WAVENUMBER-EXPLICIT HP-BEM FOR HIGH FREQUENCY SCATTERING

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Abstract. For the Helmholtz equation (with wavenumber k) and analytic curves or surfaces Γ we analyze the Galerkin discretization of classical combined field integral equations in an L^2-setting. We give abstract conditions on the approximation properties of the ansatz space that ensure stability and quasi-optimality of the Galerkin method. Special attention is paid to the hp-version of the boundary element method (hp-BEM). Under the assumption of polynomial growth of the solution operator we show stability and quasi-optimality of the hp-BEM if the following scale resolution condition is satisfied: the polynomial degree p is at least $O(\log k)$ and kh/p is bounded by a number that is sufficiently small, but independent of k. Under this assumption, the constant in the quasi-optimality estimate is independent of k. Numerical examples in 2D illustrate the theoretical results and even suggest that in many cases quasi-optimality is given under the weaker condition that kh/p is sufficiently small.

Key words. high order boundary element method, high frequency scattering, combined field integral equation, Helmholtz equation

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1. Introduction. Acoustic and electromagnetic scattering problems are often treated with boundary integral equation methods. In a time-harmonic acoustic setting, popular boundary integral operators (BIOs) are the combined field integral operators, namely, those usually attributed to Burton & Miller, [10] (see (1.5)) and those commonly associated with the names of Brakhage & Werner [4], Leis [16], and Panič [25] (see (1.4)). The present paper is devoted to the study of discretizations of these two combined field BIOs for the case of smooth (more precisely: analytic) geometries paying special attention to the situation of large wavenumbers k.
In the present context of smooth geometries, the combined field operators A_k and A_k' are $L^2(\Gamma)$-invertible and compact perturbations of the identity. At first glance, therefore, the stability and convergence theory of Galerkin discretizations of the combined field boundary integral equations does not seem to pose difficulties since general functional analytic arguments yield *asymptotic* quasi-optimality. However, these general arguments give no indication of how the wavenumber k enters in the estimates and, in particular, affects the onset of quasi-optimal convergence. The recent k-explicit regularity theory for Helmholtz BIOs developed in [21] allows us be explicit at this point for the hp-version of the BEM in Corollaries 3.18, 3.21: For analytic geometries and under the assumption that the solution operator for the combined field BIO grows at most polynomially in the wavenumber k, a scale resolution condition of the form

$$\frac{kh}{p} \text{ sufficiently small} \qquad \text{and} \qquad p \geq C \log k \tag{1.1}$$

ensures quasi-optimality of the hp-BEM. We stress that, by [8], the assumption of polynomial growth of the norm of the inverse of the combined field BIO is ensured for star-shaped domains so that the present paper provides a complete k-explicit convergence theory for the case of star-shaped domains with analytic boundary. It is

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worth rephrasing the scale resolution condition (1.1) as follows: If the approximation order p is selected as $p = O(\log k)$, then the onset of quasi-optimality is achieved for $h = O(p/k)$, i.e., for a fixed number of degrees of freedom per wavelength. The numerical results of Section 4 illustrate that indeed a scale resolution condition of the form (1.1) ensures quasi-optimality of the hp-BEM. The side condition $p = O(\log k)$ in (1.1) may be viewed as expressing the possibility of "pollution", i.e., the possibility that the onset of quasi-optimality of the method is k-dependent. Nevertheless, our numerical experiments show that the weaker condition "kh/p small" alone is often sufficient for quasi-optimality of the hp-BEM. Put differently: in contrast to the finite element method, the BEM does not appear to be very susceptible to "pollution."

To the authors' knowledge, the only other k-explicit stability analysis for discretizations of combined field BIOs is provided in [11], where the special cases of circular or spherical geometries are studied; in that setting the double layer and single layer operators can be diagonalized simultaneously by Fourier techniques, which allows [11] to show that the combined field BIOs are even L^2-elliptic. Recently, [28] has introduced a novel BIO that is closely related to the operator A'_k studied here. For star-shaped domains, [28] shows k-uniform ellipticity in L^2, thus leading to the propect of k-robust numerical methods based on this BIO.

The results of the present paper have counterparts in the context of differential equations and finite element methods. Decomposition results analogous to those of [21] have recently been obtained in [12, 22, 23] for several Helmholtz boundary value problems. A k-explicit convergence theory for the hp-version of the finite element method has also been developed in [12, 22, 23] using similar techniques; also there, the key scale resolution condition on the mesh size h and the approximation order p takes the form (1.1).

The present paper analyzes the classical hp-BEM for high frequency scattering problems. This approach mandates a scale resolution condition of the form "kh/p sufficiently small" and thus for problems in $\mathbb{R}^d$, the problem size N will scale at least like $N = O(k^{d-1})$. To circumvent or mitigate this scale resolution condition, integral equation methods that are based on non-polynomial ansatz spaces have attracted significant attention in recent years; we refer to the survey [5] for an up-to-date account. While these non-standard methods can perform very impressively, their stability is typically not analyzed; notable exceptions are the analysis of [11] for the special case of a circular/spherical geometry and numerical methods based on the new BIO introduced in [28] for star-shaped domains.

The paper is organized as follows: The remainder of this first section introduces general notation and various boundary integral operators. Section 2 collects the relevant results from [21] and rephrases them in a simplified form suitable for our L^2-based analysis. Section 3 shows how the regularity theory of Section 2 permits a k-explicit stability and convergence analysis of the hp-BEM. We acknowledge here that our technique, which derives the stability of the method from approximation results for suitable adjoint problems, has previously been used in the literature, for example, in [20, 22, 23] and notably [2] in a BEM-context. In Section 4 finally, we present numerical results for the hp-BEM in 2D.

1.1. Notation and General Assumptions.

1.1.1. General Notation. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded Lipschitz domain with a connected boundary. Throughout this work we assume that $\Gamma := \partial\Omega$ is analytic. Furthermore, we assume for the case $d = 2$ the scaling condition $\operatorname{diam} \Omega < 1$. We set $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}$. Throughout the paper, we assume that the open ball $B_R := B_R(0)$ of radius R around the origin contains $\overline{\Omega}$, i.e., $\overline{\Omega} \subset B_R$. We set $\Omega_R := (\Omega \cup \Omega^+) \cap B_R = B_R \setminus \Gamma$. We will denote by γ_0^{int} and γ_0^{ext} the interior and exterior trace operator on Γ. The interior and exterior co-normal derivative operators are denoted by γ_1^{int}, γ_1^{ext}, i.e., for sufficiently smooth functions u, we set $\gamma_1^{int} u := \gamma_0^{int} \nabla u \cdot \vec{n}$ and $\gamma_1^{ext} u := \gamma_0^{ext} \nabla u \cdot \vec{n}$, where, in both cases $\vec{n}$ is the unit normal vector point out of Ω. As is standard, we introduce the jump operators

$$[u] = \gamma_0^{ext} u - \gamma_0^{int} u, \qquad [\partial_n u] = \gamma_1^{ext} u - \gamma_1^{int} u.$$

For linear operators $\widetilde{\mathcal{A}}$ that map into spaces of piecewise defined functions, we define the operators $[\widetilde{\mathcal{A}}]$ and $[\partial_n \widetilde{\mathcal{A}}]$ in an analogous way, e.g., $[\widetilde{\mathcal{A}}]\varphi = [\widetilde{\mathcal{A}}\varphi]$. Sobolev spaces H^s are defined in the standard way, [1, 30]. We stress, however, that if an open set $\omega \subset \mathbb{R}^d$ consists of $m \in \mathbb{N}$ components of connectedness ω_i, $i = 1, \ldots, m$, then the space $H^s(\omega)$ can be identified with the product space $\prod_{i=1}^m H^s(\omega_i)$ equipped with the norm $\left(\sum_{i=1}^m \|u\|^2_{H^s(\omega_i)}\right)^{1/2}$. Sets of analytic functions will play a very important role in our theory. We therefore introduce the following definition.

Definition 1.1. *For an open set T and constants $\widetilde{C}$, $\widetilde{\gamma} > 0$ we set*

$$\mathfrak{A}(\widetilde{C}, \widetilde{\gamma}, T) := \{f \in L^2(T) \,|\, \|\nabla^n f\|_{L^2(T)} \leq \widetilde{C}\widetilde{\gamma}^n \max\{n+1, |k|\}^n \qquad \forall n \in \mathbb{N}_0\}.$$

Here, $|\nabla^n u(x)|^2 = \displaystyle\sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = n} \frac{n!}{\alpha!} |D^\alpha u(x)|^2.$

Tubular neighborhoods T of Γ are open sets with $T \supset \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \Gamma) < \varepsilon\}$ for some $\varepsilon > 0$.

Throughout the paper, we will use the assumption that

$$k \geq k_0 > 0;$$

all constants depend implicitly on k_0. The restriction to positive k is solely done for notational convenience: In view of the relations (1.3) and Lemma 3.1, analogous results hold for $k < 0$. The combined field operators A_k and A_k' depend on a coupling parameter η; we will select $\eta \sim k$. All our estimates depend implicitly on the constants hidden in the $\sim$-notation, i.e., on the constant constant C_η of (1.6).

1.1.2. Layer Potentials. Boundary element methods (BEM) and, more generally, integral equation methods for partial differential equations have a long history going back well over 100 years. In the last decade, several monographs [13,19,24,26,29] have appeared, which are excellent references for many aspects that the present paper merely touches on.

We define the Helmholtz kernel G_k by

$$G_k(x,y) := \begin{cases} \frac{\mathbf{i}}{4} H_0^{(1)}(k|x-y|), & d = 2, \\ \frac{e^{\mathbf{i}k|x-y|}}{4\pi|x-y|}, & d = 3, \end{cases} \qquad \text{for } k > 0,$$
$$G_k := \overline{G_{-k}} \qquad \text{for } k < 0,$$

where $H_0^{(1)}$ is the first kind Hankel function of order zero. The limiting case $k = 0$ corresponds to the Laplace operator and is defined as $G_0(x,y) = -1/(2\pi) \ln|x-y|$ for the case $d = 2$ and $G_0(x,y) = 1/(4\pi|x-y|)$ for the case $d = 3$. We define the potential operators $\widetilde{V}_k$ and $\widetilde{K}_k$ in the standard way by

$$(\widetilde{V}_k \varphi)(x) := \int_\Gamma G_k(x,y)\varphi(y)\,ds_y, \qquad (\widetilde{K}_k \varphi)(x) := \int_\Gamma \partial_{n_y} G_k(x,y)\varphi(y)\,ds_y, \quad x \in \mathbb{R}^d \backslash \Gamma.$$

From these potentials, the single layer, double layer, and adjoint double layer operators are derived as follows:

$$V_k := \gamma_0^{int}\widetilde{V}_k, \qquad K_k := \frac{1}{2}\left(\gamma_0^{int}\widetilde{K}_k + \gamma_0^{ext}\widetilde{K}_k\right), \qquad K'_k := \gamma_1^{int}\widetilde{V}_k - \frac{1}{2}\,\mathrm{Id}\,. \tag{1.2}$$

We mention in passing that for $k \neq 0$, the potentials $\widetilde{V}_k$ and $\widetilde{K}_k$ are solutions of the homogeneous Helmholtz equation on $\mathbb{R}^d \setminus \Gamma$; for $k > 0$ they satisfy the *outgoing* Sommerfeld radiation condition while for $k < 0$, they satisfy the *incoming* radiation condition.

Since G_{-k} and G_k are related by $G_{-k} = \overline{G_k}$, we have for all $k \in \mathbb{R}$ for the $L^2(\Gamma)$ scalar product and all φ, $\psi \in H^{1/2}(\Gamma)$ (cf. also [9, p. 42])

$$(V_k\varphi, \psi)_{L^2(\Gamma)} = (\varphi, V_{-k}\psi)_{L^2(\Gamma)}, \tag{1.3a}$$

$$(K_k\varphi, \psi)_{L^2(\Gamma)} = (\varphi, K'_{-k}\psi)_{L^2(\Gamma)}, \tag{1.3b}$$

i.e., the adjoints of V_k and K_k are V_{-k} and K'_{-k}, respectively.

1.1.3. Combined Field Boundary Integral Operators. The integral operators A_k proposed independently by Brakhage & Werner, [4], Leis [16], and Panič [25] and the operator A'_k introduced by Burton & Miller, [10], are defined as follows for a coupling parameter $\eta \in \mathbb{R} \setminus \{0\}$:

$$A_k = \frac{1}{2} + K_k - \mathbf{i}\eta V_k, \tag{1.4}$$

$$A'_k = \frac{1}{2} + K'_k + \mathbf{i}\eta V_k. \tag{1.5}$$

In order to avoid keeping track of the precise dependence of various constants on η, we assume throughout this paper that, for some fixed $C_\eta > 0$,

$$C_\eta^{-1}|k| \leq |\eta| \leq C_\eta|k|. \tag{1.6}$$

On smooth surfaces, it is well-known, [7,10], that the operators A_k and A'_k are invertible as operators acting on $L^2(\Gamma)$. In fact, A_k is invertible as an operator on $H^s(\Gamma)$, $s \geq 0$, and A'_k is invertible as an operator on $H^s(\Gamma)$, $s \geq -1/2$.

In general, little is known about the k-dependence of the norms of their inverses. A notable exception are star-shaped geometries:

Lemma 1.2 ([8]). *Let the Lipschitz domain Ω be star-shaped. Then there exists a constant $C > 0$ independent of $k \geq k_0$ such that for the operators A_k, A'_k there holds*

$$\|A_k^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)} = \|(A'_{-k})^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)} \leq C.$$

Lemma 1.2 requires Ω to be star-shaped for the k-uniform bounds. In general, bounds for $\|A_k^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)}$ and $\|(A'_k)^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)}$ strongly depend on k and the geometry. As an example, we mention the case of "trapping domains" where, for certain wavenumbers $k \in \mathbb{R}$, the norms $\|A_k^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)}$ and $\|(A'_k)^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)}$ can be very large. Indeed, [3] exhibit geometries and sequences of wavenumbers $(k_m)_{m=0}^\infty$ with $k_m \to \infty$ such that $\|A_k^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)} \geq Cb^{-bk_m}$ for some C, $b > 0$ independent of m.

Our stability analysis of high order Galerkin BEM in Corollaries 3.18, 3.21 will depend on estimates for $\|A_k^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)}$ or $\|(A'_k)^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)}$. We will stipulate polynomial (in k) bounds for these norms. This assumption captures many cases of practical interest but, of course, fails to be valid for certain critical wavenumbers of trapping domains. We point out, however, that computations for these cases are very challenging regardless of the numerical method employed.

1.2. Galerkin Discretization. Associated with the operators A_k and A'_k are the sesquilinear forms a_k and a'_k (which are linear in the first and anti-linear in the second argument) given by

$$
\begin{aligned}
a_k(u,v) &:= (A_k u, v)_0 = \frac{1}{2}(u,v)_0 + (K_k u, v)_0 - \mathbf{i}\eta (V_k u, v)_0,\\
a'_k(u,v) &:= (A'_k u, v)_0 = \frac{1}{2}(u,v)_0 + (K'_k u, v)_0 + \mathbf{i}\eta (V_k u, v)_0.
\end{aligned}
$$

Here and in the following, the short-hand $(\cdot,\cdot)_0$ denotes the $L^2(\Gamma)$-inner product. Given $f \in L^2(\Gamma)$ we study the operator equations $A_k u = f$ and $A'_k u = f$. For closed subspaces $X_N \subset L^2(\Gamma)$, these operator equations are discretized as follows:

$$
\text{find } u_N \in X_N \text{ s.t.} \qquad a_k(u_N, v) = (f,v)_0 \qquad \forall v \in X_N, \tag{1.7}
$$

$$
\text{find } u'_N \in X_N \text{ s.t.} \qquad a'_k(u'_N, v) = (f,v)_0 \qquad \forall v \in X_N. \tag{1.8}
$$

Since A_k and A'_k are compact perturbations of the identity operator, unique solvability of (1.7), (1.8) and quasi-optimality is given if X_N is sufficiently large (see, e.g., [26, Thm. 4.2.9]). The purpose of the present paper is to make the k-dependence of the required approximation properties of X_N explicit.

2. Regularity. Subsections 2.1 and 2.2 collect results from [21]. These results, however, are simplified to be directly applicable to the L^2-convergence theory, which is the focus of the present paper.

2.1. Decomposition of A_k and A'_k.

2.1.1. Decomposition of A_k. The following lemma is derived from [21, Thm. 2.1, Rem. 2.2]:

LEMMA 2.1 (decomposition of A_k). *Fix $q \in (0,1)$ and $\alpha \in \mathbb{R}$. Then the operator A_k can be written as*

$$
A_k = \frac{1}{2} + K_0 + \mathbf{i}\alpha V_0 + R_A + k[\widetilde{\mathcal{A}}_A]
$$

where the linear operators $R_A : L^2(\Gamma) \to H^1(\Gamma)$ and $\widetilde{\mathcal{A}}_A : L^2(\Gamma) \to C^\infty(\Omega_R)$ satisfy for some constant $C > 0$, which is independent of $k \geq k_0$ and q, and constants C_q, γ_q, which are independent of $k \geq k_0$ (but depend on the chosen q),

$$
\begin{aligned}
&\|R_A\|_{H^1(\Gamma)\leftarrow L^2(\Gamma)} \leq Ck, \qquad \|R_A\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)} \leq q,\\
&\widetilde{\mathcal{A}}_A \varphi \in \mathfrak{A}(C_q k \|\varphi\|_{L^2(\Gamma)}, \gamma_q, \Omega_R) \qquad \forall \varphi \in L^2(\Gamma).
\end{aligned}
$$

REMARK 2.2. Our reason for permitting the choice $\alpha \neq 0$ in the decomposition of Lemma 2.1 is that the operator $1/2 + K_0$ has a one-dimensional kernel, [14, Thm. 6.20]. However, the operator $1/2 + K_0 - \mathbf{i}V_0$ is invertible (see, e.g., [18, Appendix D]). We will see below that it is convenient to work with a decomposition of A_k whose leading term is invertible. ∎

The next lemma is derived from [21, Thm. 2.7, Rem. 2.8]:

LEMMA 2.3 (decomposition of A'_k). *Fix $q \in (0,1)$ and $\alpha \in \mathbb{R}$. Then the operator A'_k can be written in the form*

$$
A'_k = \frac{1}{2} + K'_0 + \mathbf{i}\alpha V_0 + R_{A'} + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}]
$$

where the linear operators $R_{A'} : L^2(\Gamma) \to H^1(\Gamma)$ *and* $\widetilde{\mathcal{A}}_{A'} : L^2(\Gamma) \to C^\infty(\Omega_R)$ *satisfy for some constant* C*, which is independent of* $k \geq k_0$ *and* q*, and some constants* C_q*,* γ_q*, which are independent of* $k \geq k_0$ *(but depend on the chosen* q*),*

$$\|R_{A'}\|_{H^1(\Gamma)\leftarrow L^2(\Gamma)} \leq Ck, \qquad \|R_{A'}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)} \leq q,$$
$$\widetilde{\mathcal{A}}_{A',i}\varphi \in \mathfrak{A}(C_q k\|\varphi\|_{L^2(\Gamma)}, \gamma_q, \Omega_R) \qquad \forall \varphi \in L^2(\Gamma), \quad i \in \{1,2\}.$$

Remark 2.4. As in Remark 2.2, the operator $1/2 + K_0'$ has a one-dimensional kernel, [14, Thm. 6.20]. However, the operator $1/2 + K_0' + \mathbf{i}V_0$ is invertible (see, e.g., [18, Appendix D]). ∎

2.2. Decomposition of A_k^{-1} and $(A_k')^{-1}$. The following result is taken from [21, Cor. 7.3] and exploits the fact that A_k is $L^2(\Gamma)$-invertible:

Lemma 2.5. *Let* T *be a tubular neighborhood of* Γ*, and* $\gamma_g > 0$*. Then there exist constants* C*,* $\gamma > 0$ *such that for every* $g \in \mathfrak{A}(C_g, \gamma_g, T \setminus \Gamma)$*,* $C_g > 0$*, the solution* $\varphi \in L^2(\Gamma)$ *of* $A_k\varphi = [g]$ *induces a potential* $u = \widetilde{K}_k\varphi - \mathbf{i}\eta\widetilde{V}_k\varphi$ *with*

$$\varphi = [u], \qquad u \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R), \quad C_\varphi := C_g k(1 + k^{5/2}\|A_k^{-1}\|_{L^2\leftarrow L^2}).$$

For the proof of the next result, we refer to [21, Cor. 7.5] and make use of the fact that A_k' is $L^2(\Gamma)$-invertible:

Lemma 2.6. *Let* Γ *be analytic,* T *be a tubular neighborhood of* Γ*, and* $\gamma_g > 0$*. Then there exist constants* C*,* $\gamma > 0$ *independent of* $k \geq k_0$ *such that for all* $g_1 \in \mathfrak{A}(C_{g_1}, \gamma_g, T \setminus \Gamma)$*,* $g_2 \in \mathfrak{A}(C_{g_2}, \gamma_g, T \setminus \Gamma)$*,* C_{g_1}*,* $C_{g_2} > 0$*, the solution* $\varphi \in L^2(\Gamma)$ *of* $A_k'\varphi = k[g_1] + [\partial_n g_2]$ *induces a potential* $u = \widetilde{V}_k\varphi$ *with the following properties:*

$$-\varphi = [\partial_n u], \qquad u \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R), \quad C_\varphi := (C_{g_1} + C_{g_2})\left(1 + k^{5/2}\|(A_k')^{-1}\|_{L^2\leftarrow L^2}\right).$$

Since A_k is $L^2(\Gamma)$-invertible, we have the following decomposition result for A_k^{-1} of [21, Thm. 2.4]:

Theorem 2.7 (decomposition of A_k^{-1}). *Let* Γ *be analytic. Then there exist constants* C*,* $\gamma > 0$ *independent of* $k \geq k_0$ *and a decomposition of* A_k^{-1} *of the form*

$$A_k^{-1} = A_Z + [\widetilde{\mathcal{A}}_{A^{-1}}]$$

where the linear operators $A_Z : L^2(\Gamma) \to L^2(\Gamma)$ *and* $\widetilde{\mathcal{A}}_{A^{-1}} : L^2(\Gamma) \to C^\infty(\Omega_R)$ *satisfy*

$$\|A_Z\|_{L^2\leftarrow L^2} \leq C,$$
$$\widetilde{\mathcal{A}}_{A^{-1}}f \in \mathfrak{A}(CC_f, \gamma, \Omega_R), \qquad C_f := k^3\left(1 + k^{5/2}\|A_k^{-1}\|_{L^2\leftarrow L^2}\right)\|f\|_{L^2(\Gamma)} \quad \forall f \in L^2(\Gamma).$$

Finally, we have an analogous result for the adjoint $(A_k')^{-1}$ (see [21, Thm. 2.11]):

Theorem 2.8 (decomposition of $(A_k')^{-1}$). *Let* Γ *be analytic. Then there exist constants* C*,* $\gamma > 0$ *independent of* $k \geq k_0$ *and a decomposition of* $(A_k')^{-1}$ *of the form*

$$(A_k')^{-1} = A_Z' + [\partial_n\widetilde{\mathcal{A}}_{A',inv}]$$

with linear operators $A_Z' : L^2(\Gamma) \to L^2(\Gamma)$ *and* $\widetilde{\mathcal{A}}_{A',inv} : L^2(\Gamma) \to C^\infty(\Omega_R)$ *satisfying*

$$\|A_Z'\|_{L^2\leftarrow L^2} \leq C,$$
$$\widetilde{\mathcal{A}}_{A',inv}f \in \mathfrak{A}(CC_f, \gamma, \Omega_R), \quad C_f := \left(1 + k^{5/2}\|(A_k')^{-1}\|_{L^2\leftarrow L^2}\right)\|f\|_{L^2(\Gamma)} \quad \forall f \in L^2(\Gamma).$$

3. L^2-Stability and Convergence. Our stability and convergence theory rests on viewing the operators A_k and A'_k as perturbation of the zero-th order operators A_0 and A'_0 given by:

$$A_0 = +1/2 + K_0 - \mathbf{i}V_0, \tag{3.1}$$

$$A'_0 = +1/2 + K'_0 + \mathbf{i}V_0 \tag{3.2}$$

We view these operators as operators acting on $L^2(\Gamma)$ and note that the operator A'_0 is the $L^2(\Gamma)$-adjoint of the operator A_0. We refer to Section 5, where we discuss the possibility of viewing A_k and A'_k as perturbations of the $\frac{1}{2}$ Id.

3.1. Regularity Properties of Auxiliary Adjoint Problems. In view of (1.3), we have

$$a_k(u,v) = \overline{a'_{-k}(v,u)} \qquad \forall u, v \in L^2(\Gamma),$$

i.e., A_k is the $L^2(\Gamma)$-adjoint of $A_{-k}{}'$:

Lemma 3.1. *For every $k \in \mathbb{R} \setminus \{0\}$ the operators A_k and A'_{-k} are $L^2(\Gamma)$-adjoints of each other.*

We recall from Lemmata 2.1, 2.3 that the operators $A_k - A_0$ and $A'_{-k} - A'_0$ can be decomposed into two parts, namely, a part that is arbitrarily small as an operator $L^2(\Gamma) \to L^2(\Gamma)$ and an operator that maps into a class of analytic functions. In view of this observation and the fact that the operators A_k^{-1} and $(A'_{-k})^{-1}$ can, by Theorems 2.7, 2.8, be decomposed into a zero-th order operator (which is uniformly bounded in k) and an operator that maps into a class of analytic functions, we can can formulate the following result:

Lemma 3.2. *Let Γ be analytic. Let q, $q' \in (0,1)$ be given. Then*

$$A_k^{-1}(A_k - A_0) = T_A + [\widetilde{\mathcal{A}}_{k,A,inv}],$$
$$(A'_{-k})^{-1}(A'_{-k} - A'_0) = T_{A'} + [\partial_n \widetilde{\mathcal{A}}_{-k,A',inv}],$$

where for some C, $\gamma > 0$ independent of $k \geq k_0$ (but depending on q, q') and all $\varphi \in L^2(\Gamma)$

$$\|T_A\|_{L^2 \leftarrow L^2} \leq q, \qquad \|T_{A'}\|_{L^2 \leftarrow L^2} \leq q',$$
$$\widetilde{\mathcal{A}}_{k,A,inv}\varphi \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R), \qquad C_\varphi = k^3(1 + k^{5/2}\|A_k^{-1}\|_{L^2 \leftarrow L^2})\|\varphi\|_{L^2(\Gamma)},$$
$$\widetilde{\mathcal{A}}_{-k,A',inv}\varphi \in \mathfrak{A}(CC'_\varphi, \gamma, \Omega_R), \qquad C'_\varphi = k(1 + k^{5/2}\|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2})\|\varphi\|_{L^2(\Gamma)}.$$

Proof. We first prove the decomposition result for $(A'_{-k})^{-1}(A'_{-k} - A'_0)$. From Lemma 2.3 and Theorem 2.8 we get

$$(A'_{-k})^{-1} = A'_Z + [\partial_n \widetilde{\mathcal{A}}_{A',inv}],$$
$$A'_{-k} - A'_0 = R_{A'} + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}].$$

Hence, we obtain

$$(A'_{-k})^{-1}(A'_{-k} - A'_0) = A'_Z R_{A'} + [\partial_n \widetilde{\mathcal{A}}_{A',inv}]R_{A'} + (A'_{-k})^{-1}\left(k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}]\right).$$

We set $T_{A'} := A'_Z R_{A'}$. From Theorem 2.8 we know that $\|A'_Z\|_{L^2 \leftarrow L^2}$ is bounded uniformly in k. Lemma 2.3 tells us that $\|R_{A'}\|_{L^2 \leftarrow L^2}$ can be made arbitrarily small.

Hence, $T_{A'}$ has the desired property. For $\varphi \in L^2(\Gamma)$ we get from Theorem 2.8 and Lemma 2.3

$$\widetilde{\mathcal{A}}_{A',inv} R_{A'}\varphi \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R), \qquad C_\varphi = \left(1 + k^{5/2} \|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2}\right) \|\varphi\|_{L^2(\Gamma)},$$
$$\widetilde{\mathcal{A}}_{A',1}\varphi, \quad \widetilde{\mathcal{A}}_{A',2}\varphi \in \mathfrak{A}(Ck\|\varphi\|_{L^2(\Gamma)}, \gamma, \Omega_R).$$

Lemma 2.6 then allows us to define the operators $\widetilde{\mathcal{A}}_{-k,A',inv}$ with the stated properties. The decomposition of $A_k^{-1}(A_k - A_0)$ is performed in an analogous way by making use of Theorem 2.7, Lemma 2.1, and Lemma 2.5: We can write

$$A_k^{-1} = A_Z + [\widetilde{\mathcal{A}}_{A^{-1}}], \qquad A_k - A_0 = R_A + k[\widetilde{\mathcal{A}}_A].$$

Therefore, $A_k^{-1}(A_k - A_0) = A_Z R_A + [\widetilde{\mathcal{A}}_{A^{-1}}]R_A + A_k^{-1}k[\widetilde{\mathcal{A}}_A]$. Again, we set $T_A := A_Z R_A$ and see that its norm can be made arbitrarily small. The properties of $\widetilde{\mathcal{A}}_{A^{-1}}$ given in Theorem 2.7 and those of $\widetilde{\mathcal{A}}_A$ given in Lemma 2.1 together with Lemma 2.5 then imply the result. ▯

3.2. Abstract Convergence Analysis. For the approximation space $X_N \subset L^2(\Gamma)$ we denote by $\Pi_N^{L^2} : L^2(\Gamma) \to X_N$ the $L^2(\Gamma)$-projection onto X_N. It will be useful to quantify the approximation of analytic functions from the space X_N:

DEFINITION 3.3. *Let T be a fixed tubular neighborhood of Γ. For every $\gamma > 0$, $N \in \mathbb{N}$, and k define $\eta_1(N,k)$, $\eta_2(N,k,\gamma)$, $\eta(N,k,\gamma)$ by*

$$\eta_1(N,k,\gamma) := \sup\{\|k[u] - \Pi_N^{L^2} k[u]\|_{L^2(\Gamma)} \,|\, u \in \mathfrak{A}(1,\gamma,T \setminus \Gamma)\},$$
$$\eta_2(N,k,\gamma) := \sup\{\|[\partial_n u] - \Pi_N^{L^2} [\partial_n u]\|_{L^2(\Gamma)} \,|\, u \in \mathfrak{A}(1,\gamma,T \setminus \Gamma)\},$$
$$\eta(N,k,\gamma) := \eta_1(N,k,\gamma) + \eta_2(N,k,\gamma).$$

We point out that, by linearity, we have for functions $u \in \mathfrak{A}(C_u, \gamma, T \setminus \Gamma)$ the bound $\|k[u] - \Pi_N^{L^2} k[u]\|_{L^2(\Gamma)} \leq C_u \eta_1(N,k,\gamma)$ and an analogous estimate for $\|[\partial_n u] - \Pi_N^{L^2} [\partial_n u]\|_{L^2(\Gamma)}$.

We will also need stability properties of the spaces X_N for the operators A_0 and A_0'; for future reference we formulate these as assumptions:

ASSUMPTION 3.4. *The space X_N satisfies a uniform discrete inf-sup condition for the operator $1/2 + K_0 - \mathbf{i}V_0$, i.e., there exists $\gamma_0 > 0$ independent of N such that*

$$0 < \gamma_0 \leq \inf_{0 \neq u \in X_N} \sup_{0 \neq v \in X_N} \frac{|((1/2 + K_0 - \mathbf{i}V_0)u, v)_0|}{\|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)}}. \tag{3.3}$$

The inf-sup condition (3.3) is equivalent to

$$0 < \gamma_0 \leq \inf_{0 \neq u \in X_N} \sup_{0 \neq v \in X_N} \frac{|((1/2 + K_0' + \mathbf{i}V_0)u, v)_0|}{\|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)}}, \tag{3.4}$$

with the same constant $\gamma_0 > 0$.

REMARK 3.5. For the present case of smooth surfaces Γ, the operators $K_0 : L^2(\Gamma) \to L^2(\Gamma)$ and $V_0 : L^2(\Gamma) \to L^2(\Gamma)$ are compact. Hence, Assumption 3.4 is satisfied, for example, for standard *hp*-BEM spaces, when the discretization is sufficiently fine. ▪

We close this subsection with two approximation results.

LEMMA 3.6. *Let Γ be analytic. Fix $q \in (0,1)$. Then there exists $\gamma > 0$, which is independent of $k \geq k_0$ (but depends on q), such that*

$$\|(\mathrm{Id} - \Pi_N^{L^2})(A'_{-k} - A'_0)\|_{L^2 \leftarrow L^2} \leq q + Ck\eta(N,k,\gamma),$$
$$\|(\mathrm{Id} - \Pi_N^{L^2})(A'_{-k})^{-1}(A'_{-k} - A'_0)\|_{L^2 \leftarrow L^2} \leq q + Ck\Big\{1 + k^{5/2}\|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2}\Big\}\eta_2(N,k,\gamma).$$

Proof. Lemma 2.3 implies $A'_{-k} - A'_0 = R_{A'} + k[\widetilde{A}_{A',1}] + [\partial_n \widetilde{A}_{A',2}]$ where $\|R_{A'}\|_{L^2 \leftarrow L^2}$ can be made arbitrarily small. Thus, $\|(\mathrm{Id} - \Pi_N^{L^2})R_{A'}\|_{L^2 \leftarrow L^2} \leq \|R_{A'}\|_{L^2 \leftarrow L^2}$ can be made arbitrarily small. The $L^2(\Gamma)$-approximation of the remaining terms $k[\widetilde{\mathcal{A}}_{A',1}]$ and $[\partial_n \widetilde{\mathcal{A}}_{A',2}]$ directly lead to the stated estimate.
From Lemma 3.2 we get the decomposition $(A'_{-k})^{-1}(A'_{-k} - A'_0) = T_{A'} + [\partial_n \widetilde{\mathcal{A}}_{-k,A',inv}]$, where $\|T_{A'}\|_{L^2 \leftarrow L^2}$ can be made arbitrarily small. The $L^2(\Gamma)$-approximation of the second term $[\partial_n \widetilde{\mathcal{A}}_{-k,A',inv}]$ then leads to the stated estimate. ∎

LEMMA 3.7. *Let Γ be analytic. Fix $q \in (0,1)$. Then there exists $\gamma > 0$, which is independent of $k \geq k_0$ (but depends on q), such that*

$$\|(\mathrm{Id} - \Pi_N^{L^2})(A_k - A_0)\|_{L^2 \leftarrow L^2} \leq q + Ck\eta_1(N,k,\gamma),$$
$$\|(\mathrm{Id} - \Pi_N^{L^2})A_k^{-1}(A_k - A_0)\|_{L^2 \leftarrow L^2} \leq q + Ck^2\left\{1 + k^{5/2}\|A_k^{-1}\|_{L^2 \leftarrow L^2}\right\}\eta_1(N,k,\gamma).$$

Proof. The proof follows the lines of Lemma 3.6. The estimate for $\|(\mathrm{Id} - \Pi_N^{L^2})(A_k - A_0)\|_{L^2 \leftarrow L^2}$ follows from Lemma 2.1. Lemma 3.2 finally leads to the second bound. ∎

3.2.1. The Case of the Operator A_k.

At the heart of our analysis is the following quasi-optimality result:

THEOREM 3.8. *Let Γ be analytic, η_1, η_2 be given by Definition 3.3, and let Assumption 3.4 hold. Then there exist constants ε, $\gamma > 0$ independent of $k \geq k_0$ such that under the assumption*

$$k\eta_1(N,k,\gamma) \leq \varepsilon, \qquad k\left(1 + k^{5/2}\|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2}\right)\eta_2(N,k,\gamma) \leq \varepsilon \tag{3.5}$$

the following is true: If $u \in L^2(\Gamma)$ and $u_N \in X_N$ are two functions that satisfy the Galerkin orthogonality

$$a_k(u - u_N, v) = 0 \qquad \forall v \in X_N, \tag{3.6}$$

then with γ_0 as stated in Assumption 3.4

$$\|u - u_N\|_{L^2(\Gamma)} \leq 2\left(1 + \frac{\|A_0\|_{L^2 \leftarrow L^2}}{\gamma_0}\right) \inf_{w_N \in X_N} \|u - w_N\|_{L^2(\Gamma)}. \tag{3.7}$$

Proof. We introduce the abbreviation $e := u - u_N$. Let $w_N \in X_N$ be arbitrary. Then by the triangle inequality

$$\|e\|_{L^2(\Gamma)} \leq \|u - w_N\|_{L^2(\Gamma)} + \|u_N - w_N\|_{L^2(\Gamma)}. \tag{3.8}$$

Hence, we have to estimate $\|u_N - w_N\|_{L^2(\Gamma)}$. By the discrete inf-sup condition we can find a $v_N \in X_N$ with $\|v_N\|_{L^2(\Gamma)} = 1$ and $\gamma_0\|u_N - w_N\|_{L^2(\Gamma)} \leq (A_0(u_N - w_N), v_N)_0$.

With the Galerkin orthogonality $(A_k(u-u_N), v_N)_0 = 0$, we then obtain

$$\begin{aligned}\gamma_0\|u_N - w_N\|_{L^2(\Gamma)} &\leq ((A_0 - A_k)(u_N - w_N), v_N)_0 + (A_k(u_N - w_N), v_N)_0 \\ &= ((A_0 - A_k)(u_N - w_N), v_N)_0 + (A_k(u - w_N), v_N)_0 \\ &= ((A_k - A_0)e, v_N)_0 + (A_0(u - w_N), v_N)_0 \\ &\leq \|A_0\|_{L^2 \leftarrow L^2}\|u - w_N\|_{L^2(\Gamma)} + ((A_k - A_0)e, v_N)_0. \end{aligned} \tag{3.9}$$

In order to treat the term $((A_k - A_0)e, v_N)_0$ we define $\psi \in L^2(\Gamma)$ by

$$((A_k - A_0)z, v_N)_0 = (z, A'_{-k}\psi)_0 \qquad \forall z \in L^2(\Gamma). \tag{3.10}$$

Lemma 3.1 tells us

$$\psi = (A'_{-k})^{-1}(A'_{-k} - A'_0)v_N. \tag{3.11}$$

By selecting $z = e$ in (3.10), using Galerkin orthogonality satisfied by the error e and orthogonality properties of $\Pi_N^{L^2}$ we obtain

$$\begin{aligned}&((A_k - A_0)e, v_N)_0 = (e, A'_{-k}\psi)_0 = (A_k e, \psi)_0 = (A_k e, \psi - \Pi_N^{L^2}\psi)_0 \\ &= (A_0 e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k - A_0)e, \psi - \Pi_N^{L^2}\psi)_0 \\ &= (A_0 e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k - A_0)e - \Pi_N^{L^2}(A_k - A_0)e, \psi - \Pi_N^{L^2}\psi)_0.\end{aligned}$$

Hence, from (3.11) and $\|v_N\|_{L^2(\Gamma)} = 1$

$$\begin{aligned}|((A_k - A_0)e, v_N)_0| \leq &\left\{\|A_0\|_{L^2\leftarrow L^2} + \|(\mathrm{Id} - \Pi_N^{L^2})(A_k - A_0)\|_{L^2\leftarrow L^2}\right\} \\ &\times \|(\mathrm{Id} - \Pi_N^{L^2})(A'_{-k})^{-1}(A'_{-k} - A'_0)\|_{L^2\leftarrow L^2}\,\|e\|_{L^2(\Gamma)}.\end{aligned}$$

From Lemmata 3.6, 3.7 we get for arbitrary $q \in (0,1)$

$$\begin{aligned}|((A_k - A_0)e, v_N)_0| \leq &\{\|A_0\|_{L^2\leftarrow L^2} + q + Ck\eta_1(N,k,\gamma)\} \\ &\times \left\{q + Ck\left(1 + k^{5/2}\|(A'_{-k})^{-1}\|_{L^2\leftarrow L^2}\right)\eta_2(N,k,\gamma)\right\}\,\|e\|_{L^2(\Gamma)}.\end{aligned} \tag{3.12}$$

Select now $q \in (0,1)$ such that $(\|A_0\|_{L^2\leftarrow L^2} + q)q < \gamma_0/2$. Then the constants C and γ in (3.12) are fixed and independent of $k \geq k_0$. We may then select $\varepsilon > 0$ independent of k such that the assumption (3.5) guarantees that the product of the two curly braces in (3.12) is bounded by $\gamma_0/2$. Combining (3.8), (3.9), and (3.12) then yields

$$\|e\|_{L^2(\Gamma)} \leq \left(1 + \frac{\|A_0\|_{L^2\leftarrow L^2}}{\gamma_0}\right)\|u - w_N\|_{L^2(\Gamma)} + \frac{1}{2}\|e\|_{L^2(\Gamma)},$$

which leads to the desired estimate. ☐

Theorem 3.8 provides quasi-optimality under the assumption that $u_N \in X_N$ exists. However, the discrete inf-sup condition follows easily from Theorem 3.8. In particular, we obtain that the discrete inf-sup constant is, up to a constant which is independent of k, and N, the inf-sup constant for the continuous problem. This is a consequence of the following, general result:

Theorem 3.9. *Let X be a Hilbert space with norm $\|\cdot\|_X$. Let $X_N \subset X$ be a finite-dimensional subspace. Let $a : X \times X \to \mathbb{C}$ be a continuous sesquilinear form that satisfies the following two conditions*

$$0 < \gamma_a \leq \inf_{0\neq u\in X}\sup_{0\neq v\in X}\frac{|a(u,v)|}{\|u\|_X\|v\|_X}, \qquad \sup_{u\in X}|a(u,v)| > 0 \quad \forall 0 \neq v \in X.$$

Let $C_{qopt} > 0$ be such that any pair $(u, u_N) \in X \times X_N$ satisfying

$$a(u - u_N, v) = 0 \qquad \forall v \in X_N$$

enjoys the best approximation property

$$\|u - u_N\|_X \leq C_{qopt} \inf_{v \in X_N} \|u - v\|_X.$$

Then the discrete inf-sup condition holds, i.e.,

$$\inf_{0 \neq u \in X_N} \sup_{0 \neq v \in X_N} \frac{|a(u, v)|}{\|u\|_X \|v\|_X} =: \gamma_N \geq \gamma_a \frac{1}{1 + C_{qopt}} > 0.$$

Proof. We first show that the restriction of the sesquilinear form a to $X_N \times X_N$ induces an injective operator $X_N \to X_N'$. To see this, let $u_N \in X_N$ satisfy $a(u_N, v) = 0$ for all $v \in X_N$. Our assumption is then applicable to the pair $(u, u_N) = (0, u_N)$, and we get $\|u_N\|_X = \|u - u_N\|_X \leq C_{qopt} \inf_{v \in X_N} \|u - v\|_X \leq C_{qopt} \|u\|_X = 0$. By dimension arguments, therefore, the Galerkin projection operator $P_N : X \to X_N$ given by

$$a(u - P_N u, v) = 0 \qquad \forall v \in X_N$$

is well-defined. Additionally, the quasi-optimality assumption produces the stability result $\|P_N u\|_X \leq \|u\|_X + \|u - P_N u\|_X \leq (1 + C_{qopt}) \|u\|_X$. Since

$$\inf_{0 \neq u \in X_N} \sup_{0 \neq v \in X_N} \frac{|a(u, v)|}{\|u\|_X \|v\|_X} = \inf_{0 \neq v \in X_N} \sup_{0 \neq u \in X_N} \frac{|a(u, v)|}{\|u\|_X \|v\|_X}, \tag{3.13}$$

we will just compute the second inf-sup constant in (3.13). To that end, we remark that the equality in (3.13) also holds with X_N replaced with the full space X (cf., e.g., [27, Prop. 1.22]), and we get for arbitrary $0 \neq v \in X_N$

$$\sup_{0 \notin u \in X_N} \frac{|a(u, v)|}{\|u\|_X \|v\|_X} = \sup_{u \in X : P_N u \neq 0} \frac{|a(P_N u, v)|}{\|P_N u\|_X \|v\|_X} \geq \frac{1}{1 + C_{qopt}} \sup_{u \in X : P_N u \neq 0} \frac{|a(P_N u, v)|}{\|u\|_X \|v\|_X}$$
$$= \frac{1}{1 + C_{qopt}} \sup_{0 \neq u \in X} \frac{|a(P_N u, v)|}{\|u\|_X \|v\|_X} = \frac{1}{1 + C_{qopt}} \sup_{0 \neq u \in X} \frac{|a(u, v)|}{\|u\|_X \|v\|_X} \geq \frac{1}{1 + C_{qopt}} \gamma_a.$$

Taking the infimum over all $v \in X_N$ concludes the argument. ☐

Combining Theorems 3.9 and 3.8 yields:

Corollary 3.10. *Assume the hypotheses of Theorem 3.8. If X_N satisfies (3.5), then (1.7) is uniquely solvable and the quasi-optimality result (3.7) is true.*

3.2.2. The Case of the Operator A_k'. The results of Section 3.2.1 for the discretization of the operator A_k have analogs for the discretization of the operator A_k'. Since the procedure is very similar to that of Section 3.2.1, we merely state the results and leave their proofs to the reader.

Theorem 3.11. *Let Γ be analytic, η_1, η be given by Definition 3.3, and let Assumption 3.4 be valid. Then there exist constants ε, $\gamma > 0$ independent of $k \geq k_0$ such that under the assumption*

$$k\eta(N, k, \gamma) \leq \varepsilon, \qquad k^2 \left(1 + k^{5/2} \|A_{-k}^{-1}\|_{L^2 \leftarrow L^2}\right) \eta_1(N, k, \gamma) \leq \varepsilon \tag{3.14}$$

the following is true: If $u \in L^2(\Gamma)$ and $u_N \in X_N$ are two functions that satisfy the Galerkin orthogonality

$$a'_k(u - u_N, v) = 0 \qquad \forall v \in X_N \tag{3.15}$$

then with γ_0 as stated in Assumption 3.4

$$\|u - u_N\|_{L^2(\Gamma)} \leq 2\left(1 + \frac{\|A'_0\|_{L^2 \leftarrow L^2}}{\gamma_0}\right) \inf_{w_N \in X_N} \|u - w_N\|_{L^2(\Gamma)}. \tag{3.16}$$

Proof. See Appendix A. □

COROLLARY 3.12. *Assume the hypotheses of Theorem 3.11. If X_N satisfies (3.14), then (1.8) is uniquely solvable and the quasi-optimality result (3.16) is true.*

3.3. Classical *hp*-BEM. The analysis of the preceding section shows that the stability and convergence analysis of discretizations of the operators A_k and A'_k can be reduced to questions of approximability. As an example of the abstract theory, we consider the classical *hp*-BEM. We restrict our attention here to a situation in which the h-dependence can be obtained by scaling arguments.

We let $\widehat{K}^{d-1} = \{x \in \mathbb{R}^{d-1} \,|\, 0 < x_i < 1,\ \sum_{i=1}^{d-1} x_i < 1\}$ and $\widehat{K}^d = \{x \in \mathbb{R}^d \,|\, 0 < x_i < 1,\ \sum_{i=1}^{d} x_i < 1\}$ be the references simplices in $\mathbb{R}^{d-1}$ and $\mathbb{R}^d$. By $\mathcal{T}$ we denote a triangulation of Γ into elements $K \in \mathcal{T}$, where the elements K are assumed to be the images of $\widehat{K}^{d-1}$ under smooth *element maps* $F_K : \widehat{K}^{d-1} \to K$. The element maps F_K are furthermore required to be C^1-diffeomorphisms between $\overline{\widehat{K}^{d-1}}$ and $\overline{K}$. For $p \in \mathbb{N}_0$, we then define the *hp*-BEM space $S^p(\mathcal{T})$ by

$$S^p(\mathcal{T}) = \{u \in L^2(\Gamma) \,|\, u|_K \circ F_K \in \mathcal{P}_p \quad \forall K \in \mathcal{T}\}, \tag{3.17}$$

where $\mathcal{P}_p$ is the vector space of all polynomials of degree p.

To motivate the class of triangulations of Assumption 3.15 below, we consider the following two examples:

EXAMPLE 3.13. *Let $d = 2$ and $\Gamma = \partial\Omega \subset \mathbb{R}^d$ be an analytic curve. Let the analytic function $R : [0,1) \to \Gamma$ be a parametrization of Γ. Denote by $\widehat{\mathcal{T}}$ a uniform mesh on $[0,1)$ with mesh size h. Define the mesh $\mathcal{T}$ by "transporting" the elements of $\widehat{\mathcal{T}}$ to Γ via R. Then the element maps F_K have the form $F_K = R \circ A_K$, where $A_K : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ is affine and invertible with $\|DA_K\|_{L^\infty(\widehat{K}^{d-1})} \leq Ch$ and $\|(DA_K)^{-1}\|_{L^\infty(\widehat{K}^{d-1}} \leq Ch^{-1}$ (the operator D denotes differentiation). Since R is a smooth bijection, the vector $DR(x) \neq 0$ for all $x \in [0,1)$; in fact, the Gramian matrix $(DR)^\top(DR)$ satisfies $\inf_{x \in [0,1)} \det(DR)^\top(DR)(x) \geq c_0 > 0$. We close this example by noting that these element maps have the form stipulated in Definition 3.15 below.*

EXAMPLE 3.14. *Let $d = 3$ and $\Gamma = \partial\Omega$ be analytic. Let $\mathcal{T}^d$ be a patchwise constructed mesh on the domain Ω as given in [22, Example 5.1]. There, the element maps $F_K : \widehat{K}^d \to K$ have the form $F_K = R_K \circ A_K$ for an affine map A_K with $\|DA_K\|_{L^\infty(\widehat{K}^d)} \leq Ch$ and $\|(DA_K)^{-1}\|_{L^\infty(\widehat{K}^d)} \leq Ch^{-1}$ and the functions R_K satisfy*

$$\|(DR_K)^{-1}\|_{L^\infty(\widetilde{K}^d)} \leq C_{\text{metric}}, \qquad \|\nabla^n R_K\|_{L^\infty(\widetilde{K}^d)} \leq C_{\text{metric}} \gamma^n n! \qquad \forall n \in \mathbb{N}_0;$$

here, $\widetilde{K}^d = A_K(\widehat{K}^d)$ is the image of the reference simplex $\widehat{K}^d$ under the affine map A_K. The mesh $\mathcal{T}^d$ on the domain Ω induces in a canonical way a mesh on $\Gamma = \partial\Omega$. This trace mesh has the form specified in the Definition 3.15 below.

The two examples motivate the following assumptions on the triangulation of Γ; we point out that, since the maps R_K map from (a subset of) $\mathbb{R}^{d-1}$ into $\mathbb{R}^d$, we capture the fact that the columns of $DR_K \in \mathbb{R}^{d\times(d-1)}$ are linearly independent with the aid of the Gramian matrix $(DR_K)^\top(DR_K)$:

Definition 3.15 (quasi-uniform triangulation). *A triangulation $\mathcal{T}_h$ of the analytic manifold Γ is said to be a quasi-uniform mesh with mesh size h if each element map $F_K : \widehat{K}^{d-1} \to K$ can be written as $F_K = R_K \circ A_K$ with the following properties: $A_K : \widehat{K}^{d-1} \to \mathbb{R}^{d-1}$ is an affine bijection between $\widehat{K}^{d-1}$ and $\widetilde{K} := A_K(\widehat{K}^{d-1})$; the map R_K is a bijection between $\widetilde{K}$ and K; the maps A_K and R_K satisfy for constants C_{affine}, C_{metric}, $\gamma_{\mathcal{T}} > 0$ independent of h and $K \in \mathcal{T}$ the following bounds:*

$$
\begin{aligned}
&\|DA_K\|_{L^\infty(\widehat{K}^{d-1})} \le C_{\mathrm{affine}} h, \qquad \|(DA_K)^{-1}\|_{L^\infty(\widehat{K}^{d-1})} \le C_{\mathrm{affine}} h^{-1},\\
&\inf_{\widetilde{x}\in\widetilde{K}} \det\left((DR_K(\widetilde{x}))^\top (DR_K)(\widetilde{x})\right) \ge C_{\mathrm{metric}}^{-2}, \quad \|\nabla^n R_K\|_{L^\infty(\widetilde{K})} \le C_{\mathrm{metric}} \gamma_{\mathcal{T}}^n n! \quad \forall n \in \mathbb{N}_0.
\end{aligned}
$$

Lemma 3.16. *Let Γ be analytic. Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Γ with mesh size h in the sense of Definition 3.15. Fix a tubular neighborhood T of Γ. Let $X_N = S^p(\mathcal{T}_h)$. Let $\widetilde{C} > 0$ be fixed and assume that h, p, and k satisfy*

$$\frac{kh}{p} \le \widetilde{C}.$$

Then, for every $\gamma > 0$ there exist C, $\sigma > 0$ (independent of h, p, and $k \ge k_0$) with

$$\eta(N, k, \gamma) = \eta_1(N, k, \gamma) + \eta_2(N, k, \gamma) \le C k^{3/2} \left\{ \left(\frac{h}{\sigma + h}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right\}.$$

Proof. We only sketch the arguments for the bound on η_1, which quantifies how well the jump $k[u]$ of a piecewise analytic function can be approximated from $X_N = S^p(\mathcal{T}_h)$. Using [21, Lemma B.5], we may assume that $u|_{\Omega^+} = 0$. Denote by $\vec{n}(x)$ the outer normal vector of Ω at the point $x \in \Gamma$.

1. step: Let T_h be a tubular neighborhood of Γ of width $O(h)$ and $u \in \mathfrak{A}(C_u, \gamma_u, T \setminus \Gamma)$ for a fixed tubular neighborhood of Γ. We assume that h is small (as compared to the width of T). With the aid of the estimate $\|z\|^2_{L^2(T_h)} \le Ch\|z\|_{L^2(T)}\|z\|_{H^1(T)}$ (valid for all $z \in H^1(T)$—see, e.g., [17, Lemma 2.1]) we conclude

$$\|\nabla^n u\|_{L^2(T_h)} \le C\sqrt{kh}\, C_u (\gamma_u')^n \max\{k, n+1\}^n \qquad \forall n \in \mathbb{N}_0, \tag{3.18}$$

where the constants C, γ_u' are independent of $k \ge k_0$ and h.

2. step: The reference simplex $\widehat{K}^d$ has the form $\widehat{K}^d = \{(\widehat{x}, z) \,|\, 0 < z < 1, \widehat{x} \in (1 - z)\widehat{K}^{d-1}\}$. The element maps $F_K : \widehat{K}^{d-1} \to \Gamma$ have the form $F_K = R_K \circ A_K$. Define

$$
\begin{aligned}
A_K^d &: \widehat{K}^d \ni (\widehat{x}, z) \mapsto (A_K(\widehat{x}), hz)\,,\\
R_K^d &: \widetilde{K}^d \ni (\widetilde{x}, \widetilde{z}) \mapsto R_K(\widetilde{x}) - \widetilde{z}\vec{n}(R_K(\widetilde{x}));
\end{aligned}
$$

here $\widetilde{K}^d$ is the image of $\widehat{K}^d$ under A_K^d, and $\widetilde{x} \in \widetilde{K}$, $\widetilde{z} \in \mathbb{R}$. The assumption on A_K implies readily that $A_K^d : \widehat{K}^d \to \widetilde{K}^d$ satisfies

$$\|DA_K^d\|_{L^\infty(\widehat{K}^d)} \le Ch, \qquad \|(DA_K^d)^{-1}\|_{L^\infty(\widehat{K}^d)} \le Ch^{-1}$$

for a constant C that is independent of h. The analyticity of Γ implies furthermore that the function R_K^d satisfies for some constants C_g, γ_g that depend solely on Γ and the constants C_{metric}, $\gamma_{\mathcal{T}}$

$$\|\nabla^n R_K^d\|_{L^\infty(\widetilde{K}^d)} \leq C_g \gamma_g n! \qquad \forall n \in \mathbb{N}_0.$$

Finally, we note $DR_K^d(\widetilde{x}, \widetilde{z}) = (DR_K(\widetilde{x}) - \widetilde{z} D_{\widetilde{x}} \vec{n}(R_K(\widetilde{x})), -\vec{n}(R_K(\widetilde{x})))$ and, since the columns of $DR_K(\widetilde{x})$ are tangential to Γ at $R_K(\widetilde{x})$, we get for $(\widetilde{x}, \widetilde{z}) \in \widetilde{K}^d$

$$\det\big((DR_K^d(\widetilde{x}, \widetilde{z}))^\top (DR_K^d(\widetilde{x}, \widetilde{z})\big) = \det\big((DR_K(\widetilde{x}))^\top (DR_K(\widetilde{x}))\big) + O(\widetilde{z}) \geq C_{\mathrm{metric}}^{-2} + O(\widetilde{z}).$$

Since $\widetilde{z} = O(h)$, we conclude that for sufficiently small h, we can bound this last term from below by $C_{\mathrm{metric}}^{-2}/2$. We conclude that also $\|(DR_K^d)^{-1}\|_{L^\infty(\widetilde{K})} \leq C_g$ after suitably adjusting C_g for sufficiently small h.

3. step: The images $K^d = (R_K^d \circ A_K^d)(\widehat{K}^d)$ lie in a tubular neighborhood T_h of Γ that has width $O(h)$. Furthermore, geometric considerations imply a finite overlap property, namely, the existence of a constant $M > 0$ such that any $x \in \Omega$ is in no more than M of these sets:

$$\sup_{x \in \Omega} |\{K \in \mathcal{T}_h \,|\, x \in K^d\}| \leq M. \tag{3.19}$$

4. step: Define for each $K \in \mathcal{T}_h$ the constant

$$C_K^2 := \sum_{n \in \mathbb{N}_0} \frac{1}{(2\gamma_u' \max\{k, n\}^n)^2} \|\nabla^n u\|^2_{L^2(K^d)} \tag{3.20}$$

and note that (3.18) and (3.19) imply

$$\sum_{K \in \mathcal{T}_h} C_K^2 \leq M \sum_{n \in \mathbb{N}_0} \left(\frac{1}{2\gamma_u' \max\{k, n+1\}^n}\right)^2 \|\nabla^n u\|^2_{L^2(T_h)} \leq \frac{4}{3} C M C_u^2 k h. \tag{3.21}$$

5. step: We have $u|_{K^d} \in \mathfrak{A}(C_K, 2\gamma_u', K^d)$, and [22, Lemma C.1] implies that the function $u \circ R_K^d$ satisfies $u \circ R_K^d \in \mathfrak{A}(CC_K, \tilde{\gamma}_u, \widetilde{K}^d)$, where the constants C and $\tilde{\gamma}_u$ depend solely on γ_u', γ_g, and C_g. Since the map A_K^d is affine and $F_K^d = R_K^d \circ A_K^d$, we get for constants C, $\overline{\gamma}$ independent of k and h

$$\|\nabla^n (u \circ F_K^d)\|_{L^2(\widehat{K}^d)} \leq C C_K h^{-d/2} (\overline{\gamma} h)^n \max\{k, n\}^n \qquad \forall n \in \mathbb{N}_0.$$

Next, [22, Lemma C.2] gives for constants C, $\sigma > 0$ independent of h, p, and $k \geq k_0$

$$\inf_{\pi \in \mathcal{P}_p} \|u \circ F_K^d - \pi\|_{L^\infty(\widehat{K}^d)} \leq C C_K h^{-d/2} \left(\left(\frac{h}{\sigma + h}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right),$$

where $\mathcal{P}_p$ is the space of d-variate polynomials of degree p. Hence, taking the trace on the $d-1$-dimensional face $\widehat{K}^{d-1}$ produces

$$\inf_{\pi \in \mathcal{P}_p} \|u \circ F_K - \pi\|_{L^\infty(\widehat{K}^{d-1})} \leq C C_K h^{-d/2} \left(\left(\frac{h}{\sigma + h}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right),$$

where $\mathcal{P}_p$ denotes the space of $d-1$-variate polynomials of degree p. Scaling back to the element K and summing over all elements $K \in \mathcal{T}_h$ yields

$$\inf_{\pi \in S^p(\mathcal{T}_h)} \|u - \pi\|^2_{L^2(\Gamma)} \leq \sum_{K \in \mathcal{T}_h} C C_K^2 h^{-d} h^{d-1} \left(\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right)^2$$
$$\leq C C_u^2 k \left(\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right)^2.$$

Recalling that that we are actually interested in the approximation of the function ku instead of u, we see that we have obtained the desired bound for η_1. □

THEOREM 3.17 (quasi-optimality for A_k). *Let Γ be analytic. Let $\mathcal{T}_h$ a quasi-uniform mesh on Γ of mesh size h in the sense of Definition 3.15. Let $X_N = S^p(\mathcal{T}_h)$. Then there exist constants C, ε, $\sigma > 0$ independent of h, k, and p such that the following is true: If the* scale resolution *condition*

$$\left\{ k^{5/2} + k^5 \|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2} \right\} \left\{ \left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right\} \leq \varepsilon \tag{3.22}$$

is satisfied, then (1.7) has a unique solution u_N, which satisfies

$$\|u - u_N\|_{L^2(\Gamma)} \leq C \inf_{v \in S^p(\mathcal{T}_h)} \|u - v\|_{L^2(\Gamma)}. \tag{3.23}$$

Proof. Combine Theorem 3.8 with Lemma 3.16. □

We now consider the case of polynomial growth of $k \mapsto \|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2}$:

COROLLARY 3.18. *Assume the hypotheses of Theorem 3.17. Assume additionally the existence of C, $\beta \geq 0$ independent of k such that*

$$\|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2} \leq C k^\beta. \tag{3.24}$$

Then there exist constants C_1, C_2 independent of h, k, and p such that for

$$\frac{hk}{p} \leq C_1 \qquad \textit{and} \qquad p \geq C_2 \log k \tag{3.25}$$

the quasi-optimality assertion (3.23) of Theorem 3.17 is true.

REMARK 3.19. Corollary 3.18 can be phrased in a different way: the onset of quasi-optimality of the BEM is guaranteed for the choice

$$p = \lceil C_2 \log k \rceil \qquad \text{and } h = C_1 \frac{p}{k}.$$

The corresponding problem size $N := \dim S^p(\mathcal{T}_h)$ is given by $N = \dim S^p(\mathcal{T}_h) \sim h^{-(d-1)} p^{d-1} \sim k^{d-1}$; i.e., the onset of quasi-optimality of the BEM is achieved with a fixed number of degrees of freedom per wavelength. ■

Results corresponding to the above ones for the operator A_k hold for the operator A'_k. We merely record the statements.

THEOREM 3.20 (quasi-optimality for A'_k). *Let Γ be analytic. Let $\mathcal{T}_h$ a quasi-uniform mesh of mesh size h in the sense of Definition 3.15. Let $X_N = S^p(\mathcal{T}_h)$. Then there* *exist constants C, ε, $\sigma > 0$ independent of h, k, and p such that the following is true: If the* scale resolution *condition*

$$\left\{k^{5/2} + k^6 \|A_{-k}^{-1}\|_{L^2 \leftarrow L^2}\right\} \left\{ \left(\frac{h}{\sigma + h}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right\} \leq \varepsilon \tag{3.26}$$

is satisfied, then (1.8) has a unique solution u_N, which satisfies

$$\|u - u_N\|_{L^2(\Gamma)} \leq C \inf_{v \in S^p(\mathcal{T}_h)} \|u - v\|_{L^2(\Gamma)}. \tag{3.27}$$

COROLLARY 3.21. *Assume the hypotheses of Theorem 3.20. Assume additionally the existence of C, $\beta \geq 0$ independent of k such that $\|A_k^{-1}\|_{L^2 \leftarrow L^2} \leq Ck^\beta$. Then there exist constants C_1, C_2 independent of h, k, and p such that for $\frac{hk}{p} \leq C_1$ and $p \geq C_2 \log k$ the quasi-optimality assertion (3.27) of Theorem 3.20 is true.*

REMARK 3.22. As in Remark 3.22, Corollary 3.21 can be phrased in a different way: the onset of quasi-optimality of the BEM is guaranteed for the choice $p = \lceil C_2 \log k \rceil$ together with $h = C_1 \frac{p}{k}$. The corresponding problem size $N := \dim S^p(\mathcal{T}_h)$ is given by $N = \dim S^p(\mathcal{T}_h) \sim h^{-(d-1)} p^{d-1} \sim k^{d-1}$; i.e., the onset of quasi-optimality of the BEM is achieved with a fixed number of degrees of freedom per wavelength. ∎

4. Numerical Results. All our numerical examples are based on the operator $A'_k = 1/2 + K'_k + \mathbf{i}\eta V_k$, where the coupling parameter is usually $\eta = k$; in a few examples, the choice $\eta = 1$ is studied. The ansatz spaces X_N are taken to be standard *hp*-BEM spaces of piecewise polynomials of degree p. Specifically, let $\mathcal{T} = \{K_i \,|\, i = 1, \dots, N\}$ be a partition of Γ into N elements and let $F_K : [-1, 1] \to \Gamma$ be the element maps. Then $S^p(\mathcal{T}) = \{u \in L^2(\Gamma) \,|\, u|_K \circ F_K \in \mathcal{P}_p \quad \forall K \in \mathcal{T}\}$. Here, $\mathcal{P}_p$ denotes the univariate polynomials of degree p. The element maps F_K are constructed as described in Example 3.13, i.e., the uniform mesh $\widehat{\mathcal{T}}$ in parameter space is transported to the curve Γ by its parametrization. The basis of $S^p(\mathcal{T})$ selected for the computations is taken to be the push-forward of the L^2-normalized Legendre polynomials on the reference element $[-1, 1]$. The BEM operators K'_k and V_k are set up with an *hp*-quadrature with $p_{max} + 2$ quadrature points in each direction per quadrature cell (usually, $p_{max} = 20$). Details of the fast quadrature technique employed are described in [15]. Systematically, the number of elements N is taken proportional to k.
Denoting by $P_{\mathcal{T},p} : L^2(\Gamma) \to S^p(\mathcal{T})$ the Galerkin projector, which is characterized by

$$a'_k(u - P_{\mathcal{T},p} u, v) = 0 \qquad \forall v \in S^p(\mathcal{T}),$$

we approximate the Galerkin error $\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2}$ by the formula

$$\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} \approx \sup_{0 \neq v \in S^{p_{max}}(\mathcal{T})} \frac{\|v - P_{\mathcal{T},p} v\|_{L^2}}{\|v\|_{L^2}}. \tag{4.1}$$

Unless stated otherwise, we select $p_{max} = 20$ for the computation of (4.1). The norm $\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2}$ represents the quasi-optimality constant as the following lemma shows, whose proof is left to the reader:

LEMMA 4.1. *Assume $P_{\mathcal{T},p} : L^2(\Gamma) \to S^p(\mathcal{T})$ to be well-defined. Then:*

$$\|(\operatorname{Id} - P_{\mathcal{T},p})u\|_{L^2(\Gamma)} \leq \|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} \inf_{v \in S^p(\mathcal{T})} \|u - v\|_{L^2(\Gamma)} \qquad \forall u \in L^2(\Gamma),$$

$$\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} = \sup \left\{ \|(\operatorname{Id} - P_{\mathcal{T},p})u\|_{L^2(\Gamma)} \colon \|u\|_{L^2(\Gamma)} \leq 1 \text{ and } u \perp S^p(\mathcal{T}) \right\},$$

where the orthogonality $u \perp S^p(\mathcal{T})$ is with respect to the $L^2(\Gamma)$ inner product.
Proof. See Appendix B. ◻
Since for smooth domains we may expect the quasi-optimality constant to be asymptotically 1 (see the discussion in Section 5 below) we do not present in our numerical examples $\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2}$ of (4.1) but instead the *Galerkin Error Measure*

$$E := \sqrt{\|\operatorname{Id} - P_{\mathcal{T},p}\|^2_{L^2 \leftarrow L^2} - 1}. \tag{4.2}$$

REMARK 4.2. The supremum in (4.1) is computed as follows: For a fixed mesh $\mathcal{T}$ the spaces $S^p(\mathcal{T}) \subset S^{p_{max}}(\mathcal{T})$ are nested. In view of the hierarchical structure of the polynomial basis, the stiffness matrix $\mathbf{A}$ for the space $S^{p_{max}}(\mathcal{T})$ has block structure

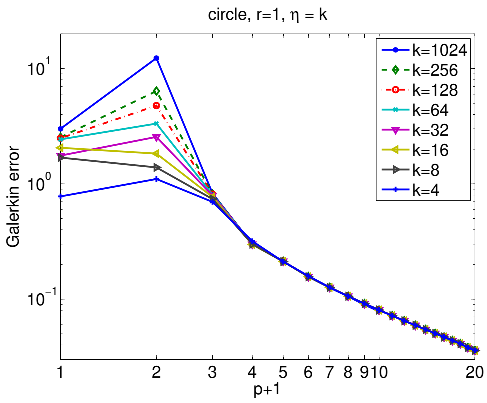
$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

where $\mathbf{A}_{11}$ is the stiffness matrix for the space $S^p(\mathcal{T})$ and $\mathbf{A}_{22}$ is the stiffness matrix for the orthogonal complement of $S^p(\mathcal{T})$ in $S^{p_{max}}(\mathcal{T})$. The mass matrices for $S^p(\mathcal{T})$ and the orthogonal complement of $S^p(\mathcal{T})$ in $S^{p_{p_{max}}}(\mathcal{T})$ are $\mathbf{M}_1$ and $\mathbf{M}_2$ with Cholesky factorizations $\mathbf{L}_1\mathbf{L}_1^\top = \mathbf{M}_1$ and $\mathbf{L}_2\mathbf{L}_2^\top = \mathbf{M}_2$. The matrix representation of $\operatorname{Id} - P_{p,\mathcal{T}} : S^{p_{max}}(\mathcal{T}) \to S^{p_{max}}(\mathcal{T})$ is then

$$\begin{pmatrix} 0 & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ 0 & \operatorname{Id} \end{pmatrix}.$$

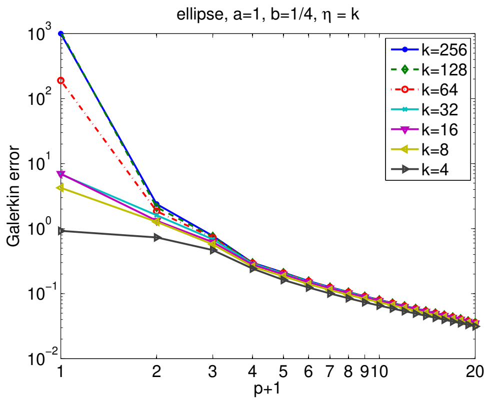
Therefore, the supremum in (4.1) is given by the norm of this last matrix if the space $S^{p_{max}}(\mathcal{T})$ is endowed with the $L^2(\Gamma)$-norm, viz., $\|\operatorname{Id} - P_{\mathcal{T},p}\|^2_{L^2 \leftarrow L^2} \approx 1 + \|\mathbf{L}_1^\top(-\mathbf{A}_{11}^{-1})\mathbf{A}_{12}\mathbf{L}_2^{-\top}\|_2^2$, where $\|\cdot\|_2$ denotes the standard spectral norm. This last norm is evaluated with the LAPACK routine ZGESVD. ∎
In our computations, we also report the extremal singular values $\sigma_{min}(\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top})$ and $\sigma_{max}(\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top})$ for $p = 10$, where $\mathbf{L}\mathbf{L}^\top = \mathbf{M}$ is the Cholesky factorization of the mass matrix $\mathbf{M}$ for the space $S^p(\mathcal{T})$ and $\mathbf{A}'$ represents the stiffness matrix for the discretization of A'_k. These numbers give a very good indication of $1/\|(A'_k)^{-1}\|_{L^2 \leftarrow L^2}$ and $\|A'_k\|_{L^2 \leftarrow L^2}$. The singular values are computed with the LAPACK-routine ZGESVD.

The examples below are selected to illustrate the theoretical results of the paper and to test its limits. The geometries of Examples 4.3 and 4.4 are a circle and an ellipse and hence fully covered by our theory (recall that $\|A_k^{-1}\|_{L^2 \leftarrow L^2} = \|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2} = O(1)$ for $\eta = k$ by [8]). The geometries in Examples 4.5, 4.6, 4.7, 4.8 are no longer star-shaped so that bounds for $\|A_k^{-1}\|_{L^2 \leftarrow L^2} = \|(A'_{-k})^{-1}\|_{L^2 \leftarrow L^2}$ are not known. In Examples 4.7, 4.8 we even leave the realm of smooth geometries; in the terminology of [6, Thm. 5.1] these geometries are "trapping domains" and the wavenumbers selected in our computations are precisely the critical wavenumbers identified there. Clearly, the choice of the coupling parameter η in (1.4) affects the norm $\|A_k^{-1}\|_{L^2 \leftarrow L^2}$ and thus, in turn, the conditions on the approximation properties of the discrete spaces X_N for quasi-optimality. We therefore also perform calculations for the choice $\eta = 1$ in Examples 4.6 and 4.8.
EXAMPLE 4.3. $\Omega = B_1(0)$ is a circle with radius $r = 1$. The mesh has $N = k$ elements of equal size. The element maps F_K are obtained with the aid of the parameterization $\{(r\cos\varphi, r\sin\varphi) \mid \varphi \in [0, 2\pi)\}$ of the circle. The coupling parameter η is selected as $\eta = k$. Fig. 4.1 shows the Galerkin Error Measure of (4.2) as a function of p; we also give an indication of $\|A'_k\|_{L^2 \leftarrow L^2}$ and $\|(A'_k)^{-1}\|_{L^2 \leftarrow L^2}$.



k k	σ_{max} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$	σ_{min} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$
4	1.26835	0.5
8	1.54632	0.5
16	1.89880	0.5
32	2.40042	0.5
64	2.98223	0.5
128	3.76487	0.5
256	4.73099	0.5
1024	7.48469	0.5

FIG. 4.1. *(see Example 4.3) Circle with radius $r = 1$, $\eta = k$. Left: Galerkin Error Measure E (see (4.2)). Right: Estimate of $\|A'_k\|_{L^2\leftarrow L^2}$ and $1/\|(A'_k)^{-1}\|_{L^2\leftarrow L^2}$.*



k	σ_{max} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$	σ_{min} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$
4	1.41593	0.489
8	1.71889	0.5
16	2.01108	0.5
32	2.64065	0.5
64	3.43955	0.5
128	4.57966	0.5
256	6.0845	0.5

FIG. 4.2. *(see Example 4.4) Ellipse with semi-axes $a = 1$ and $b = 1/4$. Left: Galerkin Error Measure E (see (4.2)). Right: Estimate of $\|A'_k\|_{L^2\leftarrow L^2}$ and $1/\|(A'_k)^{-1}\|_{L^2\leftarrow L^2}$*

EXAMPLE 4.4. Ω is an ellipse with semi-axes $a = 1$ and $b = 1/4$. The boundary Γ is parametrized in the standard way by $\{(a\cos\varphi, b\sin\varphi) \,|\, \varphi \in [0, 2\pi)\}$. The element maps are obtained by uniformly subdividing the parameter interval $[0, 2\pi)$, and the mesh has $N = k$ elements. The coupling parameter η is $\eta = k$. The numerical results are presented in Fig. 4.2.

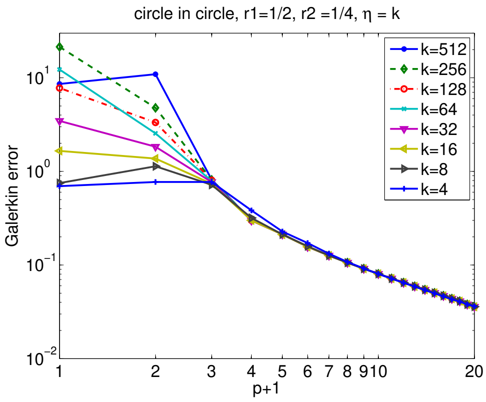
EXAMPLE 4.5. $\Omega = B_{1/2}(0) \setminus \overline{B_{1/4}(0)}$ is the annular region between two circles of radii $1/2$ and $1/4$. The normal vector appearing in the definition of K'_k always points outwards. The boundary $\partial\Omega$ is parametrized in the standard way with polar coordinates. The wave number is related to the number of elements N by $N = 2k$, and each of the two components of connectedness of $\partial\Omega$ has $N/2$ elements. The coupling parameter η is $\eta = k$. The results can be found in Fig. 4.3.

EXAMPLE 4.6. The setup is as in Example 4.5 with coupling parameter $\eta = 1$ instead of $\eta = k$. The result are presented in Fig. 4.4.

EXAMPLE 4.7. Ω is the C-shaped domain given by (see Fig. 4.5)

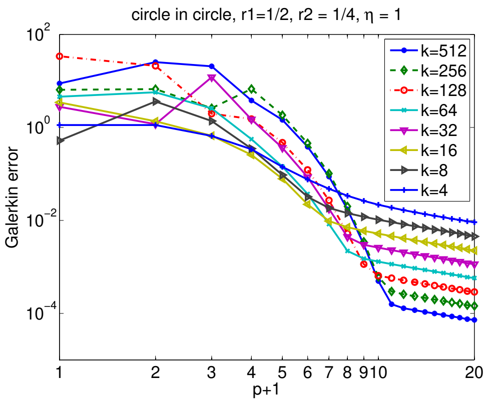
$$\Omega = ((0, r) \times (0, 2r/3)) \setminus ((r/3, 2r/3) \times (r/3, 2r/3)), \qquad r = 1/2.$$

For different values of the parmeter $m \in 3\mathbb{N}$, we select the number of elements N and



k	σ_{max} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$	σ_{min} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$
4	2.36155	0.500129
8	2.35101	0.497189
16	2.54262	0.238509
32	2.81275	0.500153
64	3.2893	0.51368_{-1}
128	3.69209	0.914729_{-1}
256	4.37155	0.884842_{-1}
512	5.1591	0.275835_{-2}

FIG. 4.3. *(see Example 4.5)* $\Omega = B_{1/2}(0) \setminus \overline{B_{1/4}(0)}$. *Left: Galerkin Error Measure E (see (4.2)). Right: Estimate of* $\|A'_k\|_{L^2\leftarrow L^2}$ *and* $1/\|(A'_k)^{-1}\|_{L^2\leftarrow L^2}$.



k	σ_{max} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$	σ_{min} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$
4	1.84018	0.304956
8	1.64098	0.147632
16	1.67512	0.102911
32	1.65914	0.603251_{-1}
64	1.70917	0.190993_{-1}
128	1.87601	0.911284_{-2}
256	1.99572	0.834516_{-2}

FIG. 4.4. *(see Example 4.6)* $\Omega = B_{1/2}(0) \setminus \overline{B_{1/4}(0)}$. *Coupling parameter* $\eta = 1$. *Left: Galerkin Error Measure E (see (4.2)). Right: Estimate of* $\|A'_k\|_{L^2\leftarrow L^2}$ *and* $1/\|(A'_k)^{-1}\|_{L^2\leftarrow L^2}$.

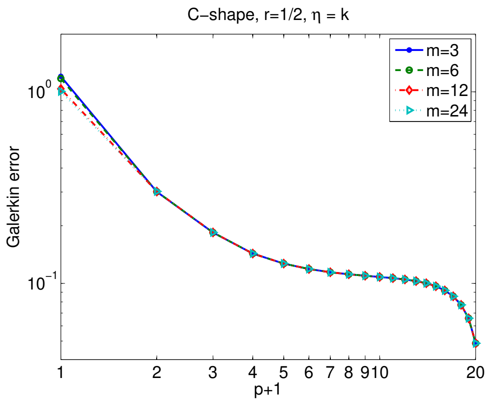
the wavenumber k according to

$$N = 20m, \qquad k = \frac{3m\pi}{r}.$$

The meshes are uniform on Γ. The coupling parameter is $\eta = k$. The results can be found in Fig. 4.5.

EXAMPLE 4.8. The setup is the same as in Example 4.7 with the exception that the coupling parameter η is chosen as $\eta = 1$ instead of $\eta = k$ and that $p_{max} = 15$ instead of $p_{max} = 20$. The numerical results can be found in Fig. 4.6.

EXAMPLE 4.9. Let $S_{R'} := \{(\rho\cos\varphi, \rho\sin\varphi)\colon 0 < \rho < R', 0 < \varphi < \omega + 2\omega'\}$, $S_R := \{(\rho\cos\varphi, \rho\sin\varphi)\colon r < \rho < R, \omega' < \varphi < \omega + \omega'\}$, and $\Omega = S_{R'} \setminus (B_r(0) \cup S_R)$ as depicted in Fig. 4.7. We select $\omega = \pi/4$, $\omega' = \pi/8$, $R = 1$, and $R' = 1.5$. The value $r \in \{0.5, 0.625, 0.75\}$. In Fig. 4.7, we present smallest singular values $\sigma_{min}(\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top})$ for different values of r and k. The calculations are done for $p = 10$ and the geometry has a fixed number of 768 elements. We stress that caution is necessary when interpreting these numbers. For example, for the case $r = 1/2$ and



m	$k = 6\pi m$	σ_{max} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$	σ_{min} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$
3	56.5487	2.721	2.24795_{-1}
6	113.097	2.99077	1.40383_{-1}
12	226.195	3.59232	7.80885_{-2}
24	452.389	4.86965	4.14679_{-2}

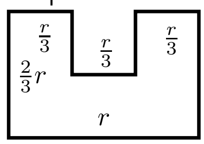
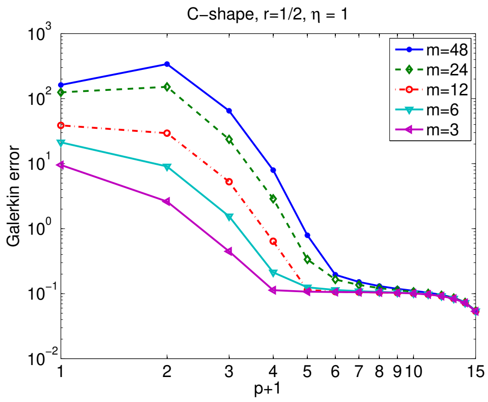


FIG. 4.5. *C-shaped domain (see Example 4.7), $\eta = k$. Left: Galerkin Error Measure E (see (4.2)). Right: Estimate of $\|A_k'\|_{L^2 \leftarrow L^2}$ and $1/\|(A_k')^{-1}\|_{L^2 \leftarrow L^2}$.*



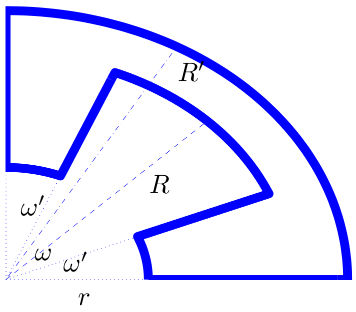
m	$k = 6\pi m$	σ_{max} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$	σ_{min} of $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$
3	56.5487	1.59341	1.24977_{-2}
6	113.097	1.71847	3.42968_{-3}
12	226.195	1.88635	1.02321_{-3}
24	452.389	2.10001	2.27319_{-4}
48	904.779	2.40774	8.30718_{-5}

FIG. 4.6. *C-shaped domain (see Example 4.8), $\eta = 1$. Left: Galerkin Error Measure E (see (4.2)). Right: Estimate of $\|A_k'\|_{L^2 \leftarrow L^2}$ and $1/\|(A_k')^{-1}\|_{L^2 \leftarrow L^2}$.*

$k = k^*$, an independent calculation (not presented) of the (complex) resonances of $\mathbb{R}^2 \setminus \overline{\Omega}$ suggested the existence of resonances near $k = k^*$ that are extremely close to the real axis, which raises doubts about the reliability of the value $\sigma_{min} = 1.6_{-11}$ given in Fig. 4.7. ■

Discussion of the Numerical Examples. Since we will also discuss the effect of the choice of the coupling parameter η, we make its appearance explicit by writing $A'_{k,\eta}$.

1. We recall that in all numerical examples the mesh size h is proportional to $1/k$. In the calculations based on smooth geometries, i.e., Examples 4.3, 4.4, 4.5, 4.6, we observe that the Galerkin Error Measure E tends to zero as $p \to \infty$. This shows that indeed, asymptotically, the quasi-optimality constant is 1 (cf. also Section 5). Closer inspection of the numerical results indicates an $O(1/p)$-behavior, which is consistent with the finite shift properties of V_0 and K_0'. It is noteworthy that in Example 4.6, where $\eta = 1$ the asymptotic behavior of the Galerkin Error Measure appears to be $O(1/(pk))$. Hence, for that geometry, the combined η and k dependence appears to be $O((1 + |\eta|)/(kp))$.



k	$r=0.5$	$r=0.625$	$r=0.75$
46.2	2.987_{-2}	1.695_{-1}	1.691_{-1}
46.4	8.372_{-2}	9.424_{-2}	9.402_{-2}
46.6	1.837_{-2}	1.837_{-2}	1.827_{-2}
46.64	3.189_{-3}	3.189_{-3}	3.147_{-3}
46.648	1.553_{-4}	1.552_{-4}	4.554_{-4}
k^*	1.642_{-11}	4.969_{-8}	4.523_{-4}
46.649	2.239_{-4}	2.224_{-4}	5.340_{-4}
46.65	6.032_{-4}	6.032_{-4}	8.059_{-4}
46.7	1.955_{-2}	1.955_{-2}	1.959_{-2}

FIG. 4.7. *Estimates* $1/\|(A'_k)^{-1}\|_{L^2\leftarrow L^2}$ *for different values of* k *and* r*;* $k^* = 46.64840949828568$.

2. In the case of a circle (Example 4.3), an ellipse (Example 4.4), and the case of an annular geometry with coupling parameter $\eta = k$ (Example 4.5) we observe that the condition

$$\frac{kh}{p} \text{ sufficiently small} \tag{4.3}$$

is already enough to ensure quasi-optimality of the Galerkin hp-BEM. The side condition $p = O(\log k)$ of (1.1) is not visible. For the special case of a circle, this absence of "pollution" may be expected in view the analysis of [2].
3. The C-shaped geometry in the Examples 4.7, 4.8 is not smooth. Hence, the operator K'_k is no longer smoothing and one cannot expect the Galerin Error Measure E of (4.2) to tend to zero. This is indeed visible in Figs. 4.5, 4.6. The sharp decrease of the the Galerkin Error Measure E for large p is likely to be a numerical artefact since E is obtained by comparing lower values of p with the result for $p_{max} = 20$ in the case of Fig. 4.5 and $p_{max} = 15$ in Fig. 4.6.
4. The work [8] shows that $\|(A'_{k,k})^{-1}\|_{L^2\leftarrow L^2}$ is bounded uniformly in k for star-shaped geometries. Indeed, the numerical results for the case of a circle (Example 4.3) and an ellipse (Example 4.4) confirm this. In contrast, the geometries of Examples 4.5 and 4.7 are not star-shaped and we observe in Figs. 4.3, 4.4, 4.5, 4.6 that $\|(A'_{k,\eta})^{-1}\|_{L^2\leftarrow L^2}$ is not bounded uniformly in k but grows algebraically. The norm $\|A'_{k,\eta}\|_{L^2\leftarrow L^2}$ is seen to grow (mildly) in k in all examples. This is in accordance with known results. For example, [11] shows $\|A'_{k,k}\|_{L^2\leftarrow L^2} = O(k^{1/3})$ for the case of a circle and [6] proves $\|A'_{k,k}\|_{L^2\leftarrow L^2} = O(k^{1/2})$ for general 2D Lipschitz domains. For the convenience of the reader, we present the tables of Figs. 4.1–4.6 in the form of graphs in Fig. 4.8.
5. For the C-shaped geometry of Examples 4.7, 4.8, a lower bound for $\|(A'_{k,\eta})^{-1}\|_{L^2\leftarrow L^2}$ is given in [6, Thm. 5.1] as

$$\|(A'_{k,\eta})^{-1}\|_{L^2\leftarrow L^2} \geq Ck^{9/10}\left(1 + \frac{|\eta|}{k}\right)^{-1}.$$

Our numerical examples (see the tables in Figs. 4.5, 4.6 or the graphs in Fig. 4.8) indicate sharpness of the lower bounds of [6, Thm. 5.1] for the case $\eta = k$.
6. In the case of circular/elliptic geometries and even in the case of the non-convex geometry of an annulus, we did not observe a "pollution" effect; in other words, quasi-optimality of the Galerkin BEM takes place as soon as kh/p is sufficiently small. The more stringent scale resolution condition (1.1) that stipulates $p = O(\log k)$ might,

however, be needed in more general situations. This is the purpose of selecting $\eta = 1$ in the Examples 4.6, 4.8. It has the effect of increasing $\|(A'_{k,\eta})^{-1}\|_{L^2 \leftarrow L^2}$, which, according to the analysis of Section 3, puts conditions on the approximation properties of the hp-BEM spaces. Indeed, the plots in Figs. 4.4, 4.6 indicate that the condition "kh/p small" alone is insufficient to ensure quasi-optimality of the Galerkin BEM.
7. The convergence theory developed here relies on estimates for $\|(A'_{k,k})^{-1}\|_{L^2 \leftarrow L^2}$. In Example 4.9, where k^* is close to a higher order "whispering gallery mode", shows that this assumption is violated for certain geometries and critical frequencies. Further examples of situations where $\|(A'_{k,k})^{-1}\|_{L^2 \leftarrow L^2}$ is large are given in [3].

5. Concluding Remarks. The convergence theory in Section 3.2 rests on two pillars, namely, the stability of the discretization of the operators A_0 or A'_0 (as stipulated in Assumption 3.4) and smoothing properties of the operators $(A'_{-k})^{-1}(A'_{-k} - A'_0)$ or $A_k^{-1}(A_k - A_0)$ as described in Lemma 3.2. In the present case of smooth geometries Γ, it is possible to make the choice $A_0 = A'_0 = \frac{1}{2}\operatorname{Id}$ instead of (3.1), (3.2). Decomposition results analogous to those of Lemma 3.2 could be then formulated by using Lemma 5.2 instead of Lemmata 2.1, 2.3. Inspection of the proofs in Section 3.2 then in turns produces the classical result that the quasi-optimality constant is effectively 1. For example, we have—analogous to Theorem 3.8—for the Galerkin error $u - u_N$ the following:
Lemma 5.1. *Let Γ be analytic. Then for every $\varepsilon' > 0$ there exist ε, $\gamma > 0$ such that the condition (3.5) implies for any pair $(u, u_N) \in L^2(\Gamma) \times X_N$ that satisfies the Galerkin orthogonality condition (3.6) the quasi-optimality estimate*

$$\|u - u_N\|_{L^2(\Gamma)} \leq (1 + \varepsilon') \inf_{w_N \in X_N} \|u - w_N\|_{L^2(\Gamma)}.$$

Proof. See Appendix C. □ We have not opted for this analytical development. Our reason is that the decompositions of Lemmata 2.1, 2.3 are based on decompositions of potentials defined on $\mathbb{R}^d \setminus \Gamma$ and then appropriate traces on Γ are taken to infer decompositions of A_k, A'_k. With this technique, the operators $\frac{1}{2} + K_0$ and $\frac{1}{2} + K'_0$ appear quite naturally, and one may hope to be able to develop a decomposition theory (and then in turn a convergence theory) for problems with piecewise smooth geometries. In contrast, the decompositions of A_k and A'_k in Lemma 5.2 below rest heavily on the smoothness of Γ, and it is not clear that a generalization to non-smooth geometries could at all be possible.
Lemma 5.2. *Let Γ be analytic. Let $q \in (0,1)$. Then the operators A_k and A'_k can be decomposed as*

$$A_k = \frac{1}{2} + R_A + k[\widetilde{\mathcal{A}}_A], \qquad A'_k = \frac{1}{2} + R_{A'} + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}],$$

where, for a constant $C > 0$ independent of $k \geq k_0$ and $q \in (0,1)$ and constants C_q, $\gamma_q > 0$ independent of $k \geq k_0$ (but depending on q),

$$\begin{aligned}
&\|R_A\|_{H^1(\Gamma) \leftarrow L^2(\Gamma)} \leq Ck, \qquad \|R_A\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} \leq q, \\
&\widetilde{\mathcal{A}}_A \varphi \in \mathfrak{A}(C_q k \|\varphi\|_{L^2(\Gamma)}, \gamma_q, \Omega_R) \qquad \forall \varphi \in L^2(\Gamma), \\
&\|R_{A'}\|_{H^1(\Gamma) \leftarrow L^2(\Gamma)} \leq Ck, \qquad \|R_{A'}\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} \leq q, \\
&\widetilde{\mathcal{A}}_{A',i} \varphi \in \mathfrak{A}(C_q k \|\varphi\|_{L^2(\Gamma)}, \gamma_q, \Omega_R) \qquad \forall \varphi \in L^2(\Gamma), \quad i \in \{1,2\}.
\end{aligned}$$

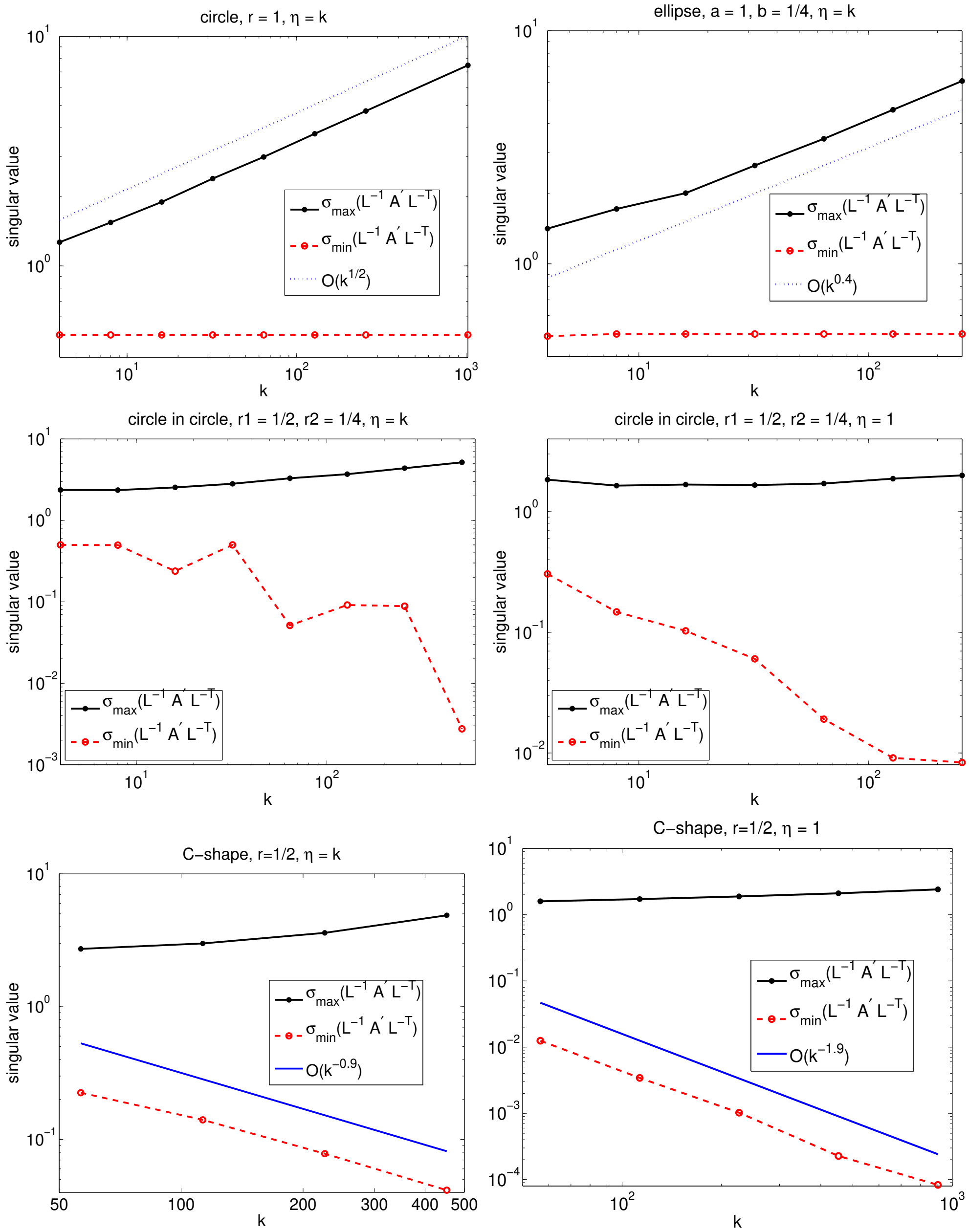


FIG. 4.8. *Extremal singular values of* $\mathbf{L}^{-1}\mathbf{A}'\mathbf{L}^{-\top}$ *for Examples 4.3 (top left), 4.4 (top right), 4.5 (middle left), 4.6 (middle right), 4.7 (bottom left), 4.8 (bottom right).*

Proof. The proof consists in using the decompositions of Lemmata 2.1, 2.3 and additional decompositions of the operator K_0 (for A_k) and K_0' (for A_k'). These latter decompositions are provided in [21, Lem. 2.3, Lem. 2.10]. ☐

Acknowledgements: We thank Simon Chandler-Wilde (University of Reading) for suggesting the geometry and the choice of k^* in Example 4.9.

Appendix A. Proof of Theorem 3.11. *Proof of Theorem 3.11:* We introduce the abbreviation $e := u - u_N$. Let $w_N \in X_N$ be arbitrary. Then by the triangle inequality

$$\|e\|_0 \leq \|u - w_N\|_0 + \|u_N - w_N\|_0. \tag{A.1}$$

Hence, we have to estimate $\|u_N - w_N\|_0$. By the discrete inf-sup condition we can find $v_N \in X_N$ with $\|v_N\|_0 = 1$ and $\gamma_0\|u_N - w_N\|_0 \leq (A_0'(u_N - w_N), v_N)_0$. With the Galerkin orthogonality $(A_k'(u - u_N), v_N)_0 = 0$, we then produce

$$\begin{aligned}\gamma_0\|u_N - w_N\|_0 &\leq ((A_0' - A_k')(u_N - w_N), v_N)_0 + (A_k'(u_N - w_N), v_N)_0 \\ &= ((A_0' - A_k')(u_N - w_N), v_N)_0 + (A_k'(u - w_N), v_N)_0 \\ &= ((A_k' - A_0')e, v_N)_0 + (A_0'(u - w_N), v_N)_0 \\ &\leq \|A_0'\|_{L^2 \leftarrow L^2}\|u - w_N\|_0 + ((A_k' - A_0')e, v_N)_0. \end{aligned} \tag{A.2}$$

In order to treat the term $((A_k' - A_0')e, v_N)_0$ we define $\psi \in L^2(\Gamma)$ by

$$((A_k' - A_0')z, v_N)_0 = (z, A_{-k}\psi)_0 \qquad \forall z \in L^2(\Gamma). \tag{A.3}$$

Lemma 3.2 tells us

$$\psi = A_{-k}^{-1}(A_{-k} - A_0)v_N \tag{A.4}$$

By selecting $z = e$ in (A.3), using Galerkin orthogonality satisfied by the error e and orthogonality properties of $\Pi_N^{L^2}$ we obtain

$$\begin{aligned}&((A_k' - A_0')e, v_N)_0 \\ &= (e, A_{-k}\psi)_0 = (A_k'e, \psi)_0 = (A_k'e, \psi - \Pi_N^{L^2}\psi)_0 \\ &= (A_0'e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k' - A_0')e, \psi - \Pi_N^{L^2}\psi)_0 \\ &= (A_0'e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k' - A_0')e - \Pi_N^{L^2}(A_k' - A_0')e, \psi - \Pi_N^{L^2}\psi)_0.\end{aligned}$$

Hence, from (A.4) and $\|v_N\|_0 = 1$

$$\begin{aligned}|((A_k' - A_0')e, v_N)_0| \leq &\left\{\|A_0'\|_{L^2\leftarrow L^2} + \|(\mathrm{Id} - \Pi_N^{L^2})(A_k' - A_0')\|_{L^2\leftarrow L^2}\right\} \\ &\times \|(\mathrm{Id} - \Pi_N^{L^2})A_{-k}^{-1}(A_{-k} - A_0)\|_{L^2\leftarrow L^2}\,\|e\|_0.\end{aligned}$$

From Lemmata 3.6, 3.7 we get for arbitrary $q \in (0, 1)$

$$\begin{aligned}|((A_k' - A_0')e, v_N)_0| \leq &\{\|A_0'\|_{L^2\leftarrow L^2} + q + Ck\eta(N, k, \gamma)\} \\ &\times \left\{q + Ck^2\left(1 + k^{5/2}\|A_{-k}^{-1}\|_{L^2\leftarrow L^2}\right)\eta_1(N, k, \gamma)\right\}\|e\|_0.\end{aligned} \tag{A.5}$$

Select now $q \in (0, 1)$ such that $(\|A_0'\|_{L^2\leftarrow L^2} + q)q < \gamma_0/2$. Then the constants C and γ in (A.5) are fixed and independent of k. We can furthermore select $\varepsilon > 0$ independent of k such that the assumption (3.14) then guarantees that the product of the two curly braces in (A.5) is bounded by $\gamma_0/2$. Combining (A.1), (A.2), and (A.5) therefore yields

$$\|e\|_0 \leq \left(1 + \frac{\|A_0'\|_{L^2\leftarrow L^2}}{\gamma_0}\right)\|u - w_N\|_0 + \frac{1}{2}\|e\|_0,$$

which leads to the desired estimate. □

Appendix B. Proof of Lemma 4.1. *Proof Lemma 4.1:* Since $P_{\mathcal{T},p}$ is well-defined, we have $P_{\mathcal{T},p}v = v$ for all $v \in S^p(\mathcal{T})$. Thus, for arbitrary $v \in S^p(\mathcal{T})$:

$$\|u - P_{\mathcal{T},p}u\|_{L^2} = \|u - v - P_{\mathcal{T},p}(u-v)\|_{L^2} \leq \|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} \|u - v\|_{L^2}.$$

Passing to the infimum over all $v \in S^p(\mathcal{T})$ gives the first estimate.
For the second estimate, let $\varepsilon > 0$ be arbitrary and write $\Pi_N^{L^2}$ for the L^2-projection onto $S^p(\mathcal{T})$. Then there exists $u_\varepsilon \in L^2(\Gamma) \setminus \{0\}$ such that $\|(\operatorname{Id} - P_{\mathcal{T},p})u_\varepsilon\|_{L^2} \geq (\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} - \varepsilon)\|u_\varepsilon\|_{L^2}$. Since $u_\varepsilon \notin S^p(\mathcal{T})$ (for otherwise $(\operatorname{Id} - P_{\mathcal{T}p})u_\varepsilon = 0$), we have $v_\varepsilon := (\operatorname{Id} - \Pi_N^{L^2})u_\varepsilon \neq 0$ with $v_\varepsilon \perp S^p(\mathcal{T})$ and

$$\begin{aligned}
\|(\operatorname{Id} - P_{\mathcal{T},p})v_\varepsilon\|_{L^2}^2 &= \|(\operatorname{Id} - P_{\mathcal{T},p})u_\varepsilon\|_{L^2}^2 \geq (\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} - \varepsilon)^2 \|u_\varepsilon\|_{L^2}^2 \\
&= (\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} - \varepsilon)^2 \left(\|v_\varepsilon\|_{L^2}^2 + \|\Pi_N^{L^2} u_\varepsilon\|_{L^2}\right) \\
&\geq (\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} - \varepsilon)^2 \|v_\varepsilon\|_{L^2}^2
\end{aligned}$$

Letting $\varepsilon \to 0$ allows us to conclude the proof. $\square$

Appendix C. Proof of Lemma 5.1. *Proof Lemma 5.1:* We employ the choice

$$A_0 = \frac{1}{2}\operatorname{Id} = A_0'.$$

and note that Lemma 5.2 shows that Lemma 3.2 is also valid with this choice of $A_0 = A_0'$ instead of that of (3.1), (3.2).
Let $e := u - u_N$ and $w_N \in X_N$ be arbitrary. With the Galerkin orthogonality $(A_k e, u_N - w_N)_0 = 0$, we then obtain

$$\begin{aligned}
\frac{1}{2}\|e\|_0^2 &= (A_0 e, e)_0 = ((A_0 - A_k)e, e)_0 + (A_k e, e)_0 = ((A_0 - A_k)e, e)_0 + (A_k e, u - w_N)_0 \\
&= ((A_0 - A_k)e, e)_0 + ((A_k - A_0)e, u - w_N)_0 + (A_0 e, u - w_N)_0 \\
&= -((A_0 - A_k)e, u_N - w_N)_0 + (A_0 e, u - w_N)_0 \\
&\leq \frac{1}{2}\|e\|_0 \|u - w_N\|_0 + ((A_k - A_0)e, u_N - w_N)_0. \qquad \text{(C.1)}
\end{aligned}$$

In order to treat the term $((A_k - A_0)e, u_N - w_N)_0$ we define $\psi \in L^2(\Gamma)$ by

$$((A_k - A_0)z, u_N - w_N)_0 = (z, A_{-k}'\psi)_0 \qquad \forall z \in L^2(\Gamma). \qquad \text{(C.2)}$$

Lemma 3.2 tells us

$$\psi = (A_{-k}')^{-1}(A_{-k}' - A_0')(u_N - w_N) \qquad \text{(C.3)}$$

By selecting $z = e$ in (C.2), using Galerkin orthogonality satisfied by the error e and orthogonality properties of $\Pi_N^{L^2}$ we obtain

$$\begin{aligned}
&((A_k - A_0)e, u_N - w_N)_0 = (e, A_{-k}'\psi)_0 = (A_k e, \psi)_0 = (A_k e, \psi - \Pi_N^{L^2}\psi)_0 \\
&= (A_0 e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k - A_0)e, \psi - \Pi_N^{L^2}\psi)_0 \\
&= (A_0 e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k - A_0)e - \Pi_N^{L^2}(A_k - A_0)e, \psi - \Pi_N^{L^2}\psi)_0.
\end{aligned}$$

Hence, from (C.3) and $\|A_0\|_{L^2\leftarrow L^2} = \frac{1}{2}$

$$|((A_k - A_0)e, u_N - w_N)_0| \leq \left\{\frac{1}{2} + \|(\mathrm{Id} - \Pi_N^{L^2})(A_k - A_0)\|_{L^2\leftarrow L^2}\right\}$$
$$\times \|(\mathrm{Id} - \Pi_N^{L^2})(A'_{-k})^{-1}(A'_{-k} - A'_0)\|_{L^2\leftarrow L^2} \|e\|_0 \|u_N - w_N\|_0.$$

From Lemmata 3.6, 3.7 we get for arbitrary $q \in (0,1)$

$$|((A_k - A_0)e, u_N - w_N)_0| \leq$$
$$\left\{\frac{1}{2} + q + Ck\eta_1(N,k,\gamma)\right\}\left\{q + C\left(1 + k^{5/2}\|(A'_{-k})^{-1}\|_{L^2\leftarrow L^2}\right)\eta(N,k,\gamma)\right\}$$
$$\times \|e\|_0\|u_N - w_N\|_0.$$

We abbreviate

$$E := \left\{\frac{1}{2} + q + Ck\eta_1(N,k,\gamma)\right\}\left\{q + C\left(1 + k^{5/2}\|(A'_{-k})^{-1}\|_{L^2\leftarrow L^2}\right)\eta(N,k,\gamma)\right\}.$$

Then, we obtain with the triangle inequality $\|u_N - w_N\|_0 \leq \|e\|_0 + \|u - w_N\|_0$

$$\frac{1}{2}\|e\|_0^2 \leq \frac{1}{2}\|e\|_0\|u - w_N\|_0 + E\|e\|_0\left(\|e\|_0 + \|u - w_N\|_0\right)$$

from which we get

$$\|e\|_0 \leq \|u - w_N\|_0 + 2E\left(\|e\|_0 + \|u - w_N\|_0\right)$$

and therefore, if $2E < 1$,

$$\|e\|_0 \leq \frac{1+2E}{1-2E}\|u - w_N\|_0.$$

The desired estimate is obtained by requiring that E is so small that

$$\frac{1+2E}{1-2E} \leq \varepsilon'.$$

This can be achieved by making q sufficiently small. This in turn fixes γ. Finally, requiring (3.5) for sufficiently stringent ε concludes the proof. $\square$

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