Wave-Number Explicit Convergence Analysis for Galerkin Discretizations of the Helmholtz Equation (extended version)

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Abstract

In this paper, we develop a new stability and convergence theory for highly indefinite elliptic partial differential equations by considering the Helmholtz equation at high wave number as our model problem. The key element in this theory is a novel k-explicit regularity theory for Helmholtz boundary value problems that is based on decomposing the solution into in two parts: the first part has the H^2-Sobolev regularity expected of elliptic PDEs but features k-independent regularity constants; the second part is an analytic function for which k-explicit bounds for all derivatives are given. This decomposition is worked out in detail for several types of boundary value problems including the case Robin boundary conditions in domains with analytic boundary and in convex polygons.

As the most important practical application we apply our full error analysis to the classical hp-version of the finite element method (hp-FEM) where the dependence on the mesh width h, the approximation order p, and the wave number k is given explicitly. In particular, under the assumption that the solution operator for Helmholtz problems grows only polynomially in k, it is shown that quasi-optimality is obtained under the conditions that kh/p is sufficiently small and the polynomial degree p is at least O(log k).

AMS Subject Classification: 35J05, 65N12, 65N30
Key Words: Helmholtz equation at high wavenumber, stability, convergence, $hp-$finite elements

1 Introduction

In this paper we analyze the numerical solution of highly indefinite boundary value problems, which arise, for example, when electromagnetic or acoustic scattering problems are modelled in the frequency domain and discretized by Galerkin methods. As our model problem we consider the Helmholtz equation at high wave numbers k.

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For low order h-version finite element methods, it is well-known that unique solvability of the discrete problem is only guaranteed under very restrictive stability conditions. More precisely, the minimal dimension N, e.g., of a $\mathcal{P}_1$ finite element space has to satisfy $N \gtrsim k^{2d}$, where $d \in \{1, 2, 3\}$ denotes the spatial dimension. In the present paper, we demonstrate that it is possible to ensure stability and quasi-optimality under the substantially relaxed condition $N \gtrsim k^d$. A different way of stating this result is that quasi-optimality of a piecewise polynomial based FEM can be achieved in a setting where (on average) the number of degrees of freedom per wavelength is independent of k. At first glance, this seems to contradict the results of [3] where it is proved that, for any (even generalized) finite element method, $N \gtrsim k^d$ is not a sufficient condition to guarantee quasi-optimality in general. However, in [3] only polynomial approximations of *fixed* order have been considered and a key result of this paper is that the polynomial order must be chosen in a wave-number dependent way in order to derive optimal stability conditions.

This quasi-optimality result hinges on two observations. Firstly, as was already exploited in [5, 26, 30, 34], the proof of quasi-optimality of Galerkin methods for Helmholtz problems can be reduced to the question of how well certain adjoint problems can be approximated from the ansatz space. Secondly, approximability questions are closely related to regularity issues. The key ingredient of the present paper, therefore, are new k-explicit regularity results for solutions of the Helmholtz equation. These regularity assertions take the form of a decomposition of the solution into a highly oscillatory, but analytic part $u_{\mathcal{A}}$ and an "elliptic" part with k-independent regularity properties. Although the constant in the regularity estimate for $u_{\mathcal{A}}$ depends critically on the wavenumber, it is the smoothness of the part $u_{\mathcal{A}}$ that can be exploited in numerical schemes. As the most important application, we illustrate this point for higher order finite element methods by showing that, for domains with analytic boundary, the condition

$$\frac{kh}{p} \text{ small} \quad \text{ together with } \quad p \geq C \log k \tag{1.1}$$

suffices to ensure stability and convergence if one makes the reasonable assumption that the solution operator for the Helmholtz problem grows at most polynomially in the wave number k. For polygonal Ω, the condition (1.1) is modified in the sense that appropriate geometric mesh refinement is required in small neighborhoods of the vertices. While an at most polynomial growth (in k) of the norm of the Helmholtz solution operator is stipulated as an extra assumption in our regularity theory (see Assumptions 4.7, 4.17 for the precise statement) we believe it to be reasonable in view of known results for special cases such as Helmholtz problems associated with star-shaped domains, [10, 28], and numerical evidence, [25].

Helmholtz problems have been studied to a considerable extent in the past decades with the ambitious goal of understanding the influence of critical parameters such as the wave number and, in the case of numerical schemes, the choice of the discretization and its parameters. Results in this direction include sharp estimates for the inf-sup constant of the continuous equations, lower estimates for the convergence rates, one-dimensional analysis by using the discrete Green's function as well as a dispersion analysis for standard and non-standard finite element discretizations (see, e.g., [2–5, 7–9, 12–15, 17–23, 26, 32–34, 38, 39] and the references therein). In spite of a large body of literature, it seems that an understanding of the behavior of numerical schemes that is fully explicit in the wave number and discretization parameters such as the mesh size h and the approximation order p is only available in very structured situations such as one-dimensional cases or fully regular tensor-product situations. The present paper considers a significantly more general situation and discusses the following three cases:

1. Bounded domains in $\mathbb{R}^d$ ($d \in \{2,3\}$) with analytic boundary and Robin boundary conditions.

2. Exterior domains in $\mathbb{R}^d$, ($d \in \{2,3\}$) with analytic boundaries and Dirichlet boundary conditions.

3. Convex two-dimensional polygons with Robin boundary conditions.

For these three cases, we show (under the above mentioned assumption of polynomial growth of the appropriate Helmholtz solution operator) quasi-optimality if the scale resolution condition (1.1) is met. The same condition has already been identified in the companion paper [30], where a simpler full space problem was analyzed.

The paper is structured as follows. In Section 2, we formulate the model Helmholtz problems and corresponding abstract Galerkin discretization. In Section 3, we briefly recapitulate the general convergence theory where the stability and convergence follows from approximability of certain adjoint problems. In Section 4, which is at the heart of the paper, we present a decomposition of the Helmholtz solution into a high-frequency and a low-frequency part. The high-frequency part is in the Sobolev space H^2 but features k-independent bounds in the H^2-norm. The low-frequency part belongs to some high-order weighted Sobolev spaces – irrespective of the fact that we only assume that the right-hand side $f \in L^2(\Omega)$ has finite regularity. The ability to decompose the solution of Helmholtz problems into such two parts appears to be a general feature. Similar decompositions are developed in [25] for the solutions of boundary integral formulations of scattering problems. Finally, in Section 5 we consider the *hp*-finite element method as an example and show that, by a suitable choice of the polynomial degree p, the condition $N \gtrsim k^d$ ensures discrete stability and optimal convergence rates.

1.1 Function Spaces and Notation

We employ standard notation concerning Sobolev spaces, [1]. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, and $k > 0$ we introduce the following k-dependent norms:

$$\|u\|^2_{\mathcal{H},\Omega} := \quad k^2\|u\|^2_{L^2(\Omega)} + |u|^2_{H^1(\Omega)}, \tag{1.2a}$$

$$\|u\|^2_{1/2,\mathcal{H},\partial\Omega} := \quad |u|^2_{H^{1/2}(\partial\Omega)} + k\|u\|^2_{L^2(\partial\Omega)}, \tag{1.2b}$$

$$\|u\|^2_{3/2,\mathcal{H},\partial\Omega} := \quad k^{-2}|u|^2_{H^{3/2}(\partial\Omega)} + \|u\|^2_{1/2,\mathcal{H},\partial\Omega}; \tag{1.2c}$$

here, the norm (1.2c) will only be employed for smooth $\partial\Omega$ so that it is indeed well-defined.

A large part of the analysis will be concerned with domains with analytic boundary or convex polygons. For ease of future reference we therefore introduce

Assumption 1.1 *$\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ is a bounded Lipschitz domain.* ***Either*** *it has an analytic boundary* ***or*** *it is a convex polygon in $\mathbb{R}^2$ with vertices A_j, $j = 1, \ldots, J$.*

For domains Ω that have a smooth boundary or are polygonal (not necessarily convex), we introduce the following short-hand:

$$H^{1/2}_{pw}(\partial\Omega) := \begin{cases} \left\{g \in L^2(\partial\Omega) : g \text{ is edgewise in } H^{1/2}\right\} & \text{if } \partial\Omega \text{ is a polygon,} \\ H^{1/2}(\partial\Omega) & \text{if } \partial\Omega \text{ is smooth.} \end{cases} \tag{1.3}$$

Furthermore, for domains satisfying Assumption 1.1, we require spaces of analytic functions, specifically, the countably normed spaces introduced in [27]. These function spaces are defined with the aid of weight functions $\Phi_{p,\overrightarrow{\beta},k}$ that we now define. For $\beta \in [0,1)$, $p \in \mathbb{N}_0$, and $k > 0$ we set

$$\Phi_{p,\beta,k}(x) = \min\left\{1, \frac{|x|}{\min\left\{1, \frac{|p|+1}{k+1}\right\}}\right\}^{p+\beta}.$$

For a polygon Ω with vertices A_j, $j = 1,\ldots,J$, and given $\overrightarrow{\beta} \in [0,1)^J$, we define

$$\Phi_{p,\overrightarrow{\beta},k}(x) \quad = \quad \prod_{j=1}^{J} \Phi_{p,\beta_j,k}(x - A_j). \tag{1.4}$$

If $\Omega \subset \mathbb{R}^d$ is not a polygon, then we set,

$$\Phi_{p,\overrightarrow{\beta},k}(x) :\equiv 1 \tag{1.5}$$

for all p and any $\overrightarrow{\beta}$. We use the symbol ∇^n to denote derivatives of order n, more precisely, for a function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, we write

$$|\nabla^n u(x)|^2 = \sum_{\alpha\in\mathbb{N}_0^d : |\alpha|=n} \frac{n!}{\alpha!} |D^\alpha u(x)|^2. \tag{1.6}$$

Definition 1.2 *Given C_u, γ, $k > 0$, we set*

$$\begin{aligned} \mathcal{B}_{\overrightarrow{\beta},k}(C_u,\gamma) \quad &:= \quad \{u \in H^1(\Omega) \mid \|u\|_{\mathcal{H},\Omega} \leq C_u k \quad \wedge & (1.7)\\ &\qquad \|\Phi_{p,\overrightarrow{\beta},k} \nabla^{p+2} u\|_{L^2(\Omega)} \leq C_u(\gamma \max\{p,k\})^{p+2} \quad \forall p \in \mathbb{N}_0\}, & (1.8) \end{aligned}$$

where the weight functions $\Phi_{p,\overrightarrow{\beta},k}$ are given by (1.4) if Ω is a polygon and by (1.5) otherwise.

Of additional interest to us will be the unit ball in $H^2(\Omega)$ and the subset of $\mathcal{B}_{\overrightarrow{\beta},k}(C_u,\gamma)$ obtained by the scaling condition $C_u = 1$:

$$\mathcal{H}_{\text{osc}}(\gamma,k) \quad : \quad = \mathcal{B}_{\overrightarrow{\beta},k}(1,\gamma), \tag{1.9}$$

$$\mathcal{H}_{\text{ell}} \quad : \quad = \left\{v \in H^2(\Omega) : \|v\|_{H^2(\Omega)} \leq 1\right\}. \tag{1.10}$$

We close this section with some general comment on constants: $C > 0$ denotes a generic constant that may have different values in different occurrences. However, it will not depend on critical parameters. We will use the symbol "$\lesssim$" to compare two quantities $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$, where C is independent of the parameters k, p, q (which will be introduced in the sequel) and – if A and B contain norms of functions – also is independent of these functions. We write $A \sim B$ if $A \lesssim B$ together with $B \lesssim A$.

2 Model Helmholtz Problems and their Discretization

We start by introducing the three model problems that will be analyzed in the paper.

2.1 Robin Boundary Conditions for a Bounded Domain

2.1.1 The Continuous Problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded Lipschitz domain. The Helmholtz equation with wavenumber

$$k \geq k_0 > 0 \tag{2.1}$$

is given by

$$-\Delta u - k^2 u = f \qquad \text{in } \Omega. \tag{2.2a}$$

As boundary conditions we consider Robin boundary conditions

$$\frac{\partial u}{\partial n} - \mathrm{i}\, k u = g \qquad \text{on } \partial\Omega. \tag{2.2b}$$

The weak form is given by

$$\text{Find } u \in H^1(\Omega): \qquad \int_\Omega \nabla u \cdot \nabla \overline{v} - k^2 u \overline{v} - \mathrm{i} \int_{\partial\Omega} k u \overline{v} = \int_\Omega f \overline{v} + \int_{\partial\Omega} g \overline{v} \qquad \forall v \in H^1(\Omega). \tag{2.3}$$

Proposition 2.1 ([26, Prop. 8.1.3]) *Let Ω be a bounded Lipschitz domain. Then, for all $f \in (H^1(\Omega))'$, $g \in H^{-1/2}(\Gamma)$, a unique solution u of problem (2.2) exists and the solution depends continuously on the data.*

2.1.2 Abstract Galerkin Discretization

The conforming Galerkin discretization of (2.3) is based on the definition of a finite dimensional subspace $S \subset H^1(\Omega)$ and given by

$$\text{Find } u_S \in S: \qquad \int_\Omega \nabla u_S \cdot \nabla \overline{v} - k^2 u_S \overline{v} - \mathrm{i} \int_{\partial\Omega} k u_S \overline{v} = \int_\Omega f \overline{v} + \int_{\partial\Omega} g \overline{v} \qquad \forall v \in S. \tag{2.4}$$

2.2 Dirichlet Boundary Conditions for an Exterior Domain

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain and let $\Omega^c := \mathbb{R}^d \backslash \overline{\Omega}$ denote its exterior. For $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_R$ for some ball B_R of radius R, we consider the problem of finding u such that

$$-\Delta u - k^2 u = f \quad \text{in } \Omega^c, \qquad u|_{\partial\Omega} = g, \tag{2.5a}$$

and the *Sommerfeld radiation condition*

$$\left| \frac{\partial u}{\partial r} - \mathrm{i}\, k u \right| = o\left(\|\mathbf{x}\|^{\frac{1-d}{2}} \right) \qquad \text{as } \|\mathbf{x}\| \to \infty \tag{2.5b}$$

is satisfied. We define the spaces

$$V_R := \left\{ u|_{\Omega^c \cap B_R} : u \in H^1(\Omega^c) \right\} \quad \text{and} \quad V_{R,0} := \left\{ u|_{\Omega^c \cap B_R} : u \in H_0^1(\Omega^c) \right\}. \tag{2.6}$$

Next, we will introduce the Dirichlet-to-Neumann operator. Let $B_R^c := \mathbb{R}^d \backslash \overline{B_R}$ and $\Gamma_R := \partial B_R$. It can be shown that, for given $h \in H^{1/2}(\Gamma_R)$, the problem:

$$\text{find } w \in H^1_{\text{loc}}(B_R^c) \text{ such that } \begin{cases} (-\Delta - k^2)\, w = 0 & \text{in } B_R^c, \\ w = h & \text{on } \Gamma_R, \\ \left| \dfrac{\partial w}{\partial r} - \mathrm{i}\, k w \right| = o\left(\|\mathbf{x}\|^{\frac{1-d}{2}} \right) & \|\mathbf{x}\| \to \infty \end{cases}$$

has a unique weak solution. The mapping $h \mapsto w$ is called the *Steklov-Poincaré operator* and denoted by $S_P : H^{1/2}(\Gamma_R) \to H^1_{\text{loc}}(B_R^c)$. The *Dirichlet-to-Neumann map* is given by $T_R := \gamma_1 S_P : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$, where $\gamma_1 := \partial/\partial n$ is the normal derivative trace operator on Γ_R. The operator T_R allows us to restrict (2.5a) to a finite domain: Find $u \in V_R$ such that

$$\begin{aligned} -\Delta u - k^2 u = f \quad & \text{in } \Omega^c \cap B_R =: \Omega_R^c, \\ u = g \quad & \text{on } \Gamma, \\ \partial u/\partial n = T_R u \quad & \text{on } \Gamma_R. \end{aligned} \tag{2.7}$$

The weak formulation to this problem is given by

$$\text{Find } u \in V_R \text{ with } u|_{\partial\Omega} = g : \quad \int_{\Omega_R^c} \left(\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v}\right) - \int_{\Gamma_R} (T_R u)\, \overline{v} = \int_{\Omega_R^c} f \overline{v} \quad \forall v \in V_{R,0}. \tag{2.8}$$

Proposition 2.2 ([10]) *Let Ω be a bounded Lipschitz domain which is star-shaped with respect to the origin. Then, (2.8) admits a unique solution $u \in V_R$ for all $g \in H^{1/2}(\Gamma)$ and $f \in V_{R,0}'$ which depends continuously on the data.*

2.2.1 Abstract Galerkin Discretization

Again the conforming Galerkin discretization is based on the definition of a finite-dimensional subspace $S \subset V_R$ and given by

$$\text{Find } u_S \in S \text{ with } u_S|_{\partial\Omega} = g_S : \quad \int_\Omega \nabla u_S \cdot \nabla \overline{v} - k^2 u_S \overline{v} - \int_{\Gamma_R} (T_R u_S)\, \overline{v} = \int_\Omega f v \quad \forall v \in S \cap V_{R,0}. \tag{2.9}$$

Here, $g_S \in S$ denotes some approximation to g in (2.7).

3 Abstract Stability and Convergence Analysis

In this section, we identify in an abstract setting conditions on the approximation properties of ansatz spaces that ensure quasi-optimality of a Galerkin discretization.

3.1 Variational Formulations and Adjoint Problems

Many Helmholtz boundary value problems can be cast in the following abstract form:

$$\text{find } u \in V \text{ s.t.} \qquad a(u, v) - b(u, v) = l(v) \qquad \forall v \in V. \tag{3.1}$$

Here, the space V is a suitable subspace of a Sobolev space $H^1(\widetilde{\Omega})$ that reflects the possible presence of essential Dirichlet boundary conditions. The sesquilinear form $a : H^1(\widetilde{\Omega}) \times H^1(\widetilde{\Omega}) \to \mathbb{C}$ has the form

$$a(u, v) := \int_{\widetilde{\Omega}} \nabla u \cdot \nabla \overline{v} - k^2 u \overline{v}, \tag{3.2}$$

and the continuous sesquilinear form b encodes the boundary conditions. Finally, l is a bounded antilinear functional on V. For example, the model problems of Section 2.1 and Section 2.2 (with additionally $g = 0$) have this form: In the setting of Section 2.1, we may choose $\widetilde{\Omega} = \Omega$, $V = H^1(\Omega)$ and $b(u,v) = \mathrm{i}\, k \int_{\partial\Omega} u\overline{v}$; in the setting of Section 2.2 with $g = 0$ we have $\widetilde{\Omega} = \Omega_R^c$, $V = V_{R,0}$, and $b(u,v) = \int_{\partial B_R} T_R u \overline{v}$.

Of interest to us will adjoint problems associated with (3.1). That is, given an antilinear functional l on V, we consider

$$\text{find } u \in V \text{ s.t.} \quad a(v,u) - b(v,u) = \overline{l(v)} \qquad \forall v \in V. \tag{3.3}$$

An important observation is that the adjoint problems for the Helmholtz problems of Sections 2.1, 2.2 are themselves Helmholtz problems:

Lemma 3.1 *(i) Denote by $S_k : (f,g) \mapsto u$ the solution operator for the problem of Section 2.1. The adjoint solution operator $S_k^\star$ for the problem:*

$$\textit{Find } z \in H^1(\Omega) \textit{ s.t.} \quad \int_\Omega \left(\nabla v \cdot \nabla \overline{z} - k^2 v \overline{z}\right) - \mathrm{i}\, k \int_{\partial\Omega} v\overline{z} = \int_\Omega v\overline{f} + \int_{\partial\Omega} v\overline{g} \qquad \forall v \in H^1(\Omega) \tag{3.4}$$

is given by $S_k^\star(f,g) = \overline{S_k(\overline{f},\overline{g})}$.

(ii) Denote by $S_k^c : (f,g) \mapsto u$ the solution operator for the problem of Section 2.2. For the special case $g = 0$, denote by $S_k^{c,\star} : f \mapsto z$ to the solution operator for the adjoint problem

$$\textit{Find } z \in V_{R,0} \textit{ s.t.} \int_{\Omega_R^c} \left(\nabla v \cdot \nabla \overline{z} - k^2 v \overline{z}\right) - \int_{\Gamma_R} T_R v \overline{z} = \int_{\Omega_R^c} v\overline{f} \qquad \forall v \in V_{R,0}. \tag{3.5}$$

Then, $S_k^{c,\star}(f) = \overline{S_k^c(\overline{f},0)}$.

Proof. We will only show (ii) since (i) is shown with similar ideas. By [30, Lemma 3.10] we have for the adjoint $T_R^\star$ (with respect to the $(\cdot,\cdot)_{L^2(\Gamma_R)}$ inner product) the representation $T_R^\star z = \overline{T_R \overline{z}}$. Hence, (3.5) is equivalent to finding $z \in V_{R,0}$ such that

$$\int_{\Omega_R^c} \nabla v \cdot \nabla \overline{z} - k^2 v \overline{z} - \int_{\Gamma_R} v T_R \overline{z} = \int_{\Omega_R^c} \overline{f} v \qquad \forall v \in V_{R,0}. \tag{3.6}$$

By replacing v with $\overline{v}$, we recognize that $\overline{z} = S_k^c(\overline{f},0)$, which concludes the proof. ∎

3.2 Abstract Stability and Convergence Analysis

It is well-known that in the context of variational problems that admit a Gårding inequality, Galerkin methods are asymptotically quasi-optimal, i.e., quasi-optimality is ensured if the ansatz space is sufficiently rich, (see [35], [6]). The following theorem restricts this general setting to one that is applicable to Helmholtz problems and formulates an abstract condition on the approximation properties of the ansatz space that guarantees quasi-optimality. In particular, the model problems of Sections 2.1 and 2.2 (with $g = 0$) are covered by the following theorem.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^d$, $d \geq \{2,3\}$ be a bounded Lipschitz domain. Let $V \subset H^1(\Omega)$ be a closed subspace of $H^1(\Omega)$, and let the sesquilinear form a be given by (3.2). Let the following additional hypotheses be true:*

(i) $b: V \times V \to \mathbb{C}$ is a continuous sesquilinear form with

$$|b(u,v)| \leq C_b \|u\|_{\mathcal{H},\Omega} \|v\|_{\mathcal{H},\Omega} \qquad \forall u, v \in V. \tag{3.7}$$

(ii) There exist $\theta \geq 0$ and $\gamma > 0$ such that the following Gårding inequality holds:

$$\operatorname{Re}\left(a(u,u) - b(u,u)\right) + \theta k^2 \|u\|^2_{L^2(\Omega)} \geq \gamma \|u\|^2_{\mathcal{H},\Omega} \qquad \forall u \in V. \tag{3.8}$$

(iii) The adjoint problem

$$\text{find } z \in V \text{ s.t.} \qquad a(v,z) - b(v,z) = (v,f)_{L^2(\Omega)} \qquad \forall v \in V \tag{3.9}$$

is uniquely solvable for every $f \in L^2(\Omega)$. Let $\widetilde{S}_k^\star : f \mapsto z$ denote this solution operator with (possibly k-dependent norm)

$$C_{adj} := \sup_{f \in L^2(\Omega)\setminus\{0\}} \frac{\|\widetilde{S}_k^\star f\|_{\mathcal{H},\Omega}}{\|f\|_{L^2(\Omega)}}. \tag{3.10}$$

Let $S \subset V$ be a closed subspace and define the adjoint approximability

$$\eta(S) := \sup_{f \in L^2(\Omega)\setminus\{0\}} \inf_{v \in S} \frac{\|\widetilde{S}_k^\star f - v\|_{\mathcal{H},\Omega}}{\|f\|_{L^2(\Omega)}}. \tag{3.11}$$

Then, the condition

$$\theta k \eta(S) \leq \frac{\gamma}{2(1+C_b)} \tag{3.12}$$

implies the following statements:

1. *The discrete inf-sup condition is satisfied:*

$$\inf_{u \in S\setminus\{0\}} \sup_{v \in S\setminus\{0\}} \frac{|a(u,v) - b(u,v)|}{\|u\|_{\mathcal{H},\Omega}\|v\|_{\mathcal{H},\Omega}} \geq \frac{\gamma}{2 + \gamma/(1+C_b) + 2\theta k C_{adj}} > 0. \tag{3.13}$$

2. *The Galerkin method based on S is quasi-optimal, i.e., for every $u \in \mathcal{H}$ there exists a unique $u_S \in S$ with $a(u - u_S, v) - b(u - u_S, v) = 0$ for all $v \in S$, and there holds*

$$\|u - u_S\|_{\mathcal{H},\Omega} \leq \frac{2}{\gamma}(1+C_b) \inf_{v \in S} \|u - v\|_{\mathcal{H},\Omega}, \tag{3.14}$$

$$\|u - u_S\|_{L^2(\Omega)} \leq (1+C_b)\eta(S)\|u - u_S\|_{\mathcal{H},\Omega}. \tag{3.15}$$

Proof. The proof follows very closely the proofs of [30, Thms. 4.2, 4.3]. Details can be found in Appendix B. ∎

Theorem 3.2 is applicable to the model problems of Sections 2.1, 2.2 with $\theta = 2$ and $\gamma = 1$ as we now show:

Corollary 3.3 *Let $k \geq k_0$.*

(i) For the model problem of Section 2.1 the assumptions of Theorem 3.2 are satisfied for the choices $V = H^1(\Omega)$, $\theta = 2$, $\gamma = 1$, and a constant $C_b > 0$ that depends solely on Ω.

(ii) For the model problem of Section 2.2 with $g = 0$ the assumptions of Theorem 3.2 are satisfied for the choices $V = V_{R,0}$ (see (2.6)), $\theta = 2$, $\gamma = 1$, and a constant $C_b > 0$ that depends solely on k_0 and R.

In both cases, the constant C_{adj} is finite (but possibly k-dependent) for any $k \geq k_0$.

Proof. To see (i) we note that $b(u, v) = \mathrm{i}\, k(u, v)_{L^2(\partial\Omega)}$. By [30, Cor. 3.2] C_b is bounded uniformly in k. By Lemma 3.1 and Proposition 2.1 the solvability of the adjoint problem is ensured. From $\operatorname{Re} b(u, u) = 0$, it follows that the Gårding inequality is satisfied with $\theta = 2$ and $\gamma = 1$.

To see (ii) we observe $b(u, v) = \int_{\partial B_R} T_R u \overline{v}$. [30, Lemma 3.3] give a bound for C_b that is uniform in k; additionally, [30, Lemma 3.3] provides $\operatorname{Re} b(u, u) \leq -CR^{-1} \|u\|^2_{L^2(\partial B_R)} \leq 0$ so that again $\theta = 2$ and $\gamma = 1$ are valid choices. The unique solvability of the adjoint problem follows again by Lemma 3.1 and Proposition 2.2. ■

The usefulness of Theorem 3.2 rests on the ability to quantify the adjoint approximability $\eta(S)$ in terms of the wavenumber k and properties of the approximation space S. Since $\eta(S)$ depends on the solution operator $S_k^\star$ of some adjoint Helmholtz problems, we need a regularity for these operators in which the influence of k is made explicit. This is the purpose of the following Section 4. There, we construct for the model problems of Section 2 for every $f \in L^2(\Omega)$ a stable splitting $\widetilde{S}_k^\star f = C_{k,\mathcal{A}}(f) u_{\mathcal{A},f} + C_{H^2}(f) u_{H^2}(f)$, where $u_{\mathcal{A},f} \in \mathcal{H}_{\mathrm{osc}}(\gamma, k)$ and $u_{H^2,f} \in \mathcal{H}_{\mathrm{ell}}$ and $C_{k,\mathcal{A}}(f)$ and $C_{H^2}(f)$ are constants; we recall that the spaces $\mathcal{H}_{\mathrm{osc}}(\gamma, k)$ and $\mathcal{H}_{\mathrm{ell}}$ are introduced in (1.9), (1.10). Furthermore, we show that

$$C_{H^2} := \sup_{f \in L^2(\Omega) \setminus \{0\}} |C_{H^2}(f)| < \infty, \qquad C_{k,\mathcal{A}} := \sup_{f \in L^2(\Omega) \setminus \{0\}} |C_{k,\mathcal{A}}(f)| < \infty.$$

Accepting this decomposition result for the moment, we can formulate

Lemma 3.4 *The adjoint approximability (3.11) is bounded by*

$$\eta(S) \leq C_{k,\mathcal{A}} \eta_{\mathcal{A}}(S) + C_{H^2} \eta_{H^2}(S), \tag{3.16}$$

where

$$\eta_{\mathcal{A}}(S) := \sup_{v \in \mathcal{H}_{\mathrm{osc}}(\gamma,k) \setminus \{0\}} \inf_{w \in S} \|v - w\|_{\mathcal{H}} \quad \text{and} \quad \eta_{H^2}(S) := \sup_{\substack{v \in H^2(\Omega) \\ \|v\|_{H^2(\Omega)} = 1}} \inf_{w \in S} \|v - w\|_{\mathcal{H}}. \tag{3.17}$$

Proof. Follows by the triangle inequality. We refer to [30, Lemma 5.10], where a similar calculation is worked out. ■

The important conclusion of Lemma 3.4 is that the stability and convergence estimates for Helmholtz problems follow from two types of approximation properties: $\eta_{\mathcal{A}}(S)$ measures the approximability of the Galerkin space S for analytic, highly oscillating functions and $\eta_{H^2}(S)$ measures the standard approximation property of S for H^2-functions. We mention at this point that our analysis in Section 4 will show that the constant C_{H^2} in (3.16) can be bounded uniformly in k and that $C_{k,\mathcal{A}}$ in (3.16) will have—due to our assumptions—a polynomial growth in k. We emphasize that estimates for $\eta_{\mathcal{A}}(S)$, $\eta_{H^2}(S)$ involve neither any stability nor any regularity issues for Helmholtz problems. Finally, we point out that Lemma 3.1 shows that the regularity properties of the adjoint problems for our model problems of Sections 2.1, 2.2 can be inferred from those of the "original" model problems. The focus of the present paper is therefore the regularity of these "original" problems.

4 Stable Decompositions of the Helmholtz Solutions

4.1 Preliminaries

In this section, we will develop the theoretical tools which will be used for the regularity estimates of the Helmholtz problems.

4.1.1 Frequency Splitting

The key role for proving the refined regularity results are played by a frequency splitting of the right-hand side and some estimates of the solution operators applied to the high and low frequency part of the right-hand side. We start with introducing the frequency splitting. For functions on $\mathbb{R}^d$ the splitting is defined via the Fourier transform and, for functions on closed surfaces of finite domains, it is defined via the composition of a lifting operator of the boundary data with the frequency splitting for functions in $\mathbb{R}^d$. Recall the definition of the Fourier transform for functions with compact support

$$\hat{u}\left(\xi\right) = \mathcal{F}\left(u\right)\left(\xi\right) = \left(2\pi\right)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\,\mathrm{i}\langle \xi, x\rangle}\, u\left(x\right) dx \qquad \forall \xi \in \mathbb{R}^d$$

and the inversion formula

$$u\left(x\right) = \mathcal{F}^{-1}\left(\hat{u}\right)\left(x\right) = \left(2\pi\right)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x, \xi\rangle}\, \hat{u}\left(\xi\right) d\xi \qquad \forall x \in \mathbb{R}^d.$$

- For functions $f \in L^2\left(\mathbb{R}^d\right)$ the *high frequency filter* $H_{\mathbb{R}^d}$ and the *low frequency filter* $L_{\mathbb{R}^d}$ are defined by

  $$\mathcal{F}(L_{\mathbb{R}^d} f) = \chi_{\eta k}\mathcal{F}(f), \qquad \mathcal{F}(H_{\mathbb{R}^d} f) = (1-\chi_{\eta k})\mathcal{F}(f), \tag{4.1a}$$

  where $\chi_{\eta k}$ is the characteristic function of the ball $B_{\eta k}(0)$.

- Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $E_\Omega : L^2(\Omega) \to L^2(\mathbb{R}^d)$ be the extension operator of Stein, [37, Chap. VI]. Then for $f \in L^2\left(\Omega\right)$ we set

  $$L_\Omega f := \left(L_{\mathbb{R}^d}\left(E_\Omega f\right)\right)\big|_\Omega \quad \text{and} \quad H_\Omega f := \left(H_{\mathbb{R}^d}\left(E_\Omega f\right)\right)\big|_\Omega. \tag{4.1b}$$

- Let $\partial\Omega$ be smooth or (in 2D) polygonal. We remind the reader of the space $H^{1/2}_{pw}(\partial\Omega)$ introduced in (1.3) and define operators $H^N_{\partial\Omega}$ and $L^N_{\partial\Omega}$ as follows. For smooth boundaries, there exists a lifting operator G^N with the mapping property $G^N : H^s(\partial\Omega) \to H^{3/2+s}(\Omega)$ for every $s > 0$ and $\partial_n G^N g = g$. For polygonal domains, we have the existence of a simplified lifting operator $G^N : H^{1/2}_{pw}(\partial\Omega) \to H^2(\Omega)$ with $\partial_n G^N g = g$ (see, e.g., Lemma A.1 for details).

  We then define $H^N_{\partial\Omega}$ and $L^N_{\partial\Omega}$ as follows:

  $$H^N_{\partial\Omega}(g) := \partial_n H_\Omega(G^N(g)), \qquad L^N_{\partial\Omega}(g) := \partial_n L_\Omega(G^N(g)). \tag{4.1c}$$

  In particular, for both smooth domains and polygons, we have $H^N_{\partial\Omega} : H^{1/2}_{pw}\left(\partial\Omega\right) \to H^{1/2}_{pw}(\partial\Omega)$ and $L^N_{\partial\Omega} : H^{1/2}_{pw}\left(\partial\Omega\right) \to H^{1/2}_{pw}(\partial\Omega)$.

Remark 4.1 *One has significant freedom in the choice of the lifting operator G^N. Here, we selected G^N independent of k. For the Dirichlet problem in Sec. 4.3 we will select the corresponding lifting operator G^D in a k-dependent manner. This could likewise be done here and would alter the k-dependence for the "analytic" part in the decomposition result Theorem 4.9.*

Lemma 4.2 *Let $\eta > 1$ be the parameter appearing in the definition of $H_{\mathbb{R}^d}$ in (4.1a). Then, the frequency splitting via (4.1a) satisfies for all $0 \le s' \le s$ the estimates*

$$\|H_{\mathbb{R}^d} f\|_{H^{s'}(\mathbb{R}^d)} \le \tilde{C}_{s',s} (\eta k)^{s'-s} \|f\|_{H^s(\mathbb{R}^d)} \qquad \forall f \in H^s(\mathbb{R}^d), \tag{4.2}$$

$$\|H_\Omega f\|_{H^{s'}(\Omega)} \le \tilde{C}_{s',s} (\eta k)^{s'-s} \|f\|_{H^s(\Omega)} \qquad \forall f \in H^s(\Omega). \tag{4.3}$$

If $\partial\Omega$ is smooth, then the operator $H^N_{\partial\Omega}$ satisfies for $0 \le s' \le s$

$$\|H^N_{\partial\Omega} g\|_{H^{s'}(\partial\Omega)} \le C_{s',s} (\eta k)^{s'-s} \|g\|_{H^s(\partial\Omega)}. \tag{4.4}$$

For smooth or polygonal $\partial\Omega$, we have for $s' \in \{0, 1/2\}$ and $s = 1/2$

$$\|H^N_{\partial\Omega} g\|_{H^{s'}_{pw}(\partial\Omega)} \le C(\eta k)^{s'-s} \|g\|_{H^{1/2}_{pw}(\partial\Omega)}. \tag{4.5}$$

In particular, in (4.2)–(4.5) one can select, for any $s' < s$ and any $q \in (0,1)$ a parameter η such that $C\eta^{-(s-s')} \le q < 1$.

Proof. For $s \ge s'$ and $f \in H^s(\mathbb{R}^d)$ it holds

$$\begin{aligned}
\|H_{\mathbb{R}^d} f\|^2_{H^{s'}(\mathbb{R}^d)} &\le C_{s'} \int_{\mathbb{R}^d \setminus B_{\eta k}(0)} \left(1 + \|\xi\|^{2s'}\right) |\mathcal{F}(f)|^2 \\
&\le C_{s'} \sup_{r \ge \eta k} \frac{1 + r^{2s'}}{1 + r^{2s}} \int_{\mathbb{R}^d \setminus B_{\eta k}(0)} \left(1 + \|\xi\|^{2s}\right) |\mathcal{F}(f)|^2 \le \tilde{C}_{s',s} (\eta k)^{2(s'-s)} \|f\|^2_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

The corresponding estimate for H_Ω follows from the properties of $H_{\mathbb{R}^d}$ and the continuity properties of the Stein extension operator E_Ω. We mention in passing that this argument also works for Lipschitz domains Ω.

The estimate (4.4) for the case of smooth $\partial\Omega$ and $0 < s' \le s$ follow from the continuity properties of the trace operator. The limiting case $s' = 0$ is shown by a multiplicative trace inequality by observing that for $\zeta > 1/2$ we have $\|u\|_{L^2(\partial\Omega)} \lesssim \|u\|^{1-1/(2\zeta)}_{L^2(\Omega)} \|u\|^{1/(2\zeta)}_{H^\zeta(\Omega)}$ (see, e.g., [29, Thm. A.2] for a short proof). Using this with $\zeta := s + 1/2$ and recalling the definition of $H^N_{\partial\Omega}$ as in (4.1c)) we get

$$\begin{aligned}
\|H^N_{\partial\Omega} g\|_{L^2(\partial\Omega)} &\lesssim \|\nabla H_\Omega G^N g\|^{1-1/(2s+1)}_{L^2(\Omega)} \|\nabla H_\Omega G^N g\|^{1/(2s+1)}_{H^{s+1/2}(\Omega)} \\
&\lesssim (\eta k)^{-(s+1/2)(1-1/(2s+1))} \|G^N g\|_{H^{s+3/2}(\Omega)} \lesssim (\eta k)^{-s} \|G^N g\|_{H^{s+3/2}(\Omega)} \\
&\lesssim (\eta k)^{-s} \|g\|_{H^s(\Omega)}.
\end{aligned}$$

Finally, we consider the case of polygonal domains $\Omega \subset \mathbb{R}^2$. The result follows by the same arguments as above if one observes that the mapping $v \mapsto \partial_n v$ maps $H^2(\Omega)$ into $H^{1/2}_{pw}(\partial\Omega)$. ■

The low frequency part represents an analytic function as can be seen from the following lemma.

Lemma 4.3 *The low frequency parts of the splittings (4.1a), (4.1b) satisfy*

$$\|\nabla^p L_{\mathbb{R}^d} f\|_{L^2(\mathbb{R}^d)} \leq (\eta k)^p \|f\|_{L^2(\mathbb{R}^d)} \quad \forall p \in \mathbb{N}_0 \quad \forall f \in L^2(\mathbb{R}^d), \tag{4.6}$$

$$\|\nabla^p L_\Omega f\|_{L^2(\Omega)} \leq C (\eta k)^p \|f\|_{L^2(\Omega)} \quad \forall p \in \mathbb{N}_0 \quad \forall f \in L^2(\Omega). \tag{4.7}$$

The constant C in (4.7) is independent of p, η, and k. If $f \in H^s(\Omega)$ for some $s \geq 0$, then the following stronger estimates are valid:

$$\|\nabla^p L_\Omega f\|_{L^2(\Omega)} \leq C (\eta k)^{p-s} \|f\|_{H^s(\Omega)} \quad \forall f \in H^s(\Omega) \qquad \forall p \in \mathbb{N}_0, \quad p \geq s. \tag{4.8}$$

Again, the constant $C > 0$ is independent of p, η, and k.

For $s > 0$ the operator $L^N_{\partial\Omega}$ is obtained as the normal trace to $\partial\Omega$ of an entire function, viz., $L^N_{\partial\Omega} g = n \cdot \nabla L_\Omega(G^N g)|_{\partial\Omega}$, where the entire function $L_\Omega G^N g$ satisfies:

- *if $\partial\Omega$ is smooth and $g \in H^s(\partial\Omega)$ for some $s > 0$, then*

$$\|L_\Omega G^N g\|_{H^{3/2+s}(\Omega)} \lesssim \|g\|_{H^s(\partial\Omega)},$$
$$\|\nabla^p L_\Omega G^N g\|_{L^2(\Omega)} \lesssim (\eta k)^{p-3/2-s} \|g\|_{H^s(\partial\Omega)} \qquad \forall p \in \mathbb{N}_0, \quad p \geq s + 3/2,$$

- *if Ω is a polygon, then*

$$\|L_\Omega G^N g\|_{H^2(\Omega)} \lesssim \|g\|_{H^{1/2}_{pw}(\partial\Omega)},$$
$$\|\nabla^{p+2} L_\Omega G^N g\|_{L^2(\Omega)} \lesssim (\eta k)^p \|g\|_{H^{1/2}_{pw}(\partial\Omega)} \qquad \forall p \in \mathbb{N}_0. \tag{4.9}$$

In particular for analytic boundaries $\partial\Omega$ we have that $L^N_{\partial\Omega} g$ is an analytic function and for polygonal Ω, the function $L^N_{\partial\Omega} g$ is piecewise analytic on $\partial\Omega$.

Proof. We recall the multinomial formula $\sum_{|\alpha|=n} \frac{n!}{\alpha!} \prod_{i=1}^d \xi_i^{2\alpha_i} = \left(\sum_{i=1}^d \xi_i^2\right)^n$. Then, by Parseval's relation we have, for all $p \in \mathbb{N}_0$,

$$\begin{aligned} \|\nabla^p L_\Omega f\|_{L^2(\Omega)} &\leq \|\nabla^p L_{\mathbb{R}^d} E_\Omega f\|_{L^2(\mathbb{R}^d)} = \sqrt{\int_{B_{\eta k}(0)} \|\xi\|^{2p} |\mathcal{F}(E_\Omega f)|^2} \\ &\leq \sqrt{\int_{B_{\eta k}(0)} \|\xi\|^{2(p-s)} \left|\|\xi\|^s \mathcal{F}(E_\Omega f)\right|^2} \leq C (\eta k)^{p-s} \|f\|_{H^s(\Omega)}. \end{aligned} \tag{4.10}$$

The estimates for $L_\Omega G^N g$ follow by definition. ■

Note that the statements of Lemma 4.2 and 4.3 imply that the splittings $f = L_\Omega f + H_\Omega f$ and $g = L^N_{\partial\Omega} g + H^N_{\partial\Omega}$ are *stable* in appropriate scales of Sobolev norms.

4.1.2 Properties of the Solution Operators N_k and S_k^Δ

We consider the regularity for two types of problems.

1.) The Helmholtz problem in the full space $\mathbb{R}^d$ with Sommerfeld radiation condition is given by: Find $U \in H^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$\left(-\Delta - k^2\right) U = f \qquad \text{in } \mathbb{R}^d \tag{4.11}$$

and the *Sommerfeld radiation condition* (2.5b) are satisfied in a weak sense (cf. [31]). Here, $\partial/\partial r$ denotes the derivative in radial direction $\mathbf{x}/\|\mathbf{x}\|$. We assume that f is local so that $\operatorname{supp} f \subset B_R$ for $R > 0$, where B_R denotes the ball with radius R about the origin. It can be shown that the problem (4.11), (2.5b) has a unique solution $U \in H^1_{\text{loc}}\left(\mathbb{R}^d\right)$ in a weak sense (cf. [24]). The solution can be represented as a acoustic volume potential

$$U(x) := (N_k f)(x) := \int_{\mathbb{R}^d} G_k(x-y) f(y) \, dy \qquad \forall x \in \mathbb{R}^d, \tag{4.12}$$

where

$$G_k(z) := \begin{cases} -\frac{\mathrm{e}^{\mathrm{i}\, k|z|}}{2\,\mathrm{i}\, k} & d=1, \\ \frac{\mathrm{i}}{4} H_0^{(1)}(k\|z\|) & d=2, \\ \frac{\mathrm{e}^{\mathrm{i}\, k\|z\|}}{4\pi\|z\|} & d=3. \end{cases}$$

In [30], it is explained that $(N_k f)|_\Omega$ is the solution operator on finite domains Ω if Dirichlet-to-Neumann boundary conditions are imposed at $\partial\Omega$.

2.) For a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary let S_k^Δ be the solution operator for the modified Helmholtz equation with Robin boundary conditions, i.e., $u = S_k^\Delta(g)$ solves

$$-\Delta u + k^2 u = 0, \quad \text{in } \Omega, \qquad \partial_n u - \mathrm{i} k u = g \quad \text{on } \partial\Omega. \tag{4.13}$$

In the following we will analyze the regularity properties of the solution operators N_k and S_k^Δ. The following lemma is a direct consequence of [30, Lemma 3.4].

Lemma 4.4 (properties of N_k) *For $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_R$, the function $u := N_k(f)$ satisfies $-\Delta u - k^2 u = f$ on B_R. Additionally, for every $q \in (0,1)$ one can select $\eta > 1$ (appearing in the definition of the operator $H_{\mathbb{R}^d}$ as in (4.1a)) such that*

$$\|N_k(H_{\mathbb{R}^d} f)\|_{\mathcal{H},B_R} \leq k^{-1} q \|f\|_{L^2(\mathbb{R}^d)}, \tag{4.14a}$$

$$\|N_k(H_{\mathbb{R}^d} f)\|_{H^2(B_R)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}. \tag{4.14b}$$

Lemma 4.5 (properties of S_k^Δ) *Let Ω be a bounded Lipschitz domain. For $g \in H^{-1/2}(\partial\Omega)$, let $u = S_k^\Delta g$ denote the solution of (4.13). Then:*

$$\|u\|_{\mathcal{H},\Omega} \lesssim \|g\|_{H^{-1/2}(\partial\Omega)}, \tag{4.15}$$

$$\|u\|_{\mathcal{H},\Omega} \lesssim k^{-1/2} \|g\|_{L^2(\partial\Omega)}, \tag{4.16}$$

$$\|u\|_{L^2(\partial\Omega)} \lesssim k^{-1} \|g\|_{L^2(\partial\Omega)}. \tag{4.17}$$

If $\partial\Omega$ is sufficiently smooth or if Ω is a convex polygon (in 2D), then the following shift theorem is true: If $g \in H^{1/2}(\partial\Omega)$ if $\partial\Omega$ is smooth or if $g \in H^{1/2}_{pw}(\partial\Omega)$ if Ω a convex polygon, then

$$\|u\|_{H^2(\Omega)} \lesssim \|g\|_{H^{1/2}_{pw}(\partial\Omega)} + k^{1/2} \|g\|_{L^2(\partial\Omega)}. \tag{4.18}$$

Proof. The function u satisfies

$$\int_\Omega \nabla u \cdot \nabla \overline{v} + k^2 \int_\Omega u \overline{v} - \mathrm{i} k \int_{\partial\Omega} u \overline{v} = \int_{\partial\Omega} g \overline{v} \qquad \forall v \in H^1(\Omega).$$

Taking $v = u$ and considering the real and imaginary parts separately yields immediately the bounds (4.15), (4.17), (4.16).

Since u satisfies

$$-\Delta u + k^2 u = 0, \qquad \partial_n u = g + \mathrm{i}ku$$

the standard shift theorem (which is applicable for smooth $\partial\Omega$ and convex polygons with piecewise $H^{1/2}$-Neumann data, [16, Cor. 4.4.3.8]) gives

$$\|u\|_{H^2(\Omega)} \lesssim k^2\|u\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)} + k\|u\|_{H^{1/2}(\partial\Omega)}.$$

Using (4.16) we get (4.18). ∎

Lemma 4.6 (properties of $S_k^\Delta \circ H^N_{\partial\Omega}$) *Let Ω have a smooth boundary or let Ω be a convex polygon. Let $q \in (0,1)$, and let S_k^Δ be the solution operator for (4.13). Then there exists $\eta > 1$ defining the high frequency filter $H^N_{\partial\Omega}$ such that for every $g \in H^{1/2}_{pw}(\partial\Omega)$ there holds*

$$\begin{aligned} \|S_k^\Delta(H^N_{\partial\Omega} g)\|_{\mathcal{H},\Omega} &\leq qk^{-1}\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \\ \|S_k^\Delta(H^N_{\partial\Omega} g)\|_{H^2(\Omega)} &\lesssim \|g\|_{H^{1/2}_{pw}(\partial\Omega)}. \end{aligned}$$

Proof. The combination of (4.16) and Lemma 4.2 gives the first estimate. The second estimate follows from (4.18) and, again, Lemma 4.2. ∎

4.2 The Case of a Bounded Domain with Robin Boundary Conditions

We consider the following problem:

$$-\Delta u - k^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^d, \qquad \partial_n u - \mathrm{i}ku = g \quad \text{on } \partial\Omega. \tag{4.19}$$

Assumption 4.7 *The solution operator $(f,g) \mapsto u := S_k(f,g)$ for (4.19) grows only polynomially in k:*

$$\|u\|_{\mathcal{H},\Omega} \lesssim k^\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right) \tag{4.20}$$

for some $\alpha \geq 0$.

Remark 4.8 *Let Assumption 1.1 be valid. Then it is proved in [26] that (4.20) holds with $\alpha = 0$ if Ω is star-shaped with respect to a ball.*

The goal of this section is the proof of the following result:

Theorem 4.9 (decomposition for bounded domain) *Let Assumptions 1.1 and 4.7 be valid and let $\eta > 1$ appearing in the definition of L_Ω and $L^N_{\partial\Omega}$ be fixed. Then there exist constants C, $\gamma > 0$, $\vec{\beta} \in [0,1)^J$ independent of k such that for every $f \in L^2(\Omega)$ and $g \in H^{1/2}_{pw}(\partial\Omega)$ the solution $u = S_k(f,g)$ can be written as $u = u_{\mathcal{A}} + u_{H^2}$, where, for all $p \in \mathbb{N}_0$, it holds*

$$\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} \leq Ck^\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)}\right), \tag{4.21a}$$

$$\|\Phi_{p,\vec{\beta},k} \nabla^{p+2} u_{\mathcal{A}}\|_{L^2(\Omega)} \leq C\gamma^p k^{\alpha-1} \max\{p,k\}^{p+2} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)}\right), \tag{4.21b}$$

$$\|u_{H^2}\|_{H^2(\Omega)} + k\|u_{H^2}\|_{\mathcal{H},\Omega} \leq C\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)}\right). \tag{4.21c}$$

Concerning the weight functions $\Phi_{p,\vec{\beta},k}$, we remind the reader of our convention introduced in Section 1.1, namely, $\Phi_{p,\vec{\beta},k} \equiv 1$ if Ω has an analytic boundary.

Proof. The proof is based on Lemmata 4.14, 4.15 below. By linearity of the operator S_k it suffices to consider the decomposition of $u = S_k(f, 0)$ and $u = S_k(0, g)$ separately. Writing $f^{(0)} := f$ we get for $S_k(f^{(0)}, 0)$ from Lemma 4.14 that

$$u = u_{\mathcal{A}}^{(0)} + u_{H^2}^{(0)} + S_k(f^{(1)}, 0) \quad \text{for some } f^{(1)} \in L^2(\Omega),$$

where $u_{\mathcal{A}}^{(0)}$, $u_{H^2}^{(0)}$ satisfy the desired bounds and $\|f^{(1)}\|_{L^2(\Omega)} \leq q\|f^{(0)}\|_{L^2(\Omega)}$ for some $q \in (0, 1)$. Hence, we may iterate and can write u as a sum of series (one of analytic functions and one of H^2-functions) that can be bounded (in appropriate norms) by geometric series. For the decomposition of $S_k(0, g)$ we proceed completely analogously. ■

Remark 4.10 *For the case of polygonal Ω the Theorem 4.9 merely asserts the existence of $\overrightarrow{\beta} \in [0, 1)^J$ with the stated properties. The proof of Lemmata 4.14, 4.15 relies on [27]. A closer inspection of the proofs there reveals that for convex Ω, any $\overrightarrow{\beta} \in (0, 1)^J$ may be chosen.*

In view of Lemma 3.1, the following corollary is evident:

Corollary 4.11 *Under the hypotheses of Theorem 4.9, the statement of Theorem 4.9 holds verbatim for the adjoint solution operator $(f, g) \mapsto S_k^\star(f, g)$ (see (3.4)).*

Lemma 4.12 (analyticity of $S_k(L_\Omega f, L_{\partial\Omega}^N g)$) *Let Assumption 1.1 be valid. Then there exist constants C, $K > 0$, $\overrightarrow{\beta} \in [0, 1)^J$ independent of k such that, for every $g \in H_{pw}^{1/2}(\partial\Omega)$ and $f \in L^2(\Omega)$, the function $u_{\mathcal{A}} = S_k(L_\Omega f, L_{\partial\Omega}^N g)$ is analytic on Ω and satisfies for all $p \in \mathbb{N}_0$ the estimates*

$$\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} \leq Ck^\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{H_{pw}^{1/2}(\partial\Omega)}\right), \tag{4.22}$$

$$\|\Phi_{p,\overrightarrow{\beta},k} \nabla^{p+2} u_{\mathcal{A}}\|_{L^2(\Omega)} \leq CK^p \max\{k, p+2\}^{p+2} k^{\alpha-1} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H_{pw}^{1/2}(\partial\Omega)}\right). \tag{4.23}$$

Proof. We first restrict our attention here to the case of polygonal Ω with edges Γ_j, $j = 1, \ldots, N_\Gamma$, and remark on the case of analytic $\partial\Omega$ at the end of the proof.

Let $u := S_k(L_\Omega f, L_{\partial\Omega}^N g)$. Set $\widetilde{f} := L_\Omega f$ and $\widetilde{g} := L_{\partial\Omega}^N g = \partial_n L_\Omega G^N g$. From Lemma 4.3 we have that $\widetilde{f}$ is an entire function. Note that for any Γ_j there exists an open neighborhood T_j of $\overline{\Gamma_j}$ such that the normal $n_j : \Gamma_j \to \mathbb{S}_1$ can be extended to an analytic function $n_j^\star : T_j \to \mathbb{R}^2$ (in the present case of a polygon, this is trivial since n_j is a constant vector). We set $G_j := \langle n_j^\star, \nabla L_\Omega G^N g\rangle$ and assume that the open neighborhood T_j of $\overline{\Gamma_j}$ is such that G_j is analytic on T_j (in view of Lemma 4.3, which asserts that G_j is an entire function, this is again trivial). We note $G_j|_{\Gamma_j} = \widetilde{g}$. Furthermore, from Lemma 4.3, we have the following estimates:

$$\|\nabla^p \tilde{f}\|_{L^2(\Omega)} \lesssim (\eta k)^p \|f\|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0, \tag{4.24}$$

$$\|G_j\|_{L^2(\Omega\cap T_j)} \leq \|\nabla L_\Omega G^N g\|_{L^2(\Omega)} \leq \|L_\Omega G^N g\|_{H^2(\Omega)} \lesssim \|g\|_{H_{pw}^{1/2}(\partial\Omega)}, \tag{4.25}$$

$$\|\nabla^{p+1} G_j\|_{L^2(\Omega\cap T_j)} \lesssim \|\nabla^{p+2} L_\Omega G^N g\|_{L^2(\Omega)} \overset{(4.9)}{\lesssim} (\eta k)^p \|g\|_{H_{pw}^{1/2}(\partial\Omega)} \qquad \forall p \in \mathbb{N}_0. \tag{4.26}$$

The bounds (4.25), (4.26) for $p = 0$ together with the multiplicative trace inequality give $\|\tilde{g}\|_{L^2(\partial\Omega)} \lesssim \|g\|_{H_{pw}^{1/2}(\partial\Omega)}$. This bound together with (4.24) and Assumption 4.7 implies (4.22).

The regularity estimate (4.23) will be derived by applying [27, Prop. 5.4.5, Rem. 5.4.6] and estimating the constants therein. To that end, we set $\varepsilon := 1/k$ and note that $u_{\mathcal{A}}$ solves

$$
\begin{aligned}
-\varepsilon^2 \Delta u - u &= \varepsilon^2 \tilde{f} \quad \text{on } \Omega, \\
\varepsilon^2 \partial_n u &= \varepsilon(\varepsilon \tilde{g} + \mathrm{i}\, u) \quad \text{on } \partial\Omega
\end{aligned}
$$

Then [27, Prop. 5.4.5] is applicable with

$$
\begin{array}{lllll}
C_f = \varepsilon^2 \|f\|_{L^2(\Omega)} O(1), & C_{G_1} = \varepsilon \|g\|_{H^{1/2}(\partial\Omega)} O(1), & C_{G_2} = \varepsilon, & C_b = 0, & C_c = 1, \\
\gamma_f = O(1), & \gamma_{G_1} = O(1), & \gamma_{G_2} = O(1), & \gamma_b = 0, & \gamma_c = 0,
\end{array}
$$

resulting in the existence of constants C, $K > 0$ and $\overrightarrow{\beta} \in [0,1)^J$ with

$$
\|\Phi_{p,\overrightarrow{\beta},k} \nabla^{p+2} S_k u\|_{L^2(\Omega)} \lesssim K^{p+2} \max\{p+2, k\}^{p+2} \left(k^{-2} \|f\|_{L^2(\Omega)} + k^{-1} \|u\|_{\mathcal{H},\Omega} + k^{-1} \|g\|_{H^{1/2}_{pw}(\partial\Omega)} \right)
$$

for all $p \in \mathbb{N}_0$. Inserting (4.22) and using $\alpha \geq 0$, we arrive at (4.23).

For the case of analytic $\partial\Omega$, we proceed analogously. The main difference is that is suffices to consider a single tubular neighborhood T of $\partial\Omega$ and that the analytic extension $n^\star$ of the normal vector is no longer constant on T. Therefore, the estimate (4.26) (we write G instead of G_j) is replaced with

$$
\|\nabla^{p+1} G\|_{L^2(\Omega \cap T)} \lesssim \gamma^p \max\{p, \eta k\}^p \|g\|_{H^{1/2}_{pw}(\partial\Omega)} \quad \forall p \in \mathbb{N}_0
$$

for a constant γ that reflects the size of the domain of analyticity of $n^\star$. The remainder of the proof follows the above arguments but appeals to [27, Remark 5.4.6]. ■

Remark 4.13 *The k-dependence in the estimates of Lemma 4.12 is likely to be suboptimal for several reasons. We treated the contributions stemming from the boundary data g in a rather generous way to treat the case of domains with analytic boundary and polygons in a unified way. However, sharper estimates are available for the lifting G^N for the case of smooth domains than for the polygonal case, and therefore sharper estimates are possible for the case of analytic boundaries.*

One important motivation for our choice of the formulation of Assumption 4.7 was the fact that it holds for a class of practically relevant situations.

It will turn out that the function $u_{\mathcal{A}}$ can be approximated exponentially well, e.g., by hp finite elements so that the algebraic growth of the regularity constant can be absorbed by the exponential convergence factor and the exponential convergence of this part of the solution is preserved.

Lemma 4.14 (properties of $S_k(f,0)$) *Let Assumptions 1.1 and 4.7 be valid. Let $q \in (0,1)$. Then there exist constants C, $K > 0$, $\overrightarrow{\beta} \in [0,1)^J$ independent of k such that for every $f \in L^2(\Omega)$ the function $u = S_k(f,0)$ can be written as $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$, where*

$$
\begin{aligned}
\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} &\leq C k^\alpha \|f\|_{L^2(\Omega)}, \\
\|\Phi_{p,\overrightarrow{\beta},k} \nabla^{p+2} u_{\mathcal{A}}\|_{L^2(\Omega)} &\leq C k^{\alpha-1} K^p \max\{p+2, k\}^{p+2} \|f\|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0, \\
\|u_{H^2}\|_{\mathcal{H},\Omega} &\leq q k^{-1} \|f\|_{L^2(\Omega)}, \\
\|u_{H^2}\|_{H^2(\Omega)} &\leq C \|f\|_{L^2(\Omega)},
\end{aligned}
$$

and the remainder $\widetilde{u} = S_k(\widetilde{f}, 0)$ *satisfies*

$$-\Delta\widetilde{u} - k^2\widetilde{u} = \widetilde{f}, \qquad \partial_n\widetilde{u} - \mathrm{i}k\widetilde{u} = 0,$$

where

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq q\|f\|_{L^2(\Omega)}.$$

Proof. Define

$$u^{\mathrm{I}}_{\mathcal{A}} := S_k(L_\Omega f, 0), \qquad u^{\mathrm{I}}_{H^2} := N_k(H_\Omega f).$$

Here, the parameter η defining the filter operators L_Ω and H_Ω is still at our disposal and will be selected at the end of the proof. Then, $u^{\mathrm{I}}_{\mathcal{A}}$ satisfies the desired bounds by Lemma 4.12. Lemma 4.4 gives

$$\begin{aligned}\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} &\leq q'k^{-1}\|f\|_{L^2(\Omega)},\\ \|u^{\mathrm{I}}_{H^2}\|_{H^2(\Omega)} &\lesssim \|f\|_{L^2(\Omega)}.\end{aligned}$$

Here, the parameter $q' \in (0,1)$ depends on η and is still at our disposal.

The function $u^{\mathrm{I}} := u - (u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{I}}_{H^2})$ solves

$$-\Delta u^{\mathrm{I}} - k^2 u^{\mathrm{I}} = 0, \qquad \partial_n u^{\mathrm{I}} - \mathrm{i}ku^{\mathrm{I}} = \mathrm{i}ku^{\mathrm{I}}_{H^2} - \partial_n u^{I}_{H^2}. \tag{4.27}$$

We note with the multiplicative trace inequality

$$\begin{aligned}\|\mathrm{i}ku^{\mathrm{I}}_{H^2}\|_{L^2(\partial\Omega)} &\lesssim k\|u^{\mathrm{I}}_{H^2}\|^{1/2}_{L^2(\Omega)}\|u^{\mathrm{I}}_{H^2}\|^{1/2}_{H^1(\Omega)} \lesssim k^{1/2}\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim q'k^{-1/2}\|f\|_{L^2(\Omega)}, &&(4.28\text{a})\\ \|\mathrm{i}ku^{\mathrm{I}}_{H^2}\|_{H^{1/2}(\partial\Omega)} &\lesssim k\|u^{\mathrm{I}}_{H^2}\|_{H^1(\Omega)} \lesssim q'\|f\|_{L^2(\Omega)}, &&(4.28\text{b})\\ \left\|\partial_n u^{\mathrm{I}}_{H^2}\right\|_{L^2(\partial\Omega)} &\lesssim \left\|\nabla u^{\mathrm{I}}_{H^2}\right\|^{1/2}_{L^2(\Omega)}\left\|u^{\mathrm{I}}_{H^2}\right\|^{1/2}_{H^2(\Omega)} \lesssim \sqrt{\frac{q'}{k}}\,\|f\|_{L^2(\Omega)}, &&(4.28\text{c})\\ \left\|\partial_n u^{\mathrm{I}}_{H^2}\right\|_{H^{1/2}_{pw}(\partial\Omega)} &\lesssim \left\|u^{\mathrm{I}}_{H^2}\right\|_{H^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}. &&(4.28\text{d})\end{aligned}$$

This implies in particular

$$\|\,\mathrm{i}\,ku^{\mathrm{I}}_{H^2} - \partial_n u^{\mathrm{I}}_{H^2}\|_{H^{1/2}_{pw}(\partial\Omega)} \lesssim \|f\|_{L^2(\Omega)}. \tag{4.29}$$

Next, we define the functions $u^{\mathrm{II}}_{\mathcal{A}}$ and $u^{\mathrm{II}}_{H^2}$ by

$$\begin{aligned}u^{\mathrm{II}}_{\mathcal{A}} &: = S_k(0, L^N_{\partial\Omega}(\mathrm{i}ku^{\mathrm{I}}_{H^2} - \partial_n u^{\mathrm{I}}_{H^2})),\\ u^{\mathrm{II}}_{H^2} &: = S^{\Delta}_k(H^N_{\partial\Omega}(\mathrm{i}ku^{\mathrm{I}}_{H^2} - \partial_n u^{\mathrm{I}}_{H^2})).\end{aligned}$$

Then, the analytic part $u^{\mathrm{II}}_{\mathcal{A}}$ satisfies again the desired analyticity bounds by Lemma 4.12. For the function $u^{\mathrm{II}}_{H^2}$ we obtain from Lemma 4.6 the estimates

$$\begin{aligned}\|u^{\mathrm{II}}_{H^2}\|_{\mathcal{H},\Omega} &\leq q'k^{-1}\|\mathrm{i}ku^{\mathrm{I}}_{H^2} - \partial_n u^{\mathrm{I}}_{H^2}\|_{H^{1/2}_{pw}(\partial\Omega)} \lesssim q'k^{-1}\|f\|_{L^2(\Omega)},\\ \|u^{\mathrm{II}}_{H^2}\|_{H^2(\Omega)} &\lesssim \|\mathrm{i}ku^{\mathrm{I}}_{H^2} - \partial_n u^{\mathrm{I}}_{H^2}\|_{H^{1/2}_{pw}(\partial\Omega)} \lesssim \|f\|_{L^2(\Omega)}.\end{aligned}$$

We now set $u_{\mathcal{A}} := u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{II}}_{\mathcal{A}}$ and $u_{H^2} := u^{\mathrm{I}}_{H^2} + u^{\mathrm{II}}_{H^2}$ and conclude for the function $\widetilde{u} := u - (u_{\mathcal{A}} + u_{H^2})$ that it satisfies

$$-\Delta\widetilde{u} - k^2\widetilde{u} = \widetilde{f} := 2k^2 u^{\mathrm{II}}_{H^2}, \qquad \partial_n\widetilde{u} - \mathrm{i}k\widetilde{u} = 0.$$

For $\widetilde{f}$ we compute

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq 2k\|u^{\mathrm{II}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim q'\|f\|_{L^2(\Omega)}.$$

Hence, by selecting q' sufficiently small, we arrive at the desired bound. ∎

Lemma 4.15 *Let Assumptions 1.1, 4.7 be valid. Let $q \in (0,1)$. Then there exist constants C, $\gamma > 0$, $\overrightarrow{\beta} \in [0,1)^J$ independent of k such that for every $g \in H^{1/2}_{pw}(\partial\Omega)$ the function $u = S_k(0,g)$ can be written as $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$, where*

$$\begin{aligned} \|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} &\leq Ck^{\alpha}\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \\ \|\Phi_{p,\overrightarrow{\beta},k}\nabla^{p+2}u_{\mathcal{A}}\|_{L^2(\Omega)} &\leq Ck^{\alpha-1}\gamma^p \max\{p+2,k\}^{p+2}\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \\ \|u_{H^2}\|_{\mathcal{H},\Omega} &\leq qk^{-1}\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \\ \|u_{H^2}\|_{H^2(\Omega)} &\leq C\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \end{aligned}$$

and the remainder $\widetilde{u} = S_k(0,\widetilde{g})$ satisfies

$$-\Delta\widetilde{u} - k^2\widetilde{u} = 0 \qquad \partial_n\widetilde{u} - \mathrm{i}k\widetilde{u} = \widetilde{g},$$

where

$$\|\widetilde{g}\|_{H^{1/2}_{pw}(\partial\Omega)} \leq q\|g\|_{H^{1/2}_{pw}(\partial\Omega)}.$$

Proof. The proof is very similar to that of Lemma 4.14. Define

$$\begin{aligned} u^{\mathrm{I}}_{\mathcal{A}} &:= S_k(0, L^N_{\partial\Omega}g), \\ u^{\mathrm{I}}_{H^2} &:= S^{\Delta}_k(H^N_{\partial\Omega}g), \end{aligned}$$

where S^{Δ}_k is the solution operator for (4.13). Then $u^{\mathrm{I}}_{\mathcal{A}}$ is analytic and satisfies the desired analyticity estimates by Lemma 4.12. For $u^{\mathrm{I}}_{H^2}$ we have by Lemma 4.6

$$\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \leq q'k^{-1}\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \tag{4.30}$$

$$\|u^{\mathrm{I}}_{H^2}\|_{H^2(\Omega)} \lesssim \|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \tag{4.31}$$

where the parameter $q' < 1$ is at our disposal and depends on η defining $H^N_{\partial\Omega}$ and $L^N_{\partial\Omega}$. Hence, the function $u^{\mathrm{I}} := u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{I}}_{H^2}$ satisfies

$$-\Delta u^{\mathrm{I}} - k^2 u^{\mathrm{I}} = -2k^2 u^{\mathrm{I}}_{H^2}, \qquad \partial_n u^{\mathrm{I}} - \mathrm{i}ku^{\mathrm{I}} = g$$

together with

$$\|2k^2 u^{\mathrm{I}}_{H^2}\|_{L^2(\Omega)} \lesssim k\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim q'\|g\|_{H^{1/2}_{pw}(\partial\Omega)}. \tag{4.32}$$

Next, we define $u^{\mathrm{II}}_{\mathcal{A}}$ and $u^{\mathrm{II}}_{H^2}$ by

$$\begin{aligned} u^{\mathrm{II}}_{\mathcal{A}} &:= S_k(L_\Omega(2k^2u^{\mathrm{I}}_{H^2}), 0), \\ u^{\mathrm{II}}_{H^2} &:= N_k(H_\Omega(2k^2u^{\mathrm{I}}_{H^2})). \end{aligned}$$

Here, in order to apply the operator N_k, we extend $H_\Omega\left(2k^2u^{\mathrm{I}}_{H^2}\right)$ by zero outside of Ω. By Lemma 4.12 and (4.32), we see that $u^{\mathrm{II}}_{\mathcal{A}}$ satisfies the desired analyticity estimates. For the function $u^{\mathrm{II}}_{H^2}$, we obtain from Lemma 4.4

$$\begin{aligned} \|u^{\mathrm{II}}_{H^2}\|_{\mathcal{H},\Omega} &\leq q'k^{-1}\|2k^2u^{\mathrm{I}}_{H^2}\|_{L^2(\Omega)} \lesssim q'\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim {q'}^2k^{-1}\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \\ \|u^{\mathrm{II}}_{H^2}\|_{H^2(\Omega)} &\lesssim \|2k^2u^{\mathrm{I}}_{H^2}\|_{L^2(\Omega)} \lesssim k\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim q'\|g\|_{H^{1/2}_{pw}(\partial\Omega)}. \end{aligned}$$

We set $u_{\mathcal{A}} := u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{II}}_{\mathcal{A}}$ and $u_{H^2} := u^{\mathrm{I}}_{H^2} + u^{\mathrm{II}}_{H^2}$. Then $u_{\mathcal{A}}$ and u_{H^2} satisfy the desired estimates and $\widetilde{u} := u - (u_{\mathcal{A}} + u_{H^2})$ satisfies

$$-\Delta\widetilde{u} - k^2\widetilde{u} = 0, \qquad \partial_n\widetilde{u} - \mathrm{i}k\widetilde{u} = \widetilde{g} := \mathrm{i}ku^{\mathrm{II}}_{H^2} - \partial_n u^{\mathrm{II}}_{H^2}$$

with

$$\begin{aligned}\|\widetilde{g}\|_{H^{1/2}_{pw}(\partial\Omega)} &\leq k\|u^{\mathrm{II}}_{H^2}\|_{H^{1/2}(\partial\Omega)} + \left\|\partial_n u^{\mathrm{II}}_{H^2}\right\|_{H^{1/2}_{pw}(\Omega)} \\ &\lesssim k\|u^{\mathrm{II}}_{H^2}\|_{\mathcal{H},\Omega} + \left\|u^{\mathrm{II}}_{H^2}\right\|_{H^2(\Omega)} \lesssim q'\|g\|_{H^{1/2}_{pw}(\partial\Omega)}.\end{aligned}$$

The result follows by selecting q' sufficiently small. ■

4.3 The Exterior Dirichlet Problem

In the present section, we study the problem (2.5), or, equivalently, (2.7) of Section 2.2, which we recall for convenience:

$$-\Delta u - k^2 u = f \quad \text{in } \Omega^c, \qquad u|_{\partial\Omega} = g, \tag{4.33a}$$

$$\left|\frac{\partial u}{\partial r} - \mathrm{i}\,ku\right| = o\left(\|\mathbf{x}\|^{\frac{1-d}{2}}\right) \qquad \text{as } \|\mathbf{x}\| \to \infty. \tag{4.33b}$$

Assumption 4.16 *1. $\partial\Omega$ analytic*

2. $\operatorname{supp} f \subset B_R$ *for fixed R.*

We recall that the solution operator S^c_k for problem (2.7) and the adjoint solution operator $S^{c,\star}_k$ have been introduced in Lemma 3.1. Concerning the mapping properties of S^c_k, we will make a polynomial growth assumption:

Assumption 4.17 *The solution operator S^c_k for the Helmholtz problem (2.7) grows only polynomially in k:*

$$\|u\|_{\mathcal{H},R} \lesssim k^\alpha \left(\|f\|_{L^2(\Omega^c_R)} + k\|g\|_{1/2,\mathcal{H},\partial\Omega}\right) \tag{4.34}$$

for some $\alpha \geq 0$ where

$$\|v\|^2_{\mathcal{H},R} := k^2\|v\|^2_{L^2(\Omega^c_R)} + |v|^2_{H^1(\Omega^c_R)}.$$

Remark 4.18 *Assumption 4.17 is true with $\alpha = 0$ for star-shaped Ω. This is shown for the case $g = 0$ in [10, Lemma 3.5]. The case $g \neq 0$ can be reduced to the case $g = 0$ via a lifting argument in the standard way: Given $g \in H^{1/2}(\partial\Omega)$, Lemma 4.21 below gives a function u_g with $u_g|_{\partial\Omega} = g$ and $-\Delta u_g - k^2 u_g = -2k^2 u_g$ on $\Omega^c \cap B_{2R}$. Using a suitable cut-off function χ, the function $\tilde{u} := \chi u_g$ satisfies $\tilde{u}|_{\partial\Omega} = g$, $\tilde{u} \equiv 0$ outside a ball of radius R, $\|\tilde{u}\|_{\mathcal{H},R} \lesssim \|g\|_{1/2,\mathcal{H},\partial\Omega}$ and $\|-\Delta\tilde{u} - k^2\tilde{u}\|_{L^2(\Omega^c_R)} \lesssim k\|u_g\|_{\mathcal{H},R} \lesssim k\|g\|_{1/2,\mathcal{H},\partial\Omega}$.*

The precise k-dependence of S^c_k is hard in general. Sharper bounds than the ones stipulated in Assumption 4.17 are available for special geometries, e.g., circles and sphere in [31, Thm. 2.6.2].

The main result of this section is a decomposition result for the solution of (2.7). We show in Theorem 4.19 that the solution can be decomposed into a part with finite Sobolev regularity and k-independent regularity constants and an analytic part that features k-explicit bounds for all derivatives.

Theorem 4.19 (decomposition for exterior Dirichlet problem) *Let Assumptions 4.16 and 4.17 be true. For $f \in L^2(\Omega_R^c)$ and $g \in H^{3/2}(\partial\Omega)$ the solution $u = S_k^c(f,g)$ of (2.5a) admits a decomposition $u = u_{\mathcal{A}} + u_{H^2}$, where for all $p \geq 2$*

$$\begin{aligned}
\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega_R^c} &\lesssim k^\alpha \left(\|f\|_{L^2(\Omega_R^c)} + k\|g\|_{1/2,\mathcal{H},\partial\Omega}\right), \\
\|\nabla^p u_{\mathcal{A}}\|_{L^2(\Omega_R^c)} &\lesssim \gamma^p \max\{p,k\}^p \left(k^{\alpha-1}\|f\|_{L^2\left(\Omega_R^c\right)} + (k+k^\alpha)\|g\|_{1/2,\mathcal{H},\partial\Omega}\right), \\
\|u_{H^2}\|_{\mathcal{H},R} &\lesssim k^{-1}\|f\|_{L^2\left(\Omega_R^c\right)} + \|g\|_{1/2,\mathcal{H},\partial\Omega}, \\
\|u_{H^2}\|_{H^2(\Omega_R^c)} &\lesssim \|f\|_{L^2\left(\Omega_R^c\right)} + k\|g\|_{3/2,\mathcal{H},\partial\Omega}.
\end{aligned}$$

Proof. The proof is a consequence of the lemmata of Section 4.5 by reasoning as in the proof of Theorem 4.9. ∎

Corollary 4.20 *Theorem 4.19 holds verbatim (with $g = 0$) for the adjoint solution operator $S_k^{c,\star}$ in view of Lemma 3.1.*

The following two subsections are devoted to the details of the proof of Theorem 4.19.

4.4 k-dependent Lifting operators for Dirichlet Problems

Lemma 4.21 (Lifting operator G^D from $\partial\Omega$ to $\Omega^c \cap B_{2R}$.) *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with smooth $\partial\Omega$. Then the exists a trace lifting operator*

$$G^D : H^{1/2}(\partial\Omega) \to H^1(\Omega^c \cap B_{2R})$$

such that, for any $g \in H^{1/2}(\partial\Omega)$, the function $u_g = G^D g$ solves

$$-\Delta u_g + k^2 u_g = 0, \quad \Omega^c \cap B_{2R}, \qquad u_g|_{\partial\Omega} = g, \qquad u_g|_{\partial B_{2R}} = 0$$

and satisfies for a constant C independent of k:

$$\|u_g\|_{\mathcal{H},2R} \leq C\|g\|_{1/2,\mathcal{H},\partial\Omega}. \tag{4.35}$$

If $g \in H^{3/2}(\partial\Omega)$ then additionally

$$\|u_g\|_{H^2(\Omega_R^c)} \leq Ck\|g\|_{3/2,\mathcal{H},\partial\Omega}. \tag{4.36}$$

Proof. *1. step:* We start with an estimate on a tubular neighborhood of $\partial\Omega$. For a univariate function $v \in H^1(0,1)$, we get for $\delta \in (0,1)$ from $v(x) = u(0) + \int_0^x v'(t)\,dt$ the bound

$$\|v\|_{L^2(0,\delta)} \leq C\left(\sqrt{\delta}|v(0)| + \delta\|v'\|_{L^2(0,1)}\right), \tag{4.37}$$

where the constant $C > 0$ is independent of δ. If we introduce the set $S_\delta := \{x \in \Omega^c \mid \operatorname{dist}(x,\partial\Omega) < \delta\}$, then the univariate result (4.37) implies

$$\|v\|_{L^2(S_\delta)} \leq C\left(\sqrt{\delta}\|v\|_{L^2(\partial\Omega)} + \delta\|v\|_{H^1(\Omega^c\cap B_{2R})}\right) \qquad \forall v \in H^1(\Omega^c \cap B_{2R}), \tag{4.38}$$

where, for sufficiently small δ, the constant $C > 0$ is independent of δ. Next, we select $\delta = 1/k$ and a cut-off function $\chi \in C_0^\infty(\mathbb{R}^d)$ with $\chi \equiv 1$ on $\partial\Omega$, $\|\nabla^j \chi\|_{L^\infty(\mathbb{R}^d)} \leq Ck^j$, $j \in \{0,1\}$, $\operatorname{supp}\chi \cap \Omega^c \subset \overline{S_{1/k}}$. Then, we arrive at

$$\|\chi v\|_{\mathcal{H},R} \lesssim \left(\sqrt{k}\|v\|_{L^2(\partial\Omega)} + \|v\|_{H^1(\Omega^c \cap B_{2R})}\right) \qquad \forall v \in H^1(\Omega^c \cap B_{2R}). \tag{4.39}$$

2. step: We recall that u_g is the minimizer in the $\|\cdot\|_{\mathcal{H},2R}$-norm over all functions that satisfy the boundary conditions. Let $\widetilde{u}_g$ solve

$$-\Delta \widetilde{u}_g = 0 \qquad \text{on } \Omega^c \cap B_{2R}, \quad \widetilde{u}_g|_{\partial\Omega} = g, \qquad \widetilde{u}_g|_{\partial B_{2R}} = 0.$$

Then

$$\|\widetilde{u}_g\|_{H^1(\Omega^c \cap B_{2R})} \lesssim \|g\|_{H^{1/2}(\partial\Omega)}.$$

In view of (4.39) the function $\chi\widetilde{u}_g$ satisfies

$$\|\chi\widetilde{u}_g\|_{\mathcal{H},2R} \lesssim \sqrt{k}\|g\|_{L^2(\partial\Omega)} + \|\widetilde{u}_g\|_{H^1(\Omega^c \cap B_{2R})} \lesssim \|g\|_{1/2,\mathcal{H},\partial\Omega}.$$

This shows the bound for $\|u_g\|_{\mathcal{H},2R}$.
3. step: To get the H^2 estimate, we use elliptic regularity to conclude

$$\|u_g\|_{H^2(\Omega^c \cap B_{2R})} \lesssim k^2\|u_g\|_{L^2(\Omega^c \cap B_{2R})} + \|g\|_{H^{3/2}(\partial\Omega)} \lesssim k\|u_g\|_{\mathcal{H},2R} + \|g\|_{H^{3/2}(\partial\Omega)} \lesssim k\|g\|_{3/2,\mathcal{H},\partial\Omega},$$

which finishes the proof. ■

We define the frequency splitting of the Dirichlet traces by means of operators $L^D_{\Omega^c_R}$ and $H^D_{\Omega^c_R}$ as follows: For $g \in H^{1/2}(\partial\Omega)$, we let G^D be the trace lifting operator of Lemma 4.21 and then set

$$\begin{aligned} L^D_{\Omega^c_R} g &:= \left(L_{\Omega^c \cap B_{2R}} G^D g\right)\big|_{\Omega^c_R}, \quad & H^D_{\Omega^c_R} g &:= \left(H_{\Omega^c \cap B_{2R}} G^D g\right)\big|_{\Omega^c_R}, \\ L^D_{\partial\Omega} g &:= \left(L^D_{\Omega^c_R} g\right)\Big|_{\partial\Omega}, & H^D_{\partial\Omega} g &:= \left(H^D_{\Omega^c_R} g\right)\Big|_{\partial\Omega}. \end{aligned} \tag{4.40}$$

In view of the stability properties of the operators $L_{\Omega^c \cap B_{2R}}$, $H_{\Omega^c \cap B_{2R}}$, given by Lemmata 4.2, 4.3 we get (with $\eta > 1$ defining these operators)

$$\|L^D_{\Omega^c_R} g\|_{\mathcal{H},R} \overset{(4.8)}{\lesssim} \|G^D g\|_{\mathcal{H},R} \overset{(4.35)}{\lesssim} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \tag{4.41a}$$

$$\|\nabla^p L^D_{\Omega^c_R} g\|_{L^2(\Omega^c_R)} \overset{(4.8),(4.35)}{\lesssim} (\eta k)^p \|g\|_{1/2,\mathcal{H},\partial\Omega}, \qquad \forall p \in \mathbb{N}_0, \tag{4.41b}$$

$$\|H^D_{\Omega^c_R} g\|_{\mathcal{H},R} \overset{(4.3)}{\lesssim} \|G^D g\|_{\mathcal{H},R} \overset{(4.35)}{\lesssim} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \tag{4.41c}$$

$$\|H^D_{\Omega^c_R} g\|_{L^2(\Omega^c_R)} \overset{(4.3)}{\lesssim} (\eta k)^{-1} \|G^D g\|_{H^1(\Omega^c \cap B_{2R})} \overset{(4.35)}{\lesssim} (\eta k)^{-1} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \tag{4.41d}$$

$$\|H^D_{\Omega^c_R} g\|_{\mathcal{H},R} \overset{(4.3)}{\lesssim} (\eta k)^{-1} \|G^D g\|_{H^2(\Omega^c \cap B_{2R})} \overset{(4.36)}{\lesssim} \eta^{-1} \|g\|_{3/2,\mathcal{H},\partial\Omega}, \tag{4.41e}$$

$$\|H^D_{\Omega^c_R} g\|_{H^2(\Omega^c_R)} \overset{(4.3)}{\lesssim} \|G^D g\|_{H^2(\Omega^c \cap B_{2R})} \overset{(4.36)}{\lesssim} k\|g\|_{3/2,\mathcal{H},\partial\Omega}. \tag{4.41f}$$

Remark 4.22 *The trace theorem in the multiplicative form yields*

$$\begin{aligned} \|u\|_{1/2,\mathcal{H},\partial\Omega} &\lesssim \|u\|_{\mathcal{H},R} & \forall u \in H^1\left(\Omega^c_R\right), \\ \|u\|_{3/2,\mathcal{H},\partial\Omega} &\lesssim k^{-1}\|u\|_{H^2\left(\Omega^c_R\right)} + \|u\|_{\mathcal{H},R} & \forall u \in H^2\left(\Omega^c_R\right). \end{aligned}$$

Hence, from (4.41) it follows

$$\left\|H^D_{\partial\Omega} g\right\|_{1/2,\mathcal{H},\partial\Omega} \lesssim \left\|H^D_{\Omega^c_R} g\right\|_{\mathcal{H},R} \overset{(4.42)}{\lesssim} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \tag{4.42a}$$

$$\left\|H^D_{\partial\Omega} g\right\|_{1/2,\mathcal{H},\partial\Omega} \lesssim \left\|H^D_{\Omega^c_R} g\right\|_{\mathcal{H},R} \overset{(4.42)}{\lesssim} \eta^{-1}\|g\|_{3/2,\mathcal{H},\partial\Omega}, \tag{4.42b}$$

$$\left\|H^D_{\partial\Omega} g\right\|_{3/2,\mathcal{H},\partial\Omega} \lesssim k^{-1}\left\|H^D_{\Omega^c_R} g\right\|_{H^2(\Omega^c_R)} + \left\|H^D_{\Omega^c_R} g\right\|_{\mathcal{H},R} \overset{(4.42),(4.42)}{\lesssim} \|g\|_{3/2,\mathcal{H},\partial\Omega}. \tag{4.42c}$$

4.5 Proof of Theorem 4.19

Lemma 4.23 (analysis of $S^c(L_{\Omega^c} f, 0)$) *Let Assumptions 4.16 and 4.17 be satisfied. Let $\eta > 1$ defining the operator L_{Ω^c} be fixed. Then the function $u = S^c_k(L_{\Omega^c} f, 0)$ is analytic on an open neighborhood of $\overline{\Omega^c_R}$ and satisfies*

$$\|\nabla^p S^c_k(L_{\Omega^c} f, 0)\|_{L^2(\Omega^c_R)} \lesssim k^{\alpha-1}\gamma^p \max\{p,k\}^p \|f\|_{L^2(\Omega^c)}.$$

Proof. Note that by replacing R in (2.7) by $2R$ and denoting the corresponding solution by u_{2R} it holds $u = u_{2R}|_{\Omega^c_R}$. As a consequence it suffices to apply from [27, Section 5.5] a) the interior estimates and b) the local estimates at the boundary $\Gamma = \partial\Omega$. In other words, [27, Theorem 5.3.10] directly applies to this situation. Rewriting the equation satisfied by u as

$$-\varepsilon^2 \Delta u - u = \varepsilon^2 L_{\Omega^c} f, \qquad \varepsilon := 1/k,$$

and noting that $\varepsilon^2 L_{\Omega^c} f$ satisfies

$$\|\nabla^p (L_{\Omega^c} f)\|_{L^2(B_{2R})} \lesssim (\eta k)^p \|f\|_{L^2(\Omega^c_R)} \qquad \forall p \in \mathbb{N}_0,$$

we may apply [27, Thm. 5.3.10] with $\mathcal{E} = \varepsilon = 1/k$, $C_c = 1$, $\gamma_f = O(1)$, $C_f = O(\varepsilon^2 \|f\|_{L^2(\Omega^c_R)})$, and $k\|u\|_{L^2(\Omega^c \cap B_{2R})} + \|\nabla u\|_{L^2(\Omega^c \cap B_{2R})} \lesssim k^\alpha \|f\|_{L^2(\Omega^c_R)}$ to get

$$\begin{aligned}\|\nabla^{p+2} u\|_{L^2(\Omega^c_R)} &\lesssim K^p \max\{p+2,k\}^{p+2}\left(k^{-2}\|f\|_{L^2(\Omega^c_R)} + k^{-1}\|u\|_{\mathcal{H},R}\right)\\ &\lesssim K^p \max\{p+2,k\}^{p+2} k^{\alpha-1}\|f\|_{L^2(\Omega^c_R)} \qquad \forall p \in \mathbb{N}_0,\end{aligned}$$

where we exploited the assumption $\alpha \geq 0$. ■

Lemma 4.24 (analysis of $S^c(0, L^D_{\partial\Omega} g)$) *Let Assumptions 4.16 and 4.17 be satisfied. Let $\eta > 1$ defining $L^D_{\partial\Omega}$ be fixed. Then the function $u = S^c_k(0, L^D_{\partial\Omega} g)$ is analytic on an open neighborhood of $\overline{\Omega^c_R}$ and satisfies*

$$\begin{aligned}\|S^c_k(0, L^D_{\partial\Omega} g)\|_{\mathcal{H},R} &\lesssim k^{\alpha+1}\|g\|_{1/2,\mathcal{H},\partial\Omega},\\ \|\nabla^p S^c_k(0, L^D_{\partial\Omega} g)\|_{L^2(\Omega^c_R)} &\lesssim (k + k^\alpha)\gamma^p \max\{p,k\}^p \|g\|_{1/2,\mathcal{H},\partial\Omega} \qquad \forall p \geq 2.\end{aligned}$$

Proof. Assumption 4.17 gives us

$$\|u\|_{\mathcal{H},R} \lesssim k^{\alpha+1}\|g\|_{1/2,\mathcal{H},\partial\Omega}.$$

Next, interior regularity as derived in [27, Prop. 5.5.1] gives

$$\|\nabla^{p+2} u\|_{L^2(\Omega^c_R \setminus S)} \lesssim K^{p+2} \max\{p,k\}^{p+2} k^{-1} \|u\|_{\mathcal{H},R} \qquad \forall p \in \mathbb{N}_0,$$

where S is a tubular neighborhood of $\partial\Omega$ of width $O(1)$. These are the desired bounds away from $\partial\Omega$. For the behavior of u near $\partial\Omega$, we write $u = \tilde{u} + L^D_{\Omega^c_R} g$ and set $\tilde{f} := -\Delta L^D_{\Omega^c_R} g + k^2 L^D_{\Omega^c_R} g$. Then, (4.41) gives us

$$\begin{aligned}
\|\nabla^p L^D_{\Omega^c_R} g\|_{L^2(\Omega^c_R)} &\lesssim (\eta k)^{p-1} \|g\|_{1/2,\mathcal{H},\partial\Omega} \qquad \forall p \in \mathbb{N}_0, \\
\|\nabla^p \tilde{f}\|_{L^2(\Omega^c_R)} &\lesssim (\eta k)^{p+1} \|g\|_{1/2,\mathcal{H},\partial\Omega} \qquad \forall p \in \mathbb{N}_0.
\end{aligned}$$

Near $\partial\Omega$, the function $\tilde{u}$ satisfies $-\Delta\tilde{u} - k^2\tilde{u} = -\tilde{f}$ together with $\tilde{u}|_{\partial\Omega} = 0$. Hence, [27, Thm. 5.3.10] gives us

$$\|\nabla^{p+2}\tilde{u}\|_{L^2(S)} \lesssim \max\{p+2, k\}^{p+2} \left(k\|g\|_{1/2,\mathcal{H},\partial\Omega} + k^{-1}\|\tilde{u}\|_{\mathcal{H},R}\right) \qquad \forall p \in \mathbb{N}_0.$$

This conclusion the argument. ■

Lemma 4.25 (decomposition of $S^c(f,0)$) *Let Assumptions 4.16 and 4.17 be satisfied. Let $q \in (0,1)$ be given. Then for $f \in L^2(\Omega^c_R)$ the solution $u = S^c_k(f,0)$ can be decomposed as $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$, where $u_{\mathcal{A}}$ is analytic on $\overline{\Omega^c_R}$, $u_{H^2} \in H^2(\Omega^c_R)$, and*

$$\begin{aligned}
\|\nabla^p u_{\mathcal{A}}\|_{L^2(\Omega^c_R)} &\lesssim k^{\alpha-1}\gamma^p \max\{p,k\}^p \,\|f\|_{L^2(\Omega^c_R)} \qquad \forall p \in \mathbb{N}_0, \\
\|u_{H^2}\|_{\mathcal{H},R} &\leq qk^{-1}\|f\|_{L^2(\Omega^c_R)} \\
\|u_{H^2}\|_{H^2(\Omega^c_R)} &\lesssim \|f\|_{L^2(\Omega^c_R)}.
\end{aligned}$$

Additionally, $\widetilde{u} = S^c_k(\widetilde{f},0)$ for a function $\widetilde{f} \in L^2(\Omega^c_R)$ with $\|\widetilde{f}\|_{L^2(\Omega^c)} \leq q\|f\|_{L^2(\Omega^c)}$.

Proof. Extend f by zero to $\mathbb{R}^d$ (and denote again by f the extended function). Define for an $\eta > 1$ to be chosen below

$$u^{\mathrm{I}}_{\mathcal{A}} := S^c_k(L_{\mathbb{R}^d} f, 0), \qquad u^{\mathrm{I}}_{H^2} := N_k(H_{\mathbb{R}^d} f).$$

By Lemma 4.23 we know that $u^{\mathrm{I}}_{\mathcal{A}}$ is analytic and satisfies the desired bounds. Lemma 4.4 implies that $u^{\mathrm{I}}_{H^2}$ satisfies (by choosing η suitably)

$$\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},R} \leq qk^{-1}\|f\|_{L^2(\Omega^c_R)}, \qquad \|u^{\mathrm{I}}_{H^2}\|_{H^2(B_R)} \lesssim \|f\|_{L^2(\Omega^c_R)}.$$

The trace inequality gives us $\|u^{\mathrm{I}}_{H^2}\|_{H^{1/2}(\partial\Omega)} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}$ and the multiplicative trace inequality (cf. (4.29)) provides $\sqrt{k}\|u^{\mathrm{I}}_{H^2}\|_{L^2(\partial\Omega)} \leq \|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},R} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}$. That is, we have $\|u^{\mathrm{I}}_{H^2}\|_{1/2,\mathcal{H},\partial\Omega} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}$. Furthermore, we have from the trace estimate $\|u^{\mathrm{I}}_{H^2}\|_{H^{3/2}(\partial\Omega)} \lesssim \|u^{\mathrm{I}}_{H^2}\|_{H^2(B_R)}$ that $\|u^{\mathrm{I}}_{H^2}\|_{3/2,\mathcal{H},\partial\Omega} \lesssim k^{-1}\|f\|_{L^2(\Omega^c_R)}$. The function $u^{\mathrm{I}} := u - (u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{I}}_{H^2})$ satisfies

$$-\Delta u^{\mathrm{I}} - k^2 u^{\mathrm{I}} = 0 \quad \text{in } \Omega^c, \qquad u^{\mathrm{I}}|_{\partial\Omega} = -u^{\mathrm{I}}_{H^2}|_{\partial\Omega}, \qquad u^{\mathrm{I}} \text{ satisfies (4.33b)}.$$

Let $u^{\mathrm{II}} := G^D\, u^{\mathrm{I}}_{H^2}\big|_{\partial\Omega}$, where G^D is the lifting operator of Lemma 4.21. Then:

$$\|u^{\mathrm{II}}\|_{\mathcal{H},R} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}, \qquad \|u^{\mathrm{II}}\|_{H^2(\Omega^c_R)} \lesssim \|f\|_{L^2(\Omega^c_R)} \tag{4.43}$$

For a smooth cut-off function $\chi \in C^\infty_0(\mathbb{R}^d)$ with $\operatorname{supp}\chi \subset B_R$ and $\chi \equiv 1$ near $\partial\Omega$, we define $\widetilde{f} := \Delta(\chi u^{\mathrm{II}}) + k^2\chi u^{\mathrm{II}}$. In view of the definition of G^D, we have $-\Delta u^{\mathrm{II}} + k^2 u^{\mathrm{II}} = 0$, which allows us to obtain with the aid of (4.43)

$$\|\widetilde{f}\|_{L^2(\Omega^c_R)} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}.$$

Next, we observe that the function χu^{II} satisfies the following equations and estimates:

$$
\begin{array}{lll}
\chi u^{\mathrm{II}}|_{\partial\Omega} = u^{\mathrm{I}}_{H^2}|_{\partial\Omega}, & -\Delta \chi u^{\mathrm{II}} - k^2 \chi u^{\mathrm{II}} = -\widetilde{f} \quad \text{in } \Omega^c & \chi u^{\mathrm{II}} \text{ satisfies (4.33b)} \\
\|\chi u^{\mathrm{II}}\|_{\mathcal{H},R} \lesssim q k^{-1} \|f\|_{L^2(\Omega^c_R)}, & \|\chi u^{\mathrm{II}}\|_{H^2(\Omega^c_R)} \lesssim \|f\|_{L^2(\Omega^c_R)} &
\end{array}
$$

We now set $u_{\mathcal{A}} := u^{\mathrm{I}}_{\mathcal{A}}$, $u_{H^2} := u^{\mathrm{I}}_{H^2} + \chi u^{\mathrm{II}}$, which satisfy the desired estimates. The function $\widetilde{u} := u - u_{\mathcal{A}} - u_{H^2} = u - u^{\mathrm{I}}_{\mathcal{A}} - u^{\mathrm{I}}_{H^2} - \chi u^{\mathrm{II}}$ satisfies

$$
\widetilde{u}|_{\partial\Omega} = 0, \qquad -\Delta \widetilde{u} - k^2 \widetilde{u} = \widetilde{f} \text{ in } \Omega^c \qquad \widetilde{u} \text{ satisfies (4.33b).}
$$

Hence, $\widetilde{u} = S^c(\widetilde{f}, 0)$. Readjusting the constant q by enlarging η suitably concludes the argument. ■

Lemma 4.26 (decomposition of $S^c_k(0,g)$) *Let Assumptions 4.16 and 4.17 be satisfied. Let $q \in (0,1)$ be given. Then for $g \in H^{3/2}(\partial\Omega)$ the solution $u = S^c_k(0,g)$ can be decomposed as $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$ with*

$$
\begin{array}{rcll}
\|u_{\mathcal{A}}\|_{\mathcal{H},R} & \lesssim & k^{\alpha+1} \|g\|_{1/2,\mathcal{H},\partial\Omega}, & \\
\|\nabla^p u_{\mathcal{A}}\|_{L^2(\Omega^c_R)} & \lesssim & (k + k^{\alpha}) \gamma^p \max\{p,k\}^p \|g\|_{1/2,\mathcal{H},\partial\Omega} & \forall p \geq 2, \\
\|u_{H^2}\|_{\mathcal{H},R} & \lesssim & \|g\|_{1/2,\mathcal{H},\partial\Omega}, & \\
\|u_{H^2}\|_{\mathcal{H},R} & \leq & q \|g\|_{3/2,\mathcal{H},\partial\Omega}, & \\
\|u_{H^2}\|_{H^2(\Omega^c_R)} & \lesssim & k \|g\|_{3/2,\mathcal{H},\partial\Omega}, &
\end{array}
$$

and $\widetilde{u} = S^c_k(0, \widetilde{g})$, where $\widetilde{g}$ satisfies

$$
\|\widetilde{g}\|_{3/2,\mathcal{H},\partial\Omega} \leq q \|g\|_{3/2,\mathcal{H},\partial\Omega}, \qquad \text{and} \qquad \|\widetilde{g}\|_{1/2,\mathcal{H},\partial\Omega} \leq q \|g\|_{1/2,\mathcal{H},\partial\Omega}.
$$

Proof. For $\eta > 1$ chosen below we split $g = L^D_{\partial\Omega} g + H^D_{\partial\Omega} g$ and define

$$
u^{\mathrm{I}}_{\mathcal{A}} := S^c_k(0, L^D_{\partial\Omega} g), \qquad u^{\mathrm{I}}_{H^2} := G^D(H^D_{\partial\Omega} g),
$$

where G^D is the trace lifting operator of Lemma 4.21. The function $u^{\mathrm{I}}_{\mathcal{A}}$ satisfies the desired analytic regularity estimates (cf. Lemma 4.24). From (4.42), we get the three estimates

$$
\begin{array}{ll}
\|H^D_{\partial\Omega} g\|_{1/2,\mathcal{H},\partial\Omega} \lesssim \|g\|_{1/2,\mathcal{H},\partial\Omega}, & \|H^D_{\partial\Omega} g\|_{1/2,\mathcal{H},\partial\Omega} \lesssim q \|g\|_{3/2,\mathcal{H},\partial\Omega}, \\
\|H^D_{\partial\Omega} g\|_{3/2,\mathcal{H},\partial\Omega} \lesssim \|g\|_{3/2,\mathcal{H},\partial\Omega}, &
\end{array}
$$

where the parameter $q \in (0,1)$ depends on the choice of η is still at our disposal. Lemma 4.21 gives

$$
\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},R} \lesssim \|H^D_{\partial\Omega} g\|_{1,2,\mathcal{H},\partial\Omega} \lesssim \left\{ \begin{array}{l} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \\ q \|g\|_{3/2,\mathcal{H},\partial\Omega}, \end{array} \right. \tag{4.44a}
$$

$$
\|u^{\mathrm{I}}_{H^2}\|_{H^2(\Omega^c_R)} \lesssim k \|H^D_{\partial\Omega} g\|_{3/2,\mathcal{H},\partial\Omega} \lesssim k \|g\|_{3/2,\mathcal{H},\partial\Omega}. \tag{4.44b}
$$

Next, we let $\chi \in C^\infty_0(\mathbb{R}^d)$ with $\operatorname{supp} \chi \subset B_R$ and $\chi \equiv 1$ near $\partial\Omega$ and define $\widetilde{f} := \Delta(\chi u^{\mathrm{I}}_{H^2}) + k^2 \chi u^{\mathrm{I}}_{H^2}$ By the definition of G^D we have $-\Delta u^{\mathrm{I}}_{H^2} + u^{\mathrm{I}}_{H^2} = 0$; a calculation using (4.44a) then shows

$$
\|\widetilde{f}\|_{L^2(\Omega^c_R)} \lesssim \begin{cases} k \|g\|_{1/2,\mathcal{H},\partial\Omega}, \\ qk \|g\|_{3/2,\mathcal{H},\partial\Omega}. \end{cases} \tag{4.45}
$$

Next, the function $u^{\mathrm{II}} := u - (u^{\mathrm{I}}_{\mathcal{A}} + \chi u^{\mathrm{I}}_{H^2})$ satisfies

$$-\Delta u^{\mathrm{II}} - k^2 u^{\mathrm{II}} = \widetilde{f} \quad \text{in } \Omega^c, \qquad u^{\mathrm{II}}|_{\partial\Omega} = 0, \qquad u^{\mathrm{II}} \text{ satisfies (4.33b)}.$$

We introduce

$$u^{\mathrm{II}}_{\mathcal{A}} := S^c_k(L_{\Omega^c}\widetilde{f}, 0), \qquad\qquad u^{\mathrm{II}}_{H^2} := N_k(H_{\Omega^c}\widetilde{f}),$$

and get from Lemma 4.23 that $u^{\mathrm{II}}_{\mathcal{A}}$ is analytic with the following bounds for all $p \in \mathbb{N}_0$:

$$\|\nabla^p u^{\mathrm{II}}_{\mathcal{A}}\|_{L^2(\Omega^c_R)} \lesssim k^{\alpha-1}\gamma^p \max\{p,k\}^p \left\|L_{\Omega^c}\widetilde{f}\right\|_{L^2(\Omega^c_R)} \overset{(4.45)}{\lesssim} k^{\alpha}\gamma^p \max\{p,k\}^p \|g\|_{1/2,\mathcal{H},\partial\Omega}.$$

The function $u^{\mathrm{II}}_{H^2}$ satisfies by Lemma 4.4 and (4.45)

$$\begin{aligned}
\|u^{\mathrm{II}}_{H^2}\|_{\mathcal{H},R} &\lesssim qk^{-1}\|\widetilde{f}\|_{L^2(\Omega^c_R)} \lesssim q\|g\|_{1/2,\mathcal{H},\partial\Omega},\\
\|u^{\mathrm{II}}_{H^2}\|_{H^2(\Omega^c_R)} &\lesssim \|\widetilde{f}\|_{L^2(\Omega^c_R)} \lesssim qk\|g\|_{3/2,\mathcal{H},\partial\Omega}.
\end{aligned}$$

Set $u_{\mathcal{A}} := u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{II}}_{\mathcal{A}}$ and $u_{H^2} := \chi u^{\mathrm{I}}_{H^2} + u^{\mathrm{II}}_{H^2}$. The function $\widetilde{u} := u - (u_{\mathcal{A}} + u_{H^2})$ satisfies

$$-\Delta\widetilde{u} - k^2\widetilde{u} = 0, \qquad \widetilde{u}|_{\partial\Omega} = -u^{\mathrm{II}}_{H^2}, \qquad \widetilde{u} \text{ satisfies (4.33b)}.$$

Therefore, $\widetilde{u} = S^c(0, -u^{\mathrm{II}}_{H^2})$ and, for $s \in \{1/2, 3/2\}$,

$$\|\widetilde{u}\|_{s,\mathcal{H},\partial\Omega} = \|u^{\mathrm{II}}_{H^2}\|_{s,\mathcal{H},\partial\Omega} \lesssim q\|g\|_{s,\mathcal{H},\partial\Omega}.$$

Choosing η sufficiently large such that q is suitably small concludes the argument. ■

5 Application to hp-Finite Elements

The present section shows how the regularity theory developed in Section 4 is applicable in the context of high order finite element spaces. We proceed in two steps: Section 5.1 quantifies $\eta_{\mathcal{A}}(S)$ and $\eta_{H^2}(S)$ (see Lemma 3.4). Section 5.2 applies these results to the specific examples of Section 2.1, 2.2.

5.1 hp-FEM Approximation results for $\eta_{\mathcal{A}}(S)$ and $\eta_{H^2}(S)$

This section is devoted to the estimates of the split adjoint approximation properties $\eta_{\mathcal{A}}(S)$ and $\eta_{H^2}(S)$ in the case where S is chosen as an $hp-$finite element space.

We have performed the regularity theory in Section 4 for domains with analytic boundaries and polygons. These two cases require different types of meshes that we now introduce.

5.1.1 Domains with Analytic Boundary

We adopt the setting of [11]. The triangulation $\mathcal{T}_h$ consists of elements which are the image of the reference triangle (in 2D) or the reference tetrahedron (in 3D). We do not allow hanging nodes and assume – as is standard – that the element maps of elements sharing an edge or a face induce the same parametrization on that edge or face. The maximal mesh width is denoted by $h := \max_{K\in\mathcal{T}_h} \operatorname{diam} K$. Additionally, we make the following assumption on the element maps $F_K : \widehat{K} \to K$.

Assumption 5.1 (quasi-uniform regular triangulation) *Each element map F_K can be written as $F_K = R_K \circ A_K$, where A_K is an affine map (corresponding to the scaling $h_K = \operatorname{diam} K$ of the triangle K) and R_K is an h-independent analytic map which corresponds to the metric distortion at the possibly curved boundary. The maps R_K and A_K satisfy for constants C_{affine}, C_{metric}, $\gamma > 0$ independent of h:*

$$\begin{aligned} &\|A'_K\|_{L^\infty(\widehat{K})} \le C_{\text{affine}} h, && \|(A'_K)^{-1}\|_{L^\infty(\widehat{K})} \le C_{\text{affine}} h^{-1} \\ &\|(R'_K)^{-1}\|_{L^\infty(\widetilde{K})} \le C_{\text{metric}}, && \|\nabla^n R_K\|_{L^\infty(\widetilde{K})} \le C_{\text{metric}} \gamma^n n! \qquad \forall n \in \mathbb{N}_0. \end{aligned}$$

Here, $\widetilde{K} = A_K(\widehat{K})$.

Remark 5.2 *Triangulations satisfying Assumption 5.1 can be obtained by patchwise construction of the mesh: Let $\mathcal{T}^{macro}$ be a* fixed *triangulation (with curved elements) with analytic element maps that resolves the geometry. If the triangulation $\mathcal{T}_h$ is obtained by quasi-uniform refinements of the reference element $\widehat{K}$ and the final mesh is obtained by mapping the subdivisions of the reference element with the macro element maps, then the resulting element maps satisfy the assumptions of Assumption 5.1.*

For meshes $\mathcal{T}_h$ satisfying Assumption 5.1 with element maps F_K we denote the usual space of piecewise (mapped) polynomials by

$$S^{p,1}(\mathcal{T}_h) := \{u \in H^1(\Omega) \,|\, \forall K \in \mathcal{T}_h : u|_K \circ F_K \in \mathcal{P}_p\}, \tag{5.1}$$

where $\mathcal{P}_p$ denotes the space of polynomials of degree p.

Proposition 5.3 *Let $\partial\Omega$ be analytic. Let Assumption 5.1 be satisfied. Let the parameter $\gamma > 0$ appearing in the definition of $\mathcal{H}_{\text{osc}}(\gamma, k)$ in (1.9) be fixed. Then for $\eta_{\mathcal{A}}$, η_{H^2} introduced in Lemma 3.4 there holds*

$$\eta_{H^2}(S) \le C\frac{h}{p}\left(1 + \frac{kh}{p}\right) \qquad \eta_{\mathcal{A}}(S) \le C\left(\left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p\right)\left(1 + \frac{kh}{p}\right),$$

where C, $\sigma > 0$ are independent of k, h, p.

Proof. The proofs of both estimates are simple consequences of the procedure in [30, Thm. 5.5]. ■

5.1.2 Polygonal Domains

For simplicity, we restrict our attention here to a special situation, namely, affine, shape regular triangulations of the polygon Ω that consist of (a) quasi-uniform triangulations (with mesh size h) away from the vertices and (b) geometric meshes in an $O(h)$-neighborhood of the vertices. We mention already now that $h = O(p/k)$ will be a choice of particular interest. We denote by A_j, $j = 1, \ldots, J$, the vertices of the polygon Ω. The ball with radius ch about A_j is denoted by $B_{ch}(A_j)$.

Assumption 5.4 *For $h > 0$, $L \in \mathbb{N}$, $\sigma \in (0, 1)$ the mesh $\mathcal{T}_h(L)$ is an affine, shape regular triangulation of Ω such that:*

1. *The restriction of $\mathcal{T}_h(L)$ to $\Omega \setminus (\cup_{j=1}^J B_{ch}(A_j))$ is a quasi-uniform triangulation of that set with mesh size h. Like the shape regularity constants, the constant c is independent of h, L.*

2. *For each vertex A_j, the set restriction of $\mathcal{T}_h(L)$ to $B_h(A_j) \cap \Omega$ is a geometric mesh with grading factor $\sigma \in (0, 1)$ and L layers (see, e.g., [36] for the precise definition).*

We mention that the smallest elements in the triangulation are those abutting the vertices and they are of size $h_{min} = O(h\sigma^L)$. Furthermore, the number of elements in $\mathcal{T}_h(L)$ is given by $|\mathcal{T}_h(L)| = O(h^{-2} + L)$.

Remark 5.5 *The meshes of Assumption 5.4 are based on a geometric refinement in an $O(h)$-neighborhood of the vertices. The corresponding hp-finite element spaces with suitable choices of p, L, and h (see Theorem 5.8) can be regarded as spaces of (quasi-) minimal dimension which lead to unique solvability of the arising Galerkin discretizations and to quasi-optimal error estimates.*

Further enrichments of these finite element spaces merely need to focus on the approximability of the solution u. Good choices of the mesh $\mathcal{T}$ and the polynomial degree p of the enriched space depend on regularity properties of the solution and can be selected either in an a priori or an a posteriori way.

On the geometric meshes of Assumption 5.4, we consider the $S^{p,1}(\mathcal{T}_h(L))$ as defined in (5.1). We have the following approximation results:

Proposition 5.6 *Let $\mathcal{T}_h(L)$ be a triangulation of the polygon Ω that satisfies Assumption 5.4. Let the parameters $\gamma > 0$ and $\overrightarrow{\beta} \in [0, 1)^J$ appearing in the definition of $\mathcal{H}_{\rm osc}(\gamma, k)$ in (1.9) be fixed. Assume*

$$\frac{kh}{p} < \widetilde{C}, \tag{5.2a}$$

$$L \geq C'p, \tag{5.2b}$$

for some $\widetilde{C}, C' > 0$. Then for some c, b, $\sigma_0 > 0$ independent of h, k, p there holds

$$\eta_{H^2}(S) \leq C\frac{h}{p}, \qquad \eta_{\mathcal{A}}(S) \leq Ck\left((hk)^{1-\beta_{\max}} e^{ckh-bp} + \left(\frac{kh}{\sigma_0 p}\right)^p\right),$$

where $\beta_{\max} = \max_{j=1,\ldots,J} \beta_j$. (Recall that according to Remark 4.10 any (small) positive choice of β_j is admissible for convex domains.)

Proof. Since the meshes $\mathcal{T}_h(L)$ are finer than quasi-uniform meshes with mesh size h, the bound for $\eta_{H^2}(S)$ follows by standard arguments.

To see the bound for $\eta_{\mathcal{A}}(S)$, we apply the approximation theory of [27, Chap. 3]. Let $u \in \mathcal{H}_{\rm osc}(\gamma, k)$ and define the approximation $v \in S^{p,1}(\mathcal{T}_h(L))$ elementwise with the aid of the operator Π_p^∞ of [27, Thm. 3.2.20]: $v|_K \circ F_K := \Pi_p^\infty(u \circ F_K)$, where F_K is the element map for K. We note that the elements of $\mathcal{T}_h(L)$ can be divided into two categories, namely, those belonging to a geometric mesh near the vertices, $\mathcal{T}_j^{geo}$, $j = 1, \ldots, J$, and those in a quasi-uniform mesh $\mathcal{T}_h^{unif}$ of mesh size h.

Let us first consider the error $u - v$ near the vertices. Let S be a fixed sector with apex A_j, where A_j is a vertex of the polygon Ω. In the notation of [27, Chap. 3], the assumption $u \in \mathcal{H}_{\text{osc}}(\gamma, k)$ means $u \in \mathcal{B}^2_{\beta,1/k}(S, C_u, \gamma)$, where $C_u = O(1)$. Then, [27, Lemma 3.4.7] gives (inspection of the proof of [27, Lemma 3.4.7] shows that it is applicable with $H = O(h)$)

$$\begin{aligned}
\sum_{K \in \mathcal{T}_j^{geo}} \|u - v\|^2_{\mathcal{H},K} &\le \left(1 + \frac{k^2 h^2}{p^4}\right) \sum_{K \in \mathcal{T}_j^{geo}} p^4 \|u - v\|^2_{L^\infty(K)} + \|\nabla(u - v)\|^2_{L^2(K)} \\
&\lesssim \left(1 + \frac{k^2 h^2}{p^4}\right) k^2 \left\{(hk)^{2-2\beta_j} \, \mathrm{e}^{chk - bp} + p^7 \left(hk\sigma^L\right)^{2-2\beta_j}\right\},
\end{aligned} \tag{5.3}$$

where we applied Hölder's inequality for the first estimate. The constant $b > 0$ is independent of h, k, and p. The factor $(1 + k^2h^2/p^4)$ can be bounded in view of the assumption (5.3) and $p \ge 1$. Next, in view of the assumption on L in (5.3) we arrive at

$$\sum_{K \in \mathcal{T}_j^{geo}} \|u - v\|^2_{\mathcal{H},K} \lesssim k^2 (kh)^{2(1-\beta_j)} e^{ckh - bp}, \tag{5.4}$$

where we suitably adjusted the constant $b > 0$. This is the desired estimate for the elements in $\mathcal{T}_j^{geo}$. For the remaining elements in $\mathcal{T}_h^{unif}$, we proceed by standard approximation arguments as follows. For each $K \in \mathcal{T}_h^{unif}$ set

$$C_K^2 := \sum_{n \ge 0} \left(\frac{1}{2\gamma \max\{k, n\}}\right)^{2(n+2)} \|\Phi_{n,\vec{\beta},k} \nabla^{n+2} u\|^2_{L^2(K)}.$$

Then,

$$\sum_{K \in \mathcal{T}_h(L)} C_K^2 \le 2.$$

Consider now an element K with $\mathfrak{d} := \operatorname{dist}(K, A_j) \ge ch$ for all vertices A_j. Abbreviate $\beta := \beta_{\max}$. Then we have, for all $n \in \mathbb{N}_0$, (cf. [27, Lemma 4.2.2])

$$\|\nabla^{n+2} u\|_{L^2(K)} \le C_K (2\gamma)^{n+2} \max\{n, k\}^{n+2} \left(\max\{1, \frac{\min\{1, \frac{n+1}{k+1}\}}{\mathfrak{d}}\}\right)^{n+\beta} =: RHS. \tag{5.5}$$

By distinguishing the three cases a) $n \ge k$, b) $n \le k$ together with $n + 1 \le (k + 1)\mathfrak{d}$, and c) $n \le k$ together with $n + 1 > (k + 1)\mathfrak{d}$, we arrive at

$$RHS \lesssim C_K \min\{1, k\mathfrak{d}\}^{2-\beta} (2\gamma)^{n+2} \max\{k, n/\mathfrak{d}\}^{n+2} \quad \forall n \in \mathbb{N}_0.$$

Combining now [30, Lemma C.2] with [27, Thm. 3.2.20] gives the existence of some C, $\sigma_0 > 0$ such that for $q \in \{0, 1\}$

$$h^q \|u - v\|_{H^q(K)} \le C C_K \min\{1, k\mathfrak{d}\}^{2-\beta} \left(\left(\frac{h/\mathfrak{d}}{\sigma_0 + h/\mathfrak{d}}\right)^{p+1} + \left(\frac{kh}{\sigma_0 p}\right)^{p+1}\right). \tag{5.6}$$

We next distinguish the cases $\mathfrak{d} \geq 1/k$ and $\mathfrak{d} < 1/k$. For $\mathfrak{d} \geq 1/k$ we have in view of $\mathfrak{d} \geq ch$

$$
\begin{aligned}
\left(k+h^{-1}\right)\min\{1,k\mathfrak{d}\}^{2-\beta}\left(\frac{h/\mathfrak{d}}{\sigma_0+h/d}\right)^{p+1} &= \left(k+h^{-1}\right)\left(\frac{h/\mathfrak{d}}{\sigma_0+h/\mathfrak{d}}\right)^{p+1} \lesssim k\left(\frac{h/\mathfrak{d}}{\sigma_0+h/\mathfrak{d}}\right)^{p} \\
\lesssim \; k\left(\frac{h/\mathfrak{d}}{\sigma_0+h/\mathfrak{d}}\right)^{1-\beta}\left(\frac{h/\mathfrak{d}}{\sigma_0+h/\mathfrak{d}}\right)^{p-1+\beta} &\lesssim k\min\{1,h/\mathfrak{d}\}^{1-\beta}\left(\frac{1}{c\sigma_0+1}\right)^{p} \\
\lesssim \; k\min\{1,hk\}^{1-\beta}\left(\frac{1}{c\sigma_0+1}\right)^{p}, &
\end{aligned}
\tag{5.7}
$$

where we additionally exploited the monotonicity properties of $x \mapsto (x/(\sigma_0+x))^{p-1+\beta}$ and $p \geq 1$ together with $\beta \geq 0$. For the case $h \lesssim \mathfrak{d} < 1/k$, we have

$$
(k+h^{-1})\min\{1,k\mathfrak{d}\}^{2-\beta} \lesssim h^{-1}(k\mathfrak{d})^{2-\beta} = k(kh)^{1-\beta}\left(\frac{\mathfrak{d}}{h}\right)^{2-\beta} \lesssim k\min\{1,kh\}^{1-\beta}\left(\frac{\mathfrak{d}}{h}\right)^{2-\beta}.
$$

Exploiting again the monotonicity properties of $x \mapsto (x/(1+x))^{p-1+\beta}$ together with $p \geq 1$, $\beta \geq 0$, and $\mathfrak{d} \geq ch$ we conclude also for the case $\mathfrak{d} < 1/k$

$$
\left(k+h^{-1}\right)\min\{1,k\mathfrak{d}\}^{2-\beta}\left(\frac{h/\mathfrak{d}}{\sigma_0+h/\mathfrak{d}}\right)^{p+1} \lesssim k\min\{1,kh\}^{1-\beta}\left(\frac{1}{c\sigma_0+1}\right)^{p}. \tag{5.8}
$$

Inserting the estimates (5.7), (5.8) into (5.6), we get

$$
k\|u-v\|_{L^2(K)} + |u-v|_{H^1(K)} \lesssim C_K k\left(\min\{1,kh\}^{1-\beta}\left(\frac{1}{c\sigma_0+1}\right)^{p} + \left(\frac{kh}{\sigma_0 p}\right)^{p}\right).
$$

By summing over all elements K that are in the quasi-uniform mesh $\mathcal{T}_h^{unif}$

$$
\sqrt{\sum_{K\in\mathcal{T}_h^{unif}} \|u-v\|^2_{\mathcal{H},K}} \lesssim k\left(\min\{1,kh\}^{1-\beta}\left(\frac{1}{c\sigma_0+1}\right)^{p} + \left(\frac{kh}{\sigma_0 p}\right)^{p}\right). \tag{5.9}
$$

Combining (5.4) with (5.9) and appropriately adjusting constants proves the claim of the proposition. ■

5.2 Stability and Convergence Analysis of *hp*-FEM for the Model Problems of Section 2

In view of the oscillatory nature of solutions of Helmholtz problems, it is reasonable to expect that a minimal condition for stability is that the dimension N of the ansatz space has to satisfy $N = O(k^d)$. The next theorem shows that, indeed, the polynomial degree p and the mesh size h can be selected such that the resulting approximation spaces has dimension $N = O(k^d)$ and at the same time ensures quasi-optimality of the Galerkin FEM.

Since we will refer to the same hypotheses several times in the section, we formulate them as an assumption:

Assumption 5.7 *(i) If the model problem of Section 2.1 (cf. (2.3)) is considered, then Assumptions 1.1 and 4.7 are valid. The given data satisfy $f \in L^2(\Omega)$ and $g \in H^{1/2}_{pw}(\partial\Omega)$. The discrete formulation (2.4) is used with an hp-FEM S. If Ω has an analytic boundary, then the hp-FEM space S described in Section 5.1.1 is used; If Ω is a polygon, then the hp-FEM space S described in Section 5.1.2 is is used with the additional assumption $L = O(p)$.*

(ii) If the exterior Dirichlet problem (2.8) is considered, Assumptions 4.16 and 4.17 are valid. The given data satisfy $f \in L^2(\Omega_R^c)$ and $g = 0$ on $\partial\Omega$[1]*. The Galerkin method (2.9) with $g_S = 0$ is based on the $V_{R,0}$-conforming subspace of the hp-FEM spaces described in Section 5.1.1. The DtN-operator T_R is assumed to be realized exactly.*

Theorem 5.8 (discrete stability of *hp*-FEM) *Consider the setup of Assumption 5.7. Assume $k > k_0 > 1$. Then there exist constants δ, $\widetilde{C} > 0$ that are independent of h, p, and k such that the conditions*

$$\frac{kh}{p} \leq \delta \quad \text{and} \quad p \geq 1 + \tilde{C} \log k \tag{5.10}$$

imply the following: The discrete problem (2.4) or (2.9) has a unique solution which satisfies

$$\begin{aligned} \|u - u_S\|_{\mathcal{H}} &\leq 2(1 + C_b) \inf_{v \in S} \|u - v\|_{\mathcal{H}}, \\ \|u - u_S\|_{L^2(\Omega)} &\leq C \frac{h}{p} \inf_{v \in S} \|u - v\|_{\mathcal{H}}; \end{aligned}$$

here, C_b and C are constants that are independent of h, p, k, and f, g.

Proof. In the interest of brevity, we will not consider the case of geometric meshes and restrict our attention to cases where quasi-uniform meshes that satisfy Assumption 5.1 are appropriate.

¿From Lemma 3.4 and Proposition 5.3 we conclude that

$$\eta(S) \leq C \left\{ C_{k,\mathcal{A}} \left(\left(\frac{h}{h + \sigma_0} \right)^p + k \left(\frac{kh}{\sigma_0 p} \right)^p \right) + C_{H^2} \frac{h}{p} \right\} \left(1 + \frac{kh}{p} \right). \tag{5.11}$$

Assumption 5.7 implies via Theorems 4.9 resp. 4.19 that the constants $C_{k,\mathcal{A}}$, C_{H^2} may be assumed to have the form

$$C_{k,\mathcal{A}} = C k^{\alpha - 1} \quad \text{and} \quad C_{H^2} = C, \tag{5.12}$$

where C is independent of k, h, p. By Lemma 3.4, the stability condition (3.12) is therefore satisfied, if

$$k^\alpha \left(\frac{h}{h + \sigma_0} \right)^p + k^{\alpha+1} \left(\frac{kh}{\sigma_0 p} \right)^p + \frac{hk}{p} \leq \rho, \tag{5.13}$$

for some $\rho > 0$ that is independent of k, h, p. Without loss of generality, we may require that $\rho < 1$. By selecting δ sufficiently small, we can ensure

$$\frac{kh}{p} \leq \delta \leq \rho/3, \qquad \frac{kh}{\sigma_0 p} \leq \delta/\sigma_0 \leq 1/2.$$

Finally, since the computational domain is bounded, we have $h/(\sigma_0 + h) \leq \theta < 1$. Therefore, the left-hand side of (5.13) can be bounded by

$$k^\alpha \theta^p + k^{\alpha+1} 2^{-p} + \rho/3.$$

This can be bounded by ρ if $p \geq \widetilde{C} \log k$ for sufficiently large $\widetilde{C}$. ∎

[1] The assumption $g = 0$ is made here to avoid further consistency estimates. Note that, for the analysis, we assumed $g \in H^{3/2}(\partial\Omega)$ which can be transformed in the standard way to the case of homogeneous boundary conditions by a trace lifting of g to some function $u_g \in H^2(\Omega_R^c)$ and then modifying the right-hand side f.

Remark 5.9 *Let $k > k_0 > 1$. Selecting*

$$p := 1 + \left\lceil \tilde{C} \log k \right\rceil, \quad h := \frac{\delta p}{k}$$

(and $L = O(p)$ for the case of polygons) for the constants δ, $\widetilde{C} > 0$ of Theorem 5.8, we see that stability of the Galerkin method can be ensured with hp-FEM spaces of dimension $N := \dim S \sim (p/h)^d \sim k^d$. In other words: stability is given with a fixed number of degrees of freedom per wavelength.

Let us compare this with the lowest order FEM, i.e., the choice $p = 1$. In this case, the requirement (5.13) reads

$$k^\alpha h + k^{\alpha+1} (kh) + hk \leq \rho.$$

Even if we assume that $\alpha = 0$, this condition leads to the condition $k^2 h \lesssim 1$ so that the minimal number of unknowns of the $\mathcal{P}_1$-finite element space is $\dim S^{1,1} = O(k^{2d})$. This illustrates the substantial savings for the choice $p \approx \log k$ over the lowest order case $p = 1$.

In order to get explicit convergence rates we employ Theorem 4.9 resp. Theorem 4.19. Let

$$C_{f,g} := \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)}$$

Proposition 5.10 *Let Assumption 5.7 be satisfied and let $kh/p \lesssim 1$. For the Robin problem on bounded domains (2.3) we get*

- *for analytic domains*

$$\inf_{v \in S} \|u - v\|_{\mathcal{H}} \lesssim C_{f,g} \left\{ \frac{h}{p} + k^{\alpha-1} \left(\left(\frac{h}{h + \sigma_0} \right)^p + k \left(\frac{kh}{\sigma_0 p} \right)^p \right) \right\} \tag{5.14a}$$

- *for convex polygonal domains we assume (5.2) and obtain*

  –

$$\text{for } p \geq 1 + \tilde{C} \log k \qquad \inf_{v \in S} \|u - v\|_{\mathcal{H}} \lesssim C_{f,g} \left\{ \frac{h}{p} + k^{\alpha-1} (hk)^{1-\beta_{\min}} \mathrm{e}^{-\tilde{c} p} \right\}, \tag{5.14b}$$

  where $\beta_{\min} \in (0, 1)$ is as in (5.6) and, according to Remark 4.10, can be chosen arbitrary small.

  –

$$\text{for } p = O(1) \qquad \inf_{v \in S} \|u - v\|_{\mathcal{H}} \lesssim C_{f,g} \left(h + (hk)^{1-\beta_{\min}} k^\alpha \right). \tag{5.14c}$$

For problem (2.8), we obtain

$$\inf_{v \in S} \|u - v\|_{\mathcal{H}} \lesssim C_{f,0} \left\{ \frac{h}{p} + k^{\alpha-1} \left(\left(\frac{h}{h + \sigma_0} \right)^p + k \left(\frac{kh}{\sigma_0 p} \right)^p \right) \right\}. \tag{5.14d}$$

Proof. We consider first the Robin problem on bounded, analytic domains and define $C_{k,\mathcal{A}} := CC_{f,g}k^{\alpha-1}$ and $C_{H^2} := CC_{f,g}$, where C is as in (4.21). Theorem 4.9 defines a splitting $u = u_{\mathcal{A}} + u_{H^2}$ with the property that the scaled functions $\widetilde{u_{\mathcal{A}}}$, $\widetilde{u_{H^2}}$ given by the conditions $C_{k,\mathcal{A}}\widetilde{u_{\mathcal{A}}} = u_{\mathcal{A}}$ and $C_{H^2}\widetilde{u_{H^2}} = u_{H^2}$ satisfy $\widetilde{u_{\mathcal{A}}} \in \mathcal{H}_{\text{osc}}(\gamma, k)$ and $\widetilde{u_{H^2}} \in \mathcal{H}_{\text{ell}}$.

Hence, by arguing as for (5.11) and using Proposition 5.3 we obtain

$$\inf_{v\in S} \|u-v\|_{\mathcal{H}} \lesssim C_{f,g}\left\{\frac{h}{p} + k^{\alpha-1}\left((hk)^{1-\beta}\,\mathrm{e}^{-\tilde{c}p} + k\left(\frac{kh}{\sigma_0 p}\right)^p\right)\right\}.$$

For sufficiently large $\tilde{C} = O(1)$ (in (5.14b)) the last term can be absorbed in the second term and we obtain the assertion.

The proof of (5.14b) and (5.14c) is analogously and based on Proposition 5.6.

The proof for the problem (2.8) is analogous by using Theorem 4.19. ■

A Lifting

Lemma A.1 *Let $\Omega \subset \mathbb{R}^2$ be a polygon, where all internal angles are different from 0, π, and 2π. Then there exists a linear operator $G : H^{1/2}_{pw}(\partial\Omega) \to H^2(\Omega)$ with $\partial_n G = g$ and $\|G\|_{H^2(\Omega)} \le C\|g\|_{H^{1/2}_{pw}(\partial\Omega)}$.*

Proof. In the interest of brevity, we base the proof on the solvability theory in convex polygons.

step 1: Let T be a (convex) triangle. Then one can infer from [16, Cor. 4.4.3.8] the existence of $C_T > 0$ such that for every $g \in H^{1/2}_{pw}(\partial T)$ with $\int_{\partial T} g = 0$ there holds for the solution $u \in H^2(T)$ of

$$-\Delta u = 0 \quad \text{in } T, \qquad \partial_n u = g \quad \text{on } \partial T, \qquad \int_T u = 0$$

the *a priori* bound $\|u\|_{H^2(T)} \le C_T\|g\|_{H^{1/2}_{pw}(\partial T)}$.

step 2: Let $S = \{(r\cos\varphi \,|\, r\sin\varphi) \,|\, 0 < r < 2\delta, 0 < \varphi < \omega\}$ with edges Γ_1, Γ_2 meeting at the origin. Set $\Gamma_{1,\delta} := \{(r,0) \,|\, 0 < r < \delta\}$, $\Gamma_{2,\delta} := \{(r\cos\omega, r\sin\omega) \,|\, 0 < r < \delta\}$.

For the case of a convex sector, i.e., $0 < \omega < \pi$, it is easy to construct with the aid of the first step a bounded linear operator $L : \prod_{i=1}^2 \{u \in H^{1/2}(\Gamma_i) \,|\, \operatorname{supp} u \subset \overline{\Gamma_{i,\delta}}\} \to H^2(S)$ with $(\partial_n L(f_1, f_2))|_{\Gamma_i} = f_i$ $(i \in \{1,2\})$ $\|L(f_1,f_2)\|_{H^2(S)} \le C\sum_{i=1}^2 \|f_i\|_{H^{1/2}(\Gamma_i)}$.

For the case of a non-convex sector, i.e., $\pi < \omega < 2\pi$, let $S' := B_{2\delta}\backslash S$ and let $E : H^2(S') \to H^2(\mathbb{R}^2)$ be the extension operator of Stein, [37, Chap. VI]. Then S' is a convex sector of the form considered above. Then it is easy to check that $(E(L(-f_1,-f_2)))|_{S'} \in H^2(S')$ has the desired lifting property for S.

step 3: Localizing with the aid of partitions of unity, we can reduce the construction of the lifting to the question of liftings from an infinite line to a half space and from two edges that meet at a common vertex V to the enclosed sector. The first case is well-known (see, e.g., [16, Thm. 1.5.1.2]). The second case is covered by step 2. ■

B Proof of Theorem 3.2

We present the proof of Theorem 3.2.

Given $u \in S$, define $z \in V$ by $\theta k^2 \widetilde{S}_k^\star u$ and let $z_S \in S$ be the best approximation to z in the $\|\cdot\|_{\mathcal{H}}$-norm. Then:

$$\begin{aligned}
&\operatorname{Re}\left(a(u,u+z) - b(u,u+z)\right) = \\
&\qquad \underbrace{\operatorname{Re}\left(a(u,u) - b(u,u)\right) + \theta k^2 \|u\|^2_{L^2(\Omega)}}_{\geq \gamma \|u\|^2_{\mathcal{H}}} + \underbrace{\operatorname{Re}\left(a(u,z) - b(u,z) - \theta k^2 \|u\|^2_{L^2(\Omega)}\right)}_{=0}
\end{aligned}$$

With the preparatory consideration, we compute

$$\begin{aligned}
&\operatorname{Re}\left(a(u,u+z_S) - b(u,u+z_S)\right) = \\
&\qquad \operatorname{Re}\left(a(u,u+z) - b(u,u+z)\right) + \operatorname{Re}\left(a(u,z_S - z) - b(u,z_S - z)\right) \\
&\quad \geq \gamma \|u\|^2_{\mathcal{H}} - (1+C_b)\|u\|_{\mathcal{H}} \|z - z_S\|_{\mathcal{H}} \\
&\quad \geq \gamma \|u\|^2_{\mathcal{H}} - (1+C_b)\|u\|_{\mathcal{H}} \eta(S) \theta k^2 \|u\|_{L^2(\Omega)} \\
&\quad \geq \left(\gamma - (1+C_b)\theta k \eta(S)\right) \|u\|^2_{\mathcal{H}}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\|u + z_S\|_{\mathcal{H}} &\leq \|u\|_{\mathcal{H}} + \|z - z_S\|_{\mathcal{H}} + \|z\|_{\mathcal{H}} \\
&\leq \|u\|_{\mathcal{H}} + \theta k^2 \|u\|_{L^2(\Omega)} \eta(S) + \theta k^2 C_{adj} \|u\|_{L^2(\Omega)} \\
&\leq \left(1 + \theta k \eta(S) + \theta k C_{adj}\right) \|u\|_{\mathcal{H}}
\end{aligned}$$

so that

$$\begin{aligned}
\inf_{0 \neq u \in S} \sup_{0 \neq v \in S} \frac{\operatorname{Re} a(u,v) - b(u,v)}{\|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}} &\geq \frac{\gamma - (1+C_b)\theta k \eta(S)}{1 + \theta k \eta(S) + \theta k C_{adj}} \\
&\geq \frac{\gamma}{2 + \gamma/(1+C_b) + 2\theta k C_{adj}}
\end{aligned}$$

This shows (3.13).

We next show the L^2-bound (3.14). To that end, denote $e := u - u_S$ and define $\psi := S_k^\star e$ and let $\psi_S \in S$ be the best approximation to ψ. Then in view of the Galerkin orthogonality

$$\begin{aligned}
\|e\|^2_{L^2(\Omega)} &= a(e,\psi) - b(e,\psi) = a(e,\psi - \psi_S) - b(e, \psi - \psi_S) \\
&\leq (1+C_b)\|e\|_{\mathcal{H}} \|\psi - \psi_S\|_{\mathcal{H}} \leq (1+C_b)\eta(S)\|e\|_{\mathcal{H}} \|e\|_{L^2(\Omega)}.
\end{aligned}$$

Hence,

$$\|e\|_{L^2(\Omega)} \leq (1+C_b)\eta(S)\|e\|_{\mathcal{H}},$$

which is (3.14). To infer from this a bound for $\|e\|_{\mathcal{H}}$, we notice that Galerkin orthogonality gives for arbitrary $v \in S$

$$\begin{aligned}
\gamma \|e\|^2_{\mathcal{H}} &\leq \operatorname{Re}\left(a(e,e) - b(e,e) + \theta k^2 \|e\|^2_{L^2(\Omega)}\right) = \operatorname{Re}\left(a(e,u-v) - b(e,u-v) + \theta k^2 \|e\|^2_{L^2(\Omega)}\right) \\
&\leq (1+C_b)\|e\|_{\mathcal{H}} \|u - v\|_{\mathcal{H}} + \theta k^2 \|e\|^2_{L^2(\Omega)} \\
&\leq (1+C_b)\|e\|_{\mathcal{H}} \|u - v\|_{\mathcal{H}} + \theta k \|e\|_{L^2(\Omega)} k \|e\|_{L^2(\Omega)} \\
&\leq (1+C_b)\|e\|_{\mathcal{H}} \|u - v\|_{\mathcal{H}} + \theta (1+C_b) k \eta(S) \|e\|_{\mathcal{H}} \|e\|_{\mathcal{H}} \\
&\leq (1+C_b)\|e\|_{\mathcal{H}} \|u - v\|_{\mathcal{H}} + \gamma/2 \|e\|^2_{\mathcal{H}}.
\end{aligned}$$

We conclude the desired estimate (3.14) for $\|e\|_{\mathcal{H}}$.

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