where in each step an operator of the form $\frac{k}{\lambda_j}\mathcal{L}_h^s + \mathrm{I}$ has to be inverted. Structurally this is very similar to the operator $\mathcal{G}^{\lambda}$, except that the parameter $\lambda := \lambda_j/k$ is complex valued. Proceeding like in [BMN+18] would require simultaneous diagonalization of the matrices

$$A_{ij} := \frac{\lambda_j}{k} v_j(0)\overline{v_i(0)} + \left(v_j', v_i'\right)_{L^2(\Omega)} \qquad \text{and } B_{ij} := \left(v_j, v_i\right)_{L^2(\Omega)}.$$

Since the matrix A is not hermitean if $\operatorname{Im}(\lambda_j) \neq 0$, it is unclear whether this diagonalization can be done (in practice it appears to be the case). Instead we employ the generalized Schur-form (or QZ-decomposition; see [GVL96, Section 7.72]). It gives unitary matrices Q and Z, such that $Q^H A Z =: T$ and $Q^H B Z =: S$ are both upper triangular. Inserting this decomposition into the definition of $\frac{k}{\lambda_j}\mathcal{L}_h^s + \mathrm{I}$ and using a backward-substitution leads to a sequence of problems of the form

$$-\kappa_\ell \Delta w_\ell + w_\ell = \text{r.h.s.}$$

for $w \in H_0^1(\Omega)$ with $\kappa_\ell \in \mathbb{C}$.

Overall, Problem 4.1 can be solved by solving $\dim(\mathcal{S}^{\mathbf{r}_t}(\mathcal{T}_{(0,T)})) \times \dim(\mathcal{S}^{\mathbf{r}}(\mathcal{T}^M_{(\mathbf{0},\mathcal{Y})}))$ scalar problems posed on Ω. For the case of the simplified model problem of Section 3.3 using the method described in Corollary 4.7, this means that $\mathcal{O}(M^4)$ problems of size $\mathcal{O}(M^2)$ need to be solved.

5. Numerical Results

In this section we test the theoretical findings of the previous sections by implementing them using the finite element package NGSolve [Sch14, Sch17] for the discretization in Ω.

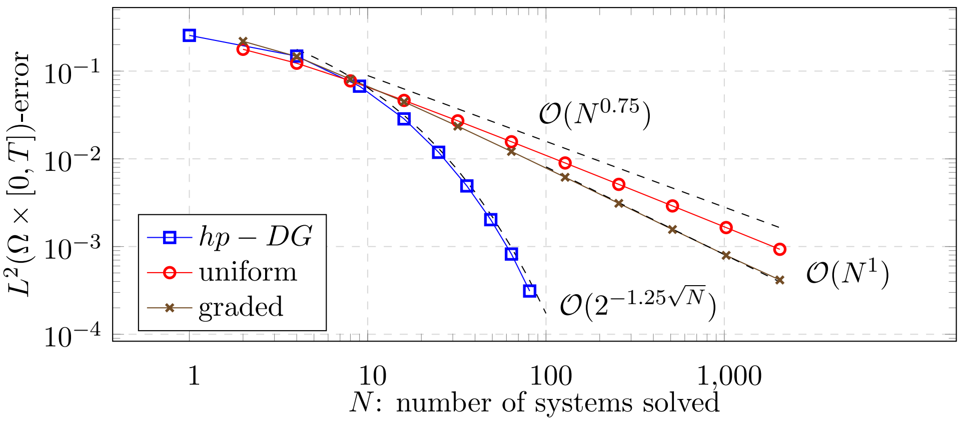


Figure 5.1: Convergence rate in the case of non-matching initial condition

5.1. Smooth solution

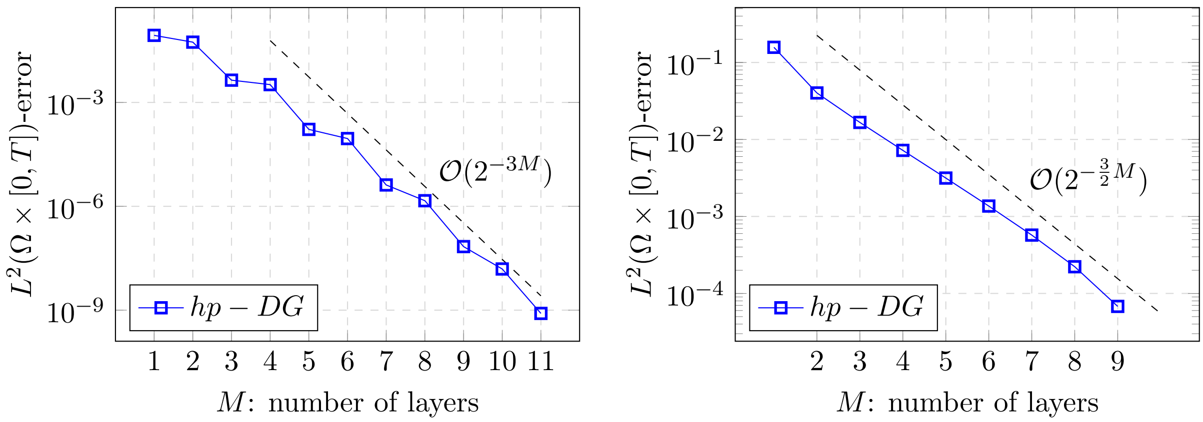
In order to verify our implementation, we consider an example which has a known exact solution. We work with the simplified model problem of Section 3.3. The initial condition is chosen as $u_0(x) := \sin(2\pi\, x)$. As an eigenvalue of the Dirichlet-Laplacian this leads to the exact solution $u(x,t) := e^{-t(2\pi)^s} \sin(2\pi\, x)$. We use $s = 0.5$ and plot our findings, applying the *hp*-DG method. As seen in Figure 5.2a, we get the predicted exponential convergence with respect to the number of refinement layers.

5.2. Singular solution

In order to verify that our method handles startup singularities robustly, we stay in the simplified setting of Section 3.3, but consider the initial condition $u_0 \equiv 1$ and set $s := 0.75$. We use the trivial right-hand side $f \equiv 0$. Since the initial condition does not satisfy any compatibility condition, we expect startup singularities. As the exact solution is unknown, we precompute a numerical solution with high accuracy using the hp-DG method described in Corollary 4.7 with $M = 13$ layers. We integrate up to the terminal time $T = 1$. Due to the predicted exponential convergence, we expect a good match of the estimated error to the (unknown) true error.

We compare different time discretization schemes. For the implicit Euler based schemes we chose a fixed polynomial degree for discretizing x and y to be $p = 8$. For the $hp - DG$ scheme we chose the same polynomial degree in each variable. As an indicator for comparing the numerical cost, we use the number of systems N we need to solve involving the nonlocal operator $\mathcal{L}_h^s$. For the implicit Euler, this is proportional to the number of timesteps. For the $hp - DG$ approach it is proportional to the number of layers M squared, i.e. $N \sim M^2$. In Figure 5.1 we compare the spacetime L^2-error to the number of such systems that need solving. We see that, as predicted, the implicit Euler method with a graded stepsize recovers the full convergence rate $\mathcal{O}(N^{-1})$ whereas a uniform approach only yields a reduced rate. It is important to point out that practical considerations may still favor using a uniform grid, as in this case the corresponding matrices can be factorized only once. This yields much faster solution times in each step. Since the reduction of order is small, the uniform approach often outperforms the graded mesh in our experience.

The best performance, as expected, is observed by the $hp - DG$ based method. It provides rapid exponential convergence of order $\mathcal{O}(e^{-b\sqrt{N}})$, confirming Theorem 4.6 and Corollary 4.7.



(a) Convergence of the *hp*-DG method for a smooth solution

(b) Convergence rate in the case of non-matching initial condition in 2D

Figure 5.2: Convergence for the $2d$ and smooth cases

5.3. A 2d example

Even though our theory does not yet fully cover the case of two spatial dimensions, we considered this problem for our numerical investigation. We chose $\Omega := (0,1)^2$, $u_0 \equiv 1$, $f \equiv 0$, $A :=$ I, $c = 0$ and $s := 1/4$. Since no known analytic solution is avaliable, we computed the approximation using $M = 10$ levels of refinement in time and used it as our reference solution. All computations were done up to the terminal time $T = 1$ and using the *hp*-DG method. For the time discretization and discretization in y, we used a geometric grid with M layers. In Ω we used a geometrically refined grid of $3M/2$ layers in accordance to Corollary 3.26.

In Figure 5.2b, we see that also in this case we get the exponential convergence with respect to the number of layers in the *hp*-refinement. This suggests that our methods could also be extended to cover this case.

A. Proof of Proposition 3.12

The following proof consists of condensed and restated results from [Tho06, Chapter 3]. We fix $t_0 > 0$ and consider the discrete backward problem

$$-\dot{z}_h + \mathcal{L}_h^s z_h = \theta, \quad \text{in } (0,t_0), \quad \text{and} \quad z_h(t_0) = 0. \tag{A.1}$$

For $\tau \in (0,t_0)$, we get by testing (A.1) with θ in the L^2-inner product and using (3.7) and (3.6):

$$\begin{aligned}\|\theta\|^2_{L^2(\Omega)} &= -\left(\dot{z}_h(\tau),\theta(\tau)\right)_{L^2(\Omega)} + \left(\mathcal{L}_h^s z_h(\tau),\theta(\tau)\right)_{L^2(\Omega)} \\ &= -\frac{d}{dt}\left(z_h(\tau),u(\tau)-u_h(\tau)\right)_{L^2(\Omega)} + \left(\rho(\tau),\dot{z}_h(\tau)\right)_{L^2(\Omega)}.\end{aligned}$$

For $0 < \varepsilon < t_0$ we get by integrating, since $z_h(t_0) = 0$:

$$\int_\varepsilon^{t_0} \|\theta(\tau)\|^2_{L^2(\Omega)} \, d\tau \le (z_h(\varepsilon), u(\varepsilon) - u_h(\varepsilon))_{L^2(\Omega)} + \int_\varepsilon^{t_0} \|\rho(\tau)\|_{L^2(\Omega)} \|\dot z_h(\tau)\|_{L^2(\Omega)} \, d\tau$$
$$\le (z_h(\varepsilon), u(\varepsilon) - u_h(\varepsilon))_{L^2(\Omega)} + \left(\int_\varepsilon^{t_0} \|\rho(\tau)\|^2_{L^2(\Omega)} \, d\tau\right)^{1/2} \left(\int_\varepsilon^{t_0} \|\dot z_h(\tau)\|^2_{L^2(\Omega)} \, d\tau\right)^{1/2}. \tag{A.2}$$

In the limit $\varepsilon \to 0$, the first term converges due to Lemma 3.10 (i) to

$$(z_h(\varepsilon), u(\varepsilon) - u_h(\varepsilon))_{L^2(\Omega)} \to (z_h(0), u_0 - u_{h,0})_{L^2(\Omega)} .$$

The following stability estimate holds for z_h by Lemma 3.10 (iv):

$$\int_0^{t_0} \|\dot z_h(\tau)\|^2_{L^2(\Omega)} \, d\tau + t_0^{-1} \|z_h(0)\|^2_{L^2(\Omega)} \lesssim \int_0^{t_0} \|\theta(\tau)\|^2_{L^2(\Omega)} \, d\tau.$$

Combining this estimate with (A.2) completes the proof of (3.8).

Proof of (3.9): For fixed $t > 0$, testing the equation (3.7) with $v := t\,\theta(t)$ and integrating over Ω gives:

$$\frac{1}{2}\frac{d}{dt}\big(t \|\theta(t)\|^2_{L^2(\Omega)}\big) + t\,(\mathcal{L}_h^s \theta(t), \theta(t))_{L^2(\Omega)} = t\,(\dot\rho(t), \theta(t))_{L^2(\Omega)} + \frac{1}{2} \|\theta(t)\|^2_{L^2(\Omega)} .$$

We integrate in t from $\varepsilon > 0$ to t and get:

$$\frac{1}{2} t \|\theta(t)\|^2_{L^2(\Omega)}\big) + \int_\varepsilon^\tau (\mathcal{L}_h^s \theta(\tau), \theta(t))_{L^2(\Omega)} \, d\tau$$
$$\le \frac{1}{2}\varepsilon \|\theta(\varepsilon)\|^2_{L^2(\Omega)} + \sqrt{\int_\varepsilon^t \tau^2 \|\dot\rho(\tau)\|^2_{L^2(\Omega)} \, d\tau} \sqrt{\int_\varepsilon^t \|\theta(\tau)\|^2_{L^2(\Omega)} \, d\tau} + \frac{1}{2}\int_\varepsilon^t \|\theta(\tau)\|^2_{L^2(\Omega)} \, d\tau.$$

We need to bound $\lim_{\varepsilon\to 0} \varepsilon \|\theta(\varepsilon)\|^2_{L^2(\Omega)}$. Writing $\theta = \Pi_h u - u_h = \rho + u - u_h$, we use the fact that u_h and u are bounded by Lemma 3.10 (i). This gives:

$$\lim_{\varepsilon\to 0} \varepsilon \|\theta(\varepsilon)\|^2_{L^2(\Omega)} \le \limsup_{\varepsilon\to 0} \varepsilon \|\rho(\varepsilon)\|^2_{L^2(\Omega)} + \limsup_{\varepsilon\to 0} \varepsilon \|u(\varepsilon) - u_h(\varepsilon)\|^2_{L^2(\Omega)} \quad \le \sup_{\tau\in(0,t)} \tau \|\rho(\tau)\|^2_{L^2(\Omega)}.$$

By using Young's inequality and (3.8), we easily obtain (3.9) from the fact that $\|\theta(t)\|^2_{\widetilde H^s(\Omega)} \lesssim (\mathcal{L}_h^s \theta(t), \theta(t))_{L^2(\Omega)}$ by Lemma 3.3.

B. Singularly perturbed problems in 1D

In this appendix we provide the details for singularly perturbed problems with a perturbation parameter in the complex plane. We recall the definition of $\mathscr{S}$ from Definition 3.6 as

$$\mathscr{S} := \mathbb{C} \setminus \left[\left\{z_0 + z : |\mathrm{Arg}(z)| \le \frac{\pi}{8}, \mathrm{Re}(z) \ge 0\right\} \cup B_{\varepsilon_0}(0)\right],$$

where z_0 is sufficiently small, and depends on the Poincaré constant of Ω.

We consider the 1D problem

$$(\mathcal{L} - z)u_z := -u_z'' + (c - z)u_z = zf \tag{B.1}$$

for $f \in L^2(\Omega)$, which is assumed to be analytic on a neighborhood $\widetilde\Omega \supset \overline\Omega$ with $\Omega = (-1, 1)$ and $u_z(-1) = u_z(1) = 0$. We assume $z \in \mathscr{S}$, as defined in Definition 3.6 and $c \ge 0$ is constant. This is also the setting of Section 3.3.

Remark B.1. *Problem* (B.1) *does not look singularly perturbed the way it is written, but by dividing by $c - z$ we get to the more common form $-\xi u'' + u = \frac{z}{\xi} f$ for $\xi := (c - z)^{-1}$. The way the problem is written here is more convenient for our applications. This is also the reason for the scaling on the right-hand side, since $z\xi^{-1} \sim 1$ for large z.*

We define the differential operator $L_z u := (\mathcal{L} - z)u$ and associated sesquilinear form

$$a_z(u, v) := \int_\Omega \nabla u \overline{\nabla v} + (c - z) \int_\Omega u \, \overline{v},$$

as well as energy norm $\|\cdot\|_{|z|}^2 := \|\nabla \cdot\|_{L^2(\Omega)}^2 + |z| \, \|\cdot\|_{L^2(\Omega)}^2$.

Lemma B.2. *For all $z \in \mathscr{S}$, the bilinear form $a_z(\cdot, \cdot)$ is bounded and elliptic in the energy norm, i.e., there exists $\theta(\xi) \in (-\pi, \pi)$ such that*

$$\|u\|_{|z|}^2 \lesssim \operatorname{Re}\left(e^{i\theta(z)} a_z(u, u)\right) \qquad \text{and} \qquad |a_z(u, v)| \lesssim \|u\|_{|z|} \|v\|_{|z|}.$$

The implied constants do not depend on ξ or u. This implies for the solution u_z to (B.1)*:*

$$\|u_z\|_{|z|} \leq C \, |z|^{1/2} \, \|f\|_{L^2(\Omega)}. \tag{B.2}$$

Proof. We start with the case $|z| \leq 3z_0$. We calculate

$$\begin{aligned}
\operatorname{Re}\left(a_z(u, u)\right) &= \|\nabla u\|_{L^2(\Omega)}^2 + (c - \operatorname{Re}(z)) \, \|u\|_{L^2(\Omega)}^2 \\
&\geq \left(1 - (1 + \varepsilon) \, |z| \, C_P^2\right) \|\nabla u\|_{L^2(\Omega)}^2 + \varepsilon \, |z| \, \|u\|_{L^2(\Omega)}^2 \\
&\gtrsim \left(1 - 3z_0[1 + |\varepsilon|] C_P^2\right) \|\nabla u\|_{L^2(\Omega)}^2 + \varepsilon \, |z| \, \|u\|_{L^2(\Omega)}^2 \\
&\gtrsim \min(1, \varepsilon) \left(\|\nabla u\|_{L^2(\Omega)}^2 + |z| \, \|u\|_{L^2(\Omega)}^2\right),
\end{aligned}$$

as long as we choose ε sufficiently small (but only depending on z_0 and C_P).

We now assume $z \in \mathscr{S}$ with $|z| > 3z_0$. By making the angle of the cone slightly smaller, we may neglect the shift by z_0 and assume that $|(\operatorname{Arg})(z)| \geq \delta > 0$ where δ only depends on z_0. See Figure B.1. For $\alpha \in \mathbb{C}$, we compute:

$$\operatorname{Re}\left(\alpha a_z(u, u)\right) = \operatorname{Re}(\alpha) \, \|\nabla u\|_{L^2(\Omega)}^2 - \operatorname{Re}(\alpha z) \, \|u\|_{L^2(\Omega)}^2$$

Thus it remains to show that we can choose α such that $\operatorname{Re}(\alpha) > 0$ and $-\operatorname{Re}(\alpha z) \sim |z|$. For $\operatorname{Arg}(z) \geq \delta > 0$, we can pick $\alpha := e^{i\frac{\pi - \delta}{2}}$. For $\operatorname{Arg}(z) \leq -\delta < 0$ we use $\alpha := e^{-i\frac{\pi-\delta}{2}}$.

The estimate (B.2) follows from the simple calculation

$$\left|(zf, v)_{L^2(\Omega)}\right| \leq |z|^{1/2} \, \|f\|_{L^2(\Omega)} \, |z|^{1/2} \, \|v\|_{L^2(\Omega)} \lesssim |z|^{1/2} \, \|f\|_{L^2(\Omega)} \, \|v\|_{|z|}$$

for all $v \in H^1(\Omega)$ and the Lax-Milgram lemma. □

The previous lemma ensures existence and uniqueness of solutions u_z. In the next one we further prove that u_z is analytic with explicit bounds on the derivative with respect to the parameter z.

Lemma B.3. *Let u_z solve* (B.1)*, and let f be analytic on $\overline{\Omega}$ and satisfy*

$$\left\|f^{(p)}\right\|_{L^\infty(\Omega)} \lesssim C_f \gamma_f^p p! \qquad \forall p \in \mathbb{N}_0. \tag{B.3}$$

Then u_z is analytic on $\overline{\Omega}$ and satisfies:

$$\left\|u_z^{(p)}\right\|_{L^2(\Omega)} \leq C K^p \max(p, \sqrt{|z|})^p \qquad \forall p \in \mathbb{N}_0. \tag{B.4}$$

Proof. The proof can be taken verbatim from [Mel97]. We note that an induction argument easily gives that u_z is smooth since $u_z \in H^1 \Rightarrow u_z'' = -zf + (z-c)u_z \in H^1$ etc. For simplicity we assume $|z| > c$, the case for small z can be shown similarly but is not of interest here.

Fix $K > \max(1, \gamma_f)$ such that

$$2\,K^{-2}\left[C_f\left(\frac{\gamma_f}{K}\right)^n + 1\right] < 1 \qquad \forall n \in \mathbb{N}_0.$$

For $p = 0,\ 1$, the estimate (B.4) follows from the Lax-Milgram lemma and Lemma B.2, as long as we choose $C > 0$ sufficiently large. We now proceed by induction on p. Differentiating the equation (B.1), we get:

$$u_z^{(p+2)} = zf^{(p)} + (z-c)u_z^{(p)},$$

or for the norm by inserting (B.3), the induction assumption (B.4) and using the assumption $c \le |\xi|$:

$$\begin{aligned}
\left\|u_z^{(p+2)}\right\|_{L^2(\Omega)} &\le |z|\left[\left\|f^{(p)}\right\|_{L^2(\Omega)} + 2\left\|u_z^{(p)}\right\|_{L^2(\Omega)}\right] \le |z|\left[2C_f\gamma_f^p p^p + 2\left\|u_\xi^{(p)}\right\|_{L^2(\Omega)}\right]\\
&\le 2\,|z|\left[C_f\gamma_f^p p^p + CK^p\max\left(p,\sqrt{|\xi|}\right)^p\right]\\
&\le C\,|z|\,K^{p+2}\max\left(p+1,\sqrt{|\xi|}\right)^p 2\,K^{-2}\left(C_f\left(\frac{\gamma_f}{K}\right)^p + 1\right)\\
&\le CK^{p+2}\max\left(p+1,\sqrt{|\xi|}\right)^{p+2}.
\end{aligned}$$

□

Lemma B.4. *Assume that f is analytic on a fixed neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$ and satisfies*

$$\left\|f^{(p)}\right\|_{L^\infty(\Omega)} \lesssim C_f\gamma_f^p p! \qquad \forall p \in \mathbb{N}_0.$$

Let u_z denote the solution to $L_u u_z = zf$ with $u_z(\pm 1) = 0$ for $z \in \mathscr{S}$.

Then there exist $C,\ \gamma,\ b > 0$ independent of z such that u_z can be decomposed as

$$u_z = w_z + u_z^{BL} + r_z$$

with the following properties:

(i) $\left\|w_z^{(p)}\right\|_{L^\infty(\Omega)} \le C\gamma^p p!$ *for all* $p \in \mathbb{N}_0$,

(ii) $\left|u_z^{BL}(z)\right| \le C\gamma^p \max(p!, |z|^{p/2})e^{-b\rho(x)\sqrt{|z|}}$ *with* $\rho(x) := \max\big(|x-1|, |x+1|\big)$,

(iii) $\|r_z\|_{|z|} \lesssim C\,|z|^{1/2}\,e^{-b\sqrt{|z|}}$,

(iv) $r_z(\pm 1) = 0$,

Proof. We first note that w.l.o.g we can assume that $|z| \ge 2\max(z_0, c)$ as for small parameters z we may chose $w_z := u_z$, $u_z^{\mathrm{BL}} := r_z := 0$ by Lemma B.3.

We define $\xi := c - z$ and $\widetilde{f} := \frac{z}{\xi}f$. Then u_z solves

$$L_\xi u_z := -\xi^{-1}u_z'' + u_z = \widetilde{f}.$$

Since $|z| \geq 2c$, it is easy to check that $|z| \sim |\xi|$ and $\left\|f^{(p)}\right\| \sim \left\|\widetilde{f}^{(p)}\right\|$ for all $p \in \mathbb{N}_0$ and any norm and thus we may exchange one for the other in estimates whenever it is convenient.

We follow [Mel02, Lemma 7.1.1] almost verbatim, the only difference is that we allow complex parameters ξ. Let $M \in \mathbb{N}_0$ be fixed, to be chosen later. We define the outer expansion as

$$w_M(x) := \sum_{j=0}^{M} \xi^{-j} \widetilde{f}^{(2j)}(x).$$

Direct calculation shows that the defect of w_M is small, i.e., it solves:

$$\widetilde{f} - L_\xi w_M = \xi^{-(M+1)} \widetilde{f}^{(2M+1)}.$$

We next construct the boundary layer function in order to fix the boundary conditions of w_M. Defining u_M^{BL} as the solution to

$$L_z u_M^{\mathrm{BL}} = 0 \qquad \text{in } \Omega, \qquad\qquad u_M^{\mathrm{BL}}(\pm 1) = w_M(\pm 1),$$

this solution can be written as

$$u_M^{\mathrm{BL}}(x) = A_M^- e^{-(1+x)\sqrt{\xi}} + A^+ e^{-(1-x)\sqrt{\xi}}.$$

(We consider the branch of the square root satisfying $\mathrm{Re}(\xi) \geq 0$). The constants $A_M^{\pm}$ can be directly computed from the boundary values by

$$\begin{pmatrix} A_M^- \\ A_M^+ \end{pmatrix} = \frac{1}{1 - e^{-4\sqrt{\xi}}} \begin{pmatrix} -1 & e^{-2\sqrt{\xi}} \\ e^{-2\sqrt{\xi}} & -1 \end{pmatrix} \begin{pmatrix} w_M(-1) \\ w_M(1) \end{pmatrix}.$$

Since for larger $|z| > 2z_0$, the set $\mathscr{S}$ does not contain the negative real axis, the parameter $\xi = c - z$ for $z \in \mathscr{S}$ avoids the poles $\sqrt{\xi} = in\pi/2$. We also have $\mathrm{Re}(\sqrt{\xi}) \geq 0$, for which $e^{-\sqrt{\xi}}$ stays bounded. Thus, we can directly see that $\left|A_M^-\right| + \left|A_M^+\right| \lesssim \|w_M\|_{L^\infty(\Omega)}$, with an implied constant independent of ξ and m.

We now need to show the exponential decay of u_M^{BL}. For this we investigate $\mathrm{Re}(\sqrt{\xi})$, as it determines the rate. Since we have assumed $|z| \geq 2z_0$ and $\mathrm{Arg}(z_0 + z) \notin (-\pi/8, \pi/8)$, the argument of $-z$ is bounded away from π by some constant $\delta > 0$. Since $c > 0$ is just an additional shift to the right, this also holds for $\xi = c - z$.

Assume first that $\mathrm{Arg}(\xi) \in (0, \pi - \delta)$. Then the square root satisfies $\mathrm{Arg}(\sqrt{\xi}) \in (0, \frac{\pi-\delta}{2})$ and $\mathrm{Re}(\sqrt{\xi}) = \sqrt{|\xi|}\cos(\mathrm{Arg}(\sqrt{\xi}) \geq \sqrt{|\xi|}\cos\left(\frac{\pi-\delta}{2}\right)$. This means

$$\left|e^{-\rho(x)\sqrt{\xi}}\right| \leq e^{-\sqrt{|\xi|}\cos\left(\frac{\pi-\delta}{2}\right)} \lesssim e^{-b\sqrt{|z|}}.$$

See Figure B.1 for the geometric considerations. The case $\mathrm{Arg}(\xi) \in (\pi + \delta, 2\pi)$ is analogous.

Taking derivatives and using the definition $\rho(x) := \max\big(|x-1|, |x+1|\big)$ we can bound:

$$\left|\left(u_M^{\mathrm{BL}}\right)^{(p)}\right| \leq C\,|\xi|^{p/2}\, e^{-\rho(x)\sqrt{|z|}}\, \|w_M\|_{L^\infty(\Omega)},$$

again with C independent of M and z. The remainder term is defined by $r_M := w_M - u_M^{\mathrm{BL}}$. It solves the elliptic problem

$$L_\xi r_M = \xi^{-(M+1)} \widetilde{f}^{(2M+1)} \qquad \text{in } \Omega, \qquad\qquad r_M^{\mathrm{BL}}(\pm 1) = 0.$$

By Lemma B.2, we get the bound

$$\|r_M\|_{|z|} \lesssim |z|^{1/2} |z|^{-(M+1)} \left\| f^{(2M+1)} \right\|_{L^2(\Omega)} \lesssim C_f |z|^{1/2} \left(\gamma_f |z|^{-1/2} (2M+2) \right)^{(2M+2)}.$$

We can also bound, using $(2j+p)^{2j+p} \leq (2j)^p p^p e^{2j+p}$:

$$\left\| w_M^{(p)} \right\|_{L^\infty(\Omega)} \leq C_f p^p e^p S(M), \quad \text{with } S(M) := \sum_{j=0}^{M} \left(e\gamma_f |\xi|^{1/2} (2j) \right)^{2j}.$$

Overall we obtain the following estimates:

$$\begin{aligned} \left\| w_M^{(p)} \right\|_{L^\infty(\Omega)} &\leq C_f p^p e^p |z|^{-1} S(M), \\ \left| \left(u_M^{\mathrm{BL}}\right)^{(p)} \right| &\leq C |z|^{p/2} e^{-b\rho(x)\sqrt{|z|}} S(M), \\ \|r_M\|_{|z|} &\leq C_f |z|^{1/2} \left(\gamma_f |z|^{-1/2} (2M+2) \right)^{(2M+2)}. \end{aligned}$$

We now choose M in order to minimize these contributions, namely we fix M such that

$$2M + 2 = \left\lfloor b |z|^{1/2} \right\rfloor \qquad \text{with} \qquad b := \frac{1}{e^2 \gamma_f}.$$

This gives $e\gamma f |\xi|^{1/2} (2M+2) \leq e^{-1}$ and $(2M+2) \geq b|z|^{1/2} - 1$. Thus

$$\begin{aligned} (e\gamma_f |z|^{1/2} (2M+2))^{2M+2} &\leq e^{-2(M+2)} \leq e e^{-b\sqrt{|z|}}, \\ S(M) &\leq C_f \sum_{j=0}^{\infty} e^{-2(2j)} \leq C_f \frac{1}{1-e^4}. \end{aligned}$$

These estimates show that the decomposition for this choice of M has all the properties stated in the theorem. □

Next, we recall a result on the approximation of solutions to singularly perturbed problems on so-called minimal meshes.

Lemma B.5. *Fix $z \in \mathscr{S}$ and let u_z solve* (B.1). *For $\kappa > 0$, we define the nodes*

$$x_0 := -1,\ x_1 := -1 + \min(0.5, \kappa),\ x_2 := 1 - \min(0.5, \kappa),\ x_3 := 1$$

and define the minimal mesh $\mathcal{T}^{\kappa}_{min} := \{(x_0, x_1),\ (x_1, x_2)\ (x_2, x_3)\}$.

Then there exist constants C, b, $\lambda_0 > 0$ such that for all $p \in \mathbb{N}$ and $\lambda \in (0, \lambda_0)$

$$\inf_{v_h \in \mathcal{S}^{p,1}\left(\mathcal{T}^{\lambda p}_{min}\right)} |z|^{-1} \|u - v_h\|^2 + \|u - v_h\|^2_{L^2(\Omega)} \leq C e^{-bp}.$$

Proof. Follows verbatim to [Mel02, Proposition 2.2.5], see also [Mel97, Theorem 16]. The main ingredients to go from the case of real parameter ξ to the complex case are given by Lemmas B.4 and B.3. □

We are now in a position to complete the proof of Theorem 3.29. As in [BMN$^+$18], we observe that a geometrically refined mesh is a refinement of the three element mesh

$$\left\{(-1, -1 + \lambda |z|^{-1/2}), \{(-1 + \lambda |z|^{-1/2}, 1 - \lambda |z|^{-1/2}), \{(1 - \lambda |z|^{-1/2}, 1)\right\}$$

for $\lambda := |z|^{1/2} \sigma^\ell$ and $\ell \leq L$. Thus we can apply Lemma B.5 to get the stated estimate. Since in one spatial dimension we have $\dim(\mathbb{V}_h^{\mathcal{X}}) = \dim\left(\mathcal{S}_0^{p,1}(\mathcal{T}^L_{(-\mathbf{1},\mathbf{1})})\right) \sim\ p \cdot L \sim L^2$ this concludes the proof.

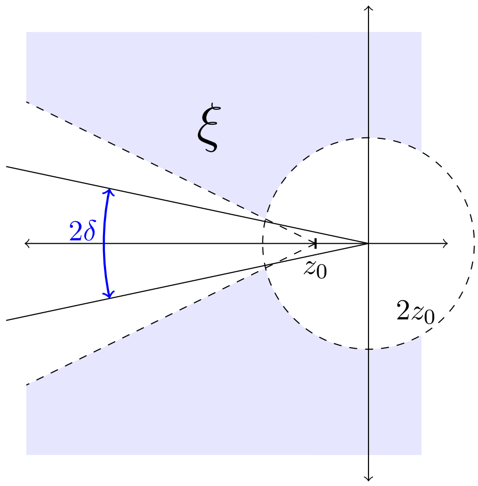


Figure B.1: The geometric situation in the proof of Lemma B.4 (for $c = 0$)

C. Polynomial liftings and interpolation spaces

In this section, we investigate under which conditions we can lift discrete functions from $\mathbb{V}_h^{\mathcal{X}}$ to functions in $\mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ in a stable way. This question is deeply related to the theory of interpolation of discrete polynomial spaces. This can be seen in the following proposition:

Proposition C.1 ([Tar07, Lemma 40.1]). *Let X_0, X_1 be Banach spaces with $X_1 \subseteq X_0$ continuously embedded. For $\theta \in (0,1)$, denote the interpolation space by $X_\theta := [X_0, X_1]_{\theta,2}$. Then the following statements hold:*

(i) If v is a X_0-valued function such that $v(t) \in X_1$ and $\dot{v}(t) \in X_0$ for all $t > 0$ and $t^{1-\theta} \|\dot{v}(t)\|_{X_0} \in L^2(\mathbb{R}_+, \frac{dt}{t})$, $t^{1-\theta} \|v(t)\|_{X_1} \in L^2(\mathbb{R}_+, \frac{dt}{t})$ then $v(0) \in X_\theta$ with

$$\|v(0)\|_{X_\theta}^2 \lesssim \int_0^\infty t^{1-2\theta} \left[\|\dot{v}(t)\|_{X_0}^2 + \|v\|_{X_1}^2\right] dt.$$

(ii) If $v_0 \in X_\theta$, there exists a function $v : \mathbb{R}_+ \to X_1$ such that $v(0) = v_0$ and

$$\int_0^\infty t^{1-2\theta} \left[\|\dot{v}(t)\|_{X_0}^2 + \|v\|_{X_1}^2\right] dt \lesssim \|v_0\|_{X_\theta}^2 .$$

Proof. This is just a special case of [Tar07, Lemma 40.1]. We note that in comparison to the statement in the book we changed the roles of X_0 and X_1. But since

$$[X_1, X_0]_{\theta,2} = [X_0, X_1]_{1-\theta,2}$$

by [Tar07, Lemma 25.4], the theorem holds in the stated form. □

The case of lifting a polynomial on $[0,1]$ to the unit square was addressed in [BDM07]. Namely, the following holds:

Proposition C.2 ([BDM07]). *Let $\mathcal{P}_0^p([0,1])$ denote the space of polynomials $u \in \mathcal{P}^p([0,1])$ with $u(0) = u(1) = 0$. Then the following statements hold:*

(i) *The interpolation norm coincides with the Sobolev norm, i.e., for all* $\theta \in (0,1)$

$$\left[\left(\mathcal{P}_0^p, \|\cdot\|_{L^2([0,1])}\right), \left(\mathcal{P}_0^p, \|\cdot\|_{H_0^1([0,1])}\right)\right]_{\theta,2} = \left(\mathcal{P}_0^p, \|\cdot\|_{\widetilde{H}^\theta([0,1])}\right)$$

with equivalent norms. The implied constant depends only on θ.

(ii) *For all* $u \in \mathcal{P}_0^p([0,1])$, *there exists a polynomial* $\mathscr{U} \in \mathcal{Q}^p([0,1]^2) := \operatorname{span}\{x^i y^i, 0 \leq i,j \leq p\}$ *such that* $\operatorname{tr}\mathscr{U} = u$, $\mathscr{U}(\cdot, y) \in \mathcal{P}_0^p([0,1])$ *for all* $y \in [0,1]$. *For* $y > 1$, $\mathscr{U}$ *can be extended by* 0 *to* $\mathbb{R}_+$ *such that* $\|\mathscr{U}\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \lesssim \|u\|_{\widetilde{H}^s([0,1])}$.

Proof. See Theorem 4.6 and Proposition 3.16 in [BDM07]. □

The previous Proposition gives a lifting to the space of polynomials in the extended variable y. Since we will be working with piecewise polynomials with a linear degree vector this is not sufficient for our needs. We need the following variation of the previous result:

Lemma C.3. *Let* $u \in \mathcal{P}_0^p([0,1])$. *Assume that the triangulation* $\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})}$ *satisfies* $\operatorname{diam}(K_0) \leq p^{-2}$, *where* K_0 *is the element at* 0. *Then there exists a lifting* $\mathscr{U}_h \in \mathbb{V}_h^{\mathcal{X},\mathcal{Y}}$ *such that*

$$\|\mathscr{U}_h\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \leq C\,\|u\|_{\widetilde{H}^s(\Omega)} \qquad \text{and} \qquad \operatorname{tr}\mathscr{U}_h = u.$$

The constant C *depends only on* s *and the mesh grading parameter* σ. *The lifting can be chosen to be piecewise linear with respect to* y.

Proof. By Propositions C.1 and C.2, there exists a lifting $\mathscr{U} \in C(\mathbb{R}_+, \mathcal{P}_0^p([0,1])$ such that

$$\|\mathscr{U}\|_{\mathring{H}^1(y^\alpha,\mathcal{C})} \leq C\,\|u\|_{\widetilde{H}^s(\Omega)}.$$

Inspecting the proof of Proposition C.1, as given in [Tar07], one can see that the lifting $\mathscr{U}$ is piecewise linear on the grid $\left(e^n\right)_{n\in\mathbb{Z}}$. By a simple rescaling, we may choose $\mathscr{U}$ as piecewise linear in y on the geometric mesh σ^n for $n \in \mathbb{Z}$. To get a function which is in the space $\mathcal{S}^{1,1}(\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})})$, we need to make two modifications: modify $\mathscr{U}$ on the element $K_0 := (0, \sigma^L)$ to also be linear and cut the function off at $\mathcal{Y}$. We define $h_0 := \operatorname{diam}(K_0) = \sigma^L$.

We define $\mathscr{U}_h(\cdot, t)$ as the linear interpolation between $u = \mathscr{U}(0)$ and $\mathscr{U}(\sigma^L)$ on K_0 and $\mathscr{U}_h = \mathscr{U}$ otherwise. We need to show:

$$\int_{K_0} y^\alpha \|\partial_y \mathscr{U}_h(y)\|^2_{L^2(\Omega)}\,dy \lesssim \|\mathscr{U}\|^2_{\mathring{H}^1(y^\alpha,\mathcal{C})}, \tag{C.1}$$

$$\int_{K_0} y^\alpha \|\nabla_x \mathscr{U}_h(y)\|^2_{L^2(\Omega)}\,dy \lesssim \|\mathscr{U}\|^2_{\mathring{H}^1(y^\alpha,\mathcal{C})}. \tag{C.2}$$

We start with the first inequality. Since $\mathscr{U}_h$ is the linear interpolant of $\mathscr{U}$, we can write $\partial_y \mathscr{U}_h = h_0^{-1}\int_0^{h_0} \partial_y \mathscr{U}(\tau)\,d\tau$. This gives:

$$\begin{aligned}
\int_{K_0} y^\alpha \|\partial_y \mathscr{U}_h(y)\|^2_{L^2(\Omega)}\,dy &\lesssim h_0^{-2}\int_{K_0} y^\alpha \left(\int_0^{h_0} \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)}\,d\tau\right)^2 dy \\
&\lesssim \underbrace{h_0^{-2}\int_{K_0} y^\alpha \left(\int_0^{y} \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)}\,d\tau\right)^2 dy}_{=:I_1} + \underbrace{h_0^{-2}\int_{K_0} y^\alpha \left(\int_y^{h_0} \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)}\,d\tau\right)^2 dy}_{=:I_2}.
\end{aligned}$$

We first investigate the term I_1. Using the fact that $y \le h_0$ and therefore $h_0^{-2} \le y^{-2}$, we get

$$I_1 \le \int_0^{h_0} y^\alpha \left(y^{-1} \int_0^y \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy \le \int_0^{h_0} y^\alpha \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)}^2 \, dy$$

by Hardy's inequality(see [Gri85, page 28]).

When investigating I_2, we distinguish $\alpha \ge 0$ and $\alpha \le 0$. For $\alpha \ge 0$ we have $y^\alpha \le \tau^\alpha$ for $y \le \tau$ and thus after applying Cauchy Schwarz to get the square into the integral:

$$\begin{aligned} h_0^{-2} \int_{K_0} y^\alpha \left(\int_y^{h_0} \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy &\le h_0^{-2} \int_{K_0} \left(\int_y^{h_0} \tau^\alpha \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy \\ &\le \int_y^{h_0} \tau^\alpha \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)}^2 \, d\tau \le \|\mathscr{U}\|_{\mathring{H}^1(y^\alpha, \mathcal{C})}^2 . \end{aligned}$$

For $\alpha \le 0$, we have $h_0^\alpha \le \tau^\alpha$ and get:

$$\begin{aligned} h_0^{-2} \int_{K_0} y^\alpha \Big(\int_y^{h_0} \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \Big)^2 dy &\le h_0^{-2} \int_{K_0} y^\alpha \left(\int_0^{h_0} h_0^{-\alpha} \tau^\alpha \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy \\ &\lesssim h_0^{-2} h_0^{-\alpha} h_0^{\alpha+1} \left(\int_0^{h_0} \tau^\alpha \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 \le \|\mathscr{U}\|_{\mathring{H}^1(y^\alpha, \mathcal{C})}^2 , \end{aligned}$$

which proves (C.1).

We now show (C.2). The proof relies on an inverse estimate and the fact that $\mathscr{U}_h$ approximates $\mathscr{U}$. We estimate:

$$\begin{aligned} \int_{K_0} y^\alpha \|\nabla_x \mathscr{U}_h(y)\|_{L^2(\Omega)}^2 \, dy &\lesssim \int_{K_0} y^\alpha \|\nabla_x \mathscr{U}_h(y) - \nabla_x \mathscr{U}(y)\|_{L^2(\Omega)}^2 \, dy + \int_{K_0} y^\alpha \|\nabla_x \mathscr{U}(y)\|_{L^2(\Omega)}^2 \, dy \\ &\lesssim \int_{K_0} y^\alpha \|\nabla_x \mathscr{U}_h(y) - \nabla_x \mathscr{U}(y)\|_{L^2(\Omega)}^2 \, dy + \|\mathscr{U}\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} . \end{aligned}$$

Since $\mathscr{U}_h(\cdot, y)$ and $\mathscr{U}(\cdot, y)$ are polynomials on $[0,1]$ for all fixed y, we can use an inverse estimate [Sch98, Theorem 3.91] to get:

$$\int_{K_0} y^\alpha \|\nabla_x \mathscr{U}_h(y) - \nabla_x \mathscr{U}(y)\|_{L^2(\Omega)}^2 \, dy \lesssim p^4 \int_{K_0} y^\alpha \|\mathscr{U}_h(y) - \mathscr{U}(y)\|_{L^2(\Omega)}^2 \, dy.$$

Since $\mathscr{U}_h - \mathscr{U}$ vanishes at $y = 0$, we can write it as $\mathscr{U}_h(y) - \mathscr{U}(y) = \int_0^y \partial_y \mathscr{U}_h(\tau) - \partial_y \mathscr{U}(\tau) \, d\tau$ and further estimate:

$$\begin{aligned} \int_{K_0} y^\alpha \|\mathscr{U}_h(y) - \mathscr{U}(y)\|_{H^1(\Omega)}^2 \, dy &\lesssim p^4 \int_{K_0} y^\alpha \left(\int_0^y \|\partial_y \mathscr{U}_h(\tau)\|_{L^2(\Omega)} + \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy \\ &\lesssim \underbrace{p^4 \int_{K_0} y^\alpha \left(\int_0^y \|\partial_y \mathscr{U}_h(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy}_{=:I_3} + \underbrace{p^4 \int_{K_0} y^\alpha \left(\int_0^y \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy}_{=:I_4} . \end{aligned}$$

The term I_3 is structurally analogous to the term (C.2) and can be estimated using the same techniques. The extra integration in τ gives an additional power of h_0^2, and we get

$$I_3 \le p^4 h_0^2 \|\mathscr{U}\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} .$$

For the term I_4, we apply Hardy's inequality and the estimate $h_0^{-2} \leq y^{-2}$ to get:

$$\begin{aligned} I_4 &= p^4 \int_{K_0} y^\alpha \left(\int_0^y \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy \lesssim p^4 h_0^2 \int_{K_0} y^\alpha \left(\frac{1}{y} \int_0^y \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)} \, d\tau \right)^2 dy \\ &\lesssim p^4 h_0^2 \int_{K_0} y^\alpha \|\partial_y \mathscr{U}(\tau)\|_{L^2(\Omega)}^2 \, dy = p^4 h_0^2 \|\mathscr{U}\|_{\mathring{H}^1(y^\alpha, \mathcal{C})} . \end{aligned}$$

Overall, since we assumed $h_0 \leq p^{-2}$, we get the stability of the modified lifting.

In order to get $\operatorname{supp} \mathscr{U}_h \subset [0, \mathcal{Y}]$, we pick the cutoff function $\varphi \in \mathcal{S}^{1,1}(\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})})$ such that $\varphi|_{K_i} = 1$ on K_i for $i = 0, \dots, |\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})}| - 1$ and $\varphi(\mathcal{Y}) = 0$. We note that the element where φ is non-constant has size $\mathcal{O}(1)$, and it can be easily checked that $\varphi \cdot \mathscr{U}_h$ is also a stable lifting of u. In order to get a function in $\mathcal{S}^{1,1}\left(\mathcal{T}^L_{\mathcal{Y}}, \mathcal{P}^p_0([0,1])\right)$ we interpolate the function in the grid points. Since $\mathscr{U}_h \cdot \varphi$ is a polynomial of degree at most 2, interpolating it down to degree 1 is stable in the L^2 and H^1 norm (see [BM97, Rem. 13.5 and (13.27)]). Away from 0, the weighted norms are equivalent to the standard ones. This shows that the "cutoff and interpolation"-procedure is stable in $\mathring{H}^1(y^\alpha, \mathcal{C})$. ☐

Remark C.4. *In higher dimensions, Lemma C.3 could also be generalized to spaces $\mathcal{S}^{p,1}(\mathcal{T}_\Omega)$ as long as $\mathcal{T}_\Omega$ is a shape regular triangulation of Ω. The main ingredient is the equivalence of the discrete interpolation norm to the Sobolev norm. This is more involved than in the 1D single element case and is part of the upcoming work [MKR18]. The requirement on $\mathcal{T}^L_{(\mathbf{0},\mathcal{Y})}$ would then read $h_0 \leq p^{-2} h_x$ where h_x is the minimum mesh width in $\mathcal{T}_\Omega$.*

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