Multilevel preconditioning for the boundary concentrated hp-FEM

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Abstract

The boundary concentrated finite element method is a variant of the hp-version of the finite element method that is particularly suited for the numerical treatment of elliptic boundary value problems with smooth coefficients and low regularity boundary conditions. For this method we present two multilevel preconditioners that lead to preconditioned stiffness matrices with condition numbers that are bounded uniformly in the problem size N. The cost of applying the preconditioners is $O(N)$. Numerical examples illustrate the efficiency of the algorithms.

1 Introduction

The boundary concentrated finite element method, introduced in [KM03], is a version of the hp-FEM that is particularly suited for solving second order elliptic boundary value problems where the differential equation has analytic coefficients but the boundary conditions have low regularity or the geometry is non-smooth. A further possible application includes its use as a subdomain solver in domain decomposition methods. The main idea of the boundary concentrated finite element method is to exploit the interior regularity of the solution by applying special types of meshes and polynomial distributions, namely, small elements with low order polynomials near the boundary and large elements with high order polynomials in the interior. Detailed analyses of the approximation properties of the boundary concentrated FEM are given in [KM03, EM04]; it is also shown in [KM03] that the stiffness matrix can be set up in optimal complexity.

The present paper focuses on efficient solution techniques for the boundary concentrated FEM. To the knowledge of the authors efficient solution methods have only been designed for two special cases, namely, two-dimensional problems and boundary value problems where Dirichlet boundary conditions are prescribed on the whole boundary. Specifically, for two-dimensional problems, it is shown in [KM02] how the LU-factorization of the stiffness matrix can be performed with work $O(N \log^8 N)$, where N is the problem size. [KM03] briefly discusses a preconditioner that relies on the fast realization of $H^{1/2}(\partial\Omega)$-like norms on the space of piecewise linear functions; in the two-dimensional case, this can be effected by Fast Fourier Transform techniques. The case of Dirichlet boundary conditions has also been recognized as being special in [KM03], since the condition number of the diagonally scaled stiffness matrix is shown to be bounded by $O(\log^\beta N)$ for some $\beta \geq 0$ (the precise value of β depends on the specific choice of the polynomial basis). These preconditioners lead to condition numbers that grow polylogarithmically in the problem size N and the work per iteration is $O(N \log N)$ instead of the desirable $O(N)$. Here, we present two multilevel preconditioners that are freed from the above mentioned limitations. They are conceptually dimension independent and applicable to a wide range of boundary conditions since they rely merely on the equivalence of the bilinear form under consideration with the $H^1(\Omega)$ inner product. For two- and three-dimensional problems, we will show that the preconditioned stiffness matrices have condition numbers bounded uniformly in the problem size N. The alphanumerical work to apply the preconditioners is $O(N)$.

Our preconditioners are based on the additive Schwarz framework as discussed, for example, in [Nep86, Osw94, TW05]. The finite element space $\mathcal{V}_N$ is decomposed as $\mathcal{V}_N = (\sum_{v\in\mathcal{V}_M} \mathcal{S}_v) + \left(\sum_{m=0}^{M} \sum_{v\in I_m} \mathcal{V}_v^m\right)$. Here, the spaces $\mathcal{S}_v$ consist of all functions of $\mathcal{V}_N$ that are supported by the patch associated with the vertex v; the second term, $\sum_{m=0}^{M} \sum_{v\in I_m} \mathcal{V}_v^m$ comprises one-dimensional spaces of piecewise linear functions on different levels and may be viewed as a form of BPX/multilevel-level diagonal scaling [BPX91, Zha92] treatment of the relevant piecewise linear functions on the finest mesh $\mathcal{T}_M$.
The analysis of our preconditioners proceeds along well-established lines by exhibiting a stable splitting and by bounding the spectral radius $\rho(\mathbf{E})$ of the matrices $\mathbf{E}$ containing the angles between the subspaces. The construction of stable splittings in the present context of *hp*-FEM relies on recent work [SMPZ05, EM04] for two- and three-dimensional problems. Our analysis of the angle between the subspaces makes use of the special refinement structure of the meshes utilized in the boundary concentrated FEM. This allows us to present simple, self-contained proofs of the fact that the spectral radius $\rho(\mathbf{E})$ is bounded uniformly in the problem size N; we mention that the techniques employed and the results obtained are similar to the multilevel analysis of the classical FEM or the BEM on locally refined meshes in [DK92, BY93, AM03]. As we will discuss in more detail in Remark 3.2 the special structure of the meshes employed in the boundary concentrated FEM links our preconditioners also to existing Schur complement based approaches that rely on solving Dirichlet problems on the domain and realizing a discrete $H^{1/2}(\partial\Omega)$-norm on the boundary.
The paper is organized as follows: We start with a brief description of the boundary concentrated FEM. Thereafter we introduce two preconditioners, formulate the main theorems concerning the properties of these preconditioners and present some numerical examples. The remainder of the paper is devoted to the proofs of the main theorems.

2 Model problem and FE-discretization

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a polygonal or polyhedral Lipschitz domain, which we assume to be scaled such that $\operatorname{diam}\Omega \leq 1$. Let $\Gamma_D \subset \partial\Omega$ be a union of edges (for $d = 2$) or faces (for $d = 3$); the case $\Gamma_D = \emptyset$ is allowed. Upon setting

$$H^1_D(\Omega) := \{u \in H^1(\Omega) \mid u|_{\Gamma_D} = 0\}$$

we consider the following model problem:

Problem 2.1 (model problem)**.** *Find $u \in H^1_D(\Omega)$ such that*

$$a(u,v) := \int_\Omega \left\langle \nabla u, \hat{\mathbf{A}}(x)\nabla v \right\rangle + a_0(x)uv\,d\Omega = f(v) \qquad \forall\, v \in H^1_D(\Omega). \tag{1}$$

Throughout the paper we assume $a_0 \in L^\infty(\Omega)$ and $\hat{\mathbf{A}} \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ is pointwise symmetric positive definite. Furthermore, we require the existence of a constant $C > 0$ such that

$$C^{-1}\|u\|^2_{H^1(\Omega)} \leq a(u,u) \leq C\|u\|^2_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega). \tag{2}$$

The discretization of Problem 2.1 is based on the *hp*-version of the finite element method. For $d \in \{2,3\}$ we consider shape-regular meshes $\mathcal{T}$ (see, e.g., [Cia76]) consisting of triangles (in 2D) or tetrahedra (in 3D). The elements $K \in \mathcal{T}$ are images of a reference element $\widehat{K}$ under the affine element maps $F_K : \widehat{K} \to K$. With each element $K \in \mathcal{T}$ we associate a polynomial degree $p_K \in \mathbb{N}$ and collect all polynomials in the degree vector $\mathbf{p}$. The *hp*-FEM is then based on the following spaces:

$$S^{\mathbf{p}}(\Omega,\mathcal{T}) := \{u \in H^1(\Omega) \mid u \circ F_K \in \mathcal{P}_{p_K}(\hat{K}) \quad \forall K \in \mathcal{T}\}, \tag{3a}$$
$$S^{\mathbf{p}}_D(\Omega,\mathcal{T}) := S^{\mathbf{p}}(\Omega,\mathcal{T}) \cap H^1_D(\Omega), \tag{3b}$$

where $\mathcal{P}_p$ denotes the vector space of all polynomials of degree p.
The boundary concentrated *hp*-FEM is a variant of the *hp*-FEM where the meshes $\mathcal{T}$ and the polynomial degree distribution $\mathbf{p}$ have a special structure (see Figs. 1, 2 for typical meshes):

Definition 2.2 (geometric mesh with boundary mesh size h and linear degree vector)**.** *A shape regular mesh $\mathcal{T}$ is called a geometric mesh with boundary mesh size h if there exist $C_1, C_2 > 0$ such that for all elements $K \in \mathcal{T}$ the element size $h_K = \operatorname{diam}(K)$ is given by*

1. $h \leq h_K \leq C_2 h$ *for all* $K \in \mathcal{T}$ *with* $\overline{K} \cap \partial\Omega \neq \emptyset$,
2. $C_1 \inf_{x \in K} \operatorname{dist}(x, \partial\Omega) \leq h_K \leq C_2 \sup_{x \in K} \operatorname{dist}(x, \partial\Omega)$ *for all* $K \in \mathcal{T}$ *with* $\overline{K} \cap \partial\Omega = \emptyset$.

A polynomial degree vector $\mathbf{p} = (p_K)_{K \in \mathcal{T}}$ is said to be a linear degree vector if

$$\left\lfloor 1 + C_1' \log \frac{h_K}{h_{min}} \right\rfloor \leq p_K \leq \left\lceil 1 + C_2' \log \frac{h_K}{h_{min}} \right\rceil \tag{4}$$

for some C_1', $C_2' > 0$ and $h_{min} := \min_{K \in \mathcal{T}} h_K$.

Remark 2.3. *For linear polynomial degree vectors we have the additional property that there exists a constant $C > 0$ such that*

$$C^{-1} \leq p_K / p_{K'} \leq C \qquad \forall K, K' \in \mathcal{T} \quad \text{with } \overline{K} \cap \overline{K}' \neq \emptyset. \tag{5}$$

The discretization of Problem 2.1 via boundary concentrated FEM then reads:

Problem 2.4 (boundary concentrated FE-problem)**.** *For $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ let $\mathcal{T}$ be a geometric mesh with boundary mesh size h and $\mathbf{p}$ be a linear polynomial degree vector. Find $u \in S_D^{\mathbf{p}}(\Omega, \mathcal{T})$ such that*

$$a(u, v) = F(v) \qquad \forall\, v \in S_D^{\mathbf{p}}(\Omega, \mathcal{T}).$$

We propose to solve the linear system of equations corresponding to the variational formulation of Problem 2.4 by the preconditioned conjugate gradient method, where the preconditioners are of the type discussed in Section 3.2.

3 Multilevel preconditioning

3.1 Setting

The preconditioner is based on the additive Schwarz framework (see, e.g., [TW05]). We decompose the *hp*-FEM space $S_D^{\mathbf{p}}(\Omega, \mathcal{T}) = \sum_{i=0}^{K} \mathcal{V}_i$ for subspaces $\mathcal{V}_i$ that will be specificed below. The action of the preconditioner $B^{-1} : (S_D^{\mathbf{p}}(\Omega, \mathcal{T}))^* \to S_D^{\mathbf{p}}(\Omega, \mathcal{T})$ is defined as $B^{-1} f := \sum_{i=0}^{K} u_i$, where the functions $u_i \in \mathcal{V}_i$ are the solutions of

$$a(u_i, v_i) = \langle f, v_i \rangle \qquad \forall v_i \in \mathcal{V}_i.$$

For the purpose of the analysis, it is useful to note that the linear operator B^{-1} is invertible (see, e.g., [TW05, Lemma 2.5]), and its inverse $B : S_D^{\mathbf{p}}(\Omega, \mathcal{T}) \to (S_D^{\mathbf{p}}(\Omega, \mathcal{T}))^*$ induces a symmetric positive definite bilinear form $b : S_D^{\mathbf{p}}(\Omega, \mathcal{T}) \times S_D^{\mathbf{p}}(\Omega, \mathcal{T}) \to \mathbb{R}$ given by (see, e.g., [TW05, Lemma 2.5]):

$$b(u, u) = \inf_{\substack{u_i \in \mathcal{V}_i \\ u = \sum_i u_i}} a(u_i, u_i). \tag{6}$$

The constants λ_{min}, λ_{max} in the two-sided bound

$$\lambda_{min} a(u, u) \leq b(u, u) \leq \lambda_{max} a(u, u) \qquad \forall u \in S_D^{\mathbf{p}}(\Omega, \mathcal{T}) \tag{7}$$

allow us to estimate the convergence behavior of the preconditioned conjugate gradient method, namely, the iterates $u^k \in S^{\mathbf{p}}_D(\Omega, \mathcal{T})$ satisfy the well-known bounds (see, e.g., [TW05, Appendix C.5])

$$\|u - u^k\|_A \leq 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k \|u - u^0\|_A, \qquad \kappa = \frac{\lambda_{max}}{\lambda_{min}}, \tag{8}$$

where $u \in S^{\mathbf{p}}_D(\Omega, \mathcal{T})$ is the sought solution, u^0 is the initial guess, and the energy norm $\|v\|_A$ is given by $\|v\|_A = \sqrt{a(v, v)}$. Thus, the goal of the design of the preconditioner is to get bounded condition number κ.

3.2 Two preconditioners for the boundary concentrated FEM

To construct a preconditioner for the linear system of equations arising in Problem 2.4 we start from a sequence of nested geometric meshes

$$\mathcal{T}_0 \subset \mathcal{T}_1 \subset \ldots \subset \mathcal{T}_M = \mathcal{T} \tag{9}$$

with element sizes $h_K = \operatorname{diam}(K)$ for $K \in \mathcal{T}_m$ given by

$$h_K \sim \begin{cases} h_0 2^{-m} & : \ \overline{K} \cap \partial\Omega \neq \emptyset \\ \operatorname{dist}(K, \partial\Omega) & : \ \text{otherwise.} \end{cases} \tag{10}$$

Furthermore, we assume that the refinement $\mathcal{T}_m \mapsto \mathcal{T}_{m+1}$ is a regular ("red") refinement for all element at the boundary, i.e.,

$$K \in \mathcal{T}_m \text{and } \overline{K} \cap \partial\Omega \neq \emptyset \implies \mathcal{T}_{m+1} \text{contains the } 2^d \text{ sons of } K \text{ obtained by regular refinement} \tag{11}$$

By V_m we denote the set of all free vertices of $\mathcal{T}_m$; that is, V_m contains all vertices of $\mathcal{T}_m$ without vertices of the Dirichlet part Γ_D of the boundary. We associate a patch

$$\omega_v^m := \cup\{K \in \mathcal{T}_m \mid v \in \overline{K}\} \tag{12}$$

with each vertex $v \in V_m$. For our subspace splitting of $S^{\mathbf{p}}_D(\Omega, \mathcal{T}_M)$ we introduce the spaces of hat functions

$$\mathcal{V}_v^m = \{u \in S^1_D(\Omega, \mathcal{T}_m) \mid u(v') = 0 \ \forall v' \in V_m \backslash \{v\}\}, \quad \forall m \in \{0, \ldots, M\}, \ \forall v \in V_m \tag{13}$$

and the patch spaces

$$\mathcal{S}_v = \{u \in S^{\mathbf{p}}_D(\Omega, \mathcal{T}_M) \mid \operatorname{supp} u \subset \overline{\omega_v^M}\} \quad \forall v \in V_M. \tag{14}$$

Now we are in the position to formulate the first preconditioner for the boundary concentrated FEM:

Theorem 3.1. *Let $\{\mathcal{T}_m\}_{m=0}^M$ be a sequence of nested geometric meshes that satisfy (9)–(11). Let $\mathbf{p} = (p_K)_{K \in \mathcal{T}_M}$ be a linear polynomial degree vector. Let the patches ω_v^m and the spaces $\mathcal{V}_v^m$, $\mathcal{S}_v$ be defined as in (12)—(14). Set $I_m^B := \{v \in V_m \mid v \in \partial\Omega\}$. Then the splitting*

$$S^{\mathbf{p}}_D(\Omega, \mathcal{T}_M) = \sum_{v \in V_M} \mathcal{S}_v + \sum_{m=0}^{M} \sum_{v \in I_m^B} \mathcal{V}_v^m \tag{15}$$

determines an ASM-preconditioner whose associated bilinear form b satisfies for some $C > 0$ independent of the problem size N:

$$C^{-1} a(u, u) \leq b(u, u) \leq C a(u, u) \qquad \forall u \in S^{\mathbf{p}}_D(\Omega, \mathcal{T}).$$

The cost of applying the preconditioner is $O(N)$.

Proof. See Section 5. □

Remark 3.2. *One popular technique to construct preconditioners is based on distinguishing between boundary and internal degrees of freedom. The building blocks in these approaches are a) a solver/preconditioner for the domain problem with homogeneous Dirichlet boundary conditions; b) a solver/preconditioner for the Schur complement that arises from eliminating the internal degrees of freedom; c) a stable extension operator* $\mathbf{E}$ *from the boundary. To see the relation of such a procedure to the preconditioner of Theorem 3.1, consider the case* $\Gamma_D = \emptyset$ *and assume that all elements at the boundary have polynomial degree* 1*. As is shown in this paper below an optimal preconditioner for the Dirichlet problem is described by the stable splitting* $S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M) \cap H_0^1(\Omega) = \sum_v (\mathcal{S}_v \cap H_0^1(\Omega))$*. The preconditioner for the Schur complement could be realized by a multilevel splitting of BPX type. Finally, in view of the special structure of the geometric meshes, the extension operator* $\mathbf{E}$ *can be chosen as the hierarchical extension operator of [HN97]. This extension operator essentially maps into* $\sum_{m=0}^M \sum_{v\in I_m^B} \mathcal{V}_v^m$*, which may therefore serve as a motivation to choose it as a component for the domain-based preconditioner of Theorem 3.1. Closely related work includes [Kuz95] (and the references there) and [BELep].*

To introduce our second preconditioner we assign two numbers to each patch ω_v^m,

$$l(\omega_v^m) := \lceil -\log_2 \operatorname{diam} \omega_v^m \rceil, \qquad g(\omega_v^m) := \min\left\{k \in \{0,\dots,M\} \mid \omega_v^k = \omega_v^m\right\}. \tag{16}$$

The first number, the level of the patch ω_v^m, is a measure of the size of the patch ω_v^m. The second number specifies the mesh $\mathcal{T}_{g(\omega_v^m)}$ in which the patch ω_v^m appears first. Note that by our scaling assumption $\operatorname{diam}\Omega \le 1$ and by the assumption on h_K

$$0 \le l(\omega_v^m) \le L := \max\{l(\omega_v^m) \mid m = 0,\dots,M, \;\; v \in V_m\} \le CM. \tag{17}$$

Furthermore, for the subspaces $\mathcal{V}_v^m$ and $\mathcal{S}_v$ we define the numbers l and g in terms of the corresponding patch, viz.,

$$l(\mathcal{V}_v^m) := l(\omega_v^m), \qquad l(\mathcal{S}_v) := l(\omega_v^M), \qquad g(\mathcal{V}_v^m) := g(\omega_v^m), \qquad g(\mathcal{S}_v) := g(\omega_v^M).$$

Theorem 3.3. *Let* $\{\mathcal{T}_m\}_{m=0}^M$ *be a sequence of nested geometric meshes that satisfy (9)–(11). Let* $\mathbf{p} = (p_K)_{K\in\mathcal{T}_M}$ *be a linear polynomial degree vector. Let the patches* ω_v^m*, the spaces* $\mathcal{V}_v^m$*,* $\mathcal{S}_v$ *and the numbers* $l(\cdot)$*,* $g(\cdot)$ *be defined as in (12)–(14), (16). Set* $I_m := \{v \in V_m \mid g(\omega_v^m) = m\}$ *and assume the existence of a constant* $C_1 > 0$ *with*

$$g(\omega_v^m) \le l(\omega_v^m) + C_1 \quad \forall m \in \{0,\dots,M\}, \; \forall v \in I_m. \tag{18}$$

Then the splitting

$$S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M) = \sum_{v\in V_M} \mathcal{S}_v + \sum_{m=0}^M \sum_{v\in I_m} \mathcal{V}_v^m \tag{19}$$

determines an ASM-preconditioner whose associated bilinear form b *satisfies for some* $C > 0$ *independent of the problem size* N*:*

$$C^{-1} a(u,u) \le b(u,u) \le C a(u,u) \qquad \forall u \in S_D^{\mathbf{p}}(\Omega, \mathcal{T}).$$

The cost of applying the preconditioner is $O(N)$*.*

Proof. See Section 5. □

Remark 3.4. *Due to the definition of* $g(\cdot)$*, the decomposition (19) could also be written in the form* $S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M) = \sum_{v\in V_M} \mathcal{S}_v + \left(\sum_{m=0}^M \sum_{v\in V_m} \mathcal{V}_v^m\right)'$*, where the prime indicates that the summation is only over pairwise distinct spaces* $\mathcal{V}_v^m$*.*

Remark 3.5. *Assumption (18) is satisfied if the refinement procedure* $\mathcal{T}_k \mapsto \mathcal{T}_{k+1}$ *affects* only *elements at or in a limited neighborhood of the boundary.*

Remark 3.6. *The preconditioner of Theorem 3.3 may be motivated from the splitting* $S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M) = \left(\sum_{v \in V_M} \mathcal{S}_v\right) + \mathcal{V}_M$*, where* $\mathcal{V}_M \subset S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M)$ *is the space of piecewise linear functions on* $\mathcal{T}_M$*. The first term represents a stable splitting as shown in this paper. For the second part,* $\mathcal{V}_M$*, the preconditioner of Theorem 3.3 may be viewed as realizing a stable multilevel decomposition of* $\mathcal{V}_M$*. This stability result could also be inferred from the work [DK92, BY93, AM03].*

Remark 3.7. *The preconditioners of Theorems 3.1, 3.3 are based on splittings into high order parts on patches of the finest mesh and one-dimensional spaces associated with vertices of different meshes. The effectivity of the preconditioners depends on number of nodes of the coarsest mesh* $\mathcal{T}_0$*, which is clearly visible in the numerical examples in Figs. 1, 2. To mitigate this effect, one could include a coarse space* $\mathcal{V}_0 = S_D^1(\Omega, \mathcal{T}_0)$ *in the preconditioner. The condition number estimates of Theorems 3.1, 3.3 remain valid. The total cost of the preconditioner is then, of course, increased by the cost of solving the coarse grid problem on* $\mathcal{T}_0$*.*

Remark 3.8. *The proofs of our Theorems reveal that the* $O(1)$ *condition numbers are special features of the boundary concentrated hp-FEM and stem from the special structure of our meshes. However, for some applications these meshes may be considered over-refined at the boundary.*

4 Numerical examples

In this section we present some numerical examples in two dimensions to confirm the theoretical results and to demonstrate the efficiency of our preconditioners. In all examples we start with a coarse grid $\mathcal{T}_0$ of the given domain and we create a sequence of hierarchically nested geometric meshes $\{\mathcal{T}_l\}_{l=0,1,\ldots}$ with boundary mesh sizes $h_l \sim 2^{-l} h_0$ by applying the following algorithm.

Algorithm 4.1 (boundary concentrated mesh refinement)**.**
input: mesh $\mathcal{T}_i$*;* *output: mesh* $\mathcal{T}_{i+1}$

1. *Subdivide all* $K \in \mathcal{T}_i$ *with* $\overline{K} \cap \partial\Omega \neq \emptyset$ *into four congruent sons (red refinement).*
2. *Subdivide all* $K \in \mathcal{T}_i$ *with more than one hanging node into four congruent sons (red refinement), and repeat this procedure as long as there exists a triangle with more than one hanging node.*
3. *Subdivide all* $K \in \mathcal{T}_i$ *with one hanging node into two sons (green refinement).*

Algorithm 4.1 refines the elements near the boundary and afterwards performs a "green closure" to reach a regular mesh. The following lemma shows that the green closure does not spread very far into the domain. This ensures then the following: firstly, the meshes $\mathcal{T}_i$, $i = 1, \ldots$ generated with Algorithm 4.1 remain shape regular (with shape regularity constants depending only on the initial triangulation $\mathcal{T}_0$) and secondly, Assumption (18) in Theorem 3.3 is valid.

Lemma 4.2 (boundary concentrated refinement)**.** *Let* $\mathcal{T}_0$ *be an arbitrary coarse grid and denote by* $\{\mathcal{T}_i\}_{i=0,1,2,\ldots}$ *the sequence of meshes created by applying Algorithm 4.1. Let* V_i *be the set of all vertices of* $\mathcal{T}_i$ *and* E_i *the set of all edges of* $\mathcal{T}_i$*. For each edge* $e \in E_i$ *we denote its endpoints by* $v_a(e), v_e(e) \in V_i$ *and for each* $K \in \mathcal{T}_i$ *we denote its corners by* $v_1(K), v_2(K), v_3(K) \in V_i$*. With*

$$W^l(\mathcal{T}_i) := \left\{(v_0, \ldots, v_l) \in (V_i)^{l+1} \;\middle|\; \exists e_j \in E_i \text{ with } \{v_a(e_j), v_e(e_j)\} = \{v_j, v_{j+1}\}\; \forall\, 0 \leq j < l\right\}$$

we denote the set of all paths of length l *in* $\mathcal{T}_i$ *and with* $v_j(w)$*,* $0 \leq j \leq l$ *we denote the* j*-th vertex of the path* $w = (v_0, \ldots, v_l) \in W^l$*. Let*

$$D(v, \mathcal{T}_i) := \min\{l \mid \exists w \in W^l(\mathcal{T}_i) \text{ with } v_0(w) \in \partial\Omega,\ v_l(w) = v\}$$

be the discrete distance of $v \in V_i$ *to the boundary* $\partial\Omega$*. Then:*

- $K \in \mathcal{T}_i$ *with* $\sum_{k=1}^{3} D(v_k(K), \mathcal{T}_i) \geq 5$ *implies* $K \in \mathcal{T}_{i+1}$*.*

- *if $K \in \mathcal{T}_i$ is subdivided into $K_{S1}, K_{S2} \in \mathcal{T}_{i+1}$ (green refinement) then $K_{S1}, K_{S2} \in \mathcal{T}_j$ for all $j > i$.*

Proof. See [Eib06]. $\square$

We apply the two preconditioners of Theorem 3.1, 3.3 to two problems. The first example is a boundary value problem on the L-shaped domain:

Example 4.3. *We consider*

$$\begin{aligned} -\Delta u = f \quad & on \quad \Omega = (0,1)^2 \backslash ([0,1] \times [-1,0]) \\ \frac{\partial u}{\partial n} = 0 \quad & on \quad \Gamma_N = (\{-1\} \times [-1,1]) \cup ([-1,1] \times \{1\}) \\ u = 0 \quad & on \quad \Gamma_D = \partial\Omega \backslash \Gamma_N \end{aligned}$$

with a right-hand side f, such that the exact solution is given by

$$u = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right) \left(1 - r^2 \cos^2 \varphi\right) \left(1 - r^2 \sin^2 \varphi\right) (1 + r\cos\phi)(1 - r\sin\phi).$$

In the second example we verify our theoretical results for a domain with a more complicated boundary:

Example 4.4. *On the snow flake domain (Fig. 2) we consider the boundary value problem*

$$\begin{aligned} -\Delta u = 1 \quad & on \quad \Omega \\ \frac{\partial u}{\partial n} = 0 \quad & on \quad \Gamma_N = \{(x,y) \in \partial\Omega \mid y < 0\} \\ u = 0 \quad & on \quad \Gamma_D = \partial\Omega \backslash \Gamma_N. \end{aligned}$$

We apply the preconditioners of Theorem 3.1, 3.3 to the boundary value problems of Examples 4.3, 4.4. The numerical results are displayed in Figs. 1, 2, respectively. In these figures, CG stands for the classical, unpreconditioned CG method, PCG-1 represents the preconditioner of Theorem 3.1, and PCG-2 indicates the preconditioner of Theorem 3.3. All iterations are started with the zero vector. We show the number of iterations required to reduce the residual by a given factor; as expected, the number of iterations of the PCG method remains practically bounded.

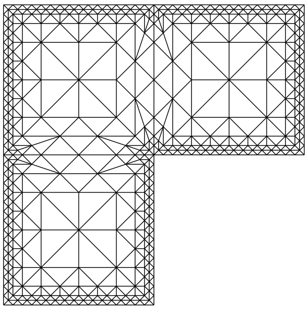
4.1 Complexity

We now turn to the complexity of the preconditioner. The application of the preconditioners discussed in Theorems 3.1 and 3.3 depends on solving the subdomain problems associated with the splittings (15), (19). We note that the subdomain problems associated with the one-dimensional spaces $\mathcal{V}_v^m$ are trivial to solve; the contribution of these problems to the total cost can be bounded by the number of vertices of all meshes. To estimate this, it suffices to bound the number of elements of all meshes:

$$T = \sum_{m=0}^{M} \sum_{K \in \mathcal{T}_m} 1.$$

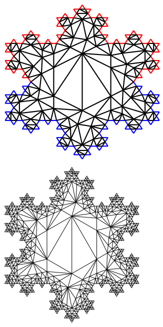
Arguing as in Lemma 4.5 below, we have $\sum_{K \in \mathcal{T}_m} 1 \leq C h_m^{-(d-1)}$. Recalling $h_m \sim h_0 2^{-m}$, we can bound $T = O(h_M^{-(d-1)})$; since the number of elements on mesh $\mathcal{T}_M$ is likewise $O(h_M^{-(d-1)})$, we conclude that T is bounded by a multiple of the number of elements of $\mathcal{T}_M$; thus $T \leq CN$.
The solution of the subdomain problems associated with the spaces $\mathcal{V}_v^M$ is more delicate. We show in the following Lemma 4.5 that the cost to solve these problems based on a Cholesky factorization is still $O(N)$:

Figure 1: (Example 4.3) Left: mesh on level 4. Right: iteration numbers to achieve $\|\underline{r}^k\|^2/\|\underline{r}^0\|^2 \leq 10^{-12}$



L	elements	p_{max}	DoF	CG	PCG-1	PCG-2
0	12	1	6	4	4	4
1	48	1	24	14	14	11
2	168	1	84	29	25	18
3	392	1	196	46	36	22
4	840	2	525	87	44	24
5	1736	3	1181	195	51	25
6	3528	3	2657	263	55	27
7	7112	4	5818	368	60	27
8	14280	5	12114	474	63	28
9	28616	6	25070	581	66	29
10	57288	6	51202	691	69	30
11	114632	7	103843	884	71	31
12	229320	8	209003	1120	74	31
13	458696	8	419807	1316	77	31
14	917448	9	841904	1596	79	33
15	1834952	10	1685928	—	81	33

Figure 2: (See Example 4.4) Left: domain on levels 0 & 1. Right: iteration numbers to reach $\|\underline{r}^k\|^2/\|\underline{r}^0\|^2 \leq 10^{-10}$



L	elements	p_{max}	DoF	CG	PCG-1	PCG-2
0	298	1	149	63	44	44
1	1054	3	619	185	78	57
2	3346	4	2096	318	79	62
3	8398	5	5227	477	104	65
4	18826	5	12963	659	106	70
5	40006	6	30024	927	112	78
6	82690	7	66228	1156	120	79
7	168382	7	141311	1644	128	80
8	340090	8	296938	2321	133	81
9	683830	9	613767	3282	138	81
10	1371634	10	1253012	4638	142	81

Lemma 4.5. *Let $\mathcal{T}_M$ be a geometric mesh with boundary mesh size h_M and $\mathbf{p}$ be a linear degree vector. Then the work W to compute the Cholesky factors of all subdomain stiffness matrices associated with the subspaces $\mathcal{S}_v$ as well as the memory requirement* Mem *to store the Cholesky factors is $O(N)$.*

Proof. We recall that the memory requirement for the Cholesky factors of an $n \times n$ matrix is $O(n^2)$ and the number of floating point operations for the factorization is $O(n^3)$. Since the mesh $\mathcal{T}_M$ is assumed to be shape regular, the number of elements sharing a vertex $v \in V_M$ is bounded by a constant that depends solely on the shape regularity constant of the mesh. Further, by (5), the polynomial degrees p_K of all elements $K \subset \omega_v^M$ are comparable. Hence, the size n_v of the stiffness matrix associated with the subspace $\mathcal{S}_v$ is bounded by $n_v \leq C p_K^d$ for an arbitrarily chosen element $K \subset \omega_v^M$. We conclude

$$\mathrm{Mem} \leq \sum_{v \in V_M} (C p_K^d)^2 \leq C \sum_{K \in \mathcal{T}_M} p_K^{2d}, \qquad \mathrm{W} \leq \sum_{v \in V_M} (C p_K^d)^3 \leq C \sum_{K \in \mathcal{T}_M} p_K^{3d}.$$

In order to bound the sums $\sum_{K \in \mathcal{T}_M} p_K^{2d}$ and $\sum_{K \in \mathcal{T}_M} p_K^{3d}$, we proceed as in [KM03, Prop. 2.7]. We will only consider the estimate for the work and write

$$\sum_{K \in \mathcal{T}_M} p_K^{3d} = \sum_{K \in \mathcal{T}_M, \overline{K} \cap \partial\Omega \neq \emptyset} p_K^{3d} + \sum_{K \in \mathcal{T}_M, \overline{K} \cap \partial\Omega = \emptyset} p_K^{3d}.$$

Since $p_K \leq C$ for all $K \in \mathcal{T}_M$ with $\overline{K} \cap \partial\Omega \neq \emptyset$ and $\mathcal{T}_M|_{\partial\Omega}$ is a quasi uniform mesh with mesh size h_M we obtain

$$\sum_{K \in \mathcal{T}_M, \overline{K} \cap \partial\Omega \neq \emptyset} p_K^{3d} \leq \sum_{K \in \mathcal{T}_M, \overline{K} \cap \partial\Omega \neq \emptyset} 1 \leq C h_M^{1-d} = O(N)$$

for the first sum. The second sum extends over all elements K with $\operatorname{dist}(K, \partial\Omega) > 0$. For these elements, we can exploit $p_K \leq 1 + C\log(h_K/h_M)$ together with $h_K \sim r(x) := \operatorname{dist}(x, \partial\Omega)$ uniformly in $x \in K$ to arrive at

$$p_K^{3d} \leq \int_K \frac{(1 + C\log(h_K/h_M))^{3d}}{\operatorname{vol}(K)} d\Omega \leq C \int_K \frac{(1 + C\log(r(x)/h_M))^{3d}}{(r(x))^d} d\Omega, \quad \forall K \in \mathcal{T}_M,\ \overline{K} \cap \partial\Omega = \emptyset.$$

Arguing as in [KM03, Prop. 2.7] we obtain

$$\sum_{K \in \mathcal{T}_M, \overline{K} \cap \partial\Omega = \emptyset} p_K^{3d} \leq C \int_{x \in \Omega, r(x) \geq C h_M} \frac{(1 + C\log(r(x)/h_M))^{3d}}{(r(x))^d} d\Omega \leq C h_M^{1-d} = O(N).$$

$\square$

Lemma 4.5 shows that the number of subdomain problems of large size is so small that recomputing the Cholesky factorization of these problems in each step of the preconditioned conjugate gradient method does not destroy the optimal complexity $O(N)$. An alternative would be to precompute the Cholesky factors before calling the PCG method. This improves the execution time of the PCG method (since only the forward and backward substitutions have to be performed) at the expense of an $O(N)$ memory requirement for the Cholesky factors. Table 1 compares the two approaches in more detail for Example 4.3. The column "assemb" gives the time for setting up the stiffness matrix; the column "CG" contains the timings for the unpreconditioned CG method; the column "PCG–memory opt." shows the timings for PCG method with respect to splitting (19) where the Cholesky factors are recomputed in each step; the column "PCG–runtime opt." finally has the timings for the PCG method where the Cholesky factors are precomputed before the PCG iteration. The columns "total" gives the total computing time for the PCG-method including the application of the preconditioner. The column "precond" shows the execution time for calling the preconditioner including the time for computing or precomputing the Cholesky factors. The column "Mem" contains the additional memory requirement to store the Choleky factors. As we can see for the case of the mesh $\mathcal{T}_{15}$ with 1685928 unknowns, precomputing the Cholesky factors leads to a significant speedup of the solution procedure from 373 seconds to 149 seconds.

Remark 4.6. *We solve the subdomain problems for the spaces $\mathcal{S}_v$ by Cholesky factorization. An alternative would be replace this direct solver with a preconditioner based on a further splitting of the spaces $\mathcal{S}_v$ as discussed in [SMPZ05].*

5 Proof of Theorems 3.1, 3.3

The proofs of the Theorems 3.1 and 3.3 rely on the following theorem from the abstract additive Schwarz theory (see [Osw94, TW05, Zha92]):

Table 1: Computing time [sec.] and additional memory [kB] for Example 4.3 and $\|\underline{r}^k\|^2/\|\underline{r}^0\|^2 \leq 10^{-12}$

				PCG - memory opt.		PCG - runtime opt.		
L	DOF	assemb.	CG	total	precond.	total	precond.	Mem
0	6	7.74e-04	4.10e-05	1.45e-04	1.08e-04	1.38e-04	1.01e-04	<1
1	24	2.85e-03	1.25e-04	4.32e-04	3.33e-04	3.81e-04	2.80e-04	<1
2	84	9.92e-03	3.35e-04	1.18e-03	9.62e-04	8.64e-04	6.38e-04	<1
3	196	2.33e-02	8.29e-04	2.59e-03	2.06e-03	1.74e-03	1.21e-03	<1
4	525	5.27e-02	3.59e-03	1.50e-02	1.35e-02	7.72e-03	6.24e-03	7
5	1181	1.12e-01	1.99e-02	6.10e-02	5.32e-02	2.01e-02	1.35e-02	31
6	2657	2.34e-01	2.65e-01	2.31e-01	1.92e-01	9.35e-02	5.37e-02	123
7	5818	4.86e-01	1.63e+00	6.48e-01	5.08e-01	2.78e-01	1.38e-01	457
8	12114	9.93e-01	5.66e+00	1.58e+00	1.21e+00	7.15e-01	3.40e-01	1174
9	25070	2.02e+00	1.62e+01	3.74e+00	2.84e+00	1.71e+00	7.99e-01	3105
10	51202	4.06e+00	4.17e+01	8.42e+00	6.42e+00	3.77e+00	1.75e+00	7435
11	103843	8.22e+00	1.12e+02	1.89e+01	1.46e+01	8.19e+00	3.82e+00	17198
12	209003	1.64e+01	2.92e+02	3.98e+01	3.09e+01	1.69e+01	7.88e+00	36577
13	419807	3.29e+01	6.98e+02	8.38e+01	6.55e+01	3.45e+01	1.64e+01	77033
14	841904	6.60e+01	1.73e+03	1.85e+02	1.46e+02	7.39e+01	3.50e+01	160464
15	1685928	1.32e+02	≈4.40e+03	3.73e+02	2.95e+02	1.49e+02	7.13e+01	326792

Proposition 5.1. *Let $\mathcal{V}$ be a Hilbert space and $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \mapsto \mathbb{R}$ symmetric positive definite. Let*

$$\mathcal{V} = \sum_{i=0}^{K} \mathcal{V}_i \tag{20}$$

be a not necessarily direct splitting of $\mathcal{V}$ into subspaces $\mathcal{V}_i$ and $\mathbf{E} \in \mathbb{R}^{K\times K}$ be the symmetric matrix whose entries e_{ij}, $1 \leq i, j \leq K$, are given by

$$e_{ij} = \sup_{u\in\mathcal{V}_i} \sup_{v\in\mathcal{V}_j} \frac{|a(u,v)|}{\sqrt{a(u,u)}\sqrt{a(v,v)}} \in [0,1].$$

Furthermore, let $C_0 > 0$ be a constant such that

$$\min\left\{ \sum_{i=0}^{K} a(u_i,u_i) \;\middle|\; u = \sum_{i=0}^{K} u_i, \;\; u_i \in \mathcal{V}_i \right\} \leq C_0^2 a(u,u) \qquad \forall\, u \in \mathcal{V}.$$

Then, the splitting (20) defines an ASM preconditioner corresponding to a bilinear form b that satisfies

$$\frac{a(u,u)}{1+\rho(\mathbf{E}))} \leq b(u,u) \leq C_0^2 a(u,u) \qquad \forall u \in \mathcal{V}, \tag{21}$$

where $\rho(\mathbf{E})$ denotes the spectral radius of $\mathbf{E}$.

Proposition 5.1 reduces the analysis of the preconditioners defined in Theorems 3.1, 3.3 to estimating the spectral radii $\rho(\mathbf{E})$ and the constants C_0 that correspond to the splittings. Getting bounds on these quantities is the purpose of the present section.
Since Assumption (11) ensures

$$I_m^B \subset I_m \qquad \text{for} \qquad m = 0,\ldots,M, \tag{22}$$

where the sets I_m^B, I_m are defined in Theorems 3.1, 3.3, respectively, the proofs of Theorem 3.1 and Theorem 3.3 can be unified. Due to (22) it suffices to construct a stable splitting for the case of Theorem 3.1 and to prove a uniform bound for the spectral radius $\rho(\mathbf{E})$ for the decomposition of Theorem 3.3. This will be done in Theorem 5.13, Corollary 5.14, and Theorem 5.18.

Remark 5.2. *In the two-dimensional situation, Assumption (11) is automatically enforced by Algorithm 4.1. We remark further that it is not difficult to design a similar refinement procedure for the three-dimensional case.*

5.1 Auxiliary lemmas

In this subsection we recapitulate elementary statements from linear algebra and prove some lemmas concerning the number of overlapping patches ω_v^m. Since the geometric meshes $\mathcal{T}_m$ have elements of greatly differing size, we have introduced the level $l(\omega_v^m)$ of a patch ω_v^m as a measure of its diameter in (16); the superscript m indicates that ω_v^m is a patch of mesh $\mathcal{T}_m$.

Lemma 5.3 (spectral norm estimates)**.**

(i) *Let* $\mathbf{A} \in \mathbb{R}^{n,m}$ *and* $\mathbf{P} \in \mathbb{R}^{n,n}, \mathbf{Q} \in \mathbb{R}^{m,m}$ *orthogonal. Then* $\|\mathbf{A}\|_2 = \|\mathbf{PAQ}\|_2 = \|\mathbf{A}^T\|_2$.

(ii) *Let* $\mathbf{A} \in \mathbb{R}^{n,m}$ *and* N_c *the number of non-zero entries per column. Then* $\|\mathbf{A}\|_2 \leq \sqrt{N_c} \max_{i=1,\dots,n} \|\mathbf{A}_{i,\cdot}\|_2$.

(iii) *Let* $\mathbf{A} = [\mathbf{A}_{ij}]_{i,j=1}^{N,M}$ *with* $\mathbf{A}_{ij} \in \mathbb{R}^{n_i,m_j}$*. Then*

$$\|\mathbf{A}\|_2 \leq \left\| \left[\|\mathbf{A}_{ij}\|_2 \right]_{i,j=1}^{N,M} \right\|_2 .$$

Proof. Well-known. ∎

Lemma 5.4. *Let* $\{\mathcal{T}_m\}_{m=0}^{M}$ *be a sequence of nested geometric meshes that satisfy (9)–(11). Let the patches* ω_v^m *and the numbers* $l(\cdot)$, $g(\cdot)$ *be as defined in (12), (16). Then there exists a constant* $C \geq 0$ *depending only on the shape regularity constants of the meshes and the boundary mesh size* h_0 *of* $\mathcal{T}_0$ *such that for all* ω_v^m, $0 \leq m \leq M$, $v \in V_m$

$$l(\omega_v^m) \leq g(\omega_v^m) + C.$$

Proof. From the definition of $g(\omega_v^m)$ together with (10) follows the existence of $C > 0$ depending only on the shape regularity constants of the meshes such that $\operatorname{diam} \omega_v^m \geq C h_0 2^{-g(\omega_v^m)}$. This implies

$$-\log_2 (\operatorname{diam} \omega_v^m) \leq -\log_2 \left(C h_0 2^{-g(\omega_v^m)} \right) = g(\omega_v^m) - \log_2(C h_0).$$

∎

Lemma 5.5. *Let* $\{\mathcal{T}_m\}_{m=0}^{M}$ *be a sequence of nested geometric meshes that satisfy (9)–(11). Let the patches* ω_v^m *and the numbers* $l(\cdot)$, $g(\cdot)$ *be as defined in (12), (16). Let Assumption (18) be valid. Then there exists a constant* $C > 0$*, depending only on the dimension* d*, the shape regularity constants of the meshes,* h_0*, and the constant of Assumption (18) such that for* $0 \leq l \leq l' \leq L$ *and any* $\omega_{v'}^{m'}$ *with* $l(\omega_{v'}^{m'}) = l'$, $g(\omega_{v'}^{m'}) = m'$

$$\#\{\omega_v^m \mid l(\omega_v^m) = l, \;\; g(\omega_v^m) = m, \;\; \omega_v^m \cap \omega_{v'}^{m'} \neq \emptyset, \; m \in \{0, \dots, M\}, \; v \in V_m\} \leq C.$$

Proof. We proceed by a packing argument and by exploiting the shape regularity of the meshes. We consider a fixed patch $w_{v'}^{m'}$ and define the set of all patches of mesh $\mathcal{T}_m$ that are on level l and intersect $\omega_{v'}^{m'}$ by

$$P_m := \{\omega_v^m \mid v \in V_m, \; l(\omega_v^m) = l, \omega_v^m \cap \omega_{v'}^{m'} \neq \emptyset\}, \qquad m \in \{0, \dots, M\}.$$

From the definition of $l(\omega)$ follows

$$2^{-(l'+1)} \leq \operatorname{diam}(\omega_{v'}^{m'}) \leq 2^{-l'} \quad \text{and} \qquad 2^{-(l+1)} \leq \operatorname{diam}(\omega_i^m) \leq 2^{-l}.$$

Hence,

$$\bigcup_{\omega_v^m \in P_m} \omega_v^m \subset B_l,$$

where B_l is a suitably chosen ball with diameter

$$2^{-l} \leq \operatorname{diam}(B_l) \leq 2^{-l'} + 2 \cdot 2^{-l} \leq 3 \cdot 2^{-l}.$$

Using the shape regularity of the mesh $\mathcal{T}_m$ and denoting by χ_E the characteristic function of a set E, we estimate the cardinality of P_m as follows:

$$\# P_m = \sum_{\omega_v^m \in P_m} 1 = \sum_{\omega_v^m \in P_m} \frac{1}{\operatorname{vol}(\omega_v^m)} \int_{B_l} \chi_{\omega_v^m} \leq \sum_{\omega_v^m \in P_m} C 2^{dl} \int_{B_l} \chi_{\omega_v^m} \leq C 2^{dl} \int_{B_l} \sum_{\omega_v^m \in P_m} \chi_{\omega_v^m},$$

where $C > 0$ is a constant depending only on the dimension d. We now use the fact that the patches $\omega_v^m \in P_m$ are all patches of the same mesh $\mathcal{T}_m$. Hence, if we deal with triangles/tetrahedra, any point $x \in \Omega$, is contained in at most $d+1$ patches. Therefore, the sum under the integral is bounded pointwise by $d+1$, and we arrive at

$$\# P_m \leq C 2^{dl} \int_{B_l} (d+1) \leq C 2^{dl} (2^{-l})^d \leq C.$$

Thus a *fortiori*,

$$\widehat{P}_m := \{\omega_v^m \mid v \in V_m,\ l(\omega_v^m) = l, g(\omega_v^m) = m, \omega_v^m \cap \omega_{v'}^{m'} \neq \emptyset\} \leq C, \quad \forall m \in \{0, \ldots, M\}.$$

Additionally, due to Lemma 5.4 and Assumption (18) we have

$$g(\omega_v^m) - C \leq l(\omega_v^m) \leq g(\omega_v^m) + C \qquad \forall \omega_v^m. \tag{23}$$

Hence, $\#\widehat{P}_m = 0$ for $|m - l(\omega_v^m)| > C$, and the claim follows from

$$\{\omega_v^m \mid l(\omega_v^m) = l,\ \ g(\omega_v^m) = m,\ \ \omega_v^m \cap \omega_{v'}^{m'} \neq \emptyset,\ m \in \{0, \ldots, M\},\ v \in V_m\} = \bigcup_{m=0}^{M} \widehat{P}_m.$$

□

Lemma 5.6. *Let $\{\mathcal{T}_m\}_{m=0}^M$ be a sequence of nested geometric meshes that satisfy (9)–(11). Let the patches ω_v^m and the numbers $l(\cdot)$, $g(\cdot)$ be as defined in (12), (16). Let Assumption (18) be valid and fix $l \in \{0, \ldots, L\}$. Then there exists a constant $C > 0$, depending only on the shape regularity constants of the meshes, h_0, and the constant of Assumption (18) such that for any $\omega_{v'}^M$:*

$$\#\{\omega_v^m \mid 0 \leq m \leq M,\ v \in V_m,\ l(\omega_v^m) = l,\ g(\omega_v^m) = m,\ \omega_v^m \cap \omega_{v'}^M \neq \emptyset\} \leq C.$$

Proof. We define

$$P_m := \{\omega_v^m \mid v \in V_m,\ l(\omega_v^m) = l, g(\omega_v^m) = m, \omega_v^m \cap \omega_{v'}^M \neq \emptyset\}, \qquad m \in \{0, \ldots, M\}.$$

Due to shape regularity $\omega_{v'}^M$ consists of at most C' elements, where C' depends only on the shape regularity constant of the mesh $\mathcal{T}_M$. Since $\omega_{v'}^M$ is a patch on the finest mesh and since we assume nestedness of the meshes $\mathcal{T}_m$ (cf. (9)) we can conclude that for each mesh $\mathcal{T}_m$ and every $K \subset \omega_{v'}^M$ there exists a unique $K' \in \mathcal{T}_m$ with $K \subset K'$. Hence

$$\#\{K' \in \mathcal{T}_m \mid K' \cap \omega_{v'}^M \neq \emptyset\} \leq C'.$$

Noting that an element $K' \in \mathcal{T}_m$ is contained in at most $d+1$ patches of $\mathcal{T}_m$, we get

$$\# P_m \leq (d+1) C' \quad \text{for all } m \in \{0, \ldots, M\}.$$

Moreover, from (23) we obtain $\#P_m = 0$ for $|m - l(\omega_i^m)| > \tilde{C}$ and hence the claim follows from

$$\#\{\omega_v^m \mid 0 \leq m \leq M,\ v \in V_m,\ l(\omega_v^m) = l,\ g(\omega_v^m) = m,\ \omega_v^m \cap \omega_{v'}^M \neq \emptyset\} = \sum_{m=0}^{M} \#P_m.$$

$\square$

Lemma 5.7. *Let $\{\mathcal{T}_m\}_{m=0}^M$ be a sequence of nested geometric meshes that satisfy (9)–(11). Let the patches ω_v^m and the numbers $l(\cdot)$, $g(\cdot)$ be defined as in (12), (16). Then there exists a constant $C > 0$, depending only on the constants of the meshes such that for all ω_v^m, $0 \leq m \leq M$, $v \in V_m$ and $w_{v'}^M$, $v' \in V_M$*

$$l(\omega_v^m) - l(\omega_{v'}^M) > C \quad \Rightarrow \quad \omega_v^m \cap \omega_{v'}^M = \emptyset.$$

Proof. We consider a fixed patch $\omega_{v'}^M$ and distinguish two cases.

- Let $\overline{\omega_{v'}^M} \cap \partial\Omega = \emptyset$. Then from $l(\omega_{v'}^M) = l' \in \{0, \dots, L\}$ follows

  $$2^{-(l'+1)} \leq \operatorname{diam}(\omega_{v'}^M) \leq 2^{-l'}$$

  and by shape regularity of the mesh $\mathcal{T}_M$ together with (10) we obtain

  $$\inf_{x \in \omega_{v'}^M} \operatorname{dist}(x, \partial\Omega) \geq C_1 2^{-l'}.$$

  On the other hand, $l(\omega_v^m) = l \in \{0, \dots, L\}$ implies

  $$\sup_{x \in \omega_v^m} \operatorname{dist}(x, \partial\Omega) \leq C_2 2^{-l}.$$

  Hence, the condition $\omega_{v'}^m \cap \omega_v^M \neq \emptyset$ requires

  $$C_1 2^{-l'} \leq C_2 2^{-l} \ \Leftrightarrow\ 2^{l-l'} \leq C_2 C_1^{-1} \ \Leftrightarrow\ l - l' \leq \log_2(C_2) - \log_2(C_1) =: C'.$$

- Let $\overline{\omega_{v'}^M} \cap \partial\Omega \neq \emptyset$. Then, since all elements at the boundary are of similar size, $l(\omega_{v'}^M) = l' \in \{L - \widetilde{C}, \dots, L\}$ and we always have $l(\omega_v^m) - l(\omega_{v'}^M) \leq L - (L - \widetilde{C}) = \widetilde{C}$.

Taking $C := \max\{C', \widetilde{C}\}$ allows us to conclude the proof. $\square$

5.2 Estimating the angles between the spaces

Lemma 5.8. *Let $\{\mathcal{T}_m\}_{m=0}^M$ be a sequence of nested geometric meshes that satisfy (9)–(11). Let the patches ω_v^m be defined as in (12). Then there exists a constant $C > 0$ depending only on the shape regularity constants of the meshes and the coefficients $\hat{\mathbf{A}}$, a_0 of the bilinear form $a(\cdot,\cdot)$ such that for any $U, U' \in \{\mathcal{S}_v \mid v \in V_M\} \cup \{\mathcal{V}_v^m \mid m = 0, \dots, M,\ v \in V_m\}$ with corresponding patches ω_v^m and $\omega_{v'}^{m'}$ (i.e., supp$\{u \in U\} = \omega_v^m$, supp$\{u' \in U'\} = \omega_{v'}^{m'}$) we have*

$$e_{UU'}^2 := \sup_{u \in U} \sup_{u' \in U'} \frac{|a(u,u')|^2}{a(u,u)a(u',u')} \leq \min\left\{1, C\frac{\operatorname{vol}(\omega_v^m \cap \omega_{v'}^{m'})}{\operatorname{vol}(w_v^m)} p^{2d}\right\},$$

where p denotes the maximum polynomial degree of $u \in U$.

Proof. By the standard Cauchy-Schwarz inequality, we get

$$|a(u,u')|^2 \leq a_{\omega_v^m \cap \omega_{v'}^{m'}}(u,u)a(u',u'), \tag{24}$$

with

$$\begin{aligned}
a_{\omega_v^m \cap \omega_{v'}^{m'}}(u,u) &= \int_{\omega_v^m \cap \omega^{m'}} \langle \nabla u, A \nabla u \rangle + a_0 uu \, \mathrm{d}\Omega \\
&\leq \max\left\{ \|a_0\|_{L^\infty(\Omega)}, \|A\|_{L^\infty(\Omega)} \right\} \left(\|\nabla u\|^2_{L^2(\omega_v^m \cap \omega_{v'}^{m'})} + \|u\|^2_{L^2(\omega_v^m \cap \omega_{v'}^{m'})} \right) \\
&\leq C_{\hat{\mathbf{A}}, a_0} \operatorname{vol}\left(\omega_v^m \cap \omega_{v'}^{m'} \right) \left(\|\nabla u\|^2_{L^\infty(\omega_v^m)} + \|u\|^2_{L^\infty(\omega_v^m)} \right).
\end{aligned}$$

Since u is a piecewise polynomial of degree p on ω_v^m and since the mesh $\mathcal{T}_m$ is assumed to be shape regular, we may apply an inverse estimate to arrive at

$$\|u\|^2_{L^\infty(\omega_v^m)} \leq C p^{2d} \left(\operatorname{vol}(\omega_v^m) \right)^{-1} \|u\|^2_{L^2(\omega_v^m)},$$

which in turn yields

$$a_{\omega_v^m \cap \omega_{v'}^{m'}}(u,u) \leq C_{\hat{\mathbf{A}}, a_0} p^{2d} \frac{\operatorname{vol}\left(\omega_v^m \cap \omega_{v'}^{m'} \right)}{\operatorname{vol}(\omega_v^m)} \|u\|^2_{H^1(\omega_v^m)} \leq C_{\hat{\mathbf{A}}, a_0} p^{2d} \frac{\operatorname{vol}\left(\omega_v^m \cap \omega_{v'}^{m'} \right)}{\operatorname{vol}(\omega_v^m)} a(u,u).$$

The claim follows from inserting this bound in (24). $\square$

5.3 Estimating the spectral radius

In this section we want to derive a bound for the spectral radii $\rho(\mathbf{E})$ and $\rho(\tilde{\mathbf{E}})$, where $\mathbf{E}$ denotes the matrix containing the angles between the subspaces of splitting (19) and $\tilde{\mathbf{E}}$ the matrix containing the angles between the subspaces of splitting (15). We proceed as follows: We rearrange and subdivide the matrix $\mathbf{E}$ into submatrices $\mathbf{E}_{**}^{ll'}$ and bound the norms of these submatrices. Thus, by means of Lemma 5.3, a bound for $\rho(\mathbf{E})$ follows. The bound for $\rho(\tilde{\mathbf{E}})$ follows from the fact that $\tilde{\mathbf{E}}$ is a submatrix of $\mathbf{E}$.

Lemma 5.9. *Let the assumptions of Theorem 3.3 be valid. Let $l \leq l'$ and denote by $\mathbf{E}_{\mathcal{SS}}^{ll'} = \left[e_{\mathcal{S}_v \mathcal{S}_{v'}} \right]$ the matrix containing the angles between the subspaces $\mathcal{S}_v \in \{\mathcal{S}_v \mid v \in V_M,\ l(\mathcal{S}_v) = l\}$ and $\mathcal{S}_{v'} \in \{\mathcal{S}_{v'} \mid v' \in V_M,\ l(\mathcal{S}_{v'}) = l'\}$. Then there exist constants C, $C' > 0$ depending only on the shape regularity constant of $\mathcal{T}_M$ such that*

$$\left\| \mathbf{E}_{SS}^{ll'} \right\|_2 \leq \begin{cases} C & : \ l' - l \leq \tilde{C} \\ 0 & : \ l' - l > \tilde{C} \end{cases}$$

where $\tilde{C}$ denotes the constant of Lemma 5.7. In particular,

$$\left\| \mathbf{E}_{SS}^{ll'} \right\|_2 \leq C' 2^{(l-l')/2},$$

Proof. For $l' - l > \tilde{C}$ the claim follows directly from Lemma 5.7. For the case $l' - l \leq \tilde{C}$ we observe, that each row of $\mathbf{E}_{\mathcal{SS}}^{ll'}$ corresponds to a subspaces $\mathcal{S}_v$ with $\operatorname{supp}(\mathcal{S}_v) \subset \omega_v^M$ and each column of $\mathbf{E}_{\mathcal{SS}}^{ll'}$ corresponds to a subspace $\mathcal{S}_{v'}$ with $\operatorname{supp}(\mathcal{S}_{v'}) \subset \omega_{v'}^M$. Thus, since ω_v^M and $\omega_{v'}^M$ are patches of the same mesh and due to shape regularity, each row and each column of $\mathbf{E}_{\mathcal{SS}}^{ll'}$ has at most $O(1)$ non-zero entries, where the constant $O(1)$ depends only on the shape regularity constant of $\mathcal{T}_M$. $\square$

Lemma 5.10. *Let the assumptions of Theorem 3.3 be valid. Let $l \leq l'$ and denote by $\mathbf{E}_{\mathcal{SV}}^{ll'} = [e_{\mathcal{S}_v \mathcal{V}_{v'}^{m'}}]$ the matrix containing the angles between the spaces $\mathcal{S}_v \in \{\mathcal{S}_v \mid v \in V_M,\ l(\mathcal{S}_v) = l\}$ and $\mathcal{V}_{v'}^{m'} \in \{\mathcal{V}_{v'}^{m'} \mid 0 \leq m' \leq M,\ v \in V_m,\ l(\mathcal{V}_{v'}^{m'}) = l',\ g(\mathcal{V}_{v'}^{m'}) = m'\}$. Then there exists a constant $C > 0$ depending only on the shape regularity constants of the meshes such that*

$$\left\| \mathbf{E}_{S\mathcal{V}}^{ll'} \right\|_2 \leq \begin{cases} C & : \ l' - l \leq \tilde{C} \\ 0 & : \ l' - l > \tilde{C} \end{cases} \leq C 2^{(l-l')/2},$$

where $\tilde{C}$ denotes the constant of Lemma 5.7.

Proof. The supports of the spaces $\mathcal{S}_v$ and $\mathcal{V}_{v'}$ correspond to the patches ω_v^M and $\omega_{v'}^{m'}$ with $l(\omega_v^M) = l$, $l(\omega_{v'}^{m'}) = l'$. Thus the claim for $l' - l > \tilde{C}$ follows from Lemma 5.7. For $l' - l \leq \tilde{C}$ each row of $\mathbf{E}_{\mathcal{S}\mathcal{V}}^{ll'}$ corresponds to a subspaces $\mathcal{S}_v$ with $\operatorname{supp}(\mathcal{S}_v) \subset \omega_v^M$ and each column of $\mathbf{E}_{\mathcal{S}\mathcal{V}}^{ll'}$ corresponds to a subspaces $\mathcal{V}_{v'}^{m'}$ with $\operatorname{supp}(\mathcal{V}_{v'}^{m'}) \subset \omega_{v'}^{m'}$. If we consider a fixed subspace $\mathcal{S}_v$, Lemma 5.6 implies that there are only $O(1)$ non zero elements per row of $\mathbf{E}_{\mathcal{S}\mathcal{V}}^{ll'}$. On the other hand, if we consider a fixed column, i.e., a fixed subspace $\mathcal{V}_{v'}^{m'}$, then there are, due to $l(\mathcal{S}_v) \leq l(\mathcal{V}_{v'}) \leq l(\mathcal{S}_v) + \tilde{C}$ and an argument analogous to that of Lemma 5.5, at most $O(1)$ non zero elements in that column. Thus by Lemma 5.3

$$\left\|\mathbf{E}_{\mathcal{S}\mathcal{V}}^{ll'}\right\|_2 \leq C \max_v \|\mathbf{E}_{\mathcal{S}_v\cdot}^{ll'}\|_2 \leq C.$$

□

Lemma 5.11. *Let the assumptions of Theorem 3.3 be valid. Let $l \leq l'$ and denote by $\mathbf{E}_{\mathcal{V}\mathcal{S}}^{ll'} = \left[e_{\mathcal{V}_v^m\mathcal{S}_{v'}}\right]$ the matrix containing the angles between the spaces $\mathcal{V}_v^m \in \{\mathcal{V}_v^m \mid 0 \leq m \leq M,\ v \in V_m,\ l(\mathcal{V}_v^m) = l,\ g(\mathcal{V}_v^m) = m\}$ and $\mathcal{S}_{v'} \in \{\mathcal{S}_{v'} \mid v' \in V_M,\ l(\mathcal{S}_{v'}) = l'\}$. Then there exists a constant $C > 0$ depending only on the shape regularity constants of the meshes such that*

$$\left\|\mathbf{E}_{\mathcal{V}\mathcal{S}}^{ll'}\right\|_2 \leq C2^{(l-l')/2}.$$

Proof. Each column of $\mathbf{E}_{\mathcal{V}\mathcal{S}}^{ll'}$ corresponds to a fixed subspace $\mathcal{S}_{v'}$ with $l(\mathcal{S}_{v'}) = l' \geq l$ and support $\omega_{v'}^M$. Thus, due to Lemma 5.5 the number of non-zero entries per column of $\mathbf{E}_{\mathcal{V}\mathcal{S}}^{ll'}$ is bounded by C which depends only on the shape regularity constants of the mesh. In order to apply Lemma 5.3 we now bound the l^2-norm of the row corresponding to $\mathcal{V}_v^m$ with $l(\mathcal{V}_v^m) = l$. We have

$$\|\mathbf{E}_{\mathcal{V}_v^m,\cdot}^{ll'}\|_2^2 \leq \sum_{\mathcal{S}_{v'}:l(\mathcal{S}_{v'})=l'} |e_{\mathcal{V}_v^m\mathcal{S}_{v'}}|^2$$

The support of the functions of $\mathcal{S}_{v'}$ is the patch $\omega_{v'}^M$. Since all patches $\omega_{v'}^M$ are patches of the same mesh not more than $d+1$ can overlap. Furthermore, we observe that all patches of level l' are confined to an $O(2^{-l'})$ neighborhood of $\partial\Omega$. Noting that the spaces $\mathcal{V}_v^m$ are spaces of piecewise polynomials of degree $p = 1$, Lemma 5.8 implies

$$\sum_{\mathcal{S}_{v'}:l(\mathcal{S}_{v'})=l'} |e_{\mathcal{V}_v^m\mathcal{S}_{v'}}|^2 \leq C\frac{\operatorname{vol}(\omega_v^m \cap \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq 2^{-l'}\})}{\operatorname{vol}(\omega_v^m)}.$$

Since the patch ω_v^m is on level $l \leq l'$, elementary geometric considerations give

$$\operatorname{vol}(\omega_v^m \cap \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq 2^{-l'}\}) \leq C(2^{-l})^{d-1}2^{-l'}, \qquad \operatorname{vol}(\omega_v^m) \sim (2^{-l})^d.$$

We conclude

$$\sum_{\mathcal{S}_{v'}:l(\mathcal{S}_{v'})=l'} |e_{\mathcal{V}_v^m\mathcal{S}_{v'}}|^2 \leq C\frac{\operatorname{vol}(\omega_v^m \cap \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq 2^{-l'}\})}{\operatorname{vol}(\omega_v^m)} \leq C2^{(l-l')}.$$

Making use of Lemma 5.3 allows us to conclude the proof. □

Lemma 5.12. *Let the assumptions of Theorem 3.3 be valid. Let $l \leq l'$ and denote by $\mathbf{E}_{\mathcal{V}\mathcal{V}}^{ll'} = \left[e_{\mathcal{V}_v^m\mathcal{V}_{v'}}\right]$ the matrix containing the angles between the spaces $\mathcal{V}_v^m \in \{\mathcal{V}_v^m \mid 0 \leq m \leq M,\ v \in V_m,\ l(\mathcal{V}_v^m) = l,\ g(\mathcal{V}_v^m) = m\}$ and $\mathcal{V}_{v'}^{m'} \in \{\mathcal{V}_{v'}^{m'} \mid 0 \leq m' \leq M,\ v' \in V_{m'},\ l(\mathcal{V}_{v'}^{m'}) = l,\ g(\mathcal{V}_{v'}^{m'}) = m'\}$. Then there exists a constant $C > 0$ depending only on the shape regularity constants of the meshes such that*

$$\left\|\mathbf{E}_{\mathcal{V}\mathcal{V}}^{ll'}\right\|_2 \leq C2^{(l-l')/2}.$$

Proof. Each column of $\mathbf{E}^{ll'}_{\mathcal{V}\mathcal{V}}$ corresponds to a fixed subspace $\mathcal{V}^{m'}_{v'}$ with $l(\mathcal{V}^{m'}_{v'}) = l' \geq l$. Thus, due to Lemma 5.5 the number of non-zero entries per column of $\mathbf{E}^{ll'}_{\mathcal{V}\mathcal{V}}$ is bounded by a constant C which depends only on the shape regularity constants of the mesh. Completely analogous to the proof of Lemma 5.11 we bound the l^2-norm of the rows of $\mathbf{E}^{ll'}_{\mathcal{V}\mathcal{V}}$ by

$$\|\mathbf{E}^{ll'}_{\mathcal{V}^m_v,\cdot}\|_2^2 \leq \sum_{\mathcal{V}^{m'}_{v'}:l(\mathcal{V}^{m'}_{v'})=l'} |e_{\mathcal{V}^m_v\mathcal{V}^{m'}_{v'}}|^2 \leq C2^{(l-l')},$$

and the desired estimates follow again by Lemma 5.3. □

Theorem 5.13 (spectral radius)**.** *Let the assumptions of Theorem 3.3 be valid and let* $\mathbf{E}$ *be the matrix containing the angles between the subspaces with respect to the splitting (19). Then* $\rho(\mathbf{E}) \leq C$.

Proof. With the matrices $\mathbf{E}^{ll'}_{**}$ defined as in Lemmas 5.9–5.12 we rearrange the symmetric matrix $\mathbf{E}$ as follows:

$$\mathbf{E} = \begin{bmatrix} \begin{bmatrix} \mathbf{E}^{00}_{\mathcal{V}\mathcal{V}} & \mathbf{E}^{00}_{\mathcal{V}\mathcal{S}} \\ \mathbf{E}^{00}_{\mathcal{S}\mathcal{V}} & \mathbf{E}^{00}_{\mathcal{S}\mathcal{S}} \end{bmatrix} & \cdots & \begin{bmatrix} \mathbf{E}^{0L}_{\mathcal{V}\mathcal{V}} & \mathbf{E}^{0L}_{\mathcal{V}\mathcal{S}} \\ \mathbf{E}^{0L}_{\mathcal{S}\mathcal{V}} & \mathbf{E}^{0L}_{\mathcal{S}\mathcal{S}} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} \mathbf{E}^{L0}_{\mathcal{V}\mathcal{V}} & \mathbf{E}^{L0}_{\mathcal{V}\mathcal{S}} \\ \mathbf{E}^{L0}_{\mathcal{S}\mathcal{V}} & \mathbf{E}^{L0}_{\mathcal{S}\mathcal{S}} \end{bmatrix} & \cdots & \begin{bmatrix} \mathbf{E}^{LL}_{\mathcal{V}\mathcal{V}} & \mathbf{E}^{LL}_{\mathcal{V}\mathcal{S}} \\ \mathbf{E}^{LL}_{\mathcal{S}\mathcal{V}} & \mathbf{E}^{LL}_{\mathcal{S}\mathcal{S}} \end{bmatrix} \end{bmatrix}.$$

With the abbreviations $e^{ll'}_{**} := \|\mathbf{E}^{ll'}_{**}\|_2$ Lemma 5.3 gives:

$$\rho(\mathbf{E}) \leq \left\| \left[\begin{bmatrix} e^{ll'}_{\mathcal{V}\mathcal{V}} & e^{ll'}_{\mathcal{V}\mathcal{S}} \\ e^{ll'}_{\mathcal{S}\mathcal{V}} & e^{ll'}_{\mathcal{S}\mathcal{S}} \end{bmatrix} \right]_{l,l'=0}^{L} \right\|_2 .$$

For $l \leq l'$ we have (see Lemmas 5.9—5.12)

$$e^{ll'}_{**} \leq C2^{(l-l')/2}$$

and due to the symmetry of $\mathbf{E}$ we obtain

$$e^{ll'}_{**} \leq C\left(\sqrt{2}\right)^{-|l-l'|} \qquad \forall\, l,l' \in \{0,\ldots,L\}.$$

Now, the assertions follow from

$$\rho(\mathbf{E}) \leq \max_{l=0}^{L} \sum_{l'=0}^{L} C\sqrt{2}^{-|l-l'|} \leq C.$$

□

Corollary 5.14 (spectral radius)**.** *Let the assumptions of Theorem 3.1 be valid and let* $\tilde{\mathbf{E}}$ *be the matrix containing the angles between the subspaces with respect to the splitting (15). Then* $\rho(\tilde{\mathbf{E}}) \leq C$.

Proof. Since (22) the matrix $\tilde{\mathbf{E}}$ is a submatrix of the non-negative matrix $\mathbf{E}$ of Theorem 5.13 and $\rho(\tilde{\mathbf{E}}) \leq \rho(\mathbf{E}) \leq C$ follows. □

5.4 Construction of a stable splitting

The aim of this subsection is to prove a stable splitting for the case of Theorem 3.1.
We proceed as follows: First we construct a stable splitting for $u \in S^1_0(\Omega, \mathcal{T}_M)$. Thereafter we construct a stable splitting for $u \in S^1_D(\Omega, \mathcal{T}_M)$, and finally we expand our splitting to $u \in S^{\mathbf{p}}_D(\Omega, \mathcal{T}_M)$.

Lemma 5.15 (stable splitting for $u \in S_0^1(\Omega, \mathcal{T}_M)$). *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Let $\mathcal{T}_M$ be a geometric mesh with boundary mesh size h and $u \in S_0^1(\Omega, \mathcal{T}_M) := S^1(\Omega, \mathcal{T}_M) \cap H_0^1(\Omega)$. Denote by V_M the set of all vertices of $\mathcal{T}_M$ and by ϕ_v the standard hat function on $\mathcal{T}_M$ corresponding to $v \in V_M$. Then the decomposition of a $u \in S_0^{1,1}(\Omega, \mathcal{T}_M)$ into hat functions satisfies*

$$u = \sum_{v \in V_M} u_v, \quad \text{with} \quad u_v \in \mathcal{V}_v^M \subset \mathcal{S}_v \quad \text{and} \quad \sum_{v \in V_M} a(u_v, u_v) \leq C a(u, u),$$

where $C > 0$ is independent of h.

Proof. The function $u \in S_0^{1,1}(\Omega, \mathcal{T}_M)$ can be (uniquely) written as sum of hat functions $u = \sum_{v \in V_M} \underline{u}_v \phi_v$. With $h_v := \operatorname{diam}(\operatorname{supp} \phi_v)$ we rearrange this sum as

$$u = \sum_{v \in V_M} h_v^{(d/2-1)} \underline{u}_v h_v^{(1-d/2)} \phi_v$$

and set

$$\underline{w}_v := h_v^{(d/2-1)} \underline{u}_v, \qquad \psi_v := h_v^{(d/2-1)} \phi_v.$$

Now we proceed in several steps.
1. step: We consider a fixed $K \in \mathcal{T}_M$ with vertices $v_{(K,1)}, \ldots, v_{(K,d+1)}$. Then $u|_K = \sum_{j=1}^{d+1} \underline{w}_{v_{(K,j)}} \psi_{v_{(K,j)}}|_K$. Due to shape regularity we can bound

$$\sum_{j=1}^{d+1} |\underline{w}_{v_{(K,j)}}|^2 \leq C h_K^{(d-2)} \sum_{j=1}^{d+1} |u(v_{(K,j)})|^2 \leq C h_K^{(d-2)} (d+1) \|u\|_{L^\infty(K)}^2,$$

where $C > 0$ depends solely on the shape regularity constant γ. Now, let $\hat{K}$ be the reference element and let the pull back of a function to the reference element be marked by a hat. Then we get

$$\sum_{j=1}^{d+1} |w_{v_{(K,j)}}|^2 \leq C h_K^{(d-2)} (d+1) \|\hat{u}\|_{L^\infty(\hat{K})}^2 \leq C h_K^{(d-2)} (d+1) \|\hat{u}\|_{L^2(\hat{K})}^2 \leq C h_K^{-2} \|u\|_{L^2(K)}^2. \tag{25}$$

2. step: Let $v \in V_M$ and $K \in \mathcal{T}_M$ with $K \subset \operatorname{supp} \psi_v$. Then we have

$$\|\nabla \psi_v\|_{L^2(K)}^2 \leq C h_K^{(d-2)} \|\nabla \hat{\psi}_v\|_{L^2(\hat{K})}^2 \leq C h_K^{(d-2)} h_K^{(2-d)} \leq C$$

and

$$\|\psi_v\|_{L^2(K)}^2 \leq C h_K^d \|\hat{\psi}_v\|_{L^2(\hat{K})}^2 \leq C h_K^d h_K^{(2-d)} \leq C h_K^2 \leq C$$

with $C > 0$ depending solely on the shape regularity constant.
3. step: We recall Hardy's inequality

$$\left\| \frac{v}{\operatorname{dist}(x, \partial\Omega)} \right\| \leq C \|\nabla v\|_{L^2(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Then, we combine this estimate with (25) and the assumptions on h_K to arrive at

$$\sum_{v \in V_M} |\underline{w}_v|^2 \leq \sum_{K \in \mathcal{T}_M} \sum_{j=1}^{d+1} |\underline{w}_{v_{(K,j)}}|^2 \leq C \sum_{K \in \mathcal{T}_M} h_K^{-2} \|u\|_{L^2(K)}^2 \leq C \|u / \operatorname{dist}(\cdot, \partial\Omega)\|_{L^2(\Omega)}^2 \leq C \|\nabla u\|_{L^2(\Omega)}^2.$$

Since $a(\cdot, \cdot) \sim \|\cdot\|_{H^1(\Omega)}^2$ and due to the bounds of the second step we get the desired result

$$\sum_{v \in V_M} a(u_v, u_v) = \sum_{v \in V_M} a(\underline{w}_v \psi_v, \underline{w}_v \psi_v) = \sum_{v \in V_M} \underline{w}_v^2 a(\psi_v, \psi_v) \leq C \sum_{v \in V_M} \underline{w}_v^2 \leq C \|\nabla u\|_{L^2(\Omega)}^2 \leq C a(u, u).$$

□

Remark 5.16. *A proof for the two-dimensional case of Lemma 5.15 can also be found in [Yse99, Mel01].*

Lemma 5.17 (stable splitting for $u \in S_D^1(\Omega, \mathcal{T}_M)$)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and $\{\mathcal{T}_m\}_{m=0,\dots,M}$ be a sequence of nested geometric meshes that satisfy (9)–(11). Let $I_m^B := \{v \in V_m \mid v \in \partial\Omega\}$. Then there exists for every $u \in S_D^1(\Omega, \mathcal{T}_M)$ a decomposition*

$$u = \sum_{v \in V_M} u_v^{\mathcal{S}} + \sum_{m=0}^{M} \sum_{v \in I_m^B} u_v^m, \quad \text{with } u_v^m \in \mathcal{V}_v^m \quad \text{ and } u_v^{\mathcal{S}} \in \mathcal{S}_v \cap S_D^1(\Omega, \mathcal{T}) \tag{26}$$

such that

$$\sum_{v \in V_M} a(u_v^{\mathcal{S}}, u_v^{\mathcal{S}}) + \sum_{m=0}^{M} \sum_{v \in I_m^B} a(u_v^m, u_v^m) \le Ca(u, u),$$

where $C > 0$ is independent of u, h_M, and M.

Proof. We proceed in several steps.
1. step: We choose a quasi uniform mesh $\tilde{\mathcal{T}}_0$ that coincides with $\mathcal{T}_0$ near the boundary, i.e.,

$$\{K \in \mathcal{T}_0 \mid \overline{K} \cap \partial\Omega \neq \emptyset\} \subset \tilde{\mathcal{T}}_0. \tag{27}$$

If the mesh $\mathcal{T}_0$ is deemed quasi-uniform, then we may choose $\tilde{\mathcal{T}}_0 = \mathcal{T}_0$. Alternatively, it is possible to consider the set $\Omega' := \Omega \backslash \cup \{\overline{K} \mid K \in \mathcal{T}_0 \text{ and } \overline{K} \cap \partial\Omega \neq \emptyset\}$ and to extend the quasi uniform triangulation of $\partial\Omega'$ given by $\mathcal{T}_0|_{\partial\Omega'}$ to a quasi uniform triangulation $\mathcal{T}_0'$ of Ω'. Then $\tilde{\mathcal{T}}_0 := \mathcal{T}_0' \cup \{K \in \mathcal{T}_0 \mid \overline{K} \cap \partial\Omega \neq \emptyset\}$ defines a quasi uniform mesh which satisfies (27).
2. step: Let $\tilde{\mathcal{T}}_m$ be the mesh obtained from the triangulation $\tilde{\mathcal{T}}_0$ by m steps of uniform refinement. The assumption (11) then implies that $\mathcal{T}_m$ and $\tilde{\mathcal{T}}_m$ coincide for all elements abutting the boundary $\partial\Omega$, that is

$$\{K \in \mathcal{T}_m \mid \overline{K} \cap \partial\Omega \neq \emptyset\} = \{K \in \tilde{\mathcal{T}}_m \mid \overline{K} \cap \partial\Omega \neq \emptyset\}. \tag{28}$$

We introduce the set $\tilde{V}_m$ of all vertices of $\tilde{\mathcal{T}}_m$ without the vertices on the Dirichlet boundary. The functions $\tilde{\phi}_v^m$ are taken to be the standard hat functions of the mesh $\tilde{\mathcal{T}}_m$ corresponding to the vertices $v \in \tilde{V}_m$. We set

$$\tilde{\mathcal{V}}_v^m := \operatorname{span}\{\tilde{\phi}_v^m\},$$

and point out that $\tilde{\mathcal{V}}_v^m = \mathcal{V}_v^m$ for $v \in I_m^B$ by (27).
3. step: For $u \in S_D^1(\Omega, \mathcal{T}_M)$ it is easy to find a function $\tilde{u} \in S_D^1(\Omega, \tilde{\mathcal{T}}_M) := S^1(\Omega, \tilde{\mathcal{T}}_M) \cap H_D^1(\Omega)$ such that

$$\tilde{u}|_{\partial\Omega} = u|_{\partial\Omega}, \qquad \|\tilde{u}\|_{H^1(\Omega)} \le C\|u\|_{H^1(\Omega)} \tag{29}$$

and $C > 0$ independent of u. Moreover, from the classical theory (see [Zha92], [Osw94]) follows the existence of a decomposition

$$\tilde{u} = \sum_{m=0}^{M} \sum_{v \in \tilde{V}_m} \tilde{u}_v^m, \quad \text{with} \quad \tilde{u}_v^m \in \tilde{\mathcal{V}}_v^m, \quad \sum_{m=0}^{M} \sum_{v \in \tilde{V}_m} a(\tilde{u}_v^m, \tilde{u}_v^m) \le Ca(\tilde{u}, \tilde{u}) \tag{30}$$

and $C > 0$ independent of $\tilde{u}$ and h_M.
4. step: Due to $I_m^B \subset V_m \cap \tilde{V}_m$ and (28) which implies $\mathcal{V}_v^m = \tilde{\mathcal{V}}_v^m$ for all $v \in I_m^B$, $m \in \{0, \dots, M\}$, we have

$$u^B := \sum_{m=0}^{M} \sum_{v \in I_m^B} \tilde{u}_v^m \subset S_D^1(\Omega, \mathcal{T}_M) \quad \text{with} \quad u^B|_{\partial\Omega} = u|_{\partial\Omega}. \tag{31}$$

From (29) in combination with (30) we get

$$\sum_{m=0}^{M} \sum_{v \in I_m^B} a(\tilde{u}_v^m, \tilde{u}_v^m) \le Ca(\tilde{u}, \tilde{u}) \le Ca(u, u). \tag{32}$$

Furthermore, we obtain

$$\begin{aligned} a(u^B, u^B) &= a(\sum_{m=0}^{M} \sum_{v\in I_m^B} \tilde{u}_v^m, \sum_{m=0}^{M} \sum_{v\in I_m^B} \tilde{u}_v^m) \\ &\le C \sum_{m=0}^{M} \sum_{v\in I_m^B} \sum_{m'=0}^{M} \sum_{v'\in I_{m'}^B} e_{vv'}^{mm'} \sqrt{a(\tilde{u}_v^m, \tilde{u}_v^m)} \sqrt{a(\tilde{u}_{v'}^{m'}, \tilde{u}_{v'}^{m'})}, \end{aligned}$$

where $e_{vv'}^{mm'}$ denotes the angle between the spaces $\tilde{\mathcal{V}}_v^m \equiv \mathcal{V}_v^m$ and $\tilde{\mathcal{V}}_{v'}^{m'} \equiv \mathcal{V}_{v'}^{m'}$. Defining the matrix $\mathbf{E} := [e_{vv'}^{mm'}]$ which contains all the angles, we know from the previous section that $\rho(\mathbf{E}) \le C$. Thus, we arrive at

$$a(u^B, u^B) \le \rho(\mathbf{E}) \sum_{m=0}^{M} \sum_{v\in I_m^B} \left(\sqrt{a(\tilde{u}_v^m, \tilde{u}_v^m)}\right)^2 \le C \sum_{m=0}^{M} \sum_{v\in I_m^B} \left(\sqrt{a(\tilde{u}_v^m, \tilde{u}_v^m)}\right)^2 \le Ca(u,u).$$

5. step: Consider $u := u^B + u^H$ with u^B given as above and $u^H := u - u^B \in S_0^1(\Omega, \mathcal{T}_M)$. Then

$$a(u^H, u^H) = a(u,u) + a(u^B, u^B) - 2a(u, u^B) \le Ca(u,u) + 2\sqrt{a(u,u)a(u^B,u^B)} \le Ca(u,u)$$

and the stable splitting for u follows as a combination of the stable splitting for u^H given by Lemma 5.15 and the stable splitting of u^B given by (31). ☐

Now, in the final step we construct a stable splitting for $u \in S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M)$.

Theorem 5.18 (stable splitting for $u \in S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M)$)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and $\{\mathcal{T}_m\}_{m=0,\dots,M}$ be a sequence of nested geometric meshes which satisfy (9)–(11). Let $\mathbf{p} = (p_K)_{K\in\mathcal{T}_M}$ be a polynomial degree distribution on $\mathcal{T}_M$ that satisfies (5). Let the patches ω_v^m and the spaces $\mathcal{V}_v^m$, $\mathcal{S}_v$ be defined as in Section 3.2. Set $I_m^B := \{v \in V_m \mid v \in \partial\Omega\}$. Then there exists $C > 0$ and for all $u \in S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M)$ a decomposition*

$$u = \sum_{v\in V_M} u_{\mathcal{S}_v} + \sum_{m=0}^{M} \sum_{v\in I_m^B} u_v^m$$

with $u_{\mathcal{S}_v} \in \mathcal{S}_v$, $u_v^m \in \mathcal{V}_v^m$ such that

$$\sum_{v\in V_M} a(u_{\mathcal{S}_v}, u_{\mathcal{S}_v}) + \sum_{m=0}^{M} \sum_{v\in I_m^B} a(u_v^m, u_v^m) \le Ca(u,u).$$

Proof. From [SMPZ05] (see Remark 5.19) we know the existence of a stable splitting

$$u = u_D + \sum_{v\in V_M} u_{\mathcal{S}_v} \quad \text{with} \quad a(u_D, u_D) + \sum_{v\in V_M} a(u_{\mathcal{S}_v}, u_{\mathcal{S}_v}) \le Ca(u,u) \quad \forall u \in S_D^{\mathbf{p}}(\Omega, \mathcal{T}_M), \tag{33}$$

where $u_D \in S_D^1(\mathcal{T}_M, \Omega)$ and $u_{\mathcal{S}_v} \in \mathcal{S}_v$. From Lemma 5.17 we get for arbitrary $u_D \in S_D^1(\mathcal{T}_M, \Omega)$

$$u_D = \sum_{v\in V_M} \hat{u}_{\mathcal{S}_v} + \sum_{m=0}^{M} \sum_{v\in I_m^B} u_v^m, \quad \text{with} \quad \sum_{v\in V_M} a(\hat{u}_{\mathcal{S}_v}, \hat{u}_{\mathcal{S}_v}) + \sum_{m=0}^{M} \sum_{v\in I_m^B} a(u_v^m, u_v^m) \le Ca(u_D, u_D), \tag{34}$$

where $\hat{u}_{\mathcal{S}_v} \in \mathcal{S}_v$ and $u_v^m \in \mathcal{V}_v^m$. Thus

$$u = \sum_{v\in V_M} (u_{\mathcal{S}_v} + \hat{u}_{\mathcal{S}_v}) + \sum_{m=0}^{M} \sum_{v\in I_m^B} u_v^m$$

with $(\hat{u}_{\mathcal{S}_v} + u_{\mathcal{S}_v}) \in \mathcal{S}_v$, $u_v^m \in \mathcal{V}_v^m$ and

$$\begin{aligned}
&\sum_{v\in V_M} a(u_{\mathcal{S}_v} + \hat{u}_{\mathcal{S}_v}, u_{\mathcal{S}_v} + \hat{u}_{\mathcal{S}_v}) + \sum_{m=0}^{M}\sum_{v\in I_m^B} a(u_v^m, u_v^m) \\
&\quad \le \quad 2\sum_{v\in V_M} a(u_{\mathcal{S}_v}, u_{\mathcal{S}_v}) + 2\sum_{v\in V_M} a(\hat{u}_{\mathcal{S}_v}, \hat{u}_{\mathcal{S}_v}) + \sum_{m=0}^{M}\sum_{v\in I_m^B} a(u_v^m, u_v^m) \\
&\quad \le \quad Ca(u_D, u_D) + 2\sum_{v\in V_M} a(u_{\mathcal{S}_v}, u_{\mathcal{S}_v}) \le Ca(u,u).
\end{aligned}$$

□

Remark 5.19. *[SMPZ05] considers only uniform polynomial degree distributions. However, a consideration of the proofs reveals that an extension to non-uniform polynomial degree distributions is possible.*

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