Analysis, Modeling and Simulation of Multiscale Problems

Two-Scale Regularity for Homogenization Problems with Non-Smooth Fine Scale Geometries

A.-M. Matache & J.M. Melenk

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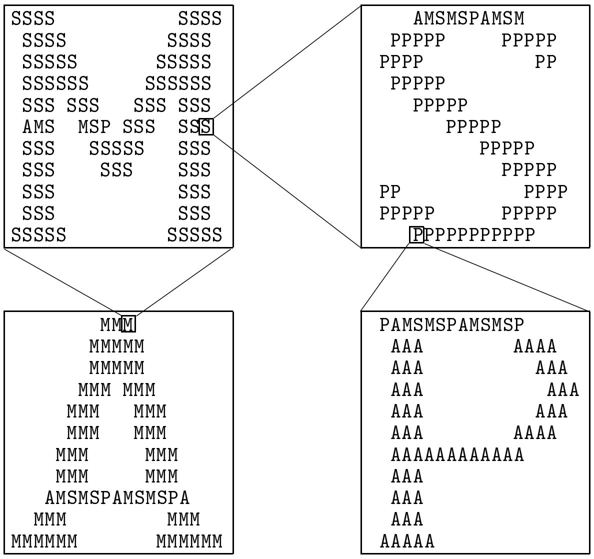

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Two-Scale Regularity for Homogenization Problems with Non-Smooth Fine Scale Geometry[1]

A.-M. Matache[2] and J.M. Melenk[3]

Abstract

Elliptic problems on unbounded domains with periodic coefficients and geometries are analyzed and two-scale regularity results for the solution are given. These are based on a detailed analysis in weighted Sobolev spaces of the so-called unit-cell problem, in which the critical parameters (the period ε, the wavenumber t, and the differentiation order) enter explicitly.

1 Introduction

The mathematical understanding of solutions to elliptic equations with rapidly oscillating coefficients is increasingly becoming important as witnessed by today's search for methods that can accurately simulate the behavior of modern materials such as composites. Typically, it is not only the averaged, "homogenized" solution that is of interest but also effects on the fine scale are relevant: for example, maximal stresses, which are responsible for material failure, arise at points where the interface between different materials is not smooth. For a class of elliptic problems with periodic coefficients and piecewise smooth fine-scale geometry, the present paper provides a two-scale regularity theory that captures both the effects on the large, global scale as well as on the fine scale.
The starting point of our analysis are representation formulas (see (2.5)) for a class of problems on unbounded domains. Key to an understanding of this integral representation is the integral kernel, which is the solution of a parameter-dependent unit-cell problem. In the present paper a detailed analysis of the regularity properties of this parameter-dependent problem is presented for the case when the unit-cell has piecewise smooth geometry (Theorems 3.7, 4.5). The representation formula allows us then to develop a new two-scale regularity theory, where the scale separation is realized by interpreting the solution as mappings defined on the physical domain and having values in weighted Sobolev spaces of periodic functions. The choice of the weighted Sobolev norm is given by our analysis of the unit-cell problem (Sections 4.1, 4.2). The present paper generalizes results of [11] where only the case of smooth fine-scale geometries was analyzed.
The regularity theory of the present paper has been formulated with a view to support the design and the understanding of new numerical methods for elliptic problems with periodic coefficients, e.g., those of [10, 8, 11]. Such non-standard methods are required as simulation tools for problems with highly oscillatory coefficients: on the one hand, standard numerical methods, which have to be based on full resolution for reliable results, cannot be employed due to computational costs; on the other hand, classical homogenization methods, which are based on averaging, fail to catch critical behaviors on the

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[2]Seminar für Angewandte Mathematik, ETH Zürich, CH–8092 Zürich
[3]MPI für Mathematik in den Naturwissenschaften, Inselstr. 22–26 D–04103 Leipzig

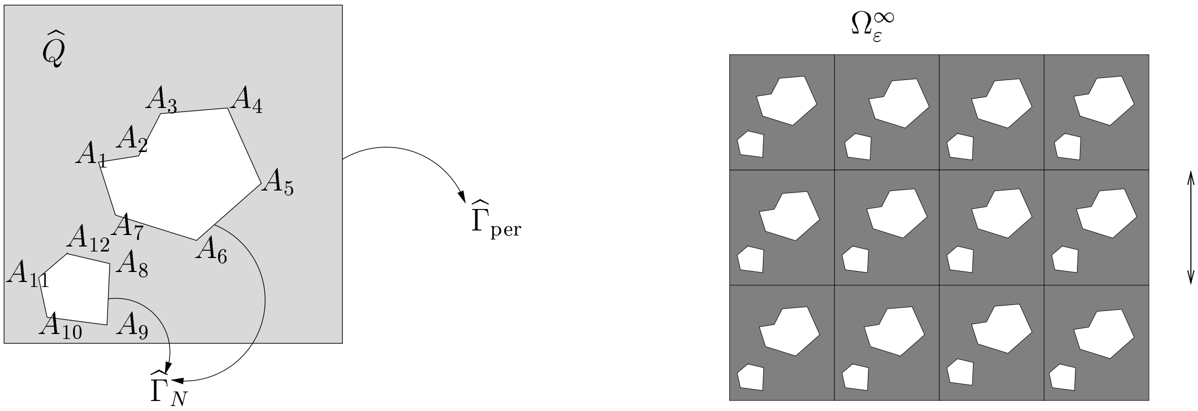

Figure 1: Unit-cell $\widehat{Q}$ (left) and tiling of $\mathbb{R}^2$ by unit-cells $\widehat{Q}$ scaled to size ε (right).

fine scale even if higher order correctors are included in the model. Our results have immediate bearing on new methods such as the generalized FEM proposed in [10, 8] and the two-scale FEM of [11]. The first method, the generalized FEM, relies on numerically solving a unit-cell problem in a preprocessing step by the *hp*-version of the finite element method. The appropriate mesh design for efficiently solving this problem can be inferred from our regularity results in Theorems 3.7, 4.5. The second method, the two-scale FEM of [11], is an energy projection on the tensor product of a standard finite-element space and a space of periodic functions. Robust convergence proofs for this method depends on a two-scale regularity theory that was developed for *smooth* fine scale geometries in [11]. The present paper extends this regularity theory to piecewise smooth fine scale geometries in Section 4.1 and develops in Section 5 the corresponding two-scale FEM approximation theory.

The paper is organized as follows: In Section 2, we introduce the class of elliptic problems with periodic fine scale geometry. For the most part, we will consider homogeneous Neumann boundary conditions; however, the corresponding results for Dirichlet or mixed boundary conditions are listed in Section 4.2. Section 3 is devoted to an analysis of the unit-cell problem. Section 4.1 presents our two-scale regularity results, and in Section 5 we illustrate how functions with two-scale regularity properties can be approximated from spaces that are tensor products of standard FEM spaces and space of periodic functions that are adapted to a non-smooth fine scale geometry. A variety of numerical methods Satz
for multiscale problems have been proposed and analyzed in the literature. The most einge-
prominent examples are based on asymptotic expansions in the parameter ε such as fuegt
classical homogenization theory [4] and Bloch wave decompositions (see, e.g., [6, 5] and the references therein). These approaches, however, rely on the assumption that ε is small. In contrast, the generalized FEM and the two-scale FEM are not restricted to small values of the parameter ε. In the concluding Section 6 we elaborate on these differences in more detail.

2 Two-Scale Representation

2.1 Problem formulation and notation

Let $\widehat{Q} \subset [0,1]^2$ be a Lipschitz domain with boundary $\partial\widehat{Q} = \widehat{\Gamma}_{\text{per}} \cup \widehat{\Gamma}_N$, where $\widehat{\Gamma}_{\text{per}} = \partial\widehat{Q} \cap \partial[0,1]^2$ and $\widehat{\Gamma}_N = \partial\widehat{Q} \setminus \widehat{\Gamma}_{\text{per}}$. (See Fig. 1, left.) We assume $\text{dist}(\widehat{\Gamma}_{\text{per}}, \widehat{\Gamma}_N) > 0$. Let further $\Omega_\varepsilon^\infty$ (cf. Fig. 1, right) denote the infinite periodic lattice with periodically repeating pattern $\varepsilon\widehat{Q}$ of size $\varepsilon \in (0,1]$, and let $\Gamma_{N,\varepsilon}^\infty$ denote the interior cavities' boundaries:

$$\Omega_\varepsilon^\infty := \bigcup_{k\in\mathbb{Z}^2} \varepsilon(k+\widehat{Q}), \qquad \Gamma_{N,\varepsilon}^\infty := \bigcup_{k\in\mathbb{Z}^2} \varepsilon(k+\widehat{\Gamma}_N). \tag{2.1}$$

We consider the regularity properties of the solution u^ε of the following problem:

$$L^\varepsilon\left(\frac{x}{\varepsilon}, \partial_x\right) u^\varepsilon := -\nabla \cdot \left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) + a_0\left(\frac{x}{\varepsilon}\right) u^\varepsilon = f(x) \quad \text{in } \Omega_\varepsilon^\infty, \tag{2.2}$$

$$\partial_{n_a} u^\varepsilon := n \cdot \left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = 0 \quad \text{on } \Gamma_{N,\varepsilon}^\infty; \tag{2.3}$$

here, $\varepsilon > 0$ is a small parameter. We assume that the matrix a is uniformly symmetric positive definite and that a_0 is uniformly positive, i.e., for some $\gamma > 0$

$$\xi^\top a(y)\xi \geq \gamma|\xi|^2, \quad a_0(y) \geq \gamma \quad \forall\, \xi \in \mathbb{R}^2\ \forall\, y \in \widehat{Q}. \tag{2.4}$$

Additionally a and a_0 are required to be analytic on $\overline{\widehat{Q}}$ and 1-periodic in each direction (thus analytic across the periodic boundary). The geometry $\partial\widehat{Q}$ is assumed to be piecewise analytic (i.e., $\partial\widehat{Q}$ consists of finitely many analytic arcs).

On bounded domains $\Omega \subset \mathbb{R}^2$, we employ standard Sobolev spaces, [1]. For non-negative integers k and the full space $\mathbb{R}^2$ the semi-norm $|f|_{H^k(\mathbb{R}^2)}$ is defined by

$$|f|^2_{H^k(\mathbb{R}^2)} := \int_{\mathbb{R}^2} |t|^{2k}\, |\hat{f}(t)|^2\, dt,$$

where $\hat{f}$ denotes the Fourier transform of f. As is customary, we define the full norm by $\|f\|^2_{H^k(\mathbb{R}^2)} := \sum_{j=0}^k |f|^2_{H^j(\mathbb{R}^2)}$. Banach space-valued Sobolev spaces will be denoted $H^k(\mathbb{R}^2, V)$. Finally, for $p \in \mathbb{N}_0$ we employ the notation of [14] and define the expression $|\nabla^p u|$ by $|\nabla^p u|^2 = \sum_{\alpha\in\mathbb{N}_0^2 : |\alpha|=p} \frac{|\alpha|!}{\alpha!} |\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} u|^2$.

2.2 Scale separation for u^ε

For any $f \in L^2(\mathbb{R}^2)$, (2.2)–(2.3) admits a unique solution $u^\varepsilon \in H^1(\Omega_\varepsilon^\infty)$. We will exploit that u^ε can be expressed as (see [8, 13])

$$u^\varepsilon(x) = \frac{1}{2\pi} \int\limits_{t\in\mathbb{R}^2} \hat{f}(t)\psi(x,\varepsilon,t)\; dt \quad x \in \Omega_\varepsilon^\infty, \tag{2.5}$$

where the kernel $\psi(x,\varepsilon,t)$ is the distributional solution of

$$L^\varepsilon \psi = e^{\mathbf{i}t\cdot x} \text{ on } \Omega_\varepsilon^\infty, \quad n \cdot (a(x/\varepsilon)\nabla\psi) = 0 \text{ on } \Gamma_{N,\varepsilon}^\infty. \tag{2.6}$$

To characterize precisely the solution of (2.6) in $\Omega_\varepsilon^\infty$, we introduce weighted Sobolev spaces $H_\nu^0(\Omega_\varepsilon^\infty)$ and $H_\nu^1(\Omega_\varepsilon^\infty)$ of complex-valued functions with exponential weights depending on a real parameter $\nu \in \mathbb{R}$by

$$H_\nu^0(\Omega_\varepsilon^\infty) = \left\{v \in L^2(\Omega_\varepsilon^\infty) \mid ve^{\nu|x|} \in L^2(\Omega_\varepsilon^\infty)\right\}, \qquad H_\nu^1(\Omega_\varepsilon^\infty) = \left\{v \in L^2(\Omega_\varepsilon^\infty) \mid ve^{\nu|x|}, \nabla ve^{\nu|x|} \in L^2(\Omega_\varepsilon^\infty)\right\}. \tag{2.7}$$

We note that for $\nu > 0$ there holds $H^1_\nu \subset H^1_0 = H^1 \subset H^1_{-\nu}$. Let us introduce the following sesquilinear form $\Psi(\varepsilon)[\cdot,\cdot] : H^1_{-\nu}(\Omega^\infty_\varepsilon) \times H^1_\nu(\Omega^\infty_\varepsilon) \to \mathbb{C}$:

$$\Psi(\varepsilon)[u,v] = \int\limits_{\Omega^\infty_\varepsilon} \left\{ \left(a\left(\frac{x}{\varepsilon}\right) \nabla_x u(x) \right) \cdot \overline{\nabla_x v(x)} + a_0\left(\frac{x}{\varepsilon}\right) u(x)\overline{v(x)} \right\} dx. \tag{2.8}$$

For all $\varepsilon > 0$ and for $\nu > 0$ sufficiently small, $\Psi(\varepsilon)$ is bounded and 'coercive' with respect to $H^1_{-\nu}(\Omega^\infty_\varepsilon) \times H^1_\nu(\Omega^\infty_\varepsilon)$, in the sense that the *inf-sup* stability condition holds, [8, 13]:

Proposition 2.1 *There exists a $\nu_0 > 0$ such that for all $\nu \in (0,\nu_0)$ and all $\varepsilon > 0$ the variational problem*

$$\begin{aligned} &\textit{given } f \in \left(H^1_\nu(\Omega^\infty_\varepsilon)\right)^*, \textit{ find} \\ &u^\varepsilon \in H^1_{-\nu}(\Omega^\infty_\varepsilon) \;:\; \Psi(\varepsilon)[u^\varepsilon, v] = \langle f, v\rangle_{(H^1_\nu(\Omega^\infty_\varepsilon))^* \times H^1_\nu(\Omega^\infty_\varepsilon)} \qquad \forall\, v \in H^1_\nu(\Omega^\infty_\varepsilon), \end{aligned} \tag{2.9}$$

admits a unique weak solution $u^\varepsilon \in H^1_{-\nu}(\Omega^\infty_\varepsilon)$ and satisfies the a-priori estimate

$$\|u^\varepsilon\|_{H^1_{-\nu}(\Omega^\infty_\varepsilon)} \le (1/\gamma)\|f\|_{(H^1_\nu(\Omega^\infty_\varepsilon))^*}.$$

Moreover, u^ε admits the representation (2.5) where the integral is understood as a Bochner integral of $H^1_{-\nu}$-valued functions.

$\psi(x,\varepsilon,t)$ is the weak solution of (2.9) with respect to the functional $f = e^{\mathbf{i}t\cdot x} \in (H^1_\nu(\Omega^\infty_\varepsilon))^*$. By Proposition 2.1 we know that

$$\left\| \psi(x,\varepsilon,t) \right\|_{H^1_{-\nu}(\Omega^\infty_\varepsilon)} \le 1/\gamma \|e^{\mathbf{i}t\cdot x}\|_{(H^1_\nu(\Omega^\infty_\varepsilon))^*} \le \frac{1}{\gamma\nu}.$$

Problem (2.2) has separated scales, a slow variable x and a fast variable $y = x/\varepsilon$, in the following sense: the kernel ψ in (2.6) (which represents fine scale response to the coarse scale excitation $e^{\mathbf{i}t\cdot x}$) can be written in separated form $\psi(x,\varepsilon,t) = e^{\mathbf{i}t\cdot x}\phi(x/\varepsilon,\varepsilon,t)$ where $\phi(y,\varepsilon,t)$ is the solution of the so-called *unit-cell problem* [8, 13]: $\phi(\cdot,\varepsilon,t) \in H^1_{\mathrm{per}}(\widehat{Q})$

$$\mathcal{L}(\varepsilon,t,y;\partial_y)\phi := e^{-\mathbf{i}\varepsilon t\cdot y} L^\varepsilon(y,\varepsilon^{-1}\partial_y) e^{\mathbf{i}\varepsilon t\cdot y}\phi = 1 \quad \text{in } \widehat{Q} \tag{2.10}$$

$$\mathcal{B}(\varepsilon,t,y;\partial_y)\phi := e^{-\mathbf{i}\varepsilon t\cdot y} n \cdot \left(a(y)\nabla_y (e^{\mathbf{i}\varepsilon t\cdot y}\phi) \right) = 0 \quad \text{on } \widehat{\Gamma}_N. \tag{2.11}$$

Unlike ψ, the kernel ϕ is computable by solving the unit-cell problem (2.10)–(2.11) numerically, for example by the finite element method.

3 Regularity of the Integral Kernel ϕ

This section provides analytic regularity assertions for the solution ϕ of the unit-cell problem (2.10). The coefficients in (2.10) are analytic and the (internal) boundary of $\widehat{Q}$ consists by assumption of analytic arcs. The solution ϕ is therefore analytic on $\widehat{Q}$ and in fact analytic up to boundary of $\widehat{Q}$ with the exception of the (internal) corners of $\widehat{Q}$. As is well-known, analyticity of a function can be characterized in terms of the growth of the derivatives as the differentiation order tends to infinity. Our technical tool to capture this growth in dependence on the critical parameters, namely, ε, t, and the distance to the nearest singularity, are the countably normed spaces $\mathcal{B}^2_{\beta,\mathcal{E}}$ that we define in Definition 3.1.

3.1 Analytic regularity of parameter-dependent problems

3.1.1 Notation

Let A_i, $i = 1, \dots, J$, be the internal vertices of $\widehat{Q}$. We define functions r_i as

$$r_i(x) := \operatorname{dist}(x, A_i), \qquad i = 1, \dots, J.$$

With each vertex A_i, we associate an "exponent" $\beta_i \in [0,1)$ and write $\beta = (\beta_1, \dots, \beta_J) \in [0,1)^J$. For each $p \in \mathbb{N}_0$ and $\mathcal{E} > 0$ we define the weight function

$$w_{p,\beta,\mathcal{E}}(x) := \prod_{i=1}^{J} \left(\min \left\{ 1, \frac{r_i(x)}{\min\{1, (p+1)\mathcal{E}\}} \right\} \right)^{p+\beta_i}. \tag{3.1}$$

Using this weight functions $w_{p,\beta,\mathcal{E}}$, we can define the weighted Sobolev spaces and spaces of analytic functions as follows:

Definition 3.1 *For $l \in \mathbb{N}$, $p \in \mathbb{N}_0$ the spaces $H^{p+l,l}_{\beta,\mathcal{E}}(\widehat{Q})$ are the completion of $C^\infty(\overline{\widehat{Q}})$ under the norm $\|\cdot\|_{H^{p+l,l}_{\beta,\mathcal{E}}(\widehat{Q})}$ given by*

$$\|u\|^2_{H^{p+l,l}_{\beta,\mathcal{E}}(\widehat{Q})} := \sum_{k=0}^{l-1} \mathcal{E}^{2k} \|\nabla^k u\|^2_{L^2(\widehat{Q})} + \sum_{k=0}^{p} \mathcal{E}^{2(k+2)} \|w_{k,\beta,\mathcal{E}} \nabla^{k+l} u\|^2_{L^2(\widehat{Q})}. \tag{3.2}$$

For C_u, $\gamma_u > 0$ the countably normed space $\mathcal{B}^2_{\beta,\mathcal{E}}(C_u, \gamma_u)$ of analytic functions is defined as:

$$u \in \mathcal{B}^2_{\beta,\mathcal{E}}(C_u,\gamma_u) \Longleftrightarrow \begin{cases} \|u\|_{L^2(\widehat{Q})} + \mathcal{E}\|\nabla u\|_{L^2(\widehat{Q})} \le C_u \quad \text{and} \\ \|w_{p,\beta,\mathcal{E}} \nabla^{p+2} u\|_{L^2(\widehat{Q})} \le C_u \gamma_u^p \max\{p+1, \mathcal{E}^{-1}\}^{p+2} \quad \forall p \in \mathbb{N}_0 . \end{cases} \tag{3.3}$$

The following lemma shows that the countably normed spaces $\mathcal{B}^2_{\beta,\mathcal{E}}$ do not change significantly if $\mathcal{E}$ is replaced with $\mathcal{E}' \sim \mathcal{E}$:

Lemma 3.2 *Let $c \in (0,1]$, $\beta \in [0,1)^J$, C_u, $\gamma_u > 0$. Then there exist C'_u, $\gamma'_u > 0$ such that for every $\mathcal{E}$, $\mathcal{E}'$ with $0 < c\mathcal{E}' \le \mathcal{E} \le \mathcal{E}' \le 1$ we have*

$$u \in \mathcal{B}^2_{\beta,\mathcal{E}}(C_u,\gamma_u) \quad \Longrightarrow \quad u \in \mathcal{B}^2_{\beta,\mathcal{E}'}(C'_u,\gamma'_u), \tag{3.4}$$

$$u \in \mathcal{B}^2_{\beta,\mathcal{E}'}(C_u,\gamma_u) \quad \Longrightarrow \quad u \in \mathcal{B}^2_{\beta,\mathcal{E}}(C'_u,\gamma'_u). \tag{3.5}$$

Proof: We note that the weight functions $w_{p,\beta,\mathcal{E}}$ are monotonically decreasing in $\mathcal{E}$ for fixed p, β, x:

$$w_{p,\beta,\mathcal{E}}(x) \ge w_{p,\beta,\mathcal{E}'}(x) \qquad \forall \mathcal{E} \le \mathcal{E}'. \tag{3.6}$$

The implication (3.4) then follows with the aid of (3.6) and the additional observation

$$\begin{aligned} \max\{p+1, 1/\mathcal{E}\} &= \max\{p+1, (1/\mathcal{E}')(\mathcal{E}'/\mathcal{E})\} \le \max\{p+1, 1/\mathcal{E}'\} \max\{1, \mathcal{E}'/\mathcal{E}\} \\ &\le \max\{p+1, 1/\mathcal{E}'\}\, 1/c. \end{aligned}$$

To show (3.5), we note that the assumption $c\mathcal{E}' \le \mathcal{E}$ implies with the definition of the weight function $w_{p,\beta,\mathcal{E}}$

$$w_{p,\beta,\mathcal{E}}(x) \le w_{p,\beta,\mathcal{E}'}(x) \prod_{j=1}^{J} c^{-(p+\beta_j)} \qquad \forall x \in \widehat{Q} \quad \forall p \in \mathbb{N}_0 .$$

This estimate together with the trivial bound $\max\{p+1, 1/\mathcal{E}'\} \le \max\{p+1, 1/\mathcal{E}\}$ then gives (3.5).

□

3.1.2 A parameter-dependent elliptic problem: regularity assumptions

We consider the regularity of solutions of the following parameter-dependent elliptic equation:

$$
\begin{aligned}
-\zeta^2 \nabla \cdot (a \nabla u) + b \cdot \nabla u + cu &= f \quad \text{on } \widehat{Q}, && (3.7a)\\
\zeta \partial_{n_a} u &= G + Hu \quad \text{on } \widehat{\Gamma}_N, && (3.7b)
\end{aligned}
$$

where a is the matrix appearing in the unit-cell problem; the $\mathbb{C}^2$-valued function b and the $\mathbb{C}$-valued function c are assumed analytic on the closure of $\widehat{Q}$, i.e., they satisfy for suitable C_a, C_f, C_b, $C_c > 0$, and $\gamma > 0$

$$
\begin{aligned}
&\|\nabla^p a\|_{L^\infty(\widehat{Q})} \le C_a \gamma^p p!, \qquad \|\nabla^p b\|_{L^\infty(\widehat{Q})} \le C_b \gamma^p p! \quad \forall p \in \mathbb{N}_0, && (3.8a)\\
&\|\nabla^p c\|_{L^\infty(\widehat{Q})} \le C_c \gamma^p p!, \qquad \|\nabla^p f\|_{L^\infty(\widehat{Q})} \le C_f \gamma^p p! \quad \forall p \in \mathbb{N}_0, && (3.8b)\\
&\|\nabla^p G\|_{L^\infty(\widehat{Q})} \le C_G \gamma^p p! \quad \forall p \in \mathbb{N}_0. && (3.8c)
\end{aligned}
$$

Concerning the coefficient H appearing in the boundary conditions, we assume that H is piecewise analytic on $\widehat{\Gamma}_N$, i.e., on each analytic arc Γ_i of the boundary $\widehat{\Gamma}_N$ we have

$$
\|D_t^p H\|_{L^\infty(\Gamma_i)} \le C_H \gamma^p p! \quad \forall p \in \mathbb{N}_0. \tag{3.8d}
$$

Here, the operator D_t stands for tangential differentiation obtained by viewing $H|_{\Gamma_i}$ as a function of arc length s of the boundary part Γ_i and differentiating with respect to s.
Finally, we assume that the parameter $\zeta > 0$.
Bounds on higher derivatives of the solution u of (3.7) depend critically on a parameter $\mathcal{E}$, which measures the size of ζ relative to the size of the coefficients b and c (under the implicit assumption that the eigenvalues of a are uniformly $O(1)$):

$$
\mathcal{E}^{-1} := 1 + \frac{C_b}{\zeta^2} + \frac{\sqrt{C_c}}{\zeta}. \tag{3.9}
$$

3.1.3 Weighted H^2-estimates

We have the following regularity assertion for the solution u of (3.7).

Proposition 3.3 *There exist $C > 0$ and $\beta \in [0,1)^J$ depending only on the coefficient a and the geometry $\widehat{Q}$ such that with $\mathcal{E}$ given by (3.9) any solution u of (3.7) satisfies*

$$
\begin{aligned}
\|w_{0,\beta,\mathcal{E}} \nabla^2 u\|_{L^2(\widehat{Q})} \le C\mathcal{E}^{-2} \Big(&(\mathcal{E}/\zeta)^2 \|w_{0,\beta,\mathcal{E}} f\|_{L^2(\widehat{Q})} + (\mathcal{E}/\zeta) \left[\mathcal{E} \|w_{0,\beta,\mathcal{E}} \nabla G\|_{L^2(\widehat{Q})} + \|G\|_{L^2(\widehat{Q})}\right] \\
&+ [1 + C_H(\mathcal{E}/\zeta)]\, \mathcal{E} \|\nabla u\|_{L^2(\widehat{Q})} + \left[C_c(\mathcal{E}/\zeta)^2 + C_H(\mathcal{E}/\zeta)\right] \|u\|_{L^2(\widehat{Q})}\Big).
\end{aligned}
$$

Proof: We will only sketch the proof as details can be found in [12, Chap. 5.4]. For simplicity, we assume that $\widehat{Q}$ is polygonal since the case of piecewise smooth boundaries can be inferred with mappings that locally flatten the boundary.
We consider a neighborhood of a fixed vertex A_i. We recall $r_i(x) = \operatorname{dist}(x, A_i)$ and introduce polar coordinates (r, φ) such that for ρ sufficiently small the sectors

$$
S_\rho := \{(r\cos\varphi, r\sin\varphi) \mid 0 < r < \rho,\ 0 < \varphi < \omega\}
$$

coincide with $\{x \in \widehat{Q} \mid \operatorname{dist}(x, A_i) < \rho\}$. We fix R, $R' > 0$ and assume without loss of generality $2\mathcal{E} < R' < R$ (since the case of small $\mathcal{E}$ is of interest to us). Additionally, we assume that R is chosen so small that ∂S_R can be decomposed into three parts Γ_0, Γ_ω, Γ', where $\Gamma_0 = \{(r,0) \mid 0 < r < R\}$, $\Gamma_\omega = \{(r\cos\omega, r\sin\omega) \mid 0 < r < R\}$ make up the lateral part of ∂S_R, and $\Gamma' = \partial S_R \setminus (\overline{\Gamma_0} \cup \overline{\Gamma_\omega})$ is the curved part of ∂S_R.
Next, we can find a cut-off function χ with the following properties:

$$\begin{array}{ll} \chi \equiv 1 \text{ on } \widehat{Q} \cap B_{\mathcal{E}}(A_i), & \chi \equiv 0 \text{ on } \widehat{Q} \setminus B_{2\mathcal{E}}(A_i), \\ \partial_{n_a}\chi = 0 \text{ on } \Gamma_0, \Gamma_\omega, & \|\nabla^j \chi\|_{L^\infty(S_R)} \le C\mathcal{E}^{-j}, \quad j \in \{0,1,2\}, \end{array}$$

where the constant $C > 0$ does not depend on $\mathcal{E} \in (0,1]$.
The datum H is only defined on $\widehat{\Gamma}_N$. We extend H to a function (again denoted H) on S_R using polar coordinates as

$$H(r\cos\varphi, r\sin\varphi) = \frac{\varphi}{\omega} H_\omega(r) + \left(1 - \frac{\varphi}{\omega}\right) H_0(r),$$

where $H_0(r) = H(r,0)$, $H_\omega(r) = H(r\cos\omega, r\sin\omega)$ are the restrictions of H to the edges Γ_0, Γ_ω, respectively. A calculation shows that the extension H is in a weighted $W^{1,\infty}$-space:

$$\|H\|_{L^\infty(S_R)} + \|\, r_i \nabla H\|_{L^\infty(S_R)} \le C C_H, \tag{3.10}$$

where $C > 0$ depends only on ω, R.
1. step: We show that there exist $C > 0$, $\beta_i \in [0,1)$ depending only on a, ω, and R such that a function $u \in H^1(S_R)$ solving

$$-\zeta^2 \nabla \cdot (a\nabla u) + b \cdot \nabla u + cu = f \quad \text{on } S_R, \qquad \zeta \partial_{n_a} u = G + Hu \quad \text{on } \Gamma_0, \Gamma_\omega \tag{3.11}$$

verifies

$$\begin{aligned} \|w_i \nabla^2 u\|_{L^2(S_{\mathcal{E}})} \le C\mathcal{E}^{-2} \Big\{ & (\mathcal{E}/\zeta)^2 \|w_i f\|_{L^2(S_{2\mathcal{E}})} + (\mathcal{E}/\zeta)\mathcal{E}\|w_i \nabla G\|_{L^2(S_{2\mathcal{E}})} \\ & + (\mathcal{E}/\zeta)\|G\|_{L^2(S_{2\mathcal{E}})} + (1 + (\mathcal{E}/\zeta) C_H) \left[\mathcal{E}\|\nabla u\|_{L^2(S_{2\mathcal{E}})} + \|u\|_{L^2(S_{2\mathcal{E}})}\right] \Big\}, \end{aligned} \tag{3.12}$$

where the weight function w_i is given by

$$w_i(x) := \min\left\{1, (r_i(x)/\mathcal{E})^{\beta_i}\right\}.$$

In order to show (3.12), we calculate that $\tilde{u} := u\chi$ satisfies

$$-\nabla \cdot (a\nabla \tilde{u}) = \tilde{f}, \qquad \partial_{n_a}\tilde{u} = \tilde{G} \quad \text{on } \Gamma_0, \Gamma_\omega, \quad \tilde{u} = 0 \quad \text{on } \Gamma',$$

where $\tilde{f}$, $\tilde{G}$ are given by

$$\begin{aligned} \tilde{f} &= \chi\zeta^{-2}[f - b\cdot\nabla u - cu] - 2\nabla\chi \cdot (a\nabla u) - u\nabla\cdot(a\nabla\chi), \\ \tilde{G} &= \zeta^{-1}(\chi G + H\chi u). \end{aligned}$$

From [2] we obtain the existence of $C > 0$, $\beta_i \in [0,1)$ (depending only on a, R) such that

$$\|r_i^{\beta_i} \nabla^2 \tilde{u}\|_{L^2(S_{2\mathcal{E}})} \le C\left[\|r_i^{\beta_i}\tilde{f}\|_{L^2(S_R)} + \|r_i^{\beta_i}\nabla\tilde{G}\|_{L^2(S_R)} + \|\tilde{G}\|_{L^2(S_R)}\right].$$

Since $\tilde{G}$ is supported by $B_{2\mathcal{E}}(A_i)$, we may choose the compact set S' of Lemma A.2 such that $S' \subset S_R \setminus B_{2\mathcal{E}}(A_i)$ to conclude $\|\tilde{G}\|_{L^2(S_R)} \leq C\|r_i^{\beta_i}\nabla\tilde{G}\|_{L^2(S_R)}$. We therefore obtain

$$\|r_i^{\beta_i}\nabla^2\tilde{u}\|_{L^2(S_{2\mathcal{E}})} \leq C\left[\|r_i^{\beta_i}\tilde{f}\|_{L^2(S_R)} + \|r_i^{\beta_i}\nabla\tilde{G}\|_{L^2(S_R)}\right]. \tag{3.13}$$

In order to bound the term $\|r_i^{\beta_i}\nabla\tilde{G}\|_{L^2(S_R)}$, we write $\nabla(H\chi u) = (\nabla H)\chi u + H\nabla(\chi u)$ and estimate using (3.10) and Lemma A.2 (again taking the compact set S' such that $\chi \equiv 0$ on S')

$$\begin{aligned} \|r_i^{\beta_i}\nabla(H\chi u)\|_{L^2(S_R)} &\leq CC_H\left[\|r_i^{\beta_i-1}(\chi u)\|_{L^2(S_R)} + \|r_i^{\beta_i}\nabla(\chi u)\|_{L^2(S_R)}\right] \\ &\leq CC_H\|r_i^{\beta_i}\nabla(\chi u)\|_{L^2(S_R)}. \end{aligned}$$

Inserting this bound into (3.13), we arrive at

$$\|r_i^{\beta_i}\nabla^2\tilde{u}\|_{L^2(S_{2\mathcal{E}})} \leq C\|r_i^{\beta_i}\tilde{f}\|_{L^2(S_R)} + C\zeta^{-1}\left[\|r_i^{\beta_i}\nabla(\chi G)\|_{L^2(S_R)} + CC_H\|r_i^{\beta_i}\nabla(\chi u)\|_{L^2(S_R)}\right].$$

Dividing by $\mathcal{E}^{\beta_i}$, observing $\operatorname{supp}\tilde{f}$, $\operatorname{supp}(\chi G)$, $\operatorname{supp}(\chi u) \subset B_{2\mathcal{E}}(A_i)$ in view of $\operatorname{supp}\chi \subset B_{2\mathcal{E}}(A_i)$, we obtain with the definition of w_i

$$\|w_i\nabla^2\tilde{u}\|_{L^2(S_{2\mathcal{E}})} \leq C\|w_i\tilde{f}\|_{L^2(S_R)} + C\zeta^{-1}\left[\|w_i\nabla(\chi G)\|_{L^2(S_R)} + C_H\|\nabla(\chi u)\|_{L^2(S_R)}\right]. \tag{3.14}$$

With the simple estimates

$$\frac{C_b}{\zeta^2} \leq \mathcal{E}^{-1}, \qquad \frac{C_c}{\zeta^2} \leq \mathcal{E}^{-2}, \tag{3.15}$$

and the properties of χ, we get by expanding $\tilde{f}$

$$\begin{aligned} \|w_i\nabla^2 u\|_{L^2(S_{\mathcal{E}})} \leq C\Big(&\zeta^{-2}\|w_i f\|_{L^2(S_{2\mathcal{E}})} + \mathcal{E}^{-1}\|\nabla u\|_{L^2(S_{2\mathcal{E}})} + \mathcal{E}^{-2}\|u\|_{L^2(S_{2\mathcal{E}})} \\ &+ \zeta^{-1}\left[\mathcal{E}^{-1}\|G\|_{L^2(S_{2\mathcal{E}})} + \|w_i\nabla G\|_{L^2(S_{2\mathcal{E}})} + C_H\mathcal{E}^{-1}\|u\|_{L^2(S_{2\mathcal{E}})} + C_H\|\nabla u\|_{L^2(S_{2\mathcal{E}})}\right]\Big), \end{aligned}$$

from which the bound (3.12) can be obtained.
2. step: In order to sharpen the bound (3.12), we define the average

$$\overline{u}_{2\mathcal{E}} := \frac{1}{|S_{2\mathcal{E}}|}\int_{S_{2\mathcal{E}}} u\,dx.$$

We note the Poincaré inequalities

$$\|u - \overline{u}_{2\mathcal{E}}\|_{L^2(S_{2\mathcal{E}})} \leq C\mathcal{E}\|\nabla u\|_{L^2(S_{2\mathcal{E}})}, \qquad \|\overline{u}_{2\mathcal{E}}\|_{L^2(S_{2\mathcal{E}})} \leq \|u\|_{L^2(S_{2\mathcal{E}})}. \tag{3.16}$$

Next, we observe that the function $\hat{u} := u - \overline{u}_{2\mathcal{E}}$ satisfies

$$-\zeta^2\nabla\cdot(a\nabla\hat{u}) + b\cdot\nabla\hat{u} + c\hat{u} = \hat{f} \quad \text{on } S_R, \qquad \partial_{n_a}\hat{u} = \hat{G} + H\hat{u} \quad \text{on } \Gamma_0,\, \Gamma_\omega$$

where the functions $\hat{f}$, $\hat{G}$ are given by

$$\hat{f} = f - c\overline{u}_{2\mathcal{E}}, \qquad \hat{G} = (G + H\overline{u}_{2\mathcal{E}}).$$

Hence the estimate (3.12) applied to $\hat{u}$ gives

$$
\begin{aligned}
\|w_i\nabla^2 u\|_{L^2(S_\mathcal{E})} \leq\ & C\mathcal{E}^{-2}\Big\{(\mathcal{E}/\zeta)^2\|f - c\overline{u}_{2\mathcal{E}}\|_{L^2(S_{2\mathcal{E}})} \\
& + (\mathcal{E}/\zeta)\mathcal{E}\|w_i\nabla(G + H\overline{u}_{2\mathcal{E}})\|_{L^2(S_{2\mathcal{E}})} + (\mathcal{E}/\zeta)\|G + H\overline{u}_{2\mathcal{E}}\|_{L^2(S_{2\mathcal{E}})} \\
& + (1 + (\mathcal{E}/\zeta)C_H)\left[\mathcal{E}\|\nabla u\|_{L^2(S_{2\mathcal{E}})} + \|u - \overline{u}_{2\mathcal{E}}\|_{L^2(S_{2\mathcal{E}})}\right]\Big\}.
\end{aligned}
$$

Inserting the inequalities (3.16) gives the desired bound, if we observe additionally

$$
\begin{aligned}
\mathcal{E}\|w_i\nabla H\overline{u}_{2\mathcal{E}}\|_{L^2(S_{2\mathcal{E}})} \leq\ & CC_H\mathcal{E}^{1-\beta_i}\|r_i^{\beta_i-1}\|_{L^2(S_{2\mathcal{E}})}|\overline{u}_{2\mathcal{E}}| \leq CC_H\mathcal{E}^{1-\beta_i}\mathcal{E}^{\beta_i}|\overline{u}_{2\mathcal{E}}| \\
\leq\ & CC_H\|\overline{u}_{2\mathcal{E}}\|_{L^2(S_{2\mathcal{E}})} \leq CC_H\|u\|_{L^2(S_{2\mathcal{E}})}.
\end{aligned}
$$

3. step: We turn to estimating $\nabla^2 u$ on $S_R \setminus S_\mathcal{E}$, which follows from standard elliptic regularity estimates on balls of size $r \sim \mathcal{E}$ as we now show. Elementary geometric considerations allow us to construct a covering of $S_{R'} \setminus S_\mathcal{E}$ by balls $B_{c\mathcal{E}}(x_j)$, $j \in \mathbb{N}$, with the following properties:

1. $\operatorname{dist}(B_{c\mathcal{E}}(x_j), A_i) > \mathcal{E}/2$ for all $j \in \mathbb{N}$;
2. Either $B_{c\mathcal{E}}(x_i) \subset S_R$ or $B_{c\mathcal{E}(x_i)} \cap S_R$ is a half-ball; in the latter case, the center $x_j \in \Gamma_0 \cup \Gamma_\omega$.
3. The balls $B_{2c\mathcal{E}}(x_i)$ have a finite overlap property: there exists $N > 0$ such that $\operatorname{card}\{j \in \mathbb{N} \,|\, x \in B_{c\mathcal{E}}(x_j)\} \leq N$ for all $x \in S_R$;

Under these assumptions, the standard regularity estimates of Lemma A.1 assert for each ball $B_{c\mathcal{E}}(x_j)$ we have using the shorthand $B_j := B_{c\mathcal{E}}(x_j) \cap S_R$, $\widehat{B}_j := B_{2c\mathcal{E}}(x_j) \cap S_R$

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2(B_j)} \leq\ & C\Big\{\zeta^{-2}\|f - b\cdot\nabla u - cu\|_{L^2(\widehat{B}_j)} \\
& + \zeta^{-1}\|\nabla(G + Hu)\|_{L^2(\widehat{B}_j)} + \zeta^{-1}\mathcal{E}^{-1}\|G + Hu\|_{L^2(\widehat{B}_j)} + \mathcal{E}^{-1}\|\nabla u\|_{L^2(\widehat{B}_j)}\Big\}.
\end{aligned}
$$

Squaring, summing over all balls, and using the overlap properties of the covering allows us to conclude

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2(S_{R'}\setminus S_\mathcal{E})} \leq\ & C\Big\{\zeta^{-2}\|f - b\cdot\nabla u - cu\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} + \zeta^{-1}\|\nabla(G + Hu)\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} \\
& + \zeta^{-1}\mathcal{E}^{-1}\|G + Hu\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} + \mathcal{E}^{-1}\|\nabla u\|_{L^2(S_R\setminus S_{\mathcal{E}/2})}\Big\}.
\end{aligned}
$$

The bound (3.10) implies $\|\nabla H\|_{L^\infty(S_R\setminus S_{\mathcal{E}/2})} \leq CC_H\mathcal{E}^{-1}$. Hence, we get together with (3.15)

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2(S_{R'}\setminus S_\mathcal{E})} \leq C\mathcal{E}^{-2}\Big\{& (\mathcal{E}/\zeta)^2\|f\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} + \mathcal{E}\|\nabla u\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} \\
& + C_c(\mathcal{E}/\zeta)^2\|u\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} + (\mathcal{E}/\zeta)\mathcal{E}\|\nabla G\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} + (\mathcal{E}/\zeta)\|G\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} \\
& + C_H(\mathcal{E}/\zeta)\left[\|u\|_{L^2(S_R\setminus S_{\mathcal{E}/2})} + \mathcal{E}\|\nabla u\|_{L^2(S_R\setminus S_{\mathcal{E}/2})}\right]\Big\}.
\end{aligned}
$$

In view of the fact that $w_i \sim 1$ on $S_R \setminus S_\mathcal{E}$, we may replace $\|\nabla^2 u\|_{L^2(S_{R'}\setminus S_\mathcal{E})}$ with $\|w_i\nabla^2 u\|_{L^2(S_{R'}\setminus S_\mathcal{E})}$ as well as $\|f\|_{L^2(S_R\setminus S_{\mathcal{E}/2})}$ with $\|w_i f\|_{L^2(S_R\setminus S_{\mathcal{E}/2})}$ and $\|\nabla G\|_{L^2(S_R\setminus S_{\mathcal{E}/2})}$ with $\|w_i\nabla G\|_{L^2(S_R\setminus S_{\mathcal{E}/2})}$.
4. step: The estimates of step 3 can also be carried out for balls and half-balls away from all vertices. Combining the above estimates then proves the proposition. □

3.1.4 Regularity in countably normed spaces

Due to the analyticity of coefficients and the right-hand side f, the solution u of (3.7) is analytic on $\widehat{Q}$. Generalizing Proposition 3.3 we have the following bounds for all derivatives:

Proposition 3.4 *There exist C, $K > 0$, and $\beta \in [0,1)^J$ depending only on $\widehat{Q}$ and a, γ of (3.8) such that with $\mathcal{E}$ given by (3.9) a solution u of (3.7) satisfies for all $p \in \mathbb{N}_0$*

$$\|w_{p,\beta,\mathcal{E}}\nabla^{p+2}u\|_{L^2(\widehat{Q})} \leq C\max\{p+1,\mathcal{E}^{-1}\}^{p+2}\Big((\mathcal{E}/\zeta)^2 C_f + (\mathcal{E}/\zeta)\, C_G \\ + [1 + C_H(\mathcal{E}/\zeta)]\,\mathcal{E}\|\nabla u\|_{L^2(\widehat{Q})} + \big[C_c(\mathcal{E}/\zeta)^2 + C_H(\mathcal{E}/\zeta)\big]\,\|u\|_{L^2(\widehat{Q})}\Big).$$

Proof: The proof is based on induction on the differentiation order; we refer to [12, Chap. 5.4], where the details are carried out. □

Remark 3.5 The assumptions (3.8) stipulate control of the data f, G in L^∞-based norms. As is shown in [12, Chap. 5.4], the weaker estimates

$$\begin{aligned} \|w_{p,\beta,\mathcal{E}}\nabla^p f\|_{L^2(\widehat{Q})} &\leq C_f\gamma^p\max\{p,\mathcal{E}^{-1}\}^p \quad \forall p\in\mathbb{N}_0, \\ \|G\|_{L^2(\widehat{Q})} &\leq C_G, \quad \|w_{p,\beta,\mathcal{E}}\nabla^{p+1}G\|_{L^2(\widehat{Q})} \leq C_G\gamma^p\max\{p,\mathcal{E}^{-1}\}^{p+1} \quad \forall p\in\mathbb{N}_0 \end{aligned}$$

suffice for Proposition 3.4 to hold.
Proposition 3.4 provides L^2-based control of the derivatives. These estimates allow us to infer pointwise bounds with the aid of [12, Thm. 4.2.23]. ▪

3.2 Low order estimates for the unit-cell problem

The main result of this section is Theorem 3.7, where the regularity of the unit cell solution ϕ of (2.10), (2.11) is characterized in terms of the countably normed spaces $\mathcal{B}^2_{\beta,\mathcal{E}}$ of Definition 3.1: We assert $\phi\in\mathcal{B}^2_{\beta,\mathcal{E}}(C,\gamma)$ for some $\beta\in[0,1)^J$, C, $\gamma>0$ independent of ε, t and characteristic length $\mathcal{E}\sim\min\{1,1/(\varepsilon|t|)\}$. This assertion will follow from Proposition 3.4. Since this proposition involves bounds on the L^2-norm and the H^1-norm of the solution on the right-hand side, we start with estimates for $\|\phi\|_{L^2(\widehat{Q})}$, $\|\nabla\phi\|_{L^2(\widehat{Q})}$ in the following lemma.

Lemma 3.6 *There exists a positive constant $C>0$ independent of ε and t such that:*

$$\|\phi\|_{0,\widehat{Q}} \leq C(1+|t|)^{-1} \quad \textit{and} \quad \|\varepsilon^{-1}\nabla_y\phi\|_{0,\widehat{Q}} \leq C. \tag{3.17}$$

Proof: Let $\chi(x,\varepsilon,t) := \mathbf{i}t\,\psi(x,\varepsilon,t)$. Then $\chi(x,\varepsilon,t)\in\big(H^1_{-\nu}(\Omega^\infty_\varepsilon)\big)^2$ solves (2.2) with right-hand side $f(x) = \mathbf{i}te^{\mathbf{i}t\cdot x}$. For $\nu>0$, the $\|\cdot\|_{(H^1_\nu(\Omega^\infty_\varepsilon))^*}$ norm of the right-hand side is uniformly bounded with respect to t and ε

$$\|\mathbf{i}te^{\mathbf{i}t\cdot x}\|_{(H^1_\nu(\Omega^\infty_\varepsilon))^*} \leq C(\nu).$$

Therefore, for all $\nu\in(0,\nu_0)$ with ν_0 as in Proposition 2.1, $\|\chi(x,\varepsilon,t)\|_{1,-\nu} \leq C\|\mathbf{i}te^{\mathbf{i}t\cdot x}\|_{(H^1_\nu(\Omega^\infty_\varepsilon))^*} \leq C(\nu)$ and

$$\|\psi(x,\varepsilon,t)\|_{1,-\nu} \leq C(1+|t|)^{-1},$$

with $C > 0$ depending only on γ and ν. Without loss of generality we may assume that $\varepsilon = 1/M$, $M \in \mathbb{N}$. Then, by the periodicity of ϕ it holds

$$
\begin{aligned}
\int_{\widehat{Q}} |\phi(y,\varepsilon,t)|^2 dy &= \varepsilon^{-2} \int_{\varepsilon\widehat{Q}} \left|\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\right|^2 dx \\
&= \sum_{k\in\{0,1,\dots,M-1\}^2} \int_{\varepsilon(\widehat{Q}+k)} \left|\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\right|^2 e^{-2\nu|x|} e^{2\nu|x|} dx \qquad (3.18)\\
&\leq C\|\psi(x,\varepsilon,t)\|_{0,-\nu}^2 \leq C(1+|t|)^{-2}.
\end{aligned}
$$

We obtain estimates for $\|\varepsilon^{-1}\nabla_y\phi\|_{0,\widehat{Q}}$ by a similar argument:

$$
\begin{aligned}
\int_{\widehat{Q}} \left|\varepsilon^{-1}\nabla_y\phi(y,\varepsilon,t)\right|^2 dy &= \int_{\varepsilon\widehat{Q}} \left|\varepsilon^{-1}\nabla_y\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\right|^2 \varepsilon^{-2} dx \\
&= \sum_{k\in\{0,1,\dots,M-1\}^2} \int_{\varepsilon(\widehat{Q}+k)} \left|\nabla_x\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\right|^2 e^{-2\nu|x|}\cdot e^{2\nu|x|} dx \\
&\leq C\left(1+|t|^2\right)\|\psi(\cdot,\varepsilon,t)\|_{1,-\nu}^2 \leq C.
\end{aligned}
$$

□

3.3 Analytic regularity for the unit-cell problem

The results of Sections 3.1, 3.2 can be applied to the unit-cell problem (2.10), (2.11). This leads to the following result:

Theorem 3.7 *Let ϕ be the solution to (2.10), (2.11). Then there exist C, $K > 0$, $\beta \in [0,1)^J$ depending only on the coefficients a, b, c, and the geometry $\widehat{Q}$ such that ϕ satisfies*

$$
\begin{aligned}
\|\phi\|_{L^2(\widehat{Q})} &\leq C\frac{1}{1+|t|}, \\
\|\nabla\phi\|_{L^2(\widehat{Q})} &\leq C\varepsilon, \\
\|w_{p,\beta,\theta}\nabla^{p+2}\phi\|_{L^2(\widehat{Q})} &\leq CK^p \max\{p,\theta^{-1}\}^{p+2}\varepsilon\theta,
\end{aligned}
$$

where

$$
\theta := \min\left\{1, \frac{1}{\varepsilon|t|}\right\}.
$$

Proof: The theorem is obtained from the following Lemmas 3.8, 3.9, where the cases $\varepsilon|t| \leq 1$ and $\varepsilon|t| > 1$ are treated separately. □

Theorem 3.7 separates the cases $\varepsilon|t| \leq 1$ and $\varepsilon|t| > 1$. This splitting is motived by the way the parameter $\varepsilon|t|$ enters in the differential equation satisfied by ϕ: Expanding equation (2.10) and the boundary conditions (2.11), we obtain

$$
-\nabla\cdot(a\nabla\phi) - 2\mathbf{i}\varepsilon a\nabla\phi\cdot t + (\varepsilon^2 t^\top a t + \varepsilon^2 a_0 - \mathbf{i}\varepsilon(\nabla\cdot a)\cdot t)\phi = \varepsilon^2 \quad \text{on } \widehat{Q}, \qquad (3.19a)
$$

$$
n\cdot(a\nabla_y)\phi = -n\cdot(a\mathbf{i}\varepsilon t)\phi \quad \text{on } \widehat{\Gamma}_N. \qquad (3.19b)
$$

We observe that the critical parameter in this equation is $\varepsilon|t|$: For moderate value of $\varepsilon|t|$, the problem is regularly perturbed, and standard elliptic regularity can be expected to hold. For large $\varepsilon|t|$, however, the problem is singularly perturbed and the dependence on εt can be characterized with the techniques of Section 3.1. We will therefore treat the cases "$\varepsilon|t|$ large" and "$\varepsilon|t|$ small " separately in the following two subsections.

3.3.1 The regular perturbation case $\varepsilon|t| \leq 1$

Lemma 3.8 *Let ε, t satisfy $\varepsilon|t| \leq 1$. Then there exist $\beta \in [0,1)^J$, C, $K > 0$ depending only on the analyticity constants of a, a_0, $\partial\widehat{Q}$ such that*

$$\begin{aligned} \|\phi\|_{L^2(\widehat{Q})} &\leq C\frac{1}{1+|t|}, \\ \|\nabla\phi\|_{L^2(\widehat{Q})} &\leq C\varepsilon, \\ \|w_{p,\beta,1}\nabla^{p+2}\phi\|_{L^2(\widehat{Q})} &\leq CK^p p!\varepsilon \qquad \forall p \in \mathbb{N}_0. \end{aligned}$$

Proof: The first two estimates follow from Lemma 3.6. For the estimate for high order derivatives, we apply Proposition 3.4 to (3.19) with $\zeta = 1$; in this case the constants C_b, C_c appearing in (3.8) are $O(1)$, and the result then follows from Lemma 3.2, since

$$1 \leq \mathcal{E}^{-1} = 1 + C_b + \sqrt{C_c} \leq C.$$

□

3.3.2 The singular perturbation case $\varepsilon|t| > 1$

In order to treat the singularly perturbed case $\varepsilon|t| > 1$, we rewrite the unit-cell problem (3.19) in the following form:

$$-\frac{1}{\varepsilon^2|t|^2}\nabla\cdot(a\nabla\phi) - 2\frac{1}{\varepsilon^2|t|^2}\mathbf{i}\varepsilon a\nabla\phi\cdot t \tag{3.20a}$$
$$+\left(\frac{t^\top at}{|t|^2} + \frac{1}{|t|^2}a_0 - \frac{1}{\varepsilon^2|t|^2}\mathbf{i}\varepsilon(\nabla\cdot a)\cdot t\right)\phi = \frac{1}{|t|^2} \qquad \text{on } \widehat{Q},$$
$$\frac{1}{\varepsilon|t|}n\cdot(a\nabla_y)\phi = -\frac{n\cdot(a\mathbf{i}\varepsilon t)}{\varepsilon|t|}\phi \quad \text{on } \widehat{\Gamma}_N. \tag{3.20b}$$

We then apply Proposition 3.4 to (3.20) with $\zeta = (\varepsilon|t|)^{-1}$. In view of the assumption $\varepsilon|t| > 1$ and the general assumption $\varepsilon \in (0,1]$, the constants C_b, C_c appearing in (3.8) can then be chosen as

$$C_b = C\zeta, \qquad C_c = C,$$

for a constant $C > 0$ independent of ε, $|t|$. Thus, the relative diffusivity $\mathcal{E}$ appearing in Proposition 3.4 satisfies

$$c_1\,\varepsilon|t| \leq \mathcal{E}^{-1} \leq c_2\,\varepsilon|t| \tag{3.21}$$

for two constants c_1, $c_2 > 0$ independent of ε and t. Then we can formulate

Lemma 3.9 *Under the assumption $\varepsilon|t| \geq 1$ the solution ϕ of (3.19) satisfies for some C, $K > 0$, $\beta \in [0,1)^J$ depending only on a, a_0, $\partial\widehat{Q}$ the following bounds:*

$$\begin{aligned}
\|\phi\|_{L^2(\widehat{Q})} &\leq C\frac{1}{1+|t|}, \\
\|\nabla\phi\|_{L^2(\widehat{Q})} &\leq C\varepsilon, \\
\|w_{p,\beta,|\varepsilon t|^{-1}}\nabla^{p+2}\phi\|_{L^2(\widehat{Q})} &\leq CK^p \max\{p, \varepsilon|t|\}^{p+2}\frac{1}{1+|t|}.
\end{aligned}$$

Proof: The L^2 and H^1-bounds follow from Lemma 3.6. For the bounds on higher derivatives, we note that the constant C_f of (3.8) satisfies $C_f = O(|t|^{-2}) = O((1+|t|)^{-1})$ and that $C_H = O(1)$. Applying now Proposition 3.4 to (3.20) the stated bounds follow, since the relative diffusivity $\mathcal{E}$ appearing in the statement of Proposition 3.4 can be replaced with $1/(\varepsilon|t|)$ in view of (3.21) and Lemma 3.2. □

4 Two-Scale Regularity of u^ε

We use the regularity for the kernel function ϕ of the preceding section to develop a two-scale regularity theory for the solutions of (2.2), (2.3). The solution u^ε can be expressed in terms of the unfolded solution U^ε by

$$u^\varepsilon(x) = U^\varepsilon(x,y)|_{y=x/\varepsilon} \tag{4.1}$$

where the two-scale function U^ε is given by

$$U^\varepsilon(x,y) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{f}(t)e^{\mathbf{i}t\cdot x}\phi(y,\varepsilon,t)\,dt; \tag{4.2}$$

this representation follows from (2.5) with $\psi(x,\varepsilon,t) = e^{\mathbf{i}t\cdot x}\phi(y,\varepsilon,t)|_{y=x/\varepsilon}$. Our main result in this section is the regularity assertion Theorem 4.3 for this two-scale function U^ε. We show that U^ε depends smoothly on ε and, viewing U^ε as a mapping

$$\begin{aligned}
\mathbb{R}^2 &\to H^1_{\mathrm{per}}(\widehat{Q}) \\
x &\mapsto U^\varepsilon(x,\cdot)
\end{aligned}$$

we even have $U^\varepsilon \in H^k(\mathbb{R}^2, H^1_{\mathrm{per}}(\widehat{Q}) \cap H^{p+2,2}_{\beta,1}(\widehat{Q}))$, where $k \in \mathbb{N}$ depends on the right-hand side f only.

4.1 Two-scale shift theorem

In order to characterize the function U^ε of (4.2), we write it as the sum $U_1^\varepsilon + U_2^\varepsilon$ where

$$U_1^\varepsilon(x,y) := \frac{1}{2\pi}\int_{|t|<1/\varepsilon} \hat{f}(t)e^{\mathbf{i}t\cdot x}\phi(y,\varepsilon,t)\,dt, \tag{4.3}$$

$$U_2^\varepsilon(x,y) := \frac{1}{2\pi}\int_{|t|>1/\varepsilon} \hat{f}(t)e^{\mathbf{i}t\cdot x}\phi(y,\varepsilon,t)\,dt. \tag{4.4}$$

Theorems 4.1, 4.2 provide regularity results for these two functions. For U_1^ε we have the following regularity assertion:

Theorem 4.1 *Let $k \in \mathbb{N}$ and let $\beta \in [0,1)^J$ be given by Theorem 3.7. Then there exist $C > 0$ (depending only on k and the constants C, $K > 0$ appearing in Theorem 3.7) and $K > 0$ such that the function U_1^ε of (4.3) satisfies for $f \in H^k(\mathbb{R}^2)$*

$$\begin{aligned}
\|U_1^\varepsilon\|_{H^k(\mathbb{R}^2, L^2_{\mathrm{per}}(\widehat{Q}))} &\le C\varepsilon^{-l}\|f\|_{H^{k-l}(\mathbb{R}^2)}, \quad l = 0,\dots,k-1,\\
\|U_1^\varepsilon\|_{H^k(\mathbb{R}^2, H^1_{\mathrm{per}}(\widehat{Q})/\mathbb{R})} &\le C\varepsilon^{1-l}\|f\|_{H^{k-l}(\mathbb{R}^2)}, \quad l = 0,\dots,k,\\
\|U_1^\varepsilon\|_{H^k(\mathbb{R}^2, H^{p+2,2}_{\beta,1}(\widehat{Q})\mathbb{R})} &\le CK^p p!\varepsilon^{1-l}\|f\|_{H^{k-l}(\mathbb{R}^2)}, \quad l = 0,\dots,k, \quad p \in \mathbb{N}_0,
\end{aligned}$$

where the spaces $H^1_{\mathrm{per}}(\widehat{Q})/\mathbb{R}$ and $H^{p+2,2}_{\beta,1}(\widehat{Q})/\mathbb{R}$ are equipped in the standard way with the norms

$$\begin{aligned}
\|u\|_{H^1_{\mathrm{per}}(\widehat{Q})/\mathbb{R}} &= \|\nabla u\|_{L^2_{\mathrm{per}}(\widehat{Q})}\\
\|u\|_{H^{p+2,2}_{\beta,1}(\widehat{Q})/\mathbb{R}} &= \left(\|\nabla u\|^2_{L^2_{\mathrm{per}}(\widehat{Q})} + \sum_{k=0}^{p}\|w_{k,\beta,1}\nabla^{k+2}u\|^2_{L^2(\widehat{Q})}\right)^{1/2}
\end{aligned}$$

Proof: We show only the third bound as the first two estimates are proved similarly. Using Parseval's identity and Lemma 3.8, we estimate for $0 \le s \le p$

$$\begin{aligned}
\|\nabla_x^k w_{s,\beta,1}(y)\nabla_y^{s+2}U_1^\varepsilon\|^2_{L^2(\mathbb{R}^2, L^2_{\mathrm{per}}(\widehat{Q}))} &= \int_{|t|<1/\varepsilon} |t|^{2k}|\hat{f}(t)|^2\|w_{s,\beta,1}\nabla_y^{s+2}\phi\|^2_{L^2_{\mathrm{per}}(\widehat{Q})}\,dt\\
&\le C(K^s s!)^2\int_{|t|<1/\varepsilon}|t|^{2k}|\hat{f}(t)|^2\varepsilon^2\,dt\\
&\le C(K^s s!)^2\varepsilon^{2(1-l)}|f|^2_{H^{k-l}(\mathbb{R}^2)}, \qquad l = 0,\dots,k,
\end{aligned}$$

where we used $\varepsilon \le |t|^{-1}$. Summing over $s \in \{0,\dots,p\}$ then gives the desired bound. □

The next theorem provides bounds on U_2^ε:

Theorem 4.2 *Let $k \in \mathbb{N}$. Let $\beta \in [0,1)^J$ be given by Theorem 3.7. Then there exists $C > 0$ (depending only on k and the constants C, $K > 0$ appearing in Theorem 3.7) such that the function U_2^ε of (4.4) satisfies*

$$\begin{aligned}
\|U_2^\varepsilon\|_{H^k(\mathbb{R}^2, L^2_{\mathrm{per}}(\widehat{Q}))} &\le Co(1)\|f\|_{H^{k-1}(\mathbb{R}^2)},\\
\|U_2^\varepsilon\|_{H^k(\mathbb{R}^2, H^1_{\mathrm{per}}(\widehat{Q}))} &\le C\varepsilon o(1)\|f\|_{H^k(\mathbb{R}^2)},\\
\|U_2^\varepsilon\|_{H^k(\mathbb{R}^2, H^{p+2,2}_{\beta,1}(\widehat{Q}))} &\le CK^p o(1)\left[p!\|f\|_{H^{k-1}(\mathbb{R}^2)} + \varepsilon^{p+2}|f|_{H^{k+p+1}(\mathbb{R}^2)}\right] \quad \forall p \in \mathbb{N}_0.
\end{aligned}$$

Here the constant C depends on k, and K depends on δ, a, a_0. The Landau symbol $o(1)$ denotes a function of ε such that

$$\begin{aligned}
o(1) &\le 1,\\
\lim_{\varepsilon\to 0} o(1) &= 0.
\end{aligned}$$

Proof: We will mostly focus on the last estimate, but mention that variants of the first two estimates are possible. In particular, we can also estimate

$$\begin{aligned}
\|U_2^\varepsilon\|^2_{H^k(\mathbb{R}^2, L^2_{\mathrm{per}}(\widehat{Q}))} &\le C\int_{|t|>1/\varepsilon}|t|^{2k}|\hat{f}|^2\frac{1}{1+|t|^2}\,dt \le C\varepsilon^2 C\int_{|t|>1/\varepsilon}|t|^{2k}|\hat{f}|^2\,dt \le C\varepsilon^2|f|^2_{H^k(\mathbb{R}^2)}\\
\|U_2^\varepsilon\|^2_{H^k(\mathbb{R}^2, H^1_{\mathrm{per}}(\widehat{Q})/\mathbb{R})} &\le C\varepsilon^2\int_{|t|>1/\varepsilon}|t|^{2k}|\hat{f}|^2\,dt \le C\varepsilon^2\int_{|t|>1/\varepsilon}|t|^{2(k+p+1)}|t|^{-2(p+1)}|\hat{f}|^2\,dt\\
&\le C\varepsilon^{2(p+2)}\int_{|t|>1/\varepsilon}|t|^{2(k+p+1)}|\hat{f}|^2\,dt \le C\varepsilon^{2(p+2)}|f|^2_{H^{k+p+1}(\mathbb{R}^2)}
\end{aligned}$$

We will again only show the last estimate as the first two are proved similarly. With Parseval's identity, monotonicity properties of the weight function $\mathcal{E} \mapsto w_{p,\beta,\mathcal{E}}$ (cf. (3.6)) and Lemma 3.9 Note added in proof

$$\begin{aligned}
\|\nabla_x^k w_{s,\beta,1} \nabla_y^{s+2} U_2^\varepsilon\|^2_{L^2(\mathbb{R}^2, L^2_{\mathrm{per}}(\widehat{Q}))} &= \int_{|t|>1/\varepsilon} |t|^{2k} |\hat{f}(t)|^2 \|w_{s,\beta,1} \nabla_y^{s+2} \phi\|^2_{L^2_{\mathrm{per}}(\widehat{Q})} \, dt \\
&\leq \int_{|t|>1/\varepsilon} |t|^{2k} |\hat{f}(t)|^2 \|w_{s,\beta,\mathcal{E}} \nabla_y^{s+2} \phi\|^2_{L^2_{\mathrm{per}}(\widehat{Q})} \, dt \\
&\leq CK^{2s} \int_{|t|>1/\varepsilon} |t|^{2k} |\hat{f}(t)|^2 \left(\max\{s!, (\varepsilon|t|)^{s+2}\right)^2 \frac{1}{|t|^2} \, dt.
\end{aligned}$$

Since we can bound

$$\left(\max\{s!, (\varepsilon|t|)^{s+2}\}\right)^2 \leq (s!)^2 + (\varepsilon|t|)^{2(s+2)},$$

we get

$$\begin{aligned}
&\|\nabla_x^k w_{s,\beta,1} \nabla_y^{s+2} U_2^\varepsilon\|^2_{L^2(\mathbb{R}^2, L^2_{\mathrm{per}}(\widehat{Q}))} \leq \\
&\qquad CK^{2s} \left[(s!)^2 \int_{|t|>1/\varepsilon} |t|^{2k-2} |\hat{f}(t)|^2 \, dt + \varepsilon^{2s+4} \int_{|t|>1/\varepsilon} |t|^{2k+2s+2} |\hat{f}(t)|^2 \, dt \right].
\end{aligned}$$

The first integral is bounded by

$$\int_{|t|>1/\varepsilon} |t|^{2k-2} |\hat{f}(t)|^2 \, dt = \int_{|t|>1/\varepsilon} |t|^{2(k-1+m)} |t|^{-2m} \, |\hat{f}(t)|^2 \, dt \leq \varepsilon^{2m} o(1) |f|^2_{H^{k-1+m}(\mathbb{R}^2)}.$$

For the second integral, we use $|t| \geq 1/\varepsilon \geq 1$ and bound for $0 \leq s \leq p$

$$\begin{aligned}
\varepsilon^{2s+4} \int_{|t|>1/\varepsilon} |t|^{2k+2s+2} |\hat{f}(t)|^2 \, dt &= \varepsilon^{2s+4} \int_{|t|>1/\varepsilon} |t|^{2k+2p+2} |\hat{f}(t)|^2 \, |t|^{2(s-p)} \, dt \\
&\leq \varepsilon^{2s+4+2(p-s)} \int_{|t|>1/\varepsilon} |t|^{2k+2p+2} |\hat{f}(t)|^2 \, dt \\
&\leq C\varepsilon^{2(p+2)} o(1) |f|^2_{H^{k+p+1}(\mathbb{R}^2)},
\end{aligned}$$

where the factor $o(1)$ expresses the fact that

$$\lim_{\varepsilon \to 0} \frac{1}{\|f\|^2_{H^{k+p+1}(\mathbb{R}^2)}} \int_{|t|\geq 1/\varepsilon} |t|^{2(k+p+1)} |\hat{f}(t)|^2 \, dt = 0.$$

Thus,

$$\begin{aligned}
&\|U_2^\varepsilon\|^2_{H^k(\mathbb{R}^2, H^{p+2,2}_{\beta,1}(\widehat{Q}))} = \|U_2^\varepsilon\|^2_{H^k(\mathbb{R}^2, H^1_{\mathrm{per}}(\widehat{Q}))} + \sum_{s=0}^{p} \|\nabla_x^k w_{s,\beta,1} \nabla_y^{s+2} U_2^\varepsilon\|^2_{L^2(\mathbb{R}^2, L^2_{\mathrm{per}}(\widehat{Q}))} \\
&\leq CK^{2p} o(1) \left[\|f\|^2_{H^{k-1}(\mathbb{R}^2)} + \sum_{s=0}^{p} (s!)^2 \|f\|^2_{H^{k-1}(\mathbb{R}^2)} + (p+1)\varepsilon^{2(p+2)} |f|^2_{H^{k+p+1}(\mathbb{R}^2)} \right] \\
&\leq CK^{2p} o(1) \left[(p!)^2 \|f\|^2_{H^{k-1}(\mathbb{R}^2)} + \varepsilon^{2(p+2)} |f|^2_{H^{k+p+1}(\mathbb{R}^2)} \right].
\end{aligned}$$

□

The above results concerning the terms U_1^ε, U_2^ε can be combined into the following statement for U^ε:

Theorem 4.3 *Let $k \in \mathbb{N}$ and $\beta \in [0,1)^J$ be given by Theorem 3.7. Then there exists $C > 0$ (depending only on k and the constants C, $K > 0$ appearing in Theorem 3.7) such that for $f \in H^k(\mathbb{R}^2)$ the function U^ε of (4.2) satisfies*

$$\|U^\varepsilon\|_{H^k(\mathbb{R}^2, H^{p+2,2}_{\beta,1}(\hat{Q}))} \leq CK^p \left[p!\|f\|_{H^{k-1}(\mathbb{R}^2)} + \varepsilon^{p+2}|f|_{H^{k+p+1}(\mathbb{R}^2)}\right],$$
$$\|\varepsilon^{-1}\nabla_y U^\varepsilon\|_{H^k(\mathbb{R}^2, H^{p+1,1}_{\beta,1}(\hat{Q}))} \leq CK^p \left[p!\|f\|_{H^k(\mathbb{R}^2)} + \varepsilon^{p+1}|f|_{H^{k+p+1}(\mathbb{R}^2)}\right].$$

Proof: Follows from Lemmas 3.8, 3.9 and the preceding two theorems. For the first estimate, an additional term $\varepsilon\|f\|_{H^k(\mathbb{R}^2)}$ would appear from a direct application of Theorem 4.2. However, the following observations allow for its removal:

$$\begin{aligned}
\varepsilon^2 \int_{|t|<1/\varepsilon} |t|^{2k}|\hat{f}|^2\,dt &\leq \int_{|t|<1/\varepsilon} |t|^{2k}|t|^{-2}|\hat{f}|^2\,dt \leq |f|^2_{H^{k-1}(\mathbb{R}^2)},\\
\varepsilon^2 \int_{|t|>1/\varepsilon} |t|^{2k}|\hat{f}|^2\,dt &= \varepsilon^2 \int_{|t|>1/\varepsilon} |t|^{2(k+p+1)}|t|^{-2(p+1)}|\hat{f}|^2\,dt\\
&\leq \varepsilon^{2+2(p+1)} \int_{|t|>1/\varepsilon} |t|^{2(k+1+p)}|\hat{f}|^2\,dt \leq \varepsilon^{2+2(p+1)}|f|^2_{H^{k+p+1}(\mathbb{R}^2)}
\end{aligned}$$

For the second estimate, the direct application of Theorem 4.2 would lead to a term $\varepsilon^{-1}\|f\|_{H^{k-1}(\mathbb{R}^2)}$. Inspection of the proof of Theorem 4.2, however, shows that the term $\|f\|_{H^{k-1}(\mathbb{R}^2)}$ can be sharpened to

$$\int_{|t|>1/\varepsilon} |t|^{2(k-1)}|\hat{f}|^2\,dt = \int_{|t|>1/\varepsilon} |t|^{2k}|t|^{-2}|\hat{f}|^2\,dt \leq \varepsilon^2 \int_{|t|>1/\varepsilon} |t|^{2k}|\hat{f}|^2\,dt,$$

which then leads to the desired bound. $\square$

Note added in proof

4.2 Remarks on Dirichlet and mixed boundary conditions

So far we have considered the case of Neumann boundary conditions (2.3). An analogous theory can be developed for other kinds of boundary conditions. We discuss in the present section the case where Dirichlet boundary conditions are imposed on parts of the internal boundary. We show that in this case, the estimates on ϕ and likewise the two-scale regularity results of Section 4.1 can be improved.
The internal boundary of $\widehat{Q}$ is split into two parts $\widehat{\Gamma}'_N$ and $\widehat{\Gamma}_D$, and we consider the function ϕ that solves

$$\begin{aligned}
e^{-\mathbf{i}\varepsilon t\cdot y}L^\varepsilon(y, \varepsilon^{-1}\partial_y)e^{\mathbf{i}\varepsilon t\cdot y}\phi &= 1 \quad \text{on } \widehat{Q}, && (4.5a)\\
e^{-\mathbf{i}\varepsilon t\cdot y}n\cdot(a(y)\nabla(e^{\mathbf{i}\varepsilon t\cdot y}\phi)) &= 0 \quad \text{on } \widehat{\Gamma}'_N, && (4.5b)\\
\phi &= 0 \quad \text{on } \widehat{\Gamma}_D. && (4.5c)
\end{aligned}$$

4.2.1 Estimates for the unit-cell Dirichlet problem

Lemma 4.4 *Let $|\Gamma_D| > 0$. Then there exists $\delta > 0$ (depending only on a, a_0 and the Poincaré constant of $\widehat{Q}$) such that the solution ϕ of (4.5) satisfies*

$$\|\phi\|_{H^1(\widehat{Q})} \leq C\varepsilon^2,$$

provided that

$$\varepsilon|t| < \delta. \tag{4.6}$$

Proof: We multiply the weak form of (4.5a) by $\overline{\phi}$ and integrate by parts:

$$\int_{\widehat{Q}} (a(y)\nabla\phi)\cdot\overline{\nabla\phi}\,dy = \int_{\widehat{Q}} \mathbf{i}\varepsilon(a(y)t)\cdot\nabla\phi\overline{\phi} + \mathbf{i}\varepsilon(a(y)\nabla\phi)\cdot t\overline{\phi}$$
$$-\int_{\widehat{Q}} (\varepsilon^2 t^\top a(y)t + \varepsilon^2 a_0(y) - \mathbf{i}\varepsilon(\nabla\cdot a)\cdot t)\phi\overline{\phi}\,dy + \int_{\widehat{Q}} \varepsilon^2\overline{\phi}\,dy.$$

Thus, for some $c_1, c_2 > 0$ we obtain

$$c_1|\phi|^2_{H^1(\widehat{Q})} \le 2c_2\varepsilon|t||\phi|_{H^1(\widehat{Q})}\|\phi\|_{L^2(\widehat{Q})}$$
$$+ (c_2\varepsilon^2|t|^2 + c_2\varepsilon^2 + \varepsilon|t|\|a\|_{W^{1,\infty}(\widehat{Q})})\|\phi\|^2_{L^2(\widehat{Q})} + \varepsilon^2\|\phi\|_{L^2(\widehat{Q})}.$$

If C_{P} denotes the Poincaré constant of the domain $\widehat{Q}$ (with Dirichlet boundary conditions prescribed on $\widehat{\Gamma}_D$), there exists $\delta = \delta(c_1, c_2, \|a\|_{W^{1,\infty}(\widehat{Q})}) > 0$ such that for all $\varepsilon|t| < \delta$

$$|\phi|^2_{H^1(\widehat{Q})} \le C\varepsilon^2\|\phi\|_{L^2(\widehat{Q})} \le CC_{\mathrm{P}}\varepsilon^2|\phi|_{H^1(\widehat{Q})}.$$

This concludes the proof of the lemma. □

Analytic regularity results analogous to those of Theorem 3.7 are formulated in the following Theorem 4.5 for the solution of (4.5). The main difference is that the use of Lemma 4.4 can replace appeals to Lemma 3.6 thus allowing for improvements in the case $\varepsilon|t|$ small:

Theorem 4.5 *Let $|\widehat{\Gamma}_D| > 0$ and let $\delta > 0$ be given by Lemma 4.4. Then there exist C, $K > 0$ and $\beta \in [0,1)^J$ such that the solution ϕ to (4.5) satisfies the following:*

1. *If ε, t satisfy $\varepsilon|t| \le \delta$, then*

$$\begin{aligned} \|\phi\|_{H^1(\widehat{Q})} &\le C\varepsilon^2, \\ \|w_{p,\beta,1}\nabla^{p+2}\phi\|_{L^2(\widehat{Q})} &\le CK^p p!\varepsilon^2 \qquad \forall p\in\mathbb{N}_0. \end{aligned}$$

2. *If ε, t satisfy $\varepsilon|t| > \delta$, then*

$$\begin{aligned} \|\phi\|_{L^2(\widehat{Q})} &\le C\frac{1}{1+|t|}, \\ \|\nabla\phi\|_{L^2(\widehat{Q})} &\le C\varepsilon, \\ \|w_{p,\beta,|\varepsilon t|^{-1}}\nabla^{p+2}\phi\|_{L^2(\widehat{Q})} &\le CK^p\max\{p,\varepsilon|t|\}^{p+2}\frac{1}{1+|t|}. \end{aligned}$$

Proof: The H^1-estimates follow from Lemmas 3.6, 4.4. The high order estimates follow from Proposition 3.4. □

4.2.2 Two-scale regularity for Dirichlet problems

The improved estimates of Theorem 4.5 for the unit-cell problem with Dirichlet or mixed boundary conditions allow for better estimates for the two-scale regularity developed in Section 4.1. As in the case of pure Neumann boundary conditions, the two-scale function U^ε can be split into U_1^ε, U_2^ε given by

$$\begin{aligned} U_1^\varepsilon(x,y) &:= \frac{1}{2\pi}\int_{|t|<\delta/\varepsilon} \hat{f}(t)e^{\mathbf{i}t\cdot x}\phi(y,\varepsilon,t)\,dt, \\ U_2^\varepsilon(x,y) &:= \frac{1}{2\pi}\int_{|t|>\delta/\varepsilon} \hat{f}(t)e^{\mathbf{i}t\cdot x}\phi(y,\varepsilon,t)\,dt. \end{aligned}$$

where $\delta > 0$ is given by Lemma 4.4 and ϕ is the solution to (4.5).
The function U_2^ε satisfies the same estimates as its counterpart for the case of Neumann boundary conditions, that is, U_2^ε satisfies the bounds given in Theorem 4.2. Reasoning as in the proof of Theorem 4.1, one can show that U_1^ε satisfies

$$\begin{aligned}
\|U_1^\varepsilon\|_{H^k(\mathbb{R}^2,L^2_{per}(\widehat{Q}))} &\leq C\varepsilon^{2-l}\|f\|_{H^{k-l}(\mathbb{R}^2)}, \quad l = 0,\dots,k, \\
\|U_1^\varepsilon\|_{H^k(\mathbb{R}^2,H^1_{per}(\widehat{Q}))} &\leq C\varepsilon^{2-l}\|f\|_{H^{k-l}(\mathbb{R}^2)}, \quad l = 0,\dots,k, \\
\|U_1^\varepsilon\|_{H^k(\mathbb{R}^2,H^{p+2,2}_{\beta,1}(\widehat{Q}))} &\leq CK^p p!\varepsilon^{2-l}\|f\|_{H^{k-l}(\mathbb{R}^2)}, \qquad l = 0,\dots,k.
\end{aligned}$$

In particular, we obtain

Theorem 4.6 *Let $k \in \mathbb{N}$, $k \geq 2$ and $\beta \in [0,1)^J$ be given by Theorem 4.5. Let U^ε be given by (4.2). Then there exist $C > 0$ and $K > 0$ (depending on k and the constants C, K of Theorem 4.5) such that*

$$\|U^\varepsilon\|_{H^k(\mathbb{R}^2,H^{p+2,2}_{\beta,1}(\widehat{Q}))} \leq CK^p\left[p!\|f\|_{H^{k-2}(\mathbb{R}^2)} + o(1)p!\|f\|_{H^{k-1}(\mathbb{R}^2)} + \varepsilon^{p+2}|f|_{H^{k+p+1}(\mathbb{R}^2)}\right].$$

$$\|\varepsilon^{-1}\nabla_y U^\varepsilon\|_{H^k(\mathbb{R}^2,H^{p+1,1}_{\beta,1}(\widehat{Q}))} \leq CK^p\left[p!\|f\|_{H^{k-1}(\mathbb{R}^2)} + o(1)p!\|f\|_{H^{k}(\mathbb{R}^2)} + \varepsilon^{p+1}|f|_{H^{k+p+1}(\mathbb{R}^2)}\right].$$

For the case of infinite, periodic lattices Theorems 4.3 and 4.6 characterize the regularity of the unfolded solution U^ε. The regularity of the solution u^ε of (2.2), (2.3) can be inferred from that of U^ε in view of the relationship (4.1).

5 Approximation of functions with two-scale regularity

In the previous section we saw that uniform regularity of u^ε in dependence on ε could be properly expressed in terms of the two-scale Sobolev spaces $H^k(\mathbb{R}^2, H^{s,2}_{\beta,1}(\widehat{Q}))$. The purpose of the present section is to show that regularity assertions of this type can be used to guide the construction of finite dimensional spaces from which rapidly oscillating functions can be approximated robustly, i.e., the rate of convergence is independent of ε. We present such a space V_N in (5.4) below and provide in Theorem 5.2 an interpolation operator for the approximation of functions with two-scale regularity. Such approximation results in combination with quasi-optimality of finite element methods, or, more generally, projection methods, lead to robust finite element convergence based on the two-scale finite element space (5.4).
We assume that the domain $\Omega \subset \mathbb{R}^2$ is axiparallel and, for simplicity of exposition, that the unit-cell $\widehat{Q}$ is polygonal.

5.1 Two-scale finite element spaces

The two-scale finite element spaces that we employ are based on tensor products of finite element spaces that are defined on Ω (the "coarse" scale) and the of ε-periodic finite element space designed to capture the effects on the fine scale.

5.1.1 Macro finite element spaces

We assume that the domain Ω is axiparallel, and we take $\mathcal{T}_H$ as the uniform triangulation of Ω of squares of side length H. We take as macro FE space in Ω the standard FE space $S^{p,1}(\Omega, \mathcal{T}_H)$ of continuous, piecewise polynomials of degree p on the mesh $\mathcal{T}_H$. Let $I_{p,\mathcal{T}_H}$ be a projector $I_{p,\mathcal{T}_H} : H^{k+1}(\Omega) \to S^{p,1}(\Omega, \mathcal{T}_H)$ with the property

$$\sum_{\alpha_j \in \{0,1\}} H^{|\alpha|} \|D^\alpha (v - I_{p,\mathcal{T}_H} v)\|_{L^2(K)} \leq C(p,k) H^{k+1} |v|_{H^{k+1}(K)} \quad \forall K \in \mathcal{T}_H. \tag{5.1}$$

Such operators can be constructed as elementwise tensor product interpolants, [11].

5.1.2 Fine scale finite element spaces

For the approximation of function of $H^k(\Omega, H^{s,2}_{\beta,1}(\widehat{Q}))$, spaces $S^{\mu,1}_{\mathrm{per}}(\widehat{Q}, \widehat{\mathcal{T}}^{\mathrm{rad}}_h)$ and projectors $I_{\mu,\widehat{\mathcal{T}}^{\mathrm{rad}}_h}$ are required that are suitable for the approximation of functions of $H^{s,2}_{\beta,1}(\widehat{Q}) \cap H^1_{\mathrm{per}}(\widehat{Q})$. Here, $\mu \in \mathbb{N}$ stands for the 'micro' polynomial degree, and $S^{\mu,1}_{\mathrm{per}}(\widehat{Q}, \widehat{\mathcal{T}}^{\mathrm{rad}}_h)$ is the space of periodic continuous, piecewise polynomials of degree μ on the (periodic) radical mesh $\widehat{\mathcal{T}}^{\mathrm{rad}}_h(\widehat{Q})$.
Radical meshes $\mathcal{T}^{\mathrm{rad}}_h(\widehat{Q})$ are designed for the approximation of functions that are smooth up to the boundary with the exception of a few points, where they have singularities; such functions are described with the aid of the weighted Sobolev spaces $H^{p+l,l}_{\beta,1}$.
A radical mesh $\widehat{\mathcal{T}}^{\mathrm{rad}}_h(\widehat{Q})$ consists of shape regular triangles T, whose diameters h_T satisfy the following conditions for a fixed vector $\alpha \in [0,1)^J$ and $h > 0$:

1. if $A_j \in \overline{T}$ for some $j \in \{1, \ldots, J\}$, then $h_T \sim h^{1/(1-\alpha_j)}$;
2. if $A_j \notin \overline{T}$ for all $j \in \{1, \ldots, J\}$, then

   $$c_1 h \inf_{x \in T} w_{0,\alpha,1}(x) \leq h_T \leq c_2 h \sup_{x \in T} w_{0,\alpha,1}(x),$$

   for some fixed constants c_1, c_2.

The exponents α_i, $i = 1, \ldots, J$, are chosen in dependence on the singularity exponents β_i, $i = 1, \ldots, J$, as well as the polynomial degree μ. For the case $\mu = 1$, one can take $\alpha = \beta$, and it is shown in [3] that

$$\|v - I_{1,\widehat{\mathcal{T}}^{\mathrm{rad}}_h} v\|_{L^2(\widehat{Q})} + h|v - I_{1,\widehat{\mathcal{T}}^{\mathrm{rad}}_h} v|_{H^1(\widehat{Q})} \leq C h^2 \|v\|_{H^{2,2}_{\beta,1}(\widehat{Q})}, \tag{5.2}$$

where $I_{1,\widehat{\mathcal{T}}^{\mathrm{rad}}_h}$ denotes the piecewise linear interpolant with respect to the 'micro' FE space $S^1(\widehat{Q}, \widehat{\mathcal{T}}^{\mathrm{rad}}_h)$. For $\mu \geq 2$, one can construct analogously operators $I_{\mu,\widehat{\mathcal{T}}^{\mathrm{rad}}_h}$ with

$$\|v - I_{\mu,\widehat{\mathcal{T}}^{\mathrm{rad}}_h} v\|_{L^2(\widehat{Q})} + h|v - I_{\mu,\widehat{\mathcal{T}}^{\mathrm{rad}}_h} v|_{H^1(\widehat{Q})} \leq C h^{\min(\mu,s)+1} \|v\|_{H^{s+1,2}_{\beta,1}(\widehat{Q})}. \tag{5.3}$$

For complexity considerations involving graded meshes and the operator $I_{\mu,\widehat{\mathcal{T}}^{\mathrm{rad}}_h}$, it is important to note that the number of elements in the mesh $\mathcal{T}^{\mathrm{rad}}_h$ is proportional to h^{-2}, i.e.,

$$h^{-2} \sim |\widehat{\mathcal{T}}^{\mathrm{rad}}_h|.$$

5.1.3 Two-scale finite element spaces

We take as the two-scale finite element space the space

$$V_N := \mathcal{R}^\varepsilon \left(S^p(\Omega, \mathcal{T}_H) \otimes S^\mu_{\mathrm{per}}(\widehat{Q}, \mathcal{T}^{\mathrm{rad}}_h) \right), \tag{5.4}$$

where the restriction operator $\mathcal{R}^\varepsilon$ is given by $(\mathcal{R}^\varepsilon U)(x) = U(x,y)|_{y=x/\varepsilon}$. The elements of V_N have the form

$$u^\varepsilon_{\mathrm{FE}}(x) = \sum_{i,j} u_{ij} N_i(x) M_j(x/\varepsilon),$$

for some $u_{ij} \in \mathbb{R}$, where the functions N_i, M_j are basis functions of the spaces $S^p(\Omega, \mathcal{T}_H)$, $S^\mu_{\mathrm{per}}(\widehat{Q}, \mathcal{T}^{\mathrm{rad}}_h)$, respectively.

5.2 Approximation from two-scale finite element spaces

We have seen that the solution u^ε may be interpreted as $u^\varepsilon(x) = U^\varepsilon(x, x/\varepsilon)$, where U^ε is defined on $\Omega \times \widehat{Q}$. This suggests to use hp-interpolants on Ω and $\widehat{Q}$ to approximate U^ε in $\Omega \times \widehat{Q}$ and take traces.

The interpolation error e^ε_I has the form

$$e^\varepsilon_I(x) := E^\varepsilon_I(x, x/\varepsilon), \quad E^\varepsilon_I(x,y) = U^\varepsilon(x,y) - U^\varepsilon_I(x,y), \quad x \in \Omega, \ y \in \Omega^\infty_\varepsilon \tag{5.5}$$

in which the two-scale interpolant U^ε_I is given by

$$U^\varepsilon_I(x,y) = I^x_{p,\mathcal{T}_H} \otimes I^y_{\mu,\widehat{\mathcal{T}}^{\mathrm{rad}}_h} U^\varepsilon(x,y).$$

Here, $I^x_{p,\mathcal{T}_H}$ denotes the piecewise polynomial of degree p interpolant in Ω as in (5.1) and $I^y_{\mu,\widehat{\mathcal{T}}^{\mathrm{rad}}_h}$ the $S^\mu_{\mathrm{per}}(\widehat{Q}, \widehat{\mathcal{T}}^{\mathrm{rad}}_h)$ interpolant in $H^1_{\mathrm{per}}(\widehat{Q})$.

For the convergence analysis of the two-scale FEM we will need the following result [11] on traces in Sobolev spaces of mixed order. Let $D \subset \mathbb{R}^2$ be a bounded domain. For a function $f(\cdot,\cdot) : D \times D \to \mathbb{R}$, we denote by $(\mathcal{R}f)(x) = f(x,y)|_{y=x} : D \to \mathbb{R}$ its restriction to the diagonal $\{(x,y) \in D \times D \mid x = y\}$. Furthermore, for multiindices α, $\beta \in \mathbb{N}^n_0$ we denote by $\mathcal{H}^{\alpha,\beta}(D \times D)$ the following Sobolev spaces of mixed order

$$\mathcal{H}^{\alpha,\beta}(D \times D) := \{ f \in L^2(D \times D) \ : \ D^\gamma_x D^\delta_y f \in L^2(D \times D) \quad \forall\, \gamma \le \alpha, \ \delta \le \beta \},$$

where the inequalities $\gamma \le \alpha$, $\delta \le \beta$ have to be understood componentwise.

Lemma 5.1 ([11]) *Write $\mathbf{1} := (1,1)$. Then for any fixed pair of multiindices $\alpha, \beta \in \mathbb{N}^2_0$ with $\alpha + \beta = \mathbf{1}$ the restriction operator $\mathcal{R} : \mathcal{H}^{\alpha,\beta}(D \times D) \to L^2(D)$ is continuous, i.e., there exists a constant $C > 0$ such that*

$$\|\mathcal{R}f\|_{L^2(D)} \le C \|f\|_{\mathcal{H}^{\alpha,\beta}(D\times D)}, \quad \forall\, f \in \mathcal{H}^{\alpha,\beta}(D \times D).$$

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^2$ be an axiparallel rectangle. Assume that u^ε has the following two-scale regularity in $\Omega_\varepsilon := \Omega \cap \Omega^\infty_\varepsilon$: $u^\varepsilon(x) = U^\varepsilon(x, x/\varepsilon)$, with U^ε being 1-periodic in $y \in \widehat{Q}$ and $U^\varepsilon \in H^{k+1}(\Omega, H^{s+1,2}_{\beta,1}(\widehat{Q}))$. For $p, \mu, k, s \ge 1$ and $H/\varepsilon \in \mathbb{N}$ we then have for the interpolation error e^ε_I of (5.5)*

$$\begin{aligned} \|e^\varepsilon_I\|_{H^1(\Omega_\varepsilon)} \le{} & C(p,k) H^{\min(p,k)} (\|\varepsilon^{-1} \nabla_y U^\varepsilon\|_{H^k(\Omega;\, L^2_{\mathrm{per}}(\widehat{Q}))} + \|U^\varepsilon\|_{H^{k+1}(\Omega;\, L^2_{\mathrm{per}}(\widehat{Q}))}) \\ & + C(\mu,s) h^{\min(\mu,s)} (\|\varepsilon^{-1} \nabla_y U^\varepsilon\|_{H^2(\Omega;\, H^{s,1}_{\beta,1}(\widehat{Q}))} + \|U^\varepsilon\|_{H^2(\Omega; H^{s+1,2}_{\beta,1}(\widehat{Q}))}). \end{aligned} \tag{5.6}$$

Proof: We split the interpolation error as

$$E_I^\varepsilon((x),y) := U^\varepsilon(x,y) - I^x_{p,\mathcal{T}_H} U^\varepsilon(x,y) + (I^x_{p,\mathcal{T}_H} U^\varepsilon(x,y) - I^x_{p,\mathcal{T}_H} \otimes I^y_{\mu,\widehat{\mathcal{T}}_h^{\mathrm{rad}}}) U^\varepsilon(x,y).$$

We estimate first the L^2 norm of the error on K and apply the trace result in Lemma 5.1 in [11] to move on full two scale interpolation error estimates

$$\begin{aligned}
\int_{K\cap\Omega_\varepsilon} |e_I^\varepsilon(x)|^2\,dx &= \varepsilon^2 \sum_{m\in\mathbb{Z}^2:\varepsilon(\widehat{Q}+m)\subset K} \int_{\widehat{Q}} \left| E_I^\varepsilon(\varepsilon(z+m),y)\Big|_{z=y} \right|^2 dz \\
&\leq C\varepsilon^2 \sum_{m\in\mathbb{Z}^2:\varepsilon(\widehat{Q}+m)\subset K} \sum_{0\leq\alpha_j\leq 1} \varepsilon^{2|\alpha|} \int_{\widehat{Q}\times\widehat{Q}} |(D_x^\alpha E_I^\varepsilon)(\varepsilon(z+m),y)|^2\,dzdy \\
&= C \sum_{0\leq\alpha_j\leq 1} \varepsilon^{2|\alpha|} \int_{(K\cap\Omega_\varepsilon)\times\widehat{Q}} |D_x^\alpha E_I^\varepsilon(x,y)|^2\,dxdy \\
&\leq C\left(\mathrm{I}_K + \mathrm{II}_K\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}_K &= \sum_{0\leq\alpha_j\leq 1} \varepsilon^{2|\alpha|} \int_{K\times\widehat{Q}} \left| D_x^\alpha \left(U^\varepsilon(x,y) - I^x_{p,\mathcal{T}_H} U^\varepsilon(x,y) \right) \right|^2\,dxdy \\
\mathrm{II}_K &= \sum_{0\leq\alpha_j\leq 1} \varepsilon^{2|\alpha|} \int_{K\times\widehat{Q}} \left| D_x^\alpha \left(I^x_{p,\mathcal{T}_H} U^\varepsilon(x,y) - (I^x_{p,\mathcal{T}_H} \otimes I^y_{\mu,\widehat{\mathcal{T}}_h^{\mathrm{rad}}}) U^\varepsilon(x,y) \right) \right|^2\,dxdy.
\end{aligned}$$

The 'macro' error I_K is estimated in view of (5.1) as follows

$$\begin{aligned}
\mathrm{I}_K &= \sum_{0\leq\alpha_j\leq 1} \varepsilon^{2|\alpha|} \int_{K\times\widehat{Q}} \left| D_x^\alpha \left(U^\varepsilon(x,y) - I^x_{p,\mathcal{T}_H} U^\varepsilon(x,y) \right) \right|^2\,dxdy \\
&\leq CH^{2(k+1)} \int_{K\times\widehat{Q}} \left| \left(D_x^{k+1} U^\varepsilon \right)(x,y) \right|^2\,dxdy.
\end{aligned}$$

Applying now the error estimates in (5.3) for the interpolation error in the 'micro' FE space $S^\mu_{\mathrm{per}}(\widehat{Q},\widehat{\mathcal{T}}_h^{\mathrm{rad}})$, the error II_K in y can be estimated as follows

$$\begin{aligned}
\mathrm{II}_K &= \sum_{0\leq\alpha_j\leq 1} \varepsilon^{2|\alpha|} \int_{K\times\widehat{Q}} \left| D_x^\alpha \left(I^x_{p,\mathcal{T}_H} U^\varepsilon(x,y) - (I^x_{p,\mathcal{T}_H} \otimes I^y_{\mu,\widehat{\mathcal{T}}_h^{\mathrm{rad}}}) U^\varepsilon(x,y) \right) \right|^2\,dxdy \\
&\leq C \sum_{0\leq\alpha_j\leq 1} \int_{K\times\widehat{Q}} \left| D_x^{|\alpha|} \left(U^\varepsilon(x,y) - I^y_{\mu,\widehat{\mathcal{T}}_h^{\mathrm{rad}}} U^\varepsilon(x,y) \right) \right|^2\,dxdy \\
&\leq Ch^{2\min(\mu,s)+2} \|U^\varepsilon\|^2_{H^2(K;\,H^{s+1,2}_{\beta,1}(\widehat{Q}))}.
\end{aligned}$$

Summing up over all elements $K \in \mathcal{T}_H$ we obtain that

$$\|e_I^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\bigg(H^{\min(p,k)+1}) \|U^\varepsilon\|_{H^{k+1}(\Omega;\,L^2_{\mathrm{per}}(\widehat{Q}))} + h^{\min(\mu,s)+1} \|U^\varepsilon\|_{H^2(\Omega;\,H^{s+1,2}_{\beta,1}(\widehat{Q}))} \bigg).$$

For the estimate of the interpolation error in the H^1-seminorm we proceed analogously:

$$
\begin{aligned}
\|\nabla_x e_I^\varepsilon(x)\|^2_{L^2(K\cap\Omega_\varepsilon)} &= \int\limits_{K\cap\Omega_\varepsilon} \left| \left((\nabla_x + \varepsilon^{-1}\nabla_y) E_I^\varepsilon\right)(x,y)\Big|_{y=\frac{x}{\varepsilon}} \right|^2 dx \\
&\le \mathrm{J}_K + \mathrm{JJ}_K,
\end{aligned}
$$

where

$$
\mathrm{J}_K = \int\limits_{K\cap\Omega_\varepsilon} \left| \left(\nabla_x E_I^\varepsilon\right)(x,y)\Big|_{y=\frac{x}{\varepsilon}} \right|^2 dx, \quad \mathrm{JJ}_K = \int\limits_{K\cap\Omega_\varepsilon} \left| \left(\varepsilon^{-1}\nabla_y E_I^\varepsilon\right)(x,y)\Big|_{y=\frac{x}{\varepsilon}} \right|^2 dx.
$$

By the trace result in Lemma 5.1 we obtain that

$$
\begin{aligned}
\mathrm{J}_K &= \varepsilon^2 \sum_{r=1}^{2} \sum_{m:\varepsilon(m+\widehat{Q})\subset K} \int\limits_{\widehat{Q}} \left| (\partial_{x_r} E_I^\varepsilon)(\varepsilon(z+m),y)\Big|_{y=z} \right|^2 dz \\
&\le C\varepsilon^2 \sum_{r=1}^{2} \sum_{m:\varepsilon(m+\widehat{Q})\subset K} \sum_{\substack{0\le\alpha_j\le 1\\ \alpha_r=1}} \varepsilon^{2|\alpha|-2} \int\limits_{\widehat{Q}} \int\limits_{\widehat{Q}} \Big[|(D_x^\alpha \partial_{y_r} E_I^\varepsilon)(\varepsilon(z+m),y)|^2 \\
&\qquad + |(D_x^\alpha E_I^\varepsilon)(\varepsilon(z+m),y)|^2 \Big] dz dy \\
&\le C \sum_{r=1}^{2} \sum_{\substack{0\le\alpha_j\le 1\\ \alpha_r=1}} \varepsilon^{2|\alpha|-2} \int\limits_{K} \int\limits_{\widehat{Q}} \Big[|(D_x^\alpha \partial_{y_r} E_I^\varepsilon)(x,y)|^2 + |(D_x^\alpha E_I^\varepsilon)(x,y)|^2 \Big] dx dy
\end{aligned}
$$

Using arguments as in the bounds for the L^2-norm we find that

$$
\begin{aligned}
\sum_{r=1}^{2} \sum_{\substack{0\le\alpha_j\le 1\\ \alpha_r=1}} \varepsilon^{2|\alpha|-2} \int\limits_{K} \int\limits_{\widehat{Q}} |(D_x^\alpha \partial_{y_r} E_I^\varepsilon)(x,y)|^2 &\le C_1 H^{2\min(p,k)} \int\limits_{K\times\widehat{Q}} \left| D_x^k(\varepsilon^{-1}\nabla_y U^\varepsilon)(x,y) \right|^2 dx dy \\
&\quad + C_2 h^{2\min(\mu,s)} \|\varepsilon^{-1}\nabla_y U^\varepsilon\|^2_{H^2(K;\, H^{s,1}_{\beta,1}(\widehat{Q}))}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{r=1}^{2} \sum_{\substack{0\le\alpha_j\le 1\\ \alpha_r=1}} \varepsilon^{2|\alpha|-2} |(D_x^\alpha E_I^\varepsilon)(x,y)|^2 dx dy &\le C_1 H^{2\min(p,k)} \int\limits_{K\times\widehat{Q}} \left| D_x^{k+1} U^\varepsilon(x,y) \right|^2 dx dy \\
&\quad + C_2 h^{2\min(\mu,s)} \|U^\varepsilon\|^2_{H^2(K;H^{s+1,2}_{\beta,1}(\widehat{Q}))}.
\end{aligned}
$$

Summing up, we obtain that

$$
\begin{aligned}
\mathrm{J}_K &\le C_1 H^{2\min(p,k)} \Bigg(\int\limits_{K\times\widehat{Q}} \left| D_x^k(\varepsilon^{-1}\nabla_y U^\varepsilon)(x,y) \right|^2 dx dy + \int\limits_{K\times\widehat{Q}} \left| D_x^{k+1} U^\varepsilon(x,y) \right|^2 dx dy \Bigg) \\
&+ C_2 h^{2\min(\mu,s)} \Bigg(\|\varepsilon^{-1}\nabla_y U^\varepsilon\|^2_{H^2(K;H^{s,1}_{\beta,1}(\widehat{Q}))} + \|U^\varepsilon\|^2_{H^2(K;H^{s+1,2}_{\beta,1}(\widehat{Q}))} \Bigg).
\end{aligned}
$$

Similar considerations for JJ_K lead to the following estimate

$$\begin{aligned}
\mathrm{JJ}_K &= \varepsilon^2 \sum_{m\in\mathbb{Z}^2:\varepsilon(\widehat{Q}+m)\subset K} \int_{\widehat{Q}} \left| (\varepsilon^{-1}\nabla_y E_I^\varepsilon)(\varepsilon(z+m),y)\Big|_{z=y} \right|^2 dz \\
&\le C\varepsilon^2 \sum_{m\in\mathbb{Z}^2:\varepsilon(\widehat{Q}+m)\subset K} \sum_{0\le\alpha_j\le 1} \varepsilon^{2|\alpha|} \int_{\widehat{Q}\times\widehat{Q}} \left|(D_x^\alpha \varepsilon^{-1}\nabla_y E_I^\varepsilon)(\varepsilon(z+m),y)\right|^2 dzdy \\
&\le C \sum_{0\le\alpha_j\le 1} \varepsilon^{2|\alpha|} \int_{K\times\widehat{Q}} \left| D_x^\alpha (\varepsilon^{-1}\nabla_y E_I^\varepsilon)(x,y)\right|^2 dxdy \\
&\le C_1 H^{2\min(p,k)} \int_{K\times\widehat{Q}} \left| \left(D_x^k(\varepsilon^{-1}\nabla_y U^\varepsilon)\right)(x,y)\right|^2 \, dxdy \\
&+ C_2 h^{2\min(\mu,s)} \|\varepsilon^{-1}\nabla_y U^\varepsilon\|^2_{H^2(K;\,H^{s,1}_{\beta,1}(\widehat{Q}))}.
\end{aligned}$$

Summing up over all elements K of the 'macro' triangulation we obtain (5.6). □

6 Concluding Remarks

Regularity results for the kernel function ϕ were developed under the assumption of piecewise analyticity of the geometry but analyticity of the coefficients. The analyticity requirement of the coefficients can be relaxed to include the case of piecewise analytic coefficients, which is relevant in practice. Then the solution is only piecewise analytic but the regularity assertions of Theorem 3.7 (and of Section 4.2) are still valid, if they are understood in a piecewise sense. Also the two-scale regularity assertions of Theorem 4.3 (and of Section 4.2.2) can be generalized to the case of piecewise analytic coefficients. Most of the tools required for these generalizations can be found in [12, Chap. 5.4, Chap. 6.4]. We remark that the case of coefficients and geometries with finite regularity is covered in [11]. The two-scale regularity theory of Section 4.1 is formulated for the infinite, periodic lattice $\Omega_\varepsilon^\infty$. In this situation a scale separation takes place in the sense that the solution u^ε can be expression in terms of the unfolded solution U^ε. If the problem (2.2), (2.3) is posed on a finite domain $\Omega_\varepsilon^\infty \cap \Omega$, then scale separation is typically lost in the vicinity of the boundary $\partial\Omega$, and a boundary layer may arise.

A variety of different approximation schemes have been proposed in the literature that aim at catching the relevant effects up to $O(\varepsilon^q)$ for chosen $q \ge 0$. These methods include the classical homogenization theory [4] and the use of Bloch approximations θ^ε, [6, 5]. These latter ones can be understood as mimicking the projection onto the first eigenmode of the solution by replacing the first Bloch coefficient of u^ε at the ε-scale by the Fourier transform of the homogenized solution u^*. This approximation is closely connected with classical homogenization theory since θ^ε captures both the first order and second order correctors of the classical homogenization theory, [5].
The approximation schemes based on classical homogenization and Bloch wave decomposition lead to error estimates of the form $O(\varepsilon^q)$ for some suitable q; that is, they rely on ε being small. For a given value of ε and accuracy requirement, this truncation error may not be sufficiently small. In contrast, the generalized FEM, [10, 8], and the two-scale FEM, [11] are projection methods that converge for any ε whenever the discretization becomes finer. Under suitable assumptions these two methods are robust, that is, the work to achieve a given accuracy is independent of ε.

The two-scale FEM is a projection method on spaces of the form (5.4); that is, the ansatz space is chosen as the tensor product of a standard FEM-space on a coarse mesh and a standard FEM-space of $\varepsilon\widehat{Q}$-periodic functions on a fine ($\varepsilon\widehat{Q}$-periodic) mesh. Thus, the unit-cell problem (2.10), (2.11) does not have to be computed explicitly, which differentiates the method from the generalized FEM and approaches based on classical homogenization of Bloch wave decomposition, where a suitable unit cell problem has to be solved (numerically) prior to actual computations. Concerning the computational cost of the two-scale FEM, we finally remark that the tensor product structure of the ansatz space can be exploited to yield a version of the two-scale FEM with the same rate of convergence (up to logarithmic factors) as the classical FEM for standard elliptic problems, [9].

A Appendix

We require the following local regularity assertion:

Lemma A.1 *For $R \in (0,1]$ set $B_R := B_R(0)$, $B_R^+ := \{(x,y)\,|\,(x,y) \in B_R,\ y > 0\}$, $\Gamma_R := B_R \cap \{(x,0)\,|\,x \in \mathbb{R}\}$. Let $a \in C^1(\overline{B_R})$ be uniformly symmetric positive definite. Then there exists $C > 0$ depending only on $\|a\|_{W^{1,\infty}(B_R)}$ with the following properties:*

1. *If $f \in L^2(B_R)$, then any solution $u \in H^1(B_R)$ of $-\nabla\cdot(a\nabla u) = f$ satisfies*
$$\|\nabla^2 u\|_{L^2(B_{R/2})} \le C\left[\|f\|_{L^2(B_R)} + R^{-1}\|\nabla u\|_{L^2(B_R)}\right].$$
2. *If $f \in L^2(B_R^+)$ and $G \in H^1(B_R^+)$, then any solution $u \in H^1(B_R^+)$ of*
$$-\nabla\cdot(a\nabla u) = f \quad \text{on } B_R^+, \qquad \partial_{n_a} u = G \quad \text{on } \Gamma_R$$
satisfies
$$\|\nabla^2 u\|_{L^2(B_{R/2}^+)} \le C\left[\|f\|_{L^2(B_R^+)} + \|\nabla G\|_{L^2(B_R^+)} + R^{-1}\|G\|_{L^2(B_R^+)} + R^{-1}\|\nabla u\|_{L^2(B_R^+)}\right].$$

Proof: The case $R = 1$ is standard; the general case $R \in (0,1]$ follows by scaling arguments. □

We also have the following Hardy-type estimate:

Lemma A.2 *Let $S_R(\omega) = \{(r\cos\varphi, r\sin\varphi)\,|\,0 < r < R,\ 0 < \varphi < \omega\}$ be a sector, $\beta \in (0,1)$, $S' \subset S_R(\omega)$ a compact subset. Then there exists a constant $C > 0$ independent of u such that*
$$\|\,|x|^{\beta-1}u\|_{L^2(S_R(\omega))} \le C\left[\|\,|x|^{\beta}\nabla u\|_{L^2(S_R(\omega))} + \|u\|_{L^2(S')}\right]$$
provided that the right-hand side is finite.

Proof: Polar coordinates are employed to reduce the proof to a one-dimensional argument: From [7, Thm. 330] (using suitable cut-off functions), we obtain for every $r_0 \in (0,R)$, $\beta \in (0,1)$ the existence of $C > 0$ independent of u such that
$$\int_0^R r^{2(\beta-1)+1}u^2(r)\,dr \le C\int_0^R r^{2\beta+1}|u'(r)|^2\,dr + \int_{r_0}^R u^2(r)r\,dr, \tag{A.1}$$
provided that u is such that the right-hand side is finite. Using polar coordinates, the desired result now follows. □

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