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On the Convergence of Filon Quadrature

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Abstract

We analyze the convergence behavior of Filon-type quadrature rules by making explicit the dependence on both k, the parameter that controls the oscillatory behavior of the integrand, and n, the number of function evaluations. We provide explicit conditions on the domain of analyticity of the integrand to ensure convergence for $n \to \infty$.

Key words: highly oscillatory integrals, Filon quadrature

1 Introduction and Filon-type quadrature

For a given $k \in \mathbb{R}$ with $|k|$ large, we seek to evaluate

$$Q(f) := \int_{-1}^{1} e^{\mathbf{i}kg(x)} f(x)\, dx. \tag{1}$$

Here, f and g are assumed to satisfy:

Assumption 1.1 *$\mathcal{G}_g \subset \mathcal{G}_f \subset \mathbb{C}$ are open neighborhoods of $[-1,1]$. The function $f \in L^\infty(\mathcal{G}_f)$ is holomorphic on $\mathcal{G}_f$, the function g is real-valued on $[-1,1]$, $g' \neq 0$ on $[-1,1]$, g and $1/g'$ are holomorphic on $\mathcal{G}_g$, and $1/g' \in L^\infty(\mathcal{G}_g)$.*

Filon-type quadrature (see [4–6]) assumes that integrals $\int_{-1}^{1} e^{\mathbf{i}kg(x)}\pi(x)\, dx$ can be evaluated for *polynomials* π. Hence, a quadrature rule for the integral (1) can be obtained by replacing the integrand $x \mapsto e^{\mathbf{i}kg(x)} f(x)$ with $x \mapsto e^{\mathbf{i}kg(x)} I_\Delta f(x)$, where $I_\Delta f$ is a polynomial (Hermite-)interpolant of f; that is, we obtain the Filon-type quadrature rule

$$Q_\Delta(f) := \int_{-1}^{1} e^{\mathbf{i}kg(x)} I_\Delta f(x)\, dx. \tag{2}$$

Throughout this note, we will associate with a sequence $\Delta = (z_0, \ldots, z_n)$ of $n+1$ nodes the (Hermite-)interpolation operator I_Δ that maps f to the polynomial $I_\Delta f \in \mathcal{P}_n$ of degree n that interpolates in the $n+1$ nodes; the implicit understanding of Hermite interpolation is that if a nodes ξ appears $m+1$ times in Δ, then $(f - I_\Delta f)^{(j)}(\xi) = 0$ for $j = 0, \ldots, m$.

For a sequence $(\Delta^n)_{n=0}^\infty$ of interpolation points $\Delta^n = (z_0^{(n)}, \ldots, z_n^{(n)})$ we can study the convergence behavior of $Q(f) - Q_{\Delta^n}(f)$ as a function of k and n. This analysis is the purpose of the present note.

It is well-known that for large $|k|$, the most important contribution to the integral comes from the endpoints. Hence, it is sensible to include as much endpoint information in the choice of the interpolant $I_\Delta f$ as possible. The extreme case is to take I_Δ as the interpolation operator of Hermite type associated with the endpoints ± 1; that is, $\Delta = \Delta_H^{2p-1} = (-1, \ldots, -1, 1, \ldots, 1)$, where each of the nodes ± 1 appears p times. The Filon quadrature $Q_{\Delta_H^{2p-1}}$, which we call "pure Filon quadrature" in this note, is shown to satisfy

$$|Q(f) - Q_{\Delta_H^{2p-1}}(f)| \le C \min\left\{q, \frac{\gamma(p+1)}{|k|}\right\}^{p+1} \tag{3}$$

for some constants C, q, $\gamma > 0$ independent of k and p (combine Thms. 2.1, 2.2). The parameter q is in general ≥ 1 so that convergence as $p \to \infty$ is not guaranteed—however, good approximations can be expected if $|k|$ is large compared to p. Theorem 2.2 gives explicit conditions on the domains of analyticity of f and g to ensure $q \in (0, 1)$, namely:

Assumption 1.2 *In addition to Assumption 1.1, the domain $\mathcal{G}_f$ satisfies*

$$\mathcal{G}_f \supset W_r^H \quad \text{for some } r > 1, \quad W_r^H := \{z \in \mathbb{C} \,|\, |z^2 - 1| < r^2\}. \tag{4}$$

If Assumption 1.2 is satisfied, then $q \in (0, 1)$ in (3) so that the pure Filon quadrature is a convergent (as $p \to \infty$) method whose *preasymptotic* behavior improves as $|k|$ becomes large.

Assumption 1.2 is seen in numerical examples (see Section 4) to be necessary for convergence of the pure Filon quadrature. Section 3.1 shows that the limitations imposed by Assumption 1.2 can be overcome with composite quadrature rules; an alternative, discussed in Section 3.2, is to insert additional quadrature points in the interior of the integration domain. Section 2.3 finally shows that by suitably clustering the quadrature points near the endpoints, Filon-type quadrature rules can be designed that do not require derivative information but still lead to convergent methods under Assumption 1.2.

Concluding this introduction, we mention that our analysis ignores several important aspects: Firstly, we do not discuss the issue of numerical stability; in particular, our numerical experiments in Section 4 are done in Maple with high precision arithmetic. Secondly, we skirt the so-called moment problem, i.e., the question of how to evaluate integrals $\int_{-1}^{1} e^{\mathbf{i}kg(x)}\pi(x)\,dx$ for polynomials π; one option is to use the classical method of steepest descent coupled with Gauß-Laguerre quadrature (see, e.g., [3] for a recent account of this procedure). Concerning notation: $B_\delta(z) \subset \mathbb{C}$ denotes an (open) ball of radius δ centered at z; for sets $A \subset \mathbb{C}$, we set $B_\delta(A) := \cup_{z\in A} B_\delta(z)$.

1.1 Reduction to interpolation error analysis

The simplest quadrature error estimate is

$$|Q(f)-Q_\Delta(f)| = |Q(f-I_\Delta f)| \leq \|f-I_\Delta f\|_{L^1(-1,1)} \leq 2\|f-I_\Delta f\|_{L^\infty(-1,1)}. \quad (5)$$

Integration by parts yields sharper bounds in terms of k:

Lemma 1.3 *Let Assumption 1.1 be valid. Set $\gamma_g := \|1/g'\|_{L^\infty(\mathcal{G}_g)}$. Let $\Delta = (z_0,\ldots,z_n) \subset [-1,1]$ be given. Let $J_0 \in \mathbb{N}_0$ be such that $(f - I_\Delta f)^{(j)}(\pm 1) = 0$ for $j = 0,\ldots,J_0-1$. Then, for every $J \in \mathbb{N}_0$ and $0 < \delta \leq \operatorname{dist}(\{\pm1\},\partial\mathcal{G}_g)$ and $0 < d \leq \operatorname{dist}([-1,1],\partial\mathcal{G}_g)$:*

$$|Q(f)-Q_\Delta(f)| \leq$$
$$\frac{\gamma_g}{|k|}\sum_{j=J_0}^{J-1}\left(\frac{j\gamma_g}{|k|\delta}\right)^j \|f-I_\Delta f\|_{L^\infty(B_\delta(\{\pm1\}))} + \left(\frac{J\gamma_g}{d|k|}\right)^J \|f-I_\Delta f\|_{L^\infty(B_d([-1,1]))}.$$

Proof: Let $\eta_0 := f - I_\Delta f$ and define the sequence $(\eta_j)_{j=0}^\infty$ of holomorphic functions by the recursion

$$\eta_{j+1} = \left(\frac{1}{g'}\eta_j\right)', \qquad j = 0,1,\ldots, \quad (6)$$

Integrating $Q(f) - Q_\Delta(f) = Q(f - I_\Delta f) = Q(\eta_0)$ by parts $J \in \mathbb{N}_0$ times gives

$$Q(\eta_0) = \frac{1}{\mathbf{i}k}\sum_{j=0}^{J-1}\left(\frac{-1}{\mathbf{i}k}\right)^j e^{\mathbf{i}kg}\frac{\eta_j}{g'}\bigg|_{-1}^{1} + \left(\frac{-1}{\mathbf{i}k}\right)^J \int_{-1}^{1} e^{\mathbf{i}kg(x)}\eta_J(x)\,dx. \quad (7)$$

The result now follows from Lemma 1.4 applied to $\eta_0 = f - I_\Delta f$. □

Lemma 1.4 *Let $G \subset \mathbb{C}$ be open and g, η_0 be holomorphic on G. Set $\gamma_g := \|1/g'\|_{L^\infty(G)}$. For $\delta > 0$ define $G_\delta := \{z \in G \mid \operatorname{dist}(z,\partial G) > \delta\}$. Starting from the function η_0, define the functions η_j, $j = 1,\ldots$ recursively by (6). Then:*

$$\|\eta_j\|_{L^\infty(G_\delta)} \leq \left(\frac{\gamma_g j}{\delta}\right)^j \|\eta_0\|_{L^\infty(G)}, \qquad j = 0,1,\ldots. \quad (8)$$

Here, we employ the convention $0^0 = 1$. If additionally $\eta^{(j)}(z_0) = 0$ for $j = 0, \ldots, p$ for a fixed $z_0 \in G$, then $\eta_j(z_0) = 0$ for $j = 0, \ldots, p$.

Proof: We proceed by induction on j. The statement is true for $j = 0$ by definition. Assuming it to be true for some $j \in \mathbb{N}_0$ and all $\delta > 0$, we get for every $x \in G_\delta$ and every $0 < \varepsilon < \delta$ by Cauchy's integral representation formula

$$|\eta_{j+1}(x)| = \left| \frac{1}{2\pi \mathbf{i}} \int_{\partial B_\varepsilon(x)} \frac{\eta_j(t)}{g'(t)(t-x)^2} \, dt \right| \leq \frac{1}{\varepsilon} \|1/g'\|_{L^\infty(G)} \|\eta_j\|_{L^\infty(G_{\delta-\varepsilon})} \tag{9}$$

$$\leq \frac{\gamma_g}{\varepsilon} \left(\frac{\gamma_g j}{\delta - \varepsilon} \right)^j \|\eta_0\|_{L^\infty(G)}. \tag{10}$$

If $j = 0$, we let $\varepsilon \to \delta$ in (9) to see that (8) is true for $j = 1$. For $j \geq 1$, we select $\varepsilon := \delta \frac{1}{j+1} < \delta$ to get from (10)

$$|\eta_{j+1}(x)| \leq \left(\frac{\gamma_g (j+1)}{\delta} \right)^{j+1} \|\eta_0\|_{L^\infty(G)}.$$

Noting that $x \in G_\delta$ is arbitrary, we can conclude the induction step. Finally, if η_0 and its derivatives up to order p vanish at z_0, then $\|\eta_0\|_{L^\infty(B_r(z_0))} \leq C r^{p+1}$ for all sufficiently small $r > 0$. Taking $G = B_r(z_0)$ and $\delta = r/2$ and letting $r \to 0$, the bound (8) then implies $\eta_j(z_0) = 0$ for $j = 0, \ldots, p$. $\square$

1.2 Polynomial interpolation

The following result is classical in polynomial interpolation of holomorphic functions and can be found, for example, in [1, Chap. IV]:

Proposition 1.5 *[Hermite] Let $f \in L^\infty(D(f))$ be holomorphic on the domain $D(f)$. Let $\Delta = (z_0, \ldots, z_n)$ and set $\omega_\Delta(z) := \prod_{i=0}^n (z - z_i)$. Then for every $z \in D(f)$ and every simple, closed Jordan curve $\mathcal{C} \subset D(f)$ with $\{z, z_0, \ldots, z_n\} \in \operatorname{Int}(\mathcal{C}) \subset D(f)$ there holds*

$$f(z) - I_\Delta f(z) = \frac{1}{2\pi \mathbf{i}} \int_{t \in \mathcal{C}} \frac{\omega_\Delta(z)}{\omega_\Delta(t)} \frac{f(t)}{t - z} \, dt. \tag{11}$$

Furthermore, for every compact $K \subset D(f)$ there exist C, $\gamma > 0$ depending solely on K and $D(f)$ such that if $\Delta \subset K$ then

$$|f(z) - I_\Delta f(z)| \leq C |\omega_\Delta(z)| \gamma^{n+1} \|f\|_{L^\infty(D(f))} \qquad \forall z \in K. \tag{12}$$

Proof: The representation (11) is taken from [1, Chap. IV]; to see (12), select a Jordan curve $\mathcal{C} \subset D(f)$ with $K \subset \operatorname{Int}(\mathcal{C})$. Then, $\operatorname{dist}(\mathcal{C}, \Delta) \geq \operatorname{dist}(\mathcal{C}, K) > 0$. Hence, $|\omega_\Delta(t)| \geq C \operatorname{dist}(\mathcal{C}, K)^{n+1}$, and (12) follows from (11). $\square$

Of particular interest here is Hermite interpolation in the endpoints given by

$$\Delta_H^{2p-1} := (\underbrace{-1, \ldots, -1}_{p \text{ times}}, \underbrace{1, \ldots, 1}_{p \text{ times}}). \tag{13}$$

The Hermite interpolation operator $I_{\Delta_H^{2p-1}} : C^{p-1}([-1,1]) \to \mathcal{P}_{2p-1}$ is characterized by the conditions

$$f^{(j)}(\pm 1) = (I_{\Delta_H^{2p-1}} f)^{(j)}(\pm 1) \qquad j = 0, \ldots, p-1. \tag{14}$$

Lemma 1.6 below will express the approximation properties of $I_{\Delta_H^{2p-1}}$ in terms of the functions ω_{2p}^H, ω^H, and the sets W_r^H:

$$\omega_{2p}^H(z) = (z+1)^p (z-1)^p = (z^2-1)^p, \qquad \omega^H(z) := |z^2-1|^{1/2}, \tag{15}$$

$$W_r^H = \{z \in \mathbb{C} \,|\, \omega^H(z) = \sqrt{|z^2-1|} < r\}. \tag{16}$$

The sets W_r^H are nested: $\operatorname{clo}(W_{r'}^H) \subset W_r^H$ for $r' < r$. We note that the interval $[-1,1]$ is only contained in the sets W_r^H for $r > 1$. Then, however, already the interval $[-\sqrt{2}, \sqrt{2}]$ is contained in W_r^H. The error representation (11) gives us:

Lemma 1.6 *Let f be holomorphic on the domain $D(f)$, and let $r > 1$ be such that $\operatorname{clo}(W_r^H) \subset D(f)$. Then for every $0 < r' < r$ there exists $C > 0$ (depending only on r, r') such that for every δ with $B_\delta(\{\pm 1\}) \subset W_{r'}^H$:*

$$\|f - I_{\Delta_H^{2p-1}} f\|_{L^\infty(W_{r'}^H)} \leq C \left(\frac{r'}{r}\right)^{2p} \|f\|_{L^\infty(W_r^H)} \qquad \forall p \in \mathbb{N}_0,$$

$$|(f - I_{\Delta_H^{2p-1}} f)(z)| \leq C \left(\frac{\delta(2+\delta)}{r^2}\right)^p \|f\|_{L^\infty(W_r^H)}, \qquad \forall z \in B_\delta(\{\pm 1\}).$$

A perturbation argument allows us to infer from Lemma 1.6 error bounds for interpolation that clusters points near the endpoints:

Lemma 1.7 *Let $1 < r' < r$ and $q \in ((r'/r)^2, 1)$. Then there exist constants δ_0, C, $\gamma > 0$, which depend only on r, r', q, such that the following is true: For any $\delta \in (0, \delta_0]$ and Δ^{2p-1} with $2p$ points satisfying*

$$\Delta^{2p-1} \textit{ has exactly } p \textit{ points in } B_\delta(-1) \textit{ and exactly points } p \textit{ in } B_\delta(1) \tag{17}$$

the interpolation error $f - I_\Delta f$ satisfies

$$\|f - I_\Delta f\|_{L^\infty(W^H_{r'})} \leq Cq^p \|f\|_{L^\infty(W^H_r)}, \tag{18}$$

$$|(f - I_\Delta f)(z)| \leq (\delta\gamma)^p \|f\|_{L^\infty(W^H_r)} \qquad \forall z \in B_\delta(\{\pm 1\}). \tag{19}$$

Proof: Let z_i, $i = 0, \ldots, p-1$ be the points of Δ^{2p-1} in $B_\delta(-1)$ and Let $\tilde{z}_i$, $i = 0, \ldots, p-1$ be the points of Δ^{2p-1} in $B_\delta(1)$. We then write for $z \neq \pm 1$:

$$\omega_{\Delta^{2p-1}}(z) = \prod_{i=0}^{p-1}(z - z_i) \prod_{i=0}^{p-1}(z - \tilde{z}_i) = (z^2 - 1)^p \prod_{i=0}^{p-1}\left(1 + \frac{-1 - z_i}{z+1}\right) \prod_{i=0}^{p-1}\left(1 + \frac{1 - \tilde{z}_i}{z-1}\right).$$

Hence, we can find $\gamma > 0$ depending only on r, r' such that for any $p \in \mathbb{N}_0$

$$(1 - \gamma\delta)^{2p} |\omega^H_{2p}(z)| \leq |\omega_{\Delta^{2p-1}}(z)| \leq (1 + \gamma\delta)^{2p} |\omega^H_{2p}(z)| \quad \forall z \in W^H_r \setminus W^H_{r'}. \tag{20}$$

Choosing δ sufficiently small, we can make $|\omega_{\Delta^{2p-1}}(z)/\omega^H_{2p}(z)|^{1/(2p)}$ arbitrarily close to 1 uniformly in $z \in W^H_r \setminus W^H_{r'}$. By the maximum modulus principle for holomorphic functions (18) is shown once $|f(z) - I_\Delta f(z)|$ can be bounded by the right-hand side of (18) for $z \in \partial W^H_{r'}$. However, this follows from (15), (20) and (11) with $\mathcal{C} = \partial W^H_r$. For (19), we insert into the error formula (11) the bound $|\omega_{\Delta^{2p-1}}(z)| \leq (2\delta)^p (2 + 2\delta)^p$ together with (20) and (15). $\square$

2 Quadrature error analysis

2.1 k-asymptotics

Theorem 2.1 (k-asymptotics) *Let Assumption 1.1 be valid and let $[-1,1] \subset K \subset \mathcal{G}_g$ be compact. Set $\gamma_g := \|1/g'\|_{L^\infty(\mathcal{G}_g)}$. Fix $\delta_0 < \min\{1, \operatorname{dist}(\{\pm 1\}, \partial\mathcal{G}_g)\}$. Then there exist constants $C, \gamma > 0$ that depend solely on K, $\operatorname{dist}(K, \partial\mathcal{G}_g) > 0$, and δ_0 such that for arbitrary interpolation points $\Delta = (z_0, \ldots, z_n)$ with $\Delta \subset K$ the following holds:*

(i) Let $p \in \mathbb{N}_0$ be such that $(f - I_\Delta f)^{(j)}(\pm 1) = 0$ for $0 \leq j \leq p-1$. Then

$$|Q(f) - Q_\Delta(f)| \leq C\gamma^{n+1} \min\left\{1, \left(\frac{\gamma_g(p+1)}{|k|}\right)^{p+1}\right\} \|f\|_{L^\infty(\mathcal{G}_g)}.$$

(ii) For $\delta > 0$ such that $\delta/|k| \leq \delta_0$ denote by $p_{-1,\delta} := \operatorname{card}\{i \mid 0 \leq i \leq n, z_i \in B_{\delta/|k|}(-1)\}$ and $p_{1,\delta} := \operatorname{card}\{i \mid 0 \leq i \leq n, z_i \in B_{\delta/|k|}(1)\}$ the number of

interpolation points in the $\delta/|k|$-neighborhoods of -1 and 1, respectively. Set $p_\delta := \min\{p_{-1,\delta}, p_{1,\delta}\}$. Then

$$|Q(f) - Q_\Delta(f)| \le C\gamma^{n+1}\frac{\gamma_g}{|k|}\min\left\{1, \left(\frac{\delta+\gamma_g(p_\delta+1)}{|k|}\right)^{p_\delta}\right\}\|f\|_{L^\infty(\mathcal{G}_g)}.$$

Proof: (5) together with (12) gives $|Q(f) - Q_\Delta(f)| \le C\gamma^{n+1}\|f\|_{L^\infty(\mathcal{G}_g)}$. To complete the proof of (i), we apply Lemma 1.3 with $J_0 = p$, $J = p+1$, $\delta = d = 1/2\,\mathrm{dist}([-1,1], \partial\mathcal{G}_g)$ and again (12) (with the compact set $\mathrm{clo}(B_d([-1,1]))$.

To see (ii), let $K' \subset \mathcal{G}_g$ be compact such that $B_{\delta_0}(\{\pm 1\}) \cup B_d([-1,1]) \subset K'$ for some $0 < d < 1/2$. (12) provides a constant (again denoted γ) such that for $z \in B_{\delta/|k|}(\{\pm 1\})$ we have $|(f - I_\Delta f)(z)| \le C(2\delta/|k|)^{p_\delta}\gamma^{n+1}\|f\|_{L^\infty(\mathcal{G}_g)}$ and $\|f - I_\Delta f\|_{L^\infty(K')} \le C\gamma^{n+1}\|f\|_{L^\infty(\mathcal{G}_g)}$. Lemma 1.3 with $J_0 = 0$, $J = 1$ gives

$$E := |Q(f) - Q_\Delta(f)| \le C\gamma^{n+1}\frac{\gamma_g}{|k|}\|f\|_{L^\infty(\mathcal{G}_g)}.$$

This is the first bound. Lemma 1.3 with $J_0 = 0$ and $J = p_\delta + 1$ gives

$$\begin{aligned} E &\le C\gamma^{n+1}\|f\|_{L^\infty(\mathcal{G}_g)}\left[\frac{\gamma_g}{|k|}\sum_{j=0}^{p_\delta}|k|^{-j}\left(\frac{\gamma_g j}{\delta/|k|}\right)^j\left(\frac{2\delta}{|k|}\right)^{p_\delta} + \left(\frac{(p_\delta+1)\gamma_g}{d|k|}\right)^{p_\delta+1}\right] \\ &\le C\gamma^{n+1}\frac{\gamma_g}{|k|}\left(\frac{\gamma_g}{|k|}\right)^{p_\delta}\|f\|_{L^\infty(\mathcal{G}_g)}\left(\sum_{j=0}^{p_\delta}\left(\frac{\gamma_g j}{\delta}\right)^j\left(\frac{2\delta}{\gamma_g}\right)^{p_\delta} + \left(\frac{p_\delta+1}{d}\right)^{p_\delta+1}\right). \end{aligned}$$

The convexity of the function $j \mapsto (j\gamma_g/\delta)^j(2\delta/\gamma_g)^{p_\delta}$ then gives us

$$\begin{aligned} E &\le C\gamma^{n+1}\frac{\gamma_g}{|k|}\left(\frac{\gamma_g}{|k|}\right)^{p_\delta}\|f\|_{L^\infty(\mathcal{G}_g)} \\ &\times\left((p_\delta+1)\max\left\{\left(\frac{2\delta}{\gamma_g}\right)^{p_\delta}, (2p_\delta)^{p_\delta}\right\} + \left(\frac{p_\delta+1}{d}\right)^{p_\delta+1}\right) \\ &\le C\gamma^{n+1}(p_\delta+1)\frac{\gamma_g}{|k|}\left(\frac{\gamma_g}{|k|}\right)^{p_\delta}\|f\|_{L^\infty(\mathcal{G}_g)}\left(\max\{\frac{2\delta}{\gamma_g}, \frac{p_\delta+1}{d}, 2p_\delta\}\right)^{p_\delta}. \end{aligned}$$

Noting $p_\delta \le n+1$ and adjusting the constant γ gives the desired bound. □

2.2 Convergent Filon quadrature

As Theorem 2.1 shows, it is advantageous to cluster interpolation points near the endpoints ± 1 for large $|k|$. Convergence (as $n \to \infty$) of Filon quadrature that is based on such interpolation point distribution can only be expected under suitable conditions on the size of the domain of analyticity of f. The following theorem shows that in the case of pure Filon quadrature the domain of analyticity $\mathcal{G}_f$ of f should contain W_1^H:

Theorem 2.2 (pure Filon quadrature) *Let Δ_H^{2p-1} be given by (13). Let Assumption 1.2 be valid. Set $\gamma_g := \|1/g'\|_{L^\infty(\mathcal{G}_g)}$. Then there exist C, $\gamma > 0$ (depending only on r and $\operatorname{dist}([-1,1], \partial\mathcal{G}_g)$) such that*

$$|Q(f) - Q_{\Delta_H^{2p-1}}(f)| \le C \left(\min \left\{ r^{-2}, \gamma \frac{\gamma_g(p+1)}{|k|} \right\} \right)^{p+1} \|f\|_{L^\infty(W_r^H)}.$$

Proof: (5) and Lemma 1.6 with $r' = 1$ (note: $[-1,1] \subset \operatorname{clo}(W_1^H)$) give the error bound $|Q(f) - Q_{\Delta_H^{2p-1}}(f)| \le Cr^{-2p}\|f\|_{L^\infty(W_r^H)}$. Adjusting the constant by the factor r^2 gives the first estimate. For the second one, select $0 < d \le \operatorname{dist}([-1,1], \partial\mathcal{G}_g)$ such that $B_d([-1,1]) \subset W_{r'}^H$ for some $r' < r$ (e.g., $r' = (1+r)/2$) and apply Lemma 1.6 to Lemma 1.3 with $J_0 = p$, $J = p+1$. □

2.3 Derivative-free Filon quadrature

Quadrature formulas that avoid knowledge of derivatives are of interest. [4] proposes to cluster the interpolation points near the endpoints ± 1. As in the case of the pure Filon quadrature, Assumption 1.2 is the key to ensure convergence as $n \to \infty$:

Theorem 2.3 *Let Assumption 1.2 be valid. Fix $q \in (1/r^2, 1)$. Set $\gamma_g := \|1/g'\|_{L^\infty(\mathcal{G}_g)}$. Then there exist δ_0, C, $\gamma > 0$ (all depending only on r, q, $\operatorname{dist}([-1,1], \partial\mathcal{G}_g)$) such that for any δ with $\delta/|k| \le \delta_0$ and any Δ^{2p-1} satisfying*

$$\Delta^{2p-1} \text{ has exactly } p \text{ points in } B_{\delta/|k|}(-1) \text{ and exactly } p \text{ points in } B_{\delta/|k|}(1) \quad (21)$$

the corresponding Filon quadrature $Q_{\Delta^{2p-1}}$ satisfies

$$|Q(f) - Q_{\Delta^{2p-1}}(f)| \le C \frac{\gamma_g}{|k|} \min \left\{ q, \gamma \frac{\delta + \gamma_g(p+1)}{|k|} \right\}^p \|f\|_{L^\infty(W_r^H)}.$$

Proof: The proof follows from the arguments given in the proof of Theorem 2.1 and an appeal to Lemma 1.7. □

The following point distribution guarantees a smooth transition from an extreme clustering near the endpoints ± 1 to the asymptotic distribution that essentially coincides with the Chebyshev points:

$$\Delta_C^{2p-1} := (z_0, \ldots, z_{p-1}, -z_{p-1}, -z_{p-2}, \ldots, -z_0), \tag{22a}$$

$$z_i = \cos\theta_i, \qquad \theta_i := i\delta_C, \qquad \delta_C := \frac{\pi}{2}\min\left\{\frac{\lambda}{|k|}, \frac{1}{p-1/2}\right\}; \tag{22b}$$

here $\lambda > 0$ is a user-chosen parameter. The Filon quadrature based on Δ_C^{2p-1} converges under the same conditions on f and g as the pure Filon quadrature:

Theorem 2.4 *Let Assumption 1.2 be valid. Then there exist $q \in (0,1)$, C, $\gamma > 0$, and an open neighborhood $\mathcal{U}$ of $[-1,1]$ (depending only on r and $\operatorname{dist}([-1,1], \partial\mathcal{G}_g)$) such that $f - I_{\Delta_C^{2p-1}} f$ with Δ_C^{2p-1} given by (22) satisfies*

$$\|f - I_{\Delta_C^{2p-1}} f\|_{L^\infty(\mathcal{U})} \le Cq^p \|f\|_{L^\infty(W_r^H)} \qquad \forall p \in \mathbb{N}_0. \tag{23}$$

Upon setting $\gamma_g := \|1/g'\|_{L^\infty(\mathcal{G}_g)}$, the Filon quadrature based on Δ_C^{2p-1} satisfies

$$|Q(f) - Q_{\Delta_C^{2p-1}}(f)| \le C\frac{\gamma_g}{|k|}\min\left\{q, \gamma\frac{\delta_C p|k| + \gamma_g(p+1)}{|k|}\right\}^p \|f\|_{L^\infty(\mathcal{G}_f)}. \tag{24}$$

Proof: See Appendix A. The result shows that the parameter λ should be chosen proportional to γ_g. The constants C, q, γ are independent of λ. □

3 Integrands with small domain of analyticity

Theorem 2.2 shows that the pure Filon quadrature converges as $n \to \infty$ provided the domain of analyticity $\mathcal{G}_f$ is sufficiently large—the numerical experiments in Section 4 indicate sharpness of the result. Theorem 2.4 shows that convergent derivative-free quadrature rules can be devised under the same regularity assumptions. As pointed out above, the condition $[-1,1] \subset W_r^H$ is only satisfied for $r > 1$, in which case already $[-\sqrt{2}, \sqrt{2}] \subset W_r^H$. Thus, the domain of analyticity of f and g cannot be an arbitrary open neighborhood of $[-1,1]$. We now discuss two options to create convergent Filon-type quadrature for integrands f whose domain of analyticity is just an open neighborhood of $[-1,1]$: In Section 3.1, we present composite Filon-type quadratures, and in Section 3.2 we insert additional quadrature points in the interval $[-1,1]$.

3.1 Composite Filon-type quadratures

Assumption 3.1 $Q_{k,p}^{ref}(f,g)$ *is a quadrature rule for integration on $[-1,1]$ such that for every f, g satisfying Assumption 1.2 there exist constants C,*

$\gamma > 0$, $q \in (0,1)$ *(all depending only on* $r > 1$ *and* $\operatorname{dist}([-1,1], \partial\mathcal{G}_g)$*) such that, upon setting* $\gamma_g := \|1/g'\|_{L^\infty(\mathcal{G}_g)}$*, there holds*

$$|Q(f) - Q^{ref}_{k,p}(f,g)| \le C \min\left\{q, \gamma \frac{\gamma_g(p+1)}{|k|}\right\}^{p+1} \|f\|_{L^\infty(\mathcal{G}_f)} \qquad \forall p \in \mathbb{N}_0. \quad (25)$$

Assumption 3.1 is satisfied in the settings of Theorems 2.2 and 2.4 (if the parameter λ is chosen proportional to γ_g). A composite quadrature rule is obtained in the usual way: For a partition $\mathcal{T}$ of $[-1,1]$ into elements K of size h_K and affine bijections $F_K : [-1,1] \to K$ the composite quadrature rule $Q_{\mathcal{T},k,p}$ is defined as

$$Q_{\mathcal{T},k,p}(f,g) := \sum_{K \in \mathcal{T}} \frac{h_K}{2} Q^{ref}_{k,p}(f|_K \circ F_K, g|_K \circ F_K).$$

Theorem 3.2 *Let* $Q^{ref}_{k,p}$ *satisfy Assumption 3.1. Let* f*,* g *satisfy Assumption 1.1. Set* $\gamma_g := \|1/g'\|_{L^\infty(\mathcal{G}_g)}$*. Assume that* $\mathcal{T}$ *satisfies, for some* $r > 1$*,*

$$F_K^{-1}(\mathcal{G}_f) \supset W^H_r \qquad \forall K \in \mathcal{T}. \quad (26)$$

Then, there exist constant C*,* $\gamma > 0$ *depending only on the constants appearing in Assumption 3.1 and* $\operatorname{dist}([-1,1], \partial\mathcal{G}_g)$ *such that*

$$|Q(f) - Q_{\mathcal{T},k,p}(f,g)| \le C \sum_{K \in \mathcal{T}} h_K \min\left\{q, \gamma \frac{\gamma_g(p+1)}{h_K |k|}\right\}^{p+1} \|f\|_{L^\infty(\mathcal{G}_f)}. \quad (27)$$

Proof: We observe that $(g|_K \circ F_K)' = \frac{h_K}{2} g' \circ F_K$. Hence, the constants γ_g appearing in (25) is adjusted by a factor $2/h_K$, which gives the claim. □

Geometric considerations give an easy sufficient condition for (26) to be met: Denoting $m_K \in K$ the mid point of the element K, then (26) is fulfilled if

$$\operatorname{clo}(B_{\frac{\sqrt{2}}{2} h_K}(m_K)) \subset \mathcal{G}_f \qquad \forall K \in \mathcal{T}. \quad (28)$$

3.2 Stabilized Filon-type quadratures

The composite quadrature that satisfies (26) can be viewed as inserting additional quadrature points in the interior of the integration domain. A similar effect can be achieved by combining Hermite interpolation with interpolation in, for example, the Gauß points. To fix ideas, denote by Δ^n_G the $n+1$ Gauß points and define, for a parameter $m \in \mathbb{N}_0$ to be chosen, the stabilized rule with points $\Delta_S^{mp+2p} := \Delta_G^{mp} \cup \Delta_H^{2p-1}$. That is, we use $mp + 2p + 1$ evaluations of f or its derivatives in the quadrature. The quadrature error then satisfies:

Theorem 3.3 (stabilized Filon-type quadrature) *Let Assumption 1.1 be valid. Set $\gamma_g := \|1/g'\|_{L^\infty(\mathcal{G}_g)}$. Then there exist constants C, $\gamma > 0$, $q \in (0,1)$, and $m \in \mathbb{N}_0$ depending only on $\mathcal{G}_f$ and $\mathcal{G}_g$ such that*

$$|Q(f) - Q_{\Delta_S^{mp+2p}}(f)| \le C \left(\min\left\{ q, \gamma \frac{\gamma_g(p+1)}{|k|} \right\} \right)^{p+1} \|f\|_{L^\infty(\mathcal{G}_f)}.$$

We note that the number of evaluations of f and its derivatives is $(2+m)p+1$.

Proof: Proceed as in the proof of Theorem 2.2. The key observation is that the asymptotic distribution of the Gauß points is known, [1, Thm. 12.4.5]. Specifically, for the Gauß points $z_i^{(n)}$, $i = 0, \dots, n$ and $\omega_{n+1}^G(z) := \prod_{i=0}^n (z - z_i^{(n)})$ we have $\lim_{n\to\infty} |\omega_{n+1}^G(z)|^{1/(n+1)} = \frac{1}{2}\rho(z)$; here, $\rho(z) > 1$ is determined by the condition $z \in \partial\mathcal{E}_{\rho(z)}$, where the ellipse $\mathcal{E}_\rho$ is given by $\mathcal{E}_\rho = \{z \in \mathbb{C} \,|\, |z-1| + |z+1| = \rho + 1/\rho\}$. For $\omega_{\Delta_S^{(m+2)p}}(z) := \omega_{2p}^H(z)\omega_{mp+1}^G(z)$ we compute

$$\omega_m^S(z) := \lim_{p\to\infty} |\omega_{\Delta_S^{(m+2)p}}(z)|^{1/(2p+mp+1)} = \left(\omega^H(z)\right)^{2/(m+2)} \left(\frac{1}{2}\rho(z)\right)^{m/(m+2)}. \quad (29)$$

The representation $\rho(z) = \zeta + \sqrt{\zeta^2 - 1}$ where $2\zeta = |z-1| + |z+1|$ shows that ω_m^S is a continuous function. Thus, the sets $W_r^S := \{z \in \mathbb{C} \,|\, \omega_m^S(z) < r\}$ are open. Note $[-1,1] \subset W_r^S$ for $r > 2^{-m/(m+2)}$. Fix $1 < \rho$ such that $\mathrm{clo}(\mathcal{E}_\rho) \subset D(f)$. Fix $1/2 < r < \rho/2$ and note that $\mathcal{E}_{\rho'} \subset \mathcal{E}_\rho$ for $\rho' < \rho$. Then (29) implies that for m sufficiently large, we have $[-1,1] \subset W_r^S \subset \mathcal{E}_\rho \subset D(f)$. The approximation properties of the interpolation operator $I_{\Delta_S^{(2+m)p}}$ now follow from Proposition 1.5. Noting $(f - I_{\Delta_S^{(2+m)p}} f)^{(j)}(\pm 1) = 0$ for $0 \le j \le p-1$ allows us to complete the proof by arguing as in the proof of Theorem 2.2. □

Remark 1 *Analogous results hold for Gauß-Lobatto or Chebyshev points.*

4 Numerical examples

All calculations in this section are done in MAPLE using a sufficient number of digits to be able to focus on the convergence properties of the Filon quadrature. In the Examples 4.1–4.3 we consider

$$g(x) = 1, \quad f_1(x) = (a - x^2)^{1/2} = (\sqrt{a} + x)^{1/2}(\sqrt{a} - x)^{1/2}, \quad a > 1. \quad (30)$$

Example 4.1 (pure Filon quadrature based on Δ_H^{2p-1}) Assumption 1.2 is only satisfied for $a > 2$. For the pure Filon quadrature, we therefore expect convergence (as $p \to \infty$) only for $a > 2$. In this case, we expect the *initial* convergence to be the more rapid the larger $|k|$ is. This is indeed visible in Fig. 1. For $a < 2$ Theorem 2.1 suggests, for a problem-dependent constant γ, rapid error decay for $p \leq \gamma|k|$ and error increase for $p > \gamma|k|$. This behavior is also visible in Fig. 1 for the case $a = 1.5$. ∎

Example 4.2 (Filon quadrature based on Δ_C^{2p-1}) Theorem 2.4 ensures uniform (in k) convergence of the method $Q_{\Delta_C^{2p-1}}$ for $a > 2$. This is visible in Fig. 2 for the case $a = 3$. For $a < 2$, Theorem 2.4 leads us to expect good results for k large compared to p and, since for $p \geq |k|/\lambda$ the points essentially coincide with the classical Chebyshev points, also good results in that regime. In the intermediate regime, the estimates of Theorem 2.4 permit large errors; indeed, these arise as shown in Fig. 2 for the case $a = 1.01$. The parameter λ appearing in (22) is chosen as $\lambda = 1$. ∎

Example 4.3 (composite pure Filon quadrature) Section 3.1 shows that composite Filon rules can make Filon quadrature applicable to integrands with singularities near the domain of integration. The condition to be satisfied is (26), or, more simply, (28). It is desirable to minimize the number of elements in the mesh $\mathcal{T}$ under the constraint (26). For the integrand given by (30), this can be achieved with geometric meshes that are refined towards the singularities of f: Let the mesh $\mathcal{T}^{geo}$ be defined by the points

$$\{-1, -1 + \sigma^i, \,|\, i = 0, \ldots, L\} \cup \{1, 1 - \sigma^i \,|\, i = 1, \ldots, L\}. \tag{31}$$

If $\sigma > (\sqrt{2} - 1)^2$, then—with the exception of the elements K abutting the endpoints ± 1—condition (26) is satisfied by all $K \in \mathcal{T}$ *regardless* of $a > 1$. Condition (26) is satisfied by the boundary elements only if L is sufficiently small. Sufficiently conditions for $\mathcal{T}^{geo}$ to perform well are therefore:

$$\sigma > (\sqrt{2} - 1)^2, \qquad L \geq L_0 \text{ with } \frac{\sqrt{2}}{2}\sigma^{L_0} < \sqrt{a} - 1 = \operatorname{dist}([-1, 1], \partial D(f)).$$

The numerical example shown in the bottom right part of Fig. 1 is done with $a = 1.01$, $\sigma = 0.2$, and $L = 4$ and a pure Filon quadrature on each element. Since we show relative errors, we mention that $Q(f) \approx 1.4$ for $k = 1$, $Q(f) \approx 0.01$ for $k = 10$, $Q(f) \approx -0.0025$ for $k = 100$. ∎

Example 4.4 (stabilized Filon quadrature) We consider the case

$$g(x) = 1, \qquad f_2(x) = (\sqrt{a} + x)^{1/2}, \qquad a > 1. \tag{32}$$

We employ the stabilized Filon quadrature Δ_{SC}^{mp+2p} based on Hermite interpo-

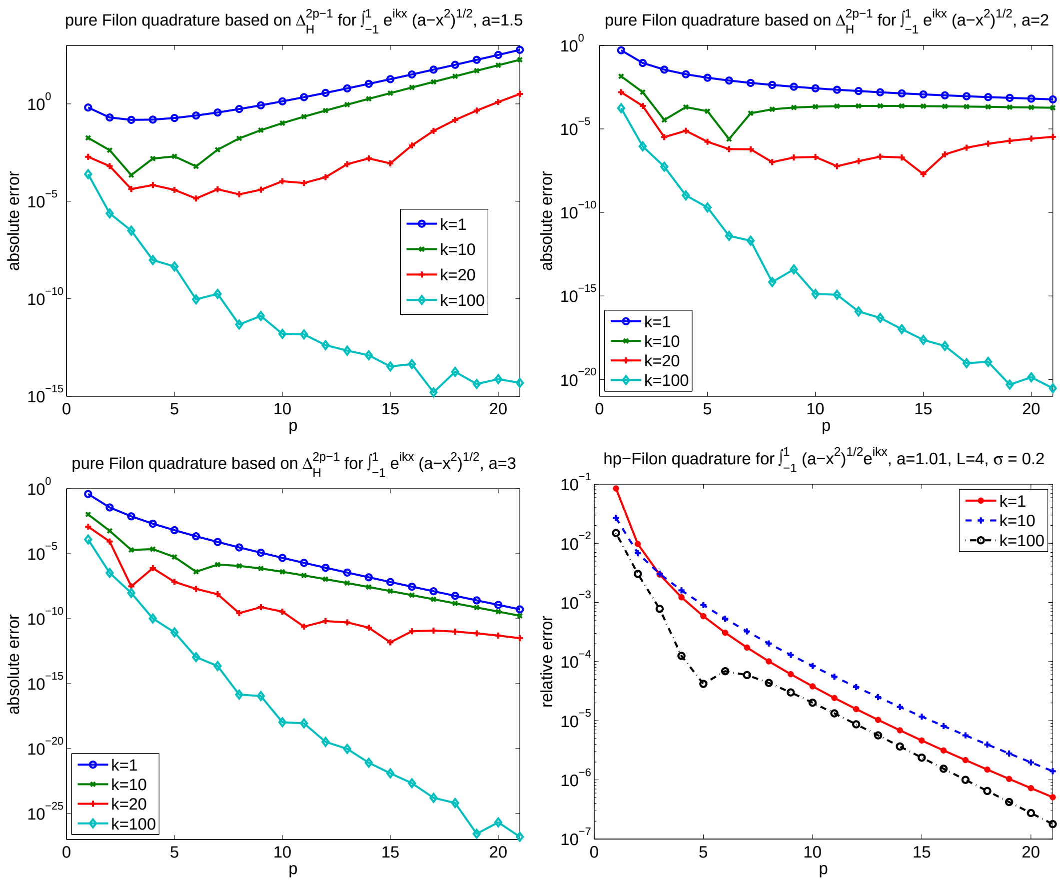

Fig. 1. Top row and bottom left: pure Filon quadrature based on Δ_H^{2p-1} for $a = 1.5$, $a = 2$, and $a = 3$. Bottom right: composite Filon quadrature with geometric mesh (31) for $L = 4$, $\sigma = 0.2$, $a = 1.01$.

lation in the endpoints and in the Chebyshev points:

$$\Delta_{SC}^{mp+2p} := \Delta_H^{2p-1} \cup \left\{ \cos\left(\frac{2i+1}{mp+1} \frac{\pi}{2} \right) \mid i = 0, \ldots, mp \right\}. \tag{33}$$

Assumption 1.2 is only satisfied for $a > 2$. Hence, convergence (as $p \to \infty$) cannot be guaranteed for $a = 1.7$ and $m = 0$; indeed Fig. 3 suggests divergence as $p \to \infty$ for $m = 0$. Convergence is ensured by selecting $m \geq 1$, which is visible in Fig. 3.
The error bound of Theorem 3.3 is of the form $u(p) := (\min\{q, \gamma(p+1)/|k|\})^{p+1}$ for some $q \in (0,1)$ and $\gamma > 0$. For q close to 1 the function u is decreasing on $(0, \frac{|k|}{e\gamma})$, increasing on $(\frac{|k|}{e\gamma}, \frac{q|k|}{\gamma})$ and decreasing on $(\frac{q|k|}{\gamma}, \infty)$. Qualitatively, such a behavior is visible in Fig. 3 for the case $a = 1.7$ and $m = 1$. It is worth noting that the range of p in which this undesirable behavior occurs is proportional to $|k|$. For sufficiently small q the function u is monotone. Indeed, the numerical experiment in the right part of Fig. 4 with $a = 4$ and $m = 1$ shows a better behavior. ■

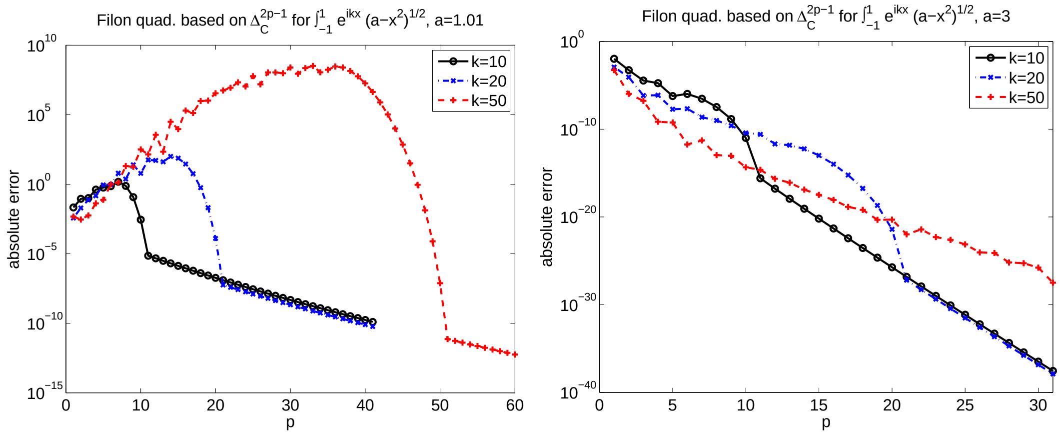

Fig. 2. Filon quadrature based on Δ_C^{2p-1} of (22) for $a = 1.01$ (left), $a = 3$ (right).

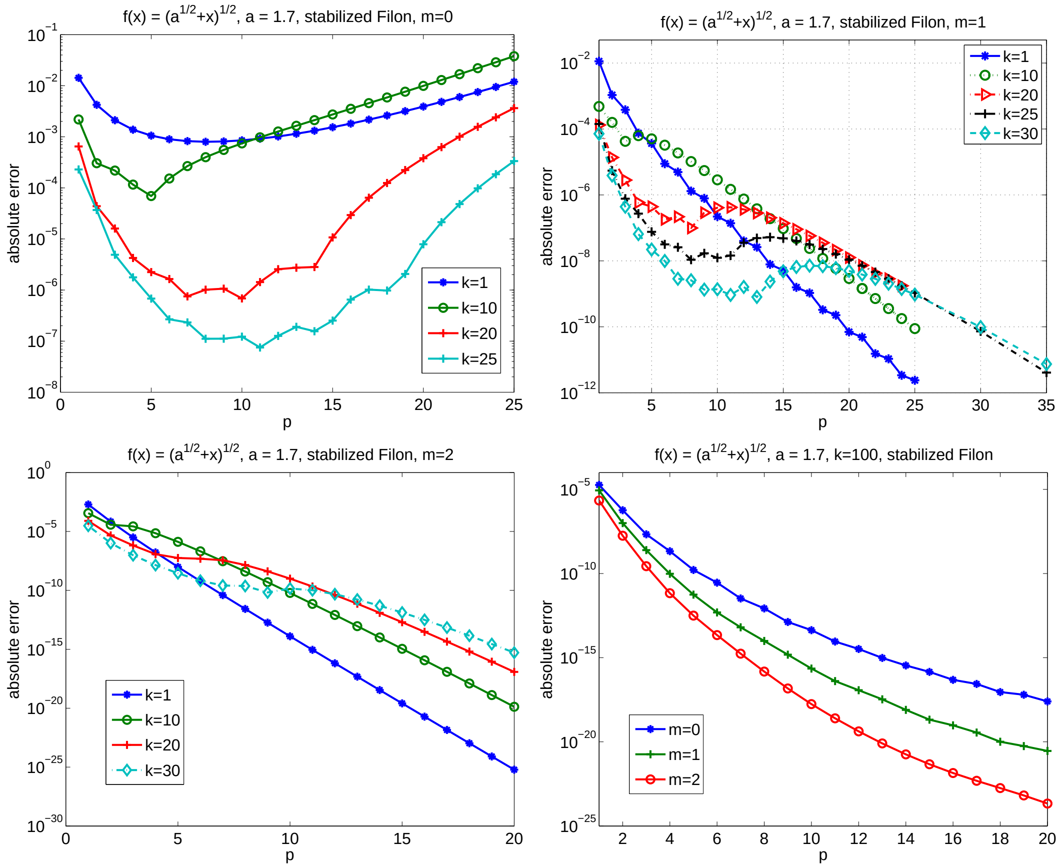

Fig. 3. Stabilized Filon quadrature based on Δ_{SC}^{mp+2p} for $a = 1.7$. Top row and bottom left: $m = 0$, $m = 1$, $m = 2$. Bottom right: $k = 100$ and $m \in \{0, 1, 2\}$.

A Proof of Theorem 2.4

Proof of Theorem 2.4: We start with the quadrature error analysis under the assumption that (23) is true. (23) together with Lemma 1.3 with $J_0 = 0$

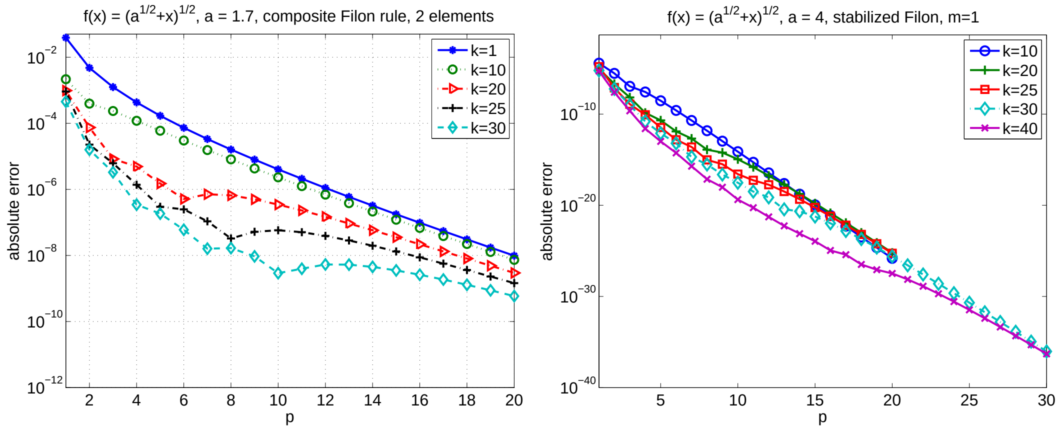

Fig. 4. Left: composite pure Filon quadrature based on 2 elements of equal length, $a = 1.7$. Right: stabilized Filon quadrature based on Δ_{SC}^{mp+2p} with $a = 4$, $m = 1$.

and $J = 1$ implies immediately implies

$$|Q(f) - Q_{\Delta_C^{2p-1}}(f)| \leq Cq^p \frac{\gamma_g}{|k|} \|f\|_{L^\infty(\mathcal{G}_f)} \qquad \forall p \in \mathbb{N}_0.$$

Next, we can apply Theorem 2.3 to infer the existence of $\delta_0 > 0$ such that for $p\delta_C \leq \delta_0$, we have

$$|Q(f) - Q_{\Delta_C^{2p-1}}(f)| \leq C\frac{\gamma_g}{|k|} \left(\gamma \frac{p\delta_C|k| + \gamma_g(p+1)}{|k|}\right)^p \|f\|_{L^\infty(\mathcal{G}_f)}.$$

Finally, for $p\delta_C > \delta_0$ we have

$$\gamma \frac{p\delta_C|k| + \gamma_g(p+1)}{|k|} \geq \gamma\delta_0 \geq \min\{q, \gamma\delta_0\} \geq q,$$

which completes the proof of the error bound (24).

It remains to show the bound (23). This is done in the classical way. First, we define for the points z_i given in (22) the functions

$$\omega_{\Delta_C^{2p-1}}(z) := \prod_{i=0}^{p-1}(z - z_i)(z + z_i) = \prod_{i=0}^{p-1}(z^2 - z_i^2) \tag{A.1}$$

The result now follows by inserting Lemma A.1 in Proposition 1.5. □

Lemma A.1 *Let the points z_i be given by (22) and let $r > 1$. Then there exist constants $C > 0$ and $q \in (0,1)$ and an open neighborhood $\mathcal{U}$ of $[-1,1]$ independent of p, k such that for all $z \in \mathcal{U}$ and all $t \in \partial W_r^H$*

$$\left|\frac{\omega_{\Delta_C^{2p-1}}(z)}{\omega_{\Delta_C^{2p-1}}(t)}\right| \leq Cq^p,$$

where $\omega_{\Delta_C^{2p-1}}$ *is defined in (A.1).*

Proof: *1. step:* From (15) we infer that for every $1 < r' < r$

$$\left|\frac{\omega_{\Delta_H^{2p-1}}(z)}{\omega_{\Delta_H^{2p-1}}(t)}\right| \le \left(\frac{r'}{r}\right)^{2p} \qquad \forall z \in \partial W_{r'}^H, \quad t \in \partial W_r^H.$$

(20) then implies the existence of $A_0 > 0$ and $q \in (0,1)$ such that $p\delta_C \le A_0$ we have

$$\left|\frac{\omega_{\Delta_C^{2p-1}}(z)}{\omega_{\Delta_C^{2p-1}}(t)}\right| \le q^p \qquad \forall z \in \partial W_{r'}^H, \quad t \in \partial W_r^H.$$

By the maximum modulus principle for holomorphic functions, the lemma is therefore true if $p\delta_C \le A_0$ for some $A_0 > 0$ depending on $r > 1$. Since by definition of δ_C, we have $p\delta_C \le \pi/2$ for all $p \in \mathbb{N}_0$, we are left with studying the case $0 < A_0 \le p\delta_C \le \pi/2$ for some fixed $A_0 > 0$.

2. step: In order to control the function

$$\omega_{p,C}(z) := \left|\omega_{\Delta_C^{2p-1}}(z)\right|^{1/(2p)}$$

we introduce the function $f_z(\theta) := \ln|z^2 - \cos^2\theta|$ to get

$$\ln \omega_{p,C}(z) = \frac{1}{2p\delta_C}\delta_C \sum_{i=0}^{p-1} \ln|z^2 - z_i^2| = \frac{1}{2p\delta_C}\int_0^{\delta_C p} f_z(\theta)\, d\theta + \frac{1}{2p\delta_C} R(z)$$

where R satisfies

$$|R(z)| \le \frac{1}{2}\delta_C p\delta_C \|f_z'\|_{L^\infty(0,p\delta_C)}.$$

3. step: We consider

$$\frac{1}{p\delta_C}\int_0^{p\delta_C} \ln|t^2 - \cos^2\theta| - \ln|z^2 - \cos^2\theta|\, d\theta$$

for $t \in \partial W_r^H$ and z in a neighborhood $\mathcal{U}$ of $[-1,1]$ to be fixed below. From $t \in \partial W_r^H$ we conclude $|t^2 - 1| = r^2$; hence, $|t^2 - \cos^2\theta| \ge |t^2-1| - (1-\cos^2\theta) = r^2 - 1 + \cos^2\theta$. Thus,

$$\frac{1}{p\delta_C}\int_0^{p\delta_C} \ln|t^2 - \cos^2\theta| - \ln|z^2 - \cos^2\theta|\, d\theta \ge \frac{1}{p\delta_C}\int_0^{p\delta_C} \ln\frac{r^2 - 1 + \cos^2\theta}{|z^2 - \cos^2\theta|}\, d\theta.$$

From Lemma A.2 we infer the existence of an open neighborhood $\mathcal{U}$ of $[-1,1]$ and a constant $\underline{c} > 0$ such that this last integral is bounded from below by $\underline{c} > 0$ uniformly in $z \in \mathcal{U}$ and $p\delta_C \in [A_0, \pi/2]$. It is now easy to show the existence of a constant $C > 0$ independent of $z \in \partial\mathcal{U}$ and $t \in \partial W_r$ such that

$$|R(z)| \le C\delta_C^2 p, \qquad |R(t)| \le C\delta_C^2 p.$$

We conclude that

$$\ln \frac{\omega_{p,C}(z)}{\omega_{p,C}(t)} \leq \frac{1}{2p\delta_C} \int_0^{p\delta_C} f_z(\theta) - f_t(\theta)\, d\theta + C\delta_C \leq -\underline{c} + C\delta_C$$

and therefore

$$\frac{|\omega_{\Delta_C^{2p-1}}(z)|}{|\omega_{\Delta_C^{2p-1}}(t)|} \leq e^{-\underline{c}2p} e^{2pC\delta_C}.$$

Since $\delta_C p \leq \pi/2$, we conclude the desired result for $z \in \partial\mathcal{U}$ and $t \in \partial W_r$. By the maximum modulus principle for holomorphic functions, we may extend the bound to all $z \in \mathcal{U}$. □

Lemma A.2 *Let $r > 1$ and $A_0 > 0$. Then there exists a constant $\underline{c} > 0$ and an open neighborhood $\mathcal{U} \subset \mathbb{C}$ of $[-1,1]$ such that*

$$\frac{1}{A}\int_0^A \ln\left|\frac{r-1+\cos^2\theta}{z^2-\cos^2\theta}\right| d\theta \geq \underline{c} > 0 \qquad \forall (A,z) \in [A_0,\pi/2]\times\mathcal{U}.$$

Proof: In view of the continuity assertion in Lemma A.3 and the fact that $(-z)^2 = z^2$, it suffices to ascertain

$$\frac{1}{A}\int_0^A \ln\left|\frac{r-1+\cos^2\theta}{z^2-\cos^2\theta}\right| d\theta > 0 \qquad \forall (A,z) \in [A_0,\pi/2]\times[0,1].$$

To see this pointwise bound, we start with some simple observations: First, if a function g is monotone increasing, then the function $A \mapsto \frac{1}{A}\int_0^A g(t)\,dt$ is likewise monotone increasing; if g is monotone decreasing, then $A \mapsto \frac{1}{A}\int_0^A g(t)\,dt$ is monotone decreasing. Next, for fixed $z \in [0,1]$, we write $z = \cos\widetilde{\theta}$ and compute the integral

$$\begin{aligned} I :=& \frac{2}{\pi}\int_0^{\pi/2} \ln|z^2-\cos^2\theta|\,d\theta = \frac{2}{4\pi}\int_0^{2\pi} \ln|\cos^2\widetilde{\theta}-\cos^2\theta|\,d\theta \\ =& \frac{2}{4\pi}\int_0^{2\pi} \ln\left|\sin(\theta+\widetilde{\theta})\right| + \ln\left|\sin(\theta-\widetilde{\theta})\right| d\theta \\ =& \frac{1}{\pi}\int_0^{2\pi} \ln|\sin x|\,dx = \frac{2}{\pi}\int_0^{\pi} \ln|\sin x|\,dx = -2\ln 2, \end{aligned} \tag{A.2}$$

where we used $\int_0^\pi \ln\sin x\,dx = -\pi\ln 2$ by [2, (4.38.19)]. Next, we observe that the function

$$\theta \mapsto h(\theta) := \left(\frac{r-1+\cos^2\theta}{z^2-\cos^2\theta}\right)^2$$

is monotone increasing on the interval $(0,\widetilde{\theta})$ and monotone decreasing on the interval $(\widetilde{\theta},\pi/2)$. Hence, we conclude immediately that for $0 < A \leq \widetilde{\theta}$ (note:

this implies $z \neq 1$):

$$\frac{1}{A}\int_0^A \ln h(\theta)\,d\theta \geq \lim_{A\to 0}\frac{1}{A}\int_0^A \ln h(\theta)\,d\theta = \ln h(0) = \ln\left(\frac{r}{z^2-1}\right)^2 > 0.$$

Additionally, the calculation in (A.2) shows that $\int_0^{\pi/2} \ln h(\theta)\,d\theta$ is independent of z. Hence, $\int_0^{\pi/2} \ln h(\theta)\,d\theta$ can be evaluated by selecting $z = 0$ to arrive at

$$\begin{aligned}
I_h := \int_0^{\pi/2} \ln h(\theta)\,d\theta &= \int_0^{\pi/2} \ln\left(\frac{r-1+\cos^2\theta}{\cos^2\theta}\right)^2 d\theta \\
&= \int_0^{\pi/2} \ln\left(1+\frac{r-1}{\cos^2\theta}\right)^2 d\theta > 0.
\end{aligned}$$

We turn to the case $\widetilde{\theta} < A \leq \pi/2$. Since h is monotone decreasing on $(\widetilde{\theta}, \pi/2)$, $\ln h$ is likewise monotone decreasing on $(\widetilde{\theta}, \pi/2)$ and therefore

$$\begin{aligned}
&\frac{1}{A}\int_0^A \ln h(\theta)\,d\theta = \frac{1}{A}\int_0^{\widetilde{\theta}} \ln h(\theta)\,d\theta + \frac{1}{A}\int_{\widetilde{\theta}}^A \ln h(\theta)\,d\theta \\
&= \frac{1}{A}\int_0^{\widetilde{\theta}} \ln h(\theta)\,d\theta + \frac{A-\widetilde{\theta}}{A}\frac{1}{A-\widetilde{\theta}}\int_{\widetilde{\theta}}^A \ln h(\theta)\,d\theta \\
&\geq \frac{1}{A}\int_0^{\widetilde{\theta}} \ln h(\theta)\,d\theta + \frac{A-\widetilde{\theta}}{A}\frac{1}{\pi/2-\widetilde{\theta}}\int_{\widetilde{\theta}}^{\pi/2} \ln h(\theta)\,d\theta \\
&= \frac{1}{A}\int_0^{\widetilde{\theta}} \ln h(\theta)\,d\theta + \frac{A-\widetilde{\theta}}{A}\frac{1}{\pi/2-\widetilde{\theta}}\left[\int_0^{\pi/2} \ln h(\theta)\,d\theta - \int_0^{\widetilde{\theta}} \ln h(\theta)\,d\theta\right] \\
&= \left[\frac{1}{A} - \frac{A-\widetilde{\theta}}{A}\frac{1}{\pi/2-\widetilde{\theta}}\right]\int_0^{\widetilde{\theta}} \ln h(\theta)\,d\theta + \frac{A-\widetilde{\theta}}{A}\frac{1}{\pi/2-\widetilde{\theta}}\int_0^{\pi/2} \ln h(\theta)\,d\theta \\
&= \frac{1}{A}\left(\frac{\pi/2-A}{\pi/2-\widetilde{\theta}}\int_0^{\widetilde{\theta}} \underbrace{\ln h(\theta)}_{>0}\,d\theta + \underbrace{\frac{A-\widetilde{\theta}}{\pi/2-\widetilde{\theta}}}_{>0}\underbrace{\int_0^{\pi/2} \ln h(\theta)\,d\theta}_{=I_h>0}\right)
\end{aligned}$$

This last expression is therefore positive. This concludes the proof. $\square$

Lemma A.3 *Let $A_0 > 0$ and $K \subset \mathbb{C}$ be compact. Then the function*

$$g : (A, z) \mapsto \frac{1}{A}\int_0^A \ln|z^2 - \cos^2\theta|\,d\theta$$

is continuous on $[A_0, \pi/2] \times K$.

Proof: *1. step:* For each $z \in K$ define $f_z : [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ by $f_z(x) := \ln|z^2 - x|$. For $\eta > 0$ define

$$f_z^\eta(x) := \begin{cases} f_z(x) & \text{if } |f_z(x)| \leq \eta \\ -\eta & \text{if } f_z(x) \leq -\eta \\ \eta & \text{if } f_z(x) \geq \eta \end{cases}$$

The Lebesgue dominated convergence theorem together with $A \geq A_0 > 0$ then easily implies that the function

$$g^\eta : (A,z) \mapsto \frac{1}{A}\int_0^A f_z^\eta(\cos^2\theta)\,d\theta = \frac{1}{A}\int_0^{\pi/2} \chi_{[0,A]} f_z^\eta(\cos^2\theta)\,d\theta$$

is continuous on $[A_0, \pi/2] \times K$. Here, χ_E denotes the characteristic function of the set E.

2. step: Denote $E_\eta := \{x \in [0,1] \,|\, |f_z(x)| \geq \eta\}$. Then

$$\begin{aligned} |g(A,z) - g^\eta(A,z)| &\leq \frac{1}{A_0}\int_0^{\pi/2} \chi_{E_\eta}(\cos^2\theta)|f_z(\cos^2\theta)|\,d\theta \\ &= \frac{1}{A_0}\int_0^1 \chi_{E_\eta}|f_z(x)|\frac{1}{\sqrt{x(1-x)}}\,dx = \frac{1}{A_0}\|\chi_{E_\eta} f_z \frac{1}{\sqrt{x(1-x)}}\|_{L^1(0,1)}. \end{aligned}$$

Fix p, q, $r \in (1,\infty)$ such that $r \in (1,2)$ and $1/p + 1/q + 1/r = 1$. The Hölder inequality then gives

$$|g(A,z) - g^\eta(A,z)| \leq \frac{1}{A_0}|E_\eta|^{1/p}\|f_z\|_{L^q(0,1)}\|\frac{1}{\sqrt{x(1-x)}}\|_{L^r(0,1)}.$$

By the choice $r \in (1,2)$ we have $\|\frac{1}{\sqrt{x(1-x)}}\|_{L^r(0,1)} < \infty$. Next, an elementary calculation shows

$$\|f_z\|_{L^q(0,1)}^q = \int_0^1 \left|\ln|z^2 - x|\right|^q\,dx \leq C$$

for a constant $C > 0$ that depends solely on K and the choice of q. Finally, for η sufficiently large (depending only on K) we have

$$|E_\eta| = |\{x : |z^2 - x| \leq e^{-\eta}\}| \leq |\{x : |\operatorname{Re} z^2 - x| \leq e^{-\eta/2}\}| \leq 2e^{-\eta/2}.$$

We conclude the existence of a constant $C > 0$ such that

$$|g(A,z) - g^\eta(A,z)| \leq Ce^{-\eta/(2p)} \qquad \forall (A,z) \in [A_0, \pi/2] \times K.$$

3. step: combining the result of the first and second step gives us that g is continuous on $[A_0, \pi/2] \times K$. $\square$

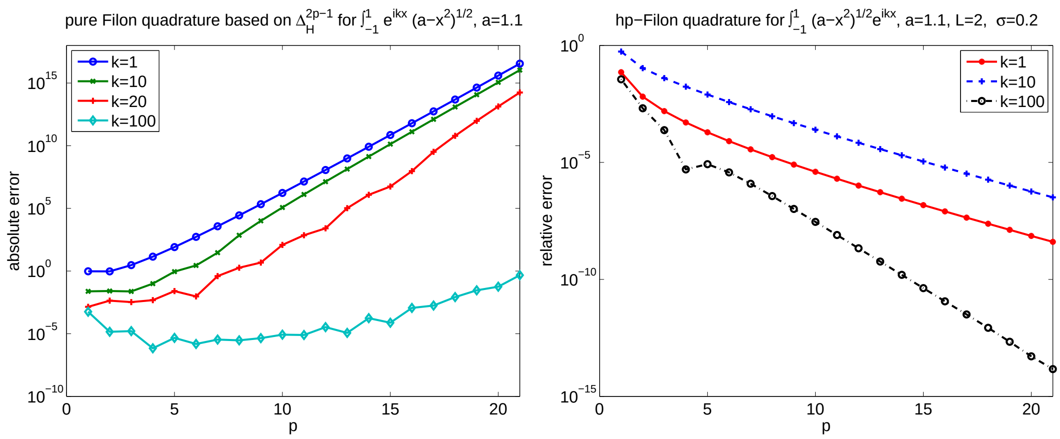

Fig. B.1. Case $a = 1.1$. Left: pure Filon quadrature $Q_{\Delta_H^{2p-1}}$. Right: Filon quadrature on geometric mesh (31) with $L = 2$, $\sigma = 0.2$

B Further numerical examples

Fig. B.1 shows the behavior of the pure Filon quadrature $Q_{\Delta_H^{2p-1}}$ for the case $a = 1.1$.

Fig. B.2 shows the behavior of the Filon quadrature based on Δ_C^{2p-1} of (22) (with $\lambda = 1$) for $a = 1.01$ and $a = 2$. Additionally, Fig. B.2 displays the behavior of $Q_{\Delta_C^{2p-1}}$ as a function of k for fixed $p \in \{1, 2, 3, 4\}$.

C The Moment Problem

There are several ways to compute efficiently the moments

$$m_n(k) := \int_{-1}^{1} e^{\mathbf{i}kx} L_n(x)\, dx.$$

One way is to employ the method of steepest descent as described in [3]. For the case $g(x) = x$, an alternative is to make use of three-term recurrence relations for orthogonal polynomials. For example, in exact arithmetic, there holds for the Legendre polynomials L_n:

$$\int_{-1}^{1} e^{\mathbf{i}kx} L_{n+1}(x)\, dx = -\frac{2n+1}{\mathbf{i}k} \int_{-1}^{1} e^{\mathbf{i}kx} L_n(x)\, dx + \int_{-1}^{1} e^{\mathbf{i}kx} L_{n-1}(x)\, dx.$$

This recursion appears to be stable for polynomial degrees up to the order of $|k|$. To illustrate this, let $m_m^{rec}(k)$ be the approximation to $m_m(k)$ obtained from the recurrence relation. We show in Table C.1 the errors $\max_{n=0,\dots,p} |m_n(k) - m_n^{rec}(k)|$ for different choices of p and k. The calculations are done in Matlab; the "exact" values $m_n(k)$ is obtained by an overkill Gaussian quadrature.

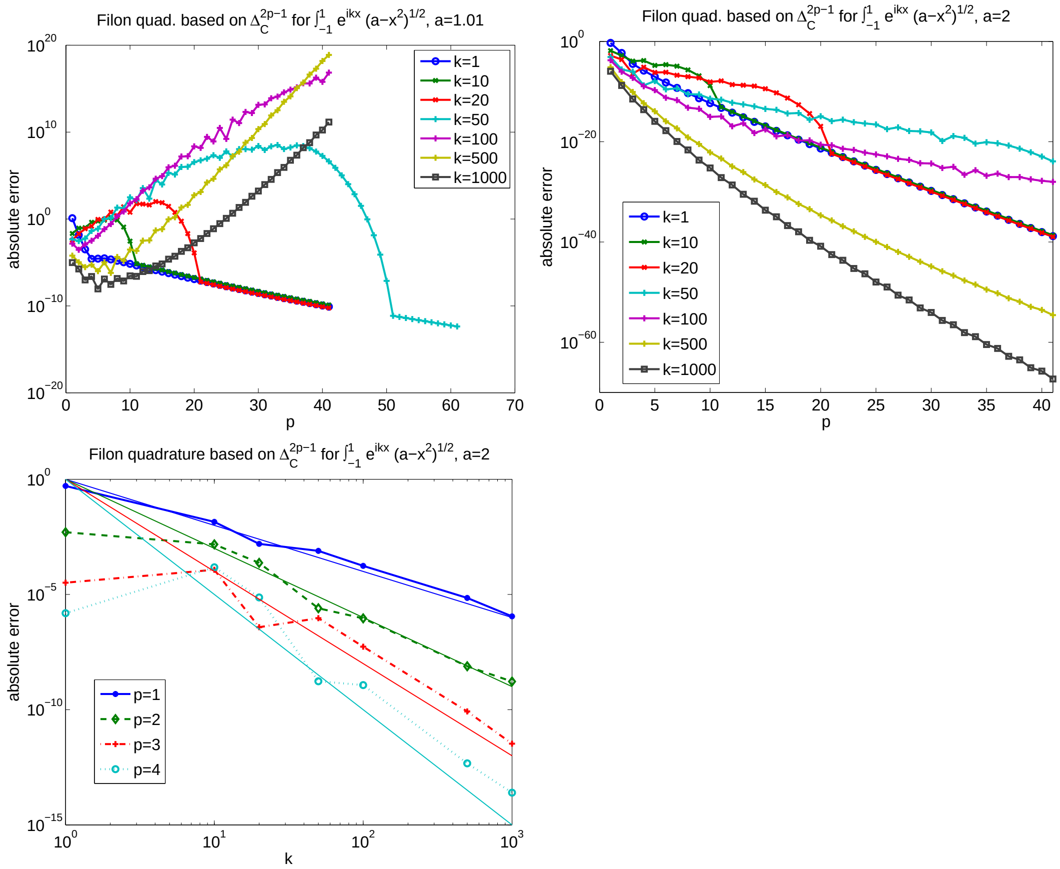

Fig. B.2. Numerical results for $Q_{\Delta_C^{2p-1}}$. Top: $a = 101$ and $a = 2$. Bottom: the case $a = 2$ for different values of fixed p.

	$k = -10$	$k = -100$
$p = \|k\| + 0$	1.6_{-16}	1.8_{-16}
$p = \|k\| + 10$	1.1_{-13}	3.4_{-16}
$p = \|k\| + 20$	3.7_{-7}	2.8_{-14}
$p = \|k\| + 30$	6.3_{1}	2.4_{-11}
$p = \|k\| + 40$	—	7.1_{-8}
$p = \|k\| + 50$	—	6.0_{-4}
$p = \|k\| + 60$	—	1.3_{1}

Table C.1
Stability of 3-term recurrence relation to determine moments.

D Stabilized Hermite interpolation

In this appendix, we work out some of the details of the proof of Theorem 3.3.

From [1, Chap. IV] we have the following result:

Lemma D.1 *Let $(\Delta^n)_{n=0}^{\infty}$ be a sequence of interpolation points such that the pointwise limit*

$$\omega(z) := \lim_{n\to\infty} |\omega_{\Delta^n}(z)|^{1/(n+1)}, \qquad \omega_{\Delta^n}(z) := \prod_{i=0}^{n}(z - z_i^{(n)})$$

exists. Set

$$W_r := \{z \in \mathbb{C} \,|\, \omega(z) < r\}$$

and assume that ∂W_r is a simple, closed Jordan curve. Assume additionally that $W_r \subset D(f)$ for some $r > 0$. Then for every $r' < r$ and $r'/r < q < 1$ there exists a constant $C > 0$ (independent of f) such that

$$\|f - I_{\Delta^n} f\|_{L^\infty(W_{r'})} \leq C_{r,r',q} \frac{1}{\operatorname{dist}(\partial W_r, W_{r'})} q^{n+1} \|f\|_{L^\infty(W_r)} \quad \forall n \in \mathbb{N}_0 \quad \text{(D.1)}$$

Denote by Δ_G^n the $n+1$ Gauß points [1] . We augment the interpolation points Δ_H^{2p-1} of (13) with Δ_G^{mp} for a parameter $m \in \mathbb{N}_0$. That is, we consider $\Delta_S^{(2+m)p} := \Delta_H^{2p-1} \cup \Delta_G^{mp}$. Define $\omega_{mp+1}^G(z) := \prod_{i=0}^{mp}(z - z_i)$. From [1, Thm. 12.4.5], we know that the Legendre polynomials L_n satisfy

$$\lim_{n\to\infty} |L_n(z)|^{1/n} = \rho \qquad \forall z \in \mathcal{E}_\rho,$$

where the ellipse $\mathcal{E}_\rho$ is defined as

$$\mathcal{E}_\rho = \{z \in \mathbb{C} \,|\, |z-1| + |z+1| = \rho + 1/\rho\}.$$

Since the leading coefficient of L_n is given by

$$\frac{(2n)!}{2^n (n!)^2} \sim 2^n$$

we get

$$\omega_n^G(z) = \frac{2^{n+1}((n+1)!)^2}{(2n+2)!} L_{n+1}(z)$$

and may conclude

$$\omega^G(z) := \lim_{n\to\infty} |\omega_n^G(z)|^{1/(n+1)} = \frac{1}{2}\rho(z), \qquad \rho(z) > 1 \text{ s.t. } z \in \partial\mathcal{E}_{\rho(z)}. \quad \text{(D.2)}$$

Elementary considerations show that ω^G can alternatively be written as

$$2\omega^G(z) = \zeta + \sqrt{\zeta^2 - 1}, \qquad \zeta = \frac{1}{2}\left\{|z-1| + |z+1|\right\}. \quad \text{(D.3)}$$

[1] analogous results can be obtained for Chebyshev or Gauß-Lobatto points

We remark that $I_{\Delta_S^{(2+m)p}} f \in \mathcal{P}_{2p+mp}$ is determined by

$$f^{(j)}(\pm 1) = (I_{\Delta_S^{(2+m)p}} f)^{(j)}(\pm 1), \quad j = 0, \dots, p-1, \tag{D.4a}$$
$$f(x_i) = (I_{\Delta_S^{(2+m)p}} f)(x_i), \quad i = 0, \dots, mp. \tag{D.4b}$$

The function $\omega_{\Delta_S^{(2+m)p}}$ associated with this interpolation operator is

$$\omega_{\Delta_S^{(2+m)p}}(z) = \omega_{2p}^H(z)\omega_{mp+1}^G(z) = (z^2-1)^p \prod_{i=0}^{mp}(z - x_i). \tag{D.5}$$

We can compute $\omega_m^S(z) := \lim_{p\to\infty} |\omega_{\Delta_S^{(2+m)p}}(z)|^{1/((2+m)p+1)}$ as follows:

$$\begin{aligned}
&\lim_{p\to\infty} |\omega_{\Delta_S^{(2+m)p}}(z)|^{1/(2p+1+mp)} = \lim_{p\to\infty} |\omega_{2p}^H(z)|^{1/(2p+1+mp)} |\omega_{mp+1}^G(z)|^{1/(2p+mp+1)} \\
&= \lim_{p\to\infty} |\sqrt{z^2-1}|^{2p/(2p+1+mp)} \lim_{p\to\infty} \left(|\omega_{mp+1}^G(z)|^{1/(mp+1)}\right)^{(mp+1)/(2p+mp+1)} \\
&= (\omega^H(z))^{2/(2+m)} (\omega^G(z))^{m/(2+m)}.
\end{aligned} \tag{D.6}$$

The parameter m allows us now to control the form of the sets $W_{r,m}^S = \{z \in \mathbb{C} \,|\, \omega_m^S(z) < r\}$:

Lemma D.2 *Let ω_m^S be as defined above. Then:*

(i) For each $z \in \mathbb{C} \setminus \{\pm 1\}$ there holds $\lim_{m\to\infty} \omega_m^S(z) = \frac{1}{2}\rho(z)$. The convergence is uniform in $z \in K$ for compact $K \subset \mathbb{C} \setminus \{\pm 1\}$.

(ii) For fixed $1 < \rho_1 < \rho < \rho_2$ and sufficiently large m we have $\mathcal{E}_{\rho_1} \subset W_{2\rho,m}^S \subset \mathcal{E}_{\rho_2}$.

Proof: For illustration purposes, we compare in Fig. D.1 the level lines of the sets W_r^H (corresponding to Hermite interpolation) with those of interpolation in the Gauß points.

For every compact $K \subset \mathbb{C} \setminus \{\pm 1\}$ we have $\lim_{m\to\infty} \left(\omega^H(z)\right)^{2/(2+m)} = 1$ *uniformly* in $z \in K$. Since $\omega^G(z) = \frac{1}{2}\rho(z)$ by (D.2) and ρ is continuous on K and bounded away from 0, the claim (i) follows.

For the second claim, we note that $\{z \in \mathbb{C} \,|\, \omega^G(z) < r\} = \mathcal{E}_{r/2}$. (ii) then follows from (i). $\square$

Lemma D.3 *One can choose $m \in \mathbb{N}_0$ sufficiently large such that the following holds: One can find constants $C > 0$, $q \in (0, 1)$ (depending on $D(f)$ and m)*

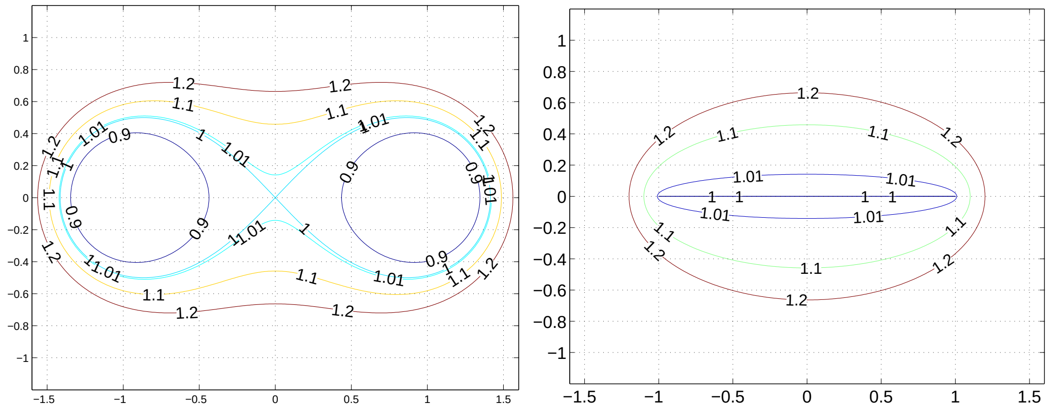

Fig. D.1. level lines of ω^H and ω^G.

such that in an open neighborhood $G \subset \mathbb{C}$ *with* $[-1,1] \subset G \subset D(f)$ *there holds*

$$\|f - I_{\Delta_S^{(2+m)p}} f\|_{L^\infty(G)} \leq C q^p \|f\|_{L^\infty(D(f))}.$$

Here, the set $\Delta^{(2+m)p}$ *has* $n = 2p + 1 + mp$ *point. In particular,*

$$(f - I_{\Delta^{(2+m)p}} f)^{(j)}(\pm 1) = 0, \qquad j = 0, \ldots, p-1.$$

Proof: Follows from Lemma D.2. □

References

[1] P.J. Davis. *Interpolation and Approximation*. Dover, 1974.

[2] I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series, and Products, corrected and enlarged edition*. Academic Press, New York, 1980.

[3] D. Huybrechs and S. Vanderwalle. The construction of cubature rules for multivariate highly oscillatory integrals. *Math. Comput.*, 76(260):1955–1980, 2007.

[4] A. Iserles and S.P. Nørsett. On quadrature methods for highly oscillatory integrals and their implementation. *BIT*, 44:755–772, 2004.

[5] A. Iserles and S.P. Nørsett. Efficient quadrature of highly oscillatory integrals using derivatives. *Proc. Roy. Soc. A*, 461:1383–1399, 2005.

[6] A. Iserles, S.P. Nørsett, and S. Olver. Highly oscillatory quadrature: The story so far. In A. Bermudez de Castro, editor, *Proceedings of ENuMath, Santiago de Compostela (2006)*, pages 97–118. Springer-Verlag, 2006.