ASC Report No. 24/2014

Robust exponential convergence of hp-FEM in balanced norms for singularly perturbed reaction-diffusion equations

J.M. Melenk and C. Xenophontos

Institute for Analysis and Scientific Computing Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-05-6

Most recent ASC Reports

23/2014 *M. Feischl, G. Gantner, and D. Praetorius*
Reliable and efficient a posteriori error estimation for adaptive IGA boundary element methods for weakly-singular integral equations

22/2014 *W. Auzinger, O. Koch, M. Thalhammer*
Defect-based local error estimators for high-order splitting methods involving three linear operators

21/2014 *A. Jüngel and N. Zamponi*
Boundedness of weak solutions to cross-diffusion systems from population dynamics

20/2014 *A. Jüngel*
The boundedness-by-entropy principle for cross-diffusion systems

19/2014 *D. Boffi, L. Gastaldi, M. Ruggeri*
Mixed formulation for interface problems with distributed Lagrange multiplier

18/2014 *M. Halla, T. Hohage, L. Nannen. J. Schöberl*
Hardy Space Infinite Elements for Time-Harmonic Wave Equations with Phase Velocities of Different Signs

17/2014 *M. Feischl, G. Gantner, and D. Praetorius*
A posteriori error estimation for adaptive IGA boundary element methods

16/2014 *M. Feischl, T. Führer, G. Mitscha-Eibl, D. Praetorius, and E.P. Stephan*
Convergence of adaptive BEM and adaptive FEM-BEM coupling for estimators without h-weighting factor

15/2014 *L. Aceto, A. Fandl, C. Magherini, E. Weinmüller*
Numerical treatment of radial Schrödinger eigenproblems defined on a semi-infinite domain

14/2014 *H. Woracek*
Reproducing kernel almost Pontryagin spaces

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-05-6

Robust exponential convergence of *hp*-FEM in balanced norms for singularly perturbed reaction-diffusion equations

J. M. Melenk
Institut für Analysis und Scientific Computing
Vienna University of Technology
Wiedner Hauptstrasse 8-10, A-1040 Wien
AUSTRIA
and
C. Xenophontos
Department of Mathematics and Statistics
University of Cyprus
P.O. BOX 20537
Nicosia 1678
CYPRUS

August 15, 2014

Abstract

The *hp*-version of the finite element method is applied to a singularly perturbed reaction-diffusion equation posed in one- and two-dimensional domains with analytic boundary. On suitably designed *Spectral Boundary Layer meshes*, robust exponential convergence in a balanced norm is shown. This balanced norm is stronger than the energy norm in that the boundary layers are $O(1)$ uniformly in the singular perturbation parameter. Robust exponential convergence in the maximum norm is also established. The theoretical findings are illustrated with two numerical experiments.

1 Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last decades (see, e.g., the books [8, 11] and the references therein). These problems typically feature boundary layers (and, more generally, also internal layers). Their resolution requires the use of strongly refined, layer-adapted meshes. In the context of fixed order methods, well-known representatives of such meshes include the Bakhvalov mesh [1] and the Shishkin mesh [14]. For the p/hp-version Finite Element Method (FEM) or for spectral methods, the *Spectral Boundary Layer mesh* [13, 3, 4] is essentially the smallest mesh that permits the resolution of boundary layers (see Definition 2.2 ahead for the 1D version and Section 3.1 for a realization in 2D).

The use of the above mentioned meshes can lead to robust convergence, i.e., convergence uniform in the singular perturbation parameter. For the reaction-diffusion equations (2.1), (3.1) under consideration here, the FEM is naturally analyzed in the *energy norm* (2.6); robust convergence of the h-FEM on Shishkin meshes can be found, for example, in [11] and robust exponential convergence on *Spectral Boundary Layer meshes* is shown in [3, 4]. The (natural) energy norm associated with this boundary value problem is rather weak in that the layer contributions are not "seen" by the energy norm; that is, the energy norm of the layer contribution vanishes as the singular perturbation parameter ε tends to zero whereas the energy norm of the smooth part of the solution does not. This has sparked the recent work [2, 9, 10] to study the convergence of the h-FEM in norms stronger than the energy norm. The analysis of [2, 9, 10] is performed in an ε-weighted H^1-norm which is *balanced* in the sense that both the smooth part and the layer part are (generically) bounded away from zero uniformly in ε. Robust convergence of fixed order methods in this balanced norm is shown in [2, 9, 10] if Shishkin meshes are employed. We show in the present work that this analysis in balanced norms can be extended to the hp-version FEM on *Spectral Boundary Layer meshes* to give robust exponential convergence of the hp-version FEM. An additional outcome of our convergence analysis in the balanced norm is the robust exponential convergence in the maximum norm.

It is worth mentioning that robust exponential convergence of the hp-FEM on *Spectral Boundary Layer meshes* in the balanced norm was shown earlier in special cases. For example, for the case of equations with constant coefficients and polynomial right-hand sides, [13] observes that the smooth part of the asymptotic expansion is again polynomial and therefore in the finite element space. It follows that a factor $\varepsilon^{1/2}$ is gained in the convergence estimate and leads to robust exponential convergence in the balanced norm. A more detailed discussion of similar effects can be found in the concluding remarks of [5] and in the section with numerical results in [6].

Let us briefly discuss the ideas underlying our analysis. Asymptotic expansions may be viewed as a tool to decompose the solution into components associated with different length scales. Roughly speaking, our analysis in balanced norms mimicks this technique on the discrete level in that the Galerkin approximation is likewise decomposed into components associated with different length scales. In total, our analysis involves the following ideas:

1. An analysis of the difference between the FEM approximation and a Galerkin approximation to a *reduced problem*.

2. A stable decomposition of the FEM space on the layer-adapted mesh into fine and coarse components. This decomposition relies essentially on strengthened Cauchy-Schwarz inequalities.

Throughout the paper we will utilize the usual Sobolev space notation $H^k(\Omega)$ to denote the space of functions on Ω with weak derivatives up to order k in $L^2(\Omega)$, equipped with the norm $\|\cdot\|_{k,\Omega}$ and seminorm $|\cdot|_{k,\Omega}$. We will also use the space $H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$, where $\partial\Omega$ denotes the boundary of Ω. The norm of the space $L^\infty(\Omega)$ of essentially bounded functions is denoted by $\|\cdot\|_{\infty,\Omega}$. The letters C, c will be used to denote generic positive constants, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence. Finally, the notation $A \lesssim B$ means the existence of a positive constant C, which is independent of the quantities A and B under consideration and of the singular perturbation parameter ε, such that $A \leq CB$.

2 The one-dimensional case

We start with the one-dimensional case as many of the ideas can be seen in this setting already.

2.1 Problem formulation and solution regularity

We consider the following model problem: Find u such that

$$-\varepsilon^2 u'' + bu = f \text{ in } I = (0,1), \tag{2.1a}$$
$$u(0) = u(1) = 0. \tag{2.1b}$$

The parameter $0 < \varepsilon \leq 1$ is given, as are the functions $b > 0$ and f, which are assumed to be analytic on $\overline{I} = [0,1]$. In particular, we assume that there exist constants C_f, γ_f, C_b, γ_b, $c_b > 0$ such that

$$\begin{cases} \left\|f^{(n)}\right\|_{\infty,I} \leq C_f \gamma_f^n n! & \forall\, n \in \mathbb{N}_0, \\ \left\|b^{(n)}\right\|_{\infty,I} \leq C_b \gamma_b^n n! & \forall\, n \in \mathbb{N}_0, \\ b(x) \geq c_b > 0 & \forall x \in \overline{I}. \end{cases} \tag{2.2}$$

The variational formulation of (2.1) reads: Find $u \in H_0^1(I)$ such that

$$\mathcal{B}_\varepsilon(u,v) = \mathcal{F}(v) \quad \forall\, v \in H_0^1(I), \tag{2.3}$$

where, with $\langle \cdot, \cdot \rangle_I$ the usual $L^2(I)$ inner product,

$$\mathcal{B}_\varepsilon(u,v) = \varepsilon^2 \langle u', v' \rangle_I + \langle bu, v \rangle_I, \tag{2.4}$$
$$\mathcal{F}(v) = \langle f, v \rangle_I. \tag{2.5}$$

It follows that the bilinear form $\mathcal{B}_\varepsilon(\cdot,\cdot)$ given by (2.4) is coercive with respect to the *energy norm*

$$\|u\|_{E,I}^2 := \mathcal{B}_\varepsilon(u,u)\,, \tag{2.6}$$

i.e.,

$$\mathcal{B}_\varepsilon(u,u) \geq \|u\|_{E,I}^2 \quad \forall\, u \in H_0^1(I)\,.$$

The solution u is analytic in I and features boundary layers at the endpoints. Its regularity was described in [3] both in terms of classical differentiability (see Proposition 2.1, (i)) as well as asymptotic expansions (see Proposition 2.1, (ii)):

Proposition 2.1 ([3]). *Assume (2.2) and let $u \in H_0^1(I)$ be the solution of (2.1) Then:*

(i) There are constants C, $K > 0$ independent of $\varepsilon \in (0,1]$ such that $\|u^{(n)}\|_{L^2(I)} \leq CK^n \max\{n+1, \varepsilon^{-1}\}^n$ for all $n \in \mathbb{N}_0$.

(ii) u can be decomposed as $u = w + u^{BL} + r$ where, for some constants C_w, γ_w, C_{BL}, γ_{BL}, C_r, γ_r, $b > 0$ independent of $\varepsilon \in (0,1]$,

$$\left\|w^{(n)}\right\|_{\infty,I} \leq C_w \gamma_w^n n^n,\ \forall\, n \in \mathbb{N}_0, \tag{2.7a}$$

$$\left|\left(u^{BL}\right)^{(n)}(x)\right| \leq C_{BL}\gamma_{BL}^n \varepsilon^{-n} e^{-b\,\mathrm{dist}(x,\partial I)/\varepsilon},\ \forall\, n \in \mathbb{N}_0, \quad \forall\, x \in I, \tag{2.7b}$$

$$\|r^{(n)}\|_{0,I} \leq C_r \varepsilon^{2-n} e^{-\gamma_r/\varepsilon},\ n \in \{0,1,2\}. \tag{2.7c}$$

2.2 High order FEM

The discrete version of the variational formulation (2.3) reads: Given $V_N \subset H_0^1(\Omega)$ find $u_{FEM} \in V_N$ such that

$$\mathcal{B}_\varepsilon(u_{FEM}, v) = \mathcal{F}(v) \quad \forall v \in V_N. \tag{2.8}$$

In order to define the FEM space V_N, let $\Delta = \{0 = x_0 < x_1 < ... < x_N = 1\}$ be an arbitrary partition of $I = (0,1)$ and set

$$I_j = [x_{j-1}, x_j]\,, \quad h_j = x_j - x_{j-1}, \quad j = 1, ..., N.$$

Also, define the reference element $I_{ST} = [-1,1]$ and note that it can be mapped onto the j^{th} element I_j by the standard affine mapping $x = Q_j(t) = \frac{1}{2}(1-t)\,x_{j-1} + \frac{1}{2}(1+t)\,x_j$. With $\Pi_p(I_{ST})$ the space of polynomials of degree $\leq p$ on I_{ST} (and with $\circ$ denoting composition of functions), we define the finite dimensional subspace as

$$\begin{aligned} \mathcal{S}^p(\Delta) &= \left\{v \in H^1(I) : v \circ Q_j^{-1} \in \Pi_{p_j}(I_{ST}), j = 1, ..., \mathcal{M}\right\}, \\ \mathcal{S}_0^p(\Delta) &= S^p(\Delta) \cap H_0^1(I). \end{aligned}$$

We restrict our attention here to constant polynomial degree p for all elements, i.e., $p_j = p$, $j = 1, \ldots, N$; clearly, more general settings with variable polynomial degree are possible. The following *Spectral Boundary Layer mesh* is essentially the minimal mesh that yields robust exponential convergence.

Definition 2.2 (Spectral Boundary Layer mesh)**.** *For $\lambda > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \leq 1$, define the Spectral Boundary Layer mesh $\Delta_{BL}(\lambda, p)$ as*

$$\Delta_{BL}(\lambda, p) := \begin{cases} \{0, \lambda p\varepsilon, 1 - \lambda p\varepsilon, 1\} & \text{if } \lambda p\varepsilon < 1/4 \\ \{0, 1\} & \text{if } \lambda p\varepsilon \geq 1/4. \end{cases}$$

The spaces $S(\lambda, p)$ and $S_0(\lambda, p)$ of piecewise polynomials of degree p are given by

$$\begin{aligned} S(\lambda, p) &:= \mathcal{S}^p(\Delta_{BL}(\lambda, p)), \\ S_0(\lambda, p) &:= \mathcal{S}_0^p(\Delta_{BL}(\lambda, p)) = S(\lambda, p) \cap H_0^1(I). \end{aligned}$$

We quote the following result from [3].

Proposition 2.3. *Assume that (2.2) holds. Let u be the solution to (2.3) and $u_{FEM} \in S_0(\lambda, p)$ be its finite element approximation based on the Spectral Boundary Layer mesh. Then, there exists $\lambda_0 > 0$ (depending only on b and f) such that for every $\lambda \in (0, \lambda_0)$ there are positive constants C, σ, independent of ε and p such that*

$$\|u_{FEM} - u\|_{E,I} \sim \|u_{FEM} - u\|_{0,I} + \varepsilon \left\|(u_{FEM} - u)'\right\|_{0,I} \leq Ce^{-\sigma p}.$$

Proposition 2.3 follows from an approximation result for the solution u of (2.3) on *Spectral Boundary Layer meshes*. The following result Lemma 2.5 slightly sharpens [3, Thm. 16] in that the approximation of the layer contribution is handled differently. This modification is due to [13] and is needed for a robust exponential convergence in the *balanced* norm. For future reference, we formulate this result as a separate lemma:

Lemma 2.4. *Let $\varepsilon \in (0, 1]$. Let the function u^{BL} satisfy on $I = (0, 1)$ the estimate*

$$|(u^{BL})^{(n)}(x)| \leq C_{BL}\gamma_{BL}^n \varepsilon^{-n} e^{-x/\varepsilon} \qquad \forall x \in I, \quad \forall n \in \mathbb{N}_0. \tag{2.9}$$

Then there are constants C, β, $\eta > 0$ (depending only on γ_{BL}) such that the following is true: Let Δ be any mesh with a mesh point $\xi \in (0, 1]$ that satisfies

$$\frac{\xi\varepsilon}{p} \leq \eta. \tag{2.10}$$

Then there exists an approximation $I_p u^{BL} \in \mathcal{S}^p(\Delta)$ with $I_p u^{BL}(0) = u^{BL}(0)$ and $I_p u^{BL}(1) = u^{BL}(1)$ having the approximation properties

$$\|u^{BL} - I_p u^{BL}\|_{\infty,(0,\xi)} + \xi^{-1/2}\|u^{BL} - I_p u^{BL}\|_{0,(0,\xi)} + \xi^{1/2}\|u^{BL} - I_p u^{BL}\|_{1,(0,\xi)}$$

$$\leq CC_{BL}\left[\frac{\xi}{p\varepsilon}e^{-\beta p} + e^{-\xi/\varepsilon}\right], \tag{2.11}$$

$$\|u^{BL} - I_p u^{BL}\|_{\infty,(\xi,1)} \leq CC_{BL}e^{-\xi/\varepsilon}, \tag{2.12}$$

$$\|u^{BL} - I_p u^{BL}\|_{0,(\xi,1)} + \varepsilon\|u^{BL} - I_p u^{BL}\|_{1,(\xi,1)} \leq CC_{BL}\sqrt{\varepsilon}e^{-\xi/\varepsilon}. \tag{2.13}$$

Proof. We will assume that $\xi \in (0, 1/2)$; in the converse, "asymptotic" case we have $\varepsilon^{-1} \lesssim p$ so that a suitable approximation on a single element may be taken (e.g., the operator $\mathcal{I}_p$ of [5]).

It suffices to assume that the mesh consists of the two elements $I_1 := (0, \xi)$ and $I_2 := (\xi, 1)$. We will construct $I_p u^{BL}$ separately on the two elements, starting with I_1.

On I_1, we construct $I_p u^{BL}$ in two steps. First, we let $\pi^1 \in \Pi_p$ be defined (on I_1) in terms of the operator $\mathcal{I}_p$ described in [5, Sec. 3.2.1]. This operator interpolates at the endpoints 0, ξ. From [5, Cor. 3.9] we get the existence of $\eta > 0$ such that the constraint (2.10) implies

$$\xi^{-1}\|\pi^1 - u^{BL}\|_{0,I_1} + |\pi^1 - u^{BL}|_{1,I_1} \leq CC_{BL}\frac{\xi^{1/2}}{p\varepsilon}e^{-\beta p}.$$

The 1D Sobolev embedding theorem in the form $\|v\|_{\infty,J} \lesssim |J|^{-1/2}\|v\|_{0,J} + |J|^{1/2}\|v'\|_{0,J}$ (where $|J|$ denotes the length of the interval J) gives

$$\xi^{-1/2}\|\pi^1 - u^{BL}\|_{\infty,I_1} + \xi^{-1}\|\pi^1 - u^{BL}\|_{0,I_1} + |\pi^1 - u^{BL}|_{1,I_1} \leq CC_{BL}\frac{\xi^{1/2}}{p\varepsilon}e^{-\beta p}.$$

Next, we modify π^1 as proposed in [13] in order to obtain a better approximation on the element I_2. We define $\pi^2 \in \Pi_p$ on I_1 as

$$\pi^2(x) := \pi^1(x) - \frac{x}{\xi}(1 - \sqrt{\varepsilon})u^{BL}(\xi),$$

so that $\pi^2(\xi) = \sqrt{\varepsilon}u^{BL}(\xi)$. In view of $|u^{BL}(\xi)| \leq C_{BL}e^{-\xi/\varepsilon}$, this modification leads to

$$\xi^{-1/2}\|\pi^2 - u^{BL}\|_{\infty,I_1} + \xi^{-1}\|\pi^2 - u^{BL}\|_{0,I_1} + |\pi^2 - u^{BL}|_{1,I_1} \leq CC_{BL}\left[\frac{\xi^{1/2}}{p\varepsilon}e^{-\beta p} + \xi^{-1/2}e^{-\xi/\varepsilon}\right].$$

We take $(I_p u^{BL})|_{I_1} = \pi^2$, and this shows (2.11). On I_2, we take $(I_p u^{BL})|_{I_2}$ as the linear interpolant between the values $\sqrt{\varepsilon}u^{BL}(\xi)$ at ξ and $u^{BL}(1)$ at 1. We immediately get

$$\|I_p u^{BL}\|_{\infty,I_2} + \|(I_p u^{BL})'\|_{\infty,I_2} \leq C\sqrt{\varepsilon}|u^{BL}(\xi)| \leq CC_{BL}\sqrt{\varepsilon}e^{-\xi/\varepsilon}. \tag{2.14}$$

Furthermore, for u^{BL} we have

$$\|u^{BL}\|_{\infty,I_2} + \varepsilon^{-1/2}\|u^{BL}\|_{0,I_2} + \sqrt{\varepsilon}\|u^{BL}\|_{1,I_2} \leq CC_{BL}e^{-\xi/\varepsilon}. \tag{2.15}$$

(2.14) and (2.15) imply, along with the triangle inequality, then (2.12), (2.13). ∎

Lemma 2.5. *Assume that (2.2) holds. Let u be the solution to (2.3). Then there are constants λ_0, C, $\beta > 0$ (depending only on the constants appearing in (2.2)) such that for every $\lambda \in (0, \lambda_0]$ there exists an approximant $I_p u \in \mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$ that satisfies*

$$\|u - I_p u\|_{\infty,I} \leq Ce^{-\beta\lambda p}, \tag{2.16a}$$

$$\|u - I_p u\|_{0,I} + \sqrt{\lambda p\varepsilon}\|(u - I_p u)'\|_{0,I} \leq Ce^{-\beta\lambda p}. \tag{2.16b}$$

Proof. The proof follows the lines of [3, Thm. 16] (which, however, was based on the piecewise Gauss-Lobatto interpolant instead of the operator $\mathcal{I}$ of [5]). There, the solution u is decomposed into the smooth part w, the boundary layer part u^{BL}, and the remainder r as in Proposition 2.1. The approximation of w and r is done as in [3, Thm. 16]. The treatment of the boundary layer part of [3, Thm. 16] is replaced with an appeal to Lemma 2.4. We remark in passing that slightly sharper estimates are possible if one studies the error $u - I_p u$ on the two elements $(0, \lambda p\varepsilon)$ and $(\lambda p\varepsilon, 1)$ separately. □

2.3 Robust exponential convergence in a balanced norm

The goal of this article is to improve on Proposition 2.3 by showing the following result:

Theorem 2.6. *Assume (2.2). Let u solve (2.3) and $u_{FEM} \in S_0(\lambda, p)$ be obtained by (2.8) based on the Spectral Boundary Layer mesh $\Delta_{BL}(\lambda, p)$. Then there exists $\lambda_0 > 0$ independent of $\varepsilon \in (0, 1]$ such that for every $\lambda \in (0, \lambda_0)$ there are constants C, $\sigma > 0$ such that*

$$\|u_{FEM} - u\|_{0,I} + \varepsilon^{1/2} \left\|(u_{FEM} - u)'\right\|_{0,I} \le Ce^{-\sigma p}. \tag{2.17}$$

The remainder of this section is devoted to the proof of Theorem 2.6. Before that, we note a consequence of Theorem 2.6:

Corollary 2.7. *Under the assumptions of Theorem 2.6 there is $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ there are constants C, $\sigma > 0$ such that*

$$\|u - u_{FEM}\|_{\infty,I} \le Ce^{-\sigma p}.$$

Proof. We first observe that standard inverse estimates yield the result when $\lambda p\varepsilon \ge 1/4$, in which case the mesh consists of a single element. Let us therefore consider the 3-element case $\lambda p\varepsilon < 1/4$. Using the boundary condition at $x = 0$ we can write

$$|u(x) - u_{FEM}(x)| = \left| \int_0^x (u(t) - u_{FEM}(t))' \, dt \right|.$$

Assume first that $x \in (0, \lambda p\varepsilon]$. Then by the Cauchy-Schwarz inequality and (2.17)

$$|u(x) - u_{FEM}(x)| \;\le\; \sqrt{\lambda p\varepsilon} \left\|(u - u_{FEM})'\right\|_{0,I} \le \sqrt{\lambda p\varepsilon} \left(C\varepsilon^{-1/2} e^{-\sigma p}\right) \le C\sqrt{\lambda p} e^{-\sigma p}.$$

The same technique works if $x \in [1-\lambda p\varepsilon, 1)$. For $x \in [\lambda p\varepsilon, 1-\lambda p\varepsilon]$, we write with the approximation $I_p u$ of Lemma 2.5 and the triangle inequality $|u(x) - u_{FEM}(x)| \le |u(x) - I_p u(x)| + |I_p u(x) - u_{FEM}(x)|$. Lemma 2.5 takes care of $|u(x) - I_p u(x)|$ while $|I_p u(x) - u_{FEM}(x)|$ is treated with the standard polynomial inverse inequality $\|I_p u - u_{FEM}\|_{\infty,[\lambda p\varepsilon, 1-\lambda p\varepsilon]} \le Cp^2 \|I_p u - u_{FEM}\|_{0,I}$ and the energy estimate of Proposition 2.3. □

Proof of Theorem 2.6

Since the desired estimate in the "asymptotic" case $\lambda p\varepsilon \geq 1/4$ is easily shown (see the formal proof of Theorem 2.6 at the end of the section) we will focus in the following analysis on the 3-element case, i.e., $\lambda p\varepsilon < 1/4$.

We begin by defining the bilinear form

$$\mathcal{B}_0(u,v) = \langle bu, v\rangle_I\,, \tag{2.18}$$

corresponding to the reduced/limit problem. We also introduce the operator $\mathcal{P}_0 : L^2(I) \to S_0(\lambda,p)$ by the orthogonality condition[1]

$$\mathcal{B}_0(u - \mathcal{P}_0 u, v) = 0 \quad \forall\, v \in S_0(\lambda,p). \tag{2.19}$$

Then, by Galerkin orthogonality satisfied by $u - u_{FEM}$ (with respect to the bilinear form $\mathcal{B}_\varepsilon$) and by $u - \mathcal{P}_0 u$ (with respect to the bilinear form $\mathcal{B}_0$) we have

$$\begin{aligned} \|u_{FEM} - \mathcal{P}_0 u\|^2_{E,I} &= \mathcal{B}_\varepsilon(u_{FEM} - \mathcal{P}_0 u, u_{FEM} - \mathcal{P}_0 u) \\ &= \mathcal{B}_\varepsilon(u - \mathcal{P}_0 u, u_{FEM} - \mathcal{P}_0 u) \\ &= \varepsilon^2 \left\langle (u - \mathcal{P}_0 u)', (u_{FEM} - \mathcal{P}_0 u)' \right\rangle_I . \end{aligned} \tag{2.20}$$

Hence

$$\varepsilon \left\|(u_{FEM} - \mathcal{P}_0 u)'\right\|_{0,I} \leq \|u_{FEM} - \mathcal{P}_0 u\|_{E,I} \leq \varepsilon \left\|(u - \mathcal{P}_0 u)'\right\|_{0,I} .$$

The triangle inequality will then allow us to infer from this the exponential convergence result (2.17) provided we can show that

$$\left\|(u - \mathcal{P}_0 u)'\right\|_{0,I} \leq C\varepsilon^{-1/2} e^{-\sigma p},$$

for some positive constants C and σ independent of ε and p. This calculation shows that we have to study the H^1-stability of the operator $\mathcal{P}_0$ on *Spectral Boundary Layer meshes*.

Asymptotic expansions are a tool to decompose the solution u into components on the different length scales. We need to mimick this on the discrete level for $\mathcal{P}_0 u$. We define (implicitly assuming $\lambda p\varepsilon < 1/4$) the layer region

$$I_\varepsilon := [0, \lambda p\varepsilon] \cup [1 - \lambda p\varepsilon, 1]$$

and the following two subspaces of $S(\lambda,p)$:

$$S_1 = \mathcal{S}^p(\Delta), \qquad \Delta = \{0,1\}, \tag{2.21}$$

$$S_\varepsilon = \{u \in S(\lambda,p) \,:\, \operatorname{supp} u \subset I_\varepsilon\}. \tag{2.22}$$

[1]Note the subtle point that $S_0(\lambda,p) \subset H^1_0(I)$; in contrast, the reduced problem doesn't involve boundary conditions.

Note that the spaces S_1 and S_ε do not carry any boundary conditions at the endpoints of I – this is a reflection of the fact that the reduced problem does not satisfy the homogeneous Dirichlet boundary conditions. It is important for the further developments to observe that for the three-element mesh of sufficiently small $\lambda p\varepsilon$, there holds $S(\lambda,p) = S_1 \oplus S_\varepsilon$. In other words, each $z \in S(\lambda,p)$ has a unique decomposition $z = z_1 + z_\varepsilon$ with $z_1 \in S_1$ and $z_\varepsilon \in S_\varepsilon$, when $\lambda p\varepsilon < 1/4$. We also have the inverse estimates

$$\begin{aligned} \|z'\|_{0,I} &\leq Cp^2\|z\|_{0,I} \quad \forall z \in S_1, && (2.23)\\ \|z'\|_{0,I} &\leq C\frac{p^2}{\lambda p\varepsilon}\|z\|_{0,I} \quad \forall z \in S_\varepsilon, && (2.24)\end{aligned}$$

by [12, Thm. 3.91]. Furthermore, we have the following strengthened Cauchy-Schwarz inequality:

Lemma 2.8 (Strengthened Cauchy-Schwarz inequality)**.** *Let $\mathcal{B}_0$ be given by (2.18). Then, there is a constant $C > 0$ depending solely on $\|b\|_{\infty,I}$ and $\inf_{x\in I} b(x)$ such that*

$$|\mathcal{B}_0\,(u,v)| \leq C\min\{1, \sqrt{\lambda p\varepsilon}p\}\,\|u\|_{0,I}\,\|v\|_{0,I_\varepsilon} \quad \forall u \in S_1, v \in S_\varepsilon.$$

Proof. The standard Cauchy-Schwarz inequality yields $|\mathcal{B}_0(u,v)| \leq \|b\|_{\infty,I}\|u\|_{0,I}\|v\|_{0,I}$, which accounts for the "1" in the minimium.

Let $I_1 = (0,\delta_1)$ and $I_2 = (0,\delta_2)$ be two intervals with $\delta_1 < \delta_2$. Consider polynomials π_1 and π_2 of degree p. Then, using an inverse inequality [12, Thm. 3.92],

$$\left|\int_{I_1} \pi_1(x)\pi_2(x)\,dx\right| \leq \int_{I_1} |\pi_1(x)|\,|\pi_2(x)|\,dx \leq C\sqrt{\frac{\delta_1}{\delta_2}}p\|\pi_1\|_{0,I_2}\|\pi_2\|_{0,I_1}.$$

The result follows by taking $\delta_1 = \lambda p\varepsilon$, $\delta_2 = 1$. □

As already mentioned, since $S(\lambda,p) = S_1 \oplus S_\varepsilon$ when $\lambda p\varepsilon < 1/4$, we can uniquely decompose $\mathcal{P}_0 u$ into components in S_1 and S_ε. The Strengthened Cauchy-Schwarz inquality of Lemma 2.8 allows us to quantify the size of these contributions:

Lemma 2.9. *There exist constants C, $c > 0$ depending solely on $\inf_{x\in I} b(x) > 0$ and $\|b\|_{\infty,I}$ such that the following is true under the assumption*

$$\sqrt{\lambda p\varepsilon}p \leq c : \qquad (2.25)$$

For each $z \in L^2(I)$, the (unique) decomposition of

$$\mathcal{P}_0 z = z_1 + z_\varepsilon$$

into the components $z_1 \in S_1$ and $z_\varepsilon \in S_\varepsilon$ satisfies

$$\begin{aligned} \|z_1\|_{0,I} &\leq C\|z\|_{0,I}, && (2.26)\\ \|z_\varepsilon\|_{0,I} &\leq C\{\|z\|_{0,I_\varepsilon} + \sqrt{\lambda p\varepsilon}p\|z\|_{0,I}\}. && (2.27)\end{aligned}$$

Proof. Write $\mathcal{P}_0 z = z_1 + z_\varepsilon$ with $z_1 \in S_1$ and $z_\varepsilon \in S_\varepsilon$. We define the auxiliary function

$$\psi_{1,\varepsilon} := \begin{cases} \left(1 - \frac{x}{\lambda p \varepsilon}\right)^p & \text{if } x \in [0, \lambda p \varepsilon] \\ 0 & \text{otherwise.} \end{cases}$$

Then supp $\psi_{1,\varepsilon} \subset [0, \lambda p\varepsilon], \psi_{1,\varepsilon}(0) = 1$ and $\|\psi_{1,\varepsilon}\|_{0,I_\varepsilon} \sim p^{-1/2}\sqrt{\lambda p \varepsilon}$. For the right endpoint we define $\psi_{2,\varepsilon}(x) := \psi_{1,\varepsilon}(1-x), x \in [1 - \lambda p\varepsilon, 1]$. We also define

$$\widetilde{z}_\varepsilon := z_\varepsilon + \psi_{1,\varepsilon} z_1(0) + \psi_{2,\varepsilon} z_1(1),$$

and note that $\mathcal{P}_0 z \in S_0(\lambda, p)$. Thus, $(z_1 + z_\varepsilon)|_{\partial\Omega} = 0$ so that $\widetilde{z}_\varepsilon \in S_\varepsilon \cap H_0^1(I) \subset S_\varepsilon \cap S_0(\lambda, p)$. Utilizing the inverse estimate [12, Thm. 3.92]

$$\|\pi\|_{\infty,I} \leq Cp \|\pi\|_{0,I} \ \forall \, \pi \in S_1,$$

we arrive at

$$\|\widetilde{z}_\varepsilon\|_{0,I} = \|\widetilde{z}_\varepsilon\|_{0,I_\varepsilon} \leq C \left\{ \|z_\varepsilon\|_{0,I_\varepsilon} + p^{1/2}\sqrt{\lambda p \varepsilon}\, \|z_1\|_{0,I} \right\}.$$

The representation $\mathcal{P}_0 z = z_1 + z_\varepsilon \in S_0(\lambda, p)$ also implies

$$\mathcal{B}_0(z_1, v_1) + \mathcal{B}_0(z_\varepsilon, v_1) = \mathcal{B}_0(\mathcal{P}_0 z, v_1) \quad \forall \, v_1 \in S_1, \tag{2.28}$$

$$\mathcal{B}_0(z_1, v_\varepsilon) + \mathcal{B}_0(z_\varepsilon, v_\varepsilon) = \mathcal{B}_0(\mathcal{P}_0 z, v_\varepsilon) = \mathcal{B}_0(z, v_\varepsilon) \quad \forall \, v_\varepsilon \in S_\varepsilon \cap S_0(\lambda, p), \tag{2.29}$$

where in (2.29) we used the fact that $\mathcal{P}_0$ is the $\mathcal{B}_0$–projection onto $S_0(\lambda, p)$. Taking $v_1 = z_1$ in (2.28) and $v_\varepsilon = \widetilde{z}_\varepsilon \in S_\varepsilon \cap S_0(\lambda, p)$ in (2.29) yields, together with the Strengthened Cauchy Schwarz inequality of Lemma 2.8,

$$\|z_1\|_{0,I}^2 \leq C\{\|\mathcal{P}^0 z\|_{0,I}\|z_1\|_{0,I} + p\sqrt{\lambda p \varepsilon}\|z_\varepsilon\|_{0,I}\|z_1\|_{0,I}\}, \tag{2.30a}$$

$$\begin{aligned} \|z_\varepsilon\|_{0,I}^2 &\leq C\{\|z\|_{0,I_\varepsilon}\|\widetilde{z}_\varepsilon\|_{0,I_\varepsilon} + p\sqrt{\lambda p\varepsilon}\|\widetilde{z}_\varepsilon\|_{0,I}\|z_1\|_{0,I} + \|z_\varepsilon\|_{0,I}\|z_1\|_{0,I}\sqrt{\lambda p \varepsilon} p^{1/2}\} \\ &\leq C\{\|z_\varepsilon\|_{0,I}\left[\|z\|_{0,I_\varepsilon} + p\sqrt{\lambda p \varepsilon}\|z_1\|_{0,I} + \sqrt{\lambda p \varepsilon}p^{1/2}\|z_1\|_{0,I}\right] \\ &\qquad + \left[\|z\|_{0,I_\varepsilon} + p\sqrt{\lambda p \varepsilon}\|z_1\|_{0,I}\right]\sqrt{\lambda p\varepsilon}p^{1/2}\|z_1\|_{0,I}\}. \end{aligned} \tag{2.30b}$$

Estimating generously $\sqrt{\lambda p \varepsilon}p^{1/2} \leq \sqrt{\lambda p \varepsilon} p$ and using an appropriate Young inequality in (2.30b) we get

$$\|z_1\|_{0,I} \leq C\{\|\mathcal{P}_0 z\|_{0,I} + p\sqrt{\lambda p \varepsilon}\|z_\varepsilon\|_{0,I}\}, \tag{2.31a}$$

$$\|z_\varepsilon\|_{0,I} \leq C\{\|z\|_{0,I_\varepsilon} + p\sqrt{\lambda p \varepsilon}\|z_1\|_{0,I}\}. \tag{2.31b}$$

Inserting (2.31b) in (2.31a), assuming that $\sqrt{\lambda p \varepsilon} p$ is sufficiently small and using the stability $\|\mathcal{P}_0 z\|_{0,I} \leq C\|z\|_{0,I}$ gives $\|z_1\|_{0,I} \leq C\|z\|_{0,I}$. Inserting this bound in (2.31b) concludes the proof. □

We are now in the position to prove the following

Lemma 2.10. *Assume (2.2), let u be the solution of (2.3) and let λ_0 be given by Lemma 2.5. Let $\lambda \in (0, \lambda_0]$ and assume that λ, p, ε satisfy (2.25). Then there exist constants C, $\beta > 0$ (independent of ε and p but dependent on λ) such that*

$$\|(u - \mathcal{P}_0 u)'\|_{0,I} \leq C\varepsilon^{-1/2} e^{-\beta p}. \tag{2.32}$$

Proof. By Lemma 2.5 we can find an approximation $I_p u \in S_0(\lambda, p)$ with

$$\|u - I_p u\|_{0,I} + \sqrt{\varepsilon}\|(u - I_p u)'\|_{0,I} \leq Ce^{-\beta p}. \tag{2.33}$$

We stress that, while the estimate (2.16) is explicit in the parameter λ, we have absorbed this dependence here in the constants C and β for simplicity of exposition.

Since $\mathcal{P}_0$ is a projection on $S_0(\lambda, p)$, we can write $u - \mathcal{P}_0 u = u - I_p u - \mathcal{P}_0(u - I_p u)$. The first term is already treated in (2.33). For the second term, $\mathcal{P}_0(u - I_p u) \in S_0(\lambda, p)$, we decompose $\mathcal{P}_0(u - I_p u) = z_1 + z_\varepsilon$ and use the inverse estimates (2.23), (2.24) to get, with Lemma 2.9,

$$\begin{aligned} \|z_1'\|_{0,I} &\lesssim p^2 \|z_1\|_{0,I} \lesssim p^2 \|(u - I_p u)\|_{0,I} \leq Ce^{-\beta p}, \\ \|z_\varepsilon'\|_{0,I} &\lesssim \frac{p^2}{\lambda p \varepsilon} \|z_\varepsilon\|_{0,I_\varepsilon} \lesssim \frac{p^2}{\lambda p \varepsilon} \left[\|(u - I_p u)\|_{0,I_\varepsilon} + \sqrt{\lambda p \varepsilon} p \|(u - I_p u)\|_{0,I} \right]. \end{aligned}$$

There are several possible ways to treat the term $\|(u - I_p u)\|_{0,I_\varepsilon}$. A rather generous approach exploits the fact that $(u - I_p u)(0) = (u - I_p u)(1) = 0$ so that a Poincaré inequality on the intervals $(0, \lambda p \varepsilon)$ and $(1 - \lambda p \varepsilon, 1)$ yields

$$\|u - I_p u\|_{0,I_\varepsilon} \leq C\lambda p \varepsilon \|(u - I_p)'\|_{0,I_\varepsilon}.$$

Hence,

$$\|z_\varepsilon'\|_{0,I} \lesssim \frac{p^2}{\lambda p \varepsilon} \left[\lambda p \varepsilon \|(u - I_p u)'\|_{0,I_\varepsilon} + \sqrt{\lambda p \varepsilon} p \|u - I_p u\|_{0,I} \right] \lesssim \varepsilon^{-1/2} e^{-\beta p}.$$

□

Proof of Theorem 2.6: In view of $\|u - u_{FEM}\|_{0,I} \leq C\|u - u_{FEM}\|_{E,I} \leq Ce^{-\sigma p}$ by Proposition 2.3, we focus on the control of $\sqrt{\varepsilon}\|(u - u_{FEM})'\|_{0,I}$. We distinguish two cases:

Case 1: Assume that (2.25) is satisfied. Then (2.32) and Lemma 2.10 yield the result.

Case 2: Assume that $\sqrt{\lambda p \varepsilon} p \geq c$ for the constant c appearing in (2.25). Then $\varepsilon \geq c^2 p^{-3} \lambda^{-1}$ so that

$$\sqrt{\varepsilon}\|(u - u_{FEM})'\|_{0,I} \leq \varepsilon^{-1/2} \|u - u_{FEM}\|_{E_\varepsilon,I} \leq \frac{1}{c}\sqrt{\lambda} p^{3/2} \|u - u_{FEM}\|_{E_\varepsilon,I} \lesssim e^{-\sigma p},$$

which concludes the proof. □

2.4 Numerical example

To illustrate the theoretical findings presented above, we show in Figure 1 the results of numerical computations for the following problem:

$$-\varepsilon^2 u''(x) + u(x) = \left(x + \frac{1}{2}\right)^{-1}, \quad x \in (0,1), \tag{2.34a}$$

$$u(0) = u(1) = 0. \tag{2.34b}$$

We use the *Spectral Boundary Layer mesh* $\Delta_{BL}(\lambda, p)$ with $\lambda = 1$ and polynomials of degree p which we increase from 1 to 4 to improve accuracy. We select $\varepsilon = 10^{-j}$, $j \in \{4, 6, 8, 10, 12\}$. We note $\dim S_0(\lambda, p) = 2 + 3(p-1)$. Since no exact solution is available, we use a reference solution and show the estimated error in the balanced norm versus the polynomial degree p in a semi-log scale. Figure 1 clearly shows the predicted robustness and exponential convergence rate.

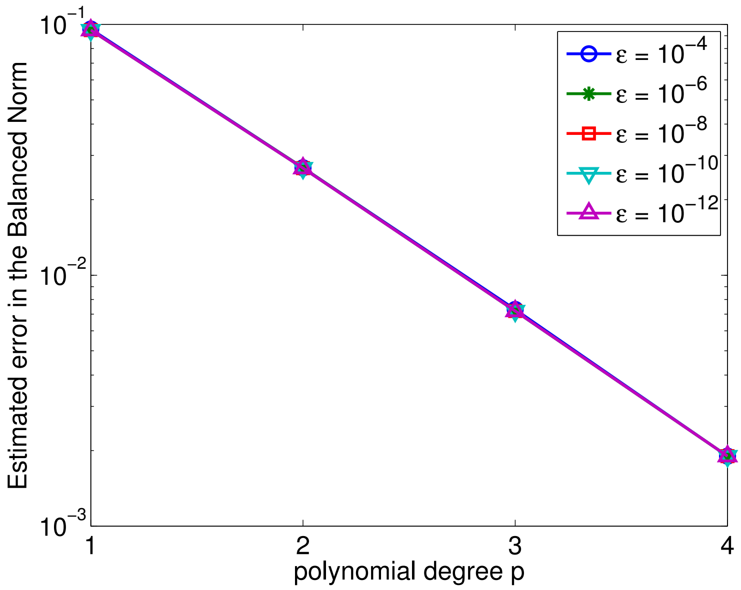

Figure 1: Balanced norm convergence on Spectral Boundary Layer meshes for (2.34).

3 The two-dimensional case

The ideas of the previous section carry over to the two-dimensional case. We consider the following boundary value problem: Find u such that

$$-\varepsilon^2 \Delta u + bu = f \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{3.1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.1b}$$

where $\varepsilon \in (0,1]$, and the functions b, f are given with $b > 0$ on $\overline{\Omega}$. We assume that the data of the problem is analytic, i.e., $\partial\Omega$ is an analytic curve and that there exist constants C_f, γ_f, C_b, γ_b, $c_b > 0$ such that

$$\begin{cases} \|\nabla^n f\|_{\infty,\Omega} \leq C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \|\nabla^n b\|_{\infty,\Omega} \leq C_b \gamma_b^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \inf_{x\in\Omega} b(x) \geq c_b > 0. \end{cases} \tag{3.2}$$

The variational formulation of (3.1a), (3.1b) reads: Find $u \in H_0^1(\Omega)$ such that

$$\mathcal{B}_\varepsilon(u,v) := \varepsilon^2 \langle \nabla u, \nabla v\rangle_\Omega + \langle bu, v\rangle_\Omega = F(v) := \langle f, v\rangle_\Omega \quad \forall v \in H_0^1(\Omega), \tag{3.3}$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the usual $L^2(\Omega)$ inner product. Again, the bilinear form $\mathcal{B}_\varepsilon$ induces the energy norm $\|\cdot\|_{E,\Omega}$ by

$$\|v\|_{E,\Omega}^2 := \mathcal{B}_\varepsilon(v,v).$$

The discrete version of (3.3) reads: find $u_{FEM} \in V_N \subset H_0^1(\Omega)$ such that (3.3) holds for all $v \in V_N \subset H_0^1(\Omega)$, with u replaced by u_{FEM}, where the subspace V_N will be defined shortly.

3.1 Meshes and spaces

Concerning the meshes and the *hp*-FEM space based on these meshes, we adopt the simplest case that generalizes our 1D analysis to 2D: The elements are (curvilinear) quadrilaterals and the needle elements required to resolve the boundary layer are obtained as mappings of needle elements of a reference configuration. This approach is discussed in more detail in [7, Sec. 3.1.2] and expanded as the notion of "patchwise structured meshes" in [4, Sec. 3.3.2].

Our *hp*-FEM spaces have the following general structure: Let $\Delta = \{\Omega_i\}_{i=1}^N$ be a mesh consisting of curvilinear quadrilaterals Ω_i, $i = 1,\ldots,N$, subject to the usual restrictions (see, e.g., [7]) and associate with each Ω_i a differentiable, bijective element mapping $M_i : S_{ST} \to \overline{\Omega}_i$, where $S_{ST} = [0,1]^2$ denotes the usual reference square. With $Q_p(S_{ST})$ the space of polynomials of degree p (in each variable) on S_{ST}, we set

$$\begin{aligned} \mathcal{S}^p(\Delta) &= \left\{u \in H^1(\Omega) : u|_{\Omega_i} = \varphi_p \circ M_i^{-1}, i = 1,...,N, \text{ for some } \varphi_p \in Q_p(S_{ST})\right\}, \\ \mathcal{S}_0^p(\Delta) &= \mathcal{S}^p(\Delta) \cap H_0^1(\Omega). \end{aligned}$$

We now describe the mesh Δ and the element maps that we will use (see Fig. 2). Our starting point is a *fixed* mesh Δ_A (the subscript "A" stands for "asymptotic") consisting of curvilinear quadrilateral elements Ω_i, $i = 1,\ldots,N'$. These elements Ω_i are the images of the reference square $S_{ST} = [0,1]^2$ under the element maps $M_{A,i}$, $i = 1,\ldots,N'$ (we added the subscript "A" to emphasize that they correspond to the asymptotic mesh Δ_A). They are assumed to satisfy the conditions (M1)–(M3) of [7] in order to ensure that the space $\mathcal{S}^p(\Delta_A)$

has suitable approximation properties. The element maps $M_{A,i}$ are assumed to be analytic with analytic inverse; that is, as in [7] we require

$$\|(M'_{A,i})^{-1}\|_{\infty,S_{ST}} \le C_1, \qquad \|D^\alpha M_{A,i}\|_{\infty,S_{ST}} \le C_2 \alpha! \gamma^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2, \quad i = 1, \dots, N'$$

for some constants C_1, C_2, $\gamma > 0$. We furthermore assume that elements do not have a single vertex on the boundary $\partial\Omega$ but only complete, single edges, i.e., the following dichotomy holds:

$$\text{either} \quad \overline{\Omega_i} \cap \partial\Omega = \emptyset \quad \text{or } \overline{\Omega_i} \cap \partial\Omega \quad \text{is a single edge of } \Omega_i. \tag{3.4}$$

Edges of curvilinear quadrilaterals are, of course, the images of the edges of S_{ST} under the element maps. For notational convenience, we assume that these edges are the image of the edge $\{0\} \times [0, 1]$ under the element map. It then follows that these elements have one edge on $\partial\Omega$ and the images of the edges $\{y = 1\}$ and $\{y = 0\}$ of S_{ST} are shared with elements that likewise have one edge on $\partial\Omega$. For notational convenience, we assume that the elements at the boundary are numbered first, i.e., they are the elements Ω_i, $i = 1, \dots, n < N'$. For a parameter $\lambda > 0$ and a degree $p \in \mathbb{N}$, the boundary layer mesh $\Delta_{BL} = \Delta_{BL}(\lambda, p)$ is defined as follows.

Definition 3.1 (Spectral Boundary Layer mesh $\Delta_{BL}(\lambda, p)$)**.** *Given parameters $\lambda > 0$, $p \in \mathbb{N}$, $\varepsilon \in (0, 1]$ and the asymptotic mesh Δ_A, the mesh $\Delta_{BL}(\lambda, p)$ is defined as follows:*

1. *$\lambda p\varepsilon \ge 1/2$. In this case we are in the asymptotic regime, and we use the coarse mesh Δ_A defined above.*
2. *$\lambda p\varepsilon < 1/2$. In this regime, we need to define so-called needle elements. This is done by splitting the elements $\Omega_i, i = 1, \dots, n$ into two elements Ω_i^{need} and Ω_i^{reg}. To that end, split the reference square S_{ST} into two elements*

$$S^{need} = [0, \lambda p\varepsilon] \times [0, 1], \qquad S^{reg} = [\lambda p\varepsilon, 1] \times [0, 1],$$

and define the elements Ω_i^{need}, Ω_i^{reg} as the images of these two elements under the element map $M_{A,i}$ and the corresponding element maps as the concatination of the affine maps

$$\begin{aligned} A^{need} &: S_{ST} \to S^{need}, \qquad (\xi, \eta) \to (\lambda p\varepsilon\xi, \eta), \\ A^{reg} &: S_{ST} \to S^{reg}, \qquad (\xi, \eta) \to (\lambda p\varepsilon + (1 - \lambda p\varepsilon)\xi, \eta) \end{aligned}$$

with the element map $M_{A,i}$, i.e., $M_i^{need} = M_{A,i} \circ A^{need}$ and $M_i^{reg} = M_{A,i} \circ A^{reg}$. Explicitly:

$$\begin{aligned} \Omega_i^{need} &= M_{A,i}\left(S^{need}\right), & \Omega_i^{reg} &= M_{A,i}\left(S^{reg}\right), \\ M_i^{need}(\xi, \eta) &= M_{A,i}\left(\lambda p\varepsilon\xi, \eta\right), & M_i^{reg}(\xi, \eta) &= M_{A,i}\left(\lambda p\varepsilon + (1 - \lambda p\varepsilon)\xi, \eta\right). \end{aligned}$$

In Figure 2 we show an example of such a mesh construction on the unit circle. In total, the mesh $\Delta_{BL}(\lambda, p)$ consists of $N = N' + n$ elements if $\lambda p\varepsilon < 1/2$. By construction, the

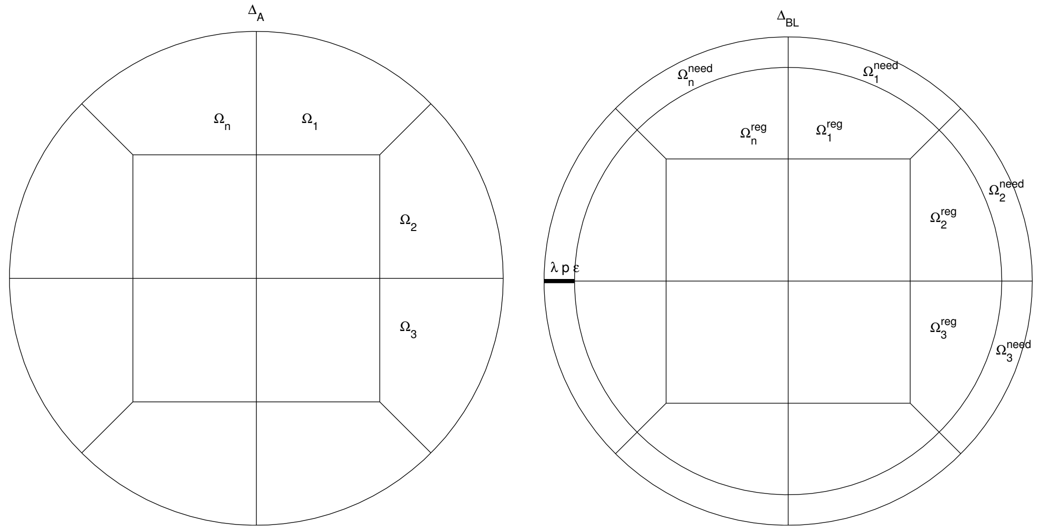

Figure 2: Example of an admissible mesh. Left: asymptotic mesh Δ_A. Right: BL mesh Δ_{BL}

resulting mesh $\Delta_{BL} = \Delta_{BL}(\lambda, p) = \left\{\Omega_1^{need}, ..., \Omega_n^{need}, \Omega_1^{reg}, ..., \Omega_n^{reg}, \Omega_{n+1}, ..., \Omega_N\right\}$ is a regular admissible mesh in the sense of [7]. Therefore, [7] gives that the space

$$S_0(\lambda, p) := \mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$$

has the following approximation properties:

Proposition 3.2 ([7])**.** *Let u be the solution to (3.3) and assume that (3.2) holds. Then there exist constants λ_0, λ_1, C, $\beta > 0$ independent of $\varepsilon \in (0, 1]$ and $p \in \mathbb{N}$ such that the following is true: For every p and every $\lambda \in (0, \lambda_0]$ with $\lambda p \geq \lambda_1$ there exists a $\pi_p u \in \mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$ such that*

$$\|u - \pi_p u\|_{\infty,\Omega} + \varepsilon^{1/2} |u - \pi_p u|_{1,\Omega} \leq Cp^2 (\ln p + 1)^2 e^{-\beta p \lambda}.$$

We mention in passing that Proposition 3.2 provides robust exponential convergence in the energy norm.

Anticipating that we will need, for the case $\lambda p \varepsilon < 1/2$, a decomposition of

$$S(\lambda, p) := \mathcal{S}^p(\Delta_{BL}(\lambda, p))$$

into two spaces reflecting the two scales present, we proceed as follows: With Δ_A the asymptotic (coarse) mesh that resolves the geometry we set

$$S_1 := \mathcal{S}^p(\Delta_A), \tag{3.5}$$

$$S_\varepsilon := \{v \in \mathcal{S}^p(\Delta_{BL}(\lambda, p)) \mid \operatorname{supp} v \subset \overline{\Omega}_{\lambda p \varepsilon}\}, \tag{3.6}$$

where the boundary layer region $\Omega_{\lambda p \varepsilon}$ is defined as

$$\Omega_{\lambda p \varepsilon} = \bigcup_{i=1}^{n} \Omega_i^{need}. \tag{3.7}$$

As in the 1D situation, our approximation space $\mathcal{S}^p(\Delta_{BL}(\lambda, p)$ can be written as a direct sum of S_1 and S_ε if $\lambda p\varepsilon < 1/2$:

Lemma 3.3. *Let $\lambda p\varepsilon < 1/2$. Then $\mathcal{S}^p(\Delta_{BL}(\lambda, p))$ is the direct sum $S_1 \oplus S_\varepsilon$. Furthermore, we have the inverse estimates*

$$\|u\|_{0,\partial\Omega_i} \leq Cp\|u\|_{0,\Omega_i} \quad \forall u \in S_1, \quad i = 1, ..., N', \tag{3.8}$$

$$|u|_{1,\Omega_i} \leq Cp^2\|u\|_{0,\Omega_i} \quad \forall u \in S_1, \quad i = 1, ..., N', \tag{3.9}$$

$$|u|_{1,\Omega_i} \leq C\frac{p^2}{\lambda p\varepsilon}\|u\|_{0,\Omega_i} \quad \forall u \in S_\varepsilon, i = 1, ..., n, \tag{3.10}$$

Proof. The claim that $\mathcal{S}^p(\Delta_{BL}(\lambda, p)) = S_1 \oplus S_\varepsilon$ follows from the way $\Delta_{BL}(\lambda, p)$ is constructed. Let $z \in \mathcal{S}^p(\Delta_{BL}(\lambda, p))$. Define $z_1 \in S_1$ as follows: For the internal elements Ω_i with $i = n+1, \ldots, N'$ take $z_1|_{\Omega_i} := z|_{\Omega_i}$. For Ω_i, $i \in \{1, \ldots, n\}$, which is further decomposed into Ω_i^{need} and Ω_i^{reg}, we consider the pull-back $\widetilde{z}_i := z|_{\Omega_i} \circ M_{A,i}$. This pull-back $\widetilde{z}_i$ is a piecewise polynomial on $S_{ST} = S^{need} \cup S^{reg}$. Define the *polynomial* $\widehat{z}_i \in Q(S_{ST})$ on the full reference element S_{ST} by the condition

$$\widehat{z}_i|_{S^{reg}} = \widetilde{z}_i|_{S^{reg}}$$

and then set $z_1|_{\Omega_i} := \widehat{z}_i \circ M_{A,i}^{-1}$; that is, the restriction $\widetilde{z}_i|_{S^{reg}}$ is extended polynomially to S_{ST}. In this way, the function z_1 is defined elementwise, and the assumptions on the element maps $M_{A,i}$ of the asymptotic mesh Δ_A ensure that $z_1 \in H^1(\Omega)$, i.e., $z_1 \in S_1$. Since by construction $z|_{\Omega_i^{reg}} = z_1|_{\Omega_i^{reg}}$ for $i = 1, \ldots, n$, we conclude that $\text{supp}(z - z_1) \subset \overline{\Omega}_{\lambda p\varepsilon}$ and therefore $z_\varepsilon := z - z_1 \in S_\varepsilon$. The construction also shows the uniqueness of the decomposition.

The inverse estimates (3.8), (3.9), (3.10) can be seen as follows. The estimate (3.9) is an easy consequence of the assumptions on the element maps $M_{A,i}$ of the asymptotic mesh Δ_A and the polynomial inverse estimates [12, Thm. 4.76]. In a similar manner, the inverse estimate (3.8), which estimates the L^2-norm on the boundary $\partial\Omega_i$ of Ω_i by the L^2-norm on Ω_i follows from a suitable application of 1D inverse estimates.

For the estimate (3.10), we note that for an element Ω_i^{need}, we can estimate for any $v \in S_\varepsilon$ again with assumptions on the element maps $M_{A,i}$

$$\|\nabla v\|_{0,\Omega_i^{need}} \sim \|\nabla(v \circ M_{A,i})\|_{0,S^{need}} \leq C\frac{p}{\lambda p\varepsilon}\|v \circ M_{A,i}\|_{0,S^{need}} \sim C\frac{p}{\lambda p\varepsilon}\|v\|_{0,S^{need}},$$

where we exploited that $v \circ M_{A,i}$ is a polynomial of degree p. □

We mention already at this point that we will quantify the contributions z_1 and z_ε of this decomposition in Lemma 3.7 below. We close this section by pointing out that in our setting, one has very good control over the element maps: There exists $C > 0$ (depending solely on the asymptotic mesh Δ_A) such that

$$\|M'_{A,i}\|_{\infty,S_{ST}} + \|(M'_{A,i})^{-1}\|_{\infty,S_{ST}} \leq C, \quad i = 1, \ldots, N', \tag{3.11a}$$

$$\|(M_i^{reg})'\|_{\infty,S_{ST}} + \|((M_i^{reg})')^{-1}\|_{\infty,S_{ST}} \leq C, \quad i = 1, \ldots, n, \tag{3.11b}$$

$$\|((M_i^{need})')^{-1}\|_{\infty,S_{ST}} \leq C\frac{1}{\lambda p\varepsilon}, \quad i = 1, \ldots, n. \tag{3.11c}$$

3.2 Robust exponential convergence in balanced norms

The main result of the paper is the following robust exponential convergence in the balance norm:

Theorem 3.4. *There is a $\lambda_0 > 0$ depending only on the functions b, f and the asymptotic mesh Δ_A such that for every $\lambda \in (0, \lambda_0]$, the hp-FEM space $\mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$ leads to a finite element approximation $u_{FEM} \in \mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$ satisfying*

$$\sqrt{\varepsilon}\|\nabla(u - u_{FEM})\|_{0,\Omega} + \|u - u_{FEM}\|_{0,\Omega} \leq Ce^{-\beta p},$$

where the constants C, $\beta > 0$ depend on the choice of λ but are independent of ε and p.

The proof is deferred to the end of the section. As a corollary, we get exponential convergenence in the maximum norm.

Corollary 3.5. *Let u be the solution of (3.3) and let $u_{FEM} \in \mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$ be its finite element approximation. Then there exist constants C, $\sigma > 0$ independent of ε and p such that*

$$\|u - u_{FEM}\|_{\infty,\Omega} \leq Ce^{-\sigma p}.$$

Proof. First we note that Proposition 3.2 provides an approximation $\pi_p u \in \mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$ with

$$\|u - \pi_p u\|_{\infty,\Omega} \leq Ce^{-\beta\lambda p}.$$

In view of the triangle inequality $\|u - u_{FEM}\|_{\infty,\Omega} \leq \|u - \pi_p u\|_{\infty,\Omega} + \|\pi_p u - u_{FEM}\|_{\infty,\Omega}$, we may focus on the term $\|\pi_p - u_{FEM}\|_{\infty,\Omega}$. It suffices to prove the result in the layer region, i.e., for the elements Ω_i^{need}, since outside $\Omega_{\lambda p\varepsilon}$ standard inverse estimates (bounding the L^∞-norm of polynomials by their L^2-norm up to powers of p) yield the desired bound in view of (3.11a), (3.11b).

For a needle element Ω_i^{need} we introduce $\widetilde{\pi}_p u := \pi_p u|_{\Omega_i^{need}} \circ M_{A,i}$ and $\widetilde{u}_{FEM} := u_{FEM}|_{\Omega_i^{need}} \circ M_{A,i}$. The polynomial inverse estimate of [12, Thm. 4.76] and an affine scaling argument (between S_{ST} and S^{need}) yield

$$\begin{aligned}\|\pi_p u - u_{FEM}\|_{\infty,\Omega_i^{need}} = \|\widetilde{\pi}_p u - \widetilde{u}_{FEM}\|_{\infty,S^{need}} &\leq C\frac{p^2}{\sqrt{\lambda p\varepsilon}}\|\widetilde{\pi}_p u - \widetilde{u}_{FEM}\|_{0,S^{need}} \\ &\sim \frac{p^2}{\sqrt{\lambda p\varepsilon}}\|\pi_p u - u_{FEM}\|_{0,\Omega_i^{need}},\end{aligned}$$

where in the last step we used the assumptions on the element maps $M_{A,i}$. The triangle inequality then gives

$$\|\pi_p u - u_{FEM}\|_{\infty,\Omega_i^{need}} \leq \frac{p^2}{\sqrt{\lambda p\varepsilon}}\left[\|\pi_p u - u\|_{0,\Omega_i^{need}} + \|u - u_{FEM}\|_{0,\Omega_i^{need}}\right]. \qquad (3.12)$$

For the first term in (3.12) we obtain from the L^∞-bound of Proposition 3.2 and the fact that $|\Omega_i^{need}| \sim \lambda p\varepsilon$

$$\|\pi_p u - u\|_{0,\Omega_i^{need}} \lesssim \sqrt{\lambda p \varepsilon} e^{-\beta p}. \tag{3.13}$$

For the second term in (3.12) we exploit the fact that $u_{FEM} = 0 = \pi_p u$ on $\partial\Omega$ and a 1D Poincaré inequality. To that end, we note that for any function $\widetilde{v} \in H^1(S^{need})$ with $v = 0$ on the edge $\{(0,y) \,|\, 0 \leq y \leq 1\}$ of $S^{need} = \{(x,y) \,|\, 0 \leq x \leq \lambda p \varepsilon, 0 \leq y \leq 1\}$, we obtain from a 1D Poincáre inequality

$$\|\widetilde{v}\|_{0,S^{need}} \leq C\sqrt{\lambda p\varepsilon} \|\partial_x \widetilde{v}\|_{0,S^{need}} \leq C\sqrt{\lambda p \varepsilon}\|\nabla \widetilde{v}\|_{0,S^{need}}. \tag{3.14}$$

Upon setting $\widetilde{v} := (u - u_{FEM})|_{\Omega_i^{need}} \circ M_{A,i}$, we may use (3.14) together with the properties of $M_{A,i}$ to get

$$\|u - u_{FEM}\|_{0,\Omega_i^{need}} \sim \|\widetilde{v}\|_{0,S^{need}} \leq C\sqrt{\lambda p\varepsilon}\|\nabla\widetilde{v}\|_{0,S^{need}} \sim \sqrt{\lambda p\varepsilon}\|\nabla(u - u_{FEM})\|_{0,\Omega_i^{need}}. \tag{3.15}$$

Combining (3.12), (3.13), (3.15) gives the desired result. □

3.3 Proof of Theorem 3.4

The proof of Theorem 3.4 parallels that of the 1D case in Section 2. We begin by defining the bilinear form for the reduced problem,

$$\mathcal{B}_0(u,v) = \langle bu, v\rangle_\Omega \,. \tag{3.16}$$

We also introduce the projection operator $\mathcal{P}_0 : L^2(\Omega) \to \mathcal{S}_0^p(\Delta_{BL}(\lambda,p))$ by the condition

$$\mathcal{B}_0\left(u - \mathcal{P}_0 u, v\right) = 0 \quad \forall v \in \mathcal{S}_0^p(\Delta_{BL}(\lambda,p)).$$

Then, by reasoning as in (2.20) with Galerkin orthogonalities, we get

$$\|u_{FEM} - \mathcal{P}_0 u\|_{E,\Omega}^2 \;\; = \;\; \varepsilon^2 \left\langle \nabla\left(u - \mathcal{P}_0 u\right), \nabla\left(u_{FEM} - \mathcal{P}_0 u\right)\right\rangle_\Omega.$$

Hence

$$\varepsilon \left\|\nabla\left(u_{FEM} - \mathcal{P}_0 u\right)\right\|_{0,\Omega} \leq \|u_{FEM} - \mathcal{P}_0 u\|_{E,\Omega} \leq \varepsilon \left\|\nabla\left(u - \mathcal{P}_0 u\right)\right\|_{0,\Omega}.$$

The key step towards showing robust exponential convergence in balanced norms is therefore to show

$$\left\|\nabla\left(u - \mathcal{P}_0 u\right)\right\|_{0,\Omega} \leq C\varepsilon^{-1/2} e^{-\sigma p},$$

for some positive constants C and σ independent of ε and p. Completely analogous to the one-dimensional case, we are therefore led to studying the H^1-stability of the projection operator $\mathcal{P}_0$ on the (admissible) mesh described in Definition 3.1.

Lemma 3.6 (Strengthened Cauchy-Schwarz inequality in 2D)**.** *Let $\mathcal{B}_0$ be given by (3.16). Then,*

$$\left|\mathcal{B}_0\left(u,v\right)\right| \leq C\min\{1, \sqrt{\lambda p \varepsilon p}\} \left\|u\right\|_{0,\Omega} \left\|v\right\|_{0,\Omega_{\lambda p\varepsilon}} \quad \forall u \in S_1, \quad v \in S_\varepsilon,$$

with S_1, S_ε given by (3.5) and (3.6), respectively. The constant $C > 0$ depends solely on $\|b\|_{\infty,\Omega}$, $\inf_{x\in\Omega} b(x) > 0$, and the element maps of the asymptotic mesh Δ_A.

Proof. We restrict our attention to the case $\lambda p\varepsilon < 1/2$ as the "1" in the minimum is a simple consequence of the Cauchy-Schwarz inequality. With $u \in S_1$, $v \in S_\varepsilon$ there holds $\mathcal{B}_0(u,v) = \iint_{\Omega_{\lambda p\varepsilon}} buv$. Fix Ω_i^{need} and recall that it is obtained from an element Ω_i ($i \in \{1,\ldots,n\}$) by a splitting, i.e., $\overline{\Omega}_i = \overline{\Omega_i^{need}} \cup \overline{\Omega_i^{reg}}$. The construction of $\Delta_{BL}(\lambda,p)$ implies that the pull-back $\pi_1 := u|_{\Omega_i} \circ M_{A,i}$ to S_{ST} is a polynomial of degree p (in each variable) whereas the pull-back $\pi_2 := v|_{\Omega_i} \circ M_{A,i}$ is a piecewise polynomial of degree p (in each variable) with $\operatorname{supp} \pi_2 \subset S^{need}$. Upon setting $\widehat{b} := b|_{\Omega_i^{need}} \circ M_{A,i}$, which is uniformly bounded on S^{need}, we calculate

$$\iint_{\Omega_i} buv\,dx\,dy = \iint_{\Omega_i^{need}} buv\,dx\,dy = \iint_{S^{need}} \pi_1(x,y)\pi_2(x,y)\,\widehat{b}\,|\det M'_{A,i}|\,dx\,dy.$$

Since $|\det M'_{A,i}|$ is bounded uniformly (in (x,y)), we obtain

$$\left|\iint_{\Omega_i^{need}} buv\right| \le C \iint_{S^{need}} |\pi_1(x,y)||\pi_2(x,y)|dxdy = C\int_0^1\int_0^{\lambda p\varepsilon} |\pi_1(x,y)||\pi_2(x,y)|dxdy.$$

Now, fix $y \in [0,1]$ and consider

$$\int_0^{\lambda p\varepsilon} |\pi_1(x,y)||\pi_2(x,y)|dx \le Cp\sqrt{\lambda p\varepsilon}\left[\int_0^1 |\pi_1(x,y)|^2 dx\right]^{1/2}\left[\int_0^{\lambda p\varepsilon} |\pi_2(x,y)|^2 dx\right]^{1/2}$$

by Lemma 2.8. Integrating in y from 0 to 1, gives

$$\int_0^1\int_0^{\lambda p\varepsilon} |\pi_1(x,y)||\pi_2(x,y)|dxdy \le Cp\sqrt{\lambda p\varepsilon}\int_0^1 \left[\int_0^1 |\pi_1(x,y)|^2 dx\right]^{1/2}\left[\int_0^{\lambda p\varepsilon} |\pi_2(x,y)|^2 dx\right]^{1/2} dy.$$

Using once more the Cauchy-Schwarz inequality, we arrive at

$$\iint_{S^{need}} |\pi_1(x,y)||\pi_2(x,y)|dxdy \le Cp\sqrt{\lambda p\varepsilon}\|\pi_1\|_{0,S_{ST}}\|\pi_2\|_{0,S^{need}}.$$

The assumptions on the element map $M_{A,i}$ allows us to infer $\|\pi_1\|_{0,S_{ST}}\|\pi_2\|_{0,S^{need}} \sim \|u\|_{0,\Omega_i}\|v\|_{0,\Omega_i^{need}}$, which concludes the proof. ∎

Lemma 3.7. *There exist constants C, $c > 0$ depending solely on $\|b\|_{\infty,\Omega}$, $\inf_{x\in\Omega} b(x) > 0$, and the element maps of the asymptotic mesh Δ_A such that the following is true under the assumption*

$$\sqrt{\lambda p\varepsilon}p \le c: \tag{3.17}$$

For each $z \in L^2(\Omega)$, the (unique) decomposition

$$\mathcal{P}_0 z = z_1 + z_\varepsilon$$

into the components $z_1 \in S_1$ and $z_\varepsilon \in S_\varepsilon$ satisfies

$$\|z_1\|_{0,\Omega} \le C\|z\|_{0,\Omega}, \tag{3.18}$$

$$\|z_\varepsilon\|_{0,\Omega} \le C\{\|z\|_{0,\Omega_{\lambda p\varepsilon}} + \sqrt{\lambda p\varepsilon}p\|z\|_{0,\Omega}\}. \tag{3.19}$$

Proof. The proof parallels that of Lemma 2.9. With Lemma 3.3 we can write $\mathcal{P}_0 z = z_1 + z_\varepsilon$. We define the auxiliary function ψ_ε on S_{ST} by

$$\psi_\varepsilon(x,y) := \begin{cases} \left(1 - \frac{2x}{\lambda p \varepsilon}\right)^p & \text{if } (x,y) \in S^{need} \\ 0 & \text{otherwise.} \end{cases}$$

Then supp $\psi_\varepsilon \subset S^{need}$, $\psi_\varepsilon(0,y) = 1$ and $\|\psi_\varepsilon\|_{0,S_{ST}} = \|\psi_\varepsilon\|_{0,S^{need}} \sim p^{-1/2}\sqrt{\lambda p \varepsilon}$. We define the function $\widetilde{z}_\varepsilon \in S_\varepsilon$ on the needle elements Ω_i^{need} by prescribing its pull-back to S^{need}:

$$(\widetilde{z}_\varepsilon|_{\Omega_i^{need}} \circ M_{A,i})(x,y) := (z_\varepsilon|_{\Omega_i^{need}} \circ M_{A,i})(x,y) + \psi_\varepsilon(x,y)(z_1|_{\Omega_i} \circ M_{A,i})(0,y), \qquad (x,y) \in S^{need};$$

here, Ω_i and Ω_i^{need} are related to each other by $\overline{\Omega_i} = \overline{\Omega_i^{need}} \cup \overline{\Omega_i^{reg}}$. It is an effect of the assumptions on the asymptotic mesh Δ_A that the elementwise defined function $\widetilde{z}_\varepsilon$ is in fact in $H^1(\Omega)$ and therefore indeed $z_\varepsilon \in S_\varepsilon$. By construction, $\widetilde{z}_\varepsilon|_{\partial\Omega} = (z_1 + z_\varepsilon)|_{\partial\Omega} = (\mathcal{P}_0 z)|_{\partial\Omega} = 0$ so that $\widetilde{z}_\varepsilon \in S_\varepsilon \cap S_0(\lambda, p)$. Noting the product structure of $(z_\varepsilon - \widetilde{z}_\varepsilon)|_{\Omega_i^{need}} \circ M_{A,i}$ on S^{need} and the above estimate on $\|\psi_\varepsilon\|_{0,S^{need}}$, we get for $\widetilde{z}_\varepsilon$ with the inverse estimate (3.8)

$$\|\widetilde{z}_\varepsilon\|_{0,\Omega} = \|\widetilde{z}_\varepsilon\|_{0,\Omega_{\lambda p\varepsilon}} \le C\left\{\|z_\varepsilon\|_{0,\Omega_{\lambda\varepsilon}} + p^{1/2}\sqrt{\lambda p \varepsilon}\,\|z_1\|_{0,\Omega}\right\}.$$

We also have in view of $\mathcal{P}_0 z = z_1 + z_\varepsilon$

$$\begin{aligned} B_0(z_1, v_1) + B_0(z_\varepsilon, v_1) &= B_0(\mathcal{P}_0 z, v_1) \quad \forall v_1 \in S_1, & (3.20)\\ B_0(z_1, v_\varepsilon) + B_0(z_\varepsilon, v_\varepsilon) &= B_0(\mathcal{P}_0 z, v_\varepsilon) = B_0(z, v_\varepsilon) \quad \forall v_\varepsilon \in S_\varepsilon \cap \mathcal{S}_0^p\left(\Delta_{BL}(\lambda,p)\right), & (3.21) \end{aligned}$$

where in (3.21) we used the fact that $\mathcal{P}_0$ is the $\mathcal{B}_0$–projection onto $\mathcal{S}_0^p\left(\Delta_{BL}(\lambda,p)\right)$. Taking $v_1 = z_1$ in (3.20) and $v_\varepsilon = \widetilde{z}_\varepsilon \in S_\varepsilon \cap \mathcal{S}_0^p\left(\Delta_{BL}(\lambda,p)\right)$ in (3.21) yields, together with the Strengthened Cauchy Schwarz inequality of Lemma 2.9, just like in the 1D case

$$\begin{aligned} \|z_1\|_{0,\Omega}^2 &\le C\left[\|\mathcal{P}_0 z\|_{0,\Omega}\|z_1\|_{0,\Omega} + \sqrt{\lambda p\varepsilon}p\|z_\varepsilon\|_{0,\Omega}\|z_1\|_{0,\Omega}\right] \\ \|z_\varepsilon\|_{0,\Omega}^2 &\le C\left[\|z\|_{0,\Omega}\|\widetilde{z}_\varepsilon\|_{0,\Omega} + \sqrt{\lambda p\varepsilon}p\|\widetilde{z}_\varepsilon\|_{0,\Omega}\|z_1\|_{0,\Omega} + \|z_\varepsilon\|_{0,\Omega}\|z_1\|_{0,\Omega}\sqrt{\lambda p\varepsilon}p^{1/2}\right], \\ &\le C\Big[\|z_\varepsilon\|_{0,\Omega_{\lambda p\varepsilon}}\{\|z\|_{0,\Omega_{\lambda p\varepsilon}} + \sqrt{\lambda p\varepsilon}p\|z_1\|_{0,\Omega} + \sqrt{\lambda p\varepsilon}p\|z_1\|_{0,\Omega}\} \\ &\qquad + \{\|z\|_{0,\Omega_{\lambda p\varepsilon}} + \sqrt{\lambda p\varepsilon}p\|z_1\|_{0,\Omega}\}\sqrt{\lambda p\varepsilon}p^{1/2}\|z_1\|_{0,\Omega}\Big]. \end{aligned}$$

Estimating $\sqrt{\lambda p\varepsilon}p^{1/2} \le \sqrt{\lambda p\varepsilon}p$ and using an appropriate Young inequality we get

$$\begin{aligned} \|z_1\|_{0,\Omega} &\le C\left[\|\mathcal{P}_0 z\|_{0,\Omega} + \sqrt{\lambda p\varepsilon}p\|z_\varepsilon\|_{0,\Omega}\right], & (3.22a)\\ \|z_\varepsilon\|_{0,\Omega} &\le C\left[\|z\|_{0,\Omega_{\lambda p\varepsilon}} + \sqrt{\lambda p\varepsilon}p\|z_1\|_{0,\Omega}\right]. & (3.22b) \end{aligned}$$

Inserting (3.22b) in (3.22a), assuming that $\sqrt{\lambda p\varepsilon}p$ is sufficiently small and using the stability $\|\mathcal{P}_0 z\|_{0,\Omega} \le C\|z\|_{0,\Omega}$ gives $\|z_1\|_{0,\Omega} \le C\|z\|_{0,\Omega}$. Inserting this bound in (3.22b) concludes the proof. □

We are now in the position to prove the following

Lemma 3.8. *Assume (3.2) and let u be the solution of (3.3). Let $\lambda_0 > 0$ be given by Proposition 3.2. Assume that $\lambda \le \lambda_0$ and that λ, p, ε satisfy (3.17). Then, for constants C, $\beta > 0$ independent of ε and p (but depending on λ)*

$$\|\nabla(u - \mathcal{P}_0 u)\|_{0,\Omega} \le C\varepsilon^{-1/2} e^{-\beta p}. \tag{3.23}$$

Proof. By Proposition 3.2 we can find an approximation $\pi_p u \in \mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$ with $(u - \pi_p u)|_{\partial\Omega} = 0$ such that

$$\sqrt{\varepsilon}\,|u - \pi_p u|_{1,\Omega} \le Cp^2 (\ln p + 1)^2 e^{-\beta\lambda p}.$$

Since $\mathcal{P}_0(u - \pi_p u) \in \mathcal{S}_0^p(\Delta_{BL}(\lambda, p))$, we decompose $\mathcal{P}_0(u - \pi_p u) = z_1 + z_\varepsilon$ and use the inverse estimates (3.9), (3.10) to get, with Lemma 3.7,

$$\begin{aligned} |z_1|_{1,\Omega} &\lesssim p^2 \|z_1\|_{0,\Omega} \lesssim p^2 \|u - \pi_p u\|_{0,\Omega} \lesssim Ce^{-bp}, & (3.24)\\ |z_\varepsilon|_{1,\Omega} &\lesssim \frac{p^2}{\lambda p\varepsilon} \|z_\varepsilon\|_{0,\Omega_{\lambda p\varepsilon}} \lesssim \frac{p^2}{\lambda p\varepsilon} \left[\|u - \pi_p u\|_{0,\Omega_{\lambda p\varepsilon}} + \sqrt{\lambda p\varepsilon} p \|u - \pi_p u\|_{0,\Omega} \right]. & (3.25) \end{aligned}$$

Let us treat the term $\|u - \pi_p u\|_{0,\Omega_{\lambda p\varepsilon}}$ above. Recall that $\overline{\Omega_{\lambda p\varepsilon}} = \cup_{i=1}^n \overline{\Omega_i^{need}}$; from (3.13) we therefore get $\|u - \pi_p u\|_{0,\Omega_{\lambda p\varepsilon}} \lesssim \sqrt{\lambda p\varepsilon} e^{-\beta p}$. Furthermore, from Proposition 3.2 we readily have $\|u - \pi_p u\|_{0,\Omega} \lesssim e^{-\beta p}$. Inserting these two estimates into (3.25) produces

$$|z_\varepsilon|_{1,\Omega} \lesssim \frac{p^2}{\lambda p\varepsilon} \sqrt{\lambda p\varepsilon} e^{-\beta p} + \sqrt{\lambda p\varepsilon p} e^{-\beta p} \lesssim \varepsilon^{-1/2} e^{-\beta p},$$

where the constant $\beta > 0$ is suitably adjusted in each estimate. The result follows. □

Proof of Theorem 3.4: Again, we focus only on the control of $\sqrt{\varepsilon}\|\nabla(u - u_{FEM})\|_{0,\Omega}$. We distinguish two cases:

Case 1: Assume that (3.17) is satisfied. Then (3.23) and Lemma 2.10 yield the result.

Case 2: Assume (3.17) is not satisfied. Then $\varepsilon \ge c^2 p^{-3} \lambda^{-1}$ so that

$$\sqrt{\varepsilon}\|\nabla(u - u_N)\|_{0,\Omega} \le \varepsilon^{-1/2} \|u - u_N\|_{E,\Omega} \le \frac{1}{c} \sqrt{\lambda} p^{3/2} \|u - u_N\|_{E,\Omega} \lesssim e^{-bp}.$$

□

3.4 Numerical example

We close with a numerical example in two dimensions: We consider the problem

$$\begin{aligned} -\varepsilon^2 \Delta u + u &= 1 \quad \text{in } \Omega := \left\{ (x, y) \,|\, 0 \le \left(\frac{x}{2}\right)^2 + y^2 < 1 \right\} \subset \mathbb{R}^2, & (3.26a)\\ u &= 0 \quad \text{on } \partial\Omega, & (3.26b) \end{aligned}$$

Deviating from the use of the boundary layer mesh $\Delta_{BL}(\lambda, p)$, we approximate the solution to this problem on a *fixed* mesh where the needle elements have width $p_{max}\varepsilon$ on the semiaxes of the ellipse, as shown in Figure 3. On this fixed mesh, we employ the p-version FEM with degrees $p = 1, \ldots, p_{max} - 1$. The reference solution, with which the FEM solutions are compared is taken as the FEM solution corresponding to $p = p_{max}$. Throughout, we take $p_{max} = 8$ and we utilize the commercial FEM code StressCheck (E.S.R.D. St. Louis, MO), which is a p-version package allowing for the polynomial degree to vary from 1 to 8.

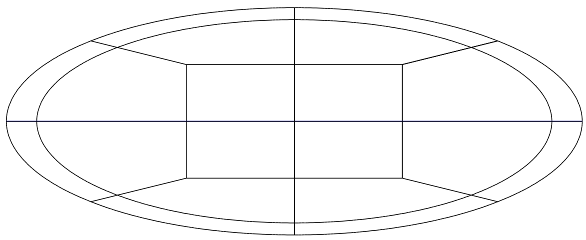

Figure 3: Mesh used for the two-dimensional example.

In Figures 4 we present the error

$$\max_{1 \le i \le M} |u(r_i) - u_{FEM}(r_i)|, \qquad M := 20,$$

versus the polynomial degree p, in a semi-log scale. The M points r_i were uniformly distributed first on the mesh line connecting the points $(1 - p_{max}\varepsilon, 0), (1, 0)$, and second on line of length of approximately $p_{max}\varepsilon$, which is the intersection of the needle element in the first quadrant and the line through the origin at an angle of 60 degrees. Figure 4 clearly shows the robust exponential convergence in the $L^\infty(\Omega)$-norm of the hp-FEM.

References

[1] N. S. Bakhvalov, *Towards optimization of methods for solving boundary value problems in the presence of boundary layers*, (in Russian), Zh. Vychisl. Mat. Mat. Fiz. Vol. 9, pp. 841–859 (1969).

[2] R. Lin and M. Stynes, *A balanced finite element method for singularly perturbed reaction-diffusion problems*, SIAM J. Numer. Anal., Vol. 50, no.5, pp. 2729–2743 (2012).

[3] J. M. Melenk, *On the robust exponential convergence of hp finite element methods for problems with boundary layers*, IMA J. Num. Anal., Vol. 17, pp. 577 – 601 (1997).

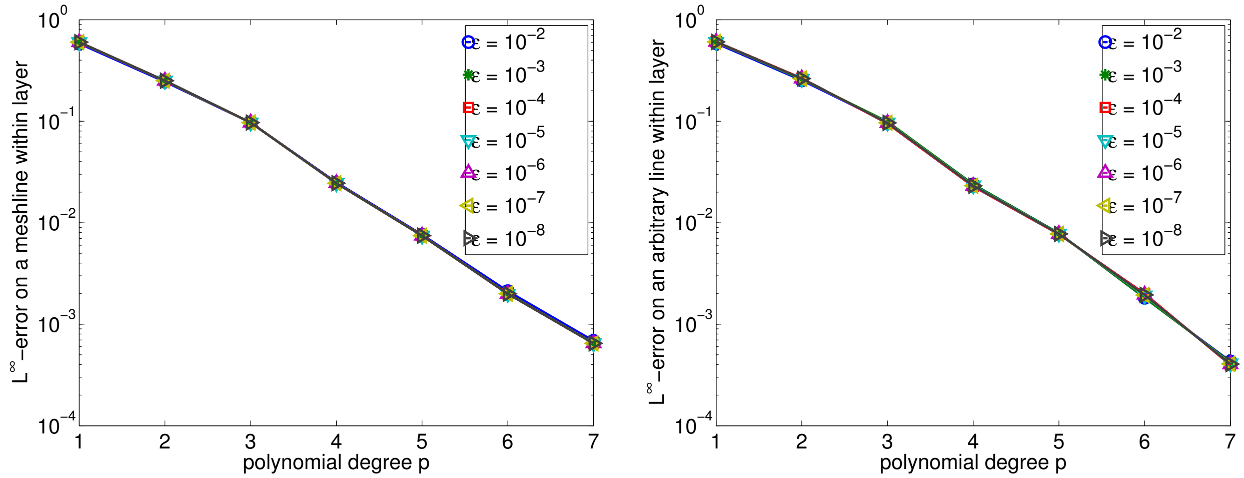

Figure 4: Maximum norm convergence of the *hp*-FEM. Left: maximum error convergence on a meshline within the layer. Right: maximum error convergence on a line within the layer that is not a meshline.

[4] J. M. Melenk, *hp-Finite Element Methods for Singular Perturbations*, Vol. 1796 of Springer Lecture Notes in Mathematics, Springer Verlag, 2002.

[5] M. J. Melenk, C. Xenophontos and L. Oberbroeckling, *Robust exponential convergence of hp-FEM for singularly perturbed systems of reaction-diffusion equations with multiple scales*, IMA J. Num. Anal., Vol. 33., No 2, pp. 609–628, (2013).

[6] M. J. Melenk, C. Xenophontos and L. Oberbroeckling, *Analytic regularity for a singularly perturbed system of reaction-diffusion equations with multiple scales*, Advances in Computational Mathematics, Vol. 39, pp. 367–394 (2013).

[7] J. M. Melenk and C. Schwab, *hp FEM for reaction diffusion equations I: Robust exponential convergence*, SIAM J. Num. Anal., Vol. 35, pp 1520–1557 (1998).

[8] J. J. Miller, E. O'Riordan, G. I. Shishkin, *Fitted Numerical Methods For Singular Perturbation Problems*, World Scientific, 1996.

[9] H. G. Roos and S. Franz, *Error estimation in a balanced norm for a convection-diffusion problems with two different boundary layers*, in press in *Calcolo*

[10] H. G. Roos and M. Schopf, *Convergence and stability in balanced norms of finite element methods on Shishkin meshes for reaction-diffusion problems*, in press in *ZAMM*

[11] H. G. Roos, M. Stynes and L. Tobiska, *Robust numerical methods for singularly perturbed differential equations*, Volume 24 of *Springer Series in Computational Mathematics*, Springer-Verlag, Berlin, 2008.

[12] C. Schwab, *p/hp Finite Element Methods*, Oxford University Press, 1998.

[13] C. Schwab and M. Suri, *The p and hp versions of the finite element method for problems with boundary layers*, Math. Comp. **65**,pp. 1403–1429 (1996).

[14] G. I. Shishkin, *Grid approximation of singularly perturbed boundary value problems with a regular boundary layer*, Sov. J. Numer. Anal. Math. Model. Vol. 4, pp. 397–417 (1989).