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Robust exponential convergence of *hp*-FEM for singularly perturbed reaction diffusion systems with multiple scales

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We consider a coupled system of two singularly perturbed reaction-diffusion equations in one dimension. Associated with the two singular perturbation parameters $0 < \varepsilon \leqslant \mu \leqslant 1$, are boundary layers of length scales $O(\varepsilon)$ and $O(\mu)$. We propose and analyze an hp finite element scheme which includes elements of size $O(\varepsilon p)$ and $O(\mu p)$ near the boundary, where p is the degree of the approximating polynomials. We show that under the assumption of analytic input data, the method yields *exponential* rates of convergence, independently of ε and μ and independently of the relative size of ε to μ. In particular, the full range $0 < \varepsilon \leqslant \mu \leqslant 1$ is covered by our analysis. Numerical computations supporting the theory are also presented.

Keywords: boundary layers; singularly perturbed systems; multiple scales;*p*-version FEM; uniform convergence.

1. Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last couple of decades (see, e.g., the books by Miller, O'Riordan and Shishkin (1996), Morton (1996), Roos, Stynes, and Tobiska (2008) and the references therein). Besides the question of stability of discretizations (e.g., in the treatment of convection-dominated problems), a main difficulty in these problems is the presence of *boundary layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter(s), is of great importance for the overall quality of the approximate solution. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires the use of layer adapted, parameter-dependent meshes. Typical representatives for the h-version of the FEM go back to Shishkin (1989) and Bakhvalov (1969); a possible choice for the p- (and hp-)version of the FEM is termed in the present work the *Spectral Boundary Layer Mesh* (Def. 3.1)

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and harks back to Schwab and Suri (1996).

In this article we consider a *system* of two coupled singularly perturbed linear reaction-diffusion equations, which have two overlapping boundary layers. In contrast to equations with a single singular perturbation parameter, systems with multiple parameters (and correspondingly multiple layers) are much less studied and understood. The problem under consideration here was studied by Madden and Stynes (2003) and by Linß and Madden (2003, 2004b) in the context of finite differences, and by Linß and Madden (2004) in the context of the h version of the FEM with piecewise linear basis functions. We refer also to Linß and Stynes (2009) for a survey on the numerical solution of systems of singularly perturbed differential equations and to Shishkin (1995) for a finite difference approximation for a problem with multiple scales. In Xenophontos and Oberbroeckling (2007) an hp FEM was presented for a coupled system of reaction-diffusion equations, and its robust exponential convergence was demonstrated via several numerical experiments. The recent regularity results of Melenk, Xenophontos and Oberbroeckling (2011) allow us to provide the mathematical justification of what was reported in Xenophontos and Oberbroeckling (2007), which is the purpose of this article.

The rest of the paper is organized as follows: In Section 2 we present the model problem and discuss the typical phenomena, along with the regularity of the solution as determined in Melenk, Xenophontos and Oberbroeckling (2011). In Section 3 we prove our main result, which is the exponential convergence of the proposed hp FEM, and in Section 4 we present the results of some numerical computations verifying our theoretical findings. Finally, in Section 5 we give some closing remarks.

We will utilize the usual Sobolev space notation $H^k(I)$ to denote the space of functions on I with $0, 1, 2, ..., k$ generalized derivatives in $L^2(I)$, equipped with the norm $\|\cdot\|_{k,I}$ and seminorm $|\cdot|_{k,I}$. For vector functions $\mathbf{U} := (u_1, u_2)^T$, we will write

$$\|\mathbf{U}\|_{k,I}^2 = \|u_1\|_{k,I}^2 + \|u_2\|_{k,I}^2 .$$

We will also use the space $H_0^1(I) = \left\{u \in H^1(I) : u|_{\partial I} = 0\right\}$, where ∂I denotes the boundary of I. The norm of the space $L^\infty(I)$ of essentially bounded functions is denoted $\|\cdot\|_{\infty,I}$. Finally, the letter C will be used to denote a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence.

2. The Model Problem and its Regularity

We consider the following model problem: Find a pair of functions (u, v) such that

$$\left\{ \begin{array}{l} -\varepsilon^2 u''(x) + a_{11}(x)u(x) + a_{12}(x)v(x) = f(x) \text{ in } I = (0,1), \\ -\mu^2 v''(x) + a_{21}(x)u(x) + a_{22}(x)v(x) = g(x) \text{ in } I = (0,1), \end{array} \right. \tag{2.1a}$$

along with the boundary conditions

$$u(0) = u(1) = 0 \ , \ v(0) = v(1) = 0. \tag{2.1b}$$

With the abbreviations

$$\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbf{E}^{\varepsilon,\mu} := \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \mu^2 \end{pmatrix}, \qquad \mathbf{A}(x) := \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix},$$

equations (2.1a)–(2.1b) may also be written in the following, more compact form:

$$L_{\varepsilon,\mu}\mathbf{U} := -\mathbf{E}^{\varepsilon,\mu}\mathbf{U}''(x) + \mathbf{A}(x)\mathbf{U} = \mathbf{F}, \qquad \mathbf{U}(0) = \mathbf{U}(1) = 0. \tag{2.2}$$

The parameters $0 < \varepsilon \leqslant \mu \leqslant 1$ are given, as are the functions f, g, and a_{ij}, $i, j \in \{1,2\}$, which are assumed to be analytic on $\overline{I} = [0,1]$. Moreover we assume that there exist constants $C_f, \gamma_f, C_g, \gamma_g, C_a, \gamma_a > 0$ such that

$$\left\{ \begin{array}{r} \left\| f^{(n)} \right\|_{\infty,I} \leqslant C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \left\| g^{(n)} \right\|_{\infty,I} \leqslant C_g \gamma_g^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \left\| a_{ij}^{(n)} \right\|_{\infty,I} \leqslant C_a \gamma_a^n n! \quad \forall\, n \in \mathbb{N}_0, \quad i,j \in \{1,2\}. \end{array} \right. \tag{2.3}$$

The variational formulation of (2.1a)–(2.1b) reads: Find $\mathbf{U} := (u,v) \in \left[H_0^1(I)\right]^2$ such that

$$B(\mathbf{U},\mathbf{V}) = F(\mathbf{V}) \quad \forall\, \mathbf{V} := (\overline{u},\overline{v}) \in \left[H_0^1(I)\right]^2, \tag{2.4}$$

where, with $\langle \cdot,\cdot \rangle_I$ the usual $L^2(I)$ inner product,

$$\begin{aligned} B(\mathbf{U},\mathbf{V}) &= \varepsilon^2 \langle u', \overline{u}' \rangle_I + \mu^2 \langle v', \overline{v}' \rangle_I + \langle a_{11}u + a_{12}v, \overline{u} \rangle_I + \langle a_{21}u + a_{22}v, \overline{v} \rangle_I, & (2.5)\\ F(\mathbf{V}) &= \langle f, \overline{u} \rangle_I + \langle g, \overline{v} \rangle_I. & (2.6) \end{aligned}$$

The matrix-valued function $\mathbf{A}$ is assumed to be pointwise positive definite (but not necessarily symmetric), i.e., for some fixed $\alpha > 0$

$$\overrightarrow{\xi}^T \mathbf{A}(x) \overrightarrow{\xi} \geqslant \alpha^2 \overrightarrow{\xi}^T \overrightarrow{\xi} \quad \forall\, \overrightarrow{\xi} \in \mathbb{R}^2 \qquad \forall x \in \overline{I}. \tag{2.7}$$

It follows that the bilinear form $B(\cdot,\cdot)$ given by (2.5) is coercive with respect to the *energy norm*

$$\|\mathbf{U}\|_{E,I}^2 \equiv \|(u,v)\|_{E,I}^2 := \varepsilon^2 |u|_{1,I}^2 + \mu^2 |v|_{1,I}^2 + \alpha^2 \left(\|u\|_{0,I}^2 + \|v\|_{0,I}^2 \right), \tag{2.8}$$

i.e.,

$$B(\mathbf{V},\mathbf{V}) \geqslant \|\mathbf{V}\|_{E,I}^2 \quad \forall\, \mathbf{V} \in \left[H_0^1(I)\right]^2. \tag{2.9}$$

This, along with the continuity of $B(\cdot,\cdot)$ and $F(\cdot)$ imply the unique solvability of (2.4). We also have the following standard *a priori* estimate

$$\|\mathbf{U}\|_{E,I} \leqslant \alpha^{-1} \sqrt{\|f\|_{0,I}^2 + \|g\|_{0,I}^2}. \tag{2.10}$$

The finite element approximation of (2.1a)–(2.1b) reads: Find $\mathbf{U}_N := (u_N, v_N)^T \in [S_N]^2 \subset \left[H_0^1(I)\right]^2$ such that

$$B(\mathbf{U}_N,\mathbf{V}) = F(\mathbf{V}) \quad \forall\, \mathbf{V} := (\overline{u},\overline{v})^T \in [S_N]^2, \tag{2.11}$$

where $[S_N]^2$ is an appropriately chosen finite dimensional subspace of $\left[H_0^1(I)\right]^2$. The unique solvability of the discrete problem (2.11) follows from (2.7), and by the well-known Galerkin orthogonality, we have

$$\|\mathbf{U} - \mathbf{U}_N\|_{E,I} \leqslant \alpha^{-1} \inf_{\mathbf{V} \in [S_N]^2} \|\mathbf{U} - \mathbf{V}\|_{E,I}. \tag{2.12}$$

As is well-known for singularly perturbed problems, the choice of the space S_N must be made carefully and in a way that depends on the layer structure of the solution $\mathbf{U}$ in order for the approximation to be *robust*, i.e., for convergence to be independent of ε and μ. As we will formalize in Theorem 2.2 below, the solution $\mathbf{U}$ has features on up to three different length scales, namely, $O(1)$, $O(\mu)$, and $O(\varepsilon)$ with the features on the $O(\varepsilon)$ and $O(\mu)$ scale being of boundary layer type. These three different length scales have to be incorporated into the approximation space, and we will do this with the *Spectral Bounday Layer mesh* below in Definition 3.1.

Our design of the *Spectral Bounday Layer mesh* hinges on the regularity theory of Melenk, Xenophontos and Oberbroeckling (2011) that we will discuss in more detail in Theorem 2.2 below. Essentially, Theorem 2.2 derives from asymptotic expansions of the solution. Such expansions rely on scale separation assumptions. For the present context of length scales $O(1)$, $O(\mu)$, and $O(\varepsilon)$, the following cases may occur:

(I) The "no scale separation case" which occurs when *neither* $\mu/1$ *nor* ε/μ is small.

(II) The "3-scale case" in which all scales are separated and occurs when $\mu/1$ is small *and* ε/μ is small.

(III) The first "2-scale case" which occurs when $\mu/1$ is *not* small *and* ε/μ is small.

(IV) The second "2-scale case" which occurs when $\mu/1$ is small *and* ε/μ is *not* small.

The concept of "small " (or "not small") mentioned above, is tied in two ways to the regularity theory in terms of asymptotic expansions. First, on the level of constructing asymptotic expansions, the decision which parameters are deemed small determines the ansatz to be made and thus the form of the expansion. Second, on the level of using asymptotic expansions for approximation purposes or the design of approximation spaces, the decision which parameters are deemed small depends on the desired accuracy, i.e., whether the remainder resulting from the asymptotic expansion can be regarded as small. We mention that a similar classification of the possible relations between the arising scales has already been in used in Shishkin (1995) for the design of a robust finite difference scheme with multiple scales.

In order to be able to describe the regularity assertions for the solution $\mathbf{U}$, we need to introduce some notation:

DEFINITION 2.1 1. We say that a function w is analytic with length scale $\nu > 0$ (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{A}(\nu, C_w, \gamma_w)$, if

$$\|w^{(n)}\|_{\infty,I} \leqslant C_w \gamma_w^n \max\{n, \nu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0.$$

2. We say that an entire function w is of L^∞-boundary layer type with length scale $\nu > 0$ (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{BL}^\infty(\nu, C_w, \gamma_w)$, if for all $x \in I$

$$|w^{(n)}(x)| \leqslant C_w \gamma_w^n \nu^{-n} e^{-\operatorname{dist}(x,\partial I)/\nu} \qquad \forall n \in \mathbb{N}_0.$$

Both definitions extend naturally to vector-valued functions by requiring that the above bounds hold componentwise.

With this notation in hand, we can formulate a regularity assertion for the solution $\mathbf{U}$ of (2.1):

THEOREM 2.2 (Melenk, Xenophontos and Oberbroeckling (2011, Thm. 2.2)) Assume *(2.3)* and *(2.7)* hold. Then there exist constants C, b, δ, q, $\gamma > 0$, independent of $0 < \varepsilon \leqslant \mu \leqslant 1$, such that the following assertions are true for the solution $\mathbf{U}$ of *(2.1)*:

(I) $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, \gamma)$.

(II) $\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^{\infty}(\delta\mu, C, \gamma)$ $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^{\infty}(\delta\varepsilon, C, \gamma)$, and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leqslant C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right]$. Furthermore, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}$ satisfies the stronger assertion $\widehat{v} \in \mathcal{BL}^{\infty}(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.

(III) If $\varepsilon/\mu \leqslant q$ then $\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{W} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(\mu, C, \gamma)$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^{\infty}(\delta\varepsilon, C, \gamma)$, and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leqslant Ce^{-b/\varepsilon}$. Furthermore, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}$ satisfies the stronger assertion $\widehat{v} \in \mathcal{BL}^{\infty}(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.

(IV) $\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^{\infty}(\delta\mu, C\sqrt{\mu/\varepsilon}, \gamma\mu/\varepsilon)$, and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leqslant C(\mu/\varepsilon)^2 e^{-b/\mu}$.

We emphasize that all decompositions of $\mathbf{U}$ stated in (I)–(IV) are valid for the solution $\mathbf{U}$ regardless of the values of $0 < \varepsilon \leqslant \mu \leqslant 1$ (caveat: in the form presented here, the case (III) requires $\varepsilon/\mu \leqslant q$); in fact, we will exploit this observation in the proof of Theorem 3.3. Nevertheless, given that the remainder $\mathbf{R}$ is only small in certain parameter regimes (and hence, only then does the corresponding decomposition provide useful information) it may be helpful to think of the cases (I)–(IV) as being associated with the parameter regimes *(I)–(IV)* mentioned earlier. These ties are further strengthened by how the decompositions of Theorem 2.2 are obtained in Melenk, Xenophontos and Oberbroeckling (2011), namely, by asymptotic expansions. In the subsections that follow we will briefly explain what the ansatz is in each Case *(II)–(IV)*.

2.1 *The three scale case (Case (II))*

Anticipating that boundary layers of length scales $O(\mu)$ and $O(\varepsilon)$ will appear at the endpoints $x = 0$ and $x = 1$, we introduce the stretched variables $\widetilde{x} = x/\mu$, $\widehat{x} = x/\varepsilon$ for the expected layers at the left endpoint $x = 0$ and variables $\widetilde{x}^R = (1-x)/\mu$, $\widehat{x}^R = (1-x)/\varepsilon$ for the expected behavior at right endpoint $x = 1$. We make the following formal ansatz for the solution $\mathbf{U}$:

$$\mathbf{U} \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{\mu}{1}\right)^i\left(\frac{\varepsilon}{\mu}\right)^j\left[\mathbf{U}_{ij}(x) + \widetilde{\mathbf{U}}^L_{ij}(\widetilde{x}) + \widehat{\mathbf{U}}^L_{ij}(\widehat{x}) + \widetilde{\mathbf{U}}^R_{ij}(\widetilde{x}^R) + \widehat{\mathbf{U}}^R_{ij}(\widehat{x}^R)\right], \tag{2.13}$$

where the functions $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}^L_{ij}$, $\widehat{\mathbf{U}}^L_{ij}$ $\widetilde{\mathbf{U}}^R_{ij}$, $\widehat{\mathbf{U}}^R_{ij}$ are to be determined by inserting the ansatz (2.13) into the boundary value problem (2.1), and equating like powers of $\mu/1$ and ε/μ. The functions $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}^L_{ij}$, $\widetilde{\mathbf{U}}^R_{ij}$, $\widehat{\mathbf{U}}^L_{ij}$, $\widehat{\mathbf{U}}^R_{ij}$ can then be determined recursively as solutions of suitable boundary value problems (see Melenk, Xenophontos and Oberbroeckling (2011) for details). The decomposition of Theorem 2.2 is obtained by truncating the asymptotic expansion (2.13) after a finite number of terms:

$$\mathbf{U}(x) = \mathbf{W}_M(x) + \widehat{\mathbf{U}}^M_{BL}(\widehat{x}) + \widehat{\mathbf{V}}^M_{BL}(\widehat{x}^R) + \widetilde{\mathbf{U}}^M_{BL}(\widetilde{x}) + \widetilde{\mathbf{V}}^M_{BL}(\widetilde{x}^R) + \mathbf{R}_M(x), \tag{2.14}$$

where

$$\mathbf{W}_M(x) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\mathbf{U}_{ij}(x) \tag{2.15}$$

denotes the outer (smooth) expansion,

$$\widehat{\mathbf{U}}_{BL}^{M}(\widehat{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \widehat{\mathbf{U}}_{ij}^{L}(\widehat{x}), \qquad \widehat{\mathbf{V}}_{BL}^{M}(\widehat{x}^R) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \widehat{\mathbf{U}}_{ij}^{R}(\widehat{x}^R), \tag{2.16}$$

denote the left and right inner (boundary layer) expansions asscociated with the variables $\widehat{x}$, $\widehat{x}^R$, respectively,

$$\widetilde{\mathbf{U}}_{BL}^{M}(\widetilde{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \widetilde{\mathbf{U}}_{ij}^{L}(\widetilde{x}), \qquad \widetilde{\mathbf{V}}_{BL}^{M}(\widetilde{x}^R) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \widetilde{\mathbf{U}}_{ij}^{R}(\widetilde{x}^R), \tag{2.17}$$

denote the left and right inner (boundary layer) expansions asscociated with the variables $\widetilde{x}$, $\widetilde{x}^R$ respectively, and the remainder $\mathbf{R}_M$ is defined such that (2.14) holds. In establishing Theorem 2.2 the choices $M_1 = O(1/\mu)$, $M_2 = O(\mu/\varepsilon)$ are made. (For full details see Melenk, Xenophontos and Oberbroeckling (2011).)

2.2 *The first two scale case (Case (III))*

We employ again the notation of the stretched variables $\widehat{x} = x/\varepsilon$ and $\widehat{x}^R = (1-x)/\varepsilon$. Since $\mu/1$ is not assumed to be small, only the scales $O(1)$ and $O(\varepsilon)$ are expected to be present in the problem. Inserting the formal ansatz

$$\mathbf{U}(x) \sim \sum_{i=0}^{\infty} \left(\frac{\varepsilon}{\mu}\right)^i \left[\mathbf{U}_i(x) + \widehat{\mathbf{U}}_i^L(\widehat{x}) + \widehat{\mathbf{U}}_i^R(\widehat{x}^R)\right], \tag{2.18}$$

into the boundary value problem (2.1) and equating like powers of ε/μ, yields recursions for the functions $\mathbf{U}$, $\widehat{\mathbf{U}}^L$ and $\widehat{\mathbf{U}}^R$. Truncating (2.18) after M terms leads to the representation

$$\mathbf{U}(x) = \mathbf{W}_M(x) + \widehat{\mathbf{U}}_{BL}^{M}(\widehat{x}) + \widehat{\mathbf{V}}_{BL}^{M}(\widehat{x}^R) + \mathbf{R}_M(x), \tag{2.19}$$

where

$$\mathbf{W}_M(x) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \mathbf{U}_i(x) \tag{2.20}$$

denotes the outer (smooth) expansion,

$$\widehat{\mathbf{U}}_{BL}^{M}(\widehat{x}) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \widehat{\mathbf{U}}_i^L(\widehat{x}), \qquad \widehat{\mathbf{V}}_{BL}^{M}(\widehat{x}^R) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \widehat{\mathbf{U}}_i^R(\widehat{x}^R) \tag{2.21}$$

denote the left and right inner (boundary layer) expansions, respectively, and the remainder $\mathbf{R}_M$ is such that (2.19) is valid. In establishing Theorem 2.2 the choice $M = O(\mu/\varepsilon)$ is made. (See Melenk, Xenophontos and Oberbroeckling (2011) for more details.)

2.3 *The second two scale case (Case (IV))*

In this case, $\mu/1$ is assumed to be small and ε/μ is not deemed small, which leads us to the ansatz

$$\mathbf{U} \sim \sum_{i=0}^{\infty} \mu^i \left[\mathbf{U}_i(x) + \widetilde{\mathbf{U}}_i^L(\widetilde{x}) + \widetilde{\mathbf{U}}_i^R(\widetilde{x}^R)\right], \tag{2.22}$$

with the stretched variables $\widetilde{x} = x/\mu$ and $\widetilde{x}^R = (1-x)/\mu$. Inserting this ansatz into the boundary value problem (2.1) and equating like powers of μ yields recursions for the functions $\mathbf{U}_i$, $\widetilde{\mathbf{U}}_i^L$, and $\widetilde{\mathbf{U}}_i^R$. The truncated series $\mathbf{W}_M(x) := \sum_{i=0}^M \mu^i \mathbf{U}_i(x)$, $\widetilde{\mathbf{U}}_{BL}^M := \sum_{i=0}^M \mu^i \mathbf{U}_i^L(\widetilde{x})$, $\widetilde{\mathbf{V}}_{BL}^M := \sum_{i=0}^M \mu^i \mathbf{U}_i^R(\widetilde{x}^R)$ yield the decomposition stated in Theorem 2.2 if $M = O(1/\mu)$. (The details are given in Melenk, Xenophontos and Oberbroeckling (2011).)

3. Approximation results

3.1 *Main results*

In this section we will describe the finite dimensional subspace $[S_N]^2$ which appears in (2.11), in order to construct an hp scheme for the approximation of the solution to (2.4). To this end, let $\Delta = \{0 = x_0 < x_1 < ... < x_{\mathcal{M}} = 1\}$ be an arbitrary partition of $I = (0,1)$ and set

$$I_j = (x_{j-1}, x_j), \quad h_j = x_j - x_{j-1}, \quad j = 1, ..., \mathcal{M}.$$

Also, define the master (or standard) element $I_{ST} = (-1,1)$, and note that it can be mapped onto the j^{th} element I_j by the linear mapping

$$x = Q_j(t) = \frac{1}{2}(1-t)\,x_{j-1} + \frac{1}{2}(1+t)\,x_j.$$

With $\Pi_p(I_{ST})$ the space of polynomials of degree $\leqslant p$ on I_{ST}, we define our finite dimensional subspace as following (with $\circ$ denoting composition of functions):

$$\begin{aligned} S_N \equiv S^p(\Delta) &= \left\{ \mathbf{V} \in [H_0^1(I)]^2 : \mathbf{V} \circ Q_j^{-1} \in \left(\Pi_{p_j}(I_{ST})\right)^2, j = 1, ..., \mathcal{M} \right\}, \\ S_0^p(\Delta) &:= S^p(\Delta) \cap [H_0^1(I)]^2. \end{aligned} \tag{3.1}$$

We restrict our attention here to constant polynomial degree p for all elements, i.e. $p_j = p\ \forall\ j$, but clearly, more general settings with variable polynomial degree are possible.

The following *Spectral Boundary Layer mesh* is essentially the minimal mesh that yields robust exponential convergence. Loosely speaking, one inserts nodes to resolve the boundary layers, i.e., upon setting $x_\varepsilon := \kappa p \varepsilon$ and $x_\mu = \kappa p \mu$, one inserts the nodes x_ε and $1 - x_\varepsilon$ if $\kappa p \varepsilon < 1/2$; the nodes x_μ and $1 - x_\mu$ are inserted if $\kappa p \mu < 1/2$. Here, $\kappa > 0$ is a user-specified parameter. The optimal choice of κ is problem-specific as it depends on the length scales of the boundary layers, and we refer the reader to Schwab and Suri (1996) for a more detailed discussion of this issue.

DEFINITION 3.1 (Spectral Boundary Layer mesh) For $\kappa > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \leqslant \mu \leqslant 1$, define the spaces $S(\kappa, p)$ of piecewise polynomials by

$$S(\kappa, p) := \begin{cases} S_0^p(\Delta); & \Delta = \{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\mu < \frac{1}{2} \\ S_0^p(\Delta); & \Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\varepsilon < \frac{1}{2} \leqslant \kappa p\mu \\ S_0^p(\Delta); & \Delta = \{0, 1\} & \text{if } \kappa p\varepsilon \geqslant \frac{1}{2} \end{cases}$$

REMARK 3.2 Strictly speaking, the method considered here is not a true hp version, since the location (and not the number) of mesh points changes as p increases. A better characterization would be a p-version FEM on a moving mesh. Obviously, additional refinement and/or using a true hp version would yield better results but at the cost of using more degrees of freedom – see Xenophontos and Oberbroeckling (2007) for a numerical comparison.

We now present the main result of this paper:

THEOREM 3.3 Let f, g and $\mathbf{A}$ be analytic on $\overline{I}$ and satisfy the conditions in *(2.3)* and *(2.7)*. Let $\mathbf{U} = (u, v)^T$ be the solution to *(2.1)*. Then there exist constants κ_0, C, $\beta > 0$ depending only on f, g, and $\mathbf{A}$ such that for each $\kappa \in (0, \kappa_0]$ there is a $\widetilde{\mathcal{I}}_p\mathbf{U} = [\mathcal{I}_p u, \mathcal{I}_p v]^T \in S(\kappa, p)$ with $\widetilde{\mathcal{I}}_p\mathbf{U} = \mathbf{U}$ on ∂I and

$$\left\| \mathbf{U} - \widetilde{\mathcal{I}}_p\mathbf{U} \right\|_{E,I} \leqslant C e^{-\beta\kappa p}.$$

Proof. The proof is given at the end of Section 3.2. $\square$

Using the above theorem and the quasioptimality result (2.12) we have the following:

COROLLARY 3.4 Let $\mathbf{U}$ be the solution to *(2.4)* and let $\mathbf{U}_N \in S_0^p(\Delta)$ be the solution to *(2.11)* based on the Spectral Boundary Layer mesh of Definition *3.1*. Then there exist constants κ_0, C, $\beta > 0$ depending *only* on the input data f, g and $\mathbf{A}$, such that for any $0 < \kappa \leqslant \kappa_0$

$$\|\mathbf{U} - \mathbf{U}_N\|_{E,I} \leqslant C e^{-\beta\kappa p}.$$

3.2 *Proof of Theorem 3.3*

3.2.1 *An approximation operator.* We start with two lemmas:

LEMMA 3.5 There exists a bounded linear operator $\mathcal{I}_p : H^1(I_{ST}) \to \Pi_p(I_{ST})$ satisfying $u(\pm 1) = \mathcal{I}_p u(\pm 1)$ with the following approximation properties: If $u \in C^\infty\left(\overline{I}_{ST}\right)$, then

$$\|u - \mathcal{I}_p u\|_{0,I_{ST}}^2 \leqslant \frac{1}{p^2}\frac{(p-s)!}{(p+s)!}\left\|u^{(s+1)}\right\|_{0,I_{ST}}^2, \ \forall\, s = 0, 1, ..., p, \tag{3.2}$$

$$\left\|(u - \mathcal{I}_p u)'\right\|_{0,I_{ST}}^2 \leqslant \frac{(p-s)!}{(p+s)!}\left\|u^{(s+1)}\right\|_{0,I_{ST}}^2, \ \forall\, s = 0, 1, ..., p. \tag{3.3}$$

Proof. The result is taken from Schwab (1998) (Cor. 3.15). $\square$

LEMMA 3.6 Let $p \in \mathbb{N}$, $\lambda \in (0, 1]$ such that $\lambda p \in \mathbb{N}$. Then

$$\frac{(p - \lambda p)!}{(p + \lambda p)!} \leqslant \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p} e^{2\lambda p + 1}.$$

Proof. This follows from Stirling's formula – see Xenophontos and Oberbroeckling (2007) (Lemma 3.1) for the details. $\square$

On the reference element I_{ST}, we have the following additional stability results.

LEMMA 3.7 Let $\mathcal{I}_p : H^1(I_{ST}) \to \Pi_p(I_{ST})$ be as in Lemma 3.5. Then, on the reference element I_{ST}, we have

$$|\mathcal{I}_p u|_{1,I_{ST}} \leqslant C|u|_{1,I_{ST}}, \qquad \|\mathcal{I}_p u\|_{0,I_{ST}} \leqslant \|u\|_{0,I_{ST}} + C\frac{1}{p}|u|_{1,I_{ST}},$$

$$\|\mathcal{I}_1 u\|_{L^\infty(I_{ST})} \leqslant \|u\|_{L^\infty(I_{ST})}, \qquad \|(\mathcal{I}_1 u)'\|_{L^\infty(I_{ST})} \leqslant C\|u\|_{L^\infty(I_{ST})}.$$

Proof. The estimates for the linear interpolant $\mathcal{I}_1$ are standard. The stability in the H^1-seminorm is a direct consequence of (3.3) with $s = 0$ and the triangle inequality. The L^2-stability follows from a triangle inequality and an approximation result:

$$\|\mathcal{I}_p u\|_{0,I_{ST}} \leqslant \|u\|_{0,I_{ST}} + \|u - \mathcal{I}_p u\|_{0,I_{ST}} \leqslant \|u\|_{0,I_{ST}} + Cp^{-1}|u|_{1,I_{ST}}.$$

$\square$

The approximation operator $\mathcal{I}_p$ on the reference element can be used to define an approximation operator in an elementwise fashion: For a mesh Δ with elements I_j, $j = 1, \ldots, \mathcal{M}$, element maps Q_j and given degree vector $\overrightarrow{p} = (p_1, \ldots, p_{\mathcal{M}})$, with $1 \leqslant p_j \leqslant p$, we define the operator $\mathcal{I}_{\overrightarrow{p},\Delta} : [H^1(I)]^2 \to S^p(\Delta)$ elementwise in the standard way with the operators $\mathcal{I}_{p_j}$, by requiring

$$(\mathcal{I}_{\overrightarrow{p},\Delta}\mathbf{V})|_{I_j} \circ Q_j = \mathcal{I}_{p_j}(\mathbf{V}|_{I_j} \circ Q_j), \qquad j = 1, \ldots, \mathcal{M}. \tag{3.4}$$

Since the operators $\mathcal{I}_{p_j}$ interpolate in the endpoints of I_{ST}, this operator is indeed well-defined. We will write $\mathcal{I}_{p,\Delta}$ if $p_j = p$ for $j = 1, \ldots, \mathcal{M}$. We point out that $p_j = 1$ corresponds to the linear interpolant. Finally, since the operators $\mathcal{I}_{\overrightarrow{p},\Delta}$ are defined elementwise, we will work with the abbreviation

$$\mathcal{I}_{p_j,I_j}\mathbf{V} := (\mathcal{I}_{\overrightarrow{p},\Delta}\mathbf{V})|_{I_j} = (\mathcal{I}_{p_j}(\mathbf{V}|_{I_j} \circ Q_j)) \circ Q_j^{-1}. \tag{3.5}$$

The following approximation result on the reference element will be one of our main tools for the proof of Theorem 3.3.

LEMMA 3.8 Let $u \in C^\infty(I_{ST})$ satisfy for some C_u, $\gamma_u > 0$, $K \geqslant 1$, $h \in (0, 1]$,

$$\|u^{(n)}\|_{0,I_{ST}} \leqslant C_u(\gamma_u h)^n \max\{n, K\}^n \qquad \forall n \in \mathbb{N}. \tag{3.6}$$

Then there exist $\eta, \beta, C > 0$ depending solely on γ_u, such that under the condition

$$\frac{hK}{p} \leqslant \eta, \tag{3.7}$$

the approximation $\mathcal{I}_p u \in \Pi_p(I_{ST})$ given by Lemma *3.5* satisfies $u(\pm 1) = (\mathcal{I}_p u)(\pm 1)$ and

$$\|u - \mathcal{I}_p u\|_{0,I_{ST}} + \|(u - \mathcal{I}_p u)'\|_{0,I_{ST}} \quad \leqslant \quad C C_u \frac{hK}{p} e^{-\beta p}.$$

Proof. First note that $\sup_{j\geqslant 1}((j+1)/j)^{2j} \leqslant e^2$. Then, in view of $hK/p \leqslant \eta$, $h \leqslant 1$ and $\lambda \leqslant 1$, we compute for $s = \lambda p$ the following:

$$\begin{aligned} h^{2(s+1)} \left(\max\{s+1, K\}\right)^{2(s+1)} &\leqslant e^2 (hK)^2 \left(\max\{hs, hK\}\right)^{2s} \leqslant e^2 (hK)^2 \left(\max\{h\lambda p, \eta p\}\right)^{2s} \\ &\leqslant e^2 (hK)^2 \left(p \max\{\lambda, \eta\}\right)^{2s}. \end{aligned}$$

This and Lemma 3.6 give

$$\begin{aligned} \frac{(p-s)!}{(p+s)!}\|u^{(s+1)}\|^2_{0,I_{ST}} &\leqslant e^2 (hK)^2 \gamma_u^2 C_u^2 \gamma_u^{2\lambda p} p^{2\lambda p} \left(\max\{\lambda, \eta\}\right)^{2\lambda p} \left[\frac{(1-\lambda)^{1-\lambda}}{(1+\lambda)^{1+\lambda}}\right]^p p^{-2\lambda p} e^{2\lambda p + 1} \\ &\leqslant C_u^2 e^3 \gamma_u^2 (hK)^2 \left[\frac{(1-\lambda)^{1-\lambda}}{(1+\lambda)^{1+\lambda}}\right]^p \left[\gamma_u e \max\{\lambda, \eta\}\right]^{2\lambda p}. \end{aligned}$$

Select now $\lambda \in (0, 1)$ and $\eta > 0$ such that $\gamma_u e \max\{\lambda, \eta\} \leqslant 1$. Since for this choice of λ we have $(1-\lambda)^{1-\lambda}/(1+\lambda)^{1+\lambda} =: q < 1$, we conclude

$$\frac{(p-s)!}{(p+s)!}\|u^{(s+1)}\|^2_{0,I_{ST}} \quad \leqslant \quad e^3 \gamma_u^2 (hK)^2 C_u^2 e q^p = e^3 \gamma_u^2 \left(\frac{hK}{p}\right)^2 C_u^2 p^2 q^p,$$

which is the desired bound, since the algebraic factor p^2 may be absorbed in the exponentially decaying one by suitably adjusting the constants. $\square$

We reformulate the approximation result of Lemma 3.8 in a form that will be convenient for the approximation of the smooth and the boundary layer parts of the expansion (the latter within the layer):

COROLLARY 3.9 Let I_j be an interval of length h_j, and let $\mathbf{V} \in C^\infty(I_j)$ satisfy for some C_u, $\gamma_u > 0$, $K \geqslant 1$,

$$\|\mathbf{V}^{(n)}\|_{L^\infty(I_j)} \leqslant C_u \gamma_u^n \max\{n, K\}^n \qquad \forall n \in \mathbb{N}.$$

Then there exist constants C, η, $\beta > 0$ depending only on γ_u, such that under the scale resolution condition

$$\frac{h_j K}{p_j} \leqslant \eta, \tag{3.8}$$

the polynomial approximation $\mathcal{I}_{p_j,I_j}\mathbf{V}$ satisfies

$$h_j^{-1}\|\mathbf{V} - \mathcal{I}_{p_j,I_j}\mathbf{V}\|_{0,I_j} + |\mathbf{V} - \mathcal{I}_{p_j,I_j}\mathbf{V}|_{1,I_j} \leqslant CC_u \frac{h_j^{1/2}K}{p_j} e^{-\beta p_j}. \tag{3.9}$$

Proof. Let $\widehat{\mathbf{V}} := \mathbf{V} \circ Q_j$, where $Q_j : I_{ST} \to I_j$ is the affine bijection. Then $\widehat{\mathbf{V}}$ satisfies

$$\|\widehat{\mathbf{V}}^{(n)}\|_{L^2(I_{ST})} \leqslant CC_u(\gamma_u h_j/2)^n \max\{n, K\}^n \qquad \forall n \in \mathbb{N}.$$

Therefore, Lemma 3.8 gives the existence of C, β, η such that under the assumption (3.8), we have

$$\|\widehat{\mathbf{V}} - \mathcal{I}_{p_j}\widehat{\mathbf{V}}\|_{0,I_{ST}} + \|\left(\widehat{\mathbf{V}} - \mathcal{I}_{p_j}\widehat{\mathbf{V}}\right)'\|_{0,I_{ST}} \leqslant CC_u \frac{h_j K}{p_j} e^{-\beta p_j}.$$

Transforming back to I_j gives the result. $\square$

The following result will be useful for the approximation of the remainder $\mathbf{R}$ and the boundary layer contributions.

LEMMA 3.10 Let I_j be an interval of length h_j and $p_j \in \mathbb{N}$. Then, for scalar functions (and analogously for vector-valued ones):

$$\|u - \mathcal{I}_{p_j,I_j}u\|_{0,I_j} \leqslant C\frac{h_j}{p_j}|u|_{1,I_j}, \tag{3.10}$$

$$|u - \mathcal{I}_{p_j,I_j}u|_{1,I_j} \leqslant C|u|_{1,I_j}, \tag{3.11}$$

$$\|u - \mathcal{I}_{1,I_j}u\|_{0,I_j} \leqslant \|u\|_{0,I_j} + Ch_j^{1/2}\|u\|_{L^\infty(I_j)}. \tag{3.12}$$

Proof. The estimates follow from Lemmas 3.5, 3.7 and standard scaling arguments. $\square$

Finally, we formulate an approximation result for the approximation of functions of boundary layer type outside the layer:

LEMMA 3.11 Let $\nu > 0$ and let u satisfy

$$|u(x)| + \nu|u'(x)| \leqslant C_u e^{-\operatorname{dist}(x,\partial I)/\nu} \qquad \forall x \in I.$$

Let $\varDelta$ be an arbitrary mesh on I with mesh points ξ and $1-\xi$, where $\xi \in (0, 1/2)$. Then, for some $C > 0$ independent of ν, the piecewise linear interpolant $\mathcal{I}_{1,\varDelta}u$ satisfies on $(\xi, 1-\xi)$:

$$\nu|u - \mathcal{I}_{1,\varDelta}u|_{1,(\xi,1-\xi)} + \|u - \mathcal{I}_{1,\varDelta}u\|_{0,(\xi,1-\xi)} \leqslant CC_u e^{-\xi/\nu}.$$

Proof. Let I_j be an element in the interval $(\xi, 1-\xi)$ of length h_j. We distinguish between the cases $h_j \leqslant \nu$ and $h_j > \nu$. In the case $h_j \leqslant \nu$, we note that (3.10), (3.11) yield

$$\|u - \mathcal{I}_{1,I_j} u\|_{0,I_j} + \nu \|(u - \mathcal{I}_{1,I_j} u)'\|_{0,I_j} \leqslant C(h_j + \nu)|u|_{1,I_j} \leqslant C\nu |u|_{1,I_j}.$$

In the converse case $h_j \geqslant \nu$, we use (3.11), (3.12), to get

$$\|u - \mathcal{I}_{1,I_j} u\|_{0,I_j} + \nu \|(u - \mathcal{I}_{1,I_j} u)'\|_{0,I_j} \quad \leqslant \quad C \left[h_j^{1/2} \|u\|_{L^\infty(I_j)} + \nu |u|_{1,I_j} \right].$$

Summation over all elements in $(\xi, 1-\xi)$ then gives the result. □

3.2.2 *Details of the proof of Theorem 3.3.* The approximation of the exact solution $\mathbf{U}$ is constructed element by element with the aid of the operator $\mathcal{I}_p$. The basic ingredient of the proof is that the four regularity assertions of Theorem 2.2 permit us to show that, on each element, the features on all length scales can either be resolved or are sufficiently small to be safely ignored. The parameter κ_0 appearing in the statement of Theorem 3.3 will be determined in the course of the proof.

We remark here that we make the simplifying assumption that

$$\frac{\varepsilon}{\mu} \leqslant q \tag{3.13}$$

with q as in Theorem 2.2. In the converse case, μ and ε are comparable, and the regularity assertion (III) follows from (IV) by suitably adjusting constants.

Case 1: $\kappa p \varepsilon \geqslant 1/2$, which implies in particular $\kappa p \mu \geqslant 1/2$ and $\kappa p \varepsilon / \mu \geqslant 1/2$. This is the no-scale separation case (from the point of view of regularity) or, equivalently, the "asymptotic case of p" (from the point of view of approximability). The mesh is given by $\Delta = \{0, 1\}$.

We may employ the regularity assertion (I) of Theorem 2.2, for the solution $\mathbf{U}$, i.e., $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, \gamma)$. The mesh Δ consists of the single element I with length 1. Corollary 3.9 then implies the existence of $\eta > 0$ (depending solely on γ) such that the condition

$$\frac{1}{p\varepsilon} \leqslant \eta, \tag{3.14}$$

implies $\|\mathbf{U} - \mathcal{I}_{p,I}\mathbf{U}\|_{H^1(I)} \leqslant Ce^{-\beta p}$, which is even stronger than what is required. The crucial condition (3.14) is easily satisfied by making sure that $\kappa_0 < \eta/2$, since then the assumption $\kappa p \varepsilon > 1/2$ produces

$$\frac{1}{p\varepsilon} = \frac{\kappa}{\kappa p \varepsilon} \leqslant 2\kappa \leqslant 2\kappa_0.$$

Case 2: $\kappa p \mu < \frac{1}{2}$ and $\kappa p \frac{\varepsilon}{\mu} < \frac{1}{2}$. This is the three-scale separation case (regularity-wise) or the "pre-asymptotic case of p" (approximability-wise). The mesh is given by $\Delta = \{0, \kappa p \varepsilon, \kappa p \mu, 1 - \kappa p \mu, 1 - \kappa p \varepsilon, 1\}$, i.e. the mesh has 5 elements $I_1, \ldots, I_5$. The statement (II) of Theorem 2.2 gives the decomposition

$$\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}, \tag{3.15}$$

where the smooth part satisfies $\mathbf{W} \in \mathcal{A}(1, C_w, \gamma_w)$, the boundary layers satisfy $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\mu, C_{\widetilde{u}}, \gamma_{\widetilde{u}})$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{\widehat{u}}, \gamma_{\widehat{u}})$, $\widehat{v}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{\widehat{v}}(\varepsilon/\mu)^2, \gamma_{\widehat{v}})$, and the remainder satisfies

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leqslant C \left[e^{-b/\mu} + e^{-b\mu/\varepsilon} \right] \leqslant Ce^{-2b\kappa p}. \tag{3.16}$$

These give $\forall\, n \in \mathbb{N}_0$,

$$\begin{aligned}
\left\|\mathbf{W}^{(n)}\right\|_{L^\infty(I)} &\leqslant C_w \gamma_w^n n^n, && (3.17)\\
\left|\widetilde{\mathbf{U}}_{BL}^{(n)}(x)\right| &\leqslant C_{\widetilde{u}} \gamma_{\widetilde{u}}^n \, (\delta\mu)^{-n}\, e^{-\operatorname{dist}(x,\partial I)/(\delta\mu)}, && (3.18)\\
\left|\widehat{\mathbf{U}}_{BL}^{(n)}(x)\right| &\leqslant C_{\widehat{u}} \gamma_{\widehat{u}}^n \, (\delta\varepsilon)^{-n}\, e^{-\operatorname{dist}(x,\partial I)/(\delta\varepsilon)}, && (3.19)\\
|\widehat{v}_{BL}^{(n)}(x)| &\leqslant C_{\widehat{v}} \left(\frac{\varepsilon}{\mu}\right)^2 \gamma_{\widehat{v}}^n (\delta\varepsilon)^{-n} e^{-\operatorname{dist}(x,\partial I)/(\delta\varepsilon)}. && (3.20)
\end{aligned}$$

We approximate $\mathbf{W}$ by $\mathcal{I}_{p,\varDelta}\mathbf{W}$, the first boundary layer function $\widetilde{\mathbf{U}}_{BL}$ by $\mathcal{I}_{(p,p,1,p,p),\varDelta}\widetilde{\mathbf{U}}_{BL}$, the second boundary layer function $\widehat{\mathbf{U}}_{BL}$ by $\mathcal{I}_{(p,1,1,1,p),\varDelta}\widehat{\mathbf{U}}_{BL}$, and the remainder $\mathbf{R}$ by its *global* linear interpolant $\mathcal{I}_{p,\{0,1\}}\mathbf{R}$.

With the aid of Corollary 3.9, it is easy to see that $\|\mathbf{W} - \mathcal{I}_{p,\varDelta}\mathbf{W}\|_{1,I} \leqslant Ce^{-\beta p}$ for some $C, \beta > 0$, independent of ε and μ. For the remainder $\mathbf{R}$, we get in view of (3.16), which gives control of $\mathbf{R}$ at the endpoint of I, that

$$\|\mathbf{R} - \mathcal{I}_{1,\{0,1\}}\mathbf{R}\|_{E,I} \leqslant \|\mathbf{R}\|_{E,I} + \|\mathcal{I}_{1,\{0,1\}}\mathbf{R}\|_{E,I} \leqslant Ce^{-2b\kappa p}.$$

We now turn to the approximation of the boundary layer contribution $\widetilde{\mathbf{U}}_{BL}$. Lemma 3.11 (with $\nu = \delta\mu$) immediately produces, for the element $I_3 = (\kappa p\mu, 1-\kappa p\mu)$, the estimate $\|\widetilde{\mathbf{U}}_{BL} - \mathcal{I}_{1,I_3}\widetilde{\mathbf{U}}_{BL}\|_{E,I_3} \leqslant Ce^{-\kappa p/\delta}$, which is exponentially small. For the small elements I_1, I_2, I_4, and I_5, we note that their length is smaller than $\kappa p\mu$. Corollary 3.9 is applicable with $K = 1/\mu$, which produces for these elements I_j, $j \in \{1, 2, 4, 5\}$,

$$h_j^{-1}\|\widetilde{\mathbf{U}}_{BL} - \mathcal{I}_{p,I_j}\widetilde{\mathbf{U}}_{BL}\|_{0,I_j} + |\widetilde{\mathbf{U}}_{BL} - \mathcal{I}_{p,I_j}\widetilde{\mathbf{U}}_{BL}|_{1,I_j} \leqslant C\frac{h_j^{1/2}}{p\mu}e^{-\beta p}, \tag{3.21}$$

if the scale resolution condition

$$\frac{\kappa p\mu}{p\mu} \leqslant \eta,$$

is satisfied, where the parameter η depends only on $\gamma_{\widetilde{u}}$. Taking κ_0 sufficiently small, this condition is satisfied. Recalling that $h_j \leqslant \kappa p\mu$, we see that (3.21) is a much stronger result than required.

We next study the approximation of $\widehat{\mathbf{U}}_{BL}$. We first consider the approximation on the elements I_2, I_3, and I_4, which are all in the interval $(\kappa p\varepsilon, 1 - \kappa p\varepsilon)$. For the $\widehat{u}$-component of $\widehat{\mathbf{U}}_{BL}$, Lemma 3.11 (with $\nu = \delta\varepsilon$, $C_u = C_{\widehat{u}}$) yields the desired exponential approximation result. For the $\widehat{v}$-component, we also apply Lemma 3.11 (with $\nu = \delta\varepsilon$, $C_u = C_{\widehat{v}}(\varepsilon/\mu)^2$) to get

$$\|\widehat{v} - \mathcal{I}_{1,\varDelta}\widehat{v}\|_{0,(\kappa p\varepsilon,1-\kappa p\varepsilon)} + \varepsilon|\widehat{v} - \mathcal{I}_{1,\varDelta}\widehat{v}|_{1,(\kappa p\varepsilon,1-\kappa p\varepsilon)} \leqslant CC_{\widehat{v}}\left(\frac{\varepsilon}{\mu}\right)^2 e^{-\kappa p/\delta}.$$

This implies

$$\|\widehat{v} - \mathcal{I}_{1,\varDelta}\widehat{v}\|_{0,(\kappa p\varepsilon,1-\kappa p\varepsilon)} + \mu|\widehat{v} - \mathcal{I}_{1,\varDelta}\widehat{v}|_{1,(\kappa p\varepsilon,1-\kappa p\varepsilon)} \leqslant CC_{\widehat{v}}\left(\frac{\varepsilon}{\mu}\right) e^{-\kappa p/\delta} \leqslant CC_{\widehat{v}}e^{-\kappa p/\delta},$$

which is the desired bound.

The approximation of $\widehat{\mathbf{U}}_{BL}$ on the remaining elements I_1 and I_5, is achieved with the aid of Corollary 3.9 in exactly the same way as $\widetilde{\mathbf{U}}_{BL}$ was approximated on I_1, I_2, I_4, and I_5. Again, for the $\widehat{v}$-component, we may exploit the fact that the bounds (3.20) feature an additional factor $(\varepsilon/\mu)^2$.

Case 3: $\kappa p\mu \geqslant 1/2$ and $\kappa p\varepsilon < 1/2$. This is the first two-scale separation case (regularity-wise) or the "semi-asymptotic case of p" (approximability-wise). The mesh is given by $\varDelta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\}$. The statement (III) of Theorem 2.2 gives the decomposition

$$\mathbf{U} = \mathbf{W} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}, \tag{3.22}$$

where $\mathbf{W} \in \mathcal{A}(\mu, C_w, \gamma_w)$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{\widehat{u}}, \gamma_{\widehat{u}})$, and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leqslant Ce^{-b/\varepsilon}$. Furthermore, the second component $\widehat{v}_{BL}$ of $\widehat{\mathbf{U}}_{BL}$ satisfies the stronger assertion $\widehat{v}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{\widehat{v}}(\varepsilon/\mu)^2, \gamma_{\widehat{v}})$. Recall that the mesh consists of three elements

$$I_1 = (0, \kappa p\varepsilon), I_2 = (\kappa p\varepsilon, 1 - \kappa p\varepsilon), I_3 = (1 - \kappa p\varepsilon, 1)\,,$$

and we approximate each component as follows: $\mathbf{W}$ is approximated by $\mathcal{I}_{p,\varDelta}\mathbf{W}$, the boundary layer $\widehat{\mathbf{U}}_{BL}$ is approximated by $\mathcal{I}_{p,1,p}\widehat{\mathbf{U}}_{BL}$, and $\mathbf{R}$ is approximated by $\mathcal{I}_{1,\varDelta}\mathbf{R}$. For the approximation of $\mathbf{W}$, we use Corollary 3.9, which yields exponential convergence, since the scale resolution condition

$$\eta \geqslant \frac{h_j}{p\mu} = \frac{h_j\kappa}{\kappa p\mu} \geqslant 2\kappa h_j,$$

can be satisfied for all elements by taking κ_0 sufficiently small.

The approximation of $\widehat{\mathbf{U}}_{BL}$ follows by the same arguments as in Case 2.

Finally, for the remainder $\mathbf{R}$, we use again stability of the piecewise linear approximation and the fact that the algebraic factor ε^{-1} can be absorbed in the exponentially small factor $e^{-b/\varepsilon}$ at the expense of slightly reducing b:

$$\begin{aligned} \|\mathbf{R} - \mathcal{I}_{1,\varDelta}\mathbf{R}\|_{E,I} &\leqslant \|\mathbf{R} - \mathcal{I}_{1,\varDelta}\mathbf{R}\|_{1,I} \leqslant C\|\mathbf{R}\|_{1,I} \leqslant C\varepsilon^{-1}\|\mathbf{R}\|_{E,I} \leqslant C\varepsilon^{-1}e^{-b/\varepsilon} \\ &\leqslant Ce^{-b'/\varepsilon} \leqslant Ce^{-2b'\kappa p}. \end{aligned}$$

Case 4: $\kappa p\varepsilon \leqslant \kappa p\mu < \frac{1}{2}$ and $\kappa p\frac{\varepsilon}{\mu} \geqslant \frac{1}{2}$. This is the second two-scale separation case or the second "semi-asymptotic case of p". The mesh is given by $\varDelta = \{0, \kappa p\varepsilon, 1 - \kappa p\varepsilon, 1\}$.

The statements (III) and (IV) of Theorem 2.2 give the decompositions

$$\begin{aligned} \mathbf{U} &= \mathbf{W}^{III} + \widehat{\mathbf{U}}_{BL}^{III} + \mathbf{R}^{III}, \\ \mathbf{U} &= \mathbf{W}^{IV} + \widetilde{\mathbf{U}}_{BL}^{IV} + \mathbf{R}^{IV}, \end{aligned}$$

where $\mathbf{W}^{IV} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL}^{IV} \in \mathcal{BL}^\infty(\delta\mu, C\sqrt{\mu/\varepsilon}, \gamma\mu/\varepsilon)$ and $\|\mathbf{R}^{IV}\|_{L^\infty(\partial I)} + \|\mathbf{R}^{IV}\|_{E,I} \leqslant C(\mu/\varepsilon)^2 e^{-b/\mu}$; furthermore, $\mathbf{W}^{III} \in \mathcal{A}(\mu, C, \gamma)$, $\widehat{\mathbf{U}}_{BL}^{III} \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$ and $\|\mathbf{R}^{III}\|_{L^\infty(\partial I)} + \|\mathbf{R}^{III}\|_{E,I} \leqslant Ce^{-b/\varepsilon}$ and additionally, the $\widehat{v}$-component of $\widehat{\mathbf{U}}_{BL}^{III}$ satisfies the stronger estimate $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$. The mesh consists of 5 elements $I_1, \ldots, I_5$. On $I_1 = (0, \kappa p\varepsilon)$ (and analogously on $I_5 = (1 - \kappa p\varepsilon, 1)$), we approximate $\mathbf{U}$ by

$$\mathcal{I}_{p,I_1}\mathbf{W}^{IV} + \mathcal{I}_{p,I_1}\widetilde{\mathbf{U}}_{BL}^{IV} + \mathcal{I}_{1,I_1}\mathbf{R}^{IV}.$$

On $I_2 = (\kappa p\varepsilon, \kappa p\mu)$ (and analogously on $I_4 = (1 - \kappa p\mu, 1 - \kappa p\varepsilon)$), we approximate $\mathbf{U}$ by

$$\mathcal{I}_{p,I_2}\mathbf{W}^{III} + \mathcal{I}_{1,I_2}\widehat{\mathbf{U}}_{BL}^{III} + \mathcal{I}_{1,I_2}\mathbf{R}^{III}.$$

For the middle element I_3, we use

$$\mathcal{I}_{p,I_3}\mathbf{W}^{IV} + \mathcal{I}_{1,I_3}\widetilde{\mathbf{U}}_{BL}^{IV} + \mathcal{I}_{1,I_3}\mathbf{R}^{IV}.$$

The approximation of the functions $\mathbf{W}^{IV}$ and $\mathbf{W}^{III}$ is done with the aid of Corollary 3.9. The interesting case is the approximation of $\mathbf{W}^{III}$ on I_2 and I_4, for which we note that the element size satisfies $h_j = \kappa p(\mu - \varepsilon) \leqslant \kappa p\mu$.

Next, we study the approximation of $\widetilde{\mathbf{U}}_{BL}^{IV}$ on the elements I_1 and I_5. In order to be able to employ Corollary 3.9, we rewrite the regularity assertion for $\widetilde{\mathbf{U}}_{BL}^{IV} \in \mathcal{BL}^\infty(\delta\mu, C\sqrt{\mu/\varepsilon}, \gamma\mu/\varepsilon)$ as follows:

$$\| \left(\widetilde{\mathbf{U}}_{BL}^{IV}\right)^{(n)} \|_{L^\infty(I_1)} \leqslant C\sqrt{\mu/\varepsilon}\left(\gamma\frac{\mu}{\varepsilon}\right)^n (\delta\mu)^{-n} \leqslant C\sqrt{\mu/\varepsilon}\left(\frac{\gamma}{\delta}\right)^n \varepsilon^{-n} \qquad \forall n \in \mathbb{N}.$$

Hence, Corollary 3.9 implies

$$\|\widetilde{\mathbf{U}}_{BL}^{IV} - \mathcal{I}_{p,I_1}\widetilde{\mathbf{U}}_{BL}^{IV}\|_{1,I_1} \leqslant C\sqrt{\mu/\varepsilon}e^{-\beta p},$$

if the scale resolution condition

$$\frac{h_1}{p_1\varepsilon} \leqslant \eta,$$

is satisfied, where η depends solely on γ/δ. In view of $h_1 = \kappa p\varepsilon$ and $p_1 = p$, this condition is satified if κ_0 is sufficiently small. Finally, the factor $\sqrt{\mu/\varepsilon}$ can be controlled since $\frac{\mu}{\varepsilon} \leqslant 2\kappa p$, and this algebraic factor can be absorbed in the exponentially decaying one.

The approximation of $\widehat{\mathbf{U}}_{BL}^{III}$ on I_2 and I_5 is achieved with Lemma 3.11 (as in Case 2, the $\widehat{u}$-component and the $\widehat{v}$-component have to be studied separately). Finally, the approximation $\widetilde{\mathbf{U}}_{BL}^{IV}$ on the middle element I_3 is covered by Lemma 3.11.

We next turn to the remainders. The stability properties of the linear interpolant stated in (3.10), (3.11) yield for each element I_j and arbitrary $\mathbf{R} \in H^1(I_j)$

$$\|\mathbf{R} - \mathcal{I}_{1,I_j}\mathbf{R}\|_{1,I_j} \leqslant C\|\mathbf{R}\|_{1,I_j}.$$

Hence, we get for arbitrary $\mathbf{R} \in H^1(I)$ that $\|\mathbf{R} - \mathcal{I}_{1,\Delta}\mathbf{R}\|_{1,I} \leqslant C\|\mathbf{R}\|_{1,I} \leqslant C\varepsilon^{-1}\|\mathbf{R}\|_{E,I}$ and thus

$$\begin{aligned} \varepsilon^{-1}\|\mathbf{R}^{IV}\|_{E,I} &\leqslant C\varepsilon^{-1}(\mu/\varepsilon)^2 e^{-b/\mu} \leqslant C(\mu/\varepsilon)^3\mu^{-1}e^{-b/\mu} \leqslant C(2\kappa p)^3 e^{-b'/\mu} \\ \varepsilon^{-1}\|\mathbf{R}^{III}\|_{E,I} &\leqslant C\varepsilon^{-1}e^{-b/\varepsilon} \leqslant Ce^{-b'/\varepsilon}. \end{aligned}$$

Using $\varepsilon \leqslant \mu$ and $2\kappa p \geqslant 1/\mu$ shows that these terms are exponentially small in κp as desired.

This completes the proof of Theorem 3.3. □

REMARK 3.12 Theorem 3.3 asserts the existence of values κ such that the Spectral Boundary Layer $S(\kappa, p)$ has good approximation properties. Some indication on how to select κ can be inferred from a more detailed analysis of the layer structure of the solution in conjunction with the results of Schwab and Suri (1996). Specifically, we recall from Schwab and Suri (1996) that on the interval $I = (0, 1)$ the function $x \mapsto e^{-x/d}$ (with $d > 0$) can be be approximated well by piecewise polynomials of degree p on the mesh with nodes $\{0, \kappa pd, 1\}$, where $\kappa \in (0, 4/e)$; in particular, $\kappa = 1$ is a good choice in this situation. In order to apply this result here, let us assume that μ and ε/μ are both sufficiently small so that the leading order terms of the asymptotic expansion provide an accurate description of the solution. By Melenk, Xenophontos and Oberbroeckling (2011, Sec. 4), we then have

$$\mathbf{U} \approx \mathbf{W} + C_1e^{-\lambda_1 x/\mu} + C_2e^{-\lambda_2 x/\varepsilon} + C_3e^{-\lambda_3(1-x)/\mu} + C_4e^{-\lambda_4(1-x)/\varepsilon} \tag{3.23}$$

with vectors $C_i \in \mathbb{R}^2$ of size $O(1)$ (additionally, the second components of C_2 and C_4 are of size $(\varepsilon/\mu)^2$). The coefficients $\lambda_i > 0$ are given by

$$\lambda_1^2 = \frac{\det \mathbf{A}(0)}{\mathbf{A}_{11}(0)}, \qquad \lambda_2^2 = \mathbf{A}_{11}(0), \qquad \lambda_3^2 = \frac{\det \mathbf{A}(1)}{\mathbf{A}_{11}(1)}, \qquad \lambda_4^2 = \mathbf{A}_{11}(1). \tag{3.24}$$

This indicates that a good choice of mesh should include points close to the points $p\mu/\lambda_1$, $1 - p\mu/\lambda_3$ to resolve the $O(\mu)$-layers and close to the points $p\varepsilon/\lambda_2$, $1 - p\varepsilon/\lambda_4$ to resolve the $O(\varepsilon)$-layers.

4. Numerical Results

In this section we present the results of numerical computations for the model problem considered in Linß and Madden (2004), Madden and Stynes (2003) and Xenophontos and Oberbroeckling (2007). The data is chosen as follows:

$$\mathbf{A} = \begin{bmatrix} 2(x+1)^2 & -(1+x^3) \\ -2\cos(\pi x/4) & 2.2e^{(1-x)} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} 2e^x \\ 10x+1 \end{bmatrix},$$

with κ appearing in the definition of the mesh taken as 1. An exact solution is not available, hence for the computations we use a reference solution $\mathbf{U}_{REF}$, obtained with a high number of degrees of freedom. Since various numerical results for this problem have already been presented in Xenophontos and Oberbroeckling (2007), we focus on issues not studied there. We will present the estimated percentage relative error in the energy norm, given by

$$100 \times \frac{\|\mathbf{U}_{REF} - \mathbf{U}_N\|_{E,I}}{\|\mathbf{U}_{REF}\|_{E,I}}, \tag{4.1}$$

versus the number of degrees of freedom N. In the last subsection we will address the issue of convergence and robustness when the maximum norm $\|\cdot\|_{\infty,I}$. is used as an error measure.

4.1 *The case of a single boundary layer, Case (III)*

In this case μ is not small but ε/μ is. It may therefore be sufficient to use a three-element mesh $\Delta_\varepsilon = \{0, p\varepsilon, 1-p\varepsilon, 1\}$. For this choice of mesh, the left graph in Figure 1 depicts the cases $\mu = 0.1, \varepsilon = 10^{-j}$, $j = 2, ..., 6$ and shows exponential convergence of the error (4.1). We note that for small polynomial degrees, the convergence is somewhat reduced since $\mu = 0.1$ is not very large.

4.2 *5 elements versus 3 elements*

We focus on the case $\varepsilon = 10^{-3}$ and $\mu = 10^{-2}$. The numerical results are shown in the right graph in Figure 1, where the error is plotted versus the polynomial degree. We compare the performance of the Spectral Boundary Layer mesh (with 5 elements) with two meshes with merely 3 elements, namely, $\Delta_\varepsilon = \{0, p\varepsilon, 1-p\varepsilon, 1\}$ (which is able to resolve the features on the $O(\varepsilon)$-scale) and $\Delta_\mu = \{0, p\mu, 1 - p\mu, 1\}$ (which is adequate to capture the features on the $O(\mu)$-scale). We recall that the problem size $N = 2(M-1)p$, where M is the number of elements (here: $M = 3$ or $M = 5$). We note that the Spectral Boundary Layer mesh with 5 elements yields exponential convergence. The other two meshes show a significantly worse performance. To qualitatively understand this behavior, we assume that μ and ε/μ are sufficiently small for asymptotic expansions to be meaningful, and we study the approximation

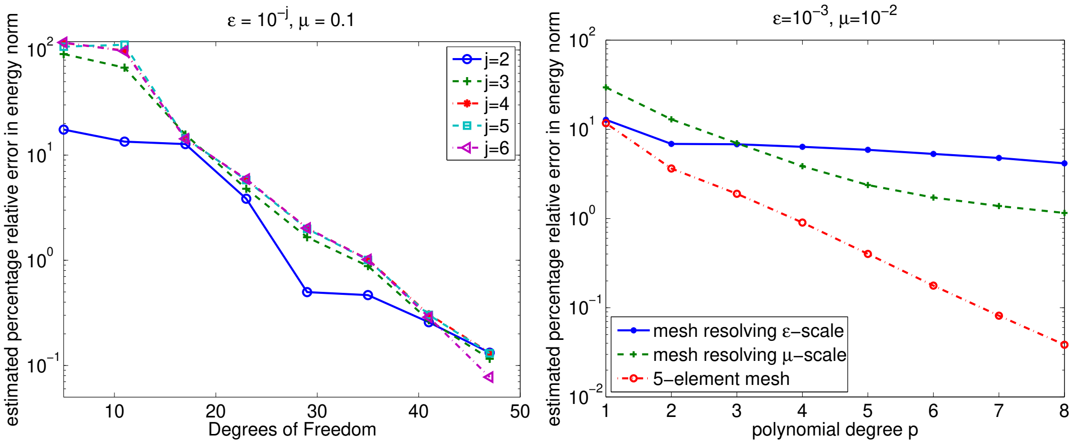

FIG. 1. Left: Energy norm convergence on the mesh $\Delta_\varepsilon = \{0, p\varepsilon, 1 - p\varepsilon, 1\}$ for the case $\varepsilon = 10^{-j}$, $j \in \{2, \ldots, 6\}$ and $\mu = 0.1$ (see Section 4.1). Right: Comparison of the meshes $\Delta = \{0, p\varepsilon, p\mu, 1 - p\mu, 1 - p\varepsilon, 1\}$, $\Delta_\varepsilon = \{0, p\varepsilon, 1 - p\varepsilon, 1\}$, and $\Delta_\mu = \{0, p\mu, 1 - p\mu, 1\}$ for $\varepsilon = 10^{-3}$, $\mu = 10^{-2}$ (see Section 4.2).

of the leading order layers, which take form (3.23) with parameters λ_i given by (3.24), i.e., $\lambda_1 \approx 2.23$, $\lambda_2 \approx 1.41$, $\lambda_3 \approx 2.82$, $\lambda_4 \approx 1.36$. The detailed analysis of Schwab and Suri (1996) leads us to expect, for the case of small p under consideration here, the following convergence behaviors: for the three-element mesh capable to resolve the features on the ε-scale, the approximation of the contributions $C_1 e^{-\lambda_1 x/\mu}$ and $C_3 e^{-\lambda_3 (1-x)/\mu}$ is achieved at the rate $O(e^{-p\lambda_{min}\varepsilon/\mu})$, where $\lambda_{min} = \min\{\lambda_1, \lambda_3\}$. For the three-element mesh that is capable of resolving the features on the μ-scale, the approximation of the contributions $C_2 e^{-\lambda_2 x/\varepsilon}$ and $C_4 e^{-\lambda_4 (1-x)/\varepsilon}$ can be done with $O(1/p)$. Figure 2 shows the performance of these meshes in doubly logarithmic and semilogarithmic scale.

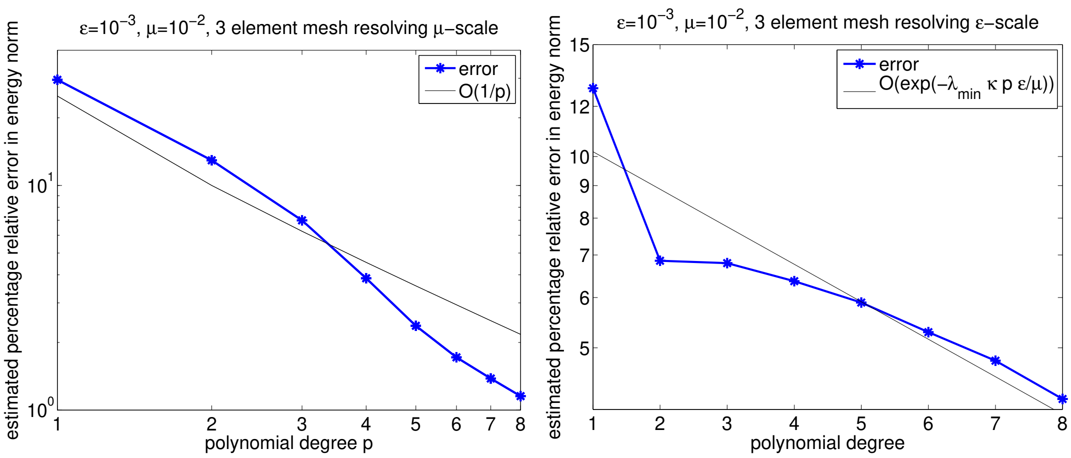

FIG. 2. Performance on the meshes Δ_μ (left, in doubly logarithmic plot) and Δ_ε (right, in semilogarithmic plot)– see Section 4.2.

4.3 *Convergence in the maximum norm*

Even though the natural energy norm associated with (2.4) was used in our analysis (and numerical experiments), we wish to comment on the performance of the proposed method when the maximum norm (4) is used as an error measure. First, we note that from the (Sobolev Embedding) inequality $\|v\|_{\infty,I} \leqslant C\|v\|_{0,I}^{1/2}\|v'\|_{0,I}^{1/2}$, we obtain bounds in the maximum norm from the estimates given in Corollary 3.4, namely,

$$\|\mathbf{U}-\mathbf{U}_N\|_{\infty,I} \leqslant C\varepsilon^{-1/2}e^{-\beta\kappa p}.$$

While this estimate is obviously ε-dependent, this dependence is rather weak since it implies exponential convergence with ε-independent constants under the weak side constraint $p \geqslant c\log\varepsilon^{-1}$ for some $c > 0$ that depends on the problem. Figure 3 shows the (estimated) percentage relative error in the maximum norm, for the three-scale separation case, i.e., when $\varepsilon \ll \mu \ll 1$. As the figure shows, the method yields exponential convergence, even in the maximum norm, with the values of the singular perturbation parameters not affecting the method's performance. See also Xenophontos and Oberbroeckling (2007) for more numerical results (in the maximum norm) and comparisons with other methods.

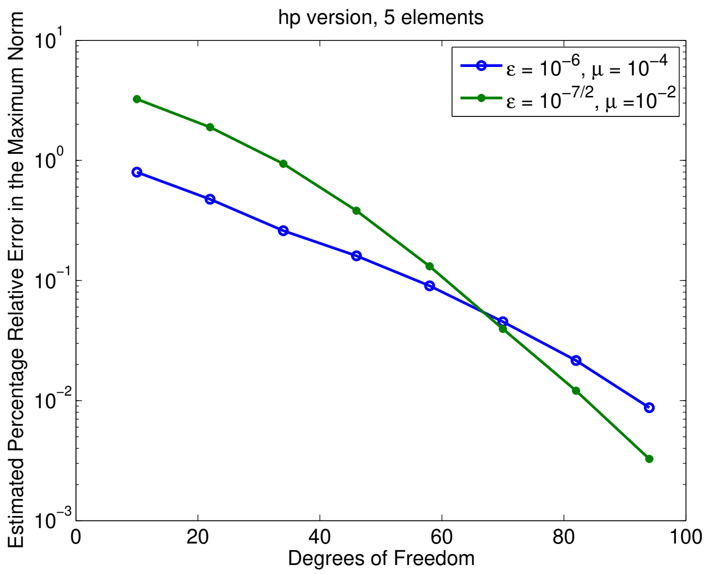

FIG. 3. Maximum norm convergence for the hp version on the Spectral Boundary Layer mesh.

5. Conclusions and Extensions

We considered a coupled system of two reaction diffusion equations with two singular perturbation parameters $0 < \varepsilon \leqslant \mu \leqslant 1$. We have proved that the hp FEM proposed in Xenophontos and Oberbroeckling (2007) for its approximation, indeed exhibits exponential rates of convergence, independently of the singular perturbation parameters ε and μ, as the degree p of the approximating polynomials is increased. The mesh design principle is simple in that for each boundary layer a small element that is capable of resolving it is inserted in the mesh. This mesh design principle is applicable beyond the problem class here, for example, in one dimension to systems with more than 2 components (and thus more than 2 layers) and convection-diffusion problems. We believe that also an extension to systems of reaction-diffusion equations in two dimensions is possible. The key ingredient of the present analysis is the detailed regularity theory of Melenk, Xenophontos and Oberbroeckling (2011). A rigorous analysis of the above extensions would require analogous regularity results. The straight-forward extension of these regularity results seems feasible but technically involved; for example, while the present system of two equations required a regularity theory distinguishing between four different cases of relationships between the scales, the extension to the problem

$$-\mathbf{E}\mathbf{U}'' + \mathbf{A}(x)\mathbf{U} = \mathbf{f} \quad \text{on } I = (0,1), \qquad \mathbf{U}(0) = \mathbf{U}(1) = 0 \in \mathbb{R}^d$$

with a positive definite diagonal matrix $\mathbf{E} = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_d)$ would require distinguishing between 2^d cases.

Inspection of the proof of the approximation result Theorem 3.3 shows that some refinements are possible. We highlight three of them:

1. Our boundary layer approximation relies on Lemma 3.11, which provides estimates for the piecewise linear approximation of boundary layer functions outside the layer. This approximation can be improved using the technique employed in Schwab and Suri (1996) (Thm. 5.1) of adding an appropriate additional piecewise linear function. The end result is then that one can construct a piecewise polynomial approximation π to a function $u \in \mathcal{BL}^\infty(\nu, C, \gamma)$ on the three-element mesh $\Delta = \{0, \kappa\nu p, 1 - \kappa\nu p, 1\}$ that satisfies

   $$\|u - \pi\|_{0,I} + (\kappa p \nu)\|(u - \pi)'\|_{0,I} \quad \leqslant \quad C\sqrt{\kappa p \nu}\, e^{-\beta\kappa p}.$$

   These boundary layer approximation results can, for example, lead to improved error bounds if the model problem (2.1) is considered with $f = g = 0$ and inhomogeneous Dirichlet boundary data. More generally, improved (with respect to μ and ε) error bounds can be expected if the leading order term of the outer expansion (of an asymptotic expansion) is contained in the approximation space. This happens, for example, for a constant $\mathbf{A}$ and polynomial right-hand sides f, g. These observations parallel the case of a scalar singularly perturbed problem analyzed in Schwab and Suri (1996) and are visible in the numerics of Xenophontos and Oberbroeckling (2007).

2. The sublayer on the $O(\varepsilon)$-scale is particularly weak in the $\widehat{v}$-component. This could be exploited further. For example, in certain parameter ranges, one could remove the mesh points $\kappa p\varepsilon$ and $1 - \kappa p\varepsilon$ in the mesh for the second component without compromising, up to certain tolerances, the accuracy of the FEM. On the other hand, this may be awkward to program, since it would require different meshes for each component of $\mathbf{U}$.

3. The proof of Theorem 3.3 relies on the regularity assertions of Theorem 2.2. There, L^∞-based estimates are given for the solution $\mathbf{U}$. The proof of Theorem 3.3, however, mostly uses L^2-based regularity estimates.

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