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MIXED CONFORMING ELEMENTS FOR THE LARGE-BODY LIMIT IN MICROMAGNETICS: A FINITE ELEMENT APPROACH

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Abstract. We introduce a stabilized conforming mixed finite element method for a macroscopic model in micromagnetics. We show well-posedness of the discrete problem for higher order elements in two and three dimensions, develop a full *a priori* analysis for lowest order elements, and discuss the extension of the method to higher order elements. We introduce a residual-based *a posteriori* error estimator and present an adaptive strategy. Numerical examples illustrate the performance of the method.

1. Introduction

Magnetic storage media, sensors, and magnetic random access memory are just a few every day examples of the importance of magnetic materials for modern life. In order to better understand the underlying physical phenomena, which take place on the micro- and even the nanoscale, and to design new devices that exploit micromagnetic effects, reliable numerical simulations are an indispensible tool. This need for simulation tools is reflected in a large body of literature. We refer to the monograph [32] and to the overview articles [12, 28, 33] as well as to the recent works [1, 4, 5, 6]. In this larger context, we study as, for example, [11, 27, 9], a specific reduced model arising in stationary micromagnetics. This model has been thoroughly analyzed mathematically in [15] and in [29, 35]. It is worth stressing that the richness of physical micromagnetic phenomena corresponds to a variety of parameter regimes that lead to other reduced models as discussed in [16].

In rigid ferromagnetic bodies $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the mathematical description of magnetization states $\boldsymbol{m} : \Omega \to \mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ goes back to the classical model by Landau and Lifshitz [8], where the magnetization solves the minimization problem (MP^α):

Problem 1.1 ($\boldsymbol{MP_\alpha}$)**.** *Given an applied field* $\boldsymbol{f} \in L^2(\Omega, \mathbb{R}^d)$*, find a minimizer* $\boldsymbol{m} \in \mathcal{A}' := H^1(\Omega, \mathbb{S}^{d-1}) := \{\boldsymbol{n} \in H^1(\Omega, \mathbb{R}^d) : |\boldsymbol{n}(x)| = 1 \text{ a.e. in } \Omega\}$ *of*

$$\inf_{\boldsymbol{m}\in\mathcal{A}'} E^\alpha_{\boldsymbol{f}}(\boldsymbol{m}), \qquad E^\alpha_{\boldsymbol{f}}(\boldsymbol{m}) := \alpha \int_\Omega |\nabla \boldsymbol{m}|^2 + \int_\Omega \varphi \circ \boldsymbol{m} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{m} + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2. \tag{1.1}$$

Here, the magnetic potential $u : \mathbb{R}^d \to \mathbb{R}$ *is linked to* $\boldsymbol{m}$ *through the magnetostatic Maxwell equation*

$$\operatorname{div}(\nabla u - \boldsymbol{m}\chi_\Omega) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{1.2}$$

The four terms in the energy functional $E^\alpha_{\boldsymbol{f}}$ of (1.1) favor different properties of the minimizer. The first term penalizes rapidly varying structures, where $\alpha > 0$ denotes a (typically very small) exchange parameter (length scale). The even function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}_{\geq 0})$ in the second term models crystallographic properties of the ferromagnet and may be non-convex. The third term with the applied exterior field $\boldsymbol{f} \in L^2(\Omega, \mathbb{R}^d)$ favors magnetizations aligned with $\boldsymbol{f}$, and the last term is a measure of $\operatorname{div} \boldsymbol{m}$; we refer to [25] for a detailed discussion of the model and its physical background.

Mathematically, Problem 1.1 (MP_α) is a non-convex, full-space minimization problem. For this problem to be numerically tractable, we have to make several simplifications, which, ultimately, will lead to our considering the saddle point problem $(SPP_{\widehat{\Omega}})$ stated in Problem 2.1 and its discrete

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version ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$) stated in Problem 3.4. Let us comment in more detail on some of the challenges that arise when trying to discretize Problem 1.1 (MP_α) directly and show how the continuous Problem 2.1 is obtained:

1. Often, the parameter α is small compared to diam(Ω). Features on this length scale are very hard to resolve numerically. A common approach is then to neglect this contribution to the energy functional $E^\alpha_{\boldsymbol{f}}$. The resulting model with energy functional $E^0_{\boldsymbol{f}}$ is known as the *large-body limit* in micromagnetics and useful, for example, to predict virgin magnetization curves; we refer to [15] for a detailed discussion of this model.

2. The minimization problem of the large-body limit is a non-convex minimization problem due to the non-convex pointwise length constraint $|\boldsymbol{m}(x)| = 1$ for almost every $x \in \Omega$ and the non-convex anisotropy density φ. Additionally, it may not have minimizers, but only infimizing sequences [26]. To overcome this difficulty, one strategy is to relax the problem by convexification [15]: We replace the anisotropy density φ by its lower convex envelope

$$\varphi^{**}(x) := \sup\left\{\phi(x) \,:\, \phi : \mathbb{R}^d \to \mathbb{R} \text{ convex and } \phi|_{\mathbb{S}^{d-1}} \le \varphi\right\} \text{ for } x \in \mathbb{B}^d := \left\{\mathbb{R}^d \,:\, |x| \le 1\right\},$$

and enlarge the set of admissible functions $\mathcal{A}'$ to

$$\mathcal{A}^* := L^2(\Omega, \mathbb{B}^d) := \left\{\boldsymbol{n} \in L^2(\Omega, \mathbb{R}^d) \,:\, |\boldsymbol{n}(x)| \le 1 \text{ a.e. in } \Omega\right\}. \tag{1.3}$$

In this new setting, the energy functional $E^0_{\boldsymbol{f}}$ attains its infimum, and its minimizers are precisely the subsequential weak limits of the infimizing sequences of the non-convex version, [15, Theorem 3.4]. We recall that by [15, Proposition 3.1] we have $\varphi^{**} \in C^1(\mathbb{B}^d)$ and φ^{**} convex. Moreover, by [15, Theorem 4.2] solving this convexified minimization problem is equivalent to solving the corresponding Euler-Lagrange equations. Finally, we point the reader to Example 1.3 below, where φ^{**} is given explicitly for the practically relevant case of "uniaxial" materials.

3. The presence of the full space $\mathbb{R}^d$ in the energy functional $E^0_{\boldsymbol{f}}$ and in the side constraint (1.2) requires special care in discretizations. Following [11], we replace the full space $\mathbb{R}^d$ in (1.1) and (1.2) by a bounded Lipschitz domain $\widehat{\Omega}$ containing $\overline{\Omega}$. Then, the integral over $\mathbb{R}^d$ in (1.1) is replaced by an integral over $\widehat{\Omega}$, and the full-space equation (1.2) simplifies to a PDE on the finite domain $\widehat{\Omega}$ which has to be supplemented by appropriate boundary conditions. For ease of presentation and following [11], we use homogeneous Dirichlet conditions.

The above modifications and simplifications lead to a convex minimization problem under a PDE-constraint, which we call ($RMP_{\widehat{\Omega}}$):

Problem 1.2 ($RMP_{\widehat{\Omega}}$)**.** *Find $u \in H^1_0(\widehat{\Omega})$ and $\boldsymbol{m} \in \mathcal{A}^*$ that minimize the energy functional*

$$E^{**}_{\boldsymbol{f},\widehat{\Omega}}(\boldsymbol{m}) := \int_\Omega \varphi^{**} \circ \boldsymbol{m} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{m} + \frac{1}{2}\int_{\widehat{\Omega}} |\nabla u|^2$$

under the side constraint

$$\operatorname{div}(\nabla u - \boldsymbol{m}\chi_\Omega) = 0 \quad \text{in } H^{-1}(\widehat{\Omega}), \tag{1.4}$$

where $H^{-1}(\widehat{\Omega}) := (H^1_0(\widehat{\Omega}))'$ denotes the dual space of $H^1_0(\widehat{\Omega})$.

Problem 1.2 is the continuous problem under consideration in this paper. We will, however, mostly use its equivalent saddle point formulation ($SPP_{\widehat{\Omega}}$) given in Problem 2.1. Existence of solutions follows by standard arguments in view of the convexity of φ^{**}. In practically relevant situations, however, the function φ^{**} is not strictly convex so that uniqueness is an issue. A prominent example is that of an "uniaxial" material such as cobalt, where the anisotropy φ favors a particular direction ("easy axis"), given as follows:

Example 1.3. *Uniaxial materials can be modelled with the aid of the* uniaxial *anisotropy density*

$$\varphi : \mathbb{S}^{d-1} \to \mathbb{R}, \quad \varphi(x) = \frac{1}{2}(1 - (\mathbf{e} \cdot x)^2), \tag{1.5}$$

where $\mathbf{e} \in \mathbb{R}^d$ *is a given fixed unit vector, called* easy axis [8, 25]. φ *favors magnetizations* $\boldsymbol{m}$ *aligned with* $\mathbf{e}$. *In this case, the lower convex envelope* φ^{**} *and its gradient* $\nabla\varphi^{**}$ *can easily be computed* [11]*:*

$$\varphi^{**}(x) = \frac{1}{2}\sum_{i=1}^{d-1}(x \cdot \mathbf{z}_i)^2, \quad x \in \mathbb{B}^d, \tag{1.6}$$

$$\nabla\varphi^{**}(x) = \sum_{i=1}^{d-1}(x \cdot \mathbf{z}_i)\,\mathbf{z}_i, \quad x \in \mathbb{B}^d, \tag{1.7}$$

where $\{\mathbf{e}, \mathbf{z}_1, \dots, \mathbf{z}_{d-1}\}$ *is an orthonormal basis of* $\mathbb{R}^d$.

A particular feature of the uniaxial case of Example 1.3 is its lack of strict convexity since it does not provide any control over the easy-axis component $\mathbf{e} \cdot \boldsymbol{m}$ of the magnetization. Nevertheless, uniqueness can be shown in the uniaxial case by exploiting the PDE-side constraint (1.4). In Proposition 2.2 below, we briefly recapitulate this non-standard argument, which can also be found in [11, Theorem 2.1] for 2D and, with a different argument, in [9, Theorem 2.2] for 2D and 3D, since the mechanisms of the proof are important for the understanding of the behavior of the discretization.

In the degenerate setting of uniaxial materials, a straightforward discretization of the Euler-Lagrange equations for Problem 1.2 using *conforming* lowest order elements loses the uniqueness assertion of the continuous problem as shown in [11]. Therefore, [11] advocates the use of lowest order *non*-conforming elements and shows existence and uniqueness for the discrete system by a direct analysis of the discrete system. The novelty of the present paper over [11] is twofold: firstly, it is based on *conforming* elements (i.e., $H^1(\widehat{\Omega})$-conforming elements for the approximation of u and $L^2(\Omega)$-conforming elements for that of $\boldsymbol{m}$) and shows that optimal order convergence can be achieved in that setting (under sufficient regularity assumptions); secondly, and more importantly, it works out a mechanism by which the existence and uniqueness assertions of the continuous problem can be transferred to the discrete problem. This opens the door to higher order discretizations.

Let us elaborate on the need of stabilization. The first order conditions for the constrained minimization Problem 1.2 ($RMP_{\widehat{\Omega}}$) take the form of a (nonlinear) saddle point problem for the primal variables $(u, \boldsymbol{m})$ and a Lagrange multiplier p. It is thus a block 2×2 system. On the continuous level, the $(1,1)$ block of this 2×2 block system has special properties on the kernel $\operatorname{Ker} b$ of the operator b associated with the linear constraint (1.4). In a straightforward (conforming) discretization, the discrete kernel $\operatorname{Ker}_N b$ is not necessarily a subspace of the continuous kernel $\operatorname{Ker} b$ and hence, the properties of the continuous problem cannot be used on the discrete level. This is rectified by our stabilization term σ since it establishes an appropriate connection between the discrete kernel $\operatorname{Ker}_N b$ and the continuous kernel $\operatorname{Ker} b$. We point out that the choice of the stabilization term is motivated by *augmented Lagrangian methods* as discussed, for example, in the classical monograph [20].

The idea of addressing the instability of conforming discretizations of the Euler-Lagrange equations for Problem 1.2 ($RMP_{\widehat{\Omega}}$) with the aid of stabilization terms has also been pursued in [22]. Whereas the stabilization term of the present paper is consistent, the stabilization term of [22] is inconsistent in that it penalizes the jump of the tangential component of the magnetization, which, however, should be allowed to be discontinuous in order to accurately reflect magnetic domains and domain walls. A completely different approach to dealing with the constraint (1.2) is taken in [9]. There, the constraint (1.2) is completely eliminated by using an explicit representation $\nabla u = \mathcal{P}\boldsymbol{m}$.

However, as the operator $\mathcal{P}$ is non-local, any discretization has to address the issue of the efficient realization of $\mathcal{P}$, for example, by matrix compression methods such as $\mathcal{H}$-matrices; see [24] for the general concept of $\mathcal{H}$-matrices and [31] for the particular application to the operator $\mathcal{P}$. As a final remark on the literature, we mention another approach to dealing with the constraint (1.2). Instead of replacing the full space $\mathbb{R}^d$ with a finite domain $\widehat{\Omega}$ and thus incur an additional modelling error, it is also possible to avoid this error using finite element–boundary element coupling techniques [3].

The remainder of this paper is structured as follows. Starting point for the discretization is the equivalent saddle point formulation ($SPP_{\widehat{\Omega}}$) for Problem 1.2 ($RMP_{\widehat{\Omega}}$), which we present in Section 2. In Section 2.1, we formulate a penalty method to enforce the side constraint $|\boldsymbol{m}(x)| \le 1$ as in [11, 22, 9]. We formulate the discrete stabilized saddle point problem in Section 3.1 as Problem 3.4 ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$) and show existence and uniqueness of solutions. Section 3.4 is devoted to the *a priori* analysis. A numerical example in Section 3.5 illustrates the *a priori* convergence results. A reliable *a posteriori* error estimator is derived Section 4.1, which forms the basis of an adaptive algorithm. A numerical example illustrating the performance of this adaptive algorithm is given Section 4.2.

2. The continuous model

We reformulate the minimizing problem ($RMP_{\widehat{\Omega}}$) as the saddle point problem ($SPP_{\widehat{\Omega}}$) as follows: With $\boldsymbol{L}^2(\Omega) := L^2(\Omega, \mathbb{R}^d)$ let $X := H^1_0(\widehat{\Omega}) \times \boldsymbol{L}^2(\Omega)$, $M := H^1_0(\widehat{\Omega})$, and let $\langle \bullet \,|\, \bullet \rangle_{\boldsymbol{L}^2(\Omega)}$ be the usual inner product in $\boldsymbol{L}^2(\Omega)$. The saddle point formulation ($SPP_{\widehat{\Omega}}$) then reads:

Problem 2.1 ($SPP_{\widehat{\Omega}}$)**.** *Find* $(u, \boldsymbol{m}; p) \in X \times M$ *and* $\lambda_{\boldsymbol{m}} \in L^2(\Omega, \mathbb{R}_{\ge 0})$ *such that*

$$a(u,\boldsymbol{m};v,\boldsymbol{n}) + b(v,\boldsymbol{n};p) = \langle \boldsymbol{f} \,|\, \boldsymbol{n} \rangle_{\boldsymbol{L}^2(\Omega)} \quad \text{for all } (v,\boldsymbol{n}) \in X, \tag{2.1}$$

$$b(u,\boldsymbol{m};q) = 0 \quad \text{for all } q \in M, \tag{2.2}$$

$$\lambda_{\boldsymbol{m}}(x)(1 - |\boldsymbol{m}(x)|) = 0 \quad \text{for almost every } x \in \Omega, \tag{2.3}$$

under the constraint $|\boldsymbol{m}(x)| \le 1$ *a.e. in* Ω*, where*

$$a(u,\boldsymbol{m};v,\boldsymbol{n}) := \langle \nabla u \,|\, \nabla v \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \nabla \varphi^{**} \circ \boldsymbol{m} + \lambda_{\boldsymbol{m}} \boldsymbol{m} \,|\, \boldsymbol{n} \rangle_{\boldsymbol{L}^2(\Omega)}, \tag{2.4}$$

$$b(u,\boldsymbol{m};p) := \langle \nabla u - \boldsymbol{m}\chi_\Omega \,|\, \nabla p \rangle_{\boldsymbol{L}(\widehat{\Omega})}. \tag{2.5}$$

We stress that (2.2) is just the weak formulation of the side constraint (1.4). Next, we show equivalence of the saddle point formulation ($SPP_{\widehat{\Omega}}$) and the minimization problem ($RMP_{\widehat{\Omega}}$) as well as existence of solutions. In general, neither the solution $(u, \boldsymbol{m}) \in X$ of ($RMP_{\widehat{\Omega}}$) nor the solution $(u, \boldsymbol{m}; p) \in X \times M$ of ($SPP_{\widehat{\Omega}}$) is unique. However, as mentioned in the introduction, for the special case of uniaxial materials (see Example 1.3), uniqueness of the magnetization $\boldsymbol{m}$ can be asserted:

Proposition 2.2 (Equivalence of ($SPP_{\widehat{\Omega}}$) and ($RMP_{\widehat{\Omega}}$) & (unique) solvability)**.** *The following statements are true:*

(i) *The relaxed minimization problem* ($RMP_{\widehat{\Omega}}$) *has solutions.*
(ii) *The minimization problem* ($RMP_{\widehat{\Omega}}$) *and the saddle point problem* ($SPP_{\widehat{\Omega}}$) *are equivalent.*
(iii) *The magnetic potential* u *and the Lagrangian* p *are uniquely determined in* ($SPP_{\widehat{\Omega}}$).
(iv) *If* φ^{**} *is given as in Example 1.3 ("uniaxial case"), then problems* ($RMP_{\widehat{\Omega}}$) *and* ($SPP_{\widehat{\Omega}}$) *are uniquely solvable.*

Proof. ***Proof of (i):*** The direct method of the calculus of variations is used in [15] to prove existence of solutions of (RMP), i.e., the relaxed minimization problem in the full space $\mathbb{R}^d$. The same argument works when $\mathbb{R}^d$ is replaced with $\widehat{\Omega}$, which implies the existence of solutions of ($RMP_{\widehat{\Omega}}$), cf. [11].

Proof of (ii): We consider the Euler-Lagrange equation of $(RMP_{\widehat{\Omega}})$:

$$
\begin{aligned}
&(2.6) \quad \langle \nabla u + \nabla\varphi^{**}\circ \boldsymbol{m} + \lambda_{\boldsymbol{m}}\boldsymbol{m} \,|\, \boldsymbol{n}\rangle_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{f}\,|\,\boldsymbol{n}\rangle_{\boldsymbol{L}^2(\Omega)} \quad &&\text{for all } \boldsymbol{n}\in \boldsymbol{L}^2(\Omega),\\
&(2.7) \quad \langle \nabla u - \boldsymbol{m}\chi_\Omega \,|\, \nabla v\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} = 0 &&\text{for all } v\in H^1_0(\widehat{\Omega}),\\
&(2.8) \quad \lambda_{\boldsymbol{m}}(x)(1-|\boldsymbol{m}(x)|) = 0 &&\text{for almost every } x\in\Omega.
\end{aligned}
$$

The equivalence of the minimization problem with the Euler-Lagrange equations (2.6) and (2.8) under the side constraint (1.2) is shown in [15, Theorem 4.2]. Inspection of the procedure there shows that this equivalence also holds under the side constraint (1.4) or (2.7). A more elementary proof of this equivalence using methods from convex analysis can be found in [2]. Next, we show the equivalence of $(SPP_{\widehat{\Omega}})$ and (2.6)—(2.8). Let $(u,\boldsymbol{m};p)\in X\times M$ and $\lambda_{\boldsymbol{m}}\in L^2(\Omega,\mathbb{R}_{\geq 0})$ be a solution of $(SPP_{\widehat{\Omega}})$. The choice $\boldsymbol{n}=\boldsymbol{0}$ in (2.1) shows $p=-u$. With this observation, (2.1) coincides with equation (2.6), and (2.2) is just (2.7). Conversely, let $(u,\boldsymbol{m})\in X$ and $\lambda_{\boldsymbol{m}}\in L^2(\Omega,\mathbb{R}_{\geq 0})$ be a solution of (2.6)—(2.8). As can easily be seen, this triple together with $p=-u$ solves $(SPP_{\widehat{\Omega}})$.

Proof of (iii): According to [11], u is unique. We recall the argument for the sake of completeness. Let $(u_i,\boldsymbol{m}_i;p_i)\in X\times M$ and $\lambda_{\boldsymbol{m}_i}\in L^2(\Omega,\mathbb{R}_{\geq 0})$, $i=1,2$, be two solutions of $(SPP_{\widehat{\Omega}})$. Subtracting equations (2.1) and (2.2) yields with the test functions $v=u_2-u_1$, $\boldsymbol{n}=\boldsymbol{m}_2-\boldsymbol{m}_1$ and $q=p_2-p_1$ the equation

$$
\begin{aligned}
(2.9) \quad &\|\nabla(u_2-u_1)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \nabla\varphi^{**}\circ\boldsymbol{m}_2 - \nabla\varphi^{**}\circ\boldsymbol{m}_1 \,|\, \boldsymbol{m}_2-\boldsymbol{m}_1\rangle_{\boldsymbol{L}^2(\Omega)}\\
&\quad + \langle \lambda_{\boldsymbol{m}_2}\boldsymbol{m}_2 - \lambda_{\boldsymbol{m}_1}\boldsymbol{m}_1 \,|\, \boldsymbol{m}_2-\boldsymbol{m}_1\rangle_{\boldsymbol{L}^2(\Omega)} = 0
\end{aligned}
$$

From the convexity of φ^{**}, we get the non-negativity of the second term. Pointwise non-negativity of the third term was proved in [11, Theorem 2.1]. Hence, all terms vanish and we deduce $u_2=u_1$. It remains to prove uniqueness of p. Suppose $(u,\boldsymbol{m};p_i)\in X\times M$ and $\lambda_{\boldsymbol{m}}\in L^2(\Omega,\mathbb{R}_{\geq 0})$, $i=1,2$, to solve $(SPP_{\widehat{\Omega}})$. From (2.1) we get

$$
b(v,\boldsymbol{n};p_1-p_2)=0 \quad \text{for all } (v,\boldsymbol{n})\in X.
$$

The desired conclusion $p_1=p_2$ follows from the fact that the bilinear form b satisfies an inf-sup condition as we now show: With the norms

$$
\|(u,\boldsymbol{m})\|_X := \left(\|\nabla u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m}\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{1/2} \quad \text{and} \quad \|q\|_M := \|\nabla q\|_{\boldsymbol{L}^2(\widehat{\Omega})},
$$

we obtain for arbitrary $q\in M\setminus\{0\}$

$$
\sup_{(u,\boldsymbol{m})\in X\setminus\{0\}} \frac{|b(u,\boldsymbol{m};q)|}{\|(u,\boldsymbol{m})\|_X\|q\|_M} \geq \frac{|b(q,0;q)|}{\|(q,0)\|_X\|q\|_M} = 1,
$$

which implies

$$
\inf_{q\in M\setminus\{0\}} \sup_{(u,\boldsymbol{m})\in X\setminus\{0\}} \frac{|b(u,\boldsymbol{m};q)|}{\|(u,\boldsymbol{m})\|_X\|q\|_M} \geq 1 > 0.
$$

Proof of (iv): Using mollifier techniques, [9] proves this assertion for the minimization problem (RMP) for the case of a the full space $\mathbb{R}^d$, $d=2$, 3. With minor modifications, the proof applies also to $(RMP_{\widehat{\Omega}})$; see also [11, Thm. 2.1] for an alternative proof for the case $d=2$. Nevertheless, we sketch the key arguments for the uniqueness of $\boldsymbol{m}$ in order to accentuate what the stabilization needs to ensure in the discrete method. Letting $(u_i,\boldsymbol{m}_i,p_i)$, $i=1$, 2 be two solutions, we proceed as above to conclude from (2.9) that $\|\nabla(u_2-u_1)\|_{L^2(\widehat{\Omega})}=0$ and $\langle\nabla\varphi^{**}\circ\boldsymbol{m}_2-\nabla\varphi^{**}\circ\boldsymbol{m}_1,\boldsymbol{m}_2-\boldsymbol{m}_1\rangle_{\boldsymbol{L}^2(\Omega)}=0$. The first equality implies $u_2=u_1$. For the second equality, we use the explicit formula for $\nabla\varphi^{**}$

given in Example 1.3 to get

$$\sum_{i=1}^{d-1} \|(\boldsymbol{m}_2 - \boldsymbol{m}_1) \cdot \boldsymbol{z}_i\|^2_{L^2(\Omega)} = 0. \tag{2.10}$$

In order to conclude that $\boldsymbol{m}_2 - \boldsymbol{m}_1 = 0$, we use the Maxwell equation (2.2): By linearity, we have $b(u_2 - u_1, \boldsymbol{m}_2 - \boldsymbol{m}_1; q) = 0$ for all $q \in M$. This means $(0, \boldsymbol{m}_2 - \boldsymbol{m}_1) \in \mathrm{Ker} b$, or, written in differential equation form

$$\operatorname{div} \boldsymbol{m}\chi_\Omega = 0 \qquad \text{in } H^{-1}(\widehat{\Omega}). \tag{2.11}$$

Combining (2.10) and (2.11) then implies $\boldsymbol{m}_2 - \boldsymbol{m}_1 = 0$. This follows by classical calculus if $\boldsymbol{m}_2 - \boldsymbol{m}_1$ is sufficiently smooth; for the present case of distributions, smoothing arguments have to employed as shown in [30, Satz 2.12] or [19, Lemma 14]. This concludes the proof. □

2.1. Penalization. The pointwise side constraint $|\boldsymbol{m}(x)| \leq 1$ is difficult to enforce numerically. We will therefore relax this condition using a penalty method as originally used in [11] and later also in [9, 22]. We assume from now on that φ^{**} is the restriction to $\mathbb{B}^d$ of a convex and continuous differentiable function defined in the full space $\mathbb{R}^d$.

Given a function $\varepsilon \in L^\infty(\Omega, \mathbb{R}_{>0})$, the penalized problem $(RMP^\varepsilon_{\widehat{\Omega}})$ is then: Find minimizer(s) $\boldsymbol{m} \in \boldsymbol{L}^2(\Omega)$ of

$$\min_{\boldsymbol{m}\in\boldsymbol{L}^2(\Omega)} E^{**,\varepsilon}_{\boldsymbol{f},\widehat{\Omega}}(u, \boldsymbol{m}), \qquad E^{**,\varepsilon}_{\boldsymbol{f},\widehat{\Omega}}(u, \boldsymbol{m}) := \int_\Omega \varphi^{**} \circ \boldsymbol{m} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{m} + \frac{1}{2} \int_{\widehat{\Omega}} |\nabla u|^2 + \frac{1}{2} \int_\Omega \frac{(|\boldsymbol{m}| - 1)^2_+}{\varepsilon} \tag{2.12}$$

where the potential $u \in H^1_0(\widehat{\Omega})$ is the unique solution of the side constraint

$$\langle \nabla u - \boldsymbol{m}\chi_\Omega \,|\, \nabla v \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \quad \text{for all } v \in H^1_0(\widehat{\Omega}). \tag{2.13}$$

Later on, the penalization parameter ε will be related to be the local mesh-size in the discrete version of (2.12). We mention that $E^{**,\varepsilon}_{\boldsymbol{f},\widehat{\Omega}}$ is convex, continuous, Gâteaux differentiable and coercive. In particular, the direct method of the calculus of variations proves that $(RMP^\varepsilon_{\widehat{\Omega}})$ has solutions, and Proposition 2.2 holds accordingly. Details are left to the reader.

3. The discrete problem

3.1. Notation. Let $\widehat{\mathcal{T}} := \{K_1, \ldots, K_{\widehat{M}}\}$ denote an affine, regular, γ-shape regular triangulation of $\widehat{\Omega}$ and its restriction to Ω, $\mathcal{T} := \widehat{\mathcal{T}}|_\Omega := \{K \in \mathcal{T} : K \subset \Omega\} = \{K_1, \ldots, K_M\}$ be a triangulation of Ω. The spaces of scalar-valued or vector-valued polynomials of (total) degree k on an element K are denoted $\mathcal{P}^k(K)$ and $\mathcal{P}^k(K; \mathbb{R}^d)$. Introduce the scalar-valued spaces

$$S^{k,1}(\widehat{\mathcal{T}}) := \{u \in H^1(\widehat{\Omega}) : u|_K \in \mathcal{P}^k(K), \text{ for all } K \in \widehat{\mathcal{T}}\} \subset H^1(\widehat{\Omega}),$$
$$S^{k,1}_0(\widehat{\mathcal{T}}) := S^{k,1}(\widehat{\mathcal{T}}) \cap H^1_0(\widehat{\Omega}),$$

of globally continuous, $\widehat{\mathcal{T}}$-piecewise polynomials of degree k. The scalar-valued and vector-valued spaces of $\mathcal{T}$-piecewise polynomials of degree k are denoted by

$$S^{k,0}(\mathcal{T}) := \{u \in L^2(\Omega) : u|_K \in \mathcal{P}^k(K), \text{ for all } K \in \mathcal{T}\} \subset L^2(\Omega),$$
$$\boldsymbol{S}^{k,0}(\mathcal{T}) := \{\boldsymbol{m} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{m}|_K \in \mathcal{P}^k(K, \mathbb{R}^d), \text{ for all } K \in \mathcal{T}\} \subset \boldsymbol{L}^2(\Omega)$$

In addition, we use the abbreviations $X^k_N := S^{k,1}_0(\widehat{\mathcal{T}}) \times \boldsymbol{S}^{k-1,0}(\mathcal{T}) \subset X$ and $M^k_N := S^{k,1}_0(\widehat{\mathcal{T}}) \subset M$ with $k \geq 1$.

3.2. An unstable saddle point formulation. We formulate now a discrete version of the saddle point problem ($SPP_{\widehat{\Omega}}$). The starting point is the minimization of the penalized energy functional $E^{**,\varepsilon}_{\boldsymbol{f},\widehat{\Omega}}(u,\boldsymbol{m})$ on the discrete space X^k_N. More specifically, the minimization problem ($RMP^{\varepsilon,N}_{\widehat{\Omega}}$) is: Find $(u_N,\boldsymbol{m}_N)\in X^k_N$ such that $E^{**,\varepsilon}_{\boldsymbol{f},\widehat{\Omega}}$ is minimized under the side constraint

$$b(u_N,\boldsymbol{m}_N;q)=0 \qquad \forall q\in M^k_N. \tag{3.1}$$

The Lagrangian associated with this constrained minimization problem is

$$\mathcal{L}^\varepsilon(u,\boldsymbol{m};p):=E^{**,\varepsilon}_{\boldsymbol{f},\widehat{\Omega}}(u,\boldsymbol{m})+b(u,\boldsymbol{m};p)\quad \text{for all } (u,\boldsymbol{m})\in X^k_N, p\in M^k_N. \tag{3.2}$$

The solution of the constrained minimization problem is the stationary point of the Lagrangian $\mathcal{L}^\varepsilon$. The derivatives of $\mathcal{L}^\varepsilon$ can be computed explicitly leading to the following formulation, if we select ε to be a $\mathcal{T}$-piecewise constant function:

Problem 3.1 (SPP$^{\varepsilon,N}_{\widehat{\Omega}}$)**.** *Let $\varepsilon\in S^{0,0}(\mathcal{T})$ and $\varepsilon>0$. Find $(u_N,\boldsymbol{m}_N;p_N)\in X^k_N\times M^k_N$ such that*

$$a_N(u_N,\boldsymbol{m}_N;v,\boldsymbol{n})+b(v,\boldsymbol{n};p_N)=\langle \boldsymbol{f}\,|\,\boldsymbol{n}\rangle_{\boldsymbol{L}^2(\Omega)} \quad \textit{for all } (v,\boldsymbol{n})\in X^k_N, \tag{3.3}$$

$$b(u_N,\boldsymbol{m}_N;q) = 0 \quad \textit{for all } q\in M^k_N, \tag{3.4}$$

where we set

$$a_N(u_N,\boldsymbol{m}_N;v,\boldsymbol{n}) := \langle\nabla u_N\,|\,\nabla v\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}+\langle\nabla\varphi^{**}\circ\boldsymbol{m}_N+\lambda_N\boldsymbol{m}_N\,|\,\boldsymbol{n}\rangle_{\boldsymbol{L}^2(\Omega)}, \tag{3.5}$$

$$\lambda_N := \frac{(|\boldsymbol{m}_N|-1)_+}{\varepsilon|\boldsymbol{m}_N|}. \tag{3.6}$$

Compared with the continuous formulation in Problem 2.1, the main difference is that the continuous Lagrange multiplier $\lambda_{\boldsymbol{m}}\in L^2(\Omega,\mathbb{R}_{\geq 0})$, characterized by the condition (2.3), is replaced by the term (3.6).

Since the minimization problem ($RMP^{\varepsilon,N}_{\widehat{\Omega}}$) has solutions, it is easy to show via the Euler-Lagrange equation that ($SPP^{\varepsilon,N}_{\widehat{\Omega}}$) also has solutions. Here, the existence and uniqueness of the Lagrange parameter p_N follows from a discrete inf-sup condition of the bilinear form b in the same way as in the proof of Lemma 2.2. Reviewing the arguments of Lemma 2.2 also shows the uniqueness of u_N. Uniqueness of $\boldsymbol{m}_N$, however, cannot be ensured by repeating the arguments of the continuous case presented in Proposition 2.2 since $\operatorname{Ker} b_N\not\subset\operatorname{Ker} b$, where we define the kernels

$$\operatorname{Ker} b:=\big\{(u,\boldsymbol{m})\in X\ :\ b(u,\boldsymbol{m};q)=0\quad\text{for all } q\in H^1_0(\widehat{\Omega})\big\},$$
$$\operatorname{Ker}_N b:=\big\{(u_N,\boldsymbol{m}_N)\in X^k_N\ :\ b(u_N,\boldsymbol{m}_N;q)=0\quad\text{for all } q\in M^k_N\big\}.$$

This lack of uniqueness expresses the fact that the discrete formulation is unstable. Stability can be obtained by adding a stabilization term as shown in the following section.

3.3. A stable saddle point formulation. In order to ensure uniqueness of a solution $(u_N,\boldsymbol{m}_N)\in X^k_N$ of the discrete problem, we add a suitable stabilization term σ to the term a_N. To that end, we define the augmented Lagrangian as

$$\mathcal{L}^{aug}(u,\boldsymbol{m};p):=E^{**,\varepsilon}_{\boldsymbol{f},\widehat{\Omega}}(u,\boldsymbol{m})+b(u,\boldsymbol{m};p)+\frac{1}{2}\sigma(u,\boldsymbol{m};u,\boldsymbol{m}), \tag{3.7}$$

where the stabilizing bilinear form σ is defined by

$$\begin{aligned}\sigma(u,\boldsymbol{m};v,\boldsymbol{n}) &:= \sum_{K\in\mathcal{T}} h_K^2 \int_K \nabla\cdot(\nabla u-\boldsymbol{m})\,\nabla\cdot(\nabla v-\boldsymbol{n}) \\ &\qquad + \sum_{E\in\mathcal{E}} h_E \int_E [(\nabla u-\boldsymbol{m}\chi_\Omega)\cdot\boldsymbol{\nu}]_E[(\nabla v-\boldsymbol{n}\chi_\Omega)\cdot\boldsymbol{\nu}]_E.\end{aligned} \tag{3.8}$$

Here, $\mathcal{E}$ denotes the set of edges ($d=2$) or faces ($d=3$) of the elements of the triangulation $\mathcal{T}$ of Ω. Moreover, $[\bullet]_E$ denotes the jump across an edge or face E and $\boldsymbol{\nu}$ the normal vector of E, i.e.

$$[(\nabla u-\boldsymbol{m}\chi_\Omega)\cdot\boldsymbol{\nu}]_E := (\nabla u-\boldsymbol{m}\chi_\Omega)|_{K'}\cdot\boldsymbol{\nu}_{K'} + (\nabla u-\boldsymbol{m}\chi_\Omega)|_{K''}\cdot\boldsymbol{\nu}_{K''}$$

on the edge (or face) $E=\overline{K'}\cap\overline{K''}\in\mathcal{E}$, which is the intersection of uniquely determined elements $K',K''\in\widehat{\mathcal{T}}$. $\boldsymbol{\nu}_{K'}$ and $\boldsymbol{\nu}_{K''}$ denote the exterior normal vectors of K' and K'' respectively. Finally, we denote with h_E and h_K the diameter of an edge (or face) E and an element K.

Lemma 3.2 (Stabilizing bilinear form)**.** *The bilinear form $\sigma(\bullet;\bullet)$ defined in* (3.8) *is positive semi-definite, symmetric, and consistent, i.e., the exact solution $(u,\boldsymbol{m})\in X$ satisfies $\sigma(u,\boldsymbol{m};v,\boldsymbol{n})=0$ for all $(v,\boldsymbol{n})\in X_N^k$. Moreover, there holds the estimate*

$$\|\mathrm{div}\,(\nabla u-\boldsymbol{m}\chi_\Omega)\|^2_{H^{-1}(\widehat{\Omega})} \lesssim \sigma(u,\boldsymbol{m};u,\boldsymbol{m}) + \|\nabla u\|^2_{\boldsymbol{L}^2(\widehat{\Omega}\setminus\Omega)} \quad \textit{for all } (u,\boldsymbol{m})\in \mathrm{Ker}_N\, b. \tag{3.9}$$

Remark 3.3. *Estimate* (3.9) *provides a connection between the discrete kernel* $\mathrm{Ker}_N\, b$ *and the continuous kernel* $\mathrm{Ker}\, b$*. It is this link between* $\mathrm{Ker}\, b$ *and* $\mathrm{Ker}_N\, b$ *that is the essential ingredient of the uniqueness assertion of Theorem 3.5 below for the stabilized discrete problem.*

Proof of Lemma 3.2. Clearly, σ is symmetric and positive semi-definite. To see the consistency, we note that (1.4) implies $\nabla u-\boldsymbol{m}\chi_\Omega\in H(\mathrm{div};\widehat{\Omega},\mathbb{R}^d)$ with $\mathrm{div}\,(\nabla u-\boldsymbol{m}\chi_\Omega)=0$. Hence, $\mathrm{div}\,(\nabla u-\boldsymbol{m})=0$ a.e. in Ω. Furthermore, since $\nabla u-\boldsymbol{m}\chi_\Omega\in H(\mathrm{div};\widehat{\Omega},\mathbb{R}^d)$, its normal trace is in $H^{-1/2}(E)$ for each edge (or face, if $d=3$) E and $[(\nabla u-\boldsymbol{m})\cdot\boldsymbol{\nu}]_E = 0\in H^{-1/2}(E)$. Hence, $[(\nabla u-\boldsymbol{m})\cdot\boldsymbol{\nu}]_E=0$ a.e. on $E\in\mathcal{E}$.

To prove the estimate (3.9), we employ the Clément interpolant operator $I_N: H_0^1(\widehat{\Omega})\to S_0^{k,1}(\widehat{\mathcal{T}})$ of [13]. Given $(u,\boldsymbol{m})\in\mathrm{Ker}_N\, b$, we estimate

$$\begin{aligned}\|\mathrm{div}\,(\nabla u-\boldsymbol{m}\chi_\Omega)\|_{H^{-1}(\widehat{\Omega})} &\le \sup_{q\in H_0^1(\widehat{\Omega})\setminus\{0\}} \frac{\left|\langle\nabla u-\boldsymbol{m}\chi_\Omega\,|\,\nabla q\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}\right|}{\|\nabla q\|_{\boldsymbol{L}^2(\widehat{\Omega})}} \\ = \sup_{q\in H_0^1(\widehat{\Omega})\setminus\{0\}} &\frac{\left|\langle\nabla u-\boldsymbol{m}\chi_\Omega\,|\,\nabla(q-I_Nq)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}\right|}{\|\nabla q\|_{\boldsymbol{L}^2(\widehat{\Omega})}} \\ \le \sup_{q\in H_0^1(\widehat{\Omega})\setminus\{0\}} &\frac{\left|\langle\nabla u-\boldsymbol{m}\,|\,\nabla(q-I_Nq)\rangle_{\boldsymbol{L}^2(\Omega)}\right| + \left|\langle\nabla u\,|\,\nabla(q-I_Nq)\rangle_{\boldsymbol{L}^2(\widehat{\Omega}\setminus\Omega)}\right|}{\|\nabla q\|_{\boldsymbol{L}^2(\widehat{\Omega})}} \\ \le \sup_{q\in H_0^1(\widehat{\Omega})\setminus\{0\}} &\frac{\sum_{K\in\mathcal{T}}\left|\langle(\nabla u-\boldsymbol{m})\cdot\boldsymbol{\nu}\,|\,q-I_Nq\rangle_{\boldsymbol{L}^2(\partial K)}\right| + C\|\nabla u\|_{\boldsymbol{L}^2(\widehat{\Omega}\setminus\Omega)}\|\nabla q\|_{\boldsymbol{L}^2(\widehat{\Omega})}}{\|\nabla q\|_{\boldsymbol{L}^2(\widehat{\Omega})}}\end{aligned}$$

Application of standard properties of the Clément interpolant yields the claimed result (3.9). We omit the details. □

We now formulate the stabilized discrete saddle point problem $(SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma})$:

Problem 3.4 ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$)**.** *Find* $(u_N, \boldsymbol{m}_N) \in X^k_N$*,* $p_N \in M^k_N$ *such that*

$$
\begin{aligned}
&(3.10) \quad a^\sigma_N(u_N, \boldsymbol{m}_N; v, \boldsymbol{n}) + b(v, \boldsymbol{n}; p_N) = \langle \boldsymbol{f} \,|\, \boldsymbol{n}\rangle_{\boldsymbol{L}^2(\Omega)} \quad \textit{for all } (v, \boldsymbol{n}) \in X^k_N,\\
&(3.11) \quad b(u_N, \boldsymbol{m}_N; q) = 0 \quad \textit{for all } q \in M^k_N
\end{aligned}
$$

with $a^\sigma_N(u_N, \boldsymbol{m}_N; v, \boldsymbol{n}) := a_N(u_N, \boldsymbol{m}_N; v, \boldsymbol{n}) + \sigma(u_N, \boldsymbol{m}_N; v, \boldsymbol{n})$*.*

The following theorem states existence and uniqueness of the solution $(u_N, \boldsymbol{m}_N, p_N)$ of the stabilized discrete saddle point problem.

Theorem 3.5 (Stability and (unique) solvability of discrete saddle point problem ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$))**.** *The following statements are true:*

(i) *The discrete problem* ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$) *has solutions.*
(ii) *The magnetic potential* u_N *and the Lagrangian* p_N *are uniquely determined in* ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$)*.*
(iii) *If* φ^{**} *is given as in Example 1.3 ("uniaxial case"), the discrete problem* ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$) *is uniquely solvable.*

Proof. Existence of solutions $(u_N, \boldsymbol{m}_N, p_N)$ for ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$) as well as uniqueness of u_n and p_N follow as in the continuous case, see Proposition 2.2. Thus, it only remains to prove uniqueness of $\boldsymbol{m}_N$ for the uniaxial case: Let $(u_{N,i}, \boldsymbol{m}_{N,i}; p_{N,i})$, for $i = 1$, 2 be solutions of ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$). We use the abbreviations $e_u := u_{N,2} - u_{N,1}$, $\boldsymbol{e_m} := \boldsymbol{m}_{N,2} - \boldsymbol{m}_{N,1}$ and $e_p := p_{N,2} - p_{N,1}$. From (3.11) we obtain

$$(3.12) \quad \langle \nabla e_u - \boldsymbol{e_m}\chi_\Omega \,|\, \nabla q\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} = 0 \quad \text{for all } q \in M^k_N,$$

and hence $(e_u, \boldsymbol{e_m}) \in \operatorname{Ker}_N b$. The key step is now to show that $(e_u, \boldsymbol{e_m}) \in \operatorname{Ker} b$ since then the same arguments as in the continuous case can be employed to show uniqueness.

Equation (3.10) with $v := e_u$ and $\boldsymbol{n} := \boldsymbol{e_m}$ together with (3.12) and $q = e_p$ shows

$$\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \sum_{i=1}^{d-1} \|\boldsymbol{e_m}\cdot \mathbf{z}_i\|^2_{L^2(\Omega)} + \langle \lambda_{N,2}\boldsymbol{m}_{N,2} - \lambda_{N,1}\boldsymbol{m}_{N,1} \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} + \sigma(e_u, \boldsymbol{e_m}; e_u, \boldsymbol{e_m}) = 0.$$

In [11, Theorem 3.1], it is shown that $(\lambda_{N,2}\boldsymbol{m}_{N,2} - \lambda_{N,1}\boldsymbol{m}_{N,1})\cdot \boldsymbol{e_m} \geq 0$ pointwise almost everywhere in Ω. Non-negativity of the bilinear form $\sigma(\bullet;\bullet)$ now leads to $e_u = 0$ and $\boldsymbol{e_m}\cdot \mathbf{z}_i = 0$ almost everywhere in $\widehat{\Omega}$ and Ω respectively. Furthermore,

$$\|\operatorname{div}(\nabla e_u - \boldsymbol{e_m}\chi_\Omega)\|^2_{H^{-1}(\widehat{\Omega})} \lesssim \sigma(e_u, \boldsymbol{e_m}; e_u, \boldsymbol{e_m}) + \|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega}\setminus\Omega)} = 0$$

shows $(e_u, \boldsymbol{e_m}) \in \operatorname{Ker} b$. In particular, we get together with $e_u = 0$

$$(3.13) \quad \operatorname{div} \boldsymbol{e_m}\chi_\Omega = 0 \quad \text{in } H^{-1}(\widehat{\Omega}) \quad \text{and hence} \quad \operatorname{div} \boldsymbol{e_m}\chi_\Omega = 0 \quad \text{in } L^2(\mathbb{R}^d).$$

This observation combined with $\boldsymbol{e_m}\cdot \mathbf{z}_i = 0$, $i = 1, \ldots, d-1$, enables us to prove $e_m\chi_\omega = 0$ on $\mathbb{R}^d$ by smoothing techniques as first noted in [30, Satz 2.12]. Hence, we have uniqueness of $\boldsymbol{m}_N$. Finally, the discrete inf-sup condition of the bilinear form b ensures uniqueness of the Lagrange multiplier p_N. □

3.4. A priori error estimation. In this section, we present a full *a priori* error analysis for the lowest order case $k = 1$, in Theorem 3.6 for general functions φ^{**} and in Theorem 3.7 for the special case of uniaxial materials given in Example 1.3. In both theorems, the continuous problem is understood to be ($SPP_{\widehat{\Omega}}$) and the discrete problem ($SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma}$). We begin with a general *a priori* estimate for arbitrary anisotropy densities φ^{**}, for which we can establish convergence $O(h + \sqrt{\varepsilon})$ for the lowest order discretization (under smoothness assumptions).

Theorem 3.6 (A priori estimate)**.** *Let* $(u, \boldsymbol{m}, p)$ *and* $(u_N, \boldsymbol{m}_N, p_N)$ *be solutions of Problem 2.1* $(SPP_{\widehat{\Omega}})$ *and Problem 3.4* $(SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma})$ *with* $k = 1$, *i.e.,* $X^1_N = S^{1,1}_0(\widehat{\mathcal{T}}) \times \boldsymbol{S}^{0,0}(\mathcal{T})$ *and* $M^1_N = S^{1,1}_0(\widehat{\mathcal{T}})$. *The following a priori estimate holds for all* $u_{\mathcal{T}}, p_{\mathcal{T}} \in S^{1,1}_0(\widehat{\mathcal{T}})$ *and for all* $\boldsymbol{m}_{\mathcal{T}} \in \boldsymbol{S}^{0,0}(\mathcal{T})$:

$$
\begin{aligned}
&\|\nabla(u - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \nabla\varphi^{**} \circ \boldsymbol{m} - \nabla\varphi^{**} \circ \boldsymbol{m}_N \,|\, \boldsymbol{m} - \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + |(u - u_N, \boldsymbol{m} - \boldsymbol{m}_N)|^2_\sigma + \|\nabla(p - p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\le C_1 C_\sigma^2 \Big\{ \Big(1 + \frac{\delta_1 + \delta_2}{\delta_1\delta_2} + C_\sigma^2\Big) \Big(\|\nabla(u - u_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m} - \boldsymbol{m}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} \Big) \\
&\quad + |(u - u_{\mathcal{T}}, \boldsymbol{m} - \boldsymbol{m}_{\mathcal{T}})|^2_\sigma + \delta_1 \|\nabla\varphi^{**} \circ \boldsymbol{m} - \nabla\varphi^{**} \circ \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \delta_2 \|\lambda_{\boldsymbol{m}}\boldsymbol{m} - \lambda_N \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + C_\gamma^2 \|p - p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} + \|\varepsilon^{1/2}\lambda_{\boldsymbol{m}}\boldsymbol{m}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\varepsilon^{1/2}\lambda_N \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} \Big\}.
\end{aligned}
\tag{3.14}
$$

The constant $C_\gamma = C_\gamma(\widehat{\Omega}, \widehat{\mathcal{T}}) > 0$ *depends only on the domain* $\widehat{\Omega}$ *and the shape regularity of the triangulation* $\widehat{\mathcal{T}}$. $C_\sigma > 0$ *is the same constant as in* (3.22). *The positive constants* $\delta_1, \delta_2 > 0$ *can be chosen arbitrarily small, and* $C_1 > 0$ *is an absolute constant. The mesh-dependent norm* $\|p - p_{\mathcal{T}}\|_{\widehat{\mathcal{T}}}$ *is defined by*

$$
\|p - p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} := \sum_{K \in \widehat{\mathcal{T}}} \Big\{ h_K^{-2} \|p - p_{\mathcal{T}}\|^2_{L^2(K)} + \|\nabla(p - p_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(K)} \Big\}.
\tag{3.15}
$$

In the uniaxial case, the upper bound is now improved to $O(h + \varepsilon)$. The power of h is optimal for lowest-order elements $k = 1$. The power of ε is empirically optimal as is seen from numerical experiments, cf. Section 3.5.

Theorem 3.7 (A priori estimate for uniaxial case)**.** *Assume in addition to the hypotheses of Theorem 3.6 that*

$$
C_0 \|\nabla\varphi^{**} \circ \boldsymbol{m}_1 - \nabla\varphi^{**} \circ \boldsymbol{m}_2\|^2_{\boldsymbol{L}^2(\Omega)} \le \langle \nabla\varphi^{**} \circ \boldsymbol{m}_1 - \nabla\varphi^{**} \circ \boldsymbol{m}_2 \,|\, \boldsymbol{m}_1 - \boldsymbol{m}_2 \rangle_{\boldsymbol{L}^2(\Omega)}.
\tag{3.16}
$$

Then there holds the a priori estimate

$$
\begin{aligned}
&\|\nabla(u - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\nabla\varphi^{**} \circ \boldsymbol{m} - \nabla\varphi^{**} \circ \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\lambda_{\boldsymbol{m}}\boldsymbol{m} - \lambda_N \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + |(u - u_N, \boldsymbol{m} - \boldsymbol{m}_N)|^2_\sigma + \|\nabla(p - p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\le 2C_2\big(1 + C_2\|\varepsilon\|_{L^\infty(\Omega)}\big) \Big\{ C_3 \Big(\|\nabla(u - u_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m} - \boldsymbol{m}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} \Big) \\
&\quad + |(u - u_{\mathcal{T}}, \boldsymbol{m} - \boldsymbol{m}_{\mathcal{T}})|^2_\sigma + C_\gamma^2 \|p - p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} + \|\lambda_{\boldsymbol{m}}\boldsymbol{m} - \Pi(\lambda_{\boldsymbol{m}}\boldsymbol{m})\|^2_{\boldsymbol{L}^2(\Omega)} \Big\} \\
&\quad + 4C_2^2 \|\varepsilon\|_{L^\infty(\Omega)} \|\varepsilon^{1/2}\lambda_{\boldsymbol{m}}\boldsymbol{m}\|^2_{\boldsymbol{L}^2(\Omega)},
\end{aligned}
\tag{3.17}
$$

where $\Pi : \boldsymbol{L}^2(\Omega) \twoheadrightarrow \boldsymbol{S}^{0,0}(\mathcal{T})$ *denotes the* $\boldsymbol{L}^2(\Omega)$*-orthogonal projection. The constants* $C_2, C_3 > 0$ *are defined by* $C_2 := 4(1 + 6C_\sigma^2)\frac{C_1 C_\sigma^2}{C_0}$ *and* $C_3 := 1 + 6(1 + 4C_\sigma^2)\frac{C_1 C_\sigma^2}{C_0} + C_\sigma^2$ *with* C_1, C_σ, $C_\gamma > 0$ *of Theorem 3.6.*

Corollary 3.8. *In addition to the assumptions of Theorems 3.6 and 3.7, assume for the solution* $(u, \boldsymbol{m}, p, \lambda_{\boldsymbol{m}})$ *of problem* $(SPP_{\widehat{\Omega}})$ *the regularity assertions* u, $p \in H^2(\widehat{\Omega})$, $\boldsymbol{m} \in H^1(\Omega, \mathbb{R}^d)$ *as well as* $\lambda_{\boldsymbol{m}}\boldsymbol{m} \in H^1(\Omega, \mathbb{R}^d)$. *Then, with* $h := \max_{K \in \widehat{\mathcal{T}}} h_K$:

$$
\begin{aligned}
&\|\nabla(u - u_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\nabla\varphi^{**} \circ \boldsymbol{m} - \nabla\varphi^{**} \circ \boldsymbol{m}_N\|_{\boldsymbol{L}^2(\Omega)} + \|\lambda_{\boldsymbol{m}}\boldsymbol{m} - \lambda_N \boldsymbol{m}_N\|_{\boldsymbol{L}^2(\Omega)} \\
&\quad + |(u - u_N, \boldsymbol{m} - \boldsymbol{m}_N)|_\sigma + \|\nabla(p - p_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})} = \mathcal{O}\big(h + \|\varepsilon\|_{L^\infty(\Omega)}\big).
\end{aligned}
\tag{3.18}
$$

Proof. Let Iu, $Ip \in S_0^{1,1}(\widehat{\mathcal{T}})$ be the piecewise linear nodal interpolants of u and p. The result then follows from (3.17) with the choices $u_{\mathcal{T}} = Iu$, $p_{\mathcal{T}} = Ip$ and $\boldsymbol{m}_{\mathcal{T}} = \Pi \boldsymbol{m}$. □

We start by formulating the Galerkin orthogonalities available to us: Subtracting (3.10) from (2.1) and (3.11) from (2.2) yields together with the consistency of σ the two relations

$$\begin{aligned} &\langle \nabla(u-u_N) \,|\, \nabla v_N \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \nabla\varphi^{**} \circ \boldsymbol{m} - \nabla\varphi^{**} \circ \boldsymbol{m}_N \,|\, \boldsymbol{n}_N \rangle_{\boldsymbol{L}^2(\Omega)} \\ &+ \langle \lambda_{\boldsymbol{m}} \boldsymbol{m} - \lambda_N \boldsymbol{m}_N \,|\, \boldsymbol{n}_N \rangle_{\boldsymbol{L}^2(\Omega)} + \langle \nabla v_N - \boldsymbol{n}_N \chi_\Omega \,|\, \nabla(p-p_N) \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \\ &+ \sigma(u-u_N, \boldsymbol{m}-\boldsymbol{m}_N; v_N, \boldsymbol{n}_N) = 0 \qquad \text{for all } (v_N, \boldsymbol{n}_N) \in X_N^k, \end{aligned} \tag{3.19}$$

and

$$\langle \nabla(u-u_N) - (\boldsymbol{m}-\boldsymbol{m}_N)\chi_\Omega \,|\, \nabla q_N \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} = 0, \qquad \text{for all } q_N \in M_N^k \tag{3.20}$$

The symmetric positive semi-definite bilinear form σ of (3.8) introduces a seminorm $|\cdot|_\sigma$ in the standard way by

$$|(u,\boldsymbol{m})|_\sigma^2 := \sigma(u,\boldsymbol{m}; u,\boldsymbol{m}).$$

We notice the Cauchy-Schwarz inequality for all $(u,\boldsymbol{m})$, $(v,\boldsymbol{n})$ with finite semi-norm:

$$\sigma(u,\boldsymbol{m}; v,\boldsymbol{n}) \le |(u,\boldsymbol{m})|_\sigma |(v,\boldsymbol{n})|_\sigma. \tag{3.21}$$

Furthermore, we have the following inverse estimate:

Lemma 3.9. *There exists $C_\sigma > 0$ depending only on the shape-regularity of $\widehat{\mathcal{T}}$ and k such that*

$$|(u_N,\boldsymbol{m}_N)|_\sigma^2 \le C_\sigma^2 (\|\nabla u_N\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)}) \qquad \text{for all } (u_N,\boldsymbol{m}_N) \in X_N^k. \tag{3.22}$$

Proof. The estimate (3.22) follows from transformation to the reference element and norm equivalence on finite dimensional spaces on the reference element. □

We will use the following abbreviations:

$$\begin{aligned} \boldsymbol{d} &:= \nabla\varphi^{**} \circ \boldsymbol{m}, & \boldsymbol{d}_N &:= \nabla\varphi^{**} \circ \boldsymbol{m}_N, \\ \boldsymbol{l} &:= \lambda_{\boldsymbol{m}} \boldsymbol{m}, & \boldsymbol{l}_N &:= \lambda_N \boldsymbol{m}_N. \end{aligned} \tag{3.23}$$

Proof of Theorem 3.6. First, in step 1 to step 8, it is convenient to look only for an estimate in the discrete constrained space $\mathrm{Ker}_N\, b$. So consider first elements $(u^\star_{\mathcal{T}}, \boldsymbol{m}^\star_{\mathcal{T}}) \in \mathrm{Ker}_N b$ instead of elements in the whole approximation space X_N^1. Later on in step 9, it will be shown how to get an estimate for arbitrary $(u_{\mathcal{T}}, \boldsymbol{m}_{\mathcal{T}}) \in X_N^1$ in the sense of a best approximation result.

Step 1: The Galerkin orthogonality (3.19) with $v_N = u^\star_{\mathcal{T}} - u_N$ and $\boldsymbol{n}_N = \boldsymbol{m}^\star_{\mathcal{T}} - \boldsymbol{m}_N$ yields

$$\begin{aligned} &\|\nabla(u^\star_{\mathcal{T}} - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{m} - \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \langle \boldsymbol{l} - \boldsymbol{l}_N \,|\, \boldsymbol{m} - \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)} + |(u^\star_{\mathcal{T}} - u_N, \boldsymbol{m}^\star_{\mathcal{T}} - \boldsymbol{m}_N)|^2_\sigma \\ &= \langle \nabla(u^\star_{\mathcal{T}} - u) \,|\, \nabla(u^\star_{\mathcal{T}} - u_N) \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{m} - \boldsymbol{m}^\star_{\mathcal{T}} \rangle_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \langle \boldsymbol{l} - \boldsymbol{l}_N \,|\, \boldsymbol{m} - \boldsymbol{m}^\star_{\mathcal{T}} \rangle_{\boldsymbol{L}^2(\Omega)} - \langle \nabla(u^\star_{\mathcal{T}} - u_N) - (\boldsymbol{m}^\star_{\mathcal{T}} - \boldsymbol{m}_N)\chi_\Omega \,|\, \nabla(p - p_N) \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \\ &\quad + \sigma(u^\star_{\mathcal{T}} - u, \boldsymbol{m}^\star_{\mathcal{T}} - \boldsymbol{m}; u^\star_{\mathcal{T}} - u_N, \boldsymbol{m}^\star_{\mathcal{T}} - \boldsymbol{m}_N). \end{aligned} \tag{3.24}$$

Step 2: We claim

$$\begin{aligned} &\left| \langle \nabla(u^\star_{\mathcal{T}} - u_N) - (\boldsymbol{m}^\star_{\mathcal{T}} - \boldsymbol{m}_N)\chi_\Omega \,|\, \nabla(p - p_N) \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \right| \\ &\le C_\gamma \Big\{ |(u^\star_{\mathcal{T}} - u_N, \boldsymbol{m}^\star_{\mathcal{T}} - \boldsymbol{m}_N)|^2_\sigma + \|\nabla(u^\star_{\mathcal{T}} - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \Big\}^{1/2} \|p - p_{\mathcal{T}}\|_{\widehat{\mathcal{T}}} \end{aligned} \tag{3.25}$$

for arbitrary $p_{\mathcal{T}} \in M_N^1 = S_0^{1,1}(\widehat{\mathcal{T}})$ and a constant $C_\gamma = C_\gamma(\widehat{\Omega}, \widehat{\mathcal{T}})$ which depends only on the domain $\widehat{\Omega}$ and the shape regularity of the triangulation $\widehat{\mathcal{T}}$.

Note first with $(u_\mathcal{T}^\star, \boldsymbol{m}_\mathcal{T}^\star), (u_N, \boldsymbol{m}_N) \in \mathrm{Ker}_N\, b$ the validity of

$$\langle \nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega \,|\, \nabla(p_\mathcal{T} - p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} = b(u_\mathcal{T}^\star - u_N, \boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N; p_\mathcal{T} - p_N) = 0.$$

With this the proof of step 2 starts by expanding

$$\begin{aligned}
&\langle \nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega \,|\, \nabla(p - p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&= \langle \nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega \,|\, \nabla(p - p_\mathcal{T})\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + b(u_\mathcal{T}^\star - u_N, \boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N; p_\mathcal{T} - p_N) \\
&= \sum_{K\in\widehat{\mathcal{T}}} \int_K \big(\nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega\big) \cdot \nabla(p - p_\mathcal{T}).
\end{aligned}$$

Using integration by parts, where $\widehat{\mathcal{E}}^{\mathrm{int}}$ denotes all edges (or faces) of elements of $\widehat{\mathcal{T}}$ that lie within $\widehat{\Omega}$, we see

$$\begin{aligned}
&= \sum_{K\in\widehat{\mathcal{T}}} -\int_K \nabla\cdot\big(\nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega\big)(p - p_\mathcal{T}) \\
&\quad + \sum_{E\in\widehat{\mathcal{E}}^{\mathrm{int}}} \int_E \big[\boldsymbol{\nu}\cdot\big(\nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega\big)\big]_E (p - p_\mathcal{T}).
\end{aligned}$$

Applying Cauchy-Schwarz inequality twice, we obtain

$$\begin{aligned}
&\le \sqrt{\sum_{K\in\widehat{\mathcal{T}}} h_K^2 \big\|\nabla\cdot\big(\nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega\big)\big\|_{L^2(K)}^2}\sqrt{\sum_{K\in\widehat{\mathcal{T}}} h_K^{-2}\|p - p_\mathcal{T}\|_{L^2(K)}^2} \\
&\quad + \sqrt{\sum_{E\in\widehat{\mathcal{E}}^{\mathrm{int}}} h_E \big\|\big[\boldsymbol{\nu}\cdot\big(\nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega\big)\big]_E\big\|_{L^2(E)}^2}\sqrt{\sum_{E\in\widehat{\mathcal{E}}^{\mathrm{int}}} h_E^{-1}\|p - p_\mathcal{T}\|_{L^2(E)}^2}.
\end{aligned}$$

By γ-shape regularity of $\widehat{\mathcal{T}}$, there holds $h_E \sim h_K$ uniformly for all $K \in \widehat{\mathcal{T}}$ and edges/faces $E \in \widehat{\mathcal{E}}^{\mathrm{int}}$ of K. The trace theorem together with a scaling argument shows for both cases $d = 2, 3$

$$h_K^{-1}\|p - p_\mathcal{T}\|_{L^2(\partial K)}^2 \lesssim h_K^{-2}\|p - p_\mathcal{T}\|_{L^2(K)}^2 + \|\nabla(p - p_\mathcal{T})\|_{\boldsymbol{L}^2(K)}^2.$$

Hence, we get

$$\sum_{E\in\widehat{\mathcal{E}}^{\mathrm{int}}} h_E^{-1}\|p - p_\mathcal{T}\|_{L^2(E)}^2 \lesssim \sum_{K\in\widehat{\mathcal{T}}} h_K^{-1}\|p - p_\mathcal{T}\|_{L^2(\partial K)}^2 \lesssim \|p - p_\mathcal{T}\|_{\widehat{\mathcal{T}}}^2.$$

Together with the definition of σ in (3.8), this leads to

$$\begin{aligned}
&\langle \nabla(u_\mathcal{T}^\star - u_N) - (\boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)\chi_\Omega \,|\, \nabla(p - p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\lesssim \Big\{ |(u_\mathcal{T}^\star - u_N, \boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)|_\sigma^2 + \|\nabla(u_\mathcal{T}^\star - u_N)\|_{\boldsymbol{L}^2(\widehat{\Omega}\setminus\bar{\Omega})}^2 \Big\}^{1/2} \|p - p_\mathcal{T}\|_{L^2(\widehat{\mathcal{T}})}.
\end{aligned}$$

This completes the proof of step 2.

Step 3: We now assert with the same constant C_γ as in step 2

$$\begin{aligned}
&\frac{1}{4}\|\nabla(u_\mathcal{T}^\star - u_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})}^2 + \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{m} - \boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} + \frac{1}{4}|(u_\mathcal{T}^\star - u_N, \boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m}_N)|_\sigma^2 \\
&\le \frac{1}{2}\|\nabla(u_\mathcal{T}^\star - u)\|_{\boldsymbol{L}^2(\widehat{\Omega})}^2 + \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{m} - \boldsymbol{m}_\mathcal{T}^\star\rangle_{\boldsymbol{L}^2(\Omega)} + \langle \boldsymbol{l} - \boldsymbol{l}_N \,|\, \boldsymbol{m} - \boldsymbol{m}_\mathcal{T}^\star\rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + C_\gamma^2\|p - p_\mathcal{T}\|_{\widehat{\mathcal{T}}}^2 + \frac{1}{2}\|\varepsilon^{1/2}\boldsymbol{l}\|_{\boldsymbol{L}^2(\Omega)}^2 - \frac{1}{2}\|\varepsilon^{1/2}\boldsymbol{l}_N\|_{\boldsymbol{L}^2(\Omega)}^2 + \frac{1}{2}|(u_\mathcal{T}^\star - u, \boldsymbol{m}_\mathcal{T}^\star - \boldsymbol{m})|_\sigma^2.
\end{aligned} \tag{3.26}$$

To see this, we use the bound

$$\frac{1}{2}\varepsilon\lambda_N^2|\boldsymbol{m}_N|^2 - \frac{1}{2}\varepsilon\lambda_{\boldsymbol{m}}^2|\boldsymbol{m}|^2 \le (\boldsymbol{l}-\boldsymbol{l}_N)\cdot(\boldsymbol{m}-\boldsymbol{m}_N) \quad \text{a.e. in } \Omega, \tag{3.27}$$

of [11, Proof of Theorem 4.3] in (3.24), insert (3.25) and apply the Young inequality to get

$$\begin{aligned}
&\|\nabla(u_{\mathcal{T}}^\star - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} + |(u_{\mathcal{T}}^\star - u_N, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}_N)|^2_\sigma \\
&\le \langle \nabla(u_{\mathcal{T}}^\star - u) \,|\, \nabla(u_{\mathcal{T}}^\star - u_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}^\star\rangle_{\boldsymbol{L}^2(\Omega)} + \langle \boldsymbol{l}-\boldsymbol{l}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}^\star\rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \big|\langle \nabla(u_{\mathcal{T}}^\star - u_N) - (\boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}_N)\chi_\Omega \,|\, \nabla(p-p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}\big| - \langle \boldsymbol{l}-\boldsymbol{l}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \sigma(u_{\mathcal{T}}^\star - u, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}; u_{\mathcal{T}}^\star - u_N, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}_N) \\
&\le \frac{1}{2}\|\nabla(u_{\mathcal{T}}^\star - u)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \frac{1}{2}\|\nabla(u_{\mathcal{T}}^\star - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}^\star\rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \langle \boldsymbol{l}-\boldsymbol{l}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}^\star\rangle_{\boldsymbol{L}^2(\Omega)} + \frac{\delta}{2}\Big\{|(u_{\mathcal{T}}^\star - u_N, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}_N)|^2_\sigma + \|\nabla(u_{\mathcal{T}}^\star - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})}\Big\} \\
&\quad + \frac{C_\gamma^2}{2\delta}\|p-p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} + \frac{1}{2}\|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \frac{1}{2}|(u_{\mathcal{T}}^\star - u, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m})|^2_\sigma + \frac{1}{2}|(u_{\mathcal{T}}^\star - u_N, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}_N)|^2_\sigma.
\end{aligned}$$

By collecting the terms $|(u_{\mathcal{T}}^\star - u_N; \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}_N)|^2_\sigma$ and $\|\nabla(u_{\mathcal{T}}^\star - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})}$ on the left-hand side and choosing $\delta = \frac{1}{2}$, we obtain (3.26).

Step 4: We show now that

$$\begin{aligned}
&\frac{1}{8}\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} + \frac{1}{4}|(u_{\mathcal{T}}^\star - u_N, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}_N)|^2_\sigma \\
&\le \frac{3}{4}\|\nabla(u-u_{\mathcal{T}}^\star)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}^\star\rangle_{\boldsymbol{L}^2(\Omega)} + \langle \boldsymbol{l}-\boldsymbol{l}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}^\star\rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + C_\gamma^2\|p-p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} + \frac{1}{2}\|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}|(u_{\mathcal{T}}^\star - u, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m})|^2_\sigma.
\end{aligned} \tag{3.28}$$

A triangle and a Young inequality show

$$\frac{1}{8}\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \le \frac{1}{4}\|\nabla(u-u_{\mathcal{T}}^\star)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \frac{1}{4}\|\nabla(u_{\mathcal{T}}^\star - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})}.$$

This together with (3.26) gives the claimed estimate.

Step 5: This step shows the validity of

$$\begin{aligned}
&\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} + |(u_{\mathcal{T}}^\star - u_N, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m}_N)|^2_\sigma \\
&\le 6\|\nabla(u-u_{\mathcal{T}}^\star)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + 4\delta_1\|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + 4\delta_2\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \frac{4(\delta_1+\delta_2)}{\delta_1\delta_2}\|\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}^\star\|^2_{\boldsymbol{L}^2(\Omega)} + 8C_\gamma^2\|p-p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} + 4\Big\{\|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\Big\} \\
&\quad + 4|(u_{\mathcal{T}}^\star - u, \boldsymbol{m}_{\mathcal{T}}^\star - \boldsymbol{m})|^2_\sigma,
\end{aligned} \tag{3.29}$$

with arbitrary small $\delta_1, \delta_2 > 0$.

Recall the convexity of φ^{**}, which guarantees $\langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} \geq 0$. Applying two times a Young inequality to the right-hand side of (3.28) shows for arbitrary positive constants δ_1, δ_2

$$\begin{aligned}
&\frac{1}{8}\Big\{\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} + |(u^\star_{\mathcal{T}}-u_N, \boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m}_N)|^2_\sigma\Big\}\\
&\leq \frac{3}{4}\|\nabla(u-u^\star_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \frac{\delta_1}{2}\|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\delta_2}{2}\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad + \Big(\frac{1}{2\delta_1}+\frac{1}{2\delta_2}\Big)\|\boldsymbol{m}-\boldsymbol{m}^\star_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} + C_\gamma^2\|p-p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}}\\
&\quad + \frac{1}{2}\|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}|(u^\star_{\mathcal{T}}-u, \boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma.
\end{aligned}$$

Multiplying through by a factor 8 gives (3.29).

Step 6: In this step, we control the error contribution $|(u-u_N, \boldsymbol{m}-\boldsymbol{m}_N)|_\sigma$ by showing

$$\begin{aligned}
&\frac{1}{2}\Big\{\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \boldsymbol{m}-\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} + |(u-u_N, \boldsymbol{m}-\boldsymbol{m}_N)|^2_\sigma\Big\}\\
&\leq 6\|\nabla(u-u^\star_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + 4\delta_1\|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + 4\delta_2\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad + \frac{4(\delta_1+\delta_2)}{\delta_1\delta_2}\|\boldsymbol{m}-\boldsymbol{m}^\star_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} + 8C_\gamma^2\|p-p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} + 4\Big\{\|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\Big\}\\
&\quad + 5|(u^\star_{\mathcal{T}}-u, \boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma,
\end{aligned} \tag{3.30}$$

This can be easily seen by adding the term $|(u^\star_{\mathcal{T}}-u, \boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma$ to both sides of (3.29). A triangle inequality now gives

$$\frac{1}{2}|(u-u_N, \boldsymbol{m}-\boldsymbol{m}_N)|^2_\sigma \leq |(u^\star_{\mathcal{T}}-u, \boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma + |(u^\star_{\mathcal{T}}-u_N, \boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m}_N)|^2_\sigma,$$

which shows (3.30).

Step 7: We now estimate $\|\nabla(p-p_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})}$. For this, we note that a special situation of the discrete LBB condition applies, namely,

$$\begin{aligned}
\|\nabla(p_{\mathcal{T}}-p_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})} &= \frac{|b(p_{\mathcal{T}}-p_N, \boldsymbol{0}; p_{\mathcal{T}}-p_N)|}{\|\nabla(p_{\mathcal{T}}-p_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})}} \leq \sup_{v_N \in S_0^{k,1}(\widehat{\Omega})} \frac{|b(v_N, \boldsymbol{0}; p_{\mathcal{T}}-p_N)|}{\|\nabla v_N\|_{\boldsymbol{L}^2(\widehat{\Omega})}}\\
&= \sup_{v_N \in S_0^{k,1}(\widehat{\Omega})} \frac{|\langle \nabla v_N \,|\, \nabla(p_{\mathcal{T}}-p)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \nabla v_N \,|\, \nabla(p-p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}|}{\|\nabla v_N\|_{\boldsymbol{L}^2(\widehat{\Omega})}}.
\end{aligned}$$

The Galerkin orthogonality (3.19) with $\boldsymbol{n}_N = \boldsymbol{0}$ shows

$$\langle \nabla v_N \,|\, \nabla(p-p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} = -\langle \nabla(u-u_N) \,|\, \nabla v_N\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} - \sigma(u-u_N, \boldsymbol{m}-\boldsymbol{m}_N; v_N, \boldsymbol{0}).$$

Using the Cauchy-Schwarz inequality together with Lemma 3.9 and the constant C_σ therein, one gets

$$\|\nabla(p_{\mathcal{T}}-p_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})} \leq \|\nabla(p_{\mathcal{T}}-p)\|_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\nabla(u-u_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})} + C_\sigma|(u-u_N, \boldsymbol{m}-\boldsymbol{m}_N)|_\sigma.$$

From now on we want to suppress non-critical coefficients arising e.g. from certain Young inequalities with a $\lesssim$ symbol. We estimate $\|\nabla(u-u_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})} + C_\sigma|(u-u_N, \boldsymbol{m}-\boldsymbol{m}_N)|_\sigma$ with (3.30). Furthermore we use from Definition (3.15) the trivial estimate $\|\nabla(p-p_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \leq \|\nabla(p-p_{\mathcal{T}})\|^2_{\widehat{\mathcal{T}}}$. Altogether we get

$$\begin{aligned}
\|\nabla(p-p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} &\le \Big\{\|\nabla(p-p_{\mathcal{T}})\|_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\nabla(p_{\mathcal{T}}-p_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})}\Big\}^2 \\
&\lesssim C_\sigma^2\Big\{\|\nabla(u-u^\star_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \delta_1\|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \delta_2\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\delta_1+\delta_2}{\delta_1\delta_2}\|\boldsymbol{m}-\boldsymbol{m}^\star_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + C_\gamma^2\|p-p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} + \|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + |(u^\star_{\mathcal{T}}-u,\boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma\Big\}.
\end{aligned}$$

Step 8: In this step, the best approximation result in the constrained space $\mathrm{Ker}_N\, b$ is stated. Note first that the results of step 6 and step 7 are valid for all $(u^\star_{\mathcal{T}},\boldsymbol{m}^\star_{\mathcal{T}})\in\mathrm{Ker}_N\, b$. Adding these results we obtain

$$\begin{aligned}
&\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle\boldsymbol{d}-\boldsymbol{d}_N\,|\,\boldsymbol{m}-\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)} + |(u-u_N,\boldsymbol{m}-\boldsymbol{m}_N)|^2_\sigma + \|\nabla(p-p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\lesssim C_\sigma^2 \inf_{(u^\star_{\mathcal{T}},\boldsymbol{m}^\star_{\mathcal{T}})\in\mathrm{Ker}_N b}\Big\{\|\nabla(u-u^\star_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \frac{\delta_1+\delta_2}{\delta_1\delta_2}\|\boldsymbol{m}-\boldsymbol{m}^\star_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} + |(u^\star_{\mathcal{T}}-u,\boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma\Big\} \\
&\quad + C_\sigma^2\Big\{\delta_1\|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \delta_2\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + C_\gamma^2\|p-p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} \\
&\quad + \|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\Big\}.
\end{aligned} \tag{3.31}$$

Step 9: As mentioned at the beginning of the proof, in this step an estimate in the whole approximation space X^1_N will be given. So let arbitrary but fixed elements $(u_{\mathcal{T}},\boldsymbol{m}_{\mathcal{T}})\in X^1_N$ be given. We consider the problem to find $(r_N,\boldsymbol{s}_N)\in X^1_N$ such that

$$b(r_N,\boldsymbol{s}_N;q_N) = b(u-u_{\mathcal{T}},\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}};q_N) \tag{3.32}$$

holds for all $q_N\in M^1_N$. Since the LBB condition for the bilinear form also holds in the discrete spaces, i.e.

$$\inf_{p\in M^1_N}\sup_{(u,\boldsymbol{m})\in X^1_N}\frac{b(u,\boldsymbol{m};p)}{\|(u,\boldsymbol{m})\|_X\|p\|_M}\ge 1,$$

there exists a unique element $(r_N,\boldsymbol{s}_N)\in(\mathrm{Ker}_N\, b)^\perp$ solving (3.32) with the additional property

$$\|(r_N,\boldsymbol{s}_N)\|_X \le \|b(u-u_{\mathcal{T}},\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}};\bullet)\|_{M^\star_N} \le \|\nabla(u-u_{\mathcal{T}})\|_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\Omega)}.$$

Since $(u,\boldsymbol{m})\in\mathrm{Ker}\, b$, (3.32) yields $(r_N+u_{\mathcal{T}},\boldsymbol{s}_N+\boldsymbol{m}_{\mathcal{T}})\in\mathrm{Ker}_N\, b$. With Lemma 3.9 and the abbreviation $\delta:=\frac{\delta_1+\delta_2}{\delta_1\delta_2}$, we get

$$\begin{aligned}
&\inf_{(u^\star_{\mathcal{T}},\boldsymbol{m}^\star_{\mathcal{T}})\in\mathrm{Ker}_N\, b}\Big\{\|\nabla(u-u^\star_{\mathcal{T}})\|^2_{\boldsymbol{L}^2\widehat{\Omega})} + \delta\|\boldsymbol{m}-\boldsymbol{m}^\star_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} + |(u^\star_{\mathcal{T}}-u,\boldsymbol{m}^\star_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma\Big\} \\
&\le \|\nabla(u-r_N-u_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \delta\|\boldsymbol{m}-\boldsymbol{s}_N-\boldsymbol{m}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} + |(r_N+u_{\mathcal{T}}-u,\boldsymbol{s}_N+\boldsymbol{m}_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma \\
&\lesssim \|\nabla(u-u_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \delta\|\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)} + |(u_{\mathcal{T}}-u,\boldsymbol{m}_{\mathcal{T}}-\boldsymbol{m})|^2_\sigma \\
&\quad + (1+C_\sigma^2)\|\nabla r_N\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + (\delta+C_\sigma^2)\|\boldsymbol{s}_N\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\lesssim \big(1+\delta+C_\sigma^2\big)\big(\|\nabla(u-u_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)}\big) + |(u-u_{\mathcal{T}},\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}})|^2_\sigma.
\end{aligned}$$

Using this last result in (3.31) gives (3.14) and concludes the proof. □

Proof of Theorem 3.7. Recall the shorthand notation in (3.23).
Step 1: With the additionally assumption (3.16), the term $\|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ on the right-hand side of

(3.14) in Proposition 3.6 can be absorbed by the left-hand side. Indeed, set e.g. $\delta_1 = \frac{C_0}{2C_1C_\sigma^2}$. This choice leads to

$$
\begin{aligned}
&\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + |(u-u_N,\boldsymbol{m}-\boldsymbol{m}_N)|^2_\sigma + \|\nabla(p-p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\le \frac{2C_1C_\sigma^2}{C_0}\Big\{\Big(1+\frac{\delta_1+\delta_2}{\delta_1\delta_2}+C_\sigma^2\Big)\Big(\|\nabla(u-u_\mathcal{T})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m}-\boldsymbol{m}_\mathcal{T}\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \\
&\quad + |(u-u_\mathcal{T},\boldsymbol{m}-\boldsymbol{m}_\mathcal{T})|^2_\sigma + \delta_2\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + C_\gamma^2\|p-p_\mathcal{T}\|^2_{\widetilde{\mathcal{T}}} \\
&\quad + \|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\Big\}.
\end{aligned}
\tag{3.33}
$$

Step 2: For $\|\Pi\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ there holds an estimate of type (3.33).

Indeed, using the $\boldsymbol{L}(\Omega)^2$-orthogonal projection, the Galerkin orthogonality (3.19) with $v_N = 0$ and $\boldsymbol{n}_N = \Pi\boldsymbol{l}-\boldsymbol{l}_N$, Lemma 3.9 and Cauchy-Schwarz inequality we get

$$
\begin{aligned}
&\|\Pi\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{l}-\boldsymbol{l}_N \,|\, \Pi\boldsymbol{l}-\boldsymbol{l}_N\rangle_{\boldsymbol{L}^2(\Omega)} \\
&= -\langle \boldsymbol{d}-\boldsymbol{d}_N \,|\, \Pi\boldsymbol{l}-\boldsymbol{l}_N\rangle_{\boldsymbol{L}^2(\Omega)} + \langle (\Pi\boldsymbol{l}-\boldsymbol{l}_N)\chi_\Omega \,|\, \nabla(p-p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\quad - \sigma(u-u_N,\boldsymbol{m}-\boldsymbol{m}_N;0,\Pi\boldsymbol{l}-\boldsymbol{l}_N) \\
&\le \Big\{\|\boldsymbol{d}-\boldsymbol{d}_N\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(p-p_N)\|_{\boldsymbol{L}^2(\widehat{\Omega})} + C_\sigma|(u-u_N,\boldsymbol{m}-\boldsymbol{m}_N)|_\sigma\Big\}\|\Pi\boldsymbol{l}-\boldsymbol{l}_N\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}
\tag{3.34}
$$

Cancelling the factor $\|\Pi\boldsymbol{l}-\boldsymbol{l}_N\|_{\boldsymbol{L}^2(\Omega)}$ on both sides and squaring the inequality gives

$$
\|\Pi\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} \le 3C_\sigma^2\Big\{\|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla(p-p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + |(u-u_N,\boldsymbol{m}-\boldsymbol{m}_N)|^2_\sigma\Big\}.
\tag{3.35}
$$

Step 3: Applying (3.33) together with (3.35) and $\|\boldsymbol{l}-\boldsymbol{l}_N\| \le \|\boldsymbol{l}-\Pi\boldsymbol{l}\| + \|\Pi\boldsymbol{l}-\boldsymbol{l}_N\|$ gives the following estimate

$$
\begin{aligned}
&\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + |(u-u_N,\boldsymbol{m}-\boldsymbol{m}_N)|^2_\sigma \\
&\quad + \|\nabla(p-p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\le C_2\Big\{\Big(1+\frac{\delta_1+\delta_2}{\delta_1\delta_2}+C_\sigma^2\Big)\Big(\|\nabla(u-u_\mathcal{T})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m}-\boldsymbol{m}_\mathcal{T}\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + |(u-u_\mathcal{T},\boldsymbol{m}-\boldsymbol{m}_\mathcal{T})|^2_\sigma \\
&\quad + C_\gamma^2\|p-p_\mathcal{T}\|^2_{\widetilde{\mathcal{T}}} + \|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{l}-\Pi\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)}\Big\},
\end{aligned}
\tag{3.36}
$$

with $C_2 := 4(1+6C_\sigma^2)\frac{C_1C_\sigma^2}{C_0}$. Note that using (3.33) would involve the term $\delta_2\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ on the right-hand side in the above estimate. We already absorbed this term from the right-hand side by setting $\delta_2 := \frac{1}{C_2}$.

Step 4: In this step, the claimed estimate (3.17) is proved. The following relation valid for all positive cconstants C was proven in [30, Lemma 2.32], see also [11]:

$$
\begin{aligned}
&C\Big\{\|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\Big\} \\
&\le C^2\Big\{\|\varepsilon\|_{L^\infty(\Omega)}\|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\varepsilon\|_{L^\infty(\Omega)}\|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\Big\} + \frac{1}{2}\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}
$$

Plugging this into (3.36) with $C = C_2$ and absorbing the term $\frac{1}{2}\|\boldsymbol{l} - \boldsymbol{l}_N\|^2_{L^2(\Omega)}$ gives

$$\begin{aligned}
&\|\nabla(u-u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + |(u-u_N,\boldsymbol{m}-\boldsymbol{m}_N)|^2_\sigma \\
&\quad + \|\nabla(p-p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\le 2C_2\Big\{\Big(1+\frac{\delta_1+\delta_2}{\delta_1\delta_2}+C_\sigma^2\Big)\Big(\|\nabla(u-u_{\mathcal{T}})\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \\
&\quad + |(u-u_{\mathcal{T}},\boldsymbol{m}-\boldsymbol{m}_{\mathcal{T}})|^2_\sigma + C_\gamma^2\|p-p_{\mathcal{T}}\|^2_{\widehat{\mathcal{T}}} + \|\boldsymbol{l}-\Pi\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)}\Big\} \\
&\quad + 2C_2^2\|\varepsilon\|_{L^\infty(\Omega)}\Big\{\|\varepsilon^{1/2}\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}\Big\}.
\end{aligned}$$

Finally, the term $\|\varepsilon^{1/2}\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ can be estimated using (3.36) resulting in the claimed bound (3.17). □

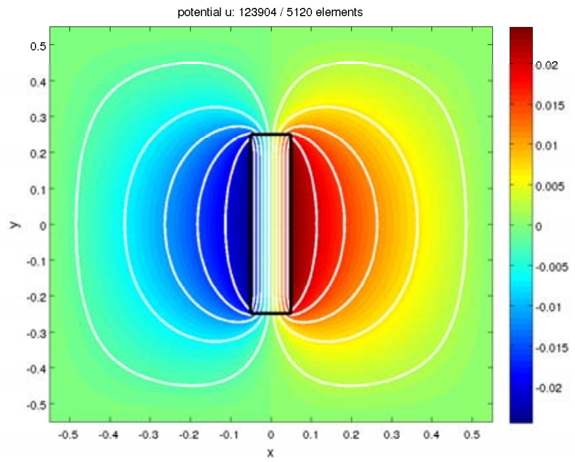
Discussion 3.10 (Proof of Theorem 3.7). *The difficulty in the proof lies in the treatment of the non-smooth non-linear terms. For interesting a priori estimates, one needs to absorb (or least estimate) the terms $\|\boldsymbol{d}-\boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} = \|\nabla\varphi^{**}\circ\boldsymbol{m} - \nabla\varphi^{**}\circ\boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ and $\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} = \|\lambda_{\boldsymbol{m}}\boldsymbol{m} - \lambda_N\boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ of the right-hand side of* (3.14)*. We achieved this for the term $\|\boldsymbol{d}-\boldsymbol{d}_N\|_{\boldsymbol{L}^2(\Omega)}$ using assumption* (3.16)*, which covers, for example, the case of uniaxial materials; similar techniques have also been employed in* [9, 11, 22]*. The key steps in the treatment of $\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ are* (3.34) (3.35)*, which rely on the use of the $\boldsymbol{n}_N = \Pi\boldsymbol{l} - \boldsymbol{l}_N$ as a test function. However, this is only possible in the lowest order case $k = 1$ in view of the representation $\boldsymbol{l}_N = \lambda_N\boldsymbol{m}_N$ and the explicit formula for λ_N given in* (3.6)*. When considering higher order elements, it is possible to use $\Pi(\boldsymbol{l}-\boldsymbol{l}_N)$ as a test function, which ultimately leads to an additional a posteriori term $\|\Pi\boldsymbol{l}_N - \boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ on the right-hand side.*

Note that estimate (3.18) is optimal with respect to the local mesh size h and suggests the choice $\varepsilon = O(h^\alpha)$ with $\alpha = 1$. Numerical experiments reveal that the choice $\alpha \in (0,1)$ dominates the error in the sense that, for smooth exact solution $(u,\boldsymbol{m})$, one observes numerically convergence $O(h^\alpha)$. Empirically, the estimate (3.18) is thus even optimal with respect to ε, and $\varepsilon = O(h)$ leads in this case to optimal convergence $O(h)$. Throughout the following experiments, we thus choose the $\mathcal{T}$-piecewise constant penalization function $\varepsilon = h$, where $h \in L^\infty(\Omega)$ is defined by $h|_K := \operatorname{diam} K$.

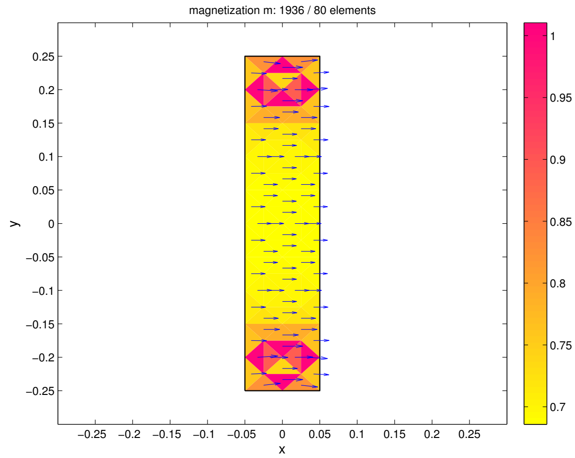
3.5. Numerical example - uniform mesh refinement. In the first numerical experiment we consider the model case of uniaxial materials, cf. (1.5)–(1.7) in two dimensions and choose a constant exterior field $\boldsymbol{f} = [0.6, 0]$ parallel to the easy axis $\mathbf{e} = [1, 0]$. Therefore, we have $\mathbf{z} = [0, 1]$. Furthermore, we choose the magnetic rod $\Omega = (-0.05, 0.05) \times (-0.25, 0.25)$ and the surrounding area $\widehat{\Omega} = (-0.55, 0.55)^2$. Up to a scaling, this example coincides with an example already studied in [11]. Fig. 1a shows the isolines of the magnetic potential u in the computational domain $\widehat{\Omega}$; Fig. 1b presents the magnetization $\boldsymbol{m}$ on a rather coarse mesh. As can be seen in Fig. 1c and Fig. 1d, the theoretical prediction for the convergence rates are verified. Although we do not control the error $\|(\boldsymbol{m}-\boldsymbol{m}_N)\cdot\mathbf{e}\|_{L^2(\Omega)}$ in the present uniaxial case (cf. (3.18)), we observe convergence $O(h)$ in the full norm $\|\boldsymbol{m}-\boldsymbol{m}_N\|_{\boldsymbol{L}^2(\Omega)}$.

In the continuous case, the Lagrange multiplier p turns out to be exactly $-u$. This relation does not hold in the discrete case. Here, we only observe convergence of ∇p towards $-\nabla u$ with almost the same rate as for the error in ∇u and ∇p.

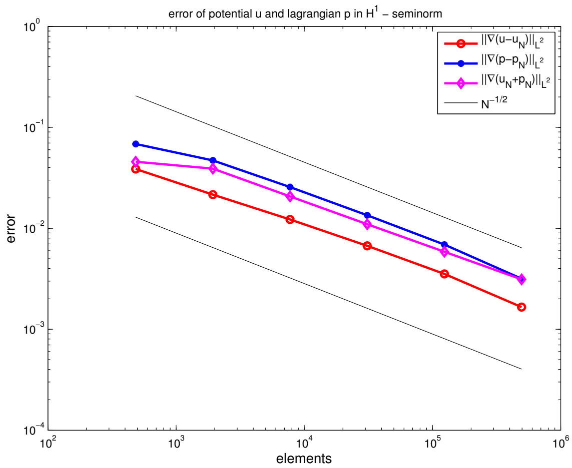
FIGURE 1. convergence analysis on uniform meshes for example of Section 3.5



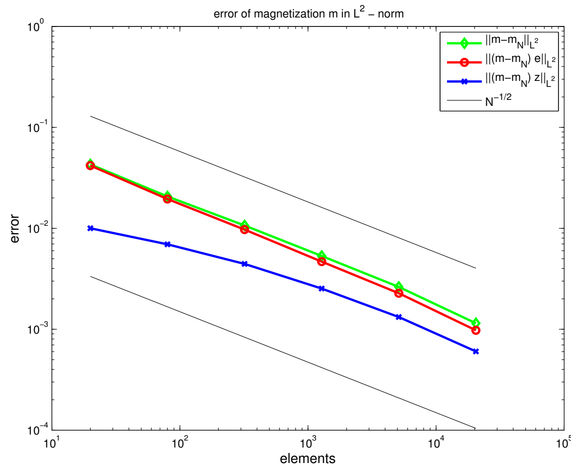
(A) potential u_N



(B) magnetization $\boldsymbol{m}_N$



(C) H^1-convergence of the potential u_N and Lagrange multiplier p_N



(D) convergence of the magnetization $\boldsymbol{m}_N$—different components

4. A POSTERIORI ANALYSIS

4.1. A posteriori error estimate. We restrict the *a posteriori* analysis of $(SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma})$ to lowest order elements in the model case of uniaxial materials (see Example 1.3 for the definition of φ^{**} and $\nabla\varphi^{**}$). The discrete spaces are $X^1_N = S^{1,1}_0(\widehat{\mathcal{T}}) \times \boldsymbol{S}^{0,0}(\mathcal{T})$ and $M^1_N = S^{1,1}_0(\widehat{\mathcal{T}})$. Before we start, we show a representation of the stabilizing bilinear form σ, whose proof is straight forward and therefore omitted:

Lemma 4.1. *For each* $(u_N, \boldsymbol{m}_N) \in X^1_N$ *there exists a unique pair* $(U_N, \boldsymbol{M}_N) \in X^1_N$ *such that*

$$\sigma(u_N, \boldsymbol{m}_N; v, \boldsymbol{n}) = \langle \nabla U_N \,|\, \nabla v \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{M}_N \,|\, \boldsymbol{n} \rangle_{\boldsymbol{L}^2(\Omega)} \quad \text{for all } (v, \boldsymbol{n}) \in X^1_N. \tag{4.1}$$

Furthermore, $\boldsymbol{M}_N$ is explicitly given by

$$\boldsymbol{M}_N|_K \cdot \mathbf{e}_j := -\frac{1}{|K|} \sum_{E \in \mathcal{E}(K)} h_E \langle [(\nabla u_N - \boldsymbol{m}_N) \cdot \boldsymbol{\nu}]_E \,|\, \mathbf{e}_j \cdot \boldsymbol{\nu}_{K,E} \rangle_{L^2(E)} \tag{4.2}$$

and $\mathbf{e}_j$ denotes the j-th canonical basis vector of $\mathbb{R}^d$, $\boldsymbol{\nu}_{K,E}$ the exterior-normal vector of the element $K \in \mathcal{T}$ along the edge (or face) $E \subset \overline{K}$ and $\mathcal{E}(K)$ the edges (or faces) of the element K. $\square$

With this notation, the main result in this section reads as follows:

Theorem 4.2 (A posteriori estimate for uniaxial case)**.** *Under the assumptions of Lemma 4.3, there holds*

$$\begin{aligned}
&\|\nabla(u - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\nabla\varphi^{**} \circ \boldsymbol{m} - \nabla\varphi^{**} \circ \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla(p - p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\quad + \|\lambda_{\boldsymbol{m}} \boldsymbol{m} - \lambda_N \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\le C\Big\{ \|\boldsymbol{f} - \Pi \boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \big|\langle \boldsymbol{f} - \Pi\boldsymbol{f} \,|\, \boldsymbol{m} - \Pi\boldsymbol{m}\rangle_{\boldsymbol{L}^2(\Omega)}\big| + \langle \varepsilon |\lambda_N \boldsymbol{m}_N| \,|\, |(\boldsymbol{f} - \Pi\boldsymbol{f}) + \boldsymbol{M}_N| \rangle_{L^2(\Omega)} \\
&\quad + \|\varepsilon \lambda_N \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \sum_{E \in \mathcal{E}^{\mathrm{int}}(\widehat{\mathcal{T}})} h_E \|[(\nabla u_N - \boldsymbol{m}_N \chi_\Omega) \cdot \boldsymbol{\nu}]_E\|^2_{L^2(E)} + \|\nabla(u_N + p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\quad + \|\boldsymbol{M}_N\|_{\boldsymbol{L}^1(\Omega)} + \big|\langle \boldsymbol{M}_N \,|\, \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)}\big| \Big\}.
\end{aligned} \tag{4.3}$$

For the proof, we use the Scott-Zhang interpolation operator $I : H^1_0(\widehat{\Omega}) \twoheadrightarrow S^{1,1}_0(\widehat{\mathcal{T}})$ of [34], which has approximation properties similar to those of the Clément interpolation operator but has the additional property of being a projection. Moreover, we use the estimate

$$-\langle \lambda_{\boldsymbol{m}} \boldsymbol{m} - \lambda_N \boldsymbol{m}_N \,|\, \boldsymbol{m} - \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)} \le \langle \varepsilon |\lambda_N \boldsymbol{m}_N| \,|\, |\lambda_{\boldsymbol{m}} \boldsymbol{m} - \lambda_N \boldsymbol{m}_N| \rangle_{L^2(\Omega)}, \tag{4.4}$$

which is part of the proof of [11, Theorem 4.3]. The main step of the proof of Theorem 4.2 is stated in the following lemma.

Lemma 4.3. *Let $(u, \boldsymbol{m}, p) \in X \times M$ and $(u_N, \boldsymbol{m}_N, p_N) \in X^1_N \times M^1_N$ be solutions of $(SPP_{\widehat{\Omega}})$ and $(SPP^{\varepsilon,N}_{\widehat{\Omega},\sigma})$ in the uniaxial case for lowest order elements. Then there holds the a posteriori estimate*

$$\begin{aligned}
&\|\nabla(u - u_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\nabla\varphi^{**} \circ \boldsymbol{m} - \nabla\varphi^{**} \circ \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla(p - p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\le C\Big\{ \big|\langle \boldsymbol{f} - \Pi\boldsymbol{f} \,|\, \boldsymbol{m} - \Pi\boldsymbol{m}\rangle_{\boldsymbol{L}^2(\Omega)}\big| + \langle \varepsilon |\lambda_N \boldsymbol{m}_N| \,|\, |(\boldsymbol{f} - \Pi\boldsymbol{f}) + \boldsymbol{M}_N| \rangle_{L^2(\Omega)} + \|\varepsilon \lambda_N \boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \sum_{E \in \mathcal{E}^{\mathrm{int}}(\widehat{\mathcal{T}})} h_E \|[(\nabla u_N - \boldsymbol{m}_N \chi_\Omega) \cdot \boldsymbol{\nu}]_E\|^2_{L^2(E)} + \|\nabla(u_N + p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\quad + \|\boldsymbol{M}_N\|_{\boldsymbol{L}^1(\Omega)} + \big|\langle \boldsymbol{M}_N \,|\, \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)}\big| \Big\},
\end{aligned} \tag{4.5}$$

where the constant C depends only on the domain $\widehat{\Omega}$ and the shape regularity constant γ.

Proof. Recall the shorthand notation of (3.23).

Step 1: Subtracting (3.10) from (2.1) and (3.11) from (2.2) yields with the abbreviations $e_u := u - u_N$, $\boldsymbol{e_m} := \boldsymbol{m} - \boldsymbol{m}_N$, $e_p := p - p_N$ for all $(v_N, \boldsymbol{n}_N, q_N) \in X^1_N \times M^1_N$ the Galerkin orthogonalities

$$\begin{aligned}
0 &= \langle \nabla e_u \,|\, \nabla v_N \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{n}_N \rangle_{\boldsymbol{L}^2(\Omega)} + \langle \boldsymbol{l} - \boldsymbol{l}_N \,|\, \boldsymbol{n}_N \rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad - \sigma(u_N, \boldsymbol{m}_N; v_N, \boldsymbol{n}_N) + \langle \nabla v_N - \boldsymbol{n}_N \chi_\Omega \,|\, \nabla e_p \rangle_{\boldsymbol{L}^2(\widehat{\Omega})},
\end{aligned} \tag{4.6}$$

$$0 = \langle \nabla e_u - \boldsymbol{e_m} \chi_\Omega \,|\, \nabla q_N \rangle_{\boldsymbol{L}^2(\widehat{\Omega})}. \tag{4.7}$$

Step 2: We subtract equation (4.6) from $\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)}$, add and subtract on the right-hand side the two terms $\langle \boldsymbol{l} \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)}$ and $\langle \nabla e_u - \boldsymbol{e_m}\chi_\Omega \,|\, \nabla p\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}$ and use (2.1) with the test functions $v := \nabla(e_u - Ie_u)$ and $\boldsymbol{n} := \boldsymbol{e_m} - \Pi\boldsymbol{e_m}$. Note now the orthogonality relations

$$\langle \boldsymbol{d}_N \,|\, \boldsymbol{e_m} - \Pi\boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} = 0, \quad \langle \boldsymbol{l}_N \,|\, \boldsymbol{e_m} - \Pi\boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} = 0, \quad \text{and}$$
$$\langle \boldsymbol{f} \,|\, \boldsymbol{e_m} - \Pi\boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{f} - \Pi\boldsymbol{f} \,|\, \boldsymbol{m} - \Pi\boldsymbol{m}\rangle_{\boldsymbol{L}^2(\Omega)}.$$

Altogether this gives

$$\begin{aligned}
&\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} \\
&= \langle \boldsymbol{f} - \Pi\boldsymbol{f} \,|\, \boldsymbol{m} - \Pi\boldsymbol{m}\rangle_{\boldsymbol{L}^2(\Omega)} - \langle \boldsymbol{l} - \boldsymbol{l}_N \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} - \langle \nabla e_u - \boldsymbol{e_m}\chi_\Omega \,|\, \nabla p\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\quad - \langle \nabla u_N \,|\, \nabla(e_u - Ie_u)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \nabla Ie_u - \Pi\boldsymbol{e_m}\chi_\Omega \,|\, \nabla p_N\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\quad + \sigma(u_N, \boldsymbol{m}_N; Ie_u, \Pi\boldsymbol{e_m}).
\end{aligned}$$

Step 3: Since $p = -u$ we get with (2.2), (3.11) and the projection property of I

$$\langle \nabla Ie_u - \Pi\boldsymbol{e_m}\chi_\Omega \,|\, \nabla p_N\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} = \langle \nabla(Ie_u - e_u) \,|\, \nabla p_N\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}.$$

Together with the approximation properties of I and application of standard finite element techniques we estimate

$$\begin{aligned}
&\langle \nabla e_u - \boldsymbol{e_m}\chi_\Omega \,|\, \nabla p\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} = \langle \nabla u_N - \boldsymbol{m}_N\chi_\Omega \,|\, \nabla e_u\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} = \langle \nabla u_N - \boldsymbol{m}_N\chi_\Omega \,|\, \nabla(e_u - Ie_u)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\le \frac{1}{2c_1} \sum_{E\in\mathcal{E}^{\mathrm{int}}(\widehat{\mathcal{T}})} h_E \|[(\nabla u_N - \boldsymbol{m}_N\chi_\Omega)\cdot\boldsymbol{\nu}]_E\|^2_{L^2(E)} + \frac{Cc_1}{2}\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})},
\end{aligned}$$

where due to the Young inequality $c_1 > 0$ may be chosen arbitrary. Together this shows the estimate

$$\begin{aligned}
&\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} \le \left|\langle \boldsymbol{f} - \Pi\boldsymbol{f} \,|\, \boldsymbol{m} - \Pi\boldsymbol{m}\rangle_{\boldsymbol{L}^2(\Omega)}\right| \\
&- \langle \boldsymbol{l} - \boldsymbol{l}_N \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2c_1} \sum_{E\in\mathcal{E}^{\mathrm{int}}(\widehat{\mathcal{T}})} h_E \|[(\nabla u_N - \boldsymbol{m}_N\chi_\Omega)\cdot\boldsymbol{\nu}]_E\|^2_{L^2(E)} \\
&+ \frac{Cc_1}{2}\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \nabla(Ie_u - e_u) \,|\, \nabla(u_N + p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})} + \sigma(u_N, \boldsymbol{m}_N; Ie_u, \Pi\boldsymbol{e_m}).
\end{aligned} \tag{4.8}$$

Step 4: We estimate $-\langle \boldsymbol{l} - \boldsymbol{l}_N \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)}$. (2.1) with $v = 0$ and (3.10) with $(v_N, \boldsymbol{n}_N) = (0, \chi_K \mathbf{e}_j)$ together with the representation of σ given by Lemma 4.1 yields

$$\boldsymbol{l} - \boldsymbol{l}_N = (\boldsymbol{f} - \Pi\boldsymbol{f}) - (\boldsymbol{d} - \boldsymbol{d}_N) + \nabla e_p + \boldsymbol{M}_N \quad \text{a.e. in } \Omega$$

and with (4.4) we get

$$\begin{aligned}
&- \langle \boldsymbol{l} - \boldsymbol{l}_N \,|\, \boldsymbol{e_m}\rangle_{\boldsymbol{L}^2(\Omega)} \le \langle \varepsilon|\boldsymbol{l}_N| \,|\, |\boldsymbol{l} - \boldsymbol{l}_N|\rangle_{L^2(\Omega)} \le \langle \varepsilon|\boldsymbol{l}_N| \,|\, |(\boldsymbol{f} - \Pi\boldsymbol{f}) + \boldsymbol{M}_N|\rangle_{L^2(\Omega)} \\
&+ \frac{c_2}{2}\|\boldsymbol{d} - \boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{c_3}{2}\|\nabla e_p\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + (\frac{1}{2c_2} + \frac{1}{2c_3})\|\varepsilon\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)}
\end{aligned} \tag{4.9}$$

again with arbitrary c_2, $c_3 > 0$.

Step 5: Next we consider the term $\sigma(u_N, \boldsymbol{m}_N; Ie_u, \Pi\boldsymbol{e_m})$. Setting $v_N = Ie_u$ and $\boldsymbol{n}_N = \boldsymbol{0}$ in (3.10) gives

$$\sigma(u_N, \boldsymbol{m}_N; Ie_u, \boldsymbol{0}) = -\langle \nabla Ie_u \,|\, \nabla(u_N + p_N)\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}$$

and hence with arbitrary $c_4 > 0$

$$
\begin{aligned}
&\sigma(u_N, \boldsymbol{m}_N; Ie_u, \Pi \boldsymbol{e_m}) = \sigma(u_N, \boldsymbol{m}_N; Ie_u, \boldsymbol{0}) + \sigma(u_N, \boldsymbol{m}_N; 0, \Pi \boldsymbol{e_m}) \\
&\le \frac{1}{2c_4} \|\nabla(u_N + p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \frac{Cc_4}{2} \|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \langle \boldsymbol{M}_N \,|\, \Pi \boldsymbol{m} - \boldsymbol{m} \rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \langle \boldsymbol{M}_N \,|\, \boldsymbol{m} - \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)} \\
&\le \frac{1}{2c_4} \|\nabla(u_N + p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \frac{Cc_4}{2} \|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{M}_N\|_{\boldsymbol{L}^1(\Omega)} + |\langle \boldsymbol{M}_N \,|\, \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)}|.
\end{aligned} \tag{4.10}
$$

Step 6: In this last step we estimate $\|\nabla e_p\|_{\boldsymbol{L}^2(\Omega)}$ and use the monotonicity of $\nabla\varphi^{**}$ and the identity $p = -u$. Firstly, we have

$$
\|\nabla e_p\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \le 2\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + 2\|\nabla(u_N + p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})}. \tag{4.11}
$$

Secondly, we substitute (4.9) and (4.10) in (4.8), multiply (4.11) by a constant $c_5 > 0$ that we will fix later on, add it to (4.8) and exploit

$$
\begin{aligned}
&\|\boldsymbol{d} - \boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{d} - \boldsymbol{d}_N \,|\, \boldsymbol{e_m} \rangle_{\boldsymbol{L}^2(\Omega)} \quad \text{(uniaxial case !) and to} \\
&\langle \nabla(Ie_u - e_u) \,|\, \nabla(u_N + p_N) \rangle_{\boldsymbol{L}^2(\widehat{\Omega})} \le \frac{Cc_6}{2} \|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \frac{1}{2c_6} \|\nabla(u_N + p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})},
\end{aligned}
$$

where $c_6 > 0$ is again arbitrary. We end up with

$$
\begin{aligned}
&\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{d} - \boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + c_5 \|\nabla e_p\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\le \left| \langle \boldsymbol{f} - \Pi \boldsymbol{f} \,|\, \boldsymbol{m} - \Pi \boldsymbol{m} \rangle_{\boldsymbol{L}^2(\Omega)} \right| + \langle \varepsilon |\boldsymbol{l}_N| \,|\, |(\boldsymbol{f} - \Pi \boldsymbol{f}) + \boldsymbol{M}_N| \rangle_{L^2(\Omega)} \\
&\quad + \left(\frac{1}{2c_2} + \frac{1}{2c_3} \right) \|\varepsilon \boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{c_2}{2} \|\boldsymbol{d} - \boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{c_3}{2} \|\nabla e_p\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\quad + \frac{1}{2c_1} \sum_{E \in \mathcal{E}^{\mathrm{int}}(\widehat{\mathcal{T}})} h_E \|[(\nabla u_N - \boldsymbol{m}_N \chi_\Omega) \cdot \boldsymbol{\nu}]_E\|^2_{L^2(E)} + \left(\frac{C(c_1 + c_4 + c_6)}{2} + 2c_5 \right) \|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} \\
&\quad + \left(\frac{1}{2c_4} + \frac{1}{2c_6} + 2c_5 \right) \|\nabla(u_N + p_N)\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \|\boldsymbol{M}_N\|_{\boldsymbol{L}^1(\Omega)} + \left| \langle \boldsymbol{M}_N \,|\, \boldsymbol{m}_N \rangle_{\boldsymbol{L}^2(\Omega)} \right|.
\end{aligned}
$$

To absorb certain right-hand side terms we choose firstly c_1, c_2, c_4, c_5 and c_6 sufficiently small to absorb $\|\boldsymbol{d} - \boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)}$ and $\|\nabla e_u\|^2_{\boldsymbol{L}^2(\widehat{\Omega})}$. Secondly, we choose $c_3 < 2c_5$ to absorb $\|\nabla e_p\|^2_{\boldsymbol{L}^2(\widehat{\Omega})}$. This gives the claimed estimate and ends the proof. □

Proof of Theorem 4.2. Because of the consistency of the stabilizing bilinear form σ, cf. Lemma 3.2, we deduce as in (3.34) and (3.35)

$$
\|\Pi \boldsymbol{l} - \boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} \le C \left(\|\boldsymbol{d} - \boldsymbol{d}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla e_p\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + |(u_N, \boldsymbol{m}_N)|^2_\sigma \right).
$$

We denote now the right-hand-side of (4.5) with $C\mathtt{est}^2$ and observe

$$
\|\Pi \boldsymbol{l} - \boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} \le C\mathtt{est}^2.
$$

We estimate now $\|\boldsymbol{l}-\Pi\boldsymbol{l}\|^2_{L^2(\Omega,\mathbb{R}^d)}$. To this end consider one more time equation (2.1) with $v=0$ and $\boldsymbol{n}=\boldsymbol{l}-\Pi\boldsymbol{l}$.

$$
\begin{aligned}
&\|\boldsymbol{l}-\Pi\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{l}-\Pi\boldsymbol{l}\,|\,\boldsymbol{l}-\Pi\boldsymbol{l}\rangle_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{l}\,|\,\boldsymbol{l}-\Pi\boldsymbol{l}\rangle_{\boldsymbol{L}^2(\Omega)}\\
&= \langle \boldsymbol{f}\,|\,\boldsymbol{l}-\Pi\boldsymbol{l}\rangle_{\boldsymbol{L}^2(\Omega)} - \langle \boldsymbol{d}\,|\,\boldsymbol{l}-\Pi\boldsymbol{l}\rangle_{\boldsymbol{L}^2(\Omega)} + \langle (\boldsymbol{l}-\Pi\boldsymbol{l})\chi_\Omega\,|\,\nabla p\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}\\
&= \langle \boldsymbol{f}-\Pi\boldsymbol{f}\,|\,\boldsymbol{l}-\Pi\boldsymbol{l}\rangle_{\boldsymbol{L}^2(\Omega)} - \langle \boldsymbol{d}-\boldsymbol{d}_N\,|\,\boldsymbol{l}-\Pi\boldsymbol{l}\rangle_{\boldsymbol{L}^2(\Omega)} + \langle (\boldsymbol{l}-\Pi\boldsymbol{l})\chi_\Omega\,|\,\nabla e_p\rangle_{\boldsymbol{L}^2(\widehat{\Omega})}\\
&\le \Big\{\|\boldsymbol{f}-\Pi\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{d}-\boldsymbol{d}_N\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla e_p\|_{\boldsymbol{L}^2(\widehat{\Omega})}\Big\}\|\boldsymbol{l}-\Pi\boldsymbol{l}\|_{\boldsymbol{L}^2(\Omega)}
\end{aligned}
$$

Cancelling $\|\boldsymbol{l}-\Pi\boldsymbol{l}\|_{\boldsymbol{L}^2(\Omega)}$ and squaring again leads to

$$\|\boldsymbol{l}-\Pi\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} \le C\Big\{\|\boldsymbol{f}-\Pi\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \mathtt{est}^2\Big\}$$

and hence

$$\|\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} = \|\boldsymbol{l}-\Pi\boldsymbol{l}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\Pi\boldsymbol{l}-\boldsymbol{l}_N\|^2_{\boldsymbol{L}^2(\Omega)} \le C\Big\{\|\boldsymbol{f}-\Pi\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \mathtt{est}^2\Big\}.$$

Adding this last result to (4.5) yields the claimed result. $\square$

4.2. Numerical example - adaptive FEM. We use a common adaptive algorithm of the type

$$\boxed{\mathtt{solve}} \to \boxed{\mathtt{estimate}} \to \boxed{\mathtt{mark}} \to \boxed{\mathtt{refine}}$$

For error estimation, we deduce from (4.3) a simplified reliable (upper) error bound of the following form:

$$\eta^2 := \sum_{K\in\widehat{\mathcal{T}}} \eta_K^2 := \sum_{K\in\widehat{\mathcal{T}}} (\eta_{1,K}^2 + \eta_{3,K}^2) + \sum_{K\in\mathcal{T}} (\eta_{2,K}^2 + \eta_{4,K}^2). \tag{4.12}$$

We make the following simplifications:

- As we are interested in constant exterior magnetization fields $\boldsymbol{f}$, the term $\|\boldsymbol{f}-\Pi\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$ vanishes.
- With the definition of λ_N given in (3.6) we can bound

$$
\begin{aligned}
&2\big|\langle \boldsymbol{M}_N\,|\,\boldsymbol{m}_N\rangle_{\boldsymbol{L}^2(\Omega)}\big| \le 2\langle |\boldsymbol{M}_N|\,|\,|\boldsymbol{m}_N|\rangle_{L^2(\Omega)}\\
&\le 2\langle |\boldsymbol{M}_N|\,|\,(|\boldsymbol{m}_N|-1)_+\rangle_{L^2(\Omega)} + 2\langle |\boldsymbol{M}_N|\,|\,1\rangle_{L^2(\Omega)}\\
&\le 2\|\boldsymbol{M}_N\|_{\boldsymbol{L}^2(\Omega)}\|(|\boldsymbol{m}_N|-1)_+\|_{L^2(\Omega)} + 2\|\boldsymbol{M}_N\|_{\boldsymbol{L}^1(\Omega)}\\
&\le \|\boldsymbol{M}_N\|^2_{\boldsymbol{L}^2(\Omega)} + \|\varepsilon\lambda_N\boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\boldsymbol{M}_N\|_{\boldsymbol{L}^1(\Omega)}.
\end{aligned}
$$

- A simple computation shows $\|\boldsymbol{M}_N\|^2_{\boldsymbol{L}^2(\Omega)} \lesssim \sum_{E\in\mathcal{E}^{\mathrm{int}}(\mathcal{T})} h_E\|[(\nabla u_N - \boldsymbol{m}\chi_\Omega)\cdot\boldsymbol{\nu}]_E\|^2_{L^2(E)}$.

With these simplifications, we define the local refinement indicators in (4.12) by

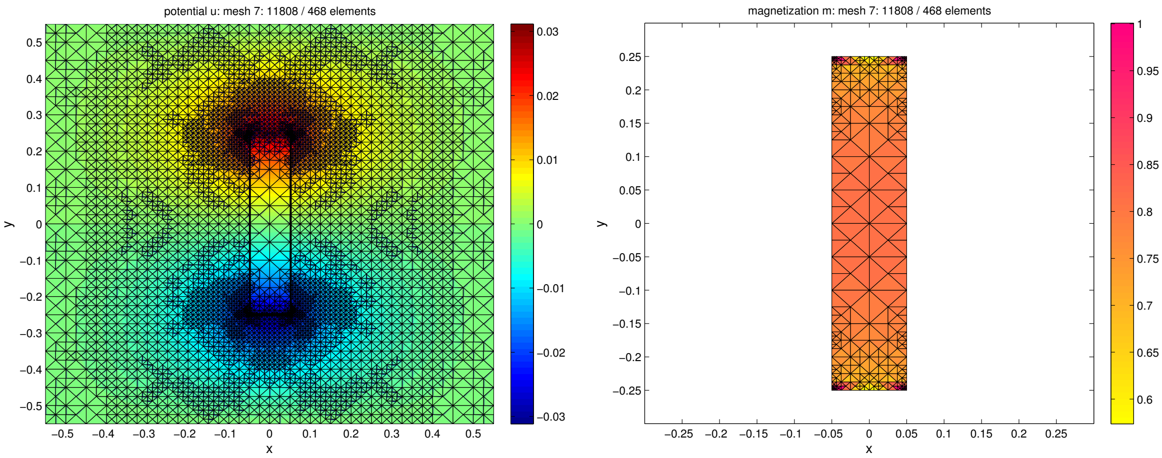
$$\eta_{1,K}^2 = \frac{1}{2}\sum_{E\in\mathcal{E}(K)\cap\mathcal{E}^{\mathrm{int}}(\widehat{\mathcal{T}})} h_E\|[(\nabla u_N - \boldsymbol{m}\chi_\Omega)\cdot\boldsymbol{\nu}]_E\|^2_{L^2(E)}, \quad \eta_{2,K}^2 = \|\boldsymbol{M}_N\|^2_{\boldsymbol{L}^1(K)},$$

$$\eta_{3,K}^2 = \|\nabla(u_N+p_N)\|^2_{\boldsymbol{L}^2(K)} \quad \text{and} \quad \eta_{4,K}^2 = \|\varepsilon\lambda_N\boldsymbol{m}_N\|^2_{\boldsymbol{L}^2(K)}.$$

For element marking, we use the strategy proposed by Dörfler in [17], i.e., for given $\theta\in(0,1]$, a smallest set $\mathscr{M}\subset\widehat{\mathcal{T}}$ is determined such that

$$\theta\sum_{K\in\widehat{\mathcal{T}}}\eta_K^2 \le \sum_{K\in\mathscr{M}}\eta_K^2,$$

FIGURE 2. convergence of adaptive algorithm: meshes



(A) mesh on $\widehat{\Omega}$ for u_N and p_N (B) mesh on Ω for magnetization

and the new mesh is obtained by refining the elements of $\mathscr{M}$ using newest vertex bisection (and performing, of course, mesh closure). In the numerical example below, we use $\theta = 0.4$.

In our second numerical experiment we choose a constant exterior field $\boldsymbol{f} = [0, 0.9]$. Again we consider the model case of uniaxial materials and choose the easy axis $\mathbf{e}$, the magnetic rod Ω, and the surrounding area $\widehat{\Omega}$ as in the first numerical experiment. Again we refer to [11].

In our numerical computation, we observe a strong mesh refinement in $\widehat{\Omega} \setminus \Omega$ towards the four corners of Ω. This is probably due to the corner singularities of u and p at the four re-entrant corners of $\widehat{\Omega} \setminus \Omega$, see Fig. 2a. Moreover, we observe in Fig. 2b some mesh refinement in Ω, which could indicate some singular behavior of $\boldsymbol{m}$. The convergence results presented in Fig. 3 suggests that the adaptive algorithm achieves the optimal convergence order $O(N^{-1/2})$, where N is the problem size, at least for $\|(\boldsymbol{m} - \boldsymbol{m}_N) \cdot \mathbf{z}\|_{L^2(\Omega)}$; this is not observed for uniform mesh-refinement for this example, see Fig. 3a and Fig. 3b. Although our *a priori* convergence theory does not provide good control over the error component $(\boldsymbol{m} - \boldsymbol{m}_N) \cdot \mathbf{e}$, the adaptive algorithm achieves almost optimal convergence of $\|(\boldsymbol{m} - \boldsymbol{m}_N) \cdot \mathbf{e}\|_{L^2(\Omega)}$ in this example.

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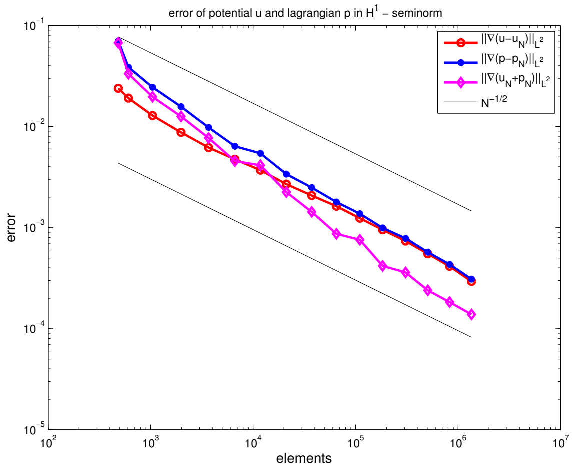
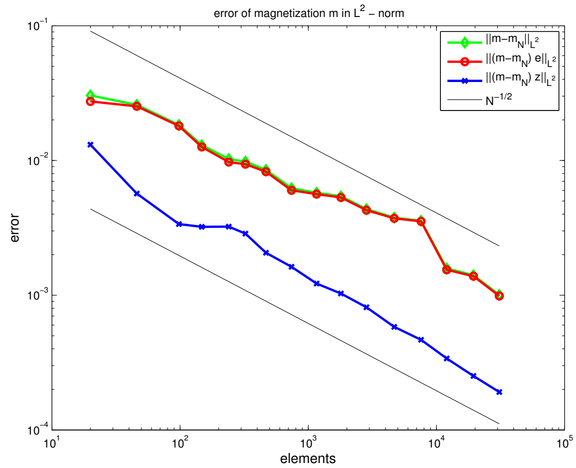
Figure 3. convergence of adaptive algorithm: errors



(A) H^1-convergence of potential u_N and Lagrange multiplier p_N



(B) convergence of different components of magnetization $\boldsymbol{m}_N$

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