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On superconvergence of Runge-Kutta convolution quadrature for the wave equation

Jens Markus Melenk, Alexander Rieder

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The semidiscretization of a sound soft scattering problem modelled by the wave equation is analyzed. The spatial treatment is done by integral equation methods. Two temporal discretizations based on Runge-Kutta convolution quadrature are compared: one relying on the incoming wave as input data and one based on its temporal derivative. The convergence rate of the latter is shown to be higher than previously established in the literature. Numerical results indicate sharpness of the analysis.

1 Introduction

Boundary element methods have established themselves as one of the standard methods when dealing with scattering problems, especially if the domain of interest is unbounded. First introduced for stationary problems, beginning with the seminal works [BH86a, BH86b] the theory for deploying these kind of methods also for time dependent problems has been steadily extended; see [Say16] for an overview. The method of convolution quadrature, introduced by Lubich in [Lub88a, Lub88b], is a convenient way of extending the stationary results to a time dependent setting.

It is well-known that the convergence rate of a Runge-Kutta convolution quadrature (as introduced in [LO93]), is determined by bounds on the convolution symbol K in the Laplace domain. Namely, a bound of the form

$$\|K(s)\| \leq C\,|s|^{\mu}\,.$$

leads to convergence rate $q+1-\mu$, as was proven in [BLM11], see also [BL11, LO93] for earlier results in this direction. Thus one might expect that changing the symbol to $s^{-1}K(s)$ would increase the convergence order by one.

When considering discretizations of the wave equation using boundary integral methods, this is not always the case. Instead, it has been observed that sometimes a “superconvergence phenomenon” appears, where the observed convergence rate surpasses those predicted, see [RSM19a, RSM19b, Rie17].

In this paper, we give a first explanation why such a phenomenon occurs in the model problem of sound soft scattering, i.e., the discretization of the Dirichlet-to-Neumann map. We expect that similar phenomena can also explain the improved convergence rate for the Neumann problem or more complex scattering problems. The proof relies on the observation that the s^{-1}-weighted Dirichlet-to-Neumann map can be decomposed into a Dirichlet-to-Impedance map plus the identity operator. For the Dirichlet-to-Impedance operator, it was observed in [Ban14] that an improved bound holds compared to the Dirichlet-to-Neumann map as long as the geometry is given by the sphere or the half-space. It is then conjectured that a similar bound holds for smooth, convex geometries. In this paper, we generalize this result to a much broader class of geometries (namely smooth or polygonal) without convexity assumption. This will then immediately give the stated improved bound for the convolution quadrature scattering problem.

We conclude that due to this phenomenon, it may often be beneficial to slightly tweak the formulation to work with an extra time derivative. In many situations, such formulations are even the natural choice, see, e.g., [BR18, BL18, BLS15]. Especially when working with the wave equation as a first order system as in [RSM19b].

Another way of looking at this phenomenon is that when using a standard formulation (see Proposition 3.2), then the discrete integral will exhibit a superconvergence effect.

We would like to point out that the present paper focuses on a semidiscretization of the problem with respect to the time variable. For practical purposes one would also have to take into account the discretization in space using boundary elements.

We also would like to note that while popular, convolution quadrature is only one possibility to apply boundary integral techniques to wave propagation problems. Notably also space-time based methods have gained popularity [GNS17, GMO+18, JR17] in recent years.

2 Model Problem

We consider a sound soft scattering problem for acoustic waves. For a bounded Lipschitz domain $\Omega^- \subseteq \mathbb{R}^d$ with $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$, the problem reads

$$\ddot{u}^{\text{tot}} = \Delta u^{\text{tot}} \quad \text{in } \Omega^+ \text{ and } \quad u^{\text{tot}}(t)|_\Gamma = 0 \quad \text{for } t > 0, \qquad u^{\text{tot}}(t) = u^{\text{inc}}(t) \quad \text{for } t \leq 0. \tag{2.1}$$

Here u^{inc} is a given incoming wave, i.e., u^{inc} also solves the wave equation, and we assume that for $t \leq 0$ it has not reached the scatterer yet. The problem can be recast by decomposing the total wave into the incoming and outgoing wave, $u^{\text{tot}} = u^{\text{inc}} + u$, where u solves:

$$\ddot{u} = \Delta u \quad \text{in } \Omega^+ \text{ and } \quad u^{\text{tot}}(t)|_\Gamma = -u^{\text{inc}}(t)|_\Gamma \quad \text{for } t > 0, \qquad u^{\text{tot}}(t) = 0 \quad \text{for } t \leq 0. \tag{2.2}$$

This will be the problem we are discretizing.

For simplicity, we consider two possible cases. Either $\Omega^- \subseteq \mathbb{R}^d$ has a smooth boundary or $\Omega^- \subseteq \mathbb{R}^2$ is a polygon. While we expect that the results and techniques can be generalized to the case of piecewise smooth geometries, such extensions would lead to a much higher level of technicality in the present paper. We focus on the exterior scattering problem as our motivation and model problem, but all of the main results also hold for the interior Dirichlet problem.

We end the section by fixing some notation. We write $H^m(\Omega^\pm)$ for the usual Sobolev spaces on Ω^+ or Ω^-. On the interface $\Gamma := \partial\Omega$ we also need fractional spaces $H^s(\Gamma)$ for $s \in [-1, 1]$, see, e.g., [McL00, AF03] for precise definitions. We also set $H^1_\Delta(\Omega^\pm) := \{u \in H^1(\Omega^\pm) : \Delta u \in L^2(\Omega^\pm)\}$. We write $\gamma^\pm : H^1(\Omega^\pm) \to H^{1/2}(\Gamma)$ for the exterior and interior trace operator, and $\partial_n^\pm : H^1_\Delta(\Omega^\pm) \to H^{-1/2}(\Gamma)$ for the normal derivative. We note that in both cases, we take the normal to point out of the bounded domain Ω^-. We write $[\![\gamma u]\!] := \gamma^+ u - \gamma^- u$ and $\{\!\{\gamma u\}\!\} := \frac{1}{2}(\gamma^+ u + \gamma^- u)$ for the trace jump and mean, and $[\![\partial_n u]\!] := \partial_n^+ u - \partial_n^- u$ for the jump of the normal derivative.

2.1 Boundary Integral Methods and Convolution Quadrature

It is well-known that scattering problems of the form presented in Section 2 can be solved by employing boundary integral methods, see [Say16] for a detailed time-domain treatment. For the frequency domain, results can be found in most textbooks on the subject, see [SS11, Ste08, McL00, GS18, HW08].

The use of boundary integral methods for discretizing the time domain scattering problem dates back to the works [BH86a, BH86b], where also important Laplace domain estimates of the form (3.2) were first shown.

For $s \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$, we introduce the single and double layer potentials

$$(\mathrm{SLP}(s)\varphi)(x) := \int_\Gamma \Phi(x-y;s)\varphi(y)\, dS(y), \tag{2.3a}$$

$$(\mathrm{DLP}(s)\psi)(x) := \int_\Gamma \partial_{\nu(y)}\Phi(x-y;s)\psi(y)\, dS(y), \tag{2.3b}$$

where Φ is the fundamental solution for the operator $-\Delta + s^2$:

$$\Phi(x;s) := \begin{cases} \frac{i}{4} H_0^{(1)}(is\,|x|) & \text{for } d=2, \\ \frac{e^{-s|x|}}{4\pi|x|}, & \text{for } d=3. \end{cases} \tag{2.4}$$

Here $H_0^{(1)}$ denotes the Hankel function of the first kind and order zero, see [McL00, Chapter 9].

Finally, we introduce the boundary integral operators engendered by the potentials:

$$V(s) := \gamma^\pm \mathrm{SLP}(s), \qquad \text{and} \qquad K(s) := \{\!\{\gamma \mathrm{DLP}(s)\}\!\}. \tag{2.5}$$

In practice, these operators can be computed via explicit representation as integrals over the boundary Γ. For sufficiently smooth functions ψ, φ the following equations hold:

$$V(s)\varphi = \int_\Gamma \Phi(\cdot,y,s)\varphi(y)\, d\Gamma(y), \qquad \text{and} \qquad K(s)\psi = \int_\Gamma \partial_{\nu(y)}\Phi(\cdot,y,s)\psi(y)\, d\Gamma(y). \tag{2.6a}$$

The operator we consider for discretizing (2.2) is the Dirichlet-to-Neumann map.

Definition 2.1. *For $s \in \mathbb{C}_+$, given $g \in H^{1/2}(\Gamma)$, let u solve*

$$-\Delta u + s^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma \qquad \text{and} \qquad \gamma^\pm u = g.$$

We then define the operators

$$\mathrm{DtN}^\pm(s)g := \partial_n^\pm u \qquad \text{and} \qquad \mathrm{DtI}^\pm(s)g := \partial_n^\pm u \pm s\gamma^\pm u = \mathrm{DtN}^\pm g \pm sg. \tag{2.7}$$

In practice, the following well known proposition gives an explicit way to calculate DtN.

Proposition 2.2 (see, e.g., [LS09, Appendix 2])**.** *The Dirichlet-to-Neumann map can be written as*

$$\mathrm{DtN}^\pm(s) = V^{-1}(s)\big(\mp \frac{1}{2} + K(s)\big). \qquad \square$$

Runge-Kutta convolution quadrature was introduced by Lubich and Ostermann in [LO93]. It provides a simple and general way of approximating convolution integrals by a high order method and has the great advantage that only the Laplace transform of the convolution symbol needs to be easily computable. We only very briefly introduce the method and notation.

Let K be a holomorphic function in a half plane $\operatorname{Re}(s) > \sigma_0 > 0$, and let $\mathscr{L}$ denote the Laplace transform and $\mathscr{L}^{-1}$ its inverse. We (formally) introduce the operational calculus by defining

$$K(\partial_t)g := \mathscr{L}^{-1}\big(K(\cdot)\mathscr{L}g\big),$$

where $g \in \operatorname{dom}(K(\partial_t))$ is such that the inverse Laplace transform exists, and the expression above is well defined.

For a Runge-Kutta method given by the Butcher tableau A, b^T, c, the convolution quadrature approximation of $K(\partial_t)$ is given at the temporal grid points $t_j := jk$ where $k > 0$ denotes the timestep size by

$$\big[K(\partial_t^k)g\big](t_{n+1}) := b^T A^{-1} \sum_{j=0}^{n} W_{n-j}\big[g\big(t_j + kc_\ell\big)\big]_{\ell=1}^{m} \quad \text{with} \quad K\left(\frac{\Delta(\zeta)}{k}\right) = \sum_{n=0}^{\infty} W_n \zeta^n. \tag{2.8}$$

The extension to operator valued functions K is straight forward.

We make the following assumptions on the Runge-Kutta method, slightly stronger than [BLM11].

Assumption 2.3. *(i) The Runge-Kutta method is A-stable with (classical) order $p \geq 1$ and stage order $q \leq p$.*

(ii) The stability function $R(z) := 1 + zb^T(\mathrm{I} - zA)^{-1}\mathbb{1}$ satisfies $|R(it)| < 1$ for $0 \neq t \in \mathbb{R}$.

(iii) The Runge-Kutta coefficient matrix A is invertible.

(iv) The method is stiffly accurate, i.e. $b^T A^{-1} = (0, \ldots, 0, 1)$.

Remark 2.4. *Assumption 2.3 is satisfied by the Radau IIA and Lobatto IIIC methods, see [HW10]. Also note that the order conditions imply that $c_m = 1$ for such methods.* ∎

Our analysis will employ the following result on Runge-Kutta convolution quadrature using Laplace domain estimates:

Proposition 2.5 ([BLM11, Theorem 3])**.** *Assume that K is holomorphic in the half plane $\operatorname{Re}(s) > \sigma_0 > 0$, and that there exist $\mu_1, \mu_2 \in \mathbb{R}$ such that $K(s)$ satisfies the following bounds for all $\delta > 0$:*

$$\begin{aligned} |K(s)| &\leq C_{\sigma_0} |s|^{\mu_1} \quad && \text{for } \operatorname{Re}(s) > \sigma_0 > 0, \\ |K(s)| &\leq C_{\sigma,\delta} |s|^{\mu_2} \quad && \text{for } \operatorname{Re}(s) > \sigma > 0 \text{ with } \operatorname{Arg}(s) \in (-\pi/2 + \delta, \pi/2 - \delta). \end{aligned}$$

Assume that the Runge-Kutta method satisfies Assumption 2.3. Let $r > \max\big(p + \mu_1, p, q+1\big)$ and $g \in C^r([0,T])$ satisfy $g(0) = \dot{g}(0) = \ldots g^{(r-1)}(0) = 0$. Then there exists $\bar{k} > 0$ such that for $0 < k < \bar{k}$,

$$\Big|K(\partial_t^k)g(t_n) - K(\partial_t)g(t_n)\Big| \leq C k^{\min(p, q+1-\mu_2)} \left(\Big|g^{(r)}(0)\Big| + \int_0^{t_n} \Big|g^{(r+1)}(\tau)\Big| \, d\tau\right).$$

The implied constant depends on t_n, σ_0, $\bar{k}$, the constants C_{σ_0}, C_δ, and the Runge-Kutta method.

3 Main results

For simpler notation introduce a symbol for the sectors in Proposition 2.5. Throughout this work we fix for $\sigma_0 > 0$ and $\delta > 0$ and set

$$\mathscr{S} := \left\{s \in \mathbb{C}, \operatorname{Re}(s) > \sigma_0, \operatorname{Arg}(s) \in (-\pi/2+\delta, \pi/2-\delta)\right\}.$$

Remark 3.1. *The choice of $\sigma_0 > 0$ and $\delta > 0$ in the definition of $\mathscr{S}$ is arbitrary, and all our estimates will hold for any pick, although all the constants will be depending on σ_0 and δ.* ▪

We are now able to state the main result of the paper. We start by stating the standard convergence result for discretizing the Dirichlet-to-Neumann map.

Proposition 3.2 (Standard method)**.** *Let $g \in C^r([0,T], H^{1/2}(\Gamma)$ for some $r > p+2$ and $g(0) = \dot{g}(0) = \dots g^{(r)}(0) = 0$. Let $\lambda := \operatorname{DtN}^{\pm}(\partial_t^k)g$ be the exact normal derivative and $\lambda^k := \operatorname{DtN}^{\pm}(\partial_t^k)g$ denote the standard CQ-approximation.*

Then the following estimate holds:

$$\left\|\lambda(t) - \lambda^k(t)\right\|_{H^{-1/2}(\Gamma)} \lesssim k^q \sum_{j=0}^{r} \sup_{\tau\in(0,T)} \|g(\tau)\|_{H^{1/2}(\Gamma)}. \tag{3.1}$$

Proof. Follows from the well known bound

$$\left\|\operatorname{DtN}^{\pm}(s)\right\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \lesssim \frac{|s|^2}{\operatorname{Re}(s)} \tag{3.2}$$

(see for example[LS09]) and Proposition 2.5. □

We will observe numerically in Section 5 that Proposition 3.2 is essentially sharp. Thus, when considering the differentiated equation, one expects an increased order by one, which follows directly from Proposition 2.5. But for the Dirichlet-to-Neumann map the increase of order is even greater, as long as one assumes slightly higher regularity of the data.

Theorem 3.3 (Method based on differentiated data)**.** *Let $g \in C^r\big([0,T], H^1(\Gamma)\big)$ for $r > p+2$ with $g(0) = \dot{g}(0) = \dots g^{(r)}(0) = 0$. Let $\lambda := \operatorname{DtN}^{\pm}(\partial_t^k)g$ be the exact normal derivative and $\lambda^k := [[\partial_t^k]^{-1} \operatorname{DtN}^{\pm}(\partial_t^k)]\dot{g}$ denote the CQ-approximation using $\dot{g}$ as input data.*

Then the following estimate holds:

$$\left\|\lambda(t) - \lambda^k(t)\right\|_{H^{-1/2}(\Gamma)} \lesssim k^{\min(q+2,p)} \sum_{j=0}^{r} \sup_{\tau\in(0,T)} \|g(\tau)\|_{H^1(\Gamma)}. \tag{3.3}$$

Proof. We apply Proposition 2.5. By linearity, we can write the Dirichlet-to-Neummann operator as

$$s^{-1}\operatorname{DtN}(s) = s^{-1}\operatorname{DtI}(s) + \mathrm{I}, \quad \text{or in the time domain} \quad \partial_t^{-1}\operatorname{DtN}(\partial_t) = \partial_t^{-1}\operatorname{DtI}(\partial_t) + \mathrm{I}(\partial_t).$$

The second operator (in frequency domain) is independent of s. It is a simple calculation that in such cases the convolution weights satisfy $W_j = \delta_{j,0}K(0)$. Thus, we have

$$\mathrm{I}(\partial_t)g(t_{n+1}) = b^T A^{-1}\left(g(t_n + kc_\ell)\right)_{\ell=1}^m.$$

Since stiff accuracy implies $b^T A^{-1} = (0, \dots, 0, 1)$ and $c_m = 1$, the operator I is reproduced exactly by the convolution quadrature. A similar decomposition was already invoked in [BLM11] to explain a superconvergence phenomenon for a scalar problem. Theorem 3.4 shows that the Dirichlet-to-Impedance map satisfies

$$\|\mathrm{DtI}(s)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \lesssim |s|^{-2} \text{ for } s \in \mathbb{C}_+, \quad \text{and} \quad \|\mathrm{DtI}(s)\|_{H^{1}(\Gamma)\to H^{-1/2}(\Gamma)} \lesssim 1 \text{ for } s \in \mathscr{S}.$$

By Proposition 2.5, this implies (3.3). ☐

While Theorem 3.3 is the main motivation for this paper, its proof is based on another result, which may be of independent interest.

Theorem 3.4. *Let $s \in \mathscr{S}$. Assume that $\Omega \subseteq \mathbb{R}^d$ is smooth or $\Omega \subseteq \mathbb{R}^2$ is a polygon. The following estimate holds for the Dirichlet-to-Neumann map:*

$$\left\|\mathrm{DtN}^{\pm}(s)g \pm sg\right\|_{H^{-1/2}(\Gamma)} \leq C \left\|g\right\|_{H^1(\Gamma)} \qquad \forall g \in H^1(\Gamma). \tag{3.4}$$

The constant C depends only on the geometry and the parameters σ_0, δ defining the sector $\mathscr{S}$.

Proof. Due to its lengthy and technical nature, we defer the proof to Section 4. ☐

Since all our results hold for both the interior and exterior problem, we can also easily treat the case of an indirect BEM formulation.

Corollary 3.5 (Indirect formulation). *Let $s \in \mathscr{S}$ and assume that $\Omega \subseteq \mathbb{R}^d$ is smooth or $\Omega \subseteq \mathbb{R}^2$ is a polygon. Then, the operator $V^{-1}(s) - 2s$ satisfies the bound*

$$\left\|V^{-1}(s)\varphi - 2s\varphi\right\|_{H^{-1/2}(\Gamma)} \leq \|\varphi\|_{H^1(\Gamma)} \qquad \forall \varphi \in H^1(\Gamma). \tag{3.5}$$

Let $g \in C^r([0,T], H^1(\Gamma)$ for $r > p+2$ with $g(0) = \dot{g}(0) = \dots g^{(r)}(0) = 0$. Let $\varphi := V^{-1}(\partial_t^k)g$ be the exact density and $\varphi^k := [[\partial_t^k]^{-1} V^{-1}(\partial_t^k)]\dot{g}$ its CQ-approximation.

Then the following estimate holds:

$$\left\|\varphi(t) - \varphi^k(t)\right\|_{H^{-1/2}(\Gamma)} \lesssim k^{\min(q+2,p)} \sum_{j=0}^{m} \sup_{\tau \in (0,T)} \|g(\tau)\|_{H^1(\Gamma)}. \tag{3.6}$$

Proof. We can write $V^{-1}(s) = \mathrm{DtN}^-(s) - \mathrm{DtN}^+(s)$. Thus the statements follows from Theorem 3.3 and Theorem 3.4. ☐

4 Proofs

The proof of Theorem 3.4 hinges on three main observations, which require some technical work to make rigorous:

1. In 1d on $\mathbb{R}_+$, the interior Dirichlet-to-Neumann map is given by $g \mapsto sg$.
2. The existing DtN-estimates poor s dependence is mainly caused by boundary layers.
3. Boundary layers are essentially a 1d phenomenon, so observation 1 applies.

4.1 Preliminaries

When working with the Helmholtz equation, it is convenient to work with $|s|$-weighted norms.

Definition 4.1. *For an open (or relatively open) set $\mathcal{O}$, parameters $s \in \mathbb{C}_+$ and $\theta \in [0,1]$, we define the weighted Sobolev norms*

$$\|u\|^2_{|s|,\theta,\mathcal{O}} := |u|^2_{H^\theta(\mathcal{O})} + |s|^{2\theta} \|u\|^2_{L^2(\mathcal{O})}. \tag{4.1}$$

and the dual norms by

$$\|u\|^2_{|s|,-\theta,\mathcal{O}} := \sup_{v \in H^\theta(\mathcal{O})} \frac{(u,v)_{L^2(\mathcal{O})}}{\|v\|_{|s|,\theta,\mathcal{O}}}.$$

We start with some well known s-explicit estimates for the (modified) Helmholtz equation.

Lemma 4.2 (Well posedness)**.** *Let $s \in \mathscr{S}$. The bilinear form*

$$a_s(u,v) := (\nabla u, \nabla v)_{L^2(\Omega^\pm)} + s^2 (u,v)_{L^2(\Omega^\pm)}$$

associated to $-\Delta + s^2$ is elliptic, i.e.,satisfies

$$\operatorname{Re}\big(\zeta a_s(u,u)\big) \geq C \|u\|^2_{|s|,1,\Omega^\pm}$$

for some scalar $\zeta \in \mathbb{C}$ with $|\zeta| = 1$.

Proof. We set $\zeta := \frac{\overline{s}}{|s|}$ and calculate:

$$\operatorname{Re}\big(\zeta a_s(u,u)\big) = \frac{\operatorname{Re}(\overline{s})}{|s|} (\nabla u, \nabla v)_{L^2(\Omega^\pm)} + \frac{\operatorname{Re}(s)}{|s|} |s|^2 (u,v)_{L^2(\Omega^\pm)}.$$

Since $\operatorname{Re}(s) \sim |s|$ in the sector $\mathscr{S}$ this concludes the proof. $\square$

Lemma 4.3 (Trace estimates)**.** *For $s \in \mathscr{S}$, let $u \in H^1(\Omega^\pm)$ satisfy*

$$-\Delta u + s^2 u = f \in L^2(\Omega^-).$$

Then the following estimates hold for the traces of u:

$$\begin{aligned}
\left\|\partial_n^\pm u\right\|_{H^{-1/2}(\Gamma)} &\lesssim |s|^{1/2} \|u\|_{|s|,1,\Omega^\pm} + |s|^{-1/2} \|f\|_{L^2(\Omega^\pm)}, \\
\left\|\gamma^\pm u\right\|_{H^{-1/2}(\Gamma)} &\lesssim |s|^{-1/2} \|u\|_{|s|,1,\Omega^\pm}, \\
\left\|\partial_n^\pm u \pm s\gamma^\pm u\right\|_{H^{-1/2}(\Gamma)} &\lesssim |s|^{1/2} \|u\|_{|s|,1,\Omega^\pm} + |s|^{-1/2} \|f\|_{L^2(\Omega^\pm)}.
\end{aligned}$$

Proof. We start with the normal derivative. For any $\xi \in H^{1/2}(\Gamma)$ and $\gamma^\pm v = \xi$ we calculate:

$$\begin{aligned}
\left\langle \partial_n^\pm u, \xi \right\rangle_\Gamma &= (f,v)_{\Omega^\pm} + (\nabla u, \nabla v)_{\Omega^\pm} + s^2 (u,v)_{\Omega^\pm} \\
&\lesssim \big(\|u\|_{|s|,1,\Omega^\pm} + |s|^{-1} \|f\|_{L^2(\Omega^\pm)} \big) \|v\|_{|s|,1,\Omega^\pm} \\
&\lesssim \big(|s|^{1/2} \|u\|_{|s|,1,\Omega^-} + |s|^{-1/2} \|f\|_{L^2(\Omega^\pm)} \big) \|\xi\|_{H^{1/2}(\Gamma)},
\end{aligned}$$

where in the last step we picked v as in [Say16, Proposition 2.5.1].

For the Dirichlet trace, we get using the multiplicative trace estimate and the same lifting v:

$$\begin{aligned}
\left\langle \gamma^\pm u, \xi \right\rangle_\Gamma &\leq \left\|\gamma^\pm u\right\|_{L^2(\Gamma)} \|\xi\|_{L^2(\Gamma)} \lesssim \|u\|^{1/2}_{L^2(\Omega^\pm)} \|u\|^{1/2}_{H^1(\Omega^\pm)} \|v\|^{1/2}_{L^2(\Omega^\pm)} \|v\|^{1/2}_{H^1(\Omega^\pm)} \\
&\leq |s|^{-1} \|u\|_{|s|,1,\Omega^\pm} \|v\|_{|s|,1,\Omega^\pm} \lesssim |s|^{-1/2} \|u\|_{|s|,1,\Omega^\pm} \|\xi\|_{H^{1/2}(\Gamma)}.
\end{aligned}$$

The estimate for the impedance trace then follows trivially. $\square$

The previous lemma shows that when using the standard Sobolev norms on the boundary, all the constants involved have some s dependence. The next lemmas show that when the use of the weighted norms introduced in Definition 4.1 avoids such dependencies:

Lemma 4.4. *The operators $\gamma^{\pm} : H^1(\Omega^{\pm}) \to H^{1/2}(\Gamma)$ have bounded right inverses $\mathcal{E}^{\pm}$ and satisfy the bounds:*

$$\left\|\gamma^{\pm}u\right\|_{|s|,1/2,\Gamma} \lesssim \|u\|_{|s|,1,\Omega^{\pm}} \qquad \text{and} \qquad \left\|\mathcal{E}^{\pm}g\right\|_{|s|,1,\Omega^{\pm}} \lesssim \|g\|_{|s|,1/2,\Gamma}. \tag{4.2}$$

Proof. The existence of the right inverse follows from [MS11, Lemma 4.22]. We note that the multiplicative trace estimate and Young's inequality give:

$$|s|^{1/2}\left\|\gamma^{\pm}u\right\|_{L^2(\Gamma)} \lesssim \left(\|u\|_{H^1(\Omega^{\pm})}|s|\,\|u\|_{L^2(\Omega^{\pm})}\right)^{1/2} \le \left(\|u\|^2_{H^1(\Omega^{\pm})} + |s|^2\|u\|^2_{L^2(\Omega^{\pm})}\right)^{1/2}.$$

Combining this with the standard trace estimate concludes the proof. □

Lemma 4.5 (Dirichlet problem)**.** *Fix $s \in \mathscr{S}$. Let $g \in H^{1/2}(\Gamma)$, $f \in L^2(\Omega^{\pm})$. Then there exists a unique solution to the problem*

$$-\Delta u + s^2 u = f \text{ in } \Omega^{\pm} \qquad \text{and} \qquad \gamma^{\pm}u = g.$$

The function satisfies the a priori bound

$$\|u\|_{|s|,1,\Omega^{\pm}} \lesssim |s|^{-1}\|f\|_{L^2(\Omega^{\pm})} + \|g\|_{|s|,1/2,\Gamma}. \tag{4.3}$$

Proof. Existence follows using the usual theory of elliptic problems. For the a priori bound, we first note that by [MS11, Lemma 4.22], there exists a lifting u_D satisfying

$$-\Delta u_D + s^2 u = 0, \qquad \|u_D\|_{|s|,1,\Omega^{\pm}} \lesssim \|g\|_{|s|,1/2,\Gamma} \qquad \text{and} \qquad \gamma^{\pm}u_D = g.$$

Thus the remainder $\widetilde{u} := u - u_D$ solves:

$$-\Delta\widetilde{u} + s^2\widetilde{u} = f$$

with homogeneous Dirichlet conditions. Since the bilinear form a_s from Lemma 4.2 is elliptic, we get

$$\begin{aligned}\|\widetilde{u}\|^2_{|s|,1,\Omega^{\pm}} &\lesssim \operatorname{Re}\left(\zeta a_s(\widetilde{u},\widetilde{u})\right) = \operatorname{Re}\left(\zeta\,(f,\widetilde{u})_{\Omega^{\pm}}\right) \le |s|^{-1}\|f\|_{L^2(\Omega^{\pm})}\Big(|s|\,\|\widetilde{u}\|_{L^2(\Omega^{\pm})}\Big)\\ &\le |s|^{-1}\|f\|_{L^2(\Omega^{\pm})}\|\widetilde{u}\|_{|s|,1,\Omega^{\pm}}.\end{aligned}$$

□

Lemma 4.6 (Neumann problem)**.** *Fix $s \in \mathscr{S}$. Let $h \in H^{-1/2}(\Gamma)$. Then there exists a unique solution to the problem*

$$-\Delta u + s^2 u = f \text{ in } \Omega^{\pm} \qquad \text{and} \qquad \partial_n^{\pm}u = h.$$

u satisfies the a priori bound

$$\|u\|_{|s|,1,\Omega^{\pm}} \lesssim \|h\|_{|s|,-1/2,\Gamma} + |s|^{-1}\|f\|_{L^2(\Omega^{\pm})}. \tag{4.4}$$

Proof. Follows easily from the weak formulation and (4.2). □

Lemma 4.7. *We also have the following trace inequality in a weaker norm than $H^{-1/2}$. If $-\Delta u + s^2 u = 0$ we can estimate:*

$$\left\|\partial_n^{-}u\right\|_{|s|,-1/2,\Gamma} \lesssim \|u\|_{|s|,1,\Omega^{-}}.$$

Proof. Follows easily from the weak definition of $\partial_n^{-}u$, the Cauchy-Schwarz inequality, and (4.2).
□

4.2 Smooth geometries

In order to prove a first version of Theorem 3.4, we consider a simplified setting of smooth geometry and Dirichlet trace. Closely following the ideas from [MS99, Mel02], we construct a lowest order boundary layer function which will be the basis for all further estimates.

Lemma 4.8 (Boundary fitted coordinates)**.** *Let $T : \mathcal{O} \subseteq \mathbb{R}^{d-1} \to \Gamma$ be a smooth local parametrization of Γ. Define $F : \mathcal{O} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^d$ as*

$$F(\widehat{x}, \rho) := -\rho\nu(\widehat{x}) + T(\widehat{x}), \tag{4.5}$$

where $\nu(\widehat{x})$ is the normal vector to Ω^- at the point $T(\widehat{x})$.

For $\varepsilon > 0$ sufficiently small, F is a smooth diffeomorphism onto $F\big(\mathcal{O} \times (-\varepsilon, \varepsilon)\big)$. It holds that $F(\mathcal{O} \times (0, \varepsilon)) \subseteq \Omega^-$ and $F(\mathcal{O} \times (-\varepsilon, 0)) \subseteq \Omega^+$. Additionally F satisfies

$$DF^{-T}(\widehat{x}, \rho) DF^{-1}(\widehat{x}, \rho) = \begin{pmatrix} \widetilde{T}(\widehat{x}) & 0 \\ 0 & 1 \end{pmatrix} + \rho\widetilde{R}(\widehat{x}, \rho), \tag{4.6}$$

where $\widetilde{T}$ and $\widetilde{R}$ are smooth.

Proof. We only show (4.6). We select a smooth orthogonal basis of the tangent space at $T(\widehat{x})$, denoted by $e_1(\widehat{x}), \dots, e_{d-1}(\widehat{x})$. This implies that $Q := \big(e_1(\widehat{x}), \dots, e_{d-1}(\widehat{x}), \nu(\widehat{x})\big)$ is orthogonal. We write the Jacobian as

$$DF(\widehat{x}, \rho) = \big(D_{\widehat{x}} T(\widehat{x}), \nu(\widehat{x})\big) + \rho D_{\widehat{x}} \nu(\widehat{x}) = \begin{pmatrix} \widetilde{T}_1 & 0 \\ 0 & 1 \end{pmatrix} Q + \rho D\nu(\widehat{x}).$$

Here $\widetilde{T}_1 := Q^{-1} D_{\widehat{x}} T(\widehat{x})$, and thus $\|\widetilde{T}_1\|_2 = \|D_{\widehat{x}} T(\widehat{x})\|_2$. We further compute:

$$\begin{aligned} DF^{-T} DF^{-1} = \big(DF DF^T\big)^{-1} &= \left(\begin{pmatrix} \widetilde{T}_1 & 0 \\ 0 & 1 \end{pmatrix} Q Q^T \begin{pmatrix} \widetilde{T}_1^T & 0 \\ 0 & 1 \end{pmatrix} + \rho R_1(\widehat{x}, \widehat{\rho}) \right)^{-1} \\ &= \left(\begin{pmatrix} \widetilde{T}_1 \widetilde{T}_1^T & 0 \\ 0 & 1 \end{pmatrix} + \rho R_1(\widehat{x}, \widehat{\rho}) \right)^{-1} \end{aligned} \tag{4.7}$$

where R_1 collects the remaining terms. For sufficiently small $\rho > 0$, depending only on $\|D_{\widehat{x}} T\|_2$ and $\|D_{\widehat{x}} \nu\|_2$ we can linearize the inverse in (4.7) to get (4.6) with $\widetilde{T} := \big(\widetilde{T}_1 \widetilde{T}_1^T\big)^{-1}$. ☐

Lemma 4.9. *Assume that Ω^- has smooth boundary Γ. Fix $s \in \mathscr{S}$. For every $u \in H^1(\Omega^-)$ solving*

$$-\Delta u + s^2 u = 0,$$

and $\gamma^- u \in H^2(\Gamma)$ there exists a function $u_{BL} \in H^1(\Omega^-)$ with the following properties:

(i) $\gamma^- u_{BL} = \gamma^- u$,

(ii) $\partial_n^- u_{BL} - s\gamma^- u_{BL} = 0$,

(iii) $-\Delta u_{BL} + s^2 u_{BL} = f$ *with*

$$\|f\|_{L^2(\Omega^-)} \lesssim |s|^{1/2} \left\|\gamma^- u\right\|_{H^1(\Gamma)} + |s|^{-1/2} \left\|\gamma^- u\right\|_{H^2(\Gamma)}. \tag{4.8}$$

(iv) For all $\varepsilon > 0$, consider the set $\Omega^-_\varepsilon := \{x \in \Omega^- : \operatorname{dist}(x,\Gamma) > \varepsilon\}$. Then, the following estimates hold for all $\ell \in \mathbb{R}$ with constants independent of s:

$$\|u_{BL}\|_{H^2(\Omega^-_\varepsilon)} \leq C_{\varepsilon,\ell} |s|^{-\ell} \|\gamma u\|_{H^2(\Gamma)} .$$

The analogous statement also holds for the exterior problem, replacing $-s$ by s in (ii).

Proof. For shorter notation, set $g := \gamma^- u$. We only show the case of the interior problem. We work in boundary fitted coordinates $(\widehat{x},\rho)$, as in Lemma 4.8. First assume, that $\operatorname{supp}(g) \subset T(\mathcal{O})$, i.e., lies in the part of the boundary parametrized by T. We note that if u solves $-\Delta u + s^2 u = f$, then $\widehat{u} := u \circ F$ solves:

$$-\nabla \cdot \left(JDF^{-T}DF\nabla\widehat{u}\right) + Js^2\widehat{u} = \widehat{f}\widehat{J}$$

with $J := \det(DF)$ (see, e.g., [Mel02, Appendix A.1.1]) and $\widehat{f} = f \circ F$. On the other hand, if $\widehat{u}_{BL}$ satisfies

$$-\nabla \cdot \left(JDF^{-T}DF\nabla\widehat{u}_{BL}\right) + Js^2\widehat{u}_{BL} = \widehat{f}_{BL},$$

then $u_{BL} := \widehat{u}_{BL} \circ F^{-1}$ solves

$$-\Delta u_{BL} + s^2 u_{BL} = f_{BL}, \qquad \text{with } f_{BL} := J^{-1}\widehat{f}_{BL} \circ F^{-1}.$$

We set $\widehat{A} := DF^{-T}DF^{-1}$, and define with $\widehat{g} := g \circ T$ the function $\widehat{u}_{BL}(\widehat{x},\rho) := e^{-s\rho}\widehat{g}(\widehat{x})$ in the boundary fitted coordinates.

By differentiating out, we note that

$$-\nabla \cdot (J\widehat{A}\nabla\widehat{u}) = \nabla_{\widehat{x}} \cdot (J\widetilde{A}\nabla\widehat{u}) + \partial_\rho(e_d^T J\widehat{A})\nabla\widehat{u} + J\widehat{A}_{d,d}\partial_\rho\nabla\widehat{u},$$

where $\widetilde{A}$ is the upper $(d-1) \times d$-block of $\widehat{A}$. Inserting (4.6), we get that $\widehat{u}_{BL}$ solves

$$-\nabla \cdot (J\widehat{A}\nabla\widehat{u}_{BL}) + Js^2\widehat{u}_{BL} = -\nabla_{\widehat{x}} \cdot (J\widetilde{A}\nabla\widehat{u}_{BL}) + \partial_\rho(e_d^T J\widehat{A})\nabla\widehat{u}_{BL} + \rho\widetilde{R}_{d,d}J\partial_\rho^2\widehat{u}_{BL} =: \widehat{f}_{BL},$$

as the leading ∂_ρ^2-term cancels with the s^2-term. Structurally, the right-hand side can be written as:

$$\widehat{f}_{BL}(\widehat{x},\rho) = \sum_{j=0}^{d-1} a_j\partial_{\widehat{x}_j}\widehat{u}_{BL} + b_j\partial_{\widehat{x}_j}\partial_\rho\widehat{u}_{BL} + \sum_{i,j=1}^{d-1} c_{ij}\partial_{\widehat{x}_i}\partial_{\widehat{x}_j}\widehat{u}_{BL} + d\partial_\rho\widehat{u}_{BL} + \rho r\partial_\rho^2\widehat{u}_{BL}.$$

with smooth functions a, b, c, d, r.

From the definition, one can easily see that $\widehat{u}_{BL}$ satisfies the estimates

$$\|\partial_\rho\widehat{u}_{BL}\|_{L^2(\mathcal{O}\times\mathbb{R}_+)} + \|\partial_\rho\nabla_{\widehat{x}}\widehat{u}_{BL}\|_{L^2(\mathcal{O}\times\mathbb{R}_+)} + \left\|\rho\partial_\rho^2\widehat{u}_{BL}\right\|_{L^2(\mathcal{O}\times\mathbb{R}_+)} \lesssim \frac{|s|}{\sqrt{\operatorname{Re}(s)}}\|\widehat{g}\|_{H^1(\mathcal{O})},$$

$$\|\nabla_{\widehat{x}}\widehat{u}_{BL}\|_{L^2(\mathcal{O}\times\mathbb{R}_+)} + \sum_{i,j=1}^{d-1}\left\|\partial_{\widehat{x}_i}\partial_{\widehat{x}_j}\widehat{u}_{BL}\right\|_{L^2(\mathcal{O}\times\mathbb{R}_+)} \lesssim \frac{1}{\sqrt{\operatorname{Re}(s)}}\|\widehat{g}\|_{H^2(\mathcal{O})}.$$

Transforming back gives (4.8) for the part of Ω^- parametrized by F. (iv) follows easily from the definition, as the exponential decay dominates all powers of $|s|$.

This allows us to smoothly cut off u_{BL} for large ρ and extend it by 0 to the whole domain. For general g, we use a smooth partition of unity to decompose g into functions with local support. □

As the next step, we lower the regularity requirement on $\gamma^- u$.

Corollary 4.10. *Assume that Ω^- has a smooth boundary Γ. Fix $s \in \mathscr{S}$. For every $u \in H^1(\Omega^-)$ with $\gamma^- u \in H^1(\Gamma)$ solving*

$$-\Delta u + s^2 u = 0,$$

there exists a function $u_{BL} \in H^1(\Omega^-)$ with the following properties:

(i) $\partial_n^- u_{BL} - s\gamma^- u_{BL} = 0,$

(ii) $\left\|\partial_n^-(u - u_{BL}) - s\big(\gamma^- u - \gamma^- u_{BL}\big)\right\|_{H^{-1/2}(\Gamma)} \lesssim \|\gamma^- u\|_{H^1(\Gamma)}\,,$

(iii) $\|u - u_{BL}\|_{|s|,1,\Omega^-} \lesssim |s|^{-1/2}\,\|\gamma^- u\|_{H^1(\Gamma)}\,,$

(iv) for all $\varepsilon > 0$, consider the set $\Omega_\varepsilon^- := \{x \in \Omega^- : \operatorname{dist}(x,\Gamma) > \varepsilon\}$. Then, the following estimates hold for all $\ell \in \mathbb{R}$ with constants independent of s:

$$\|u_{BL}\|_{H^2(\Omega_\varepsilon^-)} \le C_{\varepsilon,\ell}\,|s|^{-\ell}\,\left\|\gamma^- u\right\|_{H^1(\Gamma)}\,.$$

The analogous statement also holds in the case of the exterior problem, replacing $-s$ by s in (i).

Proof. In order to apply Lemma 4.9, we needH^2-regularity of $g := \gamma^- u$. We fix a function $\widetilde{g} \in H^2(\Gamma)$ with the following properties:

$$\|g - \widetilde{g}\|_{|s|,1/2,\Gamma} \le |s|^{-1/2}\,\|g\|_{H^1(\Gamma)} \qquad \text{and} \qquad \|\widetilde{g}\|_{H^2(\Gamma)} \lesssim |s|^1\,\|g\|_{H^1(\Gamma)}\,. \tag{4.9}$$

Such a function can be constructed via the usual mollifiers. Let $\widetilde{u}$ denote the solution to

$$-\Delta\widetilde{u} + s^2\widetilde{u} = 0 \qquad \text{and} \qquad \gamma^-\widetilde{u} = \widetilde{g}.$$

Since $\widetilde{g} \in H^2(\Gamma)$, we can apply Lemma 4.9 to construct u_{BL}. (i) then follows by construction. For (iii): we note that by Lemma 4.5 and 4.9:

$$\begin{aligned}
\|u - u_{BL}\|_{|s|,1,\Omega^-} &\lesssim \|u - \widetilde{u}\|_{|s|,1,\Omega^-} + \|\widetilde{u} - u_{BL}\|_{|s|,1,\Omega^-} \\
&\lesssim \|g - \widetilde{g}\|_{|s|,1/2,\Gamma} + |s|^{-1}\left(|s|^{1/2}\,\|\widetilde{g}\|_{H^1(\Gamma)} + |s|^{-1/2}\,\|\widetilde{g}\|_{H^2(\Gamma)}\right) \\
&\lesssim |s|^{-1/2}\,\|g\|_{H^1(\Gamma)}\,.
\end{aligned}$$

For (ii), we use Lemma 4.3 and (4.3) to get that

$$\left\|\partial_n^-(u - \widetilde{u}) - s\big(\gamma^- u - \gamma^-\widetilde{u}\big)\right\|_{H^{-1/2}(\Gamma)} \lesssim |s|^{1/2}\,\|g - \widetilde{g}\|_{|s|,1/2,\Gamma} \lesssim \|g\|_{H^1(\Gamma)}\,.$$

Similarly, we have

$$\begin{aligned}
&\left\|\partial_n^-(\widetilde{u} - u_{BL}) - s\big(\gamma^-\widetilde{u} - \gamma^- u_{BL}\big)\right\|_{H^{-1/2}(\Gamma)} \\
&\quad\lesssim |s|^{1/2}\,\|\widetilde{u} - \widetilde{u}_{BL}\|_{|s|,1,\Omega^-} + |s|^{-1/2}\left(|s|^{1/2}\,\|\widetilde{g}\|_{H^1(\Omega^-)} + |s|^{-1/2}\,\|\widetilde{g}\|_{H^2(\Omega^-)}\right) \lesssim \|g\|_{H^1(\Gamma)}\,.
\end{aligned}$$

Point (iv) directly follows from Lemma 4.9 (iv) and (4.9). ∎

4.3 Polygons

In this section, we work out what happens if the domain is non-smooth. For simplicity we restrict ourselves to the case of polygons in two dimensions. In order to match the boundary layer solutions from Lemma 4.9 at corners, we solve an appropriate transmission problem, similarly to what was done in [Mel02]. See Figure 4.1b for the geometric situation we have in mind.

We first need one additional Sobolev space. For a smooth curve Γ' and $\theta \in [0,1]$, we introduce

$$\widetilde{H}^{\theta}(\Gamma') := \left\{ u \in H^{\theta}(\Gamma') : \; \|u\|_{\widetilde{H}^{\theta}(\Gamma')} := \|u\|_{H^{\theta}(\Gamma')} + \|d_{\partial\Gamma'}^{-\theta} u\|_{L^2(\Gamma')} < \infty \right\},$$

where $d_{\partial\Gamma'}$ denotes the distance to the endpoints of Γ'.

Lemma 4.11 (Transmission problem)**.** *Let $\mathcal{O} \subset \mathbb{R}^2$ be an open Lipschitz domain. Let $\Gamma' \subset \mathcal{O}$ be a smooth interface that splits $\mathcal{O}$ into two disjoint Lipschitz domains $\mathcal{O}_1$ and $\mathcal{O}_2$.*

Given $g \in \widetilde{H}^{1/2}(\Gamma')$, $h \in H^{-1/2}(\Gamma')$, there exists a unique solution $u \in H^1(\mathcal{O}_1 \cup \mathcal{O}_2)$ to the following problem:

$$-\Delta u + s^2 u = 0 \quad \text{in } \mathcal{O}, \quad \gamma^- u = 0 \quad \text{on } \partial\mathcal{O}, \qquad [\![\gamma u]\!] = [\![\gamma g]\!] \text{ and } [\![\partial_n u]\!] = h \quad \text{across } \Gamma'.$$

Additionally, the following estimate holds:

$$\|u\|_{|s|,1,\mathcal{O}} \lesssim \|g\|_{|s|,1/2,\Gamma'} + \|d_{\partial\Gamma'}^{-1/2} g\|_{L^2(\Gamma')} + \|h\|_{|s|,-1/2,\Gamma'} .$$

Proof. Since g is assumed in $\widetilde{H}^{1/2}(\Gamma')$, we can extend it by 0 to a function $\widetilde{g} \in H^{1/2}(\partial\mathcal{O}_1)$ such that

$$\|\widetilde{g}\|_{|s|,1/2,\partial\mathcal{O}_1} \lesssim \|g\|_{|s|,1/2,\Gamma'} + \|d_{\partial\Gamma'}^{-1/2} g\|_{L^2(\Gamma')}$$

(see for example [McL00, Theorem 3.33]).

We solve a Dirichlet problem on $\mathcal{O}_1$ with data $\widetilde{g}$ to obtain u_1 and extend it by 0 to $\mathcal{O}_2$. Then we solve the following problem on $\mathcal{O}$: Find $u_2 \in H^1_0(\mathcal{O})$ such that

$$(\nabla u_2, \nabla v)_{L^2(\mathcal{O})} + s^2 (u_2, v)_{L^2(\mathcal{O})} = \langle h, \gamma_{\Gamma'} v\rangle_{\Gamma'} + (\nabla u_1, \nabla v)_{L^2(\mathcal{O})} + s^2 (u_1, v)_{L^2(\mathcal{O})} \quad \forall v \in H^1_0(\mathcal{O}),$$

where $\gamma_{\Gamma'}$ denotes the trace operator on Γ'. The function $u := u_1 + u_2$ then solves the transmission problem. The estimate follows from Lemmas 4.5 and 4.6. □

Before we can bound the functions used to match boundary layers, we must control the jump between two boundary layer solutions. We start with a very simple geometric situation.

Lemma 4.12. *Fix an opening angle $\omega \in (0, 2\pi)$. Consider the sector $S_\omega := \{(r\cos(\varphi), r\sin(\varphi)) \colon r \in (0,1), \varphi \in (0,\omega)\}$ and let $g \in H^1(-1,1)$ be given.*

Let $(\widehat{x}, \widehat{y})$ be the (right handed) Cartesian coordinate system such that positive $\widehat{y}$ denotes the distance to the axis $\{(r\sin(\omega), r\cos(\omega)) : r \in (0,1)\}$.

For $\mu > 0$, define

$$u_1(x,y) := g(x) e^{-\mu s y} \qquad \text{and} \qquad u_2(\widehat{x}, \widehat{y}) := g(\widehat{x}) e^{-\mu s \widehat{y}}.$$

Consider the interface $\Gamma' := \{r\cos(\omega/2), r\sin(\omega/2) \colon r \in (0,1)\}$. Then the following estimates hold:

$$\|u_1 - u_2\|_{|s|,1/2,\Gamma'} + \left\| d_0^{-1/2}(u_1 - u_2)\right\|_{L^2(\Gamma')} \lesssim |s|^{-1/2} \|g\|_{H^1(-1,1)}, \tag{4.10}$$

$$\left\|\partial_n^- u_1 - \partial_n^- u_2\right\|_{|s|,-1/2,\Gamma'} \lesssim |s|^{-1/2} \|g\|_{H^1(-1,1)} . \tag{4.11}$$

where the orientation of the normal is arbitrarily fixed.

Proof. We work in polar coordinates. The boundary fitted coordinates are given by

$$\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} r\cos(\vartheta) \\ r\sin(\vartheta) \end{pmatrix} \qquad \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} := \begin{pmatrix} -r\cos(\omega-\vartheta) \\ r\sin(\omega-\vartheta) \end{pmatrix} \qquad \forall r \in (0,1), \vartheta \in (0,\omega).$$

For shorter notation we introduce the constants $c_1 := \cos(\omega/2)$, $c_2 := \sin(\omega/2)$ and note $c_2 > 0$. We start with the estimate for the Dirichlet jump, and calculate on Γ':

$$[\![\gamma u]\!](r,\vartheta) := u_1(x,y) - u_2(\widehat{x},\widehat{y}) = [g(rc_1) - g(-rc_1)]\, e^{-c_2\mu s r}.$$

We estimate:

$$\begin{aligned} \|[\![\gamma u]\!]\|^2_{L^2(\Gamma')} &= \int_0^1 [g(rc_1)) - g(-rc_1)]^2\; e^{-2\operatorname{Re}(s)\mu c_2 r}\, dr = \int_0^1 \left[\int_{-r}^r g'(\tau c_1)c_1\, d\tau\right]^2 e^{-2\operatorname{Re}(s)\mu c_2 r}\, dr \\ &\lesssim \int_0^1 \|g'\|^2_{L^2(-1,1)}\, r\, e^{-2\operatorname{Re}(s)\mu c_2 r}\, dr \lesssim \frac{1}{\operatorname{Re}(s)^2}\, \|g'\|^2_{L^2(-1,1)}\,. \end{aligned}$$

An analogous computation gives:

$$\left\| d_0^{-1/2}[\![\gamma u]\!]\right\|^2_{L^2(\Gamma')} \lesssim \frac{1}{\operatorname{Re}(s)}\, \|g'\|^2_{L^2(-1,1)}\,.$$

Next we compute the tangential derivative of $[\![\gamma u]\!]$ on Γ':

$$\frac{\partial}{\partial r}[\![\gamma u]\!] = -s\mu c_2 e^{-c_2\mu s r}\big[g(rc_1) - g(-rc_1)\big] + e^{-c_2\mu s r}c_1\big[g'(rc_1) + g'(rc_1)\big].$$

The first term is handled analogously to the L^2-term. For the second term we use the crude estimate $|e^{-s\mu r c_2}| \lesssim 1$ and get:

$$\|\frac{\partial}{\partial r}[\![\gamma u]\!]\|_{L^2(\Gamma')} \lesssim \|g'\|_{L^2(-1,1)}\,.$$

Interpolating these two estimates then gives (4.10).

In polar coordinates, the normal derivative of a function can be computed as

$$\partial_{n_{\Gamma'}} u = \kappa_1(\vartheta)\frac{\partial u}{\partial r} + \kappa_2(\vartheta)\frac{1}{r}\frac{\partial u}{\partial \vartheta}.$$

for smooth functions κ_1, κ_2. Thus it is sufficient to estimate the radial and angular derivatives.

We have already handled the radial part. On Γ', we calculate for the angular derivative:

$$\frac{1}{r}\frac{\partial}{\partial\vartheta}\big(u_1 - u_2\big) = -\,\big[g'(rc_1) - g'(-rc_1)\big]\; c_2 e^{-s\mu c_2 r} - s\,[g(rc_1) - g(-rc_1)]\, \mu c_1 e^{-s\mu c_2 r}.$$

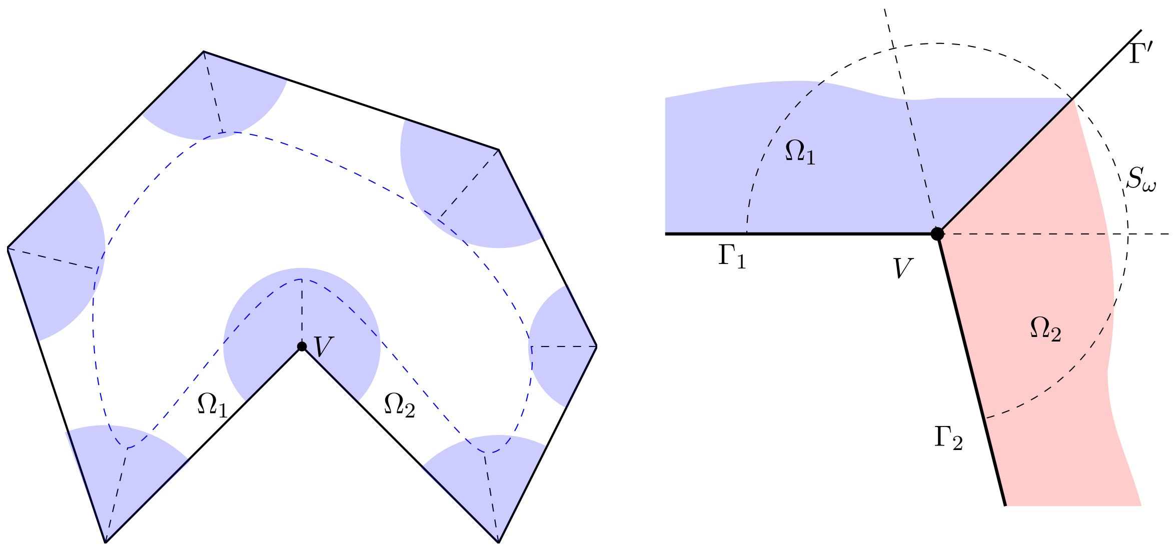
Structurally, the terms are the same as for the derivative of $[\![\gamma u]\!]$. We analogously get the estimate:

$$\|[\![\partial_n u]\!]\|_{L^2(\Gamma')} \lesssim \|g\|_{H^1(\Gamma')}\,.$$

To get to the weaker norm, we simply calculate for $\xi \in H^{1/2}(\Gamma')$:

$$\langle[\![\partial_n u]\!], \xi\rangle_\Gamma \lesssim \|[\![\partial_n u]\!]\|_{L^2(\Gamma')}\, \|\xi\|_{L^2(\Gamma')} \lesssim |s|^{-1/2}\, \|[\![\partial_n u]\!]\|_{L^2(\Gamma')}\, \|\xi\|_{|s|,1/2,\Gamma'}\,,$$

which completes the proof. □



(a) Global scheme of contributions. Subdomains for defining u_{BL} and interfaces of discontinuity. Blue: $\operatorname{supp}(u_{CL})$

(b) Geometric situation at a single corner

Figure 4.1: Geometric situation for nonsmooth domains

Thus far we only considered smooth geometries. Next we handle what happens at corners. We will do so by introducing corner layers, similarly to what was done in [Mel02, Section 7.4.3].

Theorem 4.13. *Let $\Omega^- \subseteq \mathbb{R}^2$ be a polygon, $s \in \mathscr{S}$. If u solves*

$$-\Delta u + s^2 u = 0, \qquad \gamma^- u = g \text{ for } g \in H^1(\Gamma),$$

then u can be decomposed into contributions $u = u_{BL} + u_{CL} + r$, where

(i) u_{BL} satisfies $\partial_n^- u_{BL} - s\gamma^- u_{BL} = 0$.

(ii) u_{CL} satisfies $\gamma^- u_{CL} = 0$ on Γ,

$$\|u_{CL}\|_{|s|,1,\Omega^-} \lesssim |s|^{-1/2} \|g\|_{H^1(\Gamma)}, \quad \text{and} \quad \left\|\partial_n^- u_{CL} - s\gamma^- u_{CL}\right\|_{H^{-1/2}(\Gamma)} \lesssim C \|g\|_{H^1(\Gamma)}.$$

(iii) The remainder is small in the sense

$$\left\|\partial_n^- r - s\gamma^- r\right\|_{H^{-1/2}(\Gamma)} \lesssim C \|g\|_{H^1(\Gamma)}.$$

The analogous statement holds for the exterior problem, replacing $-s$ by s in (i)-(iii).

Proof. For simplicity, we focus on a single corner, and assume $g \in H^2(\Gamma)$. The boundary consists of two straight boundaries Γ_1, Γ_2 meeting at a vertex V. We consider two subdomains with smooth boundary denoted by Ω_1^-, Ω_2^- which contain the boundary parts Γ_1 and Γ_2 respectively, and consider a sector $S_\omega = B_\kappa(V) \cap \Omega^-$ with apex V and radius κ small enough that $\partial S_\omega = (\Gamma_1 \cup \Gamma_2 \cup B_\kappa(V)) \cap \Omega^-$, i.e. the boundary can be written as Γ_1, Γ_2 and a circular segment. Let $\Gamma' \subset \overline{\Omega_1^- \cap \Omega_2^-}$ be the line bisecting the angle at V; see Figure 4.1b. We extend g to $\partial\Omega_1^-$, $\partial\Omega_2^-$ by flattening the corner and reusing the values of g.

On each Ω_i^-, $i = 1, 2$, the function u_{BL} is given by applying the construction from Lemma 4.9 to u on Ω_i^-, and using appropriate cutoff functions away from Γ to get a function which is globally defined on Ω^-. Defining u_{BL} this way leads to an discontinuity on the interface Γ', which we correct using the corner layer function. It is defined as the solution to the following transmission problem:

$$\begin{aligned} -\Delta u_{CL} + s^2 u_{CL} &= 0 \text{ on } S_\omega, \qquad u = 0 \text{ on } \partial S_\omega, \\ [\![\gamma u_{CL}]\!] &= -[\![\gamma u_{BL}]\!] \quad \text{and} \quad [\![\partial_n u_{CL}]\!] = -[\![\partial_n u_{BL}]\!] \text{ on } \Gamma'. \end{aligned}$$

We note that, up to a translation and rotation, we are in the setting of Lemma 4.12 (u_1, u_2 correspond to the two different boundary layer functions used for defining u_{BL}). Thus, we can estimate the jumps across Γ' by:

$$\|[\![\gamma u_{BL}]\!]\|_{|s|,1/2,\Gamma'} + \|d_V^{-1/2}[\![\gamma u_{BL}]\!]\|_{L^2(\Gamma')} + \|[\![\partial_n u_{BL}]\!]\|_{|s|,-1/2,\Gamma'} \lesssim |s|^{-1/2} \|g\|_{H^1(\Gamma)} .$$

By Lemma 4.11, this implies for the corner layer function:

$$\|u_{CL}\|_{|s|,1,S} \lesssim |s|^{-1/2} \|g\|_{H^1(\Omega^-)} \quad \text{and} \quad \|\partial_n u_{CL}\|_{H^{-1/2}(\Gamma)} \lesssim \|g\|_{H^1(\Omega^-)} .$$

The bound on the impedance trace follows easily from Lemma 4.3.

After repeating this construction for all boundary segments and corners of Ω^-, we have constructed functions u_{BL} and u_{CL} such that: $r := u - u_{BL} - u_{CL} \in H_0^1(\Omega^-)$ and solves

$$-\Delta r + s^2 r = f$$

with f as in (4.8). The bounds of Lemma 4.3 and 4.5 then conclude the proof if $g \in H^2(\Gamma)$. For only H^1-regular g, we employ the same smoothing strategy as in Corollary 4.10. The result for the exterior problem follows along the same lines. □

4.4 Proof of Theorem 3.4

If Ω^- has smooth boundary, we use the decomposition $u = u_{BL} + (u - u_{BL})$ with u_{BL} as in Corollary 4.10. If Ω^- is a polygon, we use the decomposition $u = u_{BL} + u_{CL} + r$ from Theorem 4.13. In both cases, the impedance trace of u_{BL} vanishes, and the impedance trace of the remainders has already been estimated, which is sufficient to prove (3.4).

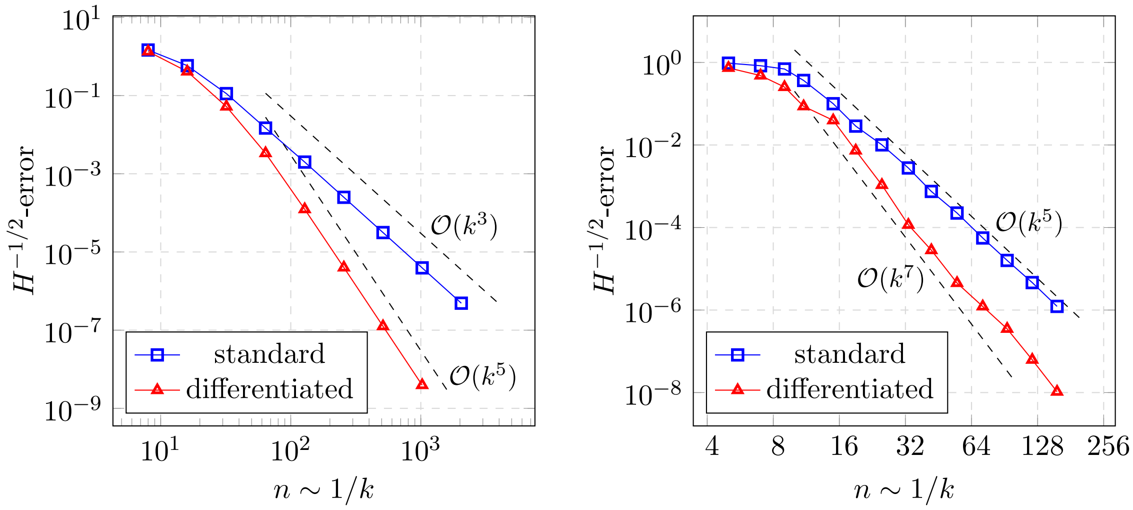
5 Numerical Examples

In this section, we compare the performance of the numerical schemes of Theorem 3.3 with the more standard method of Proposition 3.2 for an interior scattering problem. That is, we compare the Runge-Kutta convolution quadrature approximation given by

- $[\mathrm{DtN}^-(\partial_t^k)]u^{\mathrm{inc}}$, which is denoted “standard method”, and
- $[[\partial_t^k]^{-1}\,\mathrm{DtN}^-(\partial_t^k)]\dot{u}^{\mathrm{inc}}$, which is denoted “differentiated method”.

We use two different Runge-Kutta methods of the Radau IIA family, one with 3 and one with 5 stages. For the 3-stage version, we have $q = 3$ and $p = 5$. We therefore expect a convergence rate of order 3 for the standard method and full classical order 5 for the differentiated scheme.

In order to show that our theoretical estimates are sharp, we also look at the 5-stage method. There, the stage order is $q = 5$ and the classical order $p = 9$. The expected rates are therefore 5 and 7 respectively for the two numerical schemes.



(a) Comparison of the standard and differentiated method for 3-stage Radau IIA

(b) Comparison of the standard and differentiated method for 5-stage Radau IIA

Figure 5.1: Comparison of the standard and differentiated method for different RK schemes

For simplicity, we consider the interior scattering problem and prescribe an exact solution as the travelling wave

$$u(x,t) := \psi(\mathbf{d} \cdot x - t) \qquad \text{with} \quad \psi(\tau) := \cos\left(\frac{\pi\, \tau}{2}\right)\, e^{-\frac{(\tau-\tau_0)^2}{\alpha}}.$$

The wave direction is selected as $\mathbf{d} := (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and the other parameters were $\tau_0 := 4$ and $\alpha := 0.05$. We integrated until the end time $T = 12$. In order to show that the method works with the predicted rates, even for non-convex geometries, we consider the classical L-shaped geometry, given by the vertices

$$(0.5, 0),\ (1, 0),\ (1, 1),\ (0, 1),\ (0, 0.5),\ (0.5, 0.5).$$

As the space discretization, we employ a boundary element method of order 5, based on the code developed by F.-J. Sayas at the University of Delaware. A sufficiently refined grid is employed to be able to focus on the temporal error. Instead of evaluating the $H^{-1/2}$-error, we compute the quantity

$$\max_{j=0,\ldots n} \sqrt{\langle V(1)e_j, e_j\rangle_\Gamma} \qquad \text{with} \qquad e_j := \Pi_{L^2}\lambda(t_j) - \lambda^k(t_j).$$

Here Π_{L^2} denotes the L^2-orthogonal projection onto the BEM space. Since the grid is sufficiently fine and fixed, this should not impact the observed convergence rates.

In Figure 5.1, we observe that the rates from Proposition 3.2 and Theorem 3.3 are obtained as predicted. We conclude that while the fact that the rate jumps by order 2, even though the modification of the scheme is of order one, is at first surprising, this can be rigorously explained by Theorem 3.3. Observations of this type provided the main motivation for the investigations in this work.

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