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Analytic regularity for a singularly perturbed system of reaction-diffusion equations with multiple scales: proofs

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Abstract

We consider a coupled system of two singularly perturbed reaction-diffusion equations, with two small parameters $0 < \varepsilon \leq \mu \leq 1$, each multiplying the highest derivative in the equations. The presence of these parameters causes the solution(s) to have *boundary layers* which overlap and interact, based on the relative size of ε and μ. We construct full asymptotic expansions together with error bounds that cover the complete range $0 < \varepsilon \leq \mu \leq 1$. For the present case of analytic input data, we derive derivative growth estimates for the terms of the asymptotic expansion that are explicit in the perturbation parameters and the expansion order.

1 Introduction

Singularly perturbed (SP) boundary value problems (BVPs), and their numerical approximation, have received a lot of attention in the last few decades (see, e.g., the classical texts [13, 17] on asymptotic analysis and the books [10], [11], [14], whose focus is more on numerical methods for this problem class). One common feature that these problems share is the presence of *boundary layers* in the solution. In order for a numerical method, designed for the approximation of the solution to SP BVPs, to be considered *robust* it must be able to perform well independently of the singular perturbation parameter(s). To achieve this, information about the regularity of the exact solution is utilized, and in particular, bounds on the derivatives. Such information is available in the literature for scalar SP BPVs of reaction- and convection-diffusion type in one- and two-dimensions (see, e.g., [5, 7] for scalar versions of the problem studied in the present article). For *systems* of SP BVPs, the bibliography is scarce, even in one-dimension; see the relatively recent review article [4] and the references therein, for such results available to date. It is, therefore, the purpose of this article to begin filling this void; in particular, we provide the regularity theory for a system of two coupled SP linear reaction-diffusion equations, with two singular perturbation parameters. Our analysis is complete for the problem under consideration in that we derive full asymptotic expansions for all relevant cases of singular perturbation parameters and give explicit control of all derivatives of all terms appearing in the expansions. Even though this is a linear, one-dimensional problem, the methodology presented here can be the starting point for treating more difficult problems.

The regularity results obtained here are used in the companion communication [9] to prove, for the first time, exponential convergence of the *hp*-FEM for problems with multiple singular perturbation parameters. This exponential convergence result for the *hp*-FEM relies on mesh design principles firmly established for problems with a single singular perturbation parameter as discussed in [15, 16, 7, 5, 6]; the mathematical analysis of [9] shows that these mesh design principles extend to problems with multiple singular perturbation parameters and confirms the numerical results of [18] for the problem class under consideration here.

The rest of the paper is organized as follows: In Section 2 we present the model problem and discuss the typical phenomena. In Sections 3–6 we address the regularity of the solution, as it depends on the relationship between the singular perturbation parameters. The proofs of most of the results presented in these sections are technical and are relegated to Appendices A–D.

In what follows, the space of square integrable functions on an interval $I \subset \mathbb{R}$ will be denoted by $L^2(I)$, with associated inner product

$$(u, v)_I := \int_I u(x)v(x)dx.$$

We will also utilize the usual Sobolev space notation $H^k(I)$ to denote the space of functions on I with $0, 1, 2, ..., k$ generalized derivatives in $L^2(I)$, equipped with norm and seminorm $\|\cdot\|_{I,k}$ and $|\cdot|_{I,k}$, respectively. For vector functions $\mathbf{U} := (u_1(x), u_2(x))^T$, we will write

$$\|\mathbf{U}\|_{k,I}^2 = \|u_1\|_{k,I}^2 + \|u_2\|_{k,I}^2 .$$

We will also use the space

$$H_0^1(I) = \left\{ u \in H^1(I) : u|_{\partial I} = 0 \right\},$$

where ∂I denotes the boundary of I. For $z \in \mathbb{C}$, we use $\partial B_r(z)$ to denote the ball of radius r centered at z. Finally, the letter C will be used to denote a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence.

2 The Model Problem and Main Results

We consider the following model problem: Find a pair of functions $(u,v)^T$ such that

$$\left\{ \begin{array}{l} -\varepsilon^2 u''(x) + a_{11}(x)u(x) + a_{12}(x)v(x) = f(x) \text{ in } I = (0,1), \\ -\mu^2 v''(x) + a_{21}(x)u(x) + a_{22}(x)v(x) = g(x) \text{ in } I = (0,1), \end{array} \right. \tag{2.1a}$$

along with the boundary conditions

$$u(0) = u(1) = 0\ ,\ v(0) = v(1) = 0. \tag{2.1b}$$

With the abbreviations

$$\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbf{E}^{\varepsilon,\mu} := \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \mu^2 \end{pmatrix}, \qquad \mathbf{A}(x) := \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix},$$

equations (2.1a)–(2.1b) may also be written in the following, more compact form:

$$L_{\varepsilon,\mu}\mathbf{U} := -\mathbf{E}^{\varepsilon,\mu}\mathbf{U}''(x) + \mathbf{A}(x)\mathbf{U} = \mathbf{F}, \qquad \mathbf{U}(0) = \mathbf{U}(1) = 0. \tag{2.2}$$

The parameters $0 < \varepsilon \leq \mu \leq 1$ are given, as are the functions f, g, and a_{ij}, $i,j \in \{1,2\}$, which are assumed to be analytic on $\overline{I} = [0,1]$. Moreover we assume that there exist constants $C_f, \gamma_f, C_g, \gamma_g, C_a, \gamma_a > 0$ such that

$$\left\{ \begin{array}{r} \left\|f^{(n)}\right\|_{L^\infty(I)} \leq C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \left\|g^{(n)}\right\|_{L^\infty(I)} \leq C_g \gamma_g^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \left\|a_{ij}^{(n)}\right\|_{L^\infty(I)} \leq C_a \gamma_a^n n! \quad \forall\, n \in \mathbb{N}_0, i,j \in \{1,2\} \end{array} \right. . \tag{2.3}$$

The variational formulation of (2.1a)–(2.1b) reads: Find $\mathbf{U} := (u,v)^T \in \left[H_0^1(I)\right]^2$ such that

$$B(U,V) = F(V) \quad \forall\, \mathbf{V} := (\overline{u}, \overline{v}) \in \left[H_0^1(I)\right]^2, \tag{2.4}$$

where, with $\langle \cdot,\cdot \rangle_I$ the usual $L^2(I)$ inner product,

$$\begin{aligned} B(\mathbf{U},\mathbf{V}) &= \varepsilon^2 \left\langle u', \overline{u}' \right\rangle_I + \mu^2 \left\langle v', \overline{v}' \right\rangle_I + \left\langle a_{11}u + a_{12}v, \overline{u} \right\rangle_I + \left\langle a_{21}u + a_{22}v, \overline{v} \right\rangle_I, & (2.5) \\ F(\mathbf{V}) &= \left\langle f, \overline{u} \right\rangle_I + \left\langle g, \overline{v} \right\rangle_I. & (2.6) \end{aligned}$$

The matrix-valued function $\mathbf{A}$ is assumed to be pointwise positive definite, i.e., for some fixed $\alpha > 0$

$$\overrightarrow{\xi}^T \mathbf{A} \overrightarrow{\xi} \geq \alpha^2 \overrightarrow{\xi}^T \overrightarrow{\xi} \quad \forall\, \overrightarrow{\xi} \in \mathbb{R}^2 \qquad \forall x \in \overline{I}. \tag{2.7}$$

It follows that the bilinear form $B(\cdot,\cdot)$ given by (2.5) is coercive with respect to the *energy norm*

$$\|\mathbf{U}\|_{E,I}^2 \equiv \|(u,v)\|_{E,I}^2 := \varepsilon^2 |u|_{1,I}^2 + \mu^2 |v|_{1,I}^2 + \alpha^2 \left(\|u\|_{0,I}^2 + \|v\|_{0,I}^2 \right), \tag{2.8}$$

i.e.,

$$B(\mathbf{U},\mathbf{U}) \geq \|\mathbf{U}\|_{E,I}^2 \quad \forall\, \mathbf{U} \in \left[H_0^1(I)\right]^2.$$

This, along with the continuity of $B(\cdot,\cdot)$ and $F(\cdot)$, imply the unique solvability of (2.4). We also have, by the Lax-Milgram lemma, the following *a priori* estimate

$$\|\mathbf{U}\|_{E,I} \leq \max\left\{1, \frac{\|\mathbf{A}\|_{L^\infty(I)}}{\alpha}\right\} \sqrt{\|f\|_{0,I}^2 + \|g\|_{0,I}^2}. \tag{2.9}$$

For the development of certain asymptotic expansions, it will be convenient to observe that our assumption (2.7) implies (see Lemma A.2 for the proof)

$$a_{kk}(x) \geq \alpha \ \forall x \in \overline{I}, \quad k = 1,2, \tag{2.10}$$

$$\det \mathbf{A}(x) = a_{11}(x)a_{22}(x) - a_{12}(x)a_{21}(x) \geq \alpha \max\{a_{11}(x), a_{22}(x)\} \geq \alpha^2 \qquad \forall x \in \overline{I}. \tag{2.11}$$

We note that the special case when the parameters are equal, i.e. $\varepsilon = \mu$, was analyzed in [19]. In the general case considered here, there are three scales ($1 \geq \mu \geq \varepsilon$) and the regularity depends on how the scales are separated. Correspondingly, there are 4 cases:

(I) The "no scale separation case" which occurs when *neither* $\mu/1$ *nor* ε/μ is small.

(II) The "3-scale case" in which all scales are separated and occurs when $\mu/1$ is small *and* ε/μ is small.

(III) The first "2-scale case" which occurs when $\mu/1$ is not small *and* ε/μ is small.

(IV) The second "2-scale case" which occurs when $\mu/1$ is small *and* ε/μ is *not* small.

The concept of "small " (or "not small") mentioned above, is tied in two ways to our performing regularity theory in terms of asymptotic expansions. First, on the level of constructing asymptotic expansions, the decision which parameters are deemed small determines the ansatz to be made and thus the form of the expansion. Second, on the level of applying asymptotic expansions, the decision which parameters are deemed small depends on whether the remainder resulting from the asymptotic expansion can be regarded as small.

We need to introduce some notation:

Definition 2.1. *1. We say that a function w is analytic with length scale $\nu \geq 0$ (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{A}(\mu, C_w, \gamma_w)$, if*

$$\|w^{(n)}\|_{L^\infty(I)} \leq C_w \gamma_w^n \max\{n, \nu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0.$$

2. We say that that an entire function w is of L^∞-boundary layer type with length scale $\nu \geq 0$ (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{BL}^\infty(\nu, C_w, \gamma_w)$, if for all $x \in I$

$$|w^{(n)}(x)| \leq C_w \gamma_w^n \widetilde{\nu}^{-n} e^{-\operatorname{dist}(x,\partial I)/\nu} \qquad \forall n \in \mathbb{N}_0.$$

3. We say that that an entire function w is of L^2-boundary layer type with length scale $\nu > 0$ (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{BL}^2(\nu, C_w, \gamma_w)$, if for all $x \in I$

$$\|e^{\operatorname{dist}(x,\partial I)/\nu} w^{(n)}\|_{L^2(I)} \leq C_w \nu^{1/2} \gamma_w^n \nu^{-n} \qquad \forall n \in \mathbb{N}_0.$$

All three definitions extend naturally to vector-valued functions by requiring the above bounds componentwise.

In view of the length of the article, we collect the main result at this point; the four scale separation cases (I)–(IV) listed above correspond to the four cases listed in the following theorem:

Theorem 2.2. *There exist constants C, b, δ, q, $\gamma > 0$ independent of $0 < \varepsilon \le \mu \le 1$ such that the following assertions are true:*

(I) $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, \gamma)$.

(II) $\mathbf{U}$ *can be written as* $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, *where* $\mathbf{W} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\mu, C, \gamma)$ $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$, *and* $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \le C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right]$. *Additionally, the second component* $\widehat{v}$ *of* $\widehat{\mathbf{U}}$ *satisfies the sharper estimate* $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.

(III) If $\varepsilon/\mu \le q$ *then* $\mathbf{U}$ *can be written as* $\mathbf{U} = \mathbf{W} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, *where* $\mathbf{W} \in \mathcal{A}(\mu, C, \gamma)$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$, *and* $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \le Ce^{-b/\varepsilon}$. *Additionally, the second component* $\widehat{v}$ *of* $\widehat{\mathbf{U}}$ *satisfies the sharper estimate* $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.

(IV) $\mathbf{U}$ *can be written as* $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \mathbf{R}$, *where* $\mathbf{W} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\mu, C\sqrt{\mu/\varepsilon}, \gamma\mu/\varepsilon)$, *and* $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \le C\,(\mu/\varepsilon)^2\, e^{-b/\mu}$.

Proof. This result is obtained by combining Theorems 3.1, 4.1, 5.1, and 6.1 to be found in Sections 3–6. We emphasize that some results in these theorems are slightly sharper since they analyze all terms of the asymptotic expansions whereas Theorem 2.2 is obtained from the asymptotic expansions by suitably selecting the expansion order. □

3 The no scale separation case: Case I

In this case neither $\mu/1$ nor ε/μ is small, which means that the boundary layer effects are not very pronounced. By the analyticity of a_{ij}, f and g, we have that u and v are analytic. Moreover, we have the following theorem.

Theorem 3.1. *Let* (u, v) *be the solution to* (2.1a)–(2.1b) *with* $0 < \varepsilon \le \mu \le 1$. *Then there exist constants* C *and* $K > 0$, *independent of* ε *and* μ, *such that* $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, K)$ *satisfying the sharper estimate*

$$\left\|u^{(n)}\right\|_{0,I} + \left\|v^{(n)}\right\|_{0,I} \le CK^n \max\{n, \varepsilon^{-1}\}^n \quad \forall\, n \in \mathbb{N}_0. \tag{3.1}$$

Proof. The L^2-based estimate (3.1) was shown in [19] for the special case $\varepsilon = \mu$. The extension to current situation $\varepsilon \le \mu$ is straight forward. We note that the Sobolev embedding theorem in the form $\|v\|^2_{L^\infty(I)} \le C\|v\|_{L^2(I)}\|v\|_{H^1(I)}$ allows us to infer from (3.1) the assertion $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, \gamma)$ for suitable C, $\gamma > 0$ independent of ε and μ. □

4 The three scale case: Case II

In this case all scales are separated and it occurs when *both* $\mu/1$ and ε/μ are deemed small. This is arguably the most interesting (and challenging from the approximation point of view) case, since boundary layers of multiple scales appear. Additionally, this case shows most clearly the general procedure for obtaining asymptotic expansions and error bounds for problems with multiple scales. Before developing the asymptotic expansion, we formulate the main result:

Theorem 4.1. *The solution* $\mathbf{U}$ *of (2.1) can be written as* $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, *where* $\mathbf{W} \in \mathcal{A}(1, C_W, \gamma_W)$, $\widetilde{\mathbf{U}} \in \mathcal{BL}^\infty(\delta\mu, C_{BL}, \gamma_{BL})$, $\widehat{\mathbf{U}} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{BL}, \gamma_{BL})$ *for suitable constants* C_W, C_{BL}, γ_W, γ_{BL}, $\delta > 0$ *independent of* μ *and* ε. *Furthermore,* $\mathbf{R}$ *satisfies*

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|L_{\varepsilon,\mu}\mathbf{R}\|_{L^\infty(I)} \leq C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right],$$

for some constants C, $b > 0$ *independent of* μ *and* ε. *In particular,* $\|\mathbf{R}\|_{E,I} \leq C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right]$.

Additionally, the second component $\widehat{v}$ *of* $\widehat{U}$ *satisfies the sharper regularity assertion*

$$\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{BL}(\varepsilon/\mu)^2, \gamma_{BL}).$$

Proof. See Section 4.6. □

Anticipating that boundary layers of length scales $O(\mu)$ and $O(\varepsilon)$ will appear at the endpoints $x = 0$ and $x = 1$, we introduce the stretched variables $\widetilde{x} = x/\mu$, $\widehat{x} = x/\varepsilon$ for the expected layers at the left endpoint $x = 0$ and variables $\widetilde{x}^R = (1-x)/\mu$, $\widehat{x}^R = (1-x)/\varepsilon$ for the expected behavior at right endpoint $x = 1$. We make the following formal ansatz for the solution $\mathbf{U}$:

$$\mathbf{U} \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{\mu}{1}\right)^i\left(\frac{\varepsilon}{\mu}\right)^j\left[\mathbf{U}_{ij}(x) + \widetilde{\mathbf{U}}^L_{ij}(\widetilde{x}) + \widehat{\mathbf{U}}^L_{ij}(\widehat{x}) + \widetilde{\mathbf{U}}^R_{ij}(\widetilde{x}^R) + \widehat{\mathbf{U}}^R_{ij}(\widehat{x}^R)\right], \tag{4.1}$$

where the functions $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}^L_{ij}$, $\widehat{\mathbf{U}}^L_{ij}$ $\widetilde{\mathbf{U}}^R_{ij}$, $\widehat{\mathbf{U}}^R_{ij}$ will be determined shortly. The decomposition of Theorem 4.1 is obtained by truncating the asymptotic expansion (4.1) after a finite number of terms:

$$\mathbf{U}^M(x) := \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \mathbf{W}_M(x) + \widehat{\mathbf{U}}^M_{BL}(\widehat{x}) + \widehat{\mathbf{V}}^M_{BL}(\widehat{x}^R) + \widetilde{\mathbf{U}}^M_{BL}(\widetilde{x}) + \widetilde{\mathbf{V}}^M_{BL}(\widetilde{x}^R) + \mathbf{R}_M(x), \tag{4.2}$$

where

$$\mathbf{W}_M(x) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\begin{pmatrix} u_{ij}(x) \\ v_{ij}(x) \end{pmatrix}, \tag{4.3}$$

denotes the outer (smooth) expansion,

$$\widehat{\mathbf{U}}^M_{BL}(\widehat{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\begin{pmatrix} \widehat{u}^L_{ij}(\widehat{x}) \\ \widehat{v}^L_{ij}(\widehat{x}) \end{pmatrix}, \quad \widehat{\mathbf{V}}^M_{BL}(\widehat{x}^R) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\begin{pmatrix} \widehat{u}^R_{ij}(\widehat{x}^R) \\ \widehat{v}^R_{ij}(\widehat{x}^R) \end{pmatrix}, \tag{4.4}$$

denote the left and right inner (boundary layer) expansions associated with the variables $\widehat{x}$, $\widehat{x}^R$, respectively,

$$\widetilde{\mathbf{U}}^M_{BL}(\widetilde{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\begin{pmatrix} \widetilde{u}^L_{ij}(\widetilde{x}) \\ \widetilde{v}^L_{ij}(\widetilde{x}) \end{pmatrix}, \quad \widetilde{\mathbf{V}}^M_{BL}(\widetilde{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\begin{pmatrix} \widetilde{u}^R_{ij}(\widetilde{x}^R) \\ \widetilde{v}^R_{ij}(\widetilde{x}^R) \end{pmatrix}, \tag{4.5}$$

denote the left and right inner (boundary layer) expansions associated with the variables $\widetilde{x}$, $\widetilde{x}^R$ respectively, and

$$\mathbf{R}_M(x) := \begin{pmatrix} r_u(x) \\ r_v(x) \end{pmatrix} = \mathbf{U}(x) - \left(\mathbf{W}_M(x) + \widehat{\mathbf{U}}^M_{BL}(\widehat{x}) + \widehat{\mathbf{V}}^M_{BL}(\widehat{x}^R) + \widetilde{\mathbf{U}}^M_{BL}(\widetilde{x}) + \widetilde{\mathbf{V}}^M_{BL}(\widetilde{x}^R)\right) \quad (4.6)$$

denotes the remainder. Theorem 4.1 will be established by selecting $M_1 = O(1/\mu)$, $M_2 = O(\mu/\varepsilon)$.

4.1 Derivation of the asymptotic expansion

In order to derive equations for the functions $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}^L_{ij}$, $\widehat{\mathbf{U}}^L_{ij}$, $\widetilde{\mathbf{U}}^R_{ij}$, $\widehat{\mathbf{U}}^R_{ij}$, the procedure is as follows: first, the ansatz (4.1) is inserted in the differential equation (2.1a), then the scales are separated and finally recursions are obtained by equating like powers of μ and ε/μ.

In order to perform the scale separation, we need to write the differential operator $L_{\varepsilon,\mu}$ in different ways on the various scales. In particular, for the $\widetilde{x}$ and the $\widehat{x}$-scales, the coefficient $\mathbf{A}$ is written, by Taylor expansion, as

$$\mathbf{A}(x) = \sum_{k=0}^{\infty} \mu^k \mathbf{A}_k \widetilde{x}^k, \qquad \mathbf{A}_k := \mathbf{A}^{(k)}(0) = \begin{pmatrix} \frac{a_{11}^{(k)}(0)}{k!} & \frac{a_{12}^{(k)}(0)}{k!} \\ \frac{a_{21}^{(k)}(0)}{k!} & \frac{a_{22}^{(k)}(0)}{k!} \end{pmatrix}, \quad (4.7)$$

$$\mathbf{A}(x) = \sum_{k=0}^{\infty} \mu^k \left(\frac{\varepsilon}{\mu}\right)^k \mathbf{A}_k \widehat{x}^k. \quad (4.8)$$

Corresponding representations are obtained for the variables $\widetilde{x}^R$ and $\widehat{x}^R$ by expanding around the right endpoint $x = 1$. Hence, the differential operator $L_{\varepsilon,\mu}$ applied to a function depending on $\widetilde{x}$ or $\widehat{x}$ takes the following form:

$$\text{on the } \widetilde{x}\text{-scale:} \qquad -\mu^{-2}\mathbf{E}^{\varepsilon,\mu}\partial^2_{\widetilde{x}}\mathbf{U}(\widetilde{x}) + \sum_{k=0}^{\infty} \mu^k \mathbf{A}_k \widetilde{x}^k \mathbf{U}(\widetilde{x}), \quad (4.9)$$

$$\text{on the } \widehat{x}\text{-scale:} \qquad -\varepsilon^{-2}\mathbf{E}^{\varepsilon,\mu}\partial^2_{\widehat{x}}\mathbf{U}(\widehat{x}) + \sum_{k=0}^{\infty} \mu^k \left(\frac{\varepsilon}{\mu}\right)^k \mathbf{A}_k \widehat{x}^k \mathbf{U}(\widehat{x}). \quad (4.10)$$

Clearly, analogous forms exist for the operator on the $\widetilde{x}^R$ and $\widehat{x}^R$ scales. We now insert the ansatz (4.1) in the differential equation equation (2.1a), where the differential operator $L_{\varepsilon,\mu}$ takes the form given above on the fast scales $\widetilde{x}$, $\widehat{x}$, $\widetilde{x}^R$, $\widehat{x}^R$, and we separate the scales, i.e., we view the variables x, $\widetilde{x}$, $\widehat{x}$, $\widetilde{x}^R$, $\widehat{x}^R$ as independent variables. Then, we obtain

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \left[-\mathbf{E}^{\varepsilon,\mu}\mathbf{U}''_{ij} + \mathbf{A}(x)\mathbf{U}_{ij}\right] = \mathbf{F}, \quad (4.11)$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \left[-\mu^{-2}\mathbf{E}^{\varepsilon,\mu}(\widetilde{\mathbf{U}}^L_{ij})'' + \sum_{k=0}^{\infty} \mu^k \mathbf{A}_k \widetilde{x}^k \widetilde{\mathbf{U}}^L_{ij}\right] = 0, \quad (4.12)$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \left[-\varepsilon^{-2}\mathbf{E}^{\varepsilon,\mu}(\widehat{\mathbf{U}}^L_{ij})'' + \sum_{k=0}^{\infty} \varepsilon^k \mathbf{A}_k \widehat{x}^k \widehat{\mathbf{U}}^L_{ij}\right] = 0, \quad (4.13)$$

and two additional equations for $\widetilde{\mathbf{U}}^R$, $\widehat{\mathbf{U}}^R$ corresponding to the scales $\widetilde{x}^R$, $\widehat{x}^R$ that are completely analogous to (4.12), (4.13). We write

$$\mathbf{U}_{ij} = \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix}, \qquad \widetilde{\mathbf{U}}^L_{ij} = \begin{pmatrix} \widetilde{u}^L_{ij} \\ \widetilde{v}^L_{ij} \end{pmatrix}, \qquad \widehat{\mathbf{U}}^L_{ij} = \begin{pmatrix} \widehat{u}^L_{ij} \\ \widehat{v}^L_{ij} \end{pmatrix}, \tag{4.14}$$

and equate like powers of μ and ε/μ in (4.11), (4.12), (4.13) to get the following recursions:

$$-\begin{pmatrix} u''_{i-2,j-2} \\ v''_{i-2,j} \end{pmatrix} + \mathbf{A}(x)\mathbf{U}_{ij} = \mathbf{F}_{ij}, \tag{4.15a}$$

$$-\begin{pmatrix} (\widetilde{u}^L_{i,j-2})'' \\ (\widetilde{v}^L_{i,j})'' \end{pmatrix} + \sum_{k=0}^{i} \mathbf{A}_k \widetilde{x}^k \widetilde{\mathbf{U}}^L_{i-k,j} = 0, \tag{4.15b}$$

$$-\begin{pmatrix} (\widehat{u}^L_{i,j})'' \\ (\widehat{v}^L_{i,j+2})'' \end{pmatrix} + \sum_{k=0}^{\min\{i,j\}} \mathbf{A}_k \widehat{x}^k \widehat{\mathbf{U}}^L_{i-k,j-k} = 0, \tag{4.15c}$$

where we adopt the convention that if a function appears with a negative subscript, then it is assumed to be zero. Furthermore, we set

$$\mathbf{F}_{00} = \begin{pmatrix} f \\ g \end{pmatrix}, \qquad \mathbf{F}_{ij} = 0 \quad \text{if } (i,j) \neq (0,0).$$

The procedure so far leads to a formal solution $\mathbf{U}$ of the differential equation (2.1a); further boundary conditions are imposed in order to conform to the boundary conditions (2.1b), namely,

$$\mathbf{U}_{ij}(0) + \widetilde{\mathbf{U}}^L_{ij}(0) + \widehat{\mathbf{U}}^L_{ij}(0) = 0, \qquad \text{plus decay conditions for } \widetilde{\mathbf{U}}^L_{ij},\ \widehat{\mathbf{U}}^L_{ij} \text{ at } +\infty \tag{4.15d}$$

with analogous conditions at the right endpoint $x = 1$, which couple $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}^R_{ij}$, and $\widehat{\mathbf{U}}^R_{ij}$.

4.2 Analysis of the functions $\mathbf{U}_{ij}$

Since the matrix $\mathbf{A}(x)$ is invertible for every $x \in I$, equation (4.15a) may be solved for any i, j yielding

$$\begin{bmatrix} u_{0,0} \\ v_{0,0} \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} f \\ g \end{bmatrix}, \tag{4.16}$$

and for $(i,j) \neq (0,0)$

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} u''_{i-2,j-2} \\ v''_{i-2,j} \end{bmatrix}, \tag{4.17}$$

with, as mentioned above,

$$u_{ij} = 0\ ,\ v_{ij} = 0 \text{ if } i < 0 \text{ or } j < 0.$$

Note that (4.17) gives all the cases $(i,0)$ and $(0,j)$ because the right-hand side in (4.17) is known. Moreover, for each j, (4.17) allows us to compute $u_{ij}, v_{ij}\ \forall\ i$, thus (4.16)–(4.17) uniquely determine $u_{ij}, v_{ij}\ \forall\ i, j$.

We have the following lemma concerning the regularity of the functions $\mathbf{U}_{ij}$:

Lemma 4.2. *Let f, g, and $\mathbf{A}$ satisfy (2.3) and (2.7). Let u_{ij}, v_{ij} be the solutions of* (4.16), (4.17). *Then there exist positive constants C_S and K and a complex neighborhood G of the closed interval $\overline{I}$ independent of i and j such that*

$$u_{ij} = v_{ij} = 0 \;\forall\; j > i, \tag{4.18}$$

$$u_{ij} = v_{ij} = 0 \qquad \text{if } i \text{ or } j \text{ is odd}, \tag{4.19}$$

$$|u_{ij}(z)| + |v_{ij}(z)| \leq C_S \delta^{-i} K^i i^i \;\forall\; z \in G_\delta := \{z \in G\colon\ \operatorname{dist}(z, \partial G) > \delta\}. \tag{4.20}$$

Proof. The proof is by induction on i, where the estimate (4.20) follows by arguments of the type worked out in the proof of [5, Lemma 2]. For details, see Appendix D.1. □

4.3 Analysis of $\widetilde{\mathbf{U}}^L_{ij}$, $\widehat{\mathbf{U}}^L_{ij}$

4.3.1 Properties of some solution operators

Lemma 4.3. *Let $\alpha, a, b \in \mathbb{R}^+$. Then*

$$\begin{aligned}\int_x^\infty e^{-\alpha t}\,(a+bt)^i\,dt &\leq \frac{1}{\alpha}e^{-\alpha x}\sum_{\nu=0}^{i}(a+bx)^{i-\nu}\left(\frac{ib}{\alpha}\right)^\nu, \\ \int_x^\infty\int_t^\infty e^{-\alpha\tau}\,(a+b\tau)^i\,d\tau dt &\leq \frac{1}{\alpha^2}e^{-\alpha x}\sum_{\nu=0}^{i}\sum_{\ell=0}^{i-\nu}\left(\frac{ib}{\alpha}\right)^\nu\left((i-\nu)\frac{b}{\alpha}\right)^\ell(a+bx)^{i-\nu-\ell}.\end{aligned}$$

Proof. We have, after successive integrations by parts,

$$\begin{aligned}\int_x^\infty e^{-\alpha t}\,(a+bt)^i\,dt &= \frac{1}{\alpha}e^{-\alpha x}\sum_{\nu=0}^{i}(a+bx)^{i-\nu}\,i(i-1)\cdots(i-\nu+1)\left(\frac{b}{\alpha}\right)^\nu \\ &\leq \frac{1}{\alpha}e^{-\alpha x}\sum_{\nu=0}^{i}(a+bx)^{i-\nu}\left(\frac{ib}{\alpha}\right)^\nu.\end{aligned}$$

The statement about the double integral follows from this result. □

The above lemma can be formulated in the complex plane as follows.

Lemma 4.4. *Let α ,a, $b \in \mathbb{R}^+$ and v be holomorphic in the half-plane $\operatorname{Re} z > z_0$ and assume $|v(z)| \leq e^{-\alpha\operatorname{Re}(z)}\,(a+b\,|z|)^j$. Then for z with $\operatorname{Re} z > z_0$:*

$$\begin{aligned}\left|\int_z^\infty v(t)dt\right| &\leq \frac{1}{\alpha}e^{-\alpha\operatorname{Re}(z)}\sum_{\nu=0}^{j}(a+b\,|z|)^{j-\nu}\left(\frac{jb}{\alpha}\right)^\nu, \\ \left|\int_z^\infty\int_t^\infty v(\tau)d\tau dt\right| &\leq \frac{1}{\alpha^2}e^{-\alpha\operatorname{Re}(z)}\sum_{\nu=0}^{j}\sum_{\ell=0}^{j-\nu}\left(\frac{jb}{\alpha}\right)^\nu\left((j-\nu)\frac{b}{\alpha}\right)^\ell(a+b\,|z|)^{j-\nu-\ell}.\end{aligned}$$

Lemma 4.5. *Let $a > 0$, $g \in \mathbb{R}$, and f be entire satisfying*

$$|f(z)| \le C_f(q + |z|)^j \begin{cases} e^{-\underline{a}\operatorname{Re}(z)} & \textit{for } \operatorname{Re}(z) > 0 \\ e^{-\overline{a}\operatorname{Re}(z)} & \textit{for } \operatorname{Re}(z) < 0 \end{cases}$$

for some C_f, q, $\underline{a}$, $\overline{a}$, with $(a + \underline{a})q \ge 2j + 1$ and $0 < \underline{a} \le a \le \overline{a}$.

Then the solution u of

$$-u''(z) + a^2 u(z) = f(z), \qquad u(0) = g, \qquad \lim_{z\to\infty} u(z) = 0$$

satisfies the bound

$$|u(z)| \le C_f \left(\frac{1}{a}(q + |z|)^{j+1} \frac{1}{j+1} + |g| \right) \begin{cases} e^{-\underline{a}\operatorname{Re}(z)} & \textit{for } \operatorname{Re}(z) \ge 0 \\ e^{-\overline{a}\operatorname{Re}(z)} & \textit{for } \operatorname{Re}(z) < 0. \end{cases}$$

Proof. The proof follows from the appropriate modifications of [6, Lemma 7.3.6]. For details, see Appendix D.2

□

Lemma 4.6. *Let $0 < \underline{a} \le \overline{a}$. Let v be entire and satisfy for some $a > 0$, C_v, $b \ge 0$, $j \in \mathbb{N}_0$, the bound*

$$|v(z)| \le C_v(a + b|z|)^j \begin{cases} e^{-\underline{a}\operatorname{Re}(z)} & \textit{for } \operatorname{Re}(z) \ge 0 \\ e^{-\overline{a}\operatorname{Re}(z)} & \textit{for } \operatorname{Re}(z) < 0. \end{cases}$$

Assuming $\frac{jb}{a\underline{a}} < 1$, there holds

$$\left| \int_z^\infty \int_t^\infty v(\tau)\, d\tau\, dt \right| \le C_v \frac{1}{\underline{a}^2} \left(\frac{1}{1 - \frac{jb}{a\underline{a}}} \right)^2 (a + b|z|)^j \begin{cases} e^{-\underline{a}\operatorname{Re}(z)} & \textit{for } \operatorname{Re}(z) \ge 0 \\ e^{-\overline{a}\operatorname{Re}(z)} & \textit{for } \operatorname{Re}(z) < 0. \end{cases}$$

Proof. Follows essentially from Lemma 4.4. We sketch the argument for the case $\operatorname{Re}(z) < 0$. By linearity, we may assume $C_v = 1$. We start with the single integral $\int_z^\infty v(t)\, dt$, by selecting as the path of integration the line $z + \tau$, $\tau \in \mathbb{R}_+$, we get with the aid of Lemma 4.4,

$$\begin{aligned} \left| \int_z^\infty v(t)\, dt \right| &\le e^{-\overline{a}\operatorname{Re}(z)} \int_0^{-\operatorname{Re} z} e^{-\overline{a}\tau}(a + b|z| + b\tau)^j\, d\tau + e^{-\underline{a}\operatorname{Re}(z)} \int_{-\operatorname{Re} z}^\infty e^{-\underline{a}\tau}(a + b|z| + b\tau)^j\, d\tau \\ &\le e^{-\overline{a}\operatorname{Re}(z)} \int_0^\infty e^{-\underline{a}\tau}(a + b|z| + b\tau)^j\, d\tau \le e^{-\overline{a}\operatorname{Re}(z)} \frac{1}{\underline{a}} \sum_{\nu=0}^{j} (a + b|z|)^{j-\nu} \left(\frac{jb}{\underline{a}} \right)^\nu \\ &\le e^{-\overline{a}\operatorname{Re}(z)} \frac{1}{\underline{a}} (a + b|z|)^j \sum_{\nu=0}^{j} \left(\frac{jb}{\underline{a}(a + b|z|)} \right)^\nu \le e^{-\overline{a}\operatorname{Re}(z)} \frac{1}{\underline{a}} (a + b|z|)^j \frac{1}{1 - \frac{bj}{a\underline{a}}}. \end{aligned}$$

Inspection of the above derivation shows that for $\operatorname{Re}(z) \ge 0$, the same estimate holds with $e^{-\overline{a}\operatorname{Re}(z)}$ replaced by $e^{-\underline{a}\operatorname{Re}(z)}$. We may therefore repeat the argument once more for the function $z \mapsto \int_z^\infty v(t)\, dt$ to get the claimed estimate. □

Lemma 4.7. *Let the entire function v satisfy the hypotheses stated in Lemma 4.6. Then*

$$|v''(z)| \le 2e^{\overline{a}} C_v (a+b+b|z|)^j \begin{cases} e^{-\underline{a}\,\mathrm{Re}(z)} & \text{for } \mathrm{Re}(z) \ge 0 \\ e^{-\overline{a}\,\mathrm{Re}(z)} & \text{for } \mathrm{Re}(z) < 0 \end{cases} \qquad \forall z \in \mathbb{C}.$$

Proof. Follows from Cauchy's formula for derivatives by taking $\partial B_1(z)$ as the contour. ☐

Lemma 4.8. *For C_1, $\beta > 0$ and $x \ge 0$ the following estimates are valid with $\gamma = 2\max\{1, C_1^2\}$:*

$$(C_1\ell + \widehat{x})^{2\ell} \le 2^\ell (C_1\ell)^{2\ell} + 2^\ell \widehat{x}^{2\ell} \le \gamma^\ell \left(\ell^{2\ell} + \widehat{x}^{2\ell}\right), \tag{4.21}$$

$$\sup_{x>0} x^n e^{-\beta/4x} \le \left(\frac{4n}{e\beta}\right)^n. \tag{4.22}$$

Proof. The result follow from elementary considerations. ☐

4.3.2 Regularity of the functions $\widetilde{\mathbf{U}}_{ij}^L$ and $\widehat{\mathbf{U}}_{ij}^L$

We turn our attention to equations (4.15b) and (4.15c), which, after introducing appropriate boundary conditions, determine $\widetilde{u}_{ij}^L$, $\widetilde{v}_{ij}^L$ and $\widehat{u}_{ij}^L$, $\widehat{v}_{ij}^L$, respectively. These equations turn out to be systems of differential-algebraic equations (DAEs); however, their structure is such that the algebraic side constraint of the DAE can be eliminated explicitly and, additionally, we will be able to solve for 4 scalar functions sequentially instead of having to consider the coupled system. We recall that the functions $\mathbf{U}_{ij} = (u_{ij}, v_{ij})^T$ have been defined and studied in Section 4.2.

These equations may be solved by induction on j and i. For $j = 0$, we solve (4.15b) for any $(i, 0)$ by first solving for $\widetilde{u}_{i,0}^L$ and inserting it into the equation for $\widetilde{v}_{i,0}^L$. We have from (4.15b, 1st eqn)

$$\widetilde{u}_{i,0}^L = -\frac{a_{12}(0)}{a_{11}(0)}\widetilde{v}_{i,0}^L - \frac{1}{a_{11}(0)}\sum_{k=1}^{i}\frac{\widetilde{x}^k}{k!}\left[a_{11}^{(k)}(0)\widetilde{u}_{i-k,0}^L + a_{12}^{(k)}(0)\widetilde{v}_{i-k,0}^L\right], \tag{4.23}$$

which, upon inserted into (4.15b, 2nd eqn) gives

$$-\left(\widetilde{v}_{i,0}^L\right)'' + \frac{a_{11}(0)a_{22}(0) - a_{21}(0)a_{12}(0)}{a_{11}(0)}\widetilde{v}_{i,0}^L \tag{4.24a}$$

$$= \sum_{k=1}^{i}\frac{\widetilde{x}^k}{k!}\left[\left(\frac{a_{21}(0)}{a_{11}(0)}a_{11}^{(k)}(0) - a_{21}^{(k)}(0)\right)\widetilde{u}_{i-k,0}^L + \left(\frac{a_{21}(0)}{a_{11}(0)}a_{12}^{(k)}(0) - a_{22}^{(k)}(0)\right)\widetilde{v}_{i-k,0}^L\right].$$

The above second order differential equation is now posed as an equation in $(0, \infty)$ and supplemented with the two "boundary" conditions

$$\widetilde{v}_{i,0}^L(0) = -v_{i,0}(0), \qquad \widetilde{v}_{i,0}^L(\widetilde{x}) \to 0 \qquad \text{as } \widetilde{x} \to \infty. \tag{4.24b}$$

So, solving (4.24) gives us $\widetilde{v}_{i,0}^L$ and then from (4.23) we get $\widetilde{u}_{i,0}^L$. Inductively, we obtain $\widetilde{v}_{i,0}^L$ and $\widetilde{u}_{i,0}^L$ for all $i \ge 0$.

Next, we set

$$\widehat{v}_{i,0}^{L} = \widehat{v}_{i,1}^{L} = 0, \tag{4.25}$$

and we solve with $j = 0$ (4.15c, 1st eqn) for $\widehat{u}_{i,0}^{L}$ (using $\widehat{v}_{i,0}^{L} = 0$) with boundary conditions from $u_{i,0}$:

$$\begin{cases} -\left(\widehat{u}_{i,0}^{L}\right)'' + a_{11}(0)\widehat{u}_{i,0}^{L} = 0 \\ \widehat{u}_{i,0}^{L}(0) = -u_{i,0}, \qquad \widehat{u}_{i,0}^{L}(\widehat{x}) \to 0 \quad \text{for } \widehat{x} \to \infty. \end{cases} \tag{4.26}$$

Then, we solve (4.15c, 2nd eqn) for $\widehat{v}_{i,2}^{L}$:

$$\widehat{v}_{i,2}^{L}(z) = \int_z^\infty \int_t^\infty a_{21}(0)\widehat{u}_{i,0}^{L}(\tau)d\tau dt. \tag{4.27}$$

In general, assume we have performed the previous steps and we have determined $\widetilde{u}_{i,j}^{L}$, $\widetilde{v}_{i,j}^{L}$, $\widehat{u}_{i,j}^{L}$, $\widehat{v}_{i,j+2}^{L}$ for all $i \geq 0$ and second index up to j. To obtain the corresponding functions (with j replaced by $j+1$) we proceed analogously. We first solve (4.15b, 1st eqn) for $\widetilde{u}_{i,j+1}^{L}$,

$$\widetilde{u}_{i,j+1}^{L} = -\frac{a_{12}(0)}{a_{11}(0)}\widetilde{v}_{i,j+1}^{L} + \frac{\left(\widetilde{u}_{i,j-1}^{L}\right)''}{a_{11}(0)} - \frac{1}{a_{11}(0)}\sum_{k=1}^{i}\frac{\widetilde{x}^k}{k!}\left[a_{11}^{(k)}(0)\widetilde{u}_{i-k,j+1}^{L} + a_{12}^{(k)}(0)\widetilde{v}_{i-k,j+1}^{L}\right], \tag{4.28}$$

and plug it into (4.15b, 2nd eqn):

$$\begin{aligned} -\left(\widetilde{v}_{i,j+1}^{L}\right)'' + \frac{a_{11}(0)a_{22}(0) - a_{21}(0)a_{12}(0)}{a_{11}(0)}\widetilde{v}_{i,j+1}^{L} = -\frac{a_{21}(0)}{a_{11}(0)}\left(\widetilde{u}_{i,j-1}^{L}\right)'' + \\ +\sum_{k=1}^{i}\frac{\widetilde{x}^k}{k!}\left[\left(\frac{a_{21}(0)}{a_{11}(0)}a_{11}^{(k)}(0) - a_{21}^{(k)}(0)\right)\widetilde{u}_{i-k,j+1}^{L} + \left(\frac{a_{21}(0)}{a_{11}(0)}a_{12}^{(k)}(0) - a_{22}^{(k)}(0)\right)\widetilde{v}_{i-k,j+1}^{L}\right]. \end{aligned} \tag{4.29a}$$

The second order ODE, equation (4.29a), is supplemented with the boundary conditions

$$\widetilde{v}_{i,j+1}^{L}(0) = -\left(v_{i,j+1}(0) + \widehat{v}_{i,j+1}^{L}(0)\right), \qquad \widetilde{v}_{i,j+1}^{L}(\widetilde{x}) \to 0 \quad \text{for } \widetilde{x} \to \infty. \tag{4.29b}$$

This gives us $\widetilde{v}_{i,j+1}^{L}$ and in turn $\widetilde{u}_{i,j+1}^{L}$ from (4.28).

Next, we solve (4.15c, 1st eqn) for $\widehat{u}_{i,j+1}^{L}$ with boundary conditions from $u_{i,j+1}$ and $\widetilde{u}_{i,j+1}^{L}$:

$$\begin{aligned} -\left(\widehat{u}_{i,j+1}^{L}\right)'' + a_{11}(0)\widehat{u}_{i,j+1}^{L} &= a_{12}(0)\widehat{v}_{i,j+1}^{L} - \\ &- \sum_{k=1}^{\min\{i,j+1\}}\frac{\widehat{x}^k}{k!}\left(a_{11}^{(k)}(0)\widehat{u}_{i-k,j+1-k}^{L} - a_{12}^{(k)}(0)\widehat{v}_{i-k,j+1-k}^{L}\right) && (4.30a) \\ \widehat{u}_{i,j+1}^{L}(0) &= -\left(u_{i,j+1}(0) + \widetilde{u}_{i,j+1}^{L}(0)\right), && (4.30b) \\ \widehat{u}_{i,j+1}^{L}(\widehat{x}) &\to 0 \quad \text{for } \widehat{x} \to \infty. && (4.30c) \end{aligned}$$

Finally, we solve (4.15c, 2nd eqn) for $\widehat{v}_{i,j+3}^{L}$:

$$\widehat{v}_{i,j+3}^{L}(z) = \sum_{k=0}^{\min\{i,j+1\}}\frac{1}{k!}\int_z^\infty\int_t^\infty \tau^k\left\{a_{21}^{(k)}(0)\widehat{u}_{i-k,j+1-k}^{L}(\tau) + a_{22}^{(k)}(0)\widehat{v}_{i-k,j+1-k}^{L}(\tau)\right\}d\tau dt. \tag{4.31}$$

The following theorem establishes the regularity of the functions $\widetilde{u}_{i,j}^{L}$, $\widetilde{v}_{i,j}^{L}$, $\widehat{u}_{i,j}^{L}$, $\widehat{v}_{i,j}^{L}$.

Theorem 4.9. *Assume that f, g, and $\mathbf{A}$ satisfy (2.3) and (2.7). Let $\widetilde{u}^L_{i,j}$, $\widetilde{v}^L_{i,j}$, $\widehat{u}^L_{i,j}$, $\widehat{v}^L_{i,j}$ be defined recursively as above, i.e., they solve (4.23), (4.24), (4.26), (4.25), (4.27) for the case $j = 0$ and, for $j \geq 1$ (4.28), (4.29), (4.30), (4.31). Set*

$$\begin{aligned} \overline{a} &:= \max\left\{a_{11}(0),\ \frac{a_{11}(0)a_{22}(0) - a_{21}(0)a_{12}(0)}{a_{11}(0)}\right\} > 0, \\ \underline{a} &:= \min\left\{a_{11}(0),\ \frac{a_{11}(0)a_{22}(0) - a_{21}(0)a_{12}(0)}{a_{11}(0)}\right\} > 0, \\ \operatorname{Exp}(z) &:= \begin{cases} e^{-\underline{a}\operatorname{Re}(z)} & \text{for } \operatorname{Re}(z) \geq 0 \\ e^{-\overline{a}\operatorname{Re}(z)} & \text{for } \operatorname{Re}(z) < 0. \end{cases} \end{aligned}$$

Then the functions $\widetilde{u}^L_{ij}$, $\widetilde{v}^L_{ij}$, $\widehat{u}^L_{ij}$, $\widehat{v}^L_{ij}$ are entire functions and there exist positive constants $C_{\widetilde{u}}$, $C_{\widetilde{v}}$, $C_{\widehat{u}}$, $C_{\widehat{v}}$, C_i, K_i, $\overline{K}_i$, $i = 1, ..., 4$, independent of ε and μ such that

$$\left|\widetilde{u}^L_{i,j}(z)\right| \leq C_1 K_1^i \overline{K}_1^j \left(C_{\widetilde{u}}(i+j) + |z|\right)^{2(i+j)} \frac{1}{(i+j)!} \operatorname{Exp}(z), \tag{4.32}$$

$$\left|\widetilde{v}^L_{i,j}(z)\right| \leq C_2 K_2^i \overline{K}_2^j \left(C_{\widetilde{v}}(i+j) + |z|\right)^{2(i+j)} \frac{1}{(i+j)!} \operatorname{Exp}(z), \tag{4.33}$$

$$\left|\widehat{u}^L_{i,j}(z)\right| \leq C_3 K_3^i \overline{K}_3^j \left(C_{\widehat{u}}(i+j) + |z|\right)^{2(i+j)} \frac{1}{(i+j)!} \operatorname{Exp}(z), \tag{4.34}$$

$$\left|\widehat{v}^L_{i,j+2}(z)\right| \leq C_4 K_4^i \overline{K}_4^j \left(C_{\widehat{v}}(i+j) + |z|\right)^{2(i+j)} \frac{1}{(i+j)!} \operatorname{Exp}(z); \tag{4.35}$$

furthermore, $\widehat{v}^L_{i,0} = \widehat{v}^L_{i,1} \equiv 0$.

Proof. The proof is by induction on j and i. After establishing the claims for the base cases $(i, j) = (0, 0)$, $(i, j) \in \{0\} \times \mathbb{N}$, $(i, j) \in \mathbb{N} \times \{0\}$ one shows it by induction on j with induction arguments on i as parts of the induction argument in j. The structure of the equations defining $\widetilde{u}^L_{i,j}$, $\widetilde{v}^L_{i,j}$, $\widehat{u}^L_{i,j}$, $\widehat{v}^L_{i,j+2}$, is such that one can proceed successively in the induction argument on j by providing estimates for $\widehat{v}_{i,j+2}$, $\widetilde{v}_{i,j}$, $\widetilde{u}^L_{i,j}$, $\widehat{u}^L_{i,j}$ in turn. In these estimates estimates, one relies on Lemma 4.5 for the estimates for $\widetilde{v}^L_{i,j}$, $\widehat{u}^L_{i,j}$, on Lemma 4.6 for $\widehat{v}^L_{i,j+2}$, and on Lemma 4.7 for $\widetilde{u}^L_{i,j}$.

For details, see Appendix D.3. ∎

We conclude this section by showing that the boundary layer functions are in fact entire:

Corollary 4.10. *The functions $\widetilde{\mathbf{U}}^L_{i,j}$ and $\widehat{\mathbf{U}}^L_{i,j}$ are entire functions, and there exist constants C, γ_1, γ_2, $\beta > 0$ independent of i, j, n, such that for all $x \geq 0$*

$$|\widetilde{u}^{(n)}_{i,j}(x)| + |\widetilde{v}^{(n)}_{i,j}(x)| + |\widehat{u}^{(n)}_{i,j}(x)| + |\widehat{v}^{(n)}_{i,j+2}(x)| \leq C e^{-\beta x} \gamma_1^{i+j} (i+j)^{i+j} \gamma_2^n \qquad \forall n \in \mathbb{N}_0.$$

Proof. Theorem 4.9 already asserted that the boundary layer functions are entire. For the stated bound, let $n \in \mathbb{N}_0$, $x \in (0, \infty)$ and use Cauchy's integral theorem for derivatives with contour $\partial B_{n+1}(x)$. We illustrate the procedure for $\widehat{u}_{i,j}$, the other cases being similar. Theorem 4.9 then yields suitable constants C, $\gamma > 0$ such that

$$|\widehat{u}^{(n)}_{i,j}(x)| \leq C \frac{n!}{(n+1)^n} \gamma^{i+j} \frac{1}{(i+j)!} \left(C_{\widehat{u}}(i+j) + x + n + 1\right)^{2(i+j)} \operatorname{Exp}(x) e^{\overline{a}(n+1)}.$$

With the aid of Lemma 4.8, we obtain by suitably adjusting C and γ,

$$|\widehat{u}_{i,j}^{(n)}(x)| \leq C\frac{n!}{(n+1)^n}\gamma^{i+j}\frac{1}{(i+j)!}(C_{\widehat{u}}(i+j)+n+1)^{2(i+j)}e^{\overline{a}(n+1)}e^{-\underline{a}x}.$$

Using the observation $((i+j)+n+1)^{2(i+j)} \leq (i+j)^{2(i+j)}(1+(n+1)/(i+j))^{2(i+j)} \leq (i+j)^{2(i+j)}e^{2(n+1)}$, allows us to conclude the proof. $\square$

Corollary 4.10 shows that the terms defining the boundary layer contributions $\widetilde{\mathbf{U}}_{BL}^M$ and $\widehat{\mathbf{U}}_{BL}^M$ are indeed of boundary layer type. A summation argument then shows that also $\widetilde{\mathbf{U}}_{BL}^M$ and $\widehat{\mathbf{U}}_{BL}^M$ have this property provided M_1 and M_2 are not "too large", viz., $M_1 = O(\mu^{-1})$ and $M_2 = O((\mu/\varepsilon)^{-1})$:

Theorem 4.11. *There exist constants C, δ, γ, $K > 0$, independent of ε and μ, such that under the assumptions $\mu(M_1+1)K \leq 1$ and $\varepsilon/\mu(M_2+1)K \leq 1$, there holds for the boundary layer functions $\widetilde{\mathbf{U}}_{BL}^M$ and $\widehat{\mathbf{U}}_{BL}^M$ of (4.4) and (4.5), that, upon viewing $\widetilde{\mathbf{U}}_{BL}^M$ and $\widehat{\mathbf{U}}_{BL}^M$ as functions of x (via the changes of variables $\widetilde{x} = x/\mu$, $\widehat{x} = x/\varepsilon$ etc.), we have $\widetilde{\mathbf{U}}_{BL}^M \in \mathcal{BL}^\infty(\delta\mu, C, \gamma)$ and $\widehat{\mathbf{U}}_{BL}^M \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$. Furthermore, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}^M$ satisfies the stronger assertion $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.*

Proof. We do not work out the details here since structurally similar arguments can be found, for example, in the proofs of [5, Thm. 3] or [6, Thms. 7.2.2, 7.3.3]. Essentially, by inserting the bounds of Corollary 4.10 in the sums defining $\widetilde{\mathbf{U}}_{BL}^M$, $\widehat{\mathbf{U}}_{BL}^M$ and using the conditions $\mu(M_1+1)K \leq 1$ and $\varepsilon/\mu(M_2+1)K \leq 1$ for K sufficiently large, one obtains upper estimates in the form of (convergent) geometric series. We point out that the sharper estimates for the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}^M$ stems from the fact that $\widehat{v}_{i,0} = \widehat{v}_{i,1} = 0$. $\square$

4.4 Remainder estimates

In this section, we analyze $\mathbf{R}^M$. This is done by estimating the residual $L_{\varepsilon,\mu}\mathbf{R}^M - \mathbf{F}$ and then appealing to the stability estimate (2.9). We will estimate $L_{\varepsilon,\mu}\mathbf{W}_M - \mathbf{F}$, $L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M$, and $L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M$.

4.4.1 Remainder resulting from the outer expansion: control of $L_{\varepsilon,\mu}\mathbf{W}_M - \mathbf{F}$

Theorem 4.12. *Let $\mathbf{U}$ be the solution to the problem* (2.1)*. Then there exist γ, $C > 0$ depending only on f, g, and $\mathbf{A}$ such that the following is true: If M_1, $M_2 \in \mathbb{N}$ are such that $\mu M_1\gamma < 1$, then with $\mathbf{W}_M$ given by* (4.3) *we have*

$$\|L_{\varepsilon,\mu}(\mathbf{U} - \mathbf{W}_M)\|_{L^\infty(I)} \leq C\mu^2\left[\frac{1}{\mu-\varepsilon}(\mu M_1\gamma)^{M_1} + \left(\frac{\varepsilon}{\mu}\right)^{M_2+2}\right]$$

Proof. We have

$$
\begin{aligned}
L_{\varepsilon,\mu}\left(\mathbf{U}-\mathbf{W}_M\right) &= \begin{pmatrix} f \\ g \end{pmatrix} - \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j L_{\varepsilon,\mu}\begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \\
&= \begin{pmatrix} f \\ g \end{pmatrix} - \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j \begin{pmatrix} -\varepsilon^2 u_{ij}'' + a_{11}u_{ij} + a_{12}v_{ij} \\ -\mu^2 v_{ij}'' + a_{21}u_{ij} + a_{22}v_{ij} \end{pmatrix}.
\end{aligned}
$$

Defining the sets

$$
\begin{aligned}
I_u &:= \{(i,j)\colon i \le M_1, \quad j \le M_2, \quad i \ge M_1 - 1 \vee j \ge M_2 - 1\}, && (4.36)\\
I_v &:= \{(i,j)\colon i \le M_1, \quad j \le M_2, \quad M_1 - 1 \le i \le M_1\}, && (4.37)
\end{aligned}
$$

we see, after some calculations, that (4.15a) and Lemma 4.2 imply

$$
\begin{aligned}
L_{\varepsilon,\mu}\left(\mathbf{U}-\mathbf{W}_M\right) &= \begin{pmatrix} f \\ g \end{pmatrix} - \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j \left\{\begin{pmatrix} f_{ij} \\ g_{ij} \end{pmatrix} + \begin{pmatrix} u_{i-2,j-2}'' - \varepsilon^2 u_{ij}'' \\ v_{i-2,j}'' - \mu^2 v_{ij}'' \end{pmatrix}\right\} \\
&= \begin{pmatrix} \sum_{(i,j)\in I_u} \mu^{i+2}(\varepsilon/\mu)^{j+2} u_{i,j}'' \\ \sum_{(i,j)\in I_v} \mu^{i+2}(\varepsilon/\mu)^{j} v_{i,j}'' \end{pmatrix}.
\end{aligned}
$$

Hence, with the aid of Lemma 4.2 and Cauchy's Integral Theorem for derivatives, we get for a fixed $\delta > 0$ in the statement of Lemma 4.2,

$$
\begin{aligned}
\|L_{\varepsilon,\mu}(\mathbf{U}-\mathbf{W}_M)\|_{L^\infty(I)} \le\; & CC_S\Big[\mu^2 \sum_{i=0}^{M_1-2}(\mu i\delta^{-1}K)^i \sum_{j=M_2-1}^{M_2}(\varepsilon/\mu)^{j+2} + \mu^2 \sum_{i=M_1-1}^{M_1}(\mu i\delta^{-1}K)^i \sum_{j=0}^{M_2}(\varepsilon/\mu)^{j+2} \\
& + \mu^2 \sum_{i=M_1-1}^{M_1}(\mu i\delta^{-1}K)^i \sum_{j=0}^{M_2}(\varepsilon/\mu)^{j}\Big].
\end{aligned}
$$

Hence, by selecting $\gamma = \delta^{-1}K/2$ we get

$$
\begin{aligned}
\|L_{\varepsilon,\mu}(\mathbf{U}-\mathbf{W}_M)\|_{L^\infty(I)} &\le CC_S\Big[\mu^2\left(\frac{\varepsilon}{\mu}\right)^{M_2+1} + (\mu M_1\gamma)^{M_1-1}\frac{1}{1-\varepsilon/\mu}\Big] \\
&\le CC_S\mu^2\Big[\left(\frac{\varepsilon}{\mu}\right)^{M_2+1} + (\mu M_1\gamma)^{M_1}\frac{1}{\mu-\varepsilon}\Big].
\end{aligned}
$$

□

4.4.2 Remainder resulting from the inner expansion on the ε-scale: $L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M$

We next consider the inner expansions. We will only consider the contribution $\widehat{\mathbf{U}}_{BL}^M$ from the left endpoint as the contribution $\widehat{\mathbf{V}}_{BL}^M$ from the right endpoint is treated completely analogously. To simplify the notation, we drop the superscript L in $\widehat{u}_{i,j}^L$, $\widehat{v}_{i,j}^L$.

In order to simplify the ensuing calculations, we employ the convention that

$$
\widehat{u}_{ij} = \widehat{v}_{ij} = 0 \text{ for } i > M_1 \text{ or } j > M_2 \text{ and } \mathbf{A}_k = 0 \quad \forall k < 0, \tag{4.38}
$$

and let the summation on i and j in the definition of $\widehat{\mathbf{U}}_{BL}^M$ run from 0 to ∞. We recall that the differential operator $L_{\varepsilon,\mu}$ takes the form (4.10) when applied to functions depending solely on $\widehat{x}$, and compute $L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M$ (cf. (4.4) for the definition of $\widehat{\mathbf{U}}_{BL}^M$):

$$\begin{aligned} L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\begin{pmatrix} -\widehat{u}_{ij}'' \\ -\frac{\mu^2}{\varepsilon^2}\widehat{v}_{ij}''\end{pmatrix} + \sum_{i\ge 0}\sum_{j\ge 0}\sum_{k\ge 0}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\left(\frac{\varepsilon}{\mu}\right)^k\mu^k\widehat{x}^k\mathbf{A}_k\begin{pmatrix}\widehat{u}_{ij} \\ \widehat{v}_{ij}\end{pmatrix} \\ &= \sum_{i\ge 0}\sum_{j\ge 0}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\left\{\begin{pmatrix} -\widehat{u}_{ij}'' \\ -\widehat{v}_{i,j+2}''\end{pmatrix} + \sum_{k\ge 0}\widehat{x}^k\mathbf{A}_k\begin{pmatrix}\widehat{u}_{i-k,j-k} \\ \widehat{v}_{i-k,j-k}\end{pmatrix}\right\}, \end{aligned}$$

where the fact that $\widehat{v}_{k,0} = \widehat{v}_{k,1} = 0$ was used. We see that "equating like powers of μ and ε/μ " yields equation (4.15c), hence there will be no contribution to the sums for when both $i = 0, \ldots, M_1$ and $j = 0, \ldots, M_2 - 2$. Moreover, the convention (4.38) implies the following restrictions on the sums:

$$k \le \min\{i,j\}, \qquad i-k \le M_1, \qquad j-k \le M_2. \tag{4.39}$$

Thus, we have

$$L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M = \tag{4.40}$$
$$\sum_{\substack{i,j:i\ge M_1+1 \\ \text{or } j\ge M_2-1}}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\sum_{k=\max\{i-M_1,j-M_2\}}^{\min\{i,j\}}\widehat{x}^k\mathbf{A}_k\begin{pmatrix}\widehat{u}_{i-k,j-k} \\ \widehat{v}_{i-k,j-k}\end{pmatrix} - \sum_{i=0}^{M_1}\sum_{j=M_2-1}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\begin{pmatrix}\widehat{u}_{i,j}'' \\ 0\end{pmatrix}.$$

Using the estimates (4.34), (4.35) for $\widehat{u}_{i,j}$, $\widehat{v}_{i,j}$ of Theorem 4.9, one can estimate this triple sum to obtain the following result for the remainder on the positive real line:

Theorem 4.13. *There exist C, γ, $\widetilde{K} > 0$ depending only on $\mathbf{A}$, f, g such that under the assumptions*

$$0 < \gamma\widehat{x}\varepsilon \le 1 \qquad \text{and} \qquad \mu(M_1+1)\gamma \le 1 \qquad \text{and} \qquad \frac{\varepsilon}{\mu}(M_2+1)\gamma \le 1,$$

we have

$$\left|L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M(\widehat{x})\right| \le Ce^{-\underline{a}\widehat{x}/2}\left((\widetilde{K}(M_1+1)\mu)^{M_1} + (\widetilde{K}(M_2+1)\varepsilon/\mu)^{M_2-1}\right).$$

For $\gamma\widehat{x}\varepsilon > 1$ we have, under the same conditions on $(M_1+1)\mu$ and $(M_2+1)\varepsilon/\mu$, that

$$\left|L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M(\widehat{x})\right| \le Ce^{-\underline{a}\widehat{x}/2}.$$

Proof. The proof relies on using the estimates (4.34), (4.35) that are available for $\widehat{u}_{ij}$ and $\widehat{v}_{ij}$. The triple sum in (4.40) is estimated by using convexity of the function $k \mapsto \gamma^k(i-k)^{i-k}(j-k)^{j-k}$ and $k \mapsto k^k\gamma^k(i-k)^{i-k}(j-k)^{j-k}$ and by considering the two following two cases separately:

$$\begin{aligned} &(i \ \ge \ M_1+1 \ \vee \ j \ge M_2+1) \ \wedge \ (i-M_1 \le j-M_2) \ \wedge \ (j-M_2 \le k \le i) \ \wedge \ (i \le j), \\ &(i \ \ge \ M_1+1 \ \vee \ j \ge M_2+1) \ \wedge \ (i-M_1 \le j-M_2) \ \wedge \ (j-M_2 \le k \le j) \ \wedge \ (j \le i). \end{aligned}$$

For details, see Appendix D.4 □

4.4.3 Remainder resulting from the inner expansion on the μ-scale: $L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M$

In a similar fashion we may treat the left inner (boundary layer) expansion associated with $\widetilde{x} = x/\mu$ (cf. (4.4)),

$$\widetilde{\mathbf{U}}_{BL}^M(\widetilde{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \begin{pmatrix} \widetilde{u}_{ij}(\widetilde{x}) \\ \widetilde{v}_{ij}(\widetilde{x}) \end{pmatrix}, \tag{4.41}$$

where we have dropped the superscript L for notational convenience. We recall from (4.9) that for functions $\widetilde{\mathbf{U}}$ depending only on the variable $\widetilde{x}$, the differential operator $L_{\varepsilon,\mu}$ takes the form $L_{\varepsilon,\mu}\widetilde{\mathbf{U}} = -\mu^{-2}\mathbf{E}^{\varepsilon,\mu}\widetilde{\mathbf{U}}'' + \sum_{k=0}^{\infty}\mu^k\widetilde{x}^k\mathbf{A}_k\widetilde{\mathbf{U}}$. In order to simplify the ensuing calculations, we employ the convention that

$$\widetilde{u}_{ij} = \widetilde{v}_{ij} = 0 \text{ for } i > M_1 \text{ or } j > M_2 \text{ and } \mathbf{A}_k = 0 \quad \forall k < 0, \tag{4.42}$$

and let the summation in (4.41) run from 0 to ∞ for both i and j. We calculate

$$\begin{aligned} L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mu^i(\varepsilon/\mu)^j \begin{pmatrix} -(\varepsilon/\mu)^2\widetilde{u}''_{ij} \\ -\widetilde{v}''_{ij} \end{pmatrix} + \sum_{i\geq 0}\sum_{j\geq 0}\sum_{k\geq 0}\mu^i(\varepsilon/\mu)^j\mu^k\widetilde{x}^k\mathbf{A}_k \begin{pmatrix} \widetilde{u}_{ij} \\ \widetilde{v}_{ij} \end{pmatrix} \\ &= \sum_{i\geq 0}\sum_{j\geq 0}\mu^i(\varepsilon/\mu)^j \left\{ \begin{pmatrix} -\widetilde{u}''_{i,j-2} \\ -\widetilde{v}''_{ij} \end{pmatrix} + \sum_{k\geq 0}\widetilde{x}^k\mathbf{A}_k \begin{pmatrix} \widetilde{u}_{i-k,j} \\ \widetilde{v}_{i-k,j} \end{pmatrix} \right\}, \end{aligned}$$

where the convention $\widetilde{u}_{k,-2} = \widetilde{u}_{k,-1} = 0$ was used. As expected from the derivation of (4.15b), "equating like powers of μ and ε/μ" yields eqn.(4.15b), hence there will be no contribution to the sums for when both $i = 0,\dots,M_1$ and $j = 0,\dots,M_2$. Moreover, the convention (4.42) implies the following restrictions on the sums:

$$\begin{aligned} i &\leq M_1 && \text{for the terms involving } \widetilde{u}''_{i,j-2},\ \widetilde{v}''_{i,j}, && (4.43)\\ 0 \leq i-k &\leq M_1 && \text{for the sum on } k, && (4.44)\\ j &\leq M_2 && \text{for the terms involving } \widetilde{u}_{i,j},\ \widetilde{v}_{i,j},\ \widetilde{v}''_{i,j},\ \widetilde{u}_{i-k,j},\ \widetilde{v}_{i-k,j}, && (4.45)\\ j &\leq M_2 - 2 && \text{for the terms involving } \widetilde{u}''_{i,j-2}. && (4.46) \end{aligned}$$

Hence, we arrive at

$$L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M = \tag{4.47}$$
$$\sum_{i=M_1+1}^{\infty}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j \sum_{k=i-M_1}^{i}\widetilde{x}^k\mathbf{A}_k \begin{pmatrix} \widetilde{u}_{i-k,j} \\ \widetilde{v}_{i-k,j} \end{pmatrix} + \sum_{i=0}^{M_1}\sum_{j=M_2+1}^{M_2+2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j \begin{pmatrix} -\widetilde{u}''_{i,j-2} \\ 0 \end{pmatrix}.$$

Using the bounds of Theorem 5.5, these sums will be estimated, when $\widetilde{x} > 0$, in the following:

Theorem 4.14. *There exists C, γ, $\widetilde{K} > 0$ such that under the assumption*

$$0 < \gamma\widetilde{x}\mu \leq 1, \qquad \text{and} \qquad \gamma\mu(M_1+1) \leq 1 \qquad \text{and} \qquad \frac{\varepsilon}{\mu}\gamma(M_2+1) \leq 1,$$

we have

$$\left| L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M(\widetilde{x}) \right| \leq Ce^{-\underline{a}\widetilde{x}/2}\left((\widetilde{K}(M_1+1)\mu)^{M_1+1} + (\widetilde{K}(M_2+1)\varepsilon/\mu)^{M_2+1} \right).$$

For $\widetilde{\mu}\widetilde{x}\gamma > 1$ and the same assumptions on $\mu(M_1+1)$ and $\varepsilon/\mu(M_2+1)$, we have

$$\left| L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M(\widetilde{x}) \right| \leq Ce^{-\underline{a}\widetilde{x}/2}.$$

Proof. The proof is split into two cases: for $\widetilde{x}\mu\gamma > 1$, one exploits the fact that $\widetilde{\mathbf{U}}_{BL}^{M}$ (and its derivatives) is exponentially small. For the converse case $\widetilde{x}\mu\gamma \leq 1$, one bounds the sums (4.47). For details, see Appendix D.5

□

4.5 Boundary mismatch of the expansion

Theorem 4.15. *There exist constants C, b, $\gamma > 0$ such that under the assumptions*

$$\mu(M_1 + 1)\gamma \leq 1 \qquad \textit{and} \qquad \frac{\varepsilon}{\mu}(M_2 + 1)\gamma \leq 1,$$

one has

$$\|\mathbf{W}^M + \widetilde{\mathbf{U}}_{BL}^M + \widehat{\mathbf{U}}_{BL}^M + \widetilde{\mathbf{V}}_{BL}^M + \widehat{\mathbf{V}}_{BL}^M\|_{L^\infty(\partial I)} \leq C\left[e^{-b/\varepsilon} + e^{-b/\mu}\right].$$

Proof. For $\mathbf{R}_M(0)$ at the left endpoint, we note that $\mathbf{U}(0) = 0$ gives

$$\mathbf{R}_M(0) = \mathbf{U}(0) - \left[\mathbf{W}^M(0) + \widetilde{\mathbf{U}}_{BL}^M(0) + \widehat{\mathbf{U}}_{BL}^M(0) + \widetilde{\mathbf{V}}_{BL}^M(1/\mu) + \widehat{\mathbf{V}}_{BL}^M(1/\varepsilon)\right].$$

The condition (4.15d) for the boundary conditions of the left endpoint for the terms $\widehat{\mathbf{U}}_{i,j}$ and $\widetilde{\mathbf{U}}_{i,j}$, produces $\mathbf{W}^M(0) + \widetilde{\mathbf{U}}_{BL}^M(0) + \widehat{\mathbf{U}}_{BL}^M(0) = 0$. Hence, it remains to estimate $|\widetilde{\mathbf{V}}_{BL}^M(1/\mu)| + |\widehat{\mathbf{V}}_{BL}^M(1/\varepsilon)|$ which can be done based on Theorem 4.9. □

4.6 Proof of Theorem 4.1

From the estimates for the residual $L_{\varepsilon,\mu}\mathbf{R}_M$ of Theorems 4.12, 4.13, 4.14 we infer the existence of $q > 0$ such that under the assumption

$$\mu \leq q \qquad \text{and} \qquad \frac{\varepsilon}{\mu} \leq q,$$

the choice $M_1 \sim 1/\mu$ and $M_2 \sim \mu/\varepsilon$ yields

$$\|L_{\varepsilon,\mu}\mathbf{R}_M\|_{L^\infty(I)} \leq C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right],$$

where C, $b > 0$ are independent of μ and ε. Hence, by stability and Theorem 4.15, we get that $\|\mathbf{R}_M\|_{E,I} \leq C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right]$. The sharper result for the second component $\widehat{v}$ of $\widehat{\mathbf{U}}$ follows from the fact that $\widehat{v}_{i,0} = \widehat{v}_{i,1} = 0$.

It remains to formulate a decomposition of $\mathbf{U}$ for the case that $\mu \geq q$ or $\varepsilon/\mu \geq q$. In this case, we have $e^{-b/\mu}+e^{-b\mu/\varepsilon}$ is $O(1)$. Given that $\|\mathbf{U}\|_{E,I} = O(1)$, the trivial decomposition $\mathbf{U} = 0+0+0+\mathbf{R}_M$ provides the desired splitting.

5 The first two scale case: Case III

In this case it is assumed that $\mu/1$ is *not* deemed small *but* ε/μ is deemed small. The main result of this section is the following regularity assertion:

Theorem 5.1. *There exist constants C_W, γ_W, C_{BL}, γ_{BL}, δ, b, $q > 0$ independent of ε and μ such that for $\varepsilon/\mu \leq q$ the solution $\mathbf{U}$ of (2.1) can be written as $\mathbf{U} = \mathbf{W} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(\mu, C_W, \gamma_W)$, $\widehat{\mathbf{U}} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{BL}, \gamma_{BL})$. Furthermore, $\mathbf{R}$ satisfies*

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|L_{\varepsilon,\mu}\mathbf{R}\|_{L^\infty(I)} \quad \leq \quad Ce^{-b/\varepsilon}.$$

In particular, $\|\mathbf{R}\|_{E,I} \leq Ce^{-b/\varepsilon}$.

Additionally, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}$ satisfies the sharper regularity assertion

$$\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{BL}(\varepsilon/\mu)^2, \gamma_{BL}).$$

We employ again the notation of the outset of Section 4 concerning the stretched variables $\widehat{x} = x/\varepsilon$ and $\widehat{x}^R = (1-x)/\varepsilon$ and make the formal ansatz

$$\mathbf{U}(x) \sim \sum_{i=0}^{\infty} \left(\frac{\varepsilon}{\mu}\right)^i \left[\mathbf{U}_i(x) + \widehat{\mathbf{U}}_i^L(\widehat{x}) + \widehat{\mathbf{U}}_i^R(\widehat{x}^R)\right]. \tag{5.1}$$

We proceed as in Section 4 by inserting the ansatz (5.1) in the differential equation (2.1a), separating the slow (x) and the fast ($\widehat{x}$ and $\widehat{x}^L$) variables and equating like powers of ε/μ. We also recall that the differential operator $L_{\varepsilon,\mu}$ takes the form (4.10) on the $\widehat{x}$-scale. The separation of the slow and the fast variables leads to the following equations:

$$\sum_{i=0}^{\infty} \left(\frac{\varepsilon}{\mu}\right)^i \left(-\mathbf{E}^{\varepsilon,\mu}\mathbf{U}_i'' + \mathbf{A}(x)\mathbf{U}_i\right) \quad = \quad \mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{5.2a}$$

$$\sum_{i=0}^{\infty} \left(\frac{\varepsilon}{\mu}\right)^i \left(-\varepsilon^{-2}\mathbf{E}^{\varepsilon,\mu}(\widehat{\mathbf{U}}_i^L)'' + \sum_{k=0}^{\infty} \mu^k \left(\frac{\varepsilon}{\mu}\right)^k \widehat{x}^k \mathbf{A}_k \widehat{\mathbf{U}}_i^L\right) \quad = \quad 0, \tag{5.2b}$$

and an analogous equation for $\widehat{\mathbf{U}}_i^R$. Writing again

$$\widehat{\mathbf{U}}_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \qquad \widehat{\mathbf{U}}_i^L = \begin{pmatrix} \widehat{u}_i^L \\ \widehat{v}_i^L \end{pmatrix},$$

we obtain from (5.2) by equating like powers of ε/μ:

$$-\mu^2 u_{i-2}'' + a_{11}u_i + a_{12}v_i \quad = \quad f_i, \tag{5.3a}$$

$$-\mu^2 v_i'' + a_{21}u_i + a_{22}v_i \quad = \quad g_i, \tag{5.3b}$$

$$-(\widehat{u}_i^L)'' + \sum_{k=0}^{i} \mu^k \widehat{x}^k \left(\frac{a_{11}^{(k)}(0)}{k!}\widehat{u}_{i-k}^L + \frac{a_{12}^{(k)}(0)}{k!}\widehat{v}_{i-k}^L\right) \quad = \quad 0, \tag{5.3c}$$

$$-(\widehat{v}_{i+2}^L)'' + \sum_{k=0}^{i} \mu^k \widehat{x}^k \left(\frac{a_{21}^{(k)}(0)}{k!}\widehat{u}_{i-k}^L + \frac{a_{22}^{(k)}(0)}{k!}\widehat{v}_{i-k}^L\right) \quad = \quad 0, \tag{5.3d}$$

where we employed the definition of $\mathbf{A}_k$, the notation $f_0 = f$, $g_0 = g$ as well as $f_i = g_i = 0$ for $i > 0$, and the convention that function with negative subscripts are zero. Corresponding equations are satisfied the functions $\widehat{u}_i^R$ and $\widehat{v}_i^R$. The boundary condition (2.1b) is accounted for by stipulating $\mathbf{U}_i(0) + \widehat{\mathbf{U}}_i^L(0) = 0$ and $\mathbf{U}_i(1) + \widehat{\mathbf{U}}_i^R(0) = 0$ for all $i \geq 0$ and suitable decay conditions for $\widehat{\mathbf{U}}_i^L$ as $\widehat{x} \to \infty$ and, correspondingly, for $\widehat{\mathbf{U}}_i^R$ as $\widehat{x}^R \to \infty$. Rearranging the above equations and incorporating these boundary conditions, we get a a recursion of systems of DAEs in which the algebraic constraints can be accounted for explicitly. We obtain for $i = 0, 1, 2, \ldots$:

$$\left\{ \begin{array}{c} -\mu^2 v_i'' + \frac{(a_{22}a_{11} - a_{12}a_{21})}{a_{11}} v_i = g_i - \frac{a_{21}}{a_{11}} \left(f_i + \mu^2 u_{i-2}'' \right) \\ v_i(0) = -\widehat{v}_i^L(0) \ , \ v_i(1) = -\widehat{v}_i^R(0) \end{array} \right. , \tag{5.4}$$

$$u_i = \frac{1}{a_{11}} \left(f_i + \mu^2 u_{i-2}'' - a_{12} v_i \right), \tag{5.5}$$

$$\left\{ \begin{array}{c} -\left(\widehat{u}_i^L\right)'' + a_{11}(0)\widehat{u}_i^L = -\sum\limits_{k=1}^{i} \left(\frac{a_{11}^{(k)}(0)}{k!} \mu^k \widehat{x}^k \widehat{u}_{i-k}^L + \frac{a_{12}^{(k)}(0)}{k!} \widehat{x}^k \mu^k \widehat{v}_{i-k}^L \right) - a_{12}(0)\widehat{v}_i^L \\ \widehat{u}_i^L(0) = -u_i(0) \ , \ \widehat{u}_i^L \to 0 \text{ as } \widehat{x} \to \infty \end{array} \right. , \tag{5.6}$$

$$-\left(\widehat{v}_{i+2}^L\right)'' = \sum_{k=0}^{i} \left(\frac{a_{21}^{(k)}(0)}{k!} \mu^k \widehat{x}^k \widehat{u}_{i-k}^L + \frac{a_{22}^{(k)}(0)}{k!} \mu^k \widehat{x}^k \widehat{v}_{i-k}^L \right), \tag{5.7}$$

with

$$\widehat{v}_0^R = \widehat{v}_1^R = \widehat{v}_0^L = \widehat{v}_1^L = 0 \ , \ u_{-i} = 0 \ \forall \ i > 0,$$

$$f_i = \left\{ \begin{array}{l} f \ \text{ if } i = 0, \\ 0 \text{ otherwise} \end{array} \right. , \ g_i = \left\{ \begin{array}{l} g \ \text{ if } i = 0, \\ 0 \text{ otherwise} \end{array} \right. .$$

(The functions $\widehat{u}_i^R, \widehat{v}_i^R$ satisfy similar problems as (5.6) and (5.7), respectively.)

5.1 Analysis of the functions $\mathbf{U}_i$, $\widehat{\mathbf{U}}_i^L$, $\widehat{\mathbf{U}}_i^R$

5.1.1 Properties of some solution operators

Lemma 5.2. *Assume that the function c is analytic on $\overline{I}$ and $c \geq \underline{c} > 0 \ \forall \ z \in \overline{I}$. Let $\mu \in (0, 1]$. Then there exists $\gamma_0 > 0$ (independent of μ) such that for all $\gamma \geq \gamma_0$ and all $C_g > 0$ the following is true: If g satisfies*

$$\|g^{(n)}\|_{L^\infty(I)} \leq C_g \gamma^n \max\{n, \mu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0, \tag{5.8}$$

then the solution u of the boundary value problem

$$\left\{ \begin{array}{c} -\mu^2 u'' + cu = g \text{ in } I \\ u(0) = g_- \in \mathbb{R} \ , \ u(1) = g_+ \in \mathbb{R} \end{array} \right.$$

satisfies

$$\|u^{(n)}\|_{L^\infty(I)} \leq \widetilde{C} \gamma^n \max\{n, \mu^{-1}\}^n \left[C_g + |g_+| + |g_-| \right] \qquad \forall n \in \mathbb{N}_0,$$

for a constant $\widetilde{C}$ that depends solely on the function c.

Proof. This follows by inspection of the proof of [5, Thm. 1] if one replaces the energy type arguments by an appeal to the comparison principle to get L^∞-estimates.

For details, see Appendix B.1. ☐

Lemma 5.3. *Let $m \in \mathbb{N}_0$ and g be a function analytic on $\overline{I}$. Then there exist C', $\gamma_0 > 0$ depending only on g such that the following is true: If v satisfies*

$$\|v^{(n)}\|_{L^\infty(I)} \le C_v \gamma_v^n \max\{n, \mu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0,$$

for some C_v, $\mu > 0$ and $\gamma_v \ge \gamma_0$, then the function $V := g v^{(m)}$ satisfies

$$\|V^{(n)}\|_{L^\infty(I)} \le C' C_v \gamma_v^{n+m} \max\{n+m, \mu^{-1}\}^{n+m} \qquad \forall n \in \mathbb{N}_0.$$

Proof. Since g is analytic, there exist C_g, $\gamma_g > 0$ such that

$$\|g^{(n)}\|_{L^\infty(I)} \le C_g \gamma_g^n n! \qquad \forall n \in \mathbb{N}.$$

By Leibniz' rule, we get for $\gamma_v \ge \gamma_0 > \gamma_g e$ in view of $\nu^\nu \le \nu! e^\nu$:

$$\begin{aligned}
\|V^{(n)}\|_{L^\infty(I)} &\le C_v C_g \sum_{\nu=0}^{n} \binom{n}{\nu} \gamma_g^\nu \nu^\nu \gamma_v^{n+m-\nu} \max\{n+m-\nu, \mu^{-1}\}^{n+m-\nu} \\
&\le C_v C_g \gamma_v^n \sum_{\nu=0}^{n} \frac{n!}{(n-\nu)!} \left(\frac{e\gamma_g}{\gamma_v}\right)^\nu \max\{n+m, \mu^{-1}\}^{n+m-\nu} \\
&\le \frac{C_g}{1 - e\gamma_g/\gamma_v} C_v \gamma_v^n \max\{n+m, \mu^{-1}\}^{n+m}.
\end{aligned}$$

☐

Lemma 5.4. *For $0 < \widehat{\delta} \le 1/(2e)$ and $i \in \mathbb{N}$ there holds*

$$\sum_{k=0}^{i} \widehat{\delta}^k \left(\frac{i+2}{i+1-k}\right)^{i-k} \le 2e.$$

Proof. See Appendix B.2. ☐

5.1.2 Regularity of the functions $\mathbf{U}_i$, $\widehat{\mathbf{U}}_i^L$, $\widehat{\mathbf{U}}_i^R$

Theorem 5.5. *Let $v_i, u_i, \widehat{u}_i^L, \widehat{v}_i^L$ be the solutions of (5.4)–(5.7), respectively. Let $G \subset \mathbb{C}$ be a complex neighborhood of $\overline{I}$ and set $\beta = \sqrt{a_{11}(0)} \in \mathbb{R}$. Then the functions $\widehat{u}_i^L$ and $\widehat{v}_i^L$ are entire and the functions u_i, v_i are analytic in a (fixed) neighborhood of I. Furthermore, there exist positive constants γ, C_u, C_v, C_ℓ, K_ℓ, $\ell = 1, ..., 4$ independent of ε and μ, such that*

$$\widehat{v}_0^L = \widehat{v}_1^L = 0, \tag{5.9}$$

$$\left|\widehat{v}_{i+2}^L(z)\right| \le C_1 K_1^i \left(\mu + \frac{1}{i+1}\right)^i \frac{1}{i!} \left(iC_v + |z|\right)^{2i} e^{-\beta \operatorname{Re}(z)}, \tag{5.10}$$

$$\left|\widehat{u}_i^L(z)\right| \le C_2 K_2^i \left(\mu + \frac{1}{i+1}\right)^i \frac{1}{i!} \left(iC_u + |z|\right)^{2i} e^{-\beta \operatorname{Re}(z)}, \tag{5.11}$$

$$\|u_i^{(n)}\|_{L^\infty(I)} \le C_3 K_3^i \left((i+n)\mu + 1\right)^i \frac{i^i}{i!} \gamma^n \max\{n, \mu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0, \quad \forall i \in \mathbb{N}_0, \tag{5.12}$$

$$\|v_i^{(n)}\|_{L^\infty(I)} \le C_4 K_4^i \left((i+n)\mu + 1\right)^i \frac{i^i}{i!} \gamma^n \max\{n, \mu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0, \quad \forall i \in \mathbb{N}_0. \tag{5.13}$$

Furthermore, analogous estimates hold for $\widehat{u}_i^R$ and $\widehat{v}_i^R$ with β being now $a_{11}(1)$.

Proof. The proof is by induction on i. The general structure is to estimate v_i, u_i, $\widehat{u}_i$, and $\widehat{v}_{i+2}$ in turn using Lemmas 4.5, 4.6, 5.2, 5.3. For details, see Appendix B.3. ☐

We conclude this section by showing that the boundary layer functions are in fact entire:

Lemma 5.6. *Let $\beta > 0$ be as in Theorem 5.5 (i.e., $\beta = \sqrt{a_{11}(0)}$). The functions $\widehat{\mathbf{U}}_i^L$ are entire functions and there exist constants C, γ_1, $\gamma_2 > 0$ independent of i, j, n such that for all $x \ge 0$*

$$|\widehat{u}_i^{(n)}(\widehat{x})| + |\widehat{v}_{i+2}^{(n)}(\widehat{x})| \le C e^{-\beta/2\widehat{x}} \gamma_1^i (i\mu + 1)^i \gamma_2^n \qquad \forall n \in \mathbb{N}_0.$$

Proof. Similar to the proof of Corollary 4.10. ☐

By a simple summation argument we get from Lemma 5.6

Theorem 5.7. *Let β be as in Theorem 5.5 (i.e., $\beta = \sqrt{a_{11}(0)}$). There exist constants C, γ, $K > 0$ such that for $\gamma\{(M+1)\varepsilon + \frac{\varepsilon}{\mu}\} \le 1$ the function*

$$\widehat{\mathbf{U}}_{BL}^M(\widehat{x}) := \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \widehat{\mathbf{U}}_i(\widehat{x}),$$

satisfies for $\widehat{x} > 0$

$$|\left(\widehat{\mathbf{U}}_{BL}^M\right)^{(n)}(\widehat{x})| \le C e^{-\beta/2\widehat{x}} K^n \qquad \forall n \in \mathbb{N}_0.$$

An analogous result holds for $\widehat{\mathbf{V}}_{BL}^M := \sum_{i=0}^M (\varepsilon/\mu)^i \widehat{\mathbf{U}}_i^R$ with β given by $\beta = \sqrt{a_{11}(1)}$.

Proof. See Appendix B.4 for details. ☐

5.2 Remainder estimates

We now turn to estimating the remainder obtained by truncating the formal expansion (5.1). We write

$$\mathbf{U}(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \mathbf{W}_M(x) + \widehat{\mathbf{U}}_{BL}^M(\widehat{x}) + \widehat{\mathbf{V}}_{BL}^M(\widehat{x}) + \mathbf{R}_M(x), \tag{5.14}$$

where

$$\mathbf{W}_M(x) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \begin{pmatrix} u_i(x) \\ v_i(x) \end{pmatrix}, \tag{5.15}$$

denotes the outer (smooth) expansion,

$$\widehat{\mathbf{U}}_{BL}^M(\widehat{x}) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \begin{pmatrix} \widehat{u}_i^L(\widehat{x}) \\ \widehat{v}_i^L(\widehat{x}) \end{pmatrix}, \ \widehat{\mathbf{V}}_{BL}^M(\widehat{x}) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \begin{pmatrix} \widehat{u}_i^R(\widehat{x}^R) \\ \widehat{v}_i^R(\widehat{x}^R) \end{pmatrix}, \tag{5.16}$$

denote the left and right inner (boundary layer) expansions, respectively, and

$$\mathbf{R}_M(x) := \begin{pmatrix} r_u(x) \\ r_v(x) \end{pmatrix} = \mathbf{U}(x) - \left(\mathbf{W}_M(x) + \widehat{\mathbf{U}}_{BL}^M(\widehat{x}) + \widehat{\mathbf{V}}_{BL}^M(\widehat{x}^R)\right) \tag{5.17}$$

denotes the remainder. Our goal is to show that the remainder $\mathbf{R}_M(x)$ is exponentially (in ε) small. First, we need to obtain results on the other terms in (5.14). Note that by the linearity of the operator $L_{\varepsilon,\mu}$ we have

$$L_{\varepsilon,\mu}\mathbf{R}_M = L_{\varepsilon,\mu}\left(\mathbf{U} - \mathbf{W}_M\right) - L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M - L_{\varepsilon,\mu}\widehat{\mathbf{V}}_{BL}^M.$$

The terms on the right-hand side are treated separately.

5.2.1 Remainder resulting from the outer expansion: $\mathbf{F} - L_{\varepsilon,\mu}\mathbf{W}_M$

We have the following theorem.

Theorem 5.8. *Let* $\mathbf{U}$ *be the solution to the problem* (2.1)*. Then there exist* $C, \gamma > 0$ *independent of* μ, ε, *and* M *such that for* $\mathbf{W}_M$, *given by* (5.15) *we have*

$$\|L_{\varepsilon,\mu}\left(\mathbf{U} - \mathbf{W}_M\right)\|_{L^\infty(I)} \leq C\left[\left(\gamma\frac{\varepsilon}{\mu}\right)^{M+1} + \left(\gamma(M+1)\varepsilon\right)^{M+1}\right].$$

Proof. Using (5.2a) (5.4), (5.5) we obtain, after some calculations,

$$\|L_{\varepsilon,\mu}\left(\mathbf{U} - \mathbf{W}_M\right)\|_{L^\infty(I)} \quad = \quad \sum_{i=M-1}^{M} \left(\frac{\varepsilon}{\mu}\right)^{i+2} \mu^2 \begin{pmatrix} u_i'' \\ 0 \end{pmatrix}$$

From Theorem 5.5 we therefore get, with the observation $\frac{\varepsilon}{\mu}(M\mu + 1) = M\varepsilon + \frac{\varepsilon}{\mu}$, the desired result. □

5.2.2 Remainder resulting from the inner expansion: $L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M$

We now turn our attention to estimating $L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M$. Since $L_{\varepsilon,\mu}\widehat{\mathbf{V}}_{BL}^M$ is treated with similar arguments, we will not work out the details. We have:

Lemma 5.9. *The functions* $\widehat{\mathbf{U}}_{BL}^M$ *satisfy*

$$L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M = \sum_{i\geq M+1}\left(\frac{\varepsilon}{\mu}\right)^i \sum_{k=i-M}^{i} \mu^k \widehat{x}^k \mathbf{A}_k \widehat{\mathbf{U}}_{i-k}^L + \sum_{i=M-1}^{M}\left(\frac{\varepsilon}{\mu}\right)^i \begin{pmatrix} 0 \\ (\widehat{v}_{i+2}^L)'' \end{pmatrix}. \tag{5.18}$$

Proof. In order to simplify the calculation of $L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M$, we employ temporarily the convention that

$$\widehat{u}_i^L = \widehat{v}_i^L = 0 \qquad \forall i \geq M+1. \tag{5.19}$$

With this convention, we calculate (cf. (5.3c), (5.3d))

$$\begin{aligned}
L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M &= \sum_{i=0}^{\infty}\left(\frac{\varepsilon}{\mu}\right)^i \begin{pmatrix} -(\widehat{u}_i^L)'' + \sum_{k=0}^i \mu^k \widehat{x}^k \left(\frac{a_{11}^{(k)}(0)}{k!}\widehat{u}_{i-k}^L + \frac{a_{12}^{(k)}(0)}{k!}\widehat{v}_{i-k}^L\right) \\ -(\widehat{v}_{i+2}^L)'' + \sum_{k=0}^i \mu^k \widehat{x}^k \left(\frac{a_{21}^{(k)}(0)}{k!}\widehat{u}_{i-k}^L + \frac{a_{22}^{(k)}(0)}{k!}\widehat{v}_{i-k}^L\right) \end{pmatrix} \\
&= \sum_{i\geq M+1}\left(\frac{\varepsilon}{\mu}\right)^i \sum_{k=0}^{\infty} \mu^k \widehat{x}^k \mathbf{A}_k \widehat{\mathbf{U}}_{i-k}^L - \sum_{i=M-1}^{M}\left(\frac{\varepsilon}{\mu}\right)^i \sum_{k=0}^{i}\mu^k\widehat{x}^k \begin{pmatrix} 0 \\ \frac{a_{21}^{(k)}(0)}{k!}\widehat{u}_{i-k}^L + \frac{a_{22}^{(k)}(0)}{k!}\widehat{v}_{i-k}^L \end{pmatrix};
\end{aligned}$$

here, we employed additionally (5.3c), (5.3d) to see that the terms corresponding to $i \in \{0,\ldots,M-2\}$ in the first sum are zero. Finally, our convention (5.19) and the fact that $\widehat{u}_j^L = \widehat{v}_j^L = 0$ for $j < 0$ implies the restrictions

$$0 \leq i-k \leq M$$

so that we obtain

$$\sum_{i\geq M+1}\left(\frac{\varepsilon}{\mu}\right)^i \sum_{k=0}^{\infty} \mu^k \widehat{x}^k \mathbf{A}_k \widehat{\mathbf{U}}_{i-k}^L = \sum_{i\geq M+1}\left(\frac{\varepsilon}{\mu}\right)^i \sum_{k=i-M}^{i} \mu^k \widehat{x}^k \mathbf{A}_k \widehat{\mathbf{U}}_{i-k}^L,$$

which produces the double sum in (5.18). Lifting now the convention (5.19) we can use (5.3c) to replace $\sum_{k=0}^i \mu^k \widehat{x}^k (\frac{a_{21}^{(k)}(0)}{k!}\widehat{u}_{i-k}^L + \frac{a_{22}^{(k)}(0)}{k!}\widehat{v}_{i-k}^L)$ with $-\widehat{v}_{i+2}^L(\widehat{x})$. □

It remains to bound $L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M$. For that purpose, we need the following lemma:

Lemma 5.10. *Assume* $0 \leq \widehat{x}\varepsilon\gamma_A \leq X < 1$ *with* γ_A, X, C_1, $\beta > 0$ *known constants. Let* $0 < \varepsilon \leq \mu \leq 1$. *Then*

$$\begin{aligned}
&\sum_{i\geq M+1}\left(\frac{\varepsilon}{\mu}\right)^i \sum_{k=i-M}^{i} \mu^k \widehat{x}^k \gamma_A^k \left(\mu + \frac{1}{i-k+1}\right)^{i-k} \frac{1}{(i-k)!}(C_1(i-k)+\widehat{x})^{2(i-k)} e^{-\beta\widehat{x}} \\
&\qquad \leq \frac{C\mu}{1-X}(\gamma\varepsilon(M+1+1/\mu))^{M+1} e^{-\beta\widehat{x}/2},
\end{aligned}$$

where C, $\gamma > 0$ *are positive constants independent of* ε *and* μ.

Proof. The key ingredients of the proof are the estimates given in Lemma 4.8. For details, see Appendix B.5. □

Theorem 5.11. *Let $\beta > 0$ be given by Theorem 5.5, i.e., $\beta = \sqrt{a_{11}(0)}$ There exist constants C, $\gamma > 0$ such that, under the assumptions*

$$\left(\varepsilon(M+1) + \frac{\varepsilon}{\mu}\right)\gamma \leq 1 \qquad \widehat{x}\varepsilon\gamma \leq 1,$$

$\widehat{\mathbf{U}}^M$ given by (5.16) satisfies

$$\left|L_{\varepsilon,\mu}\widehat{\mathbf{U}}^M_{BL}(\widehat{x})\right| \leq C(\varepsilon(M+1+1/\mu)\gamma)^{M-1}e^{-\beta\widehat{x}}.$$

For the case $\widehat{x}\varepsilon\gamma > 1$ (but still assuming $\left(\varepsilon(M+1) + \frac{\varepsilon}{\mu}\right)\gamma \leq 1$), we have

$$\left|L_{\varepsilon,\mu}\widehat{\mathbf{U}}^M_{BL}(\widehat{x})\right| \leq Ce^{-\beta/2\widehat{x}}.$$

Proof. For the case of $\widehat{x}\varepsilon$ being sufficiently small, the starting point is (5.18), which is a sum of two contributions, namely, the double sum and the single sum (consisting of merely two terms). For the double sum, we recall that $\widehat{x} = x/\varepsilon$, so that Lemma 5.10 produces the desired estimate. For the single sum, we use the estimates of Theorem 5.5 for $\widehat{v}^L_{i+2}$. From Cauchy's Integral Formula for derivatives with contour being taken as $\partial B_1(x)$, we get

$$|(\widehat{v}^L_{i+2})''(x)| \leq C\frac{1}{(i+2)!}(C(i+3) + |\widehat{x}|)^{2(i+2)}e^{-\beta\widehat{x}}.$$

Using (4.21) and (4.22), we see that this contribution can be bounded by $C(\gamma(M+1)\varepsilon/\mu)^{M-1}$ for suitable C, $\gamma > 0$.

For the case $\widehat{x}\varepsilon\gamma > 1$, we use the exponential decay of $\widehat{\mathbf{U}}^M_{BL}$ expressed in Theorem 5.7. The factors ε^{-2}, which arise when computing $L_{\varepsilon,\mu}\widehat{\mathbf{U}}^M_{BL}$ can be absorbed by the expontially decaying term since $\varepsilon^{-1} \leq \gamma\widehat{x}$ (see Lemma 4.8). ☐

5.3 Boundary mismatch of the expansion

Theorem 5.12. *There exist constants C, b, $\gamma > 0$ such that under the assumptions*

$$\left(\varepsilon(M+1) + \frac{\varepsilon}{\mu}\right)\gamma \leq 1,$$

one has

$$\|\mathbf{W}_M + \widehat{\mathbf{U}}^M_{BL} + \widehat{\mathbf{V}}^M_{BL}\|_{L^\infty(\partial I)} \leq Ce^{-b/\varepsilon}.$$

Proof. We consider the left endpoint of I (the right endpoint is similar). By construction,

$$\|\mathbf{R}_M(0)\| \quad = \quad \left\|\mathbf{U}(0) - \left(\mathbf{W}_M(0) + \widehat{\mathbf{U}}^M_{BL}(0) + \widehat{\mathbf{V}}^M_{BL}(1/\varepsilon)\right)\right\| = \left\|\widehat{\mathbf{V}}^M(1/\varepsilon)\right\|. \tag{5.20}$$

The result now follows from Theorem 5.5. ☐

5.4 Proof of Theorem 5.1

The result of Theorem 5.1 now follows from combining Theorems 5.8, 5.11, 5.12.

6 The second two scale case: Case IV

Recall that this occurs when $\mu/1$ is small but ε/μ is *not* small. The main result is:

Theorem 6.1. *The solution* $\mathbf{U}$ *of (2.1) can be written as* $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \mathbf{R}$, *where* $\mathbf{W} \in \mathcal{A}(1, C_W, \gamma_W)$ *and* $\widetilde{\mathbf{U}} \in \mathcal{BL}^2(\delta\mu, C_{BL}, \gamma_{BL})$, *for suitable constants* C_W, C_{BL}, γ_W, γ_{BL}, $\delta > 0$ *independent of* μ *and* ε. *Furthermore,* $\mathbf{R}$ *satisfies*

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|L_{\varepsilon,\mu}\mathbf{R}\|_{L^2(I)} \;\;\leq\;\; C\,(\mu/\varepsilon)^{-2}\, e^{-b/\mu},$$

for some constants C, $b > 0$ *independent of* μ *and* ε. *In particular,* $\|\mathbf{R}\|_{E,I} \leq (\mu/\varepsilon)^2 e^{-b/\mu}$.

In this case we recall the stretched variables $\widetilde{x} = x/\mu$ and $\widetilde{x}^R = (1-x)/\mu$ and make the formal ansatz

$$\mathbf{U} \sim \sum_{i=0}^{\infty} \mu^i \left[\mathbf{U}_i(x) + \widetilde{\mathbf{U}}_i^L(\widetilde{x}) + \widetilde{\mathbf{U}}_i^R(\widetilde{x}^R)\right], \tag{6.1}$$

where again

$$\mathbf{U}_i(x) = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \qquad \mathbf{U}_i^L(\widetilde{x}) = \begin{pmatrix} \widetilde{u}_i(\widetilde{x}) \\ \widetilde{v}_i(\widetilde{x}) \end{pmatrix},$$

and an analogous definition for $\widetilde{\mathbf{U}}^R$. We also recall that the differential operator $L_{\varepsilon,\mu}$ takes the form (4.9) on the $\widetilde{x}$-scale. The separation of the slow (x) and the fast variables ($\widetilde{x}$ and $\widetilde{x}^R$) leads to the following equations:

$$\sum_{i=0}^{\infty} \mu^i \left(-\mathbf{E}^{\varepsilon,\mu}\mathbf{U}_i'' + \mathbf{A}(x)\mathbf{U}_i\right) \;=\; \mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{6.2a}$$

$$\sum_{i=0}^{\infty} \mu^i \left(-\mu^{-2}\mathbf{E}^{\varepsilon,\mu}(\widetilde{\mathbf{U}}_i^L)'' + \sum_{k=0}^{\infty} \mu^k \widetilde{x}^k \mathbf{A}_k \widetilde{\mathbf{U}}_i^L\right) \;=\; 0, \tag{6.2b}$$

and an analogous system for $\widetilde{\mathbf{U}}^R$. Next, for ε appearing in (6.2) we write $\varepsilon = \mu\frac{\varepsilon}{\mu}$ and equate like powers of μ to get with the matrix

$$\mathbf{E}^{\varepsilon/\mu,1} = \begin{pmatrix} \left(\frac{\varepsilon}{\mu}\right)^2 & 0 \\ 0 & 1 \end{pmatrix},$$

the following two recurrence relations:

$$-\mathbf{E}^{\varepsilon/\mu,1}\mathbf{U}_{i-2}'' + \mathbf{A}(x)\mathbf{U}_i \;=\; \mathbf{F}_i, \tag{6.3a}$$

$$-\mathbf{E}^{\varepsilon/\mu,1}(\widetilde{\mathbf{U}}_i^L)'' + \sum_{k=0}^{i} \widetilde{x}^{i-k}\mathbf{A}_{i-k}\widetilde{\mathbf{U}}_k^L \;=\; 0, \tag{6.3b}$$

where, as usual functions with negative index are assumed to be zero and $\mathbf{F}_i$ are defined by $\mathbf{F}_0 = (f, g)^T$ and $\mathbf{F}_i = 0$ for $i > 0$. The terms of the outer expansion, $\mathbf{U}_i$ are obtained immediately from (6.3a):

$$\mathbf{A}\mathbf{U}_i = \begin{pmatrix} f_i \\ g_i \end{pmatrix} + \begin{pmatrix} (\varepsilon/\mu)^2 u''_{i-2} \\ v''_{i-2} \end{pmatrix}. \tag{6.4}$$

The functions $\widetilde{\mathbf{U}}_i^L$ of the inner expansion are defined as the solutions of the following boundary value problems:

$$-\mathbf{E}^{\varepsilon/\mu,1}(\widetilde{\mathbf{U}}_i^L)'' + \mathbf{A}(0)\mathbf{U}_i^L = -\sum_{n=0}^{i-1} \widetilde{x}^{i-n} \widetilde{A}_{i-n}^L \widetilde{\mathbf{U}}_n^L, \tag{6.5a}$$

$$\widetilde{\mathbf{U}}_i^L(0) = -\mathbf{U}_i(0), \qquad \widetilde{\mathbf{U}}_i^L(\widetilde{x}) \to 0 \quad \text{as } \widetilde{x} \to \infty, \tag{6.5b}$$

and an analogous system for $\mathbf{U}_i^R$.

6.1 Properties of some solution operators

The functions $\mathbf{U}_i^L$ of the inner expansion are solutions to elliptic *systems*. In contrast to the previous arguments, which were based on estimates for *scalar* problems (for which strong tools such as maximum principles are readily available), we employ more general energy type arguments here to deal with the case of systems. We start by introducing exponentially weighted spaces on the half-line $(0, \infty)$, by defining the norm

$$\|u\|_{0,\beta}^2 := \int_{x=0}^{\infty} e^{2\beta x} |u(x)|^2 \, dx, \tag{6.6}$$

with the obvious interpretation in case u is vector valued. The following lemma shows that elliptic systems of the relevant type (6.5) can be solved in a setting of exponentially weighted spaces:

Lemma 6.2. *Let $\nu \in (0, 1]$ and set $\mathbf{E} := \begin{pmatrix} \nu^2 & 0 \\ 0 & 1 \end{pmatrix}$. Let $\mathbf{B} \in \mathbb{R}^2$ be positive definite, i.e., $x^\top \mathbf{B} x \geq \beta_0^2 |x|^2$ for all $x \in \mathbb{R}^2$. Then the bilinear form*

$$a(\mathbf{U}, \mathbf{V}) = \int_{x=0}^{\infty} \mathbf{U}' \cdot \mathbf{E}\mathbf{V}' + \mathbf{U} \cdot \mathbf{B}\mathbf{V} \, dx,$$

satisfies for a constant $C > 0$ that depends solely on β_0, the following inf-sup condition for all $0 < \beta < \beta_0$: For any $\mathbf{U} \in H_\beta^1(0, \infty)$, where

$$H_\beta^1(0, \infty) = \{u : ||u||_{1,\beta} < \infty\}, \tag{6.7}$$

there exists $\mathbf{V} \in H_{-\beta}^1$ with $\mathbf{V} \neq 0$ such that

$$a(\mathbf{U}, \mathbf{V}) \geq C \frac{1}{\beta_0 - \beta} \|\mathbf{U}\|_{1,\beta} \|\mathbf{V}\|_{1,-\beta}.$$

Here, we define for $\alpha \in \mathbb{R}$

$$\|\mathbf{U}\|_{1,\alpha}^2 := \int_{x=0}^{\infty} e^{2\alpha x} \left[\mathbf{U}' \cdot \mathbf{E}\mathbf{U}' + \mathbf{U} \cdot \mathbf{B}\mathbf{U}\right] \, dx. \tag{6.8}$$

Proof. Given $\mathbf{U} \in H^1_\beta(0,\infty)$ we set $\mathbf{V} := e^{2\beta x}\mathbf{U}$. Then $\mathbf{V}'(x) = 2\beta e^{2\beta x}\mathbf{U}(x) + e^{2\beta x}\mathbf{U}'(x)$ and thus

$$\begin{aligned} \|\mathbf{V}\|^2_{1,-\beta} &\leq 4(1+\frac{|\beta|}{\beta_0})^2\|\mathbf{U}\|^2_{1,\beta} \\ a(\mathbf{U},\mathbf{V}) &= \|\mathbf{U}\|^2_{1,\beta} + 2\beta\int_{x=0}^{\infty} e^{2\beta x}\mathbf{U}'\cdot\mathbf{E}\mathbf{U}\,dx \geq \left(1-\frac{|\beta|}{\beta_0}\right)\|\mathbf{U}\|^2_{1,\beta}. \end{aligned}$$

The result then follows. □

Lemma 6.3. *Let* $\mathbf{f}$ *satisfy* $\|\mathbf{f}\|_{0,\beta} < \infty$ *for some* $\beta \in [0,\beta_0)$*, and let* $\mathbf{g} \in \mathbb{R}^2$*. Then the solution* $\mathbf{U}$ *of*

$$-\mathbf{E}\mathbf{U}'' + \mathbf{B}\mathbf{U} = \mathbf{f}, \qquad \mathbf{U}(0) = \mathbf{g},$$

satisfies for a $C > 0$ *independent of* β*, the estimate*

$$\|\mathbf{U}\|_{1,\beta} \leq C(\beta_0-\beta)^{-1}\left[\|\mathbf{f}\|_{0,\beta} + |\mathbf{g}|\right].$$

Proof. Let $\beta_1 > \beta_0$ and set $\mathbf{U}_0 = \mathbf{g}e^{-\beta_1 x}$. Then $\mathbf{U}_0$ satisfies the desired estimates. The remainder $\mathbf{U}-\mathbf{U}_0$ satisfies an inhomogeneous differential equation with homogeneous boundary conditions at $x = 0$. Hence, Lemma 6.2 is applicable and yields the desired result. □

Lemma 6.4. *There exist* $\delta_0 > 0$ *and* $C_0 > 0$ *such that for every* $\delta \in (0,\delta_0]$ *and every* $m \in \mathbb{N}_0$*, there holds*

$$\sum_{n=0}^{i-1} \delta^{i-1-n}\frac{(n+m)^n}{(i+m)^i}\frac{(i+n+1+m)^{i+n+1+m}}{(2n+1+m)^{2n+1+m}} \leq C_0.$$

Proof. See Appendix C.1 □

6.2 Regularity of the functions $\mathbf{U}_i$, $\widetilde{\mathbf{U}}^L_i$, $\widetilde{\mathbf{U}}^R_i$

Lemma 6.5. *The function* $\mathbf{U}_i$ *defined by (6.4), are holomorphic in a neighborhood* G *of* $\overline{I}$ *and satisfy for some* C*,* $\widetilde{K} > 0$*,*

$$|\mathbf{U}_i(z)| \leq \widehat{C}\delta^{-i}\widehat{K}^i i^i \qquad \forall z \in G_\delta := \{z \in G \mid \operatorname{dist}(z,\partial G) > \delta\}, \qquad \forall i \geq 0.$$

Additionally, $\mathbf{U}_{2i+1} = 0$ *for all* $i \in \mathbb{N}_0$*.*

Proof. We note that $\varepsilon/\mu \leq 1$. The arguments are then analogous to those of [5, Lemma 2]. The arguments are also structurally similar to the more complicated case studied in Lemma 4.2. □

We now turn to estimates for the inner expansion functions $\widetilde{\mathbf{U}}^L_i$.

Theorem 6.6. *The functions* $\widetilde{\mathbf{U}}^L_i$ *defined by (6.5a), (6.5b) are entire functions and satisfy for all* $\beta \in (0,\beta_0)$ *(with* β_0 *given as in Lemma 6.2),*

$$\|\widetilde{\mathbf{U}}^L_i\|_{1,\beta} \leq \widetilde{C}K^i(\beta_0-\beta)^{-(2i+1)}i^i \qquad \forall i \in \mathbb{N}_0. \tag{6.9}$$

Proof. The case $i = 0$ follows from Lemma 6.3 and Lemma 6.5. For $i \geq 1$, we proceed by induction. In order to be able to apply Lemma 6.3, we define

$$\widetilde{\mathbf{F}}(\widetilde{x}) = \sum_{n=0}^{i-1} \widetilde{\mathbf{A}}_{i-n} \widetilde{x}^{i-n} \widetilde{\mathbf{U}}_n^L(\widetilde{x}).$$

Next, we estimate, for an arbitrary $\beta \in (0, \beta_0)$ and $\tilde{\beta} \in (\beta, \beta_0)$, with the aid of Lemma 4.8,

$$\begin{aligned}
\int_0^\infty e^{2\beta\widetilde{x}} \widetilde{x}^{2(i-n)} |\widetilde{\mathbf{U}}_n^L(\widetilde{x})|^2 \, d\widetilde{x} &\leq \int_0^\infty e^{-2(\tilde{\beta}-\beta)\widetilde{x}} \widetilde{x}^{2(i-n)} e^{2\tilde{\beta}\widetilde{x}} |\widetilde{\mathbf{U}}_n^L(\widetilde{x})|^2 \, d\widetilde{x} \\
&\leq \sup_{x>0} e^{-2(\tilde{\beta}-\beta)x} x^{2(n-i)} \|\widetilde{\mathbf{U}}_n^L\|_{0,\tilde{\beta}}^2 \\
&\leq e^{-2(i-n)} \left(\frac{i-n}{\tilde{\beta}-\beta} \right)^{2(i-n)} \|\widetilde{\mathbf{U}}_n^L\|_{0,\tilde{\beta}}^2 \\
&\leq \widetilde{C}^2 K^{2n} n^{2n} (\beta_0 - \tilde{\beta})^{-2(2n+1)} e^{-2(i-n)} \left(\frac{i-n}{\tilde{\beta}-\beta} \right)^{2(i-n)}.
\end{aligned}$$

Selecting $\tilde{\beta} = (\beta_0 - \beta)\kappa + \beta$ for some $\kappa \in (0, 1)$ to be chosen shortly, we get

$$\int_0^\infty e^{2\beta\widetilde{x}} \widetilde{x}^{2(i-n)} |\widetilde{\mathbf{U}}_n^L(\widetilde{x})|^2 \, d\widetilde{x} \leq$$

$$\leq \widetilde{C}^2 K^{2n} n^{2n} (\beta_0 - \beta)^{-2(i+n+1)} e^{-2(i-n)} (i-n)^{2(i-n)} \frac{1}{\kappa^{2(i-n)} (1-\kappa)^{2(2n+1)}}.$$

The choice $\kappa = \frac{i-n}{i+n+1}$ yields

$$\int_0^\infty e^{2\beta\widetilde{x}} \widetilde{x}^{2(i-n)} |\widetilde{\mathbf{U}}_n^L(x)|^2 \, dx \leq \widetilde{C}^2 K^{2n} (\beta_0 - \beta)^{-2(i+n+1)} e^{-2(i-n)} \frac{(n+i+1)^{2(i+n+1)}}{(2n+1)^{2(2n+1)}} n^{2n} \tag{6.10}$$

Hence,

$$\begin{aligned}
\|\widetilde{\mathbf{F}}\|_{0,\beta} &\leq \widetilde{C} C_A (\beta_0 - \beta)^{-2i} K^{i-1} i^i \sum_{n=0}^{i-1} \gamma_A^{i-n} K^{n-i+1} e^{-(i-n)} \frac{n^n}{i^i} \frac{(i+n+1)^{i+n+1}}{(2n+1)^{2n+1}} (\beta_0 - \beta)^{i-n-1} \\
&\leq \widetilde{C} C_A C_0 (\beta_0 - \beta)^{-2i} K^{i-1} i^i, \qquad (6.11)
\end{aligned}$$

where we appealed to Lemma 6.4 and used implicitly that K is sufficiently large. Using Lemma 6.5 for a fixed δ, we get from Lemma 6.3

$$\begin{aligned}
\|\widetilde{\mathbf{U}}_i\|_{1,\beta} &\leq (\beta_0 - \beta)^{-1} \left[\|\widetilde{\mathbf{F}}\|_{0,\beta} + \widehat{C} (\widehat{K}/\delta)^i i^i \right] \\
&\leq \widetilde{C} (\beta_0 - \beta)^{-2i-1} K^i i^i \left[K^{-1} C_A C_0 + \frac{\widehat{C}}{\widetilde{C}} (\beta_0 - \beta) \left(\frac{(\beta_0 - \beta)^2 \widehat{K}}{K} \right)^i \right].
\end{aligned}$$

The expression in square brackets can be bounded by 1 uniformly in i and $\beta \in (0, \beta_0)$ if we assume that $\widetilde{C}$ and K are sufficiently large. $\square$

We next refine the argument to include bounds on all derivatives of $\widetilde{\mathbf{U}}_i$:

Theorem 6.7. *There exist C_U, K_1, $K_2 > 0$, independent of β, ν, m and i, such that*

$$\|(\widetilde{\mathbf{U}}_i^L)^{(m)}\|_{0,\beta} \leq C_U(\beta_0 - \beta)^{-(2i+1+m)}(i+m)^i K_1^i K_2^m \nu^{-m}.$$

Proof. The cases $m = 0$ and $m = 1$ are covered by the above lemma (note: $\|\mathbf{E}^{-1}\| \leq \nu^{-2}$). The remaining cases are obtained as usual by differentiating the equation satisfied by $\widetilde{\mathbf{U}}_i^L$ and then proceed by induction on m. For details, see Appendix C.2 $\square$

We conclude this section by showing that the boundary layer functions

$$\widetilde{\mathbf{U}}_{BL}^M := \sum_{i=0}^M \mu^i \widetilde{\mathbf{U}}_i^L,$$

are entire functions.

Theorem 6.8. *Fix $\beta \in (0, \beta_0)$. There exist constants C, γ, $K > 0$ such that under the assumption $\mu(M+1)\gamma \leq 1$ there holds*

$$\left\| \frac{d^m}{d\widetilde{x}^m} \widetilde{\mathbf{U}}_{BL}^M \right\|_{0,\beta} \leq C K^m \nu^{-m} \qquad \forall m \in \mathbb{N}_0.$$

Proof. From Theorem 6.7, we see that for all $m \in \mathbb{N}_0$,

$$\begin{aligned}
\|(\widetilde{\mathbf{U}}_{BL}^M)^{(m)}\|_{0,\beta} &\leq C(\beta_0 - \beta)^{-1} \sum_{i=0}^M \mu^i (i+m)^i (\beta_0 - \beta)^{-2i} K_1^i K_2^m \nu^{-m} \\
&\leq C(\beta_0 - \beta)^{-1} K_2^m \nu^{-m} \sum_{i=0}^M (2(\beta_0 - \beta)^{-2} K_1 \mu i)^i + (2K_1 \mu m)^i \\
&\leq C(\beta_0 - \beta)^{-1} K_2^m \nu^{-m} \sum_{i=0}^M (2K_1 \mu M)^i \left((\beta_0 - \beta)^{-2i} + \left(\frac{m}{M}\right)^i \right) \leq C \widetilde{K}_2^m \nu^{-m}
\end{aligned}$$

for an appropriate $\widetilde{K}_2$ (depending on $\beta_0 - \beta$!), if we assume that μM is sufficiently small. The key observation for this fact is to note that for $m > M$ we have

$$\sum_{i=0}^M (m/M)^i \leq (M+1)(m/M)^M = m\frac{M+1}{M}(m/M)^{M-1} \leq m\frac{M+1}{M}\left(\frac{m}{M-1}\right)^{(M-1)/m)m},$$

and $n^{1/n} \to 1$ for $n \to \infty$. $\square$

6.3 Remainder estimates

6.3.1 Remainder estimates for the outer expansion: $\mathbf{F} - L_{\varepsilon,\mu}\mathbf{W}_M$

As before, the formal expansion (6.1) is truncated after M terms to yield the decomposition

$$\mathbf{U} = \sum_{i=0}^M \mu^i \mathbf{U}_i(x) + \sum_{i=0}^M \mu^i \widetilde{\mathbf{U}}_i^L(\widetilde{x}) + \sum_{i=0}^M \mu^i \widetilde{\mathbf{U}}_i^R(\widetilde{x}^R) + \mathbf{R}^M.$$

A calculation shows

$$\mathbf{F} - L_{\varepsilon,\mu} \sum_{i=0}^{M} \mu^i \mathbf{U}_i = \mu^{M+2} \mathbf{E}^{\varepsilon/\mu,1} \mathbf{U}_M''.$$

We therefore get

Theorem 6.9. *There exists $\gamma > 0$ independent of ε, μ such that for $\mu(M+1)\gamma \le 1$, there holds*

$$\|\mathbf{F} - L_{\varepsilon,\mu} \sum_{i=0}^{M} \mu^i \mathbf{U}_i\|_{L^\infty(I)} \le C(\mu(M+1)\gamma)^{M+2}.$$

Proof. The proof is analogous to that of [5, Thm. 6]. □

6.3.2 Remainder estimates for the inner expansion: $L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M$

We consider only the contribution from the left endpoint and define $\widetilde{\mathbf{U}}_{BL}^M := \sum_{i=0}^M \mu^i \widetilde{\mathbf{U}}_i^L$. A calculation shows

$$L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M = \sum_{i\ge M+1} \mu^i \sum_{k=0}^{M} \widetilde{x}^{i-k} \mathbf{A}_{i-k} \widetilde{\mathbf{U}}_k^L. \tag{6.12}$$

The following lemma provides an estimate (in an exponentially weighted space) for $L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M$ near the left endpoint:

Lemma 6.10. *There exist C, δ, $\beta > 0$, $K > 0$ such that*

$$\int_{\widetilde{x}=0}^{\delta/\mu} e^{2\beta\widetilde{x}} \left| \sum_{i\ge M+1} \mu^i \sum_{k=0}^{M} \widetilde{x}^{i-k} \mathbf{A}_{i-k} \widetilde{\mathbf{U}}_k^L(\widetilde{x}) \right|^2 d\widetilde{x} \le C(K\mu(M+1))^{2(M+1)}.$$

Proof. For fixed $\widetilde{x} > 0$ we estimate

$$\left| \sum_{i\ge M+1} \mu^i \sum_{k=0}^{M} \widetilde{x}^{i-k} \mathbf{A}_{i-k} \widetilde{\mathbf{U}}_k^L(\widetilde{x}) \right| \le C_A \sum_{k=0}^{M} |\widetilde{\mathbf{U}}_k^L(\widetilde{x})| \sum_{i=M+1}^{\infty} \gamma_A^{i-k} \widetilde{x}^{i-k} \mu^i. \tag{6.13}$$

For $\mu\widetilde{x}\gamma_A \le 1/2$, we estimate further

$$\sum_{i=M+1}^{\infty} \gamma_A^{i-k} \widetilde{x}^{i-k} \mu^i \le 2(\mu\widetilde{x}\gamma_A)^{M+1} (\widetilde{x}\gamma_A)^{-k} \le 2\mu^{M+1} (\widetilde{x}\gamma_A)^{M+1-k}.$$

Inserting this in (6.13), we see that employing the estimate (6.10) we can reason in exactly the same way as we have to reach (6.11), to get

$$\int_{\widetilde{x}=0}^{\delta/\mu} e^{2\beta\widetilde{x}} \left| \sum_{i\ge M+1} \mu^i \sum_{k=0}^{M} \widetilde{x}^{i-k} \mathbf{A}_{i-k} \widetilde{\mathbf{U}}_k^L(\widetilde{x}) \right|^2 d\widetilde{x} \le C(K\mu(M+1))^{2(M+1)},$$

which is the desired estimate. □

Lemma 6.10 provides an estimate for $L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}$ near the left endpoint; the following result provides an estimate on the whole interval I:

Theorem 6.11. *There exist C, γ, K, $b > 0$ such that for $\mu(M+1)\gamma \leq 1$, there holds*

$$\|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}\|_{L^2(I)} \leq C\sqrt{\mu}\left[(\mu(M+1)K)^{M+1} + \left(\frac{\mu}{\varepsilon}\right)^2 e^{-b/\mu}\right].$$

Proof. We merely consider the left boundary layer. Lemma 6.10 allows us to estimate for $L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}$ on the interval $(0,\delta)$ with the change of variables $x = \mu\widetilde{x}$:

$$\int_{x=0}^{\delta} |L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}|^2\,dx = \mu\int_{\widetilde{x}=0}^{\delta/\mu} |L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}(\widetilde{x})|^2\,d\widetilde{x} \leq C\mu((M+1)\mu K)^{2(M+1)}.$$

For the interval $(\delta,1)$, we note that

$$L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL} = -\mathbf{E}^{\nu,1}\left(\widetilde{\mathbf{U}}^M_{BL}\right)''(\widetilde{x}) + \mathbf{A}(x)\widetilde{\mathbf{U}}^M_{BL}(\widetilde{x}).$$

Hence, we can estimate

$$\int_{x=\delta}^{1} |L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}|^2\,dx \leq C\mu\int_{\widetilde{x}=\delta/\mu}^{\infty} |\left(\widetilde{\mathbf{U}}^M_{BL}\right)''(\widetilde{x})|^2 + |\widetilde{\mathbf{U}}^M_{BL}(\widetilde{x})|^2\,d\widetilde{x} \leq Ce^{-2\delta\beta/\mu}\left[\|\left(\widetilde{\mathbf{U}}^M_{BL}\right)''\|^2_{0,\beta} + \|\widetilde{\mathbf{U}}^M_{BL}\|^2_{0,\beta}\right].$$

With Theorem 6.8, we therefore arrive at

$$\|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}\|^2_{L^2(I)} \quad \leq \quad C\mu\left[(\mu(M+1)K)^{2(M+1)} + \nu^{-4}e^{-2\delta\beta/\mu}\right].$$

□

Remark 1. *The factor $(\mu/\varepsilon)^{-2}$ in the second term on the right-hand side of Theorem 6.11 is likely suboptimal.*

6.4 Boundary mismatch of the expansion

Theorem 6.12. *There exist constants C, b, $\gamma > 0$ such that under the assumptions*

$$\mu(M+1)\gamma \leq 1,$$

one has

$$\|\mathbf{W}_M + \widetilde{\mathbf{U}}^M_{BL} + \widetilde{\mathbf{V}}^M_{BL}\|_{L^\infty(\partial I)} \leq C\left(\frac{\mu}{\varepsilon}\right)^{1/2} e^{-b/\mu}.$$

Proof. Moreover, at the endpoints of the interval I, the remainder is small. To see this, consider the left endpoint of I (the right endpoint is similar). By construction,

$$\|\mathbf{R}_M(0)\| \quad = \quad \left\|\mathbf{U}(0) - \left(\mathbf{W}_M(0) + \widetilde{\mathbf{U}}^M_{BL}(0) + \widetilde{\mathbf{V}}^M_{BL}(1/\mu)\right)\right\| = \left\|\widetilde{\mathbf{V}}^M(1/\mu)\right\|. \qquad (6.14)$$

Theorem 6.8 informs us that $\widetilde{\mathbf{V}}_M$ is an entire function. In fact, from Theorem 6.8 and the Sobolev embedding theorem in the form $\|v\|^2_{L^\infty(\tilde I)} \le C\|v\|_{L^2(\tilde I)}\|v\|_{H^1(\tilde I)}$ applied to the interval $\tilde I = [1/\mu - 1, 1/\mu]$ of unit length, allows us to infer

$$\begin{aligned}
\|\widetilde{\mathbf{V}}^M_{BL}(1/\mu)\|^2_{L^\infty(\tilde I)} &\le C\|\widetilde{\mathbf{V}}^M_{BL}\|_{L^2(\tilde I)}\left[\|\widetilde{\mathbf{V}}^M_{BL}\|_{L^2(\tilde I)} + \|\left(\widetilde{\mathbf{V}}^M_{BL}\right)'\|_{L^2(\tilde I)}\right] \\
&\le Ce^{-2\beta/\mu}\|\widetilde{\mathbf{V}}^M_{BL}\|_{0,\beta}\left[\|\widetilde{\mathbf{V}}^M_{BL}\|_{0,\beta} + \|\left(\widetilde{\mathbf{V}}^M_{BL}\right)'\|_{0,\beta}\right] \le Ce^{-2\beta/\mu}\nu^{-1},
\end{aligned}$$

where the constant $C > 0$ depends only on the choice of β made in Theorem 6.8. Recalling the definition of ν concludes the argument. □

6.5 Proof of Theorem 6.1

The proof of Theorem 6.1 now follows by combining Theorems 6.9, 6.11, 6.12 with $M = O(1/\mu)$ and using the stability result (2.9).

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A Some miscellaneous results

Lemma A.1. *Let $\gamma > 0$ and $\eta \in (0,1)$. Then the function*

$$x \mapsto f(x) = \gamma^x(\eta x)^x,$$

is convex on $(0,\infty)$ and monotone decreasing on $(0, 1/(\eta\gamma e))$.

Proof. We only check the monotonicity assertion. To that end, we compute

$$\frac{d}{dx}\ln f(x) = \ln\gamma + 1 + \ln(\eta x)$$

and see that $\frac{d}{dx}\ln f(x) < 0$ for $x \in (0, 1/(\eta\gamma e))$. This proves the claim. □

The following lemma provides a proof for (2.11) and (2.10).

Lemma A.2. *Let $\alpha > 0$ and let $\mathbf{B} \in \mathbb{R}^2$ be such that $\vec\xi \cdot \mathbf{B}\vec\xi \ge \alpha\|\vec\xi\|^2$ for all $\vec\xi \in \mathbb{R}^2$. Then*

$$\mathbf{B}_{kk} \ge \alpha, \quad k = 1,2, \tag{A.1}$$
$$\det\mathbf{B} \ge \alpha\max\{\mathbf{B}_{11}, \mathbf{B}_{22}\}. \tag{A.2}$$

Proof. Property (A.1) follows immediately from the choice $\vec\xi = \vec e_k$, where $\vec e_k$ is the k-th unit vector.

To see property (A.2), we start by noting that $\mathbf{B}^{-1}$ is also positive definite:

$$x^T\mathbf{B}^{-1}x \overset{x=By}{=} y^T\mathbf{B}^T\mathbf{B}^{-1}\mathbf{B}y = y^T\mathbf{B}^Ty = (y^T\mathbf{B}^Ty)^T = y^T\mathbf{B}y > 0.$$

Property (A.2) follows from properties of the representation of $\mathbf{B}^{-1}$ in terms of the cofactor matrix. From

$$\mathbf{B}^{-1} = \frac{1}{\det\mathbf{B}}\mathbf{C}, \qquad \mathbf{C} := \begin{pmatrix} \mathbf{B}_{22} & -\mathbf{B}_{12} \\ -\mathbf{B}_{21} & \mathbf{B}_{11} \end{pmatrix}$$

and (A.1), we see that $\det\mathbf{B}$ is positive. The well-known fact that

$$\|\mathbf{B}^{-1}\|_2 \leq \alpha^{-1}$$

allows us to conclude

$$\left|\frac{1}{\det\mathbf{B}}\right| \|\mathbf{C}\|_2 = \|\mathbf{B}^{-1}\|_2 \leq \alpha^{-1},$$

which implies

$$\left|\frac{1}{\det(\mathbf{B})}\right| \leq \frac{1}{\alpha\|\mathbf{C}\|_2}.$$

Estimating $\|\mathbf{C}\|_2 \geq \max\{|\mathbf{C}_{11}|, |\mathbf{C}_{22}|\}$ and recalling $\det\mathbf{B} > 0$ concludes the argument. □

B Proofs for Section 5

Lemma B.1. *If u satisfies*

$$\left\|u^{(n)}\right\|_{\infty,I} \leq CK^n \max\{n, \mu^{-1}\}^n \quad \forall\, n \in \mathbb{N}_0,$$

for some positive constants C, K independent of μ, then its complex extension (denoted by $u(z)$) satisfies

$$|u(z)| \leq Ce^{\beta\,\mathrm{dist}(z,I)/\mu},$$

provided $\mathrm{dist}(z, I)$ *is sufficiently small.*

Proof. Fix $x \in I$ and let $B_r(x)$ be the ball of radius r centered at x. Then, by Taylor's theorem, we have for $z \in B_r(x)$

$$\begin{aligned} |u(z)| &\leq \sum_{k=0}^{\infty}\left|\frac{u^{(k)}(x)}{k!}(z-x)^k\right| \leq \sum_{k=0}^{\lfloor 1/\mu\rfloor} C\frac{\mu^{-k}}{k!}K^n r^k + \sum_{k=\lfloor 1/\mu\rfloor}^{\infty} C\frac{k^k}{k!}K^k r^k \\ &\leq C\sum_{k=0}^{\lfloor 1/\mu\rfloor}\left(\frac{rK}{\mu}\right)^k\frac{1}{k!} + C\sum_{k=\lfloor 1/\mu\rfloor}^{\infty} e^k K^k r^k. \end{aligned}$$

If $r < 1/(2eK)$ then the second sum above is bounded and we get

$$|u(z)| \leq Ce^{rK/\mu} + C \leq \widehat{C}e^{rK/\mu}.$$

□

B.1 Proof of Lemma 5.2

We work out the details here: We reduce to the case of homogeneous boundary conditions by introducing the linear function u_0 with $u_0(0) = g_-$ and $u_0(1) = g_+$. Then the difference $\tilde{u} := u - u_0$ solves

$$-\mu^2\tilde{u}'' + c\tilde{u} = g - cu_0 =: \tilde{g} \qquad \tilde{u}(0) = \tilde{u}(1) = 0$$

with $\|\tilde{g}\|_{L^\infty(I)} \leq C_g + |g_-| + |g_+|$. By the comparison principle (see, e.g., [2, Lemma 2.1] for the present form), we have

$$\|\tilde{u}\|_{L^\infty(I)} \leq \frac{1}{\underline{c}}\|\tilde{g}\|_{L^\infty(I)}. \tag{B.1}$$

From the differential equation, we furthermore get

$$\|\tilde{u}''\|_{L^\infty(I)} \leq \mu^{-2}\|\tilde{g}\|_{L^\infty(I)} + \frac{\|c\|_{L^\infty(I)}}{\underline{c}}\|\tilde{g}\|_{L^\infty(I)} \leq C\mu^{-2}\|\tilde{g}\|_{L^\infty(I)}, \tag{B.2}$$

where C is a constant that depends solely on c. By an interpolation inequality in Hölder spaces (see [3, Thm. 3.2.1]) we get from (B.1), (B.2), the estimate

$$\|\tilde{u}^{(1)}\|_{L^\infty(I)} \leq C\mu^{-1}\|\tilde{g}\|_{L^\infty(I)}, \tag{B.3}$$

for a constant $C > 0$ that depends only on c. Combining the above estimates together with $u = \tilde{u} + u_0$ and recalling that u_0 is linear, we have

$$\|u^{(n)}\|_{L^\infty(I)} \leq C_u \max\{n, \mu^{-1}\}^n, \qquad n \in \{0, 1, 2\}, \tag{B.4}$$

where $C_u = CC_{\tilde{g}}$ for a constant C that depends solely on the function c. Higher order estimates for u are obtained from the differential equation in the standard way. Differentiating the differential equation satisfied by u yields

$$-\mu^{-2}u^{(n+2)} = g^{(n)} - \sum_{\nu=0}^{n}\binom{n}{\nu}c^{(n-\nu)}u^{(\nu)}. \tag{B.5}$$

The analyticity of c implies the existence of C_c, $\gamma_c > 0$ such that

$$\|c^{(n)}\|_{L^\infty(I)} \leq C_c\gamma_c^n n! \qquad \forall n \in \mathbb{N}_0. \tag{B.6}$$

We claim

$$\|u^{(n)}\|_{L^\infty(I)} \leq C_u\gamma^n \max\{n, \mu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0. \tag{B.7}$$

This estimate is valid for $n \in \{0, 1, 2\}$ by (B.4). To see that it is valid in general, one proceeds by induction on n. From (B.5) we get in view (B.6)

$$\begin{aligned}
\mu^2\|u^{(n+2)}\|_{L^\infty(I)} &\leq C_{\tilde{g}}\gamma^n \max\{n, \mu^{-1}\}^n + \sum_{\nu=0}^{n}\binom{n}{\nu}C_c\gamma_c^\nu(\nu)!C_c\gamma^{n-\nu}\max\{\nu, \mu^{-1}\}^{n-\nu} \\
&\leq C_{\tilde{g}}\gamma^n \max\{n, \mu^{-1}\}^n + C_cC_u\sum_{\nu=0}^{n} n^\nu\gamma_c^\nu\gamma^{n-\nu}\max\{\nu, \mu^{-1}\}^\nu \\
&\leq C_{\tilde{g}}\gamma^n \max\{n, \mu^{-1}\}^n + C_cC_u\max\{n, \mu^{-1}\}^n\gamma^n\sum_{\nu=0}^{n}\left(\frac{\gamma_c}{\gamma}\right)^\nu \\
&\leq C_{\tilde{g}}\gamma^n \max\{n, \mu^{-1}\}^n + C_cC_u\max\{n, \mu^{-1}\}^n\gamma^n\frac{1}{1 - \gamma_c/\gamma} \\
&\leq C_u\gamma^{2+n}\max\{n, \mu^{-1}\}^n\left[\gamma^{-2}\left\{\frac{C_{\tilde{g}}}{C_u} + \frac{C_c}{1 - \gamma_c/\gamma}\right\}\right].
\end{aligned}$$

Since the expression in square brackets is bounded by 1 for $\gamma \geq \gamma_0$, if we make γ_0 sufficiently large, we have the desired estimate (B.7) for u.

B.2 Proof of Lemma 5.4

Set $\delta := 2\widehat{\delta} \leq 1/e$. Define $F : k \mapsto \delta^k \left(\frac{i+2}{i+1-k}\right)^{i-k}$ and note that

$$\ln(F) = k \ln \delta + (i-k)\left(\ln(i+2) - \ln(i+1-k)\right),$$

$$\begin{aligned}(\ln(F))' &= \ln \delta - \ln(i+2) + \ln(i+1-k) + \frac{i-k}{i+1-k} \\ &\leq \ln \delta + \ln\left(\frac{i+1-k}{i+2}\right) + 1 \leq 1 + \ln \delta.\end{aligned}$$

Since $\delta \leq 1/e$ we have $(\ln(F))' \leq 0$, and F is monotone decreasing which gives

$$\sum_{k=0}^{i} \widehat{\delta}^k \left(\frac{i+2}{i+1-k}\right)^{i-k} \leq \sum_{k=0}^{i} 2^{-k} e \leq 2e.$$

B.3 Proof of Theorem 5.5

We will only consider the inner expansions $\widehat{u}_i^L$, $\widehat{v}_i^L$ at the left endpoint of I, since for the expansions at the right endpoint the arguments are almost identical. The proof is by induction on i. For $i = 0$ we have from (5.4)–(5.7),

$$\left\{\begin{array}{c} -\mu^2 v_0'' + \frac{(a_{22}a_{11} - a_{12}a_{21})}{a_{11}} v_0 = g - \frac{a_{21}}{a_{11}} f \\ v_0(0) = v_0(1) = 0 \end{array}\right., \tag{B.8}$$

$$u_0 = \frac{1}{a_{11}}\left(f - a_{12} v_0\right), \tag{B.9}$$

$$\left\{\begin{array}{c} -\left(\widehat{u}_0^L\right)'' + a_{11}(0)\widehat{u}_0^L = 0 \\ \widehat{u}_0^L(0) = -u_0(0) \end{array}\right., \tag{B.10}$$

$$\widehat{v}_2^L = -\int_z^\infty \int_t^\infty a_{21}(0)\widehat{u}_0^L(\tau)\, d\tau\, dt. \tag{B.11}$$

Combining Lemmas 5.3 and 5.2, we can find a constant $C_4 > 0$ such that v_0 satisfies (5.13), which in turn shows that u_0 satisfies (5.12) by Lemma 5.3. The solution formula for $\widehat{u}_0^L$ then shows that (5.11) is valid. Finally, Lemma 4.6 establishes the desired bound (5.10) for $\widehat{v}_2$.

So, assume (5.10)–(5.13) hold for up to $i \geq 0$ and establish them for $i+1$. We will choose the constants C_i such that the ratios C_2/C_1, C_3/C_2, and C_4/C_3 are sufficiently small. Furthermore, the constants K_i and C_u, C_v are sufficiently large and are selected such that

$$K_1 = K_2 \qquad K_3 = K_4, \qquad C_u = C_v \tag{B.12a}$$

$$C_u^2 K_1 = K_3 \qquad C_u = C_v > 2/\min\{1/a_{11}(0), 1/a_{11}(1)\}. \tag{B.12b}$$

We start by bounding v_{i+1} and u_{i+1}. We first consider v_{i+1}, which satisfies (see eq. (5.4)):

$$\begin{cases} v''_{i+1} + \frac{(a_{22}a_{11}-a_{12}a_{21})}{a_{11}} v_{i+1} = -\mu^2 \frac{a_{21}}{a_{11}} u''_{i-1} \\ v_{i+1}(0) = -\widehat{v}^L_{i+1}(0)\ ,\ v_{i+1}(1) = -\widehat{v}^R_{i+1}(0) \end{cases} .$$

To that end, we assume, as we may, that γ is sufficiently large for Lemma 5.3 to be applicable for the function u_{i-1}. Together with the induction hypothesis, it then yields $\forall n \in \mathbb{N}_0$ (with the constant C' of Lemma 5.3 for the function $g = a_{21}/a_{11}$)

$$\|\mu^2 \left(\frac{a_{21}}{a_{11}} u''_{i-1}\right)^{(n)} \|_{L^\infty(I)} \leq \mu^2 C' C_3 K_3^{i-1}((i+1+n)\mu+1)^{i-1} \frac{(i-1)^{i-1}}{(i-1)!} \gamma^{n+2} \max\{n+2, \mu^{-1}\}^{n+2}.$$

From

$$\begin{aligned} \mu^2 \max\{n+2,\mu^{-1}\}^{n+2} &\leq \max\{n,\mu^{-1}\}^n \left(\frac{n+2}{n}\right)^{n+2} \max\{(n+2)\mu, 1\}^2 \\ &\leq C'' \max\{n,\mu^{-1}\}^n ((i+n+1)\mu+1)^2, \end{aligned}$$

for some $C'' > 0$ independent of n, we get $\forall n \in \mathbb{N}_0$

$$\|\mu^2 \left(\frac{a_{12}}{a_{11}} u''_{i-1}\right)^{(n)} \|_{L^\infty(I)} \leq C_3 \gamma^2 C' C'' K_3^{-2} K_3^{i+1}((i+1+n)\mu+1)^{i+1} \frac{(i-1)^{i-1}}{(i-1)!} \gamma^n \max\{n,\mu^{-1}\}^n.$$

The induction hypothesis for $\widehat{v}^L_{i+1}$ and $\widehat{v}^R_{i+1}$ and Lemma 5.2 therefore produce (with $\widetilde{C}$ of Lemma 5.2), for all $n \in \mathbb{N}_0$

$$\begin{aligned} \|v^{(n)}_{i+1}\|_{L^\infty(I)} &\leq \widetilde{C} \gamma^n \max\{n,\mu^{-1}\}^n \frac{(i-1)^{i-1}}{(i-1)!} \\ &\quad \times \left[C_3 K_3^{i-1}((i+1+n)\mu+1)^{i+1}\gamma^2 + 2C_1 K_1^{i-1} \left(\mu + \frac{1}{i-1}\right)^{i-1} (i-1)^{(i-1)} C_v^{2(i-1)} \right] \\ &\leq C_4 K_4^{i+1} \gamma^n \max\{n,\mu^{-1}\}^n (\mu(i+n)+1)^{i+1} \frac{(i+1)^{(i+1)}}{(i+1)!} \\ &\quad \times \left[C^{(IV)} \widetilde{C} \frac{C_3}{C_4} \gamma^2 K_3^{-2} (K_3/K_4)^{i+1} + 2C^{(IV)} \widetilde{C} \left(\frac{1}{C_v^2 K_1}\right)^2 \frac{C_1}{C_4} (C_v^2 K_1/K_4)^{i+1} \right] \\ &\leq C_4 K_4^{i+1} \gamma^n \max\{n,\mu^{-1}\}^n (\mu(i+n)+1)^{i+1} \frac{(i+1)^{(i+1)}}{(i+1)!} \\ &\quad \times \left[C^{(IV)} \widetilde{C} \frac{C_3}{C_4} \gamma^2 K_3^{-2} + 2C^{(IV)} \widetilde{C} \left(\frac{1}{C_v^2 K_1}\right)^2 \right], \end{aligned}$$

where we made use of $\frac{(i-1)^{i-1}}{(i-1)!} \leq C^{(IV)} \frac{(i+1)^{i+1}}{(i+1)!}$ and the simplifying assumptions on the relations between the constants K_i in (B.12). For the induction argument to work, we have to require that the expression in square brackets is bounded by 1:

$$\left[C^{(IV)} \widetilde{C} \frac{C_3}{C_4} \gamma^2 K_3^{-2} + 2C^{(IV)} \widetilde{C} \left(\frac{1}{C_v^2 K_1}\right)^2 \right] \overset{!}{\leq} 1. \tag{B.13}$$

We will collect further conditions on the constants C_i and K_i and see at the end of the proof that these conditions can be met.

Next, we turn to to u_{i+1}, which is given by (5.5):

$$u_{i+1}(z) = -\frac{1}{a_{11}(z)}\left[\mu^2 u''_{i-1}(z) + a_{12}(z)v_{i+1}(z)\right].$$

Lemma 5.3, the induction hypothesis and the just proved bounds for v_{i+1} then give for a constant C that depends solely on the coefficients a_{ij} of the differential operator,

$$\|u^{(n)}_{i+1}\|_{L^\infty(I)} \leq C_3 K_3^{i+1}(\mu(i+1+n)+1)^{i+1}\max\{n,\mu^{-1}\}^n \frac{(i+1)^{i+1}}{(i+1)!}\left[C\gamma^2 K_3^{-2} + C\frac{C_4}{C_3}\left(\frac{K_4}{K_3}\right)^{i+1}\right].$$

In view of $K_3 = K_4$ by (B.12), we recognize a second condition for the induction argument, namely,

$$\left[C\gamma^2 K_3^{-2} + C\frac{C_4}{C_3}\right] \overset{!}{\leq} 1. \tag{B.14}$$

Next, we consider $\widehat{u}^L_{i+1}$ which satisfies (see (5.6)):

$$\begin{cases} -\left(\widehat{u}^L_{i+1}\right)'' + a_{11}(0)\widehat{u}^L_{i+1} = -a_{12}(0)\widehat{v}^L_{i+1} - \sum_{k=1}^{i+1}\left(\frac{a_{11}^{(k)}(0)}{k!}\mu^k\widehat{x}^k\widehat{u}^L_{i+1-k} + \frac{a_{12}^{(k)}(0)}{k!}\mu^k\widehat{x}^k\widehat{v}^L_{i+1-k}\right) \\ \widehat{u}^L_{i+1}(0) = -u_{i+1}(0)\,,\ \widehat{u}^L_{i+1} \to 0 \text{ as } \widehat{x} \to \infty \end{cases}. \tag{B.15}$$

We will estimate the right-hand side of (B.15). To that end, since $1 \leq k \leq i+1$, we start with the observation

$$\begin{aligned}\mu^k\left(\mu + \frac{1}{i+2-k}\right)^{i+1-k} &\leq \mu^k\left(\mu + \frac{1}{i+2}\right)^{i+1-k}\left(\frac{i+2}{i+2-k}\right)^{i+1-k} \\ &\leq \left(\mu + \frac{1}{i+2}\right)^{i+1}\left(\frac{i+2}{i+2-k}\right)^{i+1-k},\end{aligned}$$

so that we can estimate

$$\begin{aligned}\sum_{k=1}^{i+1}\frac{|a_{11}^{(k)}(0)|}{k!}|z|^k\mu^k|\widehat{u}^L_{i+1-k}| &\leq C_a C_2 K_2^{i+1}\left(\mu + \frac{1}{i+2}\right)^{i+1}\frac{1}{i!}(C_v(i+1)+|z|)^{2i+1}e^{-\beta\,\mathrm{Re}(z)} \\ &\quad \times \sum_{k=1}^{i+1}\left(\frac{\gamma_a}{K_2}\right)^k\left(\frac{i+2}{i+2-k}\right)^{i+1-k}.\end{aligned}$$

If K_2 is sufficiently large, then the sum is bounded by $2\frac{\gamma_a}{K_2}e$ by Lemma 5.4. Analogously, we get (note that the case $i=0$ leads to empty sums)

$$\begin{aligned}&\sum_{k=0}^{i+1}\frac{|a_{12}^{(k)}(0)|}{k!}|z|^k\mu^k|\widehat{v}^L_{i+1-k}| = \sum_{k=0}^{i-1}\frac{|a_{12}^{(k)}(0)|}{k!}|z|^k\mu^k|\widehat{v}^L_{i+1-k}| \\ &\leq C_a C_1 K_1^{i-1}\left(\mu + \frac{1}{i}\right)^{i-1}\frac{1}{i!}(C_v(i-1)+|z|)^{2i-1}e^{-\beta\,\mathrm{Re}(z)}\sum_{k=0}^{i-1}\left(\frac{\gamma_a}{K_1}\right)^k\left(\frac{i}{i-k}\right)^{i-1-k} \\ &\leq K_1^{-2}C''' C_a C_1 K_1^{i+1}\left(\mu + \frac{1}{i}\right)^{i+1}\frac{1}{i!}(C_v(i+1)+|z|)^{2i+1}e^{-\beta\,\mathrm{Re}(z)},\end{aligned}$$

where we appealed again to Lemma 5.4 and used that $K_1 = K_2$ is sufficiently large to bound the sum by $2e$, and noticed additionally that

$$\left(\mu+\frac{1}{i}\right)^{i-1} \leq \left(\mu+\frac{1}{i+2}\right)^{i-1}\left(\frac{i+2}{i}\right)^{i-1} \leq \left(\mu+\frac{1}{i+2}\right)^{i+1}(i+1)^2 \underbrace{\sup_{i\geq 1}\left(\frac{i+2}{i+1}\right)^2\left(\frac{i+2}{i}\right)^{i-1}}_{=:C'''/(2e)},$$

where the last supremum is finite.

These estimates allow us to bound $\widehat{u}^L_{i+1}$ with the aid of Lemma 4.5 to arrive at

$$\begin{aligned}|\widehat{u}^L_{i+1}(z)| &\leq C_2 K_2^{i+1}\left(\mu+\frac{1}{i+2}\right)^{i+1} e^{-\beta\,\mathrm{Re}(z)}\frac{1}{(i+1)!}(C_v(i+1)+|z|)^{2(i+1)} \\ &\quad\times\left[C\frac{\gamma_a}{K_2}+K_1^{-2}\frac{C_1}{C_2}\left(\frac{K_1}{K_2}\right)^{i+1}+\frac{C_3}{C_2}\left(\frac{K_3}{C_v^2 K_1}\right)^{i+1}\right].\end{aligned}$$

Together with the simplifying assumptions (B.12) we see that for the induction argument to work, we need to require

$$\left[C\frac{\gamma_a}{K_2}+K_1^{-2}\frac{C_1}{C_2}+\frac{C_3}{C_2}\right] \overset{!}{\leq} 1. \tag{B.16}$$

Finally, for $\widehat{v}^L_{i+3}$ we have from (5.7)

$$\begin{aligned}\left|\widehat{v}^L_{i+3}(z)\right| &= \left|\int_z^\infty\int_t^\infty\sum_{k=0}^{i+1}\left(\frac{a_{21}^{(k)}(0)}{k!}\mu^k\widehat{\tau}^k\widehat{u}^L_{i+1-k}+\frac{a_{22}^{(k)}(0)}{k!}\mu^k\widehat{\tau}^k\widehat{v}^L_{i+1-k}\right)d\widehat{\tau}dt\right| \\ &\leq \sum_{k=0}^{i}\frac{1}{k!}\max\left\{\left|a_{21}^{(k)}(0)\right|,\left|a_{22}^{(k)}(0)\right|\right\}\left\{\left|\int_z^\infty\int_t^\infty\widehat{\tau}^k\widehat{u}^L_{i-k}d\widehat{\tau}dt\right|+\left|\int_z^\infty\int_t^\infty\widehat{\tau}^k\widehat{v}^L_{i-k}d\widehat{\tau}dt\right|\right\}.\end{aligned}$$

Proceeding analogously as before, we find that this last sum can be bounded as

$$|rhs(z)| \leq C_1K_1^{i+1}\left(\mu+\frac{1}{i+2}\right)^{i+1}\frac{1}{(i+1)!}(C_v(i+1)+|z|)^{2(i+1)}e^{-\beta\,\mathrm{Re}(z)}\left[C\frac{C_2}{C_1}\left(\frac{K_2}{K_1}\right)^{i+1}+CK_1^{-2}\right],$$

where the constant $C > 0$ is suitably chosen. Lemma 4.6 then gives

$$\begin{aligned}\left|\widehat{v}^L_{i+3}(z)\right| &\leq K_1^{i+1}\left(\mu+\frac{1}{i+2}\right)^{i+1}\frac{1}{(i+1)!}(C_v(i+1)+|z|)^{2(i+1)}e^{-\beta\,\mathrm{Re}(z)} \\ &\quad\times\left[C\frac{C_2}{C_1}\left(\frac{K_2}{K_1}\right)^{i+1}+CK_1^{-2}\right]\frac{1}{\beta^2}\left(\frac{1}{1-(2(i+1))/(\beta C_v(i+1))}\right)^2.\end{aligned}$$

Hence, by our assumption (B.12), we see that we can find C' (depending solely on β) such that our induction argument will work if we can satisfy

$$C'\left[C\frac{C_2}{C_1}+CK_1^{-2}\right] \overset{!}{\leq} 1. \tag{B.17}$$

In total, we have completed the induction argument if we can select the constants C_i and K_i such that (B.13), (B.14), (B.16), and (B.17) are satisfied. Inspection shows that this is the case by taking the K_i sufficiently large and appopriately controlling the ratios C_2/C_1, C_3/C_2, and C_4/C_3.

B.4 Proof of Theorem 5.7

From Lemma 5.6 and the estimates (4.21), (4.22), we get

$$\left|\left(\widehat{\mathbf{U}}_{BL}^{M}\right)^{(n)}(\widehat{x})\right| \;\leq\; Ce^{\beta\widehat{x}}\gamma_2^n\sum_{i=0}^{M}\left(\frac{\varepsilon}{\mu}\right)^i(i\mu+1)^i\gamma_1^i \leq Ce^{\beta\widehat{x}}\gamma_2^n\sum_{i=0}^{M}\left(M\varepsilon+\frac{\varepsilon}{\mu}\right)^i\gamma_1^i \leq C\gamma_2^n,$$

where, in the last step we have used the assumption that $\varepsilon(M+1+\varepsilon/\mu)$ is sufficiently small so that the sum can be estimated by a convergent geometric series.

B.5 Proof of Lemma 5.10

We start by noting that the change of summation index k to $\ell = i-k$ leads us to having to estimate

$$S := \sum_{i\geq M+1}\left(\frac{\varepsilon}{\mu}\right)^i\sum_{\ell=0}^{M}\mu^{i-\ell}\widehat{x}^{i-\ell}\gamma_A^{i-\ell}\left(\mu+\frac{1}{\ell+1}\right)^\ell\frac{1}{\ell!}(C_1\ell+\widehat{x})^{2\ell}e^{-\beta\widehat{x}}.$$

We start with the elementary observation (cf. Lemma 4.8)

$$(C_1\ell+\widehat{x})^{2\ell} \leq 2^\ell(C_1\ell)^{2\ell}+2^\ell\widehat{x}^{2\ell} \leq \gamma^\ell\left(\ell^{2\ell}+\widehat{x}^{2\ell}\right),$$

for suitable $\gamma > 0$. Next, we estimate

$$\frac{1}{\ell!}(C_1\ell+\widehat{x})^{2\ell}e^{-\beta\widehat{x}} \leq C\gamma^\ell[\ell^\ell e^\ell+\frac{1}{\ell!}\widehat{x}^{2\ell}e^{-\widehat{x}\beta/4}]e^{-3\beta\widetilde{x}/4} \leq C\tilde{\gamma}^\ell\ell^\ell e^{-3\beta\widetilde{x}/4},$$

where we suitably chose $\tilde{\gamma}$ independent of ℓ and $\widetilde{x}$. We therefore conclude

$$\begin{aligned}
&\sum_{i\geq M+1}\left(\frac{\varepsilon}{\mu}\right)^i\sum_{\ell=0}^{M}\mu^{i-\ell}\widehat{x}^{i-\ell}\gamma_A^{i-\ell}\left(\mu+\frac{1}{\ell+1}\right)^\ell\frac{1}{\ell!}(C_1\ell+\widehat{x})^{2\ell}e^{-\beta\widehat{x}}\\
\leq\;& Ce^{-3\beta\widehat{x}/4}\sum_{i\geq M+1}(\varepsilon\widehat{x}\gamma_A)^i\sum_{\ell=0}^{M}\widehat{x}^{-\ell}\gamma_A^{-\ell}\left(1+\frac{1}{\mu(\ell+1)}\right)^\ell\ell^\ell\tilde{\gamma}^\ell\\
\leq\;& Ce^{-3\beta\widehat{x}/4}\frac{1}{1-X}(\varepsilon\widehat{x}\gamma_A)^{M+1}\sum_{\ell=0}^{M}\widehat{x}^{-\ell}\gamma_A^{-\ell}\left(\ell+\frac{1}{\mu}\right)^\ell\tilde{\gamma}^\ell\\
\leq\;& \frac{C}{1-X}e^{-\beta\widehat{x}/2}\varepsilon^{M+1}\sum_{\ell=0}^{M}(\gamma_A\widehat{x})^{M+1-\ell}e^{-\beta\widehat{x}/4}\tilde{\gamma}^\ell(\ell+1/\mu)^\ell.
\end{aligned}$$

Again, elementary considerations (cf. Lemma 4.8) show $\widehat{x}^{M+1-\ell}e^{-\beta\widehat{x}/4} \leq \gamma_1^{M+1-\ell}(M+1-\ell)^{M+1-\ell}$ for suitable $\gamma_1 > 0$, so that we arrive at

$$\begin{aligned}
S \;&\leq\; \frac{C}{1-X}e^{-\beta\widehat{x}/2}\varepsilon^{M+1}\sum_{\ell=0}^{M}(\gamma_A\gamma_1)^{M+1-\ell}(M+1-\ell)^{M+1-\ell}\tilde{\gamma}^\ell(\ell+1/\mu)^\ell\\
&\leq\; \frac{C\mu}{1-X}e^{-\beta\widehat{x}/2}\varepsilon^{M+1}\gamma_2^{M+1}(M+1+1/\mu)^{M+1},
\end{aligned}$$

where, in the last step we have used the convexity of the function

$$\ell \mapsto a^{M+1-\ell}(M+1-\ell)^{M+1-\ell}(\ell+1/\mu)^\ell$$

on the intervall $[0, M]$ – see also the related Lemma D.1.

C Proofs for Section 6

C.1 Proof of Lemma 6.4

The proof consists of showing that the terms of the sum can be bounded by the terms of a convergent geometric series if δ is sufficiently small. To simplify some notation, we restrict our attention to the case $i \geq 2$ (the cases $i = 0$ and $i = 1$ being easily seen). We denote the terms of the sum by $F(n)$ and note that it suffices to study the sum $\sum_{n=1}^{i-1} F(n)$ since the case $n = 0$ produces the term

$$F(0) = \delta^{i-1} \frac{(i+1+m)^{i+1+m}}{(i+m)^i (m+1)^{m+1}} = \delta^{i-1} \left(\frac{i+1+m}{i+m} \right)^i \left(\frac{i+1+m}{m+1} \right)^{m+1} \leq \delta^{i-1} e e^i,$$

which is bounded uniformly in i if $\delta \leq \delta_0 \leq e^{-1}$.

We define, for $n \in [1, i-1]$,

$$\begin{aligned} f(n) &:= \ln F(n) \\ &= (i-1-n)\ln\delta + n\ln(n+m) + (i+n+1+m)\ln(i+n+1+m) - \\ &\quad - i\ln(i+m) - (2n+1+m)\ln(2n+1+m) \end{aligned}$$

and compute

$$f'(n) = \ln \frac{(n+m)(n+i+1+m)}{(2n+1+m)^2} - \ln\delta - 1 + \frac{n}{n+m}.$$

It is easy to see that $f'(n) \geq -1 - \ln 4 - \ln\delta$ for $n \geq 1$ so that we can find a constant $c > 0$ with $f'(n) \geq c > 0$ for all $n \geq 1$ by taking δ sufficiently small. By the mean value theorem, we therefore get for each n, that $\ln \frac{F(n+1)}{F(n)} = f(n+1) - f(n) \geq c$. Hence, $F(n) \leq e^{-c} F(n+1)$. Iterating this estimate, we get

$$F(n) \leq e^{-(i-1-n)c} F(i-1).$$

Hence,

$$\sum_{n=1}^{i-1} F(n) \leq \sum_{n=1}^{i-1} e^{-(i-1-n)c} F(i-1) \leq \frac{1}{1-e^{-c}} F(i-1).$$

The argument is concluded by noting that $F(i-1)$ is bounded uniformly in i and m as the following rearrangement shows:

$$F(i-1) = \frac{(i-1+m)^{i-1}}{(i+m)^i} \frac{(2i+m)^{2i+m}}{(2i+m-1)^{2i+m-1}} = \frac{2i+m}{i+m} \left(\frac{i-1+m}{i+m} \right)^{i-1} \left(\frac{2i+m}{2i+m-1} \right)^{2i+m-1}.$$

C.2 Proof of Theorem 6.7

Specifically, differentiating m times yields (we write $\widetilde{\mathbf{U}}_i$ instead of $\widetilde{\mathbf{U}}_i^L$) with $\mathbf{B} = \mathbf{A}(0)$

$$\begin{aligned} -\mathbf{E}\widetilde{\mathbf{U}}_i^{(m+2)} &= \mathbf{B}\widetilde{\mathbf{U}}_i^{(m)} + \sum_{n=0}^{i-1} \left(\widetilde{x}^{i-n} \mathbf{A}_{i-n} \widetilde{\mathbf{U}}_n \right)^{(m)} \\ &= \mathbf{B}\widetilde{\mathbf{U}}_i^{(m)} + \sum_{n=0}^{i-1} \mathbf{A}_{i-n} \sum_{j=0}^{\min\{m,i-n\}} \binom{m}{j} \binom{i-n}{j} j! \widetilde{x}^{i-n-j} \widetilde{\mathbf{U}}_n^{(m-j)}. \end{aligned} \tag{C.1}$$

We abbreviate

$$\mathbf{F}_n := \sum_{j=0}^{\min\{m,i-n\}} \binom{m}{j}\binom{i-n}{j} j!\widetilde{x}^{i-n-j}\widetilde{\mathbf{U}}_n^{(m-j)},$$

and we proceed as in the proof of Theorem 6.6. The induction hypothesis yields, for any $\tilde{\beta} \in (\beta, \beta_0)$,

$$\begin{aligned}
\|\widetilde{x}^{i-n-j}\widetilde{\mathbf{U}}_n^{(m-j)}\|_{0,\beta} &\leq \sup_{\widetilde{x}>0} \widetilde{x}^{i-n-j} e^{-(\tilde{\beta}-\beta)\widetilde{x}} \|\widetilde{\mathbf{U}}_n^{(m-j)}\|_{0,\tilde{\beta}} \\
&\leq C_U \left(\frac{i-n-j}{\tilde{\beta}-\beta}\right)^{i-n-j} (\beta_0-\tilde{\beta})^{-(2n+1+m)}(n+m-j)^n K_1^n K_2^{m-j}\nu^{-(m-j)}.
\end{aligned}$$

For $j < i-n$, we select $\tilde{\beta} = \beta + \kappa(\beta_0 - \beta)$ with $\kappa = \frac{i-n-j}{i+n-j+m+1}$ and get

$$\|\widetilde{x}^{i-n-j}\widetilde{\mathbf{U}}_n^{(m-j)}\|_{0,\beta} \leq \tag{C.2}$$
$$C_U(\beta_0-\beta)^{-(i+n-j+1+m)}\frac{(i+n-j+1+m)^{i+n-j+1+m}}{(2n+1+m)^{2n+1+m}}(n+m-j)^n K_1^n K_2^{m-j}\nu^{-(m-j)};$$

the induction hypothesis shows that the estimate (C.2) is also true for $j = i-n$. Therefore, by estimating $n+m-j \leq n+m$, we get

$$\begin{aligned}
\|\mathbf{F}_n\|_{0,\beta} &\leq C_U(\beta_0-\beta)^{-(i+n+1+m)}K_1^n K_2^m \nu^{-m}\frac{(n+m)^n}{(2n+1+m)^{2n+1+m}} \\
&\quad\times \sum_{j=0}^{\min\{i-n,m\}} \binom{m}{j}\binom{i-n}{j} j!(\beta_0-\beta)^j K_2^{-j}\nu^j (i+n-j+1+m)^{i+n-j+1+m}.
\end{aligned}$$

With the estimates $\binom{m}{j}j! \leq m^j$ we bound

$$\begin{aligned}
&\sum_{j=0}^{\min\{i-n,m\}} \binom{m}{j}\binom{i-n}{j} j!(\beta_0-\beta)^j K_2^{-j}\nu^j(i+n-j+1+m)^{i+n-j+1+m} \\
&\leq \sum_{j=0}^{i-n}\binom{i-n}{j}(\beta_0-\beta)^j K_2^{-j}\nu^j (i+n+1+m)^{i+n+1+m} \\
&= \left(1+(\beta_0-\beta)\nu K_2^{-1}\right)^{i-n}(i+n+1+m)^{i+n+1+m},
\end{aligned}$$

where we have recognized a binomial sum in the last equality. Hence,

$$\begin{aligned}
\sum_{n=0}^{i-1}\|\mathbf{A}_{i-n}\|_2\|\mathbf{F}_n\|_{0,\beta} &\leq C_U C_A \gamma_A (\beta_0-\beta)^{-(2i+1+m)}K_1^{i-1}K_2^m\nu^{-m} \\
&\quad \times \sum_{n=0}^{i-1}\gamma_A^{i-1-n}K_1^{n-i+1}\left(1+(\beta_0-\beta)\nu K_2^{-1}\right)^{i-n}\frac{(n+m)^n(i+n+1+m)^{i+n+1+m}}{(2n+1+m)^{2n+1+m}} \\
&\leq C_U C_0 C_A \gamma_A (\beta_0-\beta)^{-(2i+1+m)}K_1^{i-1}K_2^m \nu^{-m}(i+m)^i,
\end{aligned}$$

where we appealed to Lemma 6.4 and implicitly used that K_1 and K_2 are such that $\gamma_A K_1^{-1}(1 + (\beta_0-\beta)\nu K_2^{-1})$ is sufficiently small. Using $\|\mathbf{E}^{-1}\|_2 \leq \nu^{-2}$, we therefore get from (C.1)

$$\begin{aligned}
\|\mathbf{U}_i^{(m+2)}\|_{0,\beta} &\leq \nu^{-2}C_U(\beta_0-\beta)^{-(2i+1+m)}(i+m)^i K_1^i K_2^m \nu^{-m}\left[\|\mathbf{B}\|_2 + C_0 C_A\gamma_A K_1^{-1}\right] \\
&\leq C_U(\beta_0-\beta)^{-(2i+1+m)}(i+m)^i K_1^i K_2^{m+2}\nu^{-(m+2)}\left[K_2^{-2}\|\mathbf{B}\|_2 + K_2^{-1}C_0C_A\gamma_A K_1^{-1}\right];
\end{aligned}$$

the expression in square brackets can be bounded by 1 if K_1 and K_2 are sufficiently large.

D Proofs for Section 4

D.1 Proof of Lemma 4.2

The proof is by induction on i. For $i = 0$, equation (4.17) gives all the assertions (trivially). So, assume (4.18)–(4.20) hold and establish them for $i + 1$.

We first consider (4.18) and assuming

$$\overrightarrow{0} = \begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} u''_{i-2,j-2} \\ v''_{i-2,j} \end{bmatrix} \ \forall\, j > i,$$

we want to show that

$$\overrightarrow{0} = \begin{bmatrix} u_{i+1,j} \\ v_{i+1,j} \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} u''_{i-1,j-2} \\ v''_{i-1,j} \end{bmatrix} \ \forall\, j > i+1.$$

By the induction hypothesis and $j > i + 1 \geq i - 1$, we have $v_{i-1,j} = 0$. Also, from $j > i + 1$ we get $j - 2 = j - 1 - 1 > i - 1$, so that the induction hypothesis implies $u_{i-1,j-2} = 0$. Therefore, the right-hand side of (4.17) vanishes and thus the induction step for (4.18) is accomplished.

For (4.19) let $\mathcal{M}(\mathcal{I}) := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \colon i \leq \mathcal{I},\ i \text{ odd or } j \text{ odd}\}$. We proceed by induction on $\mathcal{I}$ by assuming (4.19) to be true up to $\mathcal{I}$. For $i \leq \mathcal{I}$ with $(i,j) \in \mathcal{M}(\mathcal{I})$, we have that $(i+1,j) \in \mathcal{M}(\mathcal{I}+1)$ implies $(i-1,j) \in \mathcal{M}(\mathcal{I})$ (either $i+1$ is odd and then $i-1$ is odd or j is odd) and additionally $(i-1,j-2) \in \mathcal{M}(\mathcal{I})$. Hence, for $(i+1,j) \in \mathcal{M}(\mathcal{I}+1)$, by the induction hypothesis, the right-hand side of (4.17) vanishes, which proves the induction step.

We finally consider (4.20) and we want to show

$$|u_{i+1,j}(z)| + |v_{i+1,j}(z)| \leq C_S \delta^{-i-1} K^{i+1} (i+1)^{i+1} \ \forall\, z \in G_\delta.$$

We set

$$C_A := \sup_{x \in I} \|\mathbf{A}^{-1}(x)\|_{\ell^1},$$

where $\|(x_1,x_2)\|_{\ell^1} := |x_1| + |x_2|$ denotes the usual ℓ^1-norm and ensure that K satisfies $2C_A/K^2 \leq 1$.

Let $\kappa \in (0,1)$. By (4.17), the induction hypothesis with $G_{(1-\kappa)\delta} \subset G_\delta$ and Cauchy's Integral Theorem, we have

$$\begin{aligned}
|u_{i+1,j}(z)| + |v_{i+1,j}(z)| &\leq C_A \left(\left| u''_{i-1,j-2}(z) \right| + \left| v''_{i-1,j}(z) \right| \right) \\
&\leq C_S C_A \frac{2}{(\kappa\delta)^2} \left((1-\kappa)\delta \right)^{-i+1} K^{i-1} (i-1)^{(i-1)} \\
&\leq C_S \delta^{-i-1} K^{i+1} (i+1)^{(i+1)} \left[\frac{1}{K^2} \frac{1}{(i+1)^2} \frac{2C_A}{\kappa^2 (1-\kappa)^{i-1}} \left(\frac{i-1}{i+1} \right)^{i-1} \right].
\end{aligned}$$

The choice $\kappa = \frac{2}{i+1}$ gives

$$|u_{i+1,j}(z)| + |v_{i+1,j}(z)| \leq C_S \delta^{-(i+1)} K^{i+1} (i+1)^{(i+1)} \left[\frac{C_A}{2K^2} \right] \leq C_S \delta^{-(i+1)} K^{i+1} (i+1)^{(i+1)}$$

by the choice of K.

D.2 Proof of Lemma 4.5

We provide some details for the case $\operatorname{Re} z < 0$. For $z \in (0, \infty)$, the use of a Green's function gives the following representation of the solution $u(z)$:

$$\begin{aligned} u(z) &= \frac{1}{2a^2}\Big(e^{-az}\int_0^{az} e^y f(y/a)\,dy + e^{az}\int_{az}^{\infty} e^{-y} f(y/a)\,dy \\ &\quad - e^{-az}\int_0^{\infty} e^{-y} f(y/a)\,dy\Big) + ge^{-az}. \end{aligned}$$

Analytic continuation then removes the restriction to $(0, \infty)$. In order to get the desired bound, we estimate each of these four terms separately. We restrict here our attention to the case of $\operatorname{Re} z < 0$.

For the first integral, we use as the path of integration the straight line connecting 0 and az to get

$$\begin{aligned} \left| e^{-az}\int_0^{az} e^y f(y/a)\,dy \right| &\le e^{-\operatorname{Re}(az)}\int_0^1 C_f(q + t|z|)^j |az|\, e^{-\operatorname{Re}(t\overline{a}z)} e^{\operatorname{Re}(taz)}\,dt \\ &\le e^{-\operatorname{Re}(\overline{a}z)}\int_0^1 C_f(q + t|z|)^j |az|\, e^{\operatorname{Re}((1-t)z(\overline{a}-a))}\,dt \\ &\le C_f e^{-\operatorname{Re}(\overline{a}z)}\frac{a}{j+1}\left\{(q + |z|)^{j+1} - q^{j+1}\right\}. \end{aligned}$$

For the third integral, we calculate with [1, eq. 8.353.5] and the incomplete Gamma-function $\Gamma(\cdot, \cdot)$,

$$\begin{aligned} \left|\int_0^{\infty} e^{-y} f(y/a)\,dy\right| &\le C_f\int_0^{\infty} e^{-y-\underline{a}/ay}(q + y/a)^j\,dy \\ &= C_f a^{-j}(1 + \underline{a}/a)^{-(j+1)} e^{aq(1+\underline{a}/a)}\Gamma(j + 1, aq(1 + \underline{a}/a)) \\ &= C_f a(a + \underline{a})^{-(j+1)} e^{q(a+\underline{a})}\Gamma(j + 1, q(a + \underline{a})). \end{aligned}$$

In view of the assumption $(a + \underline{a})q \ge 2j + 1 \ge j$, we may employ the estimate

$$|\Gamma(\alpha, \xi)| \le \frac{|e^{-\xi}\xi^{\alpha}|}{|\xi| - \alpha_0}, \qquad \alpha_0 = \max\{\alpha - 1, 0\}, \quad \operatorname{Re}(\xi) \ge 0, \quad |\xi| > \alpha_0,$$

(see, e.g., [12, Chap. 4, Sec. 10]) to arrive at

$$\left|\int_0^{\infty} e^{-y} f(y/a)\right| \le C_f a q^{j+1}\frac{1}{q(a + \underline{a}) - j}. \tag{D.1}$$

Hence, the third integral can be estimated by

$$\left| e^{-az}\int_0^{\infty} e^{-y} f(y/a)\right| \le C_f a q^{j+1}\frac{1}{q(a + \underline{a}) - j} e^{-\overline{a}\operatorname{Re}(z)}.$$

We now turn to the second integral in the representation formula for u. We split the integral as

$$\left| e^{az}\int_{az}^{\infty} e^{-y} f(y/a)\,dy\right| \le \left| e^{az}\int_0^{az} e^{-y} f(y/a)\,dy\right| + \left| e^{az}\int_0^{\infty} e^{-y} f(y/a)\,dy\right|.$$

We recognize that the second integral can be estimated using (D.1). The first integral is treated as follows:

$$\begin{aligned}
\left| e^{az} \int_0^{az} e^{-y} f(y/a)\, dy \right| &\leq C_f e^{\mathrm{Re}(az)} \int_0^1 e^{-t\,\mathrm{Re}(az)} |az| e^{-t\,\mathrm{Re}(z\overline{a})} (q + t|z|)^j \, dt \\
&\leq C_f e^{-\overline{a}\,\mathrm{Re}(z)} \int_0^1 (q + t|z|)^j a|z| e^{(1-t)(a+\overline{a})\,\mathrm{Re}\, z} \, dt \\
&\leq C_f e^{-\overline{a}\,\mathrm{Re}(z)} \frac{a}{j+1} \left[(q+|z|)^{j+1} - q^{j+1} \right].
\end{aligned}$$

Combining the above estimates and recalling $q(a + \underline{a}) - j \geq 2j + 1 - j \geq j + 1$, we conclude

$$\left| e^{-az} \int_0^{az} e^{y} f(y/a)\, dy + e^{az} \int_{az}^{\infty} e^{-y} f(y/a)\, dy - e^{-az} \int_0^{\infty} e^{-y} f(y/a)\, dy \right| \leq
2C_f e^{-\mathrm{Re}(\overline{a}z)} (q + |z|)^{j+1} \frac{a}{j+1}.$$

Combining this estimate with the obvious one for the fourth term, we arrive at the desired bound.

D.3 Proof of Theorem 4.9

We begin by setting

$$a' := \frac{\det \mathbf{A}(0)}{a_{11}(0)} = \frac{a_{11}(0) a_{22}(0) - a_{12}(0) a_{21}(0)}{a_{11}(0)},$$

and choose the constants $C_{\widetilde{u}}, C_{\widetilde{v}}, C_{\widehat{u}}, C_{\widehat{v}}, C_i, K_i, \overline{K}_i$, $i = 1, ..., 4$ to satisfy the following:

$$K_1 = K_2 = K_3 = K_4 \geq 1, \tag{D.2a}$$

$$\overline{K}_1 = \overline{K}_2 = \overline{K}_3 = \overline{K}_4 \geq 1, \tag{D.2b}$$

$$C_{\widetilde{u}} = C_{\widetilde{v}} = C_{\widehat{u}} = C_{\widehat{v}} \geq 1. \tag{D.2c}$$

Furthermore, the following requirements have to be satisfied: with the constants C_S, K of Lemma 4.2 (we assume for notational simplicity that $\delta = 1$ is admissible in Lemma 4.2) and γ_a the constant of analyticity of the data (see (2.3)):

$$
\begin{aligned}
C_2 &\geq C_S, & \text{(D.3)}\\
C_3 &\geq C_S, & \text{(D.4)}\\
C_1 &\geq \frac{|a_{12}(0)|}{a_{11}(0)} C_2, & \text{(D.5)}\\
C_1 &\geq \frac{|a_{21}(0)|}{a_{22}(0)} C_2, & \text{(D.6)}\\
C_4 &\geq \frac{|a_{21}(0)|}{\underline{a}^2} C_3, & \text{(D.7)}\\
K_1 = K_2 &> \max\{1, \gamma_a\}, & \text{(D.8)}\\
1 &\geq \left[C_a \left(1 + \frac{|a_{21}(0)|}{a_{11}(0)}\right) \frac{1}{a'} \left(1 + \frac{C_1}{C_2}\right) \frac{1}{1 - \gamma_a/K_2} \frac{\gamma_a}{K_2} + \frac{C_S}{C_2} \left(\frac{K}{K_2}\right)^{i+1} \right], & \text{(D.9)}\\
1 &\geq \left[\frac{C_2}{C_1} \frac{|a_{12}(0)|}{a_{11}(0)} + \frac{C_a}{a_{11}(0)} \frac{\gamma_a}{K_1} \frac{1}{1 - \gamma_a/K_1} \left(1 + \frac{C_2}{C_1}\right) \right], & \text{(D.10)}\\
1 &\geq \left[\frac{C_S}{C_3} \left(\frac{1}{\overline{\overline{K}}_3}\right)^{j+1} + \frac{C_1}{C_3} \left(\frac{\overline{K}_1}{\overline{\overline{K}}_3}\right)^{j+1} + \overline{K}_4^{-2} \frac{C_4}{C_3} |a_{12}(0)| \right], & \text{(D.11)}\\
C_{\widehat{u}\underline{a}} &> 4, & \text{(D.12)}\\
1 &\geq \left[\frac{C_3}{C_4} \frac{|a_{21}(0)|}{\underline{a}^2} \frac{1}{1 - 2/(C_{\widehat{u}}\underline{a})} \right], & \text{(D.13)}\\
1 &\geq \left[\frac{C_S}{C_3} \left(\frac{K}{K_3}\right)^{i+1} \right], & \text{(D.14)}\\
1 &\geq \frac{1}{\underline{a}^2} \left(\frac{1}{1 - 2/(C_{\widehat{v}}\underline{a})}\right)^2 C_a \left[\frac{C_3}{C_4} + \overline{K}_4^{-2} \right] \frac{1}{1 - \gamma_a/(K_4 \overline{K}_4)}, & \text{(D.15)}\\
1 &\geq \frac{4}{\underline{a}^2} |a_{21}(0)| \frac{C_3}{C_4} + \frac{4}{\underline{a}^2} \overline{K}_4^{-2} a_{22}(0), & \text{(D.16)}\\
1 &\geq \overline{K}_1^{-2} \frac{C_1}{C_2} \frac{2e^{\overline{a}} |a_{21}(0)|}{a' a_{22}(0)} + \frac{C_S}{C_2} \left(\frac{1}{\overline{\overline{K}}_1}\right)^{j+1} + \overline{K}_4^{-2} \frac{C_4}{C_2} \left(\frac{\overline{K}_4}{\overline{\overline{K}}_1}\right)^{j+1}, & \text{(D.17)}\\
1 &\geq \left[\frac{|a_{12}(0)|}{a_{11}(0)} \frac{C_2}{C_1} + \overline{K}_1^{-2} \frac{2e^{\overline{a}}}{a_{11}(0)} \right], & \text{(D.18)}\\
1 &\geq \Big[\frac{|a_{21}(0)|}{a_{11}(0)} 2e^{\overline{a}} \frac{C_1}{C_2} \overline{K}_2^{-2} + C_a \left(1 + \frac{|a_{21}(0)|}{a_{11}(0)}\right) \frac{\gamma_a}{K_1} \frac{1}{1 - \gamma_a/K_1} \left(\frac{C_1}{C_2} + 1\right) & \text{(D.19)}\\
&\quad + \frac{C_S}{C_2} \left(\frac{K}{K_2}\right)^i \left(\frac{1}{\overline{\overline{K}}_2}\right)^{j+1} + \overline{K}_4^{-2} \frac{C_4}{C_2} \left(\frac{K_4}{K_2}\right)^i \left(\frac{\overline{K}_4}{\overline{\overline{K}}_2}\right)^{j+1} \Big], & \text{(D.20)}\\
1 &\geq \left[\frac{|a_{12}(0)|}{a_{11}(0)} \frac{C_2}{C_1} + \overline{K}_1^{-2} \frac{2e^{\overline{a}}}{a_{11}(0)} + \frac{C_a}{a_{11}(0)} \frac{1}{1 - \gamma_a/K_1} \frac{\gamma_a}{K_1} \left(1 + \frac{C_2}{C_1}\right) \right], & \text{(D.21)}\\
1 &\geq \Big[\frac{1}{a_{11}(0)} \left\{ C_a \frac{C_4}{C_3} \overline{K}_4^{-2} + \frac{\gamma_a}{K_3 \overline{K}_3} \frac{C_a}{1 - \gamma_a/(K_3 \overline{K}_3)} \left(1 + \frac{C_4}{C_3} \overline{K}_4^{-2}\right) \right\} & \text{(D.22)}\\
&\quad + \frac{C_S}{C_3} \left(\frac{K}{K_3}\right)^i \left(\frac{1}{\overline{\overline{K}}_3}\right)^{j+1} + \frac{C_1}{C_3} \left(\frac{K_1}{K_3}\right)^i \left(\frac{\overline{K}_1}{\overline{\overline{K}}_3}\right)^{j+1} \Big].
\end{aligned}
$$

Before we proceed with the proof, we make sure that the requirements (D.3)–(D.22) can be satisfied. We make the following simplifying assumptions in addition to (D.2):

$$C_{\widetilde{u}} = C_{\widetilde{v}} = C_{\widehat{u}} = C_{\widehat{v}} > 8/\underline{a}, \tag{D.23a}$$

$$\frac{\gamma_a}{K_1} = \frac{\gamma_a}{K_2} = \frac{\gamma_a}{K_3} = \frac{\gamma_a}{K_4} \le \frac{1}{2}, \tag{D.23b}$$

$$K \le K_1 = K_2 = K_3 = K_4, \tag{D.23c}$$

$$C_2 = Q, \qquad C_1 = Q^2, \qquad C_3 = Q^3, \qquad C_4 = Q^4. \tag{D.23d}$$

Here, $Q > 0$ will be selected sufficiently large below. Then, the requirements (D.3)–(D.22) are satisfied if:

$$Q \ge C_S, \tag{D.24}$$

$$Q^3 \ge C_S, \tag{D.25}$$

$$Q^2 \ge \frac{|a_{12}(0)|}{a_{11}(0)} Q, \tag{D.26}$$

$$Q^2 \ge \frac{|a_{21}(0)|}{a_{22}(0)} Q, \tag{D.27}$$

$$Q^4 \ge \frac{|a_{21}(0)|}{\underline{a}^2} Q^3, \tag{D.28}$$

$$1 \ge \left[C_a \left(1 + \frac{|a_{21}(0)|}{a_{11}(0)}\right) \frac{2}{a'} \left(1 + \frac{Q^2}{Q}\right) \frac{\gamma_a}{K_2} + \frac{C_S}{Q} \right], \tag{D.29}$$

$$1 \ge \left[\frac{Q}{Q^2} \frac{|a_{12}(0)|}{a_{11}(0)} + \frac{C_a}{a_{11}(0)} \frac{2\gamma_a}{K_1} \left(1 + \frac{Q}{Q^2}\right) \right], \tag{D.30}$$

$$1 \ge \left[\frac{C_S}{Q^3} + \frac{Q^2}{Q^3} + \overline{K}_4^{-2} \frac{Q^4}{Q^3} |a_{12}(0)| \right], \tag{D.31}$$

$$1 \ge \left[\frac{Q^3}{Q^4} \frac{2|a_{21}(0)|}{\underline{a}^2} \right], \tag{D.32}$$

$$1 \ge \frac{C_S}{Q^3}, \tag{D.33}$$

$$1 \ge \frac{4}{\underline{a}^2} 2C_a \left[\frac{Q^3}{Q^4} + \overline{K}_4^{-2} \right], \tag{D.34}$$

$$1 \ge \frac{4}{\underline{a}^2} |a_{21}(0)| \frac{Q^3}{Q^4} + \frac{4}{\underline{a}^2} \overline{K}_4^{-2} a_{22}(0), \tag{D.35}$$

$$1 \ge \overline{K}_1^{-2} \frac{Q^2}{Q} \frac{2e^{\overline{a}} |a_{21}(0)|}{a' a_{22}(0)} + \frac{C_S}{Q} + \overline{K}_4^{-2} \frac{Q^4}{Q}, \tag{D.36}$$

$$1 \ge \left[\frac{|a_{12}(0)|}{a_{11}(0)} \frac{Q}{Q^2} + \overline{K}_1^{-2} \frac{2e^{\overline{a}}}{a_{11}(0)} \right], \tag{D.37}$$

$$1 \ge \left[\frac{|a_{21}(0)|}{a_{11}(0)} 2e^{\overline{a}} \frac{Q^2}{Q} \overline{K}_2^{-2} + C_a \left(1 + \frac{|a_{21}(0)|}{a_{11}(0)}\right) \frac{2\gamma_a}{K_1} \left(\frac{Q^2}{Q} + 1\right) + \frac{C_S}{Q} + \overline{K}_4^{-2} \frac{Q^4}{Q} \right], \tag{D.38}$$

$$1 \ge \left[\frac{|a_{12}(0)|}{a_{11}(0)} \frac{Q}{Q^2} + \overline{K}_1^{-2} \frac{2e^{\overline{a}}}{a_{11}(0)} + \frac{2C_a}{a_{11}(0)} \frac{\gamma_a}{K_1} \left(1 + \frac{Q}{Q^2}\right) \right], \tag{D.39}$$

$$1 \ge \left[\frac{1}{a_{11}(0)} \left\{ C_a \frac{Q^4}{Q^3} \overline{K}_4^{-2} + \frac{2\gamma_a}{K_3 \overline{K}_3} \left(1 + \frac{Q^4}{Q^3} \overline{K}_4^{-2}\right) \right\} + \frac{C_S}{Q^3} + \frac{C_1}{Q^3} \right]. \tag{D.40}$$

It is now easy to see that by first selecting Q sufficiently large and then choosing the parameters K_i and $\overline{K}_i$ sufficiently large, ensures the above requirements.

We now turn to the induction argument.

(Base) Case $i = j = 0$**:**

The functions $\widetilde{u}^L_{0,0}, \widetilde{v}^L_{0,0}, \widehat{u}^L_{0,0}, \widehat{v}^L_{0,2}$ ($\widehat{v}^L_{0,0} = 0$) satisfy the following:

$$\begin{cases} -\left(\widetilde{v}^L_{0,0}\right)'' + a'\widetilde{v}^L_{0,0} = 0 \\ \widetilde{v}^L_{0,0}(0) = -v_{0,0}(0)\ ,\ \widetilde{v}^L_{0,0}(\widetilde{x}) \to 0 \text{ as } \widetilde{x} \to \infty \end{cases}, \tag{D.41}$$

$$\widetilde{u}^L_{0,0} = -\frac{a_{12}(0)}{a_{11}(0)}\widetilde{v}^L_{0,0}, \tag{D.42}$$

$$\begin{cases} -\left(\widehat{u}^L_{0,0}\right)'' + a_{11}(0)\widehat{u}^L_{0,0} = 0 \\ \widehat{u}^L_{0,0}(0) = -u_{0,0}(0)\ ,\ \widehat{u}^L_{0,0}(\widehat{x}) \to 0 \text{ as } \widehat{x} \to \infty \end{cases}, \tag{D.43}$$

$$\widehat{v}^L_{0,2}(z) = \int_z^\infty \int_t^\infty a_{21}(0)\widehat{u}^L_{0,0}(\tau)d\tau dt. \tag{D.44}$$

Solution formulas for $\widetilde{v}^L_{0,0}$ and $\widehat{u}^L_{0,0}$ and Lemma 4.2 (recall that we assume that $\delta = 1$ is admissible in Lemma 4.2) give us the desired result for $\widetilde{v}^L_{0,0}$ and $\widehat{u}^L_{0,0}$ in view of the requirements (D.3), (D.4), while (D.42) gives it for $\widetilde{u}^L_{0,0}$, in view of requirement (D.6). For $\widehat{v}^L_{0,2}(z)$ we have from (D.44), the just proven result for $\widehat{u}^L_{0,0}$, and Lemma 4.6 (with $j = 0$)

$$\left|\widehat{v}^L_{0,2}(z)\right| \le |a_{21}(0)| C_3 \frac{1}{\underline{a}^2} \operatorname{Exp}(z) \le C_4 \operatorname{Exp}(z),$$

by the choice of C_4 in (D.7).

(Base) Case $j = 0$, $i > 0$**:**

The functions $\widetilde{u}^L_{i,0}, \widetilde{v}^L_{i,0}, \widehat{u}^L_{i,0}, \widehat{v}^L_{i,2}$ satisfy equations (4.23)–(4.27), respectively, for up to i. We proceed by induction on i. First, $\widetilde{v}^L_{i+1,0}$ satisfies (4.24) with i replaced by $i+1$. The right hand side of that boundary value problem satisfies, in view of the induction hypothesis and the choices $K_1 = K_2$, $C_{\widetilde{u}} = C_{\widetilde{v}} \ge 1$,

$$\begin{aligned} \left|RHS_{(4.24a)}\right| &\le C_a \left(1 + \frac{|a_{21}(0)|}{a_{11}(0)}\right) \sum_{k=1}^{i+1} \gamma_a^k \left|z\right|^k \left\{\left|\widetilde{u}^L_{i+1-k,0}(z)\right| + \left|\widetilde{v}^L_{i+1-k,0}(z)\right|\right\} \\ &\le C_a \left(1 + \frac{|a_{21}(0)|}{a_{11}(0)}\right) \sum_{k=1}^{i+1} \gamma_a^k |z|^k \frac{1}{(i+1-k)!} (C_{\widetilde{u}}(i+1-k) + |z|)^{2(i+1-k)} \times \\ &\times \left(C_1 K_1^{i+1-k} + C_2 K_2^{i+1-k}\right) \operatorname{Exp}(z) \\ &\le C_a \left(1 + \frac{|a_{21}(0)|}{a_{11}(0)}\right) (C_1 + C_2) \frac{1}{1 - \gamma_a/K_2} \frac{\gamma_a}{K_2} K_2^{i+1} \frac{1}{i!} (C_{\widetilde{u}} i + |z|)^{2i+1} \operatorname{Exp}(z), \end{aligned}$$

where we used the fact that $K_1 = K_2 > \gamma_a$ by (D.8). Lemma 4.5 and Lemma 4.2 yield, for the solution $\widetilde{v}_{i,0}^L$ of the boundary value problem (4.24),

$$\begin{aligned}
\left|\widetilde{v}_{i+1,0}^L(z)\right| &\leq \left(C_a\left(1+\frac{|a_{21}(0)|}{a_{11}(0)}\right)\frac{1}{a'}(C_1+C_2)\frac{1}{1-\gamma_a/K_2}\frac{\gamma_a}{K_2}K_2^{i+1}\frac{1}{(i+1)!}(C_{\widetilde{u}}i+|z|)^{2i+2}\right.\\
&\quad \left.+ C_S K^{i+1}(i+1)^{i+1}\right)\operatorname{Exp}(z)\\
&\leq C_2K_2^{i+1}\frac{1}{(i+1)!}(C_{\widetilde{u}}(i+1)+|z|)^{2(i+1)}\operatorname{Exp}(z)\\
&\quad \times\left[C_a\left(1+\frac{|a_{21}(0)|}{a_{11}(0)}\right)\frac{1}{a'}\left(1+\frac{C_1}{C_2}\right)\frac{1}{1-\gamma_a/K_2}\frac{\gamma_a}{K_2}+\frac{C_S}{C_2}\left(\frac{K}{K_1}\right)^{i+1}\right]. \qquad \text{(D.45)}
\end{aligned}$$

The expression in square brackets is bounded by 1 by our requirement (D.9). Next, we consider $\widetilde{u}_{i+1,0}^L$ which satisfies (4.23) with i replaced by $i+1$. We have by the induction hypothesis and (D.45),

$$\begin{aligned}
\left|\widetilde{u}_{i+1,0}^L(z)\right| &\leq \left\{\frac{|a_{12}(0)|}{a_{11}(0)}C_2K_2^{i+1}\frac{1}{(i+1)!}\left(C_{\widetilde{v}}(i+1)+|z|\right)^{2(i+1)}+\right.\\
&\quad +\sum_{k=1}^{i+1}\frac{C_a}{a_{11}(0)}\gamma_a^k|z|^k\frac{1}{(i+1-k)!}\left[C_1K_1^{i+1-k}\left(C_{\widetilde{u}}(i+1)+|z|\right)^{2(i+1-k)}\right.\\
&\quad \left.\left.+C_2K_2^{i+1-k}\left(C_{\widetilde{v}}(i+1)+|z|\right)^{2(i+1-k)}\right]\right\}\operatorname{Exp}(z)\\
&\leq C_1K_1^{i+1}\frac{1}{(i+1)!}\left(C_{\widetilde{u}}(i+1)+|z|\right)^{2(i+1)}\times\\
&\quad \times\left[\frac{C_2}{C_1}\frac{|a_{12}(0)|}{a_{11}(0)}+\frac{C_a}{a_{11}(0)}\frac{\gamma_a}{K_1}\frac{1}{1-\gamma_a/K_1}\left(1+\frac{C_2}{C_1}\right)\right]\operatorname{Exp}(z),
\end{aligned}$$

since $K_1 = K_2 > \gamma_a$ by (D.8). Again, the expression in square brackets is bounded by 1 by our requirement (D.10).

For $\widehat{u}_{i+1,0}^L$, we have by Lemma 4.5, (4.26), and Lemma 4.2,

$$\begin{aligned}
\left|\widehat{u}_{i+1,0}^L(z)\right| &\leq e^{-a_{11}(0)\operatorname{Re}(z)}\left|u_{i+1,0}(0)\right| \leq C_SK^{i+1}(i+1)^{(i+1)}\operatorname{Exp}(z)\\
&\leq C_3K_3^{i+1}(i+1)^{(i+1)}\left[\frac{C_S}{C_3}\left(\frac{K}{K_3}\right)^{i+1}\right]\operatorname{Exp}(z). \qquad \text{(D.46)}
\end{aligned}$$

In view of our requirement (D.14), the expression in square brackets is bounded by 1.

Finally, for $\widehat{v}_{i+1,2}^L$ we have from (4.27), (D.46) and Lemma 4.6, in view of $K_3 = K_4$ and $C_{\widehat{u}} = C_{\widehat{v}}$,

$$\begin{aligned}
\left|\widehat{v}_{i+1,2}^L(z)\right| &\leq \left|\int_z^\infty\int_t^\infty a_{21}(0)\widehat{u}_{i+1,0}^L(\tau)d\tau dt\right|\\
&\leq \frac{|a_{21}(0)|}{\underline{a}^2}\left(\frac{1}{1-2/(C_{\widetilde{u}}\underline{a})}\right)^2C_3K_3^{i+1}(C_{\widehat{u}}(i+1)+|z|)^{2(i+1)}\operatorname{Exp}(z)\\
&\leq C_4K_4^{i+1}(C_{\widehat{v}}(i+1)+|z|)^{2(i+1)}\operatorname{Exp}(z)\left[\frac{C_3}{C_4}\frac{|a_{21}(0)|}{\underline{a}^2}\frac{1}{1-2/(C_{\widehat{u}}\underline{a})}\right].
\end{aligned}$$

The expression in square brackets is bounded by 1 by our requirements (D.13), (D.12).

(Base) Case $i = 0$, $j > 0$**:**

The functions $\widetilde{u}^L_{0,j+1}, \widetilde{v}^L_{0,j+1}, \widehat{u}^L_{0,j+1}, \widehat{v}^L_{0,j+3}$ satisfy the following:

$$\begin{cases} -\left(\widetilde{v}^L_{0,j+1}\right)'' + a'\widetilde{v}^L_{0,j+1} = -\frac{a_{21}(0)}{a_{11}(0)}\left(\widetilde{u}^L_{0,j-1}\right)'' \\ \widetilde{v}^L_{0,j+1}(0) = -\left(v_{0,j+1}(0) + \widehat{v}^L_{0,j+1}(0)\right)\ ,\ \widetilde{v}^L_{0,j+1}(\widetilde{x}) \to 0 \text{ as } \widetilde{x} \to \infty \end{cases}, \tag{D.47}$$

$$\widetilde{u}^L_{0,j+1} = -\frac{a_{12}(0)}{a_{11}(0)}\widetilde{v}^L_{0,j+1} + \frac{\left(\widetilde{u}^L_{0,j-1}\right)''}{a_{11}(0)}, \tag{D.48}$$

$$\begin{cases} -\left(\widehat{u}^L_{0,j+1}\right)'' + a_{11}(0)\widehat{u}^L_{0,j+1} = -a_{12}(0)\widehat{v}^L_{0,j+1} \\ \widehat{u}^L_{0,j+1}(0) = -\left(u_{0,j+1}(0) + \widetilde{u}^L_{0,j+1}(0)\right)\ ,\ \widehat{u}^L_{0,j} \to 0 \text{ as } x \to \infty \end{cases}, \tag{D.49}$$

$$\widehat{v}^L_{0,j+3}(z) = \int_z^\infty\int_t^\infty \left[a_{21}(0)\widehat{u}^L_{0,j+1}(\tau) + a_{22}(0)\widehat{v}^L_{0,j+1}(\tau)\right] d\tau dt. \tag{D.50}$$

To establish the desired claims, we proceed by induction on j noting that the case $j = 0$ and $i = 0$ has been proved already. Assuming that the bounds are valid for $i = 0$ and up to j, we show them for $j + 1$.

We start with $\widehat{v}^L_{0,j+3}$. By the induction hypothesis, Lemma 4.6, and the assumption $C_{\widehat{v}}\underline{a} \geq 4$ (see requirement (D.12)) we have

$$\begin{aligned} \left|\widehat{v}^L_{0,j+3}(z)\right| \quad &\leq \quad \frac{4}{\underline{a}^2}\Big(|a_{21}(0)|C_3\overline{K}_3^{j+1}(C_{\widehat{u}}(j+1) + |z|)^{2(j+1)}\frac{1}{(j+1)!} \\ &\quad + a_{22}(0)C_4\overline{K}_4^{j-1}(C_{\widehat{v}}(j-1) + |z|)^{2(j-1)}\frac{1}{(j-1)!}\Big)\,\mathrm{Exp}(z). \end{aligned}$$

In view of $C_{\widehat{u}} = C_{\widehat{v}}$ and $\overline{K}_3 = \overline{K}_4$ we get

$$\left|\widehat{v}^L_{0,j+3}(z)\right| \quad \leq \quad C_4\overline{K}_4^{j+1}\frac{1}{(j+1)!}(C_{\widehat{v}}(j+1) + |z|)^{2(j+1)}\,\mathrm{Exp}(z)\left[\frac{4}{\underline{a}^2}|a_{21}(0)|\frac{C_3}{C_4} + \frac{4}{\underline{a}^2}|a_{22}(0)|\overline{K}_4^{-2}\right];$$

by requirement (D.16), the expression in square brackets is bounded by 1 as required.

We next turn our attention to $\widetilde{v}^L_{0,j+1}$ which satisfies (D.47) and right-hand side (RHS) satisfying

$$\left|RHS_{(D.47)}\right| \leq \frac{|a_{21}(0)|}{|a_{11}(0)|}\left|\left(\widetilde{u}^L_{0,j-1}\right)''\right|. \tag{D.51}$$

We first study the case $j = 0$. Then $\widetilde{u}^L_{0,j-1} = 0$, and Lemma 4.5 yields, together with Lemma 4.2,

$$|\widetilde{v}^L_{0,j+1}(z)|e^{-a'\operatorname{Re}(z)}C_S \leq C_2\overline{K}_2^{j+1}\,\mathrm{Exp}(z)\left[\frac{C_S}{C_2}\overline{K}_2^{-(j+1)}\right].$$

Since we assume $\overline{K}_2 \geq 1$, our requirement (D.3) implies the desired bound. Returning to (D.51) for the case $j \geq 1$, the induction hypothesis and Lemma 4.7 produce

$$\left|RHS_{(D.47)}(z)\right| \leq \frac{|a_{21}(0)|}{a_{11}(0)} 2e^{\overline{a}} C_1 \frac{\overline{K}_1^{j-1}}{(j-1)!} \left(C_{\widetilde{u}}(j-1) + |z| + 1\right)^{2(j-1)} \operatorname{Exp}(z).$$

Therefore, Lemma 4.5 together with Lemma 4.2 yields

$$\left|\widetilde{v}^L_{0,j+1}(z)\right| \leq$$

$$\begin{aligned}
&\leq \left[\frac{1}{a'}\frac{1}{2j-1} 2e^{\overline{a}} \frac{|a_{21}(0)|}{a_{22}(0)} C_1 \frac{\overline{K}_1^{j-1}}{(j-1)!} \left(C_{\widetilde{u}}(j-1) + |z| + 1\right)^{2(j-1)+1} + |v_{0,j+1}(0)| + \left|\widehat{v}^L_{0,j+1}(0)\right|\right] \operatorname{Exp}(z) \\
&\leq C_2 \overline{K}_1^{j+1} \frac{1}{(j+1)!} \left(C_{\widetilde{v}}(j+1) + |z|\right)^{2(j+1)} \\
&\quad \times \left[\overline{K}_1^{-2} \frac{C_1}{C_2} \frac{2e^{\overline{a}}|a_{21}(0)|}{a' a_{22}(0)} + \frac{C_S}{C_2}\left(\frac{1}{\overline{K}_1}\right)^{j+1} + \overline{K}_4^{-2}\frac{C_4}{C_2}\left(\frac{\overline{K}_4}{\overline{K}_1}\right)^{j+1}\right] \operatorname{Exp}(z), \qquad \text{(D.52)}
\end{aligned}$$

where, again, the expression square brackets is bounded by 1 due to our requirement (D.17).

Now, consider $\widetilde{u}^L_{0,j+1}$, which satisfies (D.48). By (D.52) and Lemma 4.7, we have

$$\begin{aligned}
\left|\widetilde{u}^L_{0,j+1}(z)\right| &\leq \Big(\frac{|a_{12}(0)|}{a_{11}(0)} C_2 \overline{K}_2^{j+1} \frac{1}{(j+1)!}\left(C_{\widetilde{v}}(j+1) + |z|\right)^{2(j+1)} \\
&\quad + \frac{2e^{\overline{a}}}{a_{11}(0)} C_1 \overline{K}_1^{j-1} \frac{1}{(j-1)!}\left(C_{\widetilde{u}}(j-1) + 1 + |z|\right)^{2(j-1)}\Big) \operatorname{Exp}(z) \\
&\leq C_1 \overline{K}_1^{j+1} \frac{1}{(j+1)!}\left(C_{\widetilde{u}}(j+1) + |z|\right)^{2(j+1)} \left[\frac{|a_{12}(0)|}{a_{11}(0)}\frac{C_2}{C_1} + \overline{K}_1^{-2}\frac{2e^{\overline{a}}}{a_{11}(0)}\right] \operatorname{Exp}(z);
\end{aligned}$$

in view of requirement (D.18), the expression in square brackets is bounded by 1 as required .

Finally, for $\widehat{u}^L_{0,j+1}$ which satisfies (D.49) we have by Lemma 4.5 (the case $j = 0$ needs special treatment in that the third term in the following estimate is not present)

$$\begin{aligned}
&\left|\widehat{u}^L_{0,j+1}(z)\right| \leq \\
&\quad \left(|u_{0,j+1}(0)| + \left|\widetilde{u}^L_{0,j+1}(0)\right| + |a_{12}(0)| C_4 \overline{K}_4^{j-1} \frac{1}{(j-1)!(2(j-1)+1)}\left(C_{\widehat{v}}(j+1) + |z|\right)^{2(j-1)+1}\right) \operatorname{Exp}(z) \\
&\leq C_3 \overline{K}_3^{j+1} \frac{1}{(j+1)!}\left(C_{\widehat{u}}(j+1) + |z|\right)^{2(j+1)} \left[\frac{C_S}{C_3}\left(\frac{1}{\overline{K}_3}\right)^{j+1} + \frac{C_1}{C_3}\left(\frac{\overline{K}_1}{\overline{K}_3}\right)^{j+1} + \overline{K}_4^{-2}\frac{C_4}{C_3}|a_{12}(0)|\right] \operatorname{Exp}(z).
\end{aligned}$$

Again, in view of our requirement (D.11), the expression in square brackets is bounded by 1.

Induction Step: We proceed by induction on j, the induction hypothesis being that the estimates (4.32)–(4.35) have been shown for all $i \in \mathbb{N}_0$ up to j and will establish them for all $i \in \mathbb{N}_0$ and $j+1$.

We start with estimating $\widehat{v}^L_{i,j+3}$. In view of the definition (4.31), we estimate with the induction hypothesis (recall $K_3 = K_4$ and $\overline{K}_3 = \overline{K}_4$; we also point out that for the case $j = 0$, the terms stemming from $\widehat{v}_{i-k,j+1-k}$ are in fact not present)

$$\begin{aligned}
&\left|\sum_{k=0}^{\min\{i,j+1\}} \frac{z^k}{k!}\left(a_{21}^{(k)}(0)\widehat{u}^L_{i-k,j+1-k}(z) + a_{22}^{(k)}(0)\widehat{v}^L_{i-k,j+1-k}(z)\right)\right| \\
&\quad\le C_a \sum_{k=0}^{\min\{i,j+1\}} \gamma_a^k \Big[C_3 K_3^{i-k}\overline{K}_3^{j+1-k}\frac{|z|^k}{(i+j+1-2k)!}(C_{\widehat{u}}(i+j+1-2k)+|z|)^{2(i+j+1-2k)} \\
&\qquad + C_4 K_4^{i-k}\overline{K}_4^{j-1-k}\frac{|z|^k}{(i+j-1-2k)!}(C_{\widehat{u}}(i+j-1-2k)+|z|)^{2(i+j-2k-1)}\Big]\operatorname{Exp}(z) \\
&\quad\le C_4 K_4^i \overline{K}_4^{j+1}\frac{(C_{\widehat{v}}(i+j+1)+|z|)^{2(i+j+1)}}{(i+j+1)!}\left[\frac{C_3}{C_4}+\overline{K}_4^{-2}\right]\frac{C_a}{1-\gamma_a/(K_4\overline{K}_4)}\operatorname{Exp}(z);
\end{aligned}$$

here, we employed observations of the form (note that $i+j-2k-1 \ge 0$)

$$\begin{aligned}
&\frac{1}{(i+j-1-2k)!}(C_{\widehat{u}}(i+j-1-2k)+|z|)^{2(i+j-2k-1)} \le \frac{1}{(i+j-1-2k)!}(C_{\widehat{u}}(i+j+1)+|z|)^{2(i+j-2k-1)} \\
&\le \frac{(i+j+1)^{2k+2}}{(i+j+1)!}(C_{\widehat{u}}(i+j+1)+|z|)^{2(i+j-2k-1)} \\
&= \frac{(i+j+1)^{2k+2}}{(i+j+1)!}(C_{\widehat{u}}(i+j+1)+|z|)^{-2k-2}(C_{\widehat{u}}(i+j+1)+|z|)^{2(i+j-k)} \\
&\le \frac{1}{(i+j+1)!}(C_{\widehat{u}}(i+j+1)+|z|)^{2(i+j)} \le \frac{1}{(i+j+1)!}(C_{\widehat{u}}(i+j+1)+|z|)^{2(i+j+1)}.
\end{aligned}$$

From (4.31) and Lemma 4.6, we therefore get

$$\begin{aligned}
\left|\widehat{v}^L_{i,j+3}(z)\right| \;&\le\; C_4 K_4^i\overline{K}_4^{j+1}\frac{(C_{\widehat{v}}(i+j+1)+|z|)^{2(i+j+1)}}{(i+j+1)!}\operatorname{Exp}(z) \\
&\quad\times \frac{1}{\underline{a}^2}\left(\frac{1}{1-2/(C_{\widehat{v}}\underline{a})}\right)^2 C_a\left[\frac{C_3}{C_4}+\overline{K}_4^{-2}\right]\frac{1}{1-\gamma_a/(K_4\overline{K}_4)}.
\end{aligned}$$

By requirement (D.15), this is the desired estimate.

We now consider $\widetilde{v}^L_{i,j+1}$ which satisfies (4.29a)–(4.29b). The required estimate is proved by induction in i, the case $i=0$ having been studied previously. The right-hand side of the boundary value problem satisfies

$$\left|RHS_{(4.29a)-(4.29b)}\right| \le \frac{|a_{21}(0)|}{a_{11}(0)}\left|\left(\widetilde{u}^L_{i,j-1}\right)''(z)\right| + C_a\left(1+\frac{|a_{21}(0)|}{a_{11}(0)}\right)\sum_{k=1}^{i}\gamma_a^k|z|^k\left(|\widetilde{u}^L_{i-k,j+1}(z)|+|\widetilde{v}^L_{i-k,j+1}(z)|\right).$$

The induction hypothesis and Lemma 4.7 produce

$$\begin{aligned}
\left|RHS_{(4.29a)-(4.29b)}\right| \;&\le\; C_2\frac{1}{(i+j)!}(C_{\widetilde{u}}(i+j)+|z|)^{2(i+j)+1}K_2^i\overline{K}_2^{j+1}\operatorname{Exp}(z) \\
&\quad\times\left[\frac{|a_{21}(0)|}{a_{11}(0)}2e^{\overline{a}}\overline{K}_2^{-2}\frac{C_1}{C_2} + C_a\left(1+\frac{|a_{21}(0)|}{a_{11}(0)}\right)\frac{\gamma_a}{K_1}\frac{1}{1-\gamma_a/K_1}\left(\frac{C_1}{C_2}+1\right)\right].
\end{aligned} \tag{D.53}$$

Lemma 4.5 and the already proven estimates for $v_{i,j+1}$ and $\widehat{v}^L_{i,j+1}$ lead to

$$\begin{aligned}
|\widetilde{v}^L_{i,j+1}(z)| &\leq C_2 \frac{1}{(i+j+1)!}(C_{\widetilde{u}}(i+j)+|z|)^{2(i+j+1)} K_2^i \overline{K}_2^{j+1} \operatorname{Exp}(z) \\
&\quad \times \Big[\frac{|a_{21}(0)|}{a_{11}(0)} 2e^{\overline{a}} \frac{C_1}{C_2} \overline{K}_2^{-2} + C_a \left(1 + \frac{|a_{21}(0)|}{a_{11}(0)}\right) \frac{\gamma_a}{K_1} \frac{1}{1-\gamma_a/K_1} \left(\frac{C_1}{C_2}+1\right) \\
&\quad + \frac{C_S}{C_2} \left(\frac{K}{K_2}\right)^i \left(\frac{1}{\overline{K}_2}\right)^{j+1} + \overline{K}_4^{-2} \frac{C_4}{C_2} \left(\frac{K_4}{K_2}\right)^i \left(\frac{\overline{K}_4}{\overline{K}_2}\right)^{j+1} \Big],
\end{aligned}$$

where the expression in square brackets is bounded by 1 due to our requirement (D.19).

Next we look at

$$\widetilde{u}^L_{i,j+1} = -\frac{a_{12}(0)}{a_{11}(0)} \widetilde{v}^L_{i,j+1} + \frac{\left(\widetilde{u}^L_{i,j-1}\right)''}{a_{11}(0)} - \frac{1}{a_{11}(0)} \sum_{k=1}^{i} \frac{\widetilde{x}^k}{k!} \left[a_{11}^{(k)}(0) \widetilde{u}^L_{i-k,j+1} + a_{12}^{(k)}(0) \widetilde{v}^L_{i-k,j+1} \right].$$

Again, we proceed by induction on i, the case $i=0$ having been handled already. From Lemma 4.7 and the induction hypotheses we get

$$\begin{aligned}
\left|\widetilde{u}^L_{i,j+1}(z)\right| &\leq C_1 K_1^i \frac{1}{(i+j+1)!} \overline{K}_1^{j+1} (C_{\widetilde{u}}(i+j+1)+|z|)^{2(i+j+1)} \operatorname{Exp}(z) \\
&\quad \times \left[\frac{|a_{12}(0)|}{a_{11}(0)} \frac{C_2}{C_1} + \overline{K}_1^{-2} \frac{2e^{\overline{a}}}{a_{11}(0)} + \frac{C_a}{a_{11}(0)} \frac{1}{1-\gamma_a/K_1} \frac{\gamma_a}{K_1} \left(1 + \frac{C_2}{C_1}\right) \right].
\end{aligned}$$

Again, the expression in square brackets is bounded by 1 in view of our requirement (D.21).

Finally, we consider $\widehat{u}^L_{i,j+1}$ which satisfies (4.30). The right-hand side of the boundary value problem satisfies

$$\begin{aligned}
\left|RHS_{(4.30a)}\right| &\leq C_a |\widehat{v}^L_{i,j+1}(z)| + C_a \sum_{k=1}^{\min\{i,j+1\}} \gamma_a^k \, |z|^k \left\{ \left|\widehat{u}^L_{i-k,j+1-k}\right| + \left|\widehat{v}^L_{i-k,j+1-k}\right| \right\} \\
&\leq C_3 K_3^i \overline{K}_3^{j+1} \frac{1}{(i+j)!} (C_{\widehat{u}}(i+j)+|z|)^{2(i+j)+1} \operatorname{Exp}(z) \\
&\quad \times \left[C_a \frac{C_4}{C_3} \overline{K}_4^{-2} + \frac{\gamma_a}{K_3 \overline{K}_3} \frac{C_a}{1-\gamma_a/(K_3\overline{K}_3)} \left(1 + \frac{C_4}{C_3} \overline{K}_4^{-2}\right) \right].
\end{aligned}$$

Lemma 4.5 together with the induction hypotheses, therefore gives us for the solution $\widehat{u}^L_{i,j+1}$ of (4.30),

$$\begin{aligned}
\left|\widehat{u}^L_{i,j+1}(z)\right| &\leq C_3 K_3^i \overline{K}_3^{j+1} \frac{1}{(i+j+1)!} (C_{\widehat{u}}(i+j+1)+|z|)^{2(i+j+1)} \operatorname{Exp}(z) \\
&\quad \times \Big[\frac{1}{a_{11}(0)} \left\{ C_a \frac{C_4}{C_3} \overline{K}_4^{-2} + \frac{\gamma_a}{K_3 \overline{K}_3} \frac{C_a}{1-\gamma_a/(K_3\overline{K}_3)} \left(1 + \frac{C_4}{C_3} \overline{K}_4^{-2}\right) \right\} \\
&\quad + \frac{C_S}{C_3} \left(\frac{K}{K_3}\right)^i \left(\frac{1}{\overline{K}_3}\right)^{j+1} + \frac{C_1}{C_3} \left(\frac{K_1}{K_3}\right)^i \left(\frac{\overline{K}_1}{\overline{K}_3}\right)^{j+1} \Big].
\end{aligned}$$

By our requirement (D.22), the expression in square brackets is bounded by 1.

D.4 Proof of Theorem 4.13

We first prove two auxiliary lemmas.

Lemma D.1. *For every* $\gamma > 0$ *the functions*

$$\begin{aligned} f_1(k) &:= \gamma^k (i-k)^{i-k} (j-k)^{j-k}, \\ f_2(k) &:= k^k \gamma^k (i-k)^{i-k} (j-k)^{j-k}, \end{aligned}$$

are convex on $(0, \min\{i,j\})$.

Proof. It is easy to check that $\frac{d^2}{dk^2} \ln f_1(k)$ and $\frac{d^2}{dk^2} \ln f_2(k) > 0$. □

Lemma D.2. *Let* $a \le b \le c \le d$*. Let* f *be non-negative and convex on* $[a,d]$*. Then*

$$\|f\|_{L^\infty(b,c)} = \max\{\min\{f(a), f(b)\}, \min\{f(c), f(d)\}\}.$$

Proof. We restrict our attention to the case $a < b < c < d$ – the general case can be proved using similar arguments. By convexity, we have $\|f\|_{L^\infty(b,c)} = \max\{f(b), f(c)\}$. We claim that

$$\max\{\min\{f(a), f(b)\}, \min\{f(c), f(d)\}\} = \max\{f(b), f(c)\}. \tag{D.54}$$

Suppose $\max\{\min\{f(a), f(b)\}, \min\{f(c), f(d)\}\} < \max\{f(b), f(c)\}$. Then

$$\min\{f(a), f(b)\} < \max\{f(b), f(c)\} \qquad \text{and} \qquad \min\{f(c), f(d)\} < \max\{f(b), f(c)\}. \tag{D.55}$$

If $f(a) < f(b)$, then we write $b = \lambda a + (1-\lambda)c$ for some $\lambda \in (0,1)$, and use convexity to get

$$f(b) = f(\lambda a + (1-\lambda)c) \le \lambda f(a) + (1-\lambda) f(c) \le \lambda f(b) + (1-\lambda) f(c),$$

from which we conclude $f(b) < f(c)$. Thus

$$f(a) < f(b) < f(c).$$

The second condition in (D.55) then produces $\min\{f(c), f(d)\} < f(c)$, from which we get $f(d) < f(c)$. On the other hand, convexity implies upon writing $c = \lambda a + (1-\lambda)d$ for some $\lambda \in (0,1)$,

$$f(c) = f(\lambda a + (1 - \lambda d)) \le \lambda f(a) + (1-\lambda) f(d) < \lambda f(c) + (1-\lambda) f(c) = f(c),$$

which is a contradiction. We conclude that $f(a) \ge f(b)$.

Next, we investigate the possibility $f(d) < f(c)$. We proceed analogously. Convexity (for the points b, c, d) implies

$$f(d) < f(c) < f(b).$$

The first condition in (D.55) then produces $\min\{f(a), f(b)\} < f(b)$ from which we get $f(a) < f(b)$. Using again convexity for the points a, b, c, yields $f(b) < f(c)$, which is the desired contradiction. We conclude that $f(c) \le f(d)$.

Combining the above, have $f(a) \ge f(b)$ and $f(c) \le f(d)$ and therefore (D.54). □

Proof of Theorem 4.13: We have to estimate the terms on the right-hand side of (4.40). A direct check shows that the estimate is valid for the special case $M_2 = 0$. Hence, we will assume $M_2 \geq 1$.

Throughout the proof, we will use that $\widehat{x}$ is real and strictly positive.

First, we estimate the double sum

$$\sum_{i=0}^{M_1} \sum_{j=M_2-1}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j |\widehat{u}''_{i,j}|.$$

Using the bounds of Theorem 4.9 and the Cauchy integral theorem for derivatives (with contour $\partial B_1(x)$), we get with the aid of Lemma 4.8,

$$|\widehat{u}''_{i,j}(x)| \leq C\widetilde{\gamma}^{i+j} i^i (j+1)^{j+1},$$

for a suitable constant $\widetilde{\gamma}$. Therefore, if $\mu\widetilde{\gamma}(M_1+1) \leq 1/2$, then

$$\sum_{i=0}^{M_1} \sum_{j=M_2-1}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j |\widehat{u}''_{i,j}| \leq C \left(\frac{\varepsilon}{\mu}\right)^{M_2-1}.$$

Hence, the double sum in (4.40) can be estimated in the desired fashion by requiring $\gamma \geq \widetilde{\gamma}/2$.

We now turn to the triple sum in (4.40). We start by writing the conditions (4.39) on the indices i, j, and k appearing in the triple sum (4.40) in a more compact form, by setting

$$\mathcal{I} := \{(i,j,k)\colon (i \geq M_1 + 1 \vee j \geq M_2 - 1) \wedge \max\{i - M_1, j - M_2\} \leq k \leq \min\{i,j\}\}. \tag{D.56}$$

Thus, the triple sum can be written as

$$S = \sum_{(i,j,k)\in\mathcal{I}} \mu^i (\varepsilon/\mu)^j \widehat{x}^k \mathbf{A}_k \begin{pmatrix} \widehat{u}_{i-k,j-k} \\ \widehat{v}_{i-k,j-k} \end{pmatrix}. \tag{D.57}$$

Using (4.34), (4.35) and $\|\mathbf{A}_k\|_{\ell^1} \leq C_A\gamma_a^k$, where $C_A := 2C_a$, we obtain with $K \geq \max\{K_3, K_4\}$, $\overline{K} \geq \max\{\overline{K}_3, \overline{K}_4\}$, $\widehat{C} \geq \max\{C_{\widehat{u}}, C_{\widehat{v}}\}$ for $\widehat{x} \geq 0$, with the aid of Lemma 4.8,

$$\begin{aligned}
\left|L_{\varepsilon,\mu}\widehat{\mathbf{U}}^M_{BL}\right| &\leq CC_A \sum_{(i,j,k)\in\mathcal{I}} \mu^i (\varepsilon/\mu)^j \widehat{x}^k \gamma_a^k K^{i-k}\overline{K}^{j-k} \frac{\left(\widehat{C}(i-k+j-k)+|\widehat{x}|\right)^{2(i-k+j-k)}}{(i-k+j-k)!} e^{-\underline{a}\widehat{x}} \\
&\leq CC_A \sum_{(i,j,k)\in\mathcal{I}} \mu^i (\varepsilon/\mu)^j K^i \overline{K}^j \frac{\widehat{x}^k\gamma_a^k}{\overline{K}^k K^k} \gamma^{i-k+j-k} (i-k)^{i-k} (j-k)^{j-k} e^{-3\underline{a}\widehat{x}/4} \\
&\leq CC_A \sum_{(i,j,k)\in\mathcal{I}} \mu^{i-k} (\varepsilon/\mu)^{j-k} K^i \overline{K}^j \frac{(\widehat{x}\varepsilon)^k\gamma_a^k}{\overline{K}^k K^k} \gamma^{i-k+j-k} (i-k)^{i-k} (j-k)^{j-k} e^{-3\underline{a}\widehat{x}/4};
\end{aligned}$$

here, we selected $\gamma > 0$ suitable in dependence on $\widehat{C}$ and $\underline{a}$. It is convenient to abbreviate

$$f(i,j,k) = \mu^{i-k} (\varepsilon/\mu)^{j-k} \widetilde{K}_1^{i-k} \widetilde{K}_2^{j-k} (\widehat{x}\varepsilon\gamma_a)^k (i-k)^{i-k} (j-k)^{j-k},$$

with $\widetilde{K}_1 := K\gamma$ and $\widetilde{K}_2 := \overline{K}\gamma$. Hence, we wish to estimate

$$e^{-3\underline{a}\widehat{x}/4} \sum_{(i,j,k)\in\mathcal{I}} f(i,j,k). \tag{D.58}$$

Next, in order unify the presentation, we consider the cases $j = M_2 - 1$ and $j = M_2$ separately. That is, we write

$$\begin{aligned}
\mathcal{I} &\subset \widetilde{\mathcal{I}} \cup \mathcal{I}_1, \\
\mathcal{I}_1 &:= \{(i,j,k)\colon j \in \{M_2-1, M_2\} \wedge \max\{i-M_1, 0\} \le k \le \min\{i,j\}\}, \\
\widetilde{\mathcal{I}} &:= \{(i,j,k)\colon (i \ge M_1+1 \vee j \ge M_2+1) \wedge \max\{i-M_1, j-M_2\} \le k \le \min\{i,j\}\},
\end{aligned}$$

and estimate the sums over $\mathcal{I}_1$ and $\widetilde{\mathcal{I}}$ separately.

The structure of the remainder of the proof is as follows:

- In *Step 1*, we estimate $e^{-\underline{a}\widehat{x}/4} \sum_{(i,j,k)\in\mathcal{I}_1} f(i,j,k)$;
- In *Step 2*, we estimate $e^{-\underline{a}\widehat{x}/4} \sum_{(i,j,k)\in\widetilde{\mathcal{I}}} f(i,j,k)$;
- Finally, in *Step 3*, we consider the case $\widehat{x}\varepsilon \ge c$ and show that then $|L_{\varepsilon,\mu} \widehat{\mathbf{U}}^M_{BL}(\widehat{x})| \le Ce^{-\beta\widehat{x}}$.

Step 1: We estimate

$$\sum_{(i,j,k)\in\mathcal{I}_1} f(i,j,k) = \sum_{i=0}^{M_1} \sum_{j=M_2-1}^{M_2} \sum_{k=0}^{\min\{i,j\}} f(i,j,k) + \sum_{i=M_1+1}^{\infty} \sum_{j=M_2-1}^{M_2} \sum_{k=i-M_1}^{\min\{i,j\}} f(i,j,k) =: S_1 + S_2,$$

using the convexity properties of the function f. Specifically, in order to estimate S_1, we first consider the case $M_1 \le M_2 - 1$.

Step 1a: Assume $M_1 \le M_2 - 1$. Then by convexity of the function $k \mapsto f(i,j,k)$ (cf. Lemma D.1),

$$\begin{aligned}
S_1 &= \sum_{i=0}^{M_1} \sum_{j=M_2-1}^{M_2} \sum_{k=0}^{i} f(i,j,k) \le \sum_{i=0}^{M_1} \sum_{j=M_2-1}^{M_2} (i+1)\max\{f(i,j,0), f(i,j,i)\} \\
&\le (M_1+1) \sum_{i=0}^{M_1} \sum_{j=M_2-1}^{M_2} \left[\mu^i (\varepsilon/\mu)^j \widetilde{K}_1^i \widetilde{K}_2^j i^i j^j + (\varepsilon/\mu)^{j-i} \widetilde{K}_2^{j-i} (\widehat{x}\varepsilon\gamma_a)^i (j-i)^{j-i}\right] \\
&\le C(M_1+1)(\varepsilon/\mu\widetilde{K}_2(M_2-1))^{M_2-1} + C(M_1+1)^2 \max\{(\widetilde{K}_2\varepsilon/\mu(M_2-1))^{M_2-1}, (\widehat{x}\varepsilon\gamma_a)^{M_1}\},
\end{aligned}$$

where we employed the assumption that μM_1 and $\varepsilon/\mu(M_2-1)$ are sufficiently small and the convexity of the second term (as a function of i). From Lemma D.2, we can bound

$$e^{-\underline{a}\widehat{x}/4} \widehat{x}^{M_1} \le C(\widehat{\gamma} M_1)^{M_1},$$

for suitable $\widehat{\gamma}$, so that we obtain together with $M_1 \le M_2 - 1$ and the trivial bound $\varepsilon \le \mu$,

$$e^{-\underline{a}\widehat{x}/4} S_1 \le C\left[\left(\frac{\varepsilon}{\mu}(M_2-1)\gamma\right)^{M_2-1} + (\mu M_1\gamma)^{M_1}\right], \tag{D.59}$$

for suitable constants C, $\gamma > 0$.

Step 1b: Analogous reasoning covers the case $M_1 = M_2$. More precisely, we write in this case

$$\sum_{i=0}^{M_1} \sum_{j=M_2-1}^{M_2} \sum_{k=0}^{\min\{i,j\}} f(i,j,k) = \sum_{i=0}^{M_1-1} \sum_{j=M_2-1}^{M_2} \sum_{k=0}^{i} f(i,j,k) + \sum_{j=M_2-1}^{M_2} \sum_{k=0}^{j} f(M_1,j,k) =: S_{1,1} + S_{1,2}.$$

For $S_{1,1}$, the reasoning of the above case ("*Step 1a*") is applicable, since $M_1 - 1 \leq M_2 - 1$, and yields

$$e^{-\underline{a}\widehat{x}/4} S_{1,1} \leq C \left[\left(\frac{\varepsilon}{\mu}(M_2-1)\gamma \right)^{M_2-1} + (\varepsilon(M_1-1)\gamma)^{M_1-1} \right] \leq C \left(\frac{\varepsilon}{\mu}(M_2-1)\gamma \right)^{M_2-1},$$

where in the last step, we used the trivial bound $\varepsilon \leq \varepsilon/\mu$ in view of $\mu \leq 1$ and the fact that $M_1 = M_2$. For $S_{1,2}$, we use convexity of f in the third argument to arrive at

$$S_{1,2} \quad \leq \quad (M_2+1) \sum_{j=M_2-1}^{M_2} \max\{f(M_1,j,0), f(M_1,j,j)\}.$$

Estimating $\max\{f(M_1,j,0), f(M_1,j,j)\} \leq f(M_1,j,0) + f(M_1,j,j)$, we get

$$f(M_1,j,0) + f(M_1,j,j) \leq \mu^{M_1} (\varepsilon/\mu)^j \widetilde{K}_1^{M_1} \widetilde{K}_2^j M_1^{M_1} j^j + \mu^{M_1-j} \widetilde{K}_1^{M_1-j} (\widehat{x}\varepsilon\gamma_a)^j (M_1-j)^{M_1-j}.$$

Using the fact that μM_1 and $\varepsilon/\mu M_2$ are sufficiently small and that $M_1 = M_2$, we get

$$\sum_{j=M_2-1}^{M_2} \max\{f(M_1,j,0), f(M_1,j,j)\} \leq$$
$$(\mu \widetilde{K}_1 M_1)^{M_1} \left(\frac{\varepsilon}{\mu} \widetilde{K}_2 (M_2-1) \right)^{M_2-1} + (\widehat{x}\varepsilon\gamma_a)^{M_1} + \mu \widetilde{K}_1 (\widehat{x}\varepsilon\gamma_a)^{M_2-1}.$$

Upon writing

$$\mu(\widehat{x}\varepsilon\gamma_a)^{M_2-1} = \left(\widehat{x}\frac{\varepsilon}{\mu}\gamma_a \right)^{M_2-1} \mu^{M_2} \leq \left(\widehat{x}\frac{\varepsilon}{\mu}\gamma_a \right)^{M_2-1},$$

we can use (D.59) to argue as in *Step 1a*.

Step 1c: We now consider S_1 for the case $M_1 \geq M_2 + 1$. We write

$$S_1 = \sum_{i=0}^{M_2-1} \sum_{j=M_2-1}^{M_2} \sum_{k=0}^{i} f(i,j,k) + \sum_{i=M_2}^{M_1} \sum_{j=M_2-1}^{M_2} \sum_{k=0}^{j} f(i,j,k).$$

Checking the arguments for the case $M_1 \leq M_2 - 1$ of *Step 1a*, we see that we can bound

$$e^{-\underline{a}\widehat{x}/4} \sum_{i=0}^{M_2-1} \sum_{j=M_2-1}^{M_2} \sum_{k=0}^{i} f(i,j,k) \leq C \left[\left(\frac{\varepsilon}{\mu}(M_2-1)\gamma \right)^{M_2-1} + (\varepsilon(M_2-1)\gamma)^{M_2-1} \right],$$

which has the desired form in view of $\mu \leq 1$.

It therefore remains to estimate $\sum_{i=M_2}^{M_1}\sum_{j=M_2-1}^{M_2}\sum_{k=0}^{j} f(i,j,k)$, which we do again by exploiting convexity of the function $k \mapsto f(i,j,k)$. We get

$$\begin{aligned}
\sum_{i=M_2}^{M_1}\sum_{j=M_2-1}^{M_2}\sum_{k=0}^{j} f(i,j,k) &\leq \sum_{i=M_2}^{M_1}\sum_{j=M_2-1}^{M_2}(j+1)\max\{f(i,j,0),f(i,j,j)\} \\
&\leq (M_2+1)\sum_{i=M_2}^{M_1}\sum_{j=M_2-1}^{M_2}\left[\mu^i\widetilde{K}_1^i(\varepsilon/\mu)^j\widetilde{K}_2^j i^i j^j + \mu^{i-j}\widetilde{K}_1^{i-j}(\widehat{x}\varepsilon\gamma_a)^j(i-j)^{i-j}\right] \\
&\leq C(M_2+1)\left[(\varepsilon/\mu(M_2-1)\gamma)^{M_2-1} + \mu(M_1+1)(\widehat{x}\varepsilon\gamma_a)^{M_2-1} + (\widehat{x}\varepsilon\gamma_a)^{M_2}\right] \\
&\leq C(M_2+1)\left[(\varepsilon/\mu(M_2-1)\gamma)^{M_2-1} + (\widehat{x}\varepsilon\gamma_a)^{M_2-1}\right],
\end{aligned}$$

where we exploited again the assumption that $\mu(M_1+1)$ and $\varepsilon/\mu(M_2+1)$ are sufficiently small so that sums can be estimated by convergent geometric series. The contribution $M_2(\widehat{x}\varepsilon\gamma_a)^{M_2-1}$ can now be estimated as before using (D.59).

Step 1d: We now turn to S_2 and start with assuming $M_1+1 \geq M_2$. Then S_2 takes the form

$$S_2 = \sum_{i=M_1+1}^{\infty}\sum_{j=M_2-1}^{M_2}\sum_{k=i-M_1}^{j} f(i,j,k) = \sum_{i=M_1+1}^{M_1+M_2}\sum_{j=M_2-1}^{M_2}\sum_{k=i-M_1}^{j} f(i,j,k).$$

Convexity of $k \mapsto f(i,j,k)$ allows us to infer

$$\begin{aligned}
\max_{k=i-M_1,\dots,j} f(i,j,k) &\leq \mu^{M_1}\widetilde{K}_1^{M_1}(\widehat{x}\varepsilon\gamma_a)^{i-M_1}(\varepsilon/\mu)^{j-i+M_1}\widetilde{K}_2^{j-i+M_1}M_1^{M_1}(j-i+M_1)^{j-i+M_1} \\
&\quad + \mu^{i-j}\widetilde{K}_1^{i-j}(\widehat{x}\varepsilon\gamma_a)^j(i-j)^{i-j}.
\end{aligned}$$

Since $0 \leq j-i+M_1 \leq M_2$, we can exploit that $\varepsilon/\mu(M_2+1)$ is sufficiently small and since $0 \leq i-j \leq M_1$, we can use that μM_1 is sufficiently small to conclude

$$S_2 \leq C(M_2+1)\left[\sum_{i=M_1+1}^{M_1+M_2}(\mu\widetilde{K}_1M_1)^{M_1}(\widehat{x}\varepsilon\gamma_a)^{i-M_1} + \sum_{j=M_2-1}^{M_2}(\widehat{x}\varepsilon\gamma_a)^j\right].$$

With Lemma 4.8, we therefore get for suitable γ', since $i-M_1 \leq M_2$,

$$\begin{aligned}
e^{-\widehat{x}/4}S_2 &\leq C(M_2+1)\left[(\mu M_1\widetilde{K}_1)^{M_1}\sum_{i=M_1+1}^{M_1+M_2}(\varepsilon M_2\gamma')^{i-M_1} + \sum_{j=M_2-1}^{M_2}(\varepsilon M_2\gamma')^j\right] \\
&\leq C(M_2+1)\left[(\mu M_1\widetilde{K}_1)^{M_1}\varepsilon M_2 + (\varepsilon M_2\gamma')^{M_2-1}\right],
\end{aligned}$$

where, in the last step, we exploited again that μM_1 and εM_2 are sufficiently small. Since $M_2 \leq M_1+1$, this last estimate has the desired form.

Step 1e: Next, we consider the case $M_1+1 < M_2$. We write

$$\begin{aligned}
S_2 &= \sum_{i=M_1+1}^{\infty}\sum_{j=M_2-1}^{M_2}\sum_{k=i-M_1}^{\min\{i,j\}} f(i,j,k) \\
&= \sum_{i=M_1}^{M_2-1}\sum_{j=M_2-1}^{M_2}\sum_{k=i-M_1}^{i} f(i,j,k) + \sum_{i=M_2}^{\infty}\sum_{j=M_2-1}^{M_2}\sum_{k=i-M_1}^{j} f(i,j,k) =: S_{2,1}+S_{2,2}.
\end{aligned}$$

We note that

$$S_{2,2} = \sum_{i=M_2}^{M_1+M_2} \sum_{j=M_2-1}^{M_2} \sum_{k=i-M_1}^{j} f(i,j,k).$$

We may bound $S_{2,2}$ using arguments similar to those of *Step 1d.* The convexity of $k \mapsto f(i,j,k)$ yields

$$\begin{aligned}\max_{k=i-M_1,\dots,i} f(i,j,k) \quad &\leq \quad \mu^{M_1} \widetilde{K}_1^{M_1} (\widehat{x}\varepsilon\gamma_a)^{i-M_1} (\varepsilon/\mu)^{j-i+M_1} \widetilde{K}_2^{j-i+M_1} M_1^{M_1} (j-i+M_1)^{j-i+M_1} + \\ &\quad + (\widehat{x}\varepsilon\gamma_a)^i.\end{aligned}$$

Since $0 \leq j-i+M_1 \leq M_1 \leq M_2$, we can estimate by convexity

$$\left(\frac{\varepsilon}{\mu}\widetilde{K}_2(j-i-M_1)\right)^{j-i-M_1} \leq \left(1 + (\varepsilon/\mu\widetilde{K}_2 M_2)^{M_2}\right) \leq C,$$

since we assume that $\varepsilon/\mu M_2$ is sufficiently small. Hence, we get

$$S_{2,2} \leq C(M_2+1) \sum_{i=M_2}^{M_1+M_2} (\mu\widetilde{K}_1 M_1)^{M_1} (\widehat{x}\varepsilon\gamma_a)^{i-M_1} + (\widehat{x}\varepsilon\gamma_a)^i.$$

With Lemma 4.8 we therefore get

$$e^{-\underline{a}\widehat{x}/4} S_{2,2} \leq C(M_2+1) \sum_{i=M_2}^{M_1+M_2} (\mu\widetilde{K}_1 M_1)^{M_1} (\varepsilon\gamma'(i-M_1))^{i-M_1} + (\varepsilon\gamma' i)^i,$$

for suitable $\gamma' > 0$. Recalling that $M_1 \leq M_2$, we obtain by assuming that εM_2 is sufficiently small, with the aid of geometric series arguments,

$$e^{-\underline{a}\widehat{x}/4} S_{2,2} \leq C(M_2+1) \left[(\mu\widetilde{K}_1 M_1)^{M_1} + (\varepsilon\gamma'(M_2+1))^{M_2}\right].$$

For $S_{2,1}$ we note

$$\begin{aligned}&\max_{k=i-M_1,\dots,i} f(i,j,k) \leq \\ &\quad (\varepsilon/\mu)^{j-i} \widetilde{K}_2^{j-i} (j-i)^{j-i} (\widehat{x}\varepsilon\gamma_a)^i + \mu^{M_1} \widetilde{K}_1^{M_1} M_1^{M_1} (\varepsilon/\mu)^{j-i+M_1} (\widehat{x}\varepsilon\gamma_a)^{i-M_1} \widetilde{K}_2^{j-i+M_1} (j-i+M_1)^{j-i+M_1}.\end{aligned}$$

We note i and j in the definition of the sum $S_{2,1}$ are such that $0 \leq j-i \leq M_2$ and $M_1 \leq j-i+M_1 \leq M_2$. Hence, this setting simplifies under the assumption that μM_1 and $\varepsilon/\mu M_2$ are sufficiently small:

$$\begin{aligned}\max_{k=i-M_1,\dots,i} f(i,j,k) \quad &\leq \quad (\widehat{x}\varepsilon\gamma_a)^i + (\mu\widetilde{K}_1 M_1)^{M_1} (\widehat{x}\varepsilon\gamma_a)^{i-M_1} (\varepsilon/\mu M_2 \widetilde{K}_2)^{M_1} \\ &\leq \quad (\widehat{x}\varepsilon\gamma_a)^i + (\mu\widetilde{K}_1 M_1)^{M_1} (\widehat{x}\varepsilon\gamma_a)^{i-M_1}.\end{aligned}$$

With the aid of Lemma 4.8, we estimate with suitable $\gamma' > 0$,

$$\begin{aligned}e^{-\underline{a}\widehat{x}/4} S_{2,1} \quad &\leq \quad (M_1+1) \sum_{i=M_1}^{M_2-1} (\varepsilon\gamma' i)^i + (\mu\widetilde{K}_1 M_1)^{M_1} (\varepsilon\gamma'(i-M_1))^{i-M_1} \\ &\leq \quad C(M_1+1) \left[(\varepsilon M_2 \gamma')^{M_1} + (\mu\widetilde{K}_1 M_1)^{M_1}\right],\end{aligned}$$

where we employed again suitable geometric series arguments. Using $\varepsilon = (\varepsilon/\mu)\mu$, we get

$$e^{-\underline{a}\widehat{x}/4} S_{2,1} \quad \leq \quad C(M_1+1)\left[\mu^{M_1}\left(\frac{\varepsilon}{\mu}M_2\gamma'\right)^{M_1} + (\mu\widetilde{K}_1 M_1)^{M_1}\right],$$

which has the desired form since $\varepsilon/\mu M_2$ is assumed to be sufficiently small.

Step 2: Before proceding, we point out that we make the assumption

$$\widehat{x}\varepsilon\gamma_a \leq \frac{1}{2}, \tag{D.60}$$

as the converse case is covered in *Step 3* below. We estimate the contribution arising from the sum over $\widetilde{\mathcal{I}}$ and show

$$e^{-\underline{a}\widehat{x}/4} \sum_{(i,j,k)\in\widetilde{\mathcal{I}}} f(i,j,k) \leq C\left[\frac{\varepsilon}{\mu}(\varepsilon M_2\gamma)^{M_2} + \mu(\mu M_1\gamma)^{M_1}\right]; \tag{D.61}$$

in fact, the reasoning below shows a slightly sharper estimate.

Noting the appearance of $\max\{i-M_1, j-M_2\}$, we have to study the cases $i-M_1 \geq j-M_2$ and the reverse case $i-M_1 < j-M_2$ separately. We restrict our attention here to the case $i-M_1 \leq j-M_2$, since the reverse case is easily obtained using the same arguments (effectively, M_1 and M_2 and μ and ε/μ reverse their roles). In this situation, we consider the following two subcases separately:

$$(i \;\; \geq \;\; M_1+1 \;\vee\; j \geq M_2+1) \;\wedge\; (i-M_1 \leq j-M_2) \;\wedge\; (j-M_2 \leq k \leq i) \;\wedge\; (i \leq j), \tag{D.62}$$

$$(i \;\; \geq \;\; M_1+1 \;\vee\; j \geq M_2+1) \;\wedge\; (i-M_1 \leq j-M_2) \;\wedge\; (j-M_2 \leq k \leq j) \;\wedge\; (j \leq i). \tag{D.63}$$

A key ingredient of our proof is the convexity assertion given in Lemma D.1. We recall that a non-negative convex function attains its maximum at the boundary (i.e., in the univariate case, at the endpoints of an interval).

Step 2a: We consider the case (D.62), which can be further subdivided into the cases $i \geq M_1+1$ and $j \geq M_2+1$.

Step 2a1: We consider the case (D.62) with the further assumption $i \geq M_1+1$. We get

$$(i \geq M_1+1) \;\wedge\; (i \leq j) \;\wedge\; (i \leq j-M_2+M_1) \;\wedge\; (j-M_2 \leq k \leq i).$$

That is, we have to estimate the triple sum

$$T := \sum_{i\geq M_1+1} \sum_{\substack{j\geq i\\ j\geq i+M_2-M_1}}^{j\leq i+M_2} \sum_{k=j-M_2}^{i} f(i,j,k),$$

where the summation index is i for the outermost sum, j for the middle sum, and k for the innermost sum. This triple sum is estimated using convexity of the argument. The innermost sum has at most $i-j+M_2+1$ terms and $k \mapsto f(i,j,k)$ is convex by Lemma D.1. Hence, we obtain

$$\begin{aligned} T \quad &\leq \quad \sum_{i\geq M_1+1} \sum_{\substack{j\geq i\\ j\geq i+M_2-M_1}}^{j\leq i+M_2} (i-j+M_2+1)\max\{f(i,j,i), f(i,j,j-M_2)\} \\ &\leq \quad (M_1+1) \sum_{i\geq M_1+1} \sum_{\substack{j\geq i\\ j\geq i+M_2-M_1}}^{j\leq i+M_2} \max\{f(i,j,i), f(i,j,j-M_2)\}, \end{aligned}$$

where, in the second step we have used the restrictions on the sum on j to bound $i-j+M_2+1 \leq M_1+1$. Writing out $f(i,j,i)$ and $f(i,j,j-M_2)$, we have

$$\begin{aligned} f(i,j,i) &= \widetilde{K}_2^{j-i}(\varepsilon/\mu)^{j-i}(\widehat{x}\varepsilon\gamma_a)^i(j-i)^{j-i}, \\ f(i,j,j-M_2) &= \mu^{i-j+M_2}(\varepsilon/\mu)^{M_2}\widetilde{K}_1^{i-j+M_2}(\widehat{x}\varepsilon\gamma_a)^{j-M_2}\widetilde{K}_2^{M_2}(i-j+M_2)^{i-j+M_2}M_2^{M_2}, \end{aligned}$$

which are again convex functions of j by Lemma D.1. Turning now to the sum on j, we see that it has at most M_2+1 terms. In view of Lemma D.2, we bound for the relevant j:

$$f(i,j,i) \leq A := \max\{f(i,j,i)|_{j=i+M_2}, f(i,j,i)|_{j=i}\} = \max\{f(i,i+M_2,i), f(i,i,i)\}, \quad \text{(D.64a)}$$

$$\begin{aligned} f(i,j,j-M_2) \leq B &:= \max\{f(i,j,j-M_2)|_{j=i+M_2}, f(i,j,j-M_2)|_{j=i+M_2-M_1}\} \\ &= \max\{f(i,i+M_2,i), f(i,i+M_2-M_1,i-M_1)\}. \quad \text{(D.64b)} \end{aligned}$$

More explicitly, these are

$$\begin{aligned} A &\leq (\widehat{x}\varepsilon\gamma_a)^i \max\{1, \widetilde{K}_2^{M_2}(\varepsilon/\mu)^{M_2}M_2^{M_2}\}, \\ B &\leq (\varepsilon/\mu)^{M_2}M_2^{M_2}\widetilde{K}_2^{M_2}\max\{(\widehat{x}\varepsilon\gamma_a)^i, (\mu\widetilde{K}_1M_1)^{M_1}(\widehat{x}\varepsilon\gamma_a)^{i-M_1}\}. \end{aligned}$$

Writing out the sum and using convexity of the argument, we can estimate

$$\begin{aligned} S &:= (M_1+1)\sum_{i\geq M_1+1}\sum_{\substack{j\geq i \\ j\geq i+M_2-M_1}}^{j\leq i+M_2}\sum_{k=j-M_2}^{i}\max\{f(i,j,i), f(i,j,j-M_2)\} \\ &\leq (M_1+1)(\min\{M_1,M_2\}+1)\sum_{i=M_1+1}^{\infty} A+B. \end{aligned}$$

Using the facts that μM_1 and $\varepsilon/\mu M_2$ are sufficiently small, we can estimate with geometric series arguments, in view of our assumption (D.60):

$$S \leq C(M_1+1)(\min\{M_1,M_2\}+1)\left[(\widehat{x}\varepsilon\gamma_a)^{M_1+1} + (\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}\widehat{x}\varepsilon\gamma_a\right].$$

Hence, using Lemma 4.8 to control $\widehat{x}^{M_1+1}$ in the first term and $\widehat{x}$ in the second term, gives with suitable γ',

$$\begin{aligned} e^{-\underline{a}\widehat{x}/4}S &\leq C(M_1+1)(\min\{M_1,M_2\}+1)\left[(\varepsilon(M_1+1)\gamma')^{M_1+1} + \varepsilon(\varepsilon/\mu\widetilde{K}_2M_2)^{M_2}\right] \\ &\leq C\left[(\varepsilon(M_1+1)\gamma'')^{M_1+1} + \frac{\varepsilon}{\mu}(\varepsilon/\mu\gamma''M_2)^{M_2}\right], \end{aligned}$$

where, in the second step, we selected γ'' suitable and used the fact that μM_1 can be controlled. Since $\varepsilon \leq \mu$, we obtain the desired estimate (D.61).

Step 2a2: The other case is $j \geq M_2+1$. This leads to

$$(j \geq M_2+1) \ \wedge\ (i \leq j) \ \wedge\ (i \leq j-M_2+M_1) \ \wedge\ (j-M_2 \leq k \leq i).$$

Writing out the sum, we have

$$T' := \sum_{j\geq M_2+1}\sum_{i\geq j-M_2}^{\substack{i\leq j \\ i\leq j-M_2+M_1}}\sum_{k=j-M_2}^{i} f(i,j,k).$$

The innermost sum has at most M_2+1 terms which, by convexity, can be estimated by

$$\max\{f(i,j,i), f(i,j,j-M_2)\}.$$

Writing out $f(i,j,i)$ and $f(i,j,j-M_2)$, we have

$$\begin{aligned} f(i,j,i) &= (\varepsilon/\mu)^{j-i}\widetilde{K}_2^{j-i}(\widehat{x}\varepsilon\gamma_a)^i(j-i)^{j-i}, \\ f(i,j,j-M_2) &= \mu^{i-j+M_2}\widetilde{K}_1^{i-j+M_2}(i-j+M_2)^{i-j+M_2}(\varepsilon/\mu)^{M_2}\widetilde{K}_2^{M_2}(\widehat{x}\varepsilon\gamma_a)^{j-M_2}M_2^{M_2}, \end{aligned}$$

which are again convex as functions of i. Hence, by Lemma D.2, we bound for the relevant j:

$$\begin{aligned} f(i,j,i) &\le A' := \max\{f(i,j,i)|_{i=j-M_2}, f(i,j,i)|_{i=j}\} \\ &= \max\{(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}(\widehat{x}\varepsilon\gamma_a)^{j-M_2}, (\widehat{x}\varepsilon\gamma_a)^j\}, \\ f(i,j,j-M_2) &\le B' := \max\{f(i,j,j-M_2)|_{i=j-M_2}, f(i,j,j-M_2)|_{i=j-M_2+M_1}\} \\ &\le (\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}(\widehat{x}\varepsilon\gamma_a)^{j-M_2}\max\{1,(\mu M_1\widetilde{K}_1)^{M_1}\}\}. \end{aligned}$$

The middle sum in T' has at most $\min\{M_1,M_2\}+1$ terms. Hence, we arrive at

$$\begin{aligned} T' &\le (M_2+1)(\min\{M_1,M_2\}+1)\sum_{j\ge M_2+1}(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}(\widehat{x}\varepsilon\gamma_a)^{j-M_2}+(\widehat{x}\varepsilon\gamma_a)^j \\ &\le C(M_2+1)(\min\{M_1,M_2\}+1)\left[(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}(\widehat{x}\varepsilon\gamma_a)+(\widehat{x}\varepsilon\gamma_a)^{M_2+1}\right], \end{aligned}$$

where, in the second step we have employed geometric sum arguments, which are applicable in view of (D.60). Reasoning as at the end of *Step 2a1*, we get for suitable γ',

$$\begin{aligned} e^{-\underline{a}\widehat{x}/4}T' &\le C(M_2+1)(\min\{M_1,M_2\}+1)\left[\varepsilon(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}+(\varepsilon(M_2+1)\gamma')^{M_2+1}\right] \\ &\le C\left[\varepsilon(\varepsilon/\mu M_2\gamma'')^{M_2}+(\varepsilon(M_2+1)\gamma'')^{M_2+1}\right], \end{aligned}$$

where, we employed the assumption that μM_1 and $\varepsilon/\mu M_2$ are sufficiently small and, in the second step, γ'' is selected appopriately. This leads to the desired estimate (D.61).

Step 2b: The case (D.63) is further subdivided into the cases $i\ge M_1+1$ and $j\ge M_2+1$.

Step 2b1: In the fist case, $i\ge M_1+1$, we get

$$(i\ge M_1+1)\ \wedge\ (j\le i)\ \wedge\ (i\le j-M_2+M_1)\ \wedge\ (j-M_2\le k\le j).$$

We immediately see that we need

$$M_2\le M_1,$$

for this set of indices to be non-empty.

Writing out the sum and using convexity of the argument, we get

$$T'' := \sum_{i\ge M_1+1}\sum_{j\ge i+M_2-M_1}^{j\le i}\sum_{k=j-M_2}^{j}f(i,j,k).$$

As before, the innermost sum has M_2+1 terms and convexity yields, for the terms of the innermost sum, the upper bound $\max\{f(i,j,j), f(i,j,j-M_2)\}$. More explicitly,

$$\begin{aligned} f(i,j,j) &= \mu^{i-j}\widetilde{K}_1^{i-j}(\widehat{x}\varepsilon\gamma_a)^j(i-j)^{i-j}, \\ f(i,j,j-M_2) &= \mu^{i-j+M_2}\widetilde{K}_1^{i-j+M_2}(\widehat{x}\varepsilon\gamma_a)^{j-M_2}(\varepsilon/\mu)^{M_2}\widetilde{K}_2^{M_2}M_2^{M_2}(i-j+M_2)^{i-j+M_2}. \end{aligned}$$

Again, we recognize the functions to be convex in j, so that for the relevant indices j we have

$$\begin{aligned} f(i,j,j) &\le A'' := \max\{f(i,j,j)|_{j=i}, f(i,j,j)|_{j=i+M_2-M_1}\} \\ &\le \max\{(\widehat{x}\varepsilon\gamma_a)^i, (\widehat{x}\varepsilon\gamma_a)^{i+M_2-M_1}(\mu(M_1-M_2)\widetilde{K}_1)^{M_1-M_2}\}, \\ f(i,j,j-M_2) &\le B'' := \max\{f(i,j,j-M_2)|_{j=i}, f(i,j,j-M_2)|_{j=i+M_2-M_1}\} \\ &\le \max\{(\mu M_2\widetilde{K}_1)^{M_2}(\widehat{x}\varepsilon\gamma_a)^{i-M_2}(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}, (\mu M_1\widetilde{K}_1)^{M_1}(\widehat{x}\varepsilon\gamma_a)^{i-M_1}(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}\}. \end{aligned}$$

The middle sum of T'' has at most M_1-M_2+1 terms. Therefore, we get with our assumption that μM_1 and $\varepsilon/\mu M_2$ are sufficiently small and the assumption (D.60),

$$\begin{aligned} T'' \le\ & (M_1-M_2+1)(M_2+1)\Big[(\widehat{x}\varepsilon\gamma_a)^{M_1+1} + (\widehat{x}\varepsilon\gamma_a)^{M_2+1}(\mu(M_1-M_2)\widetilde{K}_1)^{M_1-M_2} \\ & + (\mu M_2\widetilde{K}_1)^{M_2}(\widehat{x}\varepsilon\gamma_a)^{M_1+1-M_2}(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2} + (\mu M_1\widetilde{K}_1)^{M_1}(\widehat{x}\varepsilon\gamma_a)(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}\Big]. \end{aligned}$$

Since $M_1 \ge M_2$, we can estimate further with the aid of Lemma 4.8,

$$\begin{aligned} e^{-\underline{a}\widehat{x}/4}T'' \le\ & C(M_1-M_2+1)(M_1+1)\Big[(\varepsilon(M_1+1)\gamma')^{M_1+1} \\ & + (\varepsilon(M_2+1)\gamma')^{M_2+1}(\mu(M_1-M_2)\widetilde{K}_1)^{M_1-M_2} \\ & + (\varepsilon M_2\widetilde{K}_1\widetilde{K}_2)^{M_2}(\varepsilon(M_1-M_2+1)\gamma')^{M_1-M_2+1} + \varepsilon(\mu M_1\widetilde{K}_1)^{M_1}(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}\Big] \\ \le\ & C\left[(\varepsilon(M_1+1)\gamma')^{M_1+1} + \frac{\varepsilon}{\mu}(\varepsilon(M_2+1)\gamma')^{M_2}\right], \end{aligned}$$

where we used again that μM_1 and $\varepsilon/\mu M_2$ are sufficiently small. This leads to the desired estimate (D.61).

Step 2b2: The last case is $j \ge M_2+1$. This reads:

$$(j \ge M_2+1) \ \wedge\ (j \le i) \ \wedge\ (i \le j-M_2+M_1) \ \wedge\ (j-M_2 \le k \le j).$$

We see again that

$$M_2 \le M_1,$$

is a necessary condition for the set of indices to be non-trivial.

Writing out the sum we have

$$T''' := \sum_{j\ge M_2+1}\ \sum_{i\ge j}^{j-M_2+M_1}\ \sum_{k=j-M_2}^{j} f(i,j,k).$$

Again, we use convexity of $k \mapsto f(i,j,k)$ and observe the estimates

$$\begin{aligned} f(i,j,j) &= \mu^{i-j}\widetilde{K}_1^{i-j}(\widehat{x}\varepsilon\gamma_a)^j(i-j)^{i-j}, \\ f(i,j,j-M_2) &= \mu^{i-j+M_2}\widetilde{K}_1^{i-j+M_2}(\widehat{x}\varepsilon\gamma_a)^{j-M_2}(\varepsilon/\mu)^{M_2}\widetilde{K}_2^{M_2}M_2^{M_2}(i-j+M_2)^{i-j+M_2}. \end{aligned}$$

Since these functions are convex functions of i, we can estimate for the relevant indices j:

$$\begin{aligned}
f(i,j,j) &\le A''' := \max\{f(i,j,j)|_{i=j}, f(i,j,j)|_{i=j-M_2+M_1}\} \\
&\le (\widehat{x}\varepsilon\gamma_a)^j \max\{1, (\mu(M_1-M_2)\widetilde{K}_1)^{M_1-M_2}\}, \\
f(i,j,j-M_2) &\le B''' := \max\{f(i,j,j-M_2)|_{i=j}, f(i,j,j-M_2)|_{i=j-M_2+M_1}\} \\
&\le (\widehat{x}\varepsilon\gamma_a)^{j-M_2}(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2} \max\{(\mu M_2\widetilde{K}_1)^{M_2}, (\mu M_1\widetilde{K}_1)^{M_1}\}.
\end{aligned}$$

Furthermore, in the triple sum T''' the number of terms in the innermost sum is M_2+1 whereas the number of terms of the middle sum is bounded by M_1-M_2+1. Hence, we can bound with geometric series arguments, in view of the assumption (D.60),

$$T''' \le C(M_2+1)(M_1-M_2+1)\Big[(\widehat{x}\varepsilon\gamma_a)^{M_2+1} + (\widehat{x}\varepsilon\gamma_a)(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}\max\{(\mu M_2\widetilde{K}_1)^{M_2}, (\mu M_1\widetilde{K}_1)^{M_1}\}\Big].$$

Since $M_2 \le M_1$, we can estimate $\mu M_2\widetilde{K}_1 \le \mu M_1\widetilde{K}_1 \le 1$, where we appealed in the last step to our standing assumption that μM_1 is sufficiently small. Hence, $\max\{(\mu M_2\widetilde{K}_1)^{M_2}, (\mu M_1\widetilde{K}_1)^{M_1}\} \le 1$. Thus, we can simplify

$$T''' \le C(M_2+1)(M_1-M_2+1)\Big[(\widehat{x}\varepsilon\gamma_a)^{M_2+1} + (\widehat{x}\varepsilon\gamma_a)(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}\Big].$$

Hence, with the aid of Lemma 4.8

$$\begin{aligned}
e^{-\underline{a}\widehat{x}/4}T''' &\le C(M_2+1)(M_1-M_2+1)\Big[(\varepsilon(M_2+1)\gamma')^{M_2+1} + \varepsilon(\varepsilon/\mu M_2\widetilde{K}_2)^{M_2}\Big] \\
&\le C\frac{\varepsilon}{\mu}\Big[(\varepsilon(M_2+1)\gamma'')^{M_2} + (\varepsilon/\mu M_2\gamma'')^{M_2}\Big].
\end{aligned}$$

This leads to the desired estimate (D.61).

Step 3: We now cover the case when $\widehat{x}\varepsilon$ is bounded away from zero. Specifically, let $c>0$ be fixed and consider the case $\widehat{x}\varepsilon \ge c$. Then we have the pointwise estimate

$$\begin{aligned}
\left|\widehat{\mathbf{U}}^M_{BL}(\widehat{x})\right| &\le \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i(\varepsilon/\mu)^j\left(\left|\widehat{u}^L_{ij}(\widehat{x})\right| + \left|\widehat{v}^L_{ij}(\widehat{x})\right|\right) \\
&\le \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i(\varepsilon/\mu)^j K^i\overline{K}^j\frac{\left(\widehat{C}(i+j)+|\widehat{x}|\right)^{2(i+j)}}{(i+j)!}e^{-\underline{a}\widehat{x}} \\
&\le \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i(\varepsilon/\mu)^j K^i\overline{K}^j\gamma^{i+j}i^i j^j e^{-3\underline{a}\widehat{x}/4} \\
&\le e^{-3\underline{a}\widehat{x}/4}\sum_{i=0}^{M_1}\mu^i K^i\gamma^i i^i\sum_{j=0}^{M_2}(\varepsilon/\mu)^j\overline{K}^j\gamma^j j^j \\
&\le e^{-3\underline{a}\widehat{x}/4}\sum_{i=0}^{M_1}\left(\mu\overline{K}\gamma M_1\right)^i\sum_{j=0}^{M_2}\left((\varepsilon/\mu)K\gamma M_2\right)^j,
\end{aligned}$$

from which the desired result follows, provided $\mu K\gamma M_1 < 1$ and $(\varepsilon/\mu)\overline{K}\gamma M_2 < 1$. Completely analogously, we get bounds for the derivatives of $\widehat{\mathbf{U}}^M_{BL}$.

In view of the form of the differential operator $L_{\varepsilon,\mu}$ applied to functions of $\widehat{x}$ (see (4.10)), we get in view of $\mu \geq \varepsilon$,

$$|L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M(\widehat{x})| \leq \left[\varepsilon^2 + \mu^2\right]\varepsilon^{-2}Ce^{-3\underline{a}\widehat{x}/4} \leq C\mu^2\varepsilon^{-2}e^{-3\underline{a}\widehat{x}/4} \leq C\varepsilon^{-2}e^{-e\underline{a}\widehat{x}/4}.$$

From the assumption $\widehat{x}\varepsilon \geq c$, we see that $\varepsilon^{-2} \leq c^{-2}\widehat{x}^2$, and the factor $\widehat{x}$ can again be absorbed by the exponentially decaying $e^{-3\underline{a}\widehat{x}/4}$. □

D.5 Proof of Theorem 4.14

We first study the case

$$0 < \widetilde{x}\mu \leq 1/2. \tag{D.65}$$

The starting point is the expression for $L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M$ in (4.47). It consists of a double sum and a triple sum. We first consider the double sum. From the bounds on $\widetilde{u}_{i,j}$ of Theorem 4.9 and Cauchy's integral theorem for derivatives as well as Lemma 4.8, we get for suitable $\gamma' > 0$,

$$\begin{aligned}\sum_{i=0}^{M_1}\sum_{j=M_2+1}^{M_2+2}\mu^i(\varepsilon/\mu)^j|\widetilde{u}''_{i,j-2}(\widetilde{x})| &\leq Ce^{-\underline{a}\widetilde{x}/2}\sum_{i=0}^{M_1}\mu^i(i+M_2+1)\gamma')^{i+M_2}(\varepsilon/\mu)^{M_2+1}\\ &\leq Ce^{-\underline{a}\widetilde{x}}(\widetilde{K}\varepsilon/\mu(M_2+1))^{M_2+1},\end{aligned}$$

where, in the second step we used Lemma 4.8 again.

We next turn to the triple sum in (4.47). From (4.32), (4.33), Lemma 4.8 and $\|\mathbf{A}_k\| \leq C_A\gamma_A^k$, we obtain with $K \geq \max\{K_1, K_2\}$, $\overline{K} \geq \max\{\overline{K}_1, \overline{K}_2\}$, $\widetilde{C} \geq \max\{C_{\widetilde{u}}, C_{\widetilde{v}}\}$:

$$\begin{aligned}\left|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M\right| &\leq CC_A\sum_{i=M_1+1}^{\infty}\sum_{j=0}^{M_2}\sum_{k=i-M_1}^{i}\mu^i(\varepsilon/\mu)^j\widetilde{x}^k\gamma_A^k\overline{K}^{i-k}K^j\frac{\left(\widetilde{C}(i-k+j)+|\widetilde{x}|\right)^{2(i-k+j)}}{(i-k+j)!}e^{-\underline{a}\,\mathrm{Re}(\widetilde{x})}\\ &\leq CC_A\sum_{i=M_1+1}^{\infty}\sum_{j=0}^{M_2}\sum_{k=i-M_1}^{i}\mu^i(\varepsilon/\mu)^j\overline{K}^{i-k}K^j\widetilde{x}^k\gamma^{i+j-k}(i+j-k)^{i+j-k}e^{-\underline{a}3\widetilde{x}/4}.\end{aligned}$$

The argument is a convex function of k. Hence, we can bound for suitable $\gamma > 0$,

$$\left|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M\right| \leq Ce^{-\underline{a}3\widetilde{x}/4}(M_1+1)\sum_{i=M_1+1}^{\infty}\sum_{j=0}^{M_2}\mu^i(\varepsilon/\mu)^j(K\gamma)^j\left[j^j\widetilde{x}^i+\widetilde{x}^{i-M_1}\overline{K}^{M_1}(j+M_1)^{j+M_1}\right].$$

In view of our assumption $\widetilde{x}\mu \leq 1/2$, the outer summation on i leads to a convergent geometric series. For the inner summation, we use $(j+M_1)^{j+M_1} \leq j^jM_1^{M_1}e^{j+M_1}$, the assumptions that μM_1 and $\varepsilon/\mu M_2$ are sufficiently small and convergent geometric series arguments to get, with appropriate $\widetilde{\gamma} > 0$,

$$\left|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M\right| \leq Ce^{-\underline{a}3\widetilde{x}/4}(M_1+1)\left[(\mu\widetilde{x})^{M_1+1}+(\mu M_1\widetilde{\gamma})^{M_1}\mu\widetilde{x}\right],$$

which can be estimated in the desired fashion with the aid of Lemma 4.8:

$$\begin{aligned}\left|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}\right| &\leq Ce^{-\underline{a}\widetilde{x}/2}(M_1+1)e^{-\underline{a}\widetilde{x}/4}\left[(\mu\widetilde{x})^{M_1+1}+(\mu M_1\widetilde{\gamma})^{M_1}\mu\widetilde{x}\right]\\ &\leq Ce^{-\underline{a}\widetilde{x}/2}\left[(\mu\gamma'(M_1+1))^{M_1+1}+(\mu(M_1+1)\widetilde{\gamma})^{M_1+1}\right],\end{aligned}$$

which has the appropriate form.

We now consider the converse case $\widetilde{x}\mu \geq 1/2$. Here, we need to exploit the fact that the functions $\widetilde{\mathbf{U}}^M_{BL}$ are exponentially decaying. We observe that the same reasoning as in Step 3 of the proof of Theorem 4.13 yields

$$\left|\widetilde{\mathbf{U}}^M_{BL}(\widetilde{x})\right| \leq Ce^{-3\underline{a}\widetilde{x}/4},$$

and, by Cauchy's integral theorem for derivatives, estimates for the derivatives. The arguments of Step 3 of the proof of Theorem 4.13 therefore, yield

$$\left|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}(\widetilde{x})\right| \leq C\left[\varepsilon^2+\mu^2\right]\mu^{-2}e^{-3\underline{a}\widetilde{x}/4}. \leq Ce^{-3\underline{a}\widetilde{x}/4}.$$

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